

# EE4013 Assignment - 1

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# Contents

① Problem

② Solution-1

③ Solution-2

# Problem

Consider the following matrix:

$$X = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \end{bmatrix}$$

Calculate the absolute value of the product of the Eigen values of  $X$ .

# Solution-1

We use the following property -

$$\prod_{i=0}^{k-1} \lambda_i = \det(X)$$

where  $\lambda_i$  are the Eigen values and  $\det(.)$  is the determinant operator.

To find the determinant, we use the following recurrence relation.

$$\det(X, n) = \sum_{j=0}^{n-1} (-1)^j \times X[0][j] \times \det(\text{cof}(X, 0, j), n-1)$$

where  $\text{cof}(i, j)$  is the cofactor matrix of the position  $(i, j)$  in the matrix  $X$ .

# Flow of the recursion

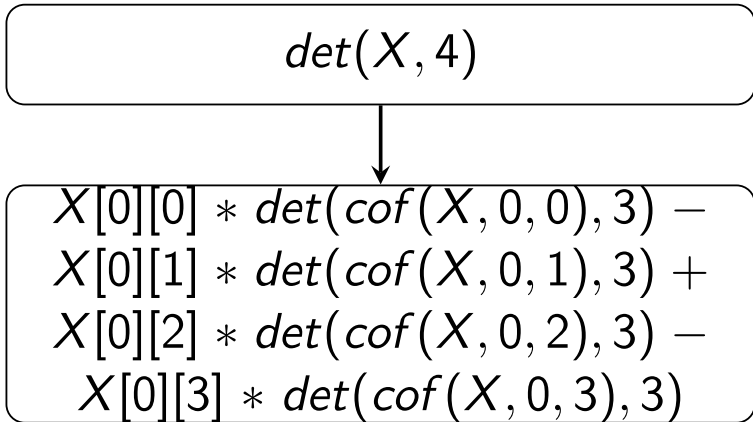


Figure 1: The flow of recursion

# Sub-problems

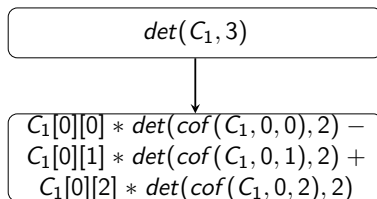


Figure 2: A  $3 \times 3$  sub-problem

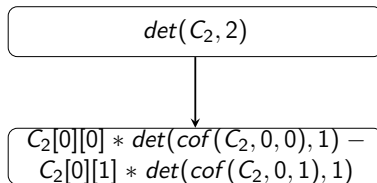


Figure 3: A  $2 \times 2$  sub-problem

# Complexity

At each stage of recursion, there are  $O(n)$  recursive calls. In each of these function calls, we compute the co-factor matrix in  $O(n^2)$  time. So, the total time complexity is  $O(n^3)$ .

## Solution-2

In this solution, we use the QR-algorithm to compute the Eigen values. The algorithm is as follows-

- Apply the QR decomposition so that

$$X = Q_0 R_0$$

where  $Q_0$  is an orthogonal matrix and  $R_0$  is an upper-triangular matrix.

- Compute

$$E_0 = Q_0^{-1} X Q_0 = Q_0^T X Q_0$$

Note that the Eigen values of  $X$  and  $E_0$  are the same.

- Next, we decompose the  $E_0$  and repeat this procedure till we obtain a triangular matrix. The Eigen values of a triangular matrix are given by the diagonal entries of the matrix.



## Householder Transformation

The reflection hyperplane can be defined by a unit vector  $u$  that is orthogonal to the hyperplane, which is its normal vector. The reflection of a vector  $x$  about this hyperplane is the linear transformation:

$$\begin{aligned}y &= x - 2(u^T x)u \\&= x - 2u(u^T x) \\&= x - 2(uu^T)x \\&= (I - 2uu^T)x\end{aligned}$$

The matrix constructed from this transformation can be expressed in terms of an outer product as:

$$H = H = I - 2uu^T$$

is called as the **Householder matrix**.

## QR decomposition

We compute the QR decomposition using the Householder transformation. Multiplying a given vector  $x$ , for example the first column of matrix  $X$ , with the Householder matrix  $H$ , which is given as

$$H = I - \frac{2}{u^T u} u u^T$$

reflects  $x$  about a plane given by its normal vector  $u$ . When the normal vector of the plane  $u$  is given as

$$u = x - \|x\|_2 e_1$$

then the transformation reflects  $x$  onto the first standard basis vector

$$e_1 = \begin{bmatrix} 1 & 0 & 0 & \dots \end{bmatrix}^T$$

which means that all entries but the first become zero.

# Algorithm

In general, if we take  $u = x - s\|x\|e_1$  where  $s = \pm 1$  and  $v = u/\|u\|$  then

$$Hx = \left( I - 2 \frac{uu^T}{u^T u} \right) x = s\|x\|e_1.$$

Let us first verify that this works:

$$\begin{aligned} u^T x &= (x - s\|x\|e_1)^T x \\ &= \|x\|^2 - sx_1\|x\| \\ u^T u &= (x - s\|x\|e_1)^T (x - s\|x\|e_1) \\ &= \|x\|^2 - 2sx_1\|x\| + \|x\|^2 \|e_1\|^2 \\ &= 2(\|x\|^2 - sx_1\|x\|) = 2u^T x \\ Hx &= x - 2u \frac{u^T x}{u^T u} = x - u = s\|x\|e_1. \end{aligned}$$

# Algorithm

As a byproduct of this calculation, note that we have

$$u^T u = -2s\|x\|u_1 = -2s\|x\|(x_1 - s\|x\|)$$

Hence, to avoid numerical cancellation errors, we take  $s = -\text{sign}(x_1)$ .

$$\implies Hx = -\text{sign}(x_1)\|x\|e_1$$

First, we multiply  $X$  with the Householder matrix  $H_1$  we obtain when we choose the first matrix column for  $X$ . This results in a matrix  $H_1X$  with zeros in the left column (except for the first row).

# Algorithm

$$H_1 X = \begin{bmatrix} -\text{sign}(X_{11})\alpha_1 & \star & \cdots & \star \\ 0 & & & \\ \vdots & & X' & \\ 0 & & & \end{bmatrix}$$

where  $\alpha_1$  is the 2-norm of the first column of  $X$ .

This can be repeated for  $X'$  (obtained from  $H_1 X$  by deleting the first row and first column), resulting in a Householder matrix  $H'_2$ .

Note that  $H'_2$  is smaller than  $H_1$ . Since we want it really to operate on  $H_1 X$  instead of  $X'$  we need to expand it to the upper left, filling in a 1, or in general:

$$H_k = \begin{bmatrix} I_{k-1} & 0 \\ 0 & H'_k \end{bmatrix}$$

# Algorithm

This is how we can, column by column, remove all sub-diagonal elements of  $A$  and thus transform it into  $R$ .

$$H_n \dots H_3 H_2 H_1 X = R$$

The product of all the Householder matrices  $H$ , for every column, in reverse order, will then yield the orthogonal matrix  $Q$ .

$$H_1 H_2 H_3 \dots H_n = Q$$

# Complexity of the algorithm

The QR decomposition step takes  $O(n^3)$  computations, because the number of multiplications at each iteration is  $O(n^2)$ , and there are  $O(n)$  iterations.

The matrix convergence is achieved in  $O(\max_{i,j} |\lambda_i/\lambda_j|^n)$  steps. Hence the total time complexity is  $O((\max_{i,j} |\lambda_i/\lambda_j|^n) \times n^3)$ .