7.5.1 证明

$$(Cu)^\wedge = Cu^\wedge C^T$$

参考PPT， $CA \times CB = C(A \times B)$

$$(Cu)^\wedge v = (Cu)\times v = (Cu)\times(CC^Tv)$$
$$= C(u \times C^T v) = Cu^\wedge C^T v$$

7.5.2 证明：

$$(Cu)^\wedge = (2\cos\phi+1)u^\wedge - u^\wedge C - C^T u^\wedge$$

教材表7-2李代数参考公式

$$(Wu)^\wedge = u^\wedge(tr(W)I - W) - W^T u^\wedge$$

左侧 $(Cu)^\wedge = tr(C)u^\wedge - u^\wedge C - C^T u^\wedge$

由于 $C = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow tr(C) = 2\cos\theta + 1$

故 $tr(C) = 2\cos\phi + 1$

上式与原式右侧相等

7.5.3 证明

$$\exp((Cu)^\wedge) = C\exp(u^\wedge)C^T$$

由公式(7.19)完成指数形式的变化 $\exp(A) = 1 + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}A^n$

$$\exp((Cu)^\wedge) = \sum_{n=0}^{\infty}\frac{1}{n!}[(Cu)^\wedge]^n = \sum_{n=0}^{\infty}\frac{1}{n!}[Cu^\wedge C^T]^n$$

因为 $[Cu^\wedge C^T]^n = (Cu^\wedge \underbrace{C^T)(C}_{I} u^\wedge \underbrace{C^T)(C}_{I} u^\wedge \underbrace{C^T)}_{I} \cdots \underbrace{(C}_{I} u^\wedge C^T) = C(u^\wedge)^n C^T$

故 $\exp((Cu)^\wedge) = \sum_{n=0}^{\infty}\frac{1}{n!}C(u^\wedge)^n C^T = C\underbrace{\sum_{n=0}^{\infty}\frac{1}{n!}(u^\wedge)^n}_{\exp(u^\wedge)}C^T$

证明: $(\mathcal{T}x)^\wedge = T x^\wedge T^{-1}$

$$T = \begin{bmatrix} C & J\rho \\ 0 & 1 \end{bmatrix},\quad T^{-1} = \begin{bmatrix} C^T & -C^T r \\ 0 & 1 \end{bmatrix},\quad \mathcal{T} = Ad(T) = \begin{bmatrix} C & (J\rho)^\wedge C \\ 0 & C \end{bmatrix}$$

等式左侧:

$$(\mathcal{T}x)^\wedge = \left(\begin{bmatrix} C & (J\rho)^\wedge C \\ 0 & C \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}\right)^\wedge = \begin{bmatrix} Cu + (J\rho)^\wedge C v \\ Cv \end{bmatrix}^\wedge = \begin{bmatrix} (Cv)^\wedge & Cu + (J\rho)^\wedge C v \\ 0 & 0 \end{bmatrix}$$

等式右侧

$$(T x^\wedge T^{-1}) = \begin{bmatrix} C & J\rho \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}^\wedge \begin{bmatrix} C^T & -C^T r \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (Cv)^\wedge & -Cv^\wedge C^T r + Cu \\ 0 & 0 \end{bmatrix}$$

这里对于 $\begin{bmatrix} u \\ v \end{bmatrix}^\wedge$ 最开始没有反应过来该怎么写，但是类比了 $\xi = \begin{bmatrix} \rho \\ \phi \end{bmatrix}$，$\xi^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ 0 & 0 \end{bmatrix}$，后来发现表 7-3 上也有 $x^\wedge = \begin{bmatrix} u \\ v \end{bmatrix}^\wedge = \begin{bmatrix} v^\wedge & u \\ 0 & 0 \end{bmatrix}$

$-Cv^\wedge C^T r = -(Cv)^\wedge r = r^\wedge C v = (J\rho)^\wedge C v$，故等式左右相等.

7.5.5 证明 $\exp((\mathcal{T}x)^\wedge) = T\exp(x^\wedge)T^{-1}$

由 $\exp((\mathcal{T}x)^\wedge) = \sum_{n=0}^{\infty} \frac{1}{n!}((\mathcal{T}x)^\wedge)^n$，$(\mathcal{T}x)^\wedge = T x^\wedge T^{-1}$

$$[(\mathcal{T}x)^\wedge]^n = (T x^\wedge \underbrace{T^{-1})(T}_{I} x^\wedge \underbrace{T^{-1})}_{I} \cdots \underbrace{(T}_{I} x^\wedge T^{-1}) \Rightarrow T(x^\wedge)^n T^{-1}$$

$$\exp((\mathcal{T}x)^\wedge) = \sum_{n=0}^{\infty}\frac{1}{n!}((\mathcal{T}x)^\wedge)^n = T\sum_{n=0}^{\infty}\frac{1}{n!}(x^\wedge)^n T^{-1} = T\exp(x^\wedge)T^{-1}$$

