1. 证明Gauss分布积分为1.

$$\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\sigma^2}}exp(-\frac{(x-\mu)^2}{2\sigma^2})dx$$

首先参考了网上的比较主流的证明方法

1) 令 $I=\int_{-\infty}^{+\infty}exp(-\frac{1}{2\sigma^2}x^2)dx$，化为二重积分的形式

$$I^2=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}exp(-\frac{1}{2\sigma^2}x^2-\frac{1}{2\sigma^2}y^2)dxdy$$

2) 将笛卡尔坐标换为极坐标： $x=r\cos\theta$，$y=r\sin\theta$

列出雅可比行列式

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial r}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r$$

因此二重积分变化为：

$$I^2=\int_0^{+\infty}\int_0^{2\pi}exp(-\frac{r^2\cos^2\theta+r^2\sin^2\theta}{2\sigma^2})r\,dr\,d\theta$$

$$=2\pi\int_0^{+\infty}exp(-\frac{r^2}{2\sigma^2})\frac{1}{2}d(r^2)$$

3) 添凑一个 $-\frac{1}{\sigma^2}$ 便于积分，$-\frac{r}{2\sigma^2}\in[-\infty,0]$

$$I^2=-2\pi\sigma^2\int_0^{-\infty}exp(-\frac{r^2}{2\sigma^2})d(-\frac{r^2}{2\sigma^2})$$，令 $z=-\frac{r^2}{2\sigma^2}$

$$=-2\pi\sigma^2\ exp(z)\Big|_0^{-\infty}=2\pi\sigma^2$$

$$I=\sqrt{2\pi\sigma^2}$$

4) 对于 $\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\sigma^2}}exp(-\frac{1}{2\sigma^2}(x-\mu)^2dx$，令 $x-\mu=y$，则.

上式等价于 $\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\sigma^2}}exp(-\frac{y^2}{2\sigma^2})dy=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{+\infty}exp(-\frac{y^2}{2\sigma^2})dy$

$$=\frac{1}{\sqrt{2\pi\sigma^2}}\cdot I=1$$

这个是在网上看见的关于多维高斯分布积分为1的证明。

1. The integral of

$$\mathcal{N}(\mathbf{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

其中有个地方我有不太明白，

这里用t去换Qz

proof.

$$\int \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} d\mathbf{x} = ?$$

$t = Qz \quad , z = Q^{-1} t$

$dz = d(Q^{-1}t) \cdot \left| \frac{\partial z}{\partial Q^{-1}}, \frac{\partial z}{\partial t} \right| = 1$???

where $d\mathbf{x} = \prod_{i=1}^{D} x_i$.

Let $\mathbf{z} = \mathbf{x} - \mu$, and noting that the Jacobian is the identity, we find

$$\int \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} d\mathbf{x} = \int \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}\mathbf{z}^{\mathrm{T}} \boldsymbol{\Sigma}^{-1}\mathbf{z}\right\} d\mathbf{z}$$

怎么得到Jacobi是一样的？

where $d\mathbf{z} = \prod_{i=1}^{D} z_i$.

Obviously, $\boldsymbol{\Sigma}$ is a symmetric and semi-positive matrix, which means that it is diagonalizable. Letting $\boldsymbol{\Sigma} = Q^T \Lambda Q, \boldsymbol{\Sigma} = Q^T \Lambda^{-1} Q$, $\mathbf{t} = Q\mathbf{z}$ and noting that the Jacobian is also identity, we get

$$\int \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}\mathbf{z}^{\mathrm{T}} \boldsymbol{\Sigma}^{-1}\mathbf{z}\right\} d\mathbf{z} = \int \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}\mathbf{t}^{\mathrm{T}} \boldsymbol{\Lambda}^{-1}\mathbf{t}\right\} d\mathbf{t}$$

where $d\mathbf{t} = \prod_{i=1}^{D} t_i$. As Λ is the eigenvalues diagonal matrix we have

$$\Lambda^{-1} = \begin{pmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1/\lambda_D \end{pmatrix}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_D$ are eigenvalues of $\boldsymbol{\Sigma}$. Finally, we find

$$\begin{aligned}
\int \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}\mathbf{t}^{\mathrm{T}} \boldsymbol{\Lambda}^{-1}\mathbf{t}\right\} d\mathbf{t} &= \int \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{D} \frac{1}{\lambda_i} t_i^2\right\} d\mathbf{t} \\
&= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \prod_i^D \int \exp\left(-\frac{1}{2}\frac{1}{\lambda_i} t_i^2\right) dt_i \\
&= \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \prod_i^D \sqrt{2\lambda_i \pi} \\
&= 1
\end{aligned}$$

