EE4013 Assignment - 1

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Contents

Problem

Consider the following matrix:

$$X = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix}$$

Calculate the absolute value of the product of the Eigen values of X.

Solution-1

We use the following property -

$$\prod_{i=0}^{k-1} \lambda_i = det(X)$$

where λ_i are the Eigen values and $det(.)$ is the determinant operator.
To find the determinant, we use the following recurrence relation.

$$det(X, n) = \sum_{j=0}^{n-1} (-1)^j \times X[0][j] \times det(cof(X, 0, j), n-1)$$

where $cof(i, j)$ is the cofactor matrix of the position (i, j) in the matrix X.

Flow of the recursion

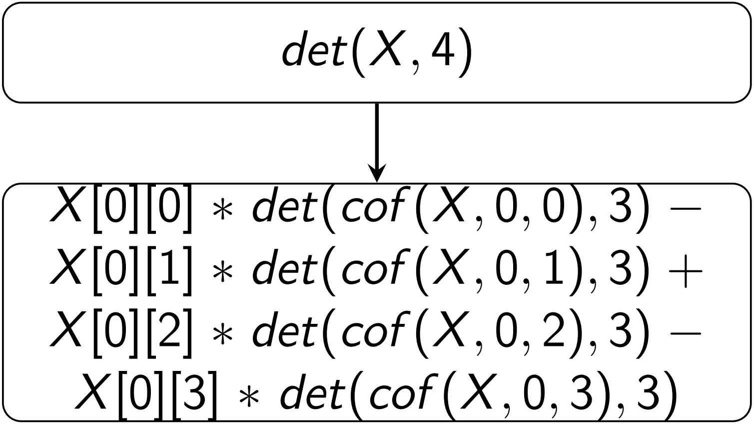

Figure 1: The flow of recursion

Sub-problems

$$det(C_1, 3)$$

$$\downarrow$$

$$C_1[0][0] * det(cof(C_1, 0, 0), 2) - C_1[0][1] * det(cof(C_1, 0, 1), 2) + C_1[0][2] * det(cof(C_1, 0, 2), 2)$$

Figure 2: A 3×3 sub-problem

$$det(C_2, 2)$$

$$\downarrow$$

$$C_2[0][0] * det(cof(C_2, 0, 0), 1) - C_2[0][1] * det(cof(C_2, 0, 1), 1)$$

Figure 3: A 2×2 sub-problem

Complexity

At each stage of recursion of depth k, there are $O(n-k)$ recursive calls. Hence, the time complexity is

$$O(n \times (n-1) \times .... \times 1) = O(n!)$$

Solution-2

In this solution, we use the QR-algorithm to compute the Eigen values. The algorithm is as follows-

- Apply the QR decomposition so that

$$X = Q_0 R_0$$

  where Q_0 is an orthogonal matrix and R_0 is an upper-triangular matrix.
- Compute

$$E_0 = Q_0^{-1} X Q_0 = Q_0^T X Q_0$$

  Note that the Eigen values of X and E_0 are the same.
- Next, we decompose the E_0 and repeat this procedure till we obtain a triangular matrix. The Eigen values of a triangular matrix are given by the diagonal entries of the matrix.

Householder Transformation

The reflection hyperplane can be defined by a unit vector u that is orthogonal to the hyperplane, which is its normal vector. The reflection of a vector x about this hyperplane is the linear transformation:

$$\begin{aligned} y &= x - 2(u^T x)u \\ &= x - 2u(u^T x) \\ &= x - 2(uu^T)x \\ &= (I - 2uu^T)x \end{aligned}$$

The matrix constructed from this transformation can be expressed in terms of an outer product as:

$$H = H = I - 2uu^T$$

is called as the **Householder matrix**.

QR decomposition

We compute the QR decomposition using the Householder transformation. Multiplying a given vector x, for example the first column of matrix X, with the Householder matrix H, which is given as

$$H = I - \frac{2}{u^T u} u u^T$$

reflects x about a plane given by its normal vector u. When the normal vector of the plane u is given as

$$u = x - \|x\|_2 e_1$$

then the transformation reflects x onto the first standard basis vector

$$e_1 = \begin{bmatrix} 1 & 0 & 0 & \dots \end{bmatrix}^T$$

which means that all entries but the first become zero.

Algorithm

In general, if we take $u = x - s\|x\|e_1$ where $s = \pm 1$ and $v = u/\|u\|$ then

$$Hx = \left(I - 2\frac{uu^T}{u^Tu}\right)x = s\|x\|e_1.$$

Let us first verify that this works:

$$\begin{aligned}
u^Tx &= (x - s\|x\|e_1)^T x \\
&= \|x\|^2 - sx_1\|x\| \\
u^Tu &= (x - s\|x\|e_1)^T (x - s\|x\|e_1) \\
&= \|x\|^2 - 2sx_1\|x\| + \|x\|^2\|e_1\|^2 \\
&= 2\left(\|x\|^2 - sx_1\|x\|\right) = 2u^Tx \\
Hx &= x - 2u\frac{u^Tx}{u^Tu} = x - u = s\|x\|e_1.
\end{aligned}$$

Algorithm

As a byproduct of this calculation, note that we have

$$u^T u = -2s\|x\|u_1 = -2s\|x\|(x_1 - s\|x\|)$$

Hence, to avoid numerical cancellation errors, we take $s = -sign(x_1)$.

$$\implies Hx = -sign(x_1)\|x\|e_1$$

First, we multiply X with the Householder matrix H_1 we obtain when we choose the first matrix column for X. This results in a matrix H_1X with zeros in the left column (except for the first row).

Algorithm

$$H_1 X = \begin{bmatrix} -sign(X_{11})\alpha_1 & \star & \cdots & \star \\ 0 & & & \\ \vdots & & X' & \\ 0 & & & \end{bmatrix}$$

where α_1 is the 2-norm of the first column of X.
This can be repeated for X' (obtained from $H_1 X$ by deleting the first row and first column), resulting in a Householder matrix H'_2. Note that H'_2 is smaller than H_1. Since we want it really to operate on $H_1 X$ instead of X' we need to expand it to the upper left, filling in a 1 , or in general:

$$H_k = \begin{bmatrix} I_{k-1} & 0 \\ 0 & H'_k \end{bmatrix}$$

Algorithm

This is how we can, column by column, remove all sub-diagonal elements of A and thus transform it into R.

$$H_n \dots H_3 H_2 H_1 X = R$$

The product of all the Householder matrices H, for every column, in reverse order, will then yield the orthogonal matrix Q.

$$H_1 H_2 H_3 \dots H_n = Q$$

Complexity of the algorithm

The QR decomposition step takes $O(n^4)$ computations, because the matrix multiplication at each iteration takes $O(n^3)$ time, and there are $O(n)$ iterations.
The matrix convergence is achieved in $O(\max_{i,j} |\lambda_i/\lambda_j|^n)$ steps.
Hence the total time complexity is $O((\max_{i,j} |\lambda_i/\lambda_j|^n) \times n^4)$.