1. 证明 PPT 17页推导

更新步　　　　预测　　　　x_{k-1} 后验

$$\underbrace{p(x_k|\check{x}_0, v_{1:k}, y_{0:k})}_{N(\hat{x}_k,\ \hat{P}_k)} = \eta\ \overbrace{\underbrace{p(y_k|x_k)}_{N(\check{y}_k+G_k(x_k-\check{x}_k),\ R'_k)}}^{\text{观测方程}} \times \int \overbrace{p(x_k|x_{k-1},v_k)}^{\text{状态转移方程}}\ \underbrace{p(x_{k-1}|\check{x}_0, v_{1:k-1}, y_{0:k-1})\,dx_{k-1}}_{N(\check{x}_k,\ F_{k-1}\hat{P}_{k-1}F_{k-1}^T+Q'_k)}$$

在第三章 高斯推断，分解部分有给出过 $p(x_k|\check{x}_0, v_{1:k}, y_{0:k})$ 基于马尔科夫假设的形式.

1) $p(x,y)$ 的联合分布为 $p(x,y)=N\left(\begin{bmatrix}\mu_x\\ \mu_y\end{bmatrix}, \begin{bmatrix}\Sigma_{xx} & \Sigma_{xy}\\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix}\right)$

将联合密度分解为两个因子的乘积（条件概率乘以边缘概率的形式）

$p(x,y)=p(x|y)\,p(y)$，对于高斯分布，可以用舒尔补 (schur complement)

推导出分解的过程. 这里是对 Σ_{yy} 舒尔补，因为 y 表示观测是已知的，x 表示待估计的状态，所以结果中表达为 Σ_{xx} 减去含 Σ_{yy} 的表达合理一些

$$\begin{bmatrix}\Sigma_{xx} & \Sigma_{xy}\\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix} = \begin{bmatrix}1 & \Sigma_{xy}\Sigma_{yy}^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\Sigma_{xx}-\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx} & 0\\ 0 & \Sigma_{yy}\end{bmatrix}\begin{bmatrix}1 & 0\\ \Sigma_{yy}^{-1}\Sigma_{yx} & 1\end{bmatrix}$$

$$\begin{bmatrix}\Sigma_{xx} & \Sigma_{xy}\\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0\\ -\Sigma_{yy}^{-1}\Sigma_{yx} & 1\end{bmatrix}\begin{bmatrix}(\Sigma_{xx}-\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx})^{-1} & 0\\ 0 & \Sigma_{yy}^{-1}\end{bmatrix}\begin{bmatrix}1 & -\Sigma_{xy}\Sigma_{yy}^{-1}\\ 0 & 1\end{bmatrix}$$

2) 联合概率密度函数 $p(x,y)$ 指数部分的二次项为：

$$\left(\begin{bmatrix}x\\ y\end{bmatrix}-\begin{bmatrix}\mu_x\\ \mu_y\end{bmatrix}\right)^T\begin{bmatrix}\Sigma_{xx} & \Sigma_{xy}\\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix}^{-1}\left(\begin{bmatrix}x\\ y\end{bmatrix}-\begin{bmatrix}\mu_x\\ \mu_y\end{bmatrix}\right)$$

$$=(x-\mu_x-\Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y))^T(\Sigma_{xx}-\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx})^{-1}(x-\mu_x-\Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y))+(y-\mu_y)\Sigma_{yy}^{-1}(y-\mu_y)$$

所以得到了 $p(x,y)=p(x|y)p(y)$ 时，

$$p(x|y)=N(\underbrace{\mu_x+\Sigma_{xy}\Sigma_{yy}^{-1}(y-\mu_y)}_{\text{均值}},\ \underbrace{\Sigma_{xx}-\Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}}_{\text{协方差}})$$

$$p(y)=N(\mu_y, \Sigma_{yy})$$

就得到了归一化积指数部分，均值，协方差.