7.5.7 证明: $x^\wedge p = p^\odot x$

$$x=\begin{bmatrix}u\\v\end{bmatrix},\ x^\wedge=\begin{bmatrix}u\\v\end{bmatrix}^\wedge=\begin{bmatrix}v^\wedge & u\\0^T & 0\end{bmatrix}$$

$$p=\begin{bmatrix}\varepsilon\\\eta\end{bmatrix},\ p^\odot=\begin{bmatrix}\varepsilon\\\eta\end{bmatrix}^\odot=\begin{bmatrix}\eta I & -\varepsilon^\wedge\\0 & 0\end{bmatrix}$$

$$x^\wedge p=\begin{bmatrix}v^\wedge & u\\0 & 0\end{bmatrix}\begin{bmatrix}\varepsilon\\\eta\end{bmatrix}=\begin{bmatrix}v^\wedge\varepsilon+u\eta\\0\end{bmatrix}$$

$$p^\odot x=\begin{bmatrix}\eta I & -\varepsilon^\wedge\\0 & 0\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}\eta u-\varepsilon^\wedge v\\0\end{bmatrix}=\begin{bmatrix}\eta u+v^\wedge\varepsilon\\0\end{bmatrix}$$

7.5.8 证明

$$p^T x^\wedge = x^T p^\circledcirc$$

$$p=\begin{bmatrix}\varepsilon\\\eta\end{bmatrix},\ p^\circledcirc=\begin{bmatrix}\varepsilon\\\eta\end{bmatrix}^\circledcirc=\begin{bmatrix}0 & \varepsilon\\-\varepsilon^\wedge & 0\end{bmatrix}$$

$$x=\begin{bmatrix}u\\v\end{bmatrix},\ x^\wedge=\begin{bmatrix}u\\v\end{bmatrix}^\wedge=\begin{bmatrix}v^\wedge & u\\0 & 0\end{bmatrix}$$

$$p^T x^\wedge=[\varepsilon^T\ \eta]\begin{bmatrix}v^\wedge & u\\0 & 0\end{bmatrix}=\begin{bmatrix}\varepsilon^T v^\wedge & \varepsilon^T u\end{bmatrix}$$

再进一步展开, 令 $\varepsilon=\begin{bmatrix}\varepsilon_x\\\varepsilon_y\\\varepsilon_z\end{bmatrix}$, $v=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$

$$x^T p^\circledcirc=\begin{bmatrix}u^T & v^T\end{bmatrix}\begin{bmatrix}0 & \varepsilon\\-\varepsilon^\wedge & 0\end{bmatrix}=\begin{bmatrix}-v^T\varepsilon^\wedge & u^T\varepsilon\end{bmatrix}$$

就可以代入证明对应项相等

7.5.11 证明: $(Tp)^\odot = Tp^\odot T^{-1}$

$$T=\begin{bmatrix}C & J\rho\\0 & 1\end{bmatrix},\ T^{-1}=\begin{bmatrix}C^T & -C^T(J\rho)^\wedge\\0 & C^T\end{bmatrix}$$

$$(Tp)^\odot=\left(\begin{bmatrix}C & J\rho\\0 & 1\end{bmatrix}\begin{bmatrix}\varepsilon\\\eta\end{bmatrix}\right)^\odot=\left(\begin{bmatrix}C\varepsilon+J\rho\eta\\\eta\end{bmatrix}\right)^\odot=\begin{bmatrix}\eta I & -(C\varepsilon+J\rho\eta)^\wedge\\0 & 0\end{bmatrix}$$

$$Tp^\odot T^{-1}=\begin{bmatrix}C & J\rho\\0 & 1\end{bmatrix}\begin{bmatrix}\eta I & -\varepsilon^\wedge\\0 & 0\end{bmatrix}\begin{bmatrix}C^T & -C^T(J\rho)^\wedge\\0 & C^T\end{bmatrix}=\begin{bmatrix}C^TC\eta & -CC^T\eta(J\rho)^\wedge-C\varepsilon^\wedge C^T\\0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}\eta I & -\eta(J\rho)^\wedge-(C\varepsilon)^\wedge\\0 & 0\end{bmatrix}=\begin{bmatrix}\eta I & -(\eta J\rho)^\wedge-(C\varepsilon)^\wedge\\0 & 0\end{bmatrix}=\begin{bmatrix}\eta I & -(J\rho\eta+C\varepsilon)^\wedge\\0 & 0\end{bmatrix}$$ 得证

7.5.12 证明：

$$((Tp)^{\odot})^T(Tp)^{\odot} = T^{-T}(p^{\odot})^T p^{\odot} T^{-1}$$

由11题已证 $(Tp)^{\odot} = Tp^{\odot}T^{-1}$

$$((Tp)^{\odot})^T = (Tp^{\odot}T^{-1})^T = T^{-T}(Tp^{\odot})^T = T^{-T}p^{\odot T}T^T$$

故左式为 $T^{-T}p^{\odot T}\underbrace{T^T T}_{=I?}p^{\odot}T^{-1}$

$$T = \begin{bmatrix} C & Jp \\ 0 & 1 \end{bmatrix} \quad T^T T = \begin{bmatrix} C^T C & C^T(Jp) \\ (Jp)^T C & (Jp)^T(Jp)+1 \end{bmatrix} \quad ?$$

$$T^T = \begin{bmatrix} C^T & 0 \\ (Jp)^T & 1 \end{bmatrix}$$