Solving 2D Geometry problems using Matrices

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Theory

- The equation of a line passing through a point 'A' and normal to a vector 'n', in matrix form is−

$$n^T(x - A) = 0 \tag{1}$$

  - The foot of the perpendicular drawn from the centre of a circle onto any chord is the mid−point of the chord. Conversely, the line joining the centre of a circle and the midpoint of a chord is perpendicular to the chord.

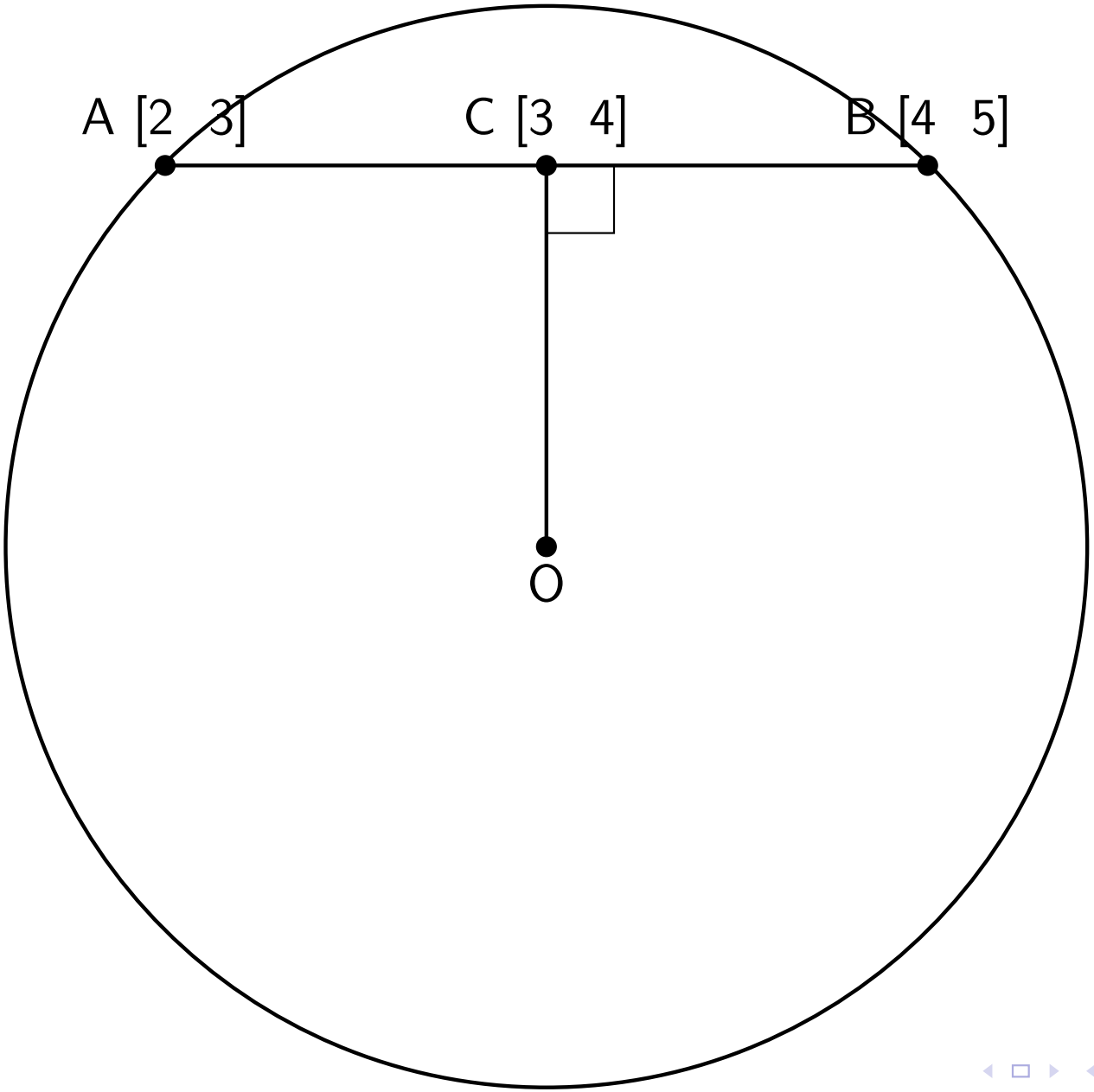
- The solution of the linear equations defined by Ax = b (where A is invertible),is

$$x = A^{-1}b \tag{2}$$

Problem

A circle passes through the points $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}4\\5\end{bmatrix}$. If its centre lies on the line $\begin{bmatrix}-1 & 4\end{bmatrix}\mathbf{x} + 3 = 0$, find its radius.