注意到均值,协方差中均有 $\Sigma_{xy}\Sigma_{yy}^{-1}$ 这一项，定义为卡尔曼增益 $K=\Sigma_{xy,k}\Sigma_{yy,k}^{-1}$（加上下标k）

则 $K_k = \Sigma_{xy,k} \Sigma_{yy,k}^{-1}$

$$\hat{P}_k = \check{P}_k - K_k \Sigma_{xy,k}^T$$

$$\hat{x}_k = \check{x}_k + K_k (y_k - \mu_{y,k})$$

将 $\mu_{y,k}$, $\Sigma_{yy,k}$, $\Sigma_{xy,k}$ 代入,可得:

$$K_k = \check{P}_k G_k^T (G_k \check{P}_k G_k^T + R_k')^{-1}$$

$$\hat{P}_k = (1 - K_k G_k) \check{P}_k$$

$$\hat{x}_k = \check{x}_k + K_k (y_k - y_{op,k} - G_k(\check{x}_k - x_{op,k}))$$

2、证明 高维（L维）高斯分布由sigma point（共2L+1个）：

$$LL^T = \Sigma_{xx} \quad (\text{cholesky},\ L\text{为下三角})$$

$$\chi_0 = \mu_x$$

$$\chi_i = \mu_x + \sqrt{L+k}\,(col_i L)$$

$$\chi_{i+L} = \mu_x - \sqrt{L+k}\,(col_i L)$$

·这些样本点满足：

$$\mu_x = \sum_{i=0}^{2L} \alpha_i \chi_i \qquad (1)$$

$$\Sigma_{xx} = \sum_{i=0}^{2L} \alpha_i (\chi_i - \mu_x)(\chi_i - \mu_x)^T \qquad (2)$$

其中：

$$\alpha_i = \begin{cases} \frac{k}{L+k} & i=0 \\ \frac{1}{2}\frac{1}{L+k} & \text{其他} \end{cases}$$

对于（1）式：

$$\sum_{i=0}^{2L} \alpha_i \chi_i = \alpha_0 \chi_0 + \sum_{i=1}^{L} \alpha_i \chi_i + \sum_{i=L+1}^{2L} \alpha_i \chi_i$$

$$= \alpha_0 \chi_0 + \frac{1}{2}\frac{1}{L+k}\left(\sum_{i=1}^{L}\chi_i + \sum_{i=L+1}^{2L}\chi_i\right)$$

$$= \alpha_0 \chi_0 + \frac{1}{2}\frac{1}{L+k}\Big(\mu_x + \sqrt{L+k}\,(col_1 L) + \mu_x + \sqrt{L+k}\,(col_2 L) \cdots\cdots + \mu_x + \sqrt{L+k}\,(col_L L)$$

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$$+ \mu_x - \sqrt{L+k}\,(col_1 L) + \mu_x - \sqrt{L+k}\,(col_2 L) \cdots\cdots + \mu_x - \sqrt{L+k}\,(col_L L)\Big)$$

$$= \alpha_0 \chi_0 + \frac{1}{2}\frac{1}{L+k}(\mu_x \cdot 2L) = \frac{k}{L+k}\cdot\mu_x + \frac{1}{2}\frac{1}{L+k}(\mu_x \cdot 2L) = \frac{(L+k)\mu_x}{L+k} = \mu_x$$

对于(2)式

$$\sum_{i=0}^{2L} \alpha_i (\chi_i-\mu_x)(\chi_i-\mu_x)^T = \alpha_0(\chi_0-\mu_x)(\chi_0-\mu_x)^T + \sum_{i=1}^{L}\alpha_i\left((\chi_i-\mu_x)(\chi_i-\mu_x)^T\right) + \sum_{i=L+1}^{2L}\alpha_i\left((\chi_i-\mu_x)(\chi_i-\mu_x)^T\right)$$

$$= 0 + \frac{1}{2(L+k)}\Big[(\sqrt{L+k}\,col_1 L)(\sqrt{L+k}\,col_1 L)^T + \cdots + (\sqrt{L+k}\,col_L L)(\sqrt{L+k}\,col_L L)^T +$$

$$(-\sqrt{L+k}\,col_1 L)(-\sqrt{L+k}\,col_1 L)^T + \cdots + (-\sqrt{L+k}\,col_L L)(-\sqrt{L+k}\,col_L L)^T\Big]$$

$$= \frac{1}{L+k}\Big[(L+k)(col_1 L\, col_1 L^T + col_2 L\, col_2 L^T + \cdots + col_L L\, col_L L^T)$$

$$= LL^T = \Sigma_{xx}$$

3、考虑如下离散时间系统

$$f:\begin{bmatrix}x_k\\y_k\\\theta_k\end{bmatrix}=\begin{bmatrix}x_{k-1}\\y_{k-1}\\\theta_{k-1}\end{bmatrix}+T\begin{bmatrix}\cos\theta_{k-1}&0\\\sin\theta_{k-1}&0\\0&1\end{bmatrix}\left(\begin{bmatrix}v_k\\w_k\end{bmatrix}+w_k\right),\quad w_k\sim N(0,Q)$$

$$g:\begin{bmatrix}r_k\\\phi_k\end{bmatrix}=\begin{bmatrix}\sqrt{x_k^2+y_k^2}\\\operatorname{atan2}(-y_k,-x_k)-\theta_k\end{bmatrix}+n_k,\quad n_k\sim N(0,R)$$

