Climb Trajectory Optimization Using Energy Height Theory

This script models and visualizes aircraft climb trajectories using a **strategy-based energy framework**. It is rooted in the concept of **specific energy height**, where the aircraft's total energy is treated as a combination of potential and kinetic components. The framework supports a wide variety of climb strategies, including linear allocation, altitude-biased exponential functions, constant-rate climbs, and eventually variable-energy climbs.

Theory Overview

1. Specific Energy Height

The total specific energy (per unit mass) of an aircraft is defined as:

$$E = h + \frac{V^2}{2g_0}$$

Where:

- h: altitude [m]
- V: true airspeed [m/s]
- g_0: standard gravitational acceleration [m/s]

This representation expresses both potential and kinetic energy in terms of an equivalent height (in meters), allowing a unified way to measure and manage aircraft energy states.

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2. Rate of Energy Gain

By differentiating the specific energy expression with respect to time, we obtain:

$$\frac{dE}{dt} = \frac{dh}{dt} + \frac{V}{g_0} \cdot \frac{dV}{dt}$$

This shows how energy gain (rate) is split between vertical motion (climb) and forward acceleration. It becomes the foundation for constructing energy-allocation strategies.

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3. Energy Split Strategy via `altitude_fraction`

To control how the aircraft divides its energy input between climb and speed, a parameter $r \in [0, 1]$ (named `altitude_fraction`) is introduced:

- r: fraction of energy allocated to altitude gain (climb)
- $1 - r$: fraction allocated to speed increase (acceleration)

Under this rule, and assuming constant total specific energy rate $\dot{E}$, we compute rates as:

$$\frac{dh}{dt} = r \cdot \dot{E}, \quad \frac{dV}{dt} = (1 - r) \cdot \dot{E} \cdot \frac{g_0}{V}$$

This method generalizes well across strategies by varying r and $\dot{E}$ independently.

4.1 Fixed Energy Linear Strategy

In this simplest case, the energy rate $\dot{E} = E_{\text{DOT}}$ remains **constant**, and the allocation ratio r is also fixed throughout the climb.

Resulting rate formulas are:

$$\frac{dh}{dt} = r \cdot E_{\text{DOT}}, \quad \frac{dV}{dt} = (1 - r) \cdot E_{\text{DOT}} \cdot \frac{g_0}{V}$$

The climb and speed rates are constant (with respect to time) for a given r, but speed gain will still cause nonlinear effects due to the $\frac{1}{V}$ dependence.

Implemented in: `StrategyProfiles.FixedEnergy.Linear`

4.2 Exponential Bias Strategy

This strategy class allows **altitude-dependent** redistribution of energy while still assuming a **constant total specific energy rate** $\dot{E} = E_{\mathrm{DOT}}$.

Unlike the linear case, the preference to climb or accelerate **evolves** with altitude, creating more realistic profiles.

4.2.1 General Methodology

Let:

- $r \in (0, 1)$: base climb preference at sea level
- h: current altitude
- h_{target}: mission-defined target altitude

We apply **exponential weighting** to adjust climb/speed priorities dynamically:

Increasing Climb Bias

$$\mathrm{climb\_weight} = r \cdot e^{h/h_{\mathrm{target}}}, \quad \mathrm{speed\_weight} = (1 - r) \cdot e^{-h/h_{\mathrm{target}}}$$

Decreasing Climb Bias

$$\mathrm{climb\_weight} = r \cdot e^{-h/h_{\mathrm{target}}}, \quad \mathrm{speed\_weight} = (1 - r) \cdot e^{h/h_{\mathrm{target}}}$$

Increasing Speed Bias

$$\mathrm{speed\_weight} = r \cdot e^{h/h_{\mathrm{target}}}, \quad \mathrm{climb\_weight} = (1 - r) \cdot e^{-h/h_{\mathrm{target}}}$$

Decreasing Speed Bias

$$\mathrm{speed\_weight} = r \cdot e^{-h/h_{\mathrm{target}}}, \quad \mathrm{climb\_weight} = (1 - r) \cdot e^{h/h_{\mathrm{target}}}$$

Weights are **normalized** to obtain valid fractions:

$$\mathrm{climb\_fraction} = \frac{\mathrm{climb\_weight}}{\mathrm{climb\_weight} + \mathrm{speed\_weight}}, \quad \mathrm{speed\_fraction} = 1 - \mathrm{climb\_fraction}$$

And finally, rates are computed as:

$$\frac{dh}{dt} = \text{climb\_fraction} \cdot E_{\text{DOT}}, \quad \frac{dV}{dt} = \text{speed\_fraction} \cdot E_{\text{DOT}} \cdot \frac{g(h)}{V}$$

Where $g(h)$ is retrieved from the atmospheric model.

4.2.2 Strategy Variants in Code

Each strategy is explicitly defined under `StrategyProfiles.FixedEnergy.Exponential`:

Strategy Function	Description
`increasing_climb`	Emphasizes climbing more as altitude increases
`decreasing_climb`	Prioritizes climb early, then transitions to speed
`increasing_speed`	Delays acceleration until higher altitudes
`decreasing_speed`	Emphasizes acceleration early in the climb

4.3 Constant Rate Strategies

Implemented in `StrategyProfiles.ConstantRates` these assume fixed speed:

Constant Speed

$$\frac{dh}{dt} = g_0, \quad \frac{dV}{dt} = 0$$

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5. Simulation Logic

The function `simulate_climb_path()` integrates the state equations over time:

- At each time step, current altitude and velocity are passed to a strategy function.
- The function returns $\frac{dh}{dt}$, $\frac{dV}{dt}$, which are used to update state variables.
- The climb ends **exactly** when $h = h_{\text{target}}$ without overshooting.