In the above equation, we should note that $|\boldsymbol{\Sigma}| = \prod_i^D \lambda_i$

习题 1.4.5.6

1. 假设 u, v 是两个相同维度的列向量，证明 $u^T v = tr(vu^T)$

基本想法就是展开去看：

假设：
$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

则 $u^T v = [u_1\ u_2 \cdots u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots u_n v_n$

对于
$$vu^T = \underset{n\times 1}{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}} \underset{1\times n}{[u_1\ u_2 \cdots u_n]} = \underset{n\times n}{\begin{bmatrix} v_1u_1 & v_1u_2 & \cdots & v_1u_n \\ v_2u_1 & v_2u_2 & \cdots & v_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ v_nu_1 & v_nu_2 & \cdots & v_nu_n \end{bmatrix}} = K$$

可以发现对于矩阵 K，~~对角线~~任意位置元素 $K_{ij} = v_i u_j$. $trace(K) = \sum_{i=1}^{n} K_{ii} = \sum_{i=1}^{n} v_i u_i = u^T v$

4. 对于高斯分布的随机变量，$x \sim N(\mu, \Sigma)$，证明 $\mu = E[x] = \int_{-\infty}^{\infty} x p(x) dx$

证明 关于高斯分布 $\int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{(x-\mu)^2}{2\sigma^2}) dx = \mu$

令 $y = x - \mu$

$$\int_{-\infty}^{+\infty} (y+\mu) \frac{1}{\sqrt{2\pi}\,\sigma} exp(-\frac{y^2}{2\sigma^2}) dy = \underbrace{\int_{-\infty}^{+\infty} \frac{y}{\sqrt{2\pi}\,\sigma} exp(\frac{-y^2}{2\sigma^2}) dy}_{\text{奇函数} = 0} + \mu \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma} exp(-\frac{y^2}{2\sigma^2}) dy}_{I=1}$$

$= \mu$

5. 对于高斯分布的随机变量，$x \sim N(\mu, \Sigma)$，请证明~~[illegible]~~

$$\Sigma = E[X] = \int_{-\infty}^{+\infty} (x-\mu)(x-\mu)^T p(x)\,dx$$

一维情况下

$$\int_{-\infty}^{+\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \int_{-\infty}^{+\infty} y^2 \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy \qquad 令 y = \sigma\sqrt{2}\,z$$

$$= \sigma\sqrt{2} \int_{-\infty}^{+\infty} (\sigma\sqrt{2}\,z)^2 \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(\sigma\sqrt{2}\,z)^2}{2\sigma^2}\right) dz \quad \cancel{[illegible]}$$

$$= \sigma\sqrt{2} \int_{-\infty}^{+\infty} \frac{\sigma\sqrt{2}}{\sqrt{\pi}} z^2 \exp(-z^2)\,dz, \quad \cancel{令 z}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} z^2 \exp(-z^2)\,dz. \text{偶函数} \Rightarrow \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{+\infty} z^2 \exp(-z^2)\,dz$$

到了这一步难以积分处理？？？ 但是我看网上有直接写答案的，有点困惑．

6. 对于归一化积仍为高斯分布证明．$X_k \sim N(\mu_k, \Sigma_k)$

$$\exp\left(-\frac{1}{2}(x-\mu^T)\Sigma^{-1}(x-\mu)\right) = \eta \prod_{k=1}^{K} \exp\left(-\frac{1}{2}(x_k-\mu_k)^T \Sigma_k^{-1}(x_k-\mu_k)\right)$$

其中： $\Sigma^{-1} = \sum_{k=1}^{K} \Sigma_k^{-1}$， $\Sigma^{-1}\mu = \sum_{k=1}^{K} \Sigma_k^{-1}\mu_k$

这个题目太抽象了．助教能不能在讲解时举个例子什么的？？？