请建立EKF方程估计移动机器人位姿，并写出雅可比 F_{k-1}，G_k 和协方差 Q_k'，R_k' 的表达式。

首先整理运动方程：标准的运动方程形式为 $x_k=x_{k-1}+v+w$，$w=\begin{bmatrix}w_{k1}\\w_{k2}\end{bmatrix}$

$$f:\begin{bmatrix}x_k\\y_k\\\theta_k\end{bmatrix}=\begin{bmatrix}x_{k-1}\\y_{k-1}\\\theta_{k-1}\end{bmatrix}+\begin{bmatrix}T\cos\theta_{k-1}(v_k+w_{k1})\\T\sin\theta_{k-1}(v_k+w_{k1})\\T(w_k+w_{k2})\end{bmatrix}=\begin{bmatrix}x_{k-1}+T\cos\theta_{k-1}(v_k+w_{k1})\\y_{k-1}+T\sin\theta_{k-1}(v_k+w_{k1})\\\theta_{k-1}+T(w_k+w_{k2})\end{bmatrix}$$

$$F_{k-1}=\frac{\partial f(x_{k-1},v_k,w_k)}{\partial x_{k-1}}$$，F_{k-1} 为在状态 x_{k-1} 处的一阶导数，状态 $x_{k-1}=\begin{bmatrix}x_{k-1}\\y_{k-1}\\\theta_{k-1}\end{bmatrix}$

$$=\begin{bmatrix}\frac{\partial f_1}{\partial x_{k-1}},&\frac{\partial f_1}{\partial y_{k-1}},&\frac{\partial f_1}{\partial\theta_{k-1}}\\\frac{\partial f_2}{\partial x_{k-1}},&\frac{\partial f_2}{\partial y_{k-1}},&\frac{\partial f_2}{\partial\theta_{k-1}}\\\frac{\partial f_3}{\partial x_{k-1}},&\frac{\partial f_3}{\partial y_{k-1}},&\frac{\partial f_3}{\partial\theta_{k-1}}\end{bmatrix}=\begin{bmatrix}1&0&-T\sin\theta_{k-1}(v_k+w_{k1})\\0&1&T\cos\theta_{k-1}(v_k+w_{k1})\\0&0&1\end{bmatrix}$$

$$\left[(x^2+y^2)^{\frac12}\right]'=\frac12(x^2+y^2)^{-\frac12}\cdot2x=\frac{x}{\sqrt{x^2+y^2}}$$

G_k 这里 $G_k=\dfrac{\partial g(x_k,n_k)}{\partial x_k}=\begin{bmatrix}\frac{x_k}{\sqrt{x_k^2+y_k^2}}&\frac{y_k}{\sqrt{x_k^2+y_k^2}}&0\\\frac{-y_k}{x_k^2+y_k^2}&\frac{x_k}{x_k^2+y_k^2}&-1\end{bmatrix}$

$$\arctan\left(\frac{y}{x}\right)'=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)'=\frac{x^2}{x^2+y^2}\cdot\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}$$

$Q_k'=E[w_k'w_k'^T]$，$w_k'=\dfrac{\partial f}{\partial w_k}\cdot w_k=\begin{bmatrix}T\cos\theta_{k-1}&0\\T\sin\theta_{k-1}&0\\0&T\end{bmatrix}\cdot\begin{bmatrix}w_{k1}\\w_{k2}\end{bmatrix}$

$$=\begin{bmatrix}T\cos\theta_{k-1}&0\\T\sin\theta_{k-1}&0\\0&T\end{bmatrix}Q_k\begin{bmatrix}T\cos\theta_{k-1}&0\\T\sin\theta_{k-1}&0\\0&T\end{bmatrix}^T$$

$R_k'=E[\cancel{R_k'R_k'^T}]$，$n_k'=\dfrac{\partial g}{\partial n_k}\cdot n_k=\begin{bmatrix}1&0\\0&1\end{bmatrix}\cdot n_k=n_k$

$$R_k'=E[n_k'n_k'^T]=E[n_kn_k^T]=R_k$$