This ensures numerical stability and accuracy of terminal conditions.

Possible Extensions:

- Thrust & drag model (based on atmosphere and engine map)
- TSFC-based fuel burn tracking
- Optimization for minimum fuel or time
- Aerodynamic constraints (e.g., max C_L, α, etc.)

Literature Survey

Summary of Climb Performance Concepts (Raymer, Chapter 17.3)

This summary consolidates key concepts and equations related to **steady climb and descent** from Daniel Raymer's *Aircraft Design: A Conceptual Approach*, with a focus on climb gradient, best angle/rate of climb, and time/fuel to climb.

Steady Climbing Flight and Climb Gradient

- **Climb gradient** G is the ratio of vertical to horizontal distance traveled.
- It is equivalent to $\sin(\gamma)$, where γ is the climb angle:

$$\gamma = \sin^{-1}\left(\frac{T-D}{W}\right) = \sin^{-1}\left(\frac{T}{W} - \frac{1}{L/D}\right) \qquad \text{(Eq. 17.38)}$$

- The **vertical velocity** or **rate of climb** V_v is:

$$V_v = V\sin(\gamma) = V\sqrt{\frac{T}{W} - \frac{1}{L/D}} \qquad \text{(Eq. 17.39)}$$

- Force balances used:

$$\sum F_x = T - D - W\sin(\gamma) \qquad \text{(Eq. 17.6)}$$

$$\sum F_z = L - W\cos(\gamma) \qquad \text{(Eq. 17.7)}$$

Graphical Method: Best Angle and Rate of Climb

- **Best rate of climb** maximizes vertical velocity V_v.
- **Best angle of climb** maximizes altitude gain per unit horizontal distance (i.e., max γ).
- Plot V_v vs airspeed (using Eq. 17.39) and superimpose thrust/drag data to identify:
  - **Peak of the curve**: Best rate of climb.
  - **Tangency from origin**: Best angle of climb.
    (Refer to Fig. 17.4 in Raymer)

Jet Aircraft: Best Climb Conditions

- For jets, thrust T is mostly constant with speed.
- Best rate of climb is found by maximizing:

$$V_v = V\left(\frac{T}{W} - \frac{\rho V^2 C_D}{2(W/S)} - \frac{2K}{\rho V}\left(\frac{W}{S}\right)\right) \qquad \text{(Eq. 17.42)}$$

- Setting $\frac{dV_v}{dV} = 0$ and solving gives:

$$V = \sqrt{\frac{W/S}{3\rho C_{D_0}}\left(\frac{T}{W} + \sqrt{\left(\frac{T}{W}\right)^2 + 12C_{D_0}K}\right)} \qquad \text{(Eq. 17.43)}$$

- Example: The B-70 has a best climb speed of 583 kt (≈1080 km/h).

Time and Fuel to Climb

- Time to climb a small height dh:

$$dt = \frac{dh}{V_v} \qquad \text{(Eq. 17.46)}$$

- Fuel burn over that time:

$$dW_f = -C_T T\, dt \qquad \text{(Eq. 17.47)}$$

- Since V_v varies with altitude, it can be linearly approximated:

$$V_v = V_{v_i} - a(h_{i+1} - h_i) \qquad \text{(Eq. 17.48)}$$

$$a = \frac{V_{v_2} - V_{v_1}}{h_2 - h_1} \qquad \text{(Eq. 17.49)}$$

- Total time and fuel between two altitudes:

$$t_{i+1} - t_i = \frac{1}{a} \ln\left(\frac{V_{v_i}}{V_{v_{i+1}}}\right) \qquad \text{(Eq. 17.50)}$$

$$\Delta W_{\text{fuel}} = -(CT)_{\text{avg}}(t_{i+1} - t_i) \qquad \text{(Eq. 17.51)}$$

- Improved accuracy can be achieved via **iteration**, updating W after each step.

Reference

Raymer, D. P. (2021). *Aircraft Design: A Conceptual Approach*, 6th ed., AIAA Education Series, Chapter 17.3.

Future Considerations and Open Questions

As development continues, several physical constraints and operational factors need to be addressed:

1. Thrust Limitations

- The climb energy rate is ultimately limited by the available engine thrust at a given altitude.
- To model this accurately, engine performance data (e.g., thrust vs. altitude) is needed.

2. TSFC Clarification

- The thrust-specific fuel consumption (TSFC) is often provided in units such as lb fuel / lb thrust / hr.
- A consistent unit system should be defined, and conversions should be handled clearly for modeling fuel flow.

3. Angle of Attack Constraints

- The maximum achievable angle of attack limits climb steepness and lift.
- For steady climb, the angle of attack can be derived using Raymer's Equation 17.38 (refer to “Aircraft Design: A Conceptual Approach”).

4. Scenario Planning

We may consider simulating under different mission or design contexts:

- Minimum fuel climb
- Minimum time climb
- Constant Mach
- Engine-out or degraded thrust condition

5. Boundary Conditions

We should consider the boundary conditions defined for each segment.