Schematic Diagram



Theoretical solution

Let A $= \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and B $= \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. Let the centre of the circle be O. The mid-point of the chord AB is C $= \frac{(A+B)}{2} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Let AB = B - A (direction vector),
which gives AB $= \begin{bmatrix} 2 & 2 \end{bmatrix}$
From the stated theory, the line joining C and O is normal to the chord AB. The equation of OC is thus -

$$AB^T(x - C) = 0. \tag{3}$$

$$\text{which gives, } \begin{bmatrix} 2 & 2 \end{bmatrix} x = 14.$$

It is given that the centre of the circle lies on the line

$$[-1\ 4]x + 3 = 0.$$

The centre is the point of intersection of OC and

$$[-1\ 4]x + 3 = 0.$$

Let $\mathrm{P} = \begin{bmatrix} -1 & 4 \\ 2 & 2 \end{bmatrix}$. Writing the equations in matrix form, we get

$$\mathrm{P\ x} = \begin{bmatrix} -3 \\ 14 \end{bmatrix}.$$

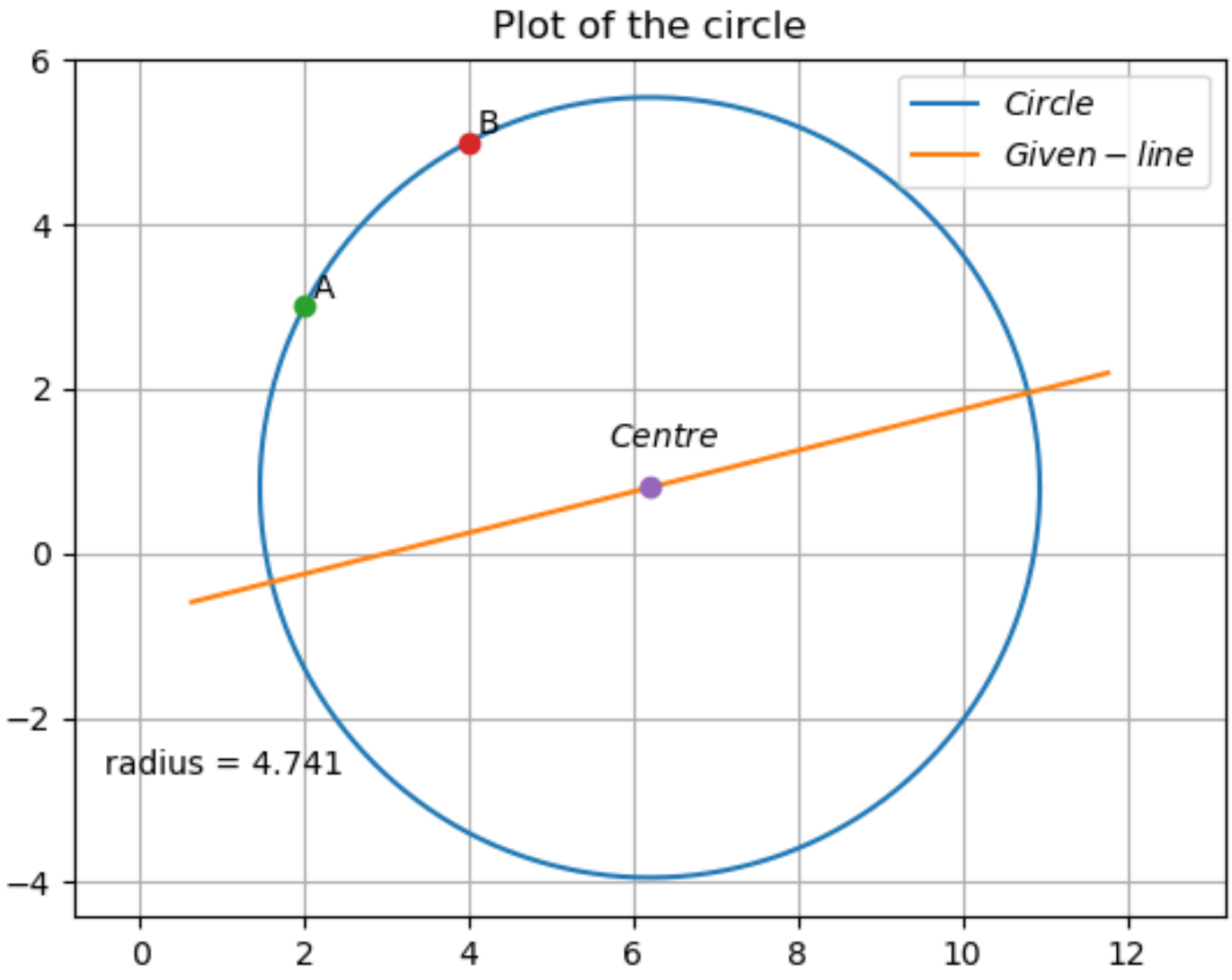
$$x = P^{-1}b \tag{4}$$

(*since P is invertible.*) $\mathrm{x} = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ 14 \end{bmatrix}$, $\mathrm{x} = \begin{bmatrix} 6.2 \\ 0.8 \end{bmatrix}$

The obtained solution is nothing but 'O'.
The radius can be obtained by computing the norm of (O - A) or (O - B).

$$\text{Radius} = ||(O - A)|| = ||[4.2 \ \ -2.2]|| = 4.741 \text{ units}$$

Plotting


Plot of the circle
Circle
Given – line
A
B
Centre
radius = 4.741
6
4
2
0
−2
−4
0
2
4
6
8
10
12