A few exercises

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In this file I put a couple of exercises around the Probability calculus.

1 Delicate question

There are 100 students and 100 sphere with a number. Every student draws a sphere and must answer with yes or no.

Sphere with number $\leq 70 \rightarrow$ delicate question.
Sphere with number $> 70 \rightarrow$ check question.

We would like to know the probability that an answer is "yes" considering that it's a delicate question. Let's assume we have 25 yes.

$$P(yes) = P(yes|delicate) \cdot P(delicate) + P(yes|check) \cdot P(check)$$

$$P(yes) \approx 0.25 \quad P(yes|check) = 0.5 \quad P(check) = 0.3 \quad P(delicate) = 0.7$$

$$P(yes|delicate) = \frac{0.25 - (0.5 \cdot 0.3)}{0.7} \approx 0.14$$

2 Books on a rack

We have 10 books and we would like to know every possible ways to put them on a rack in order by subject. The books are: $4M + 3C + 2S + 1L$

So we are trying to do permutation over these books and then here the answer:

$$(4! \cdot 3! \cdot 2! \cdot 1!) \cdot 4!$$

I supposed the subject order and I multiplied for the number of possible orders(First M, First C, etc.etc.).

3 Extraction of balls

We have 11 balls, 5 of them are white and 6 of them are black. We would like to know the probability of extracting two balls with different colour eachother with two consecutive extractions.

Possible cases $\rightarrow d_{11,2} = 11 \cdot 10 = 110$
Favourable cases $\rightarrow 6 \quad white \cdot 5 \quad black = 30$ and it's the same if I extract black before the white.

$$P(two \quad different \quad colours) = \frac{60}{110} = \frac{6}{11}$$

4 Disease test

We have a test that tells us if a person has a particolar disease or not. This test get the right answer in 99% of the cases, it gives a fake positive in 1% of the cases.

We would like to know the probability of being sick considering I had a positive response of the test.

Let's define two events:
E = positive
S = sick

99% is ok
1% is fake positive

$P(E|M) = 0.99$
$P(E|\bar{M}) = 0.01$
$P(\bar{E}|\bar{M}) = 1 - P(E|\bar{M}) = 0.99$
$P(M) = 0.005$ (We suppose this as sickness rate)

$$P(M|E) = \frac{P(M \cap E)}{P(E)} = \frac{P(E \cap M)}{P(E)} = \frac{P(E|M) \cdot P(M)}{P(E)} =$$

$$\frac{P(E|M) \cdot P(M)}{P(E|M) \cdot P(M) + P(E|\bar{M}) \cdot P(\bar{M})} = \frac{0.99 \cdot 0.005}{0.99 \cdot 0.005 + 0.01 \cdot 0.995} \approx 0.3322$$

5 Lost plane

There's a lost plane and we have 3 zones to search. We would like to know the probability that the plane has fallen in each zones considering that a research in zone 1 had a negative outcome.

So:

$$P(Z_1) = \frac{1}{3} \quad P(Z_2) = \frac{1}{3} \quad P(Z_3) = \frac{1}{3}$$

I define $\alpha_i = P(Not \;\; found | I \;\; search \;\; in \;\; Z_i)$

Let's consider the E event that happens when the plane is not found in Z_1.

So:

$$P(E|Z_1) = \alpha_1 \quad P(E|Z_2) = 1 \quad P(E|Z_3) = 1$$

$$P(E) = \sum_{i=1}^{3} P(E|R_i)P(R_i) = \frac{\alpha_i}{3} + \frac{2}{3} \quad Total \;\; probability \;\; theorem$$

$$P(R_1|E) = \frac{P(E|R_1) \cdot P(R_1)}{P(E)} = \frac{\frac{\alpha_i}{3}}{\frac{(\alpha_1+2)}{3}} = \frac{\alpha_1}{\alpha_1 + 2} \quad Bayes \;\; Theorem$$

For i = 2,3 we obtain:

$$\frac{1}{\alpha + 2}$$

6 Collection cards

There are n collection cards.

$P_i = P(I \;\; buy \;\; the \;\; i - card)$

$0 \leq P_i \leq 1$

$\sum_{i=1}^{k} P_i = 1$

I have k cards.
We would like to calculate this probability:

$$P(At \;\; least \;\; a \;\; j \;\; card | I \;\; have \;\; a \;\; i \;\; card)$$

$$P(A_j|A_i) = \frac{P(A_i \cap A_j)}{P(A_i)} = \frac{1 - (1 - P_i)^k - (1 - P_j)^k + (1 - P_i - P_j)^k}{1 - (1 - P_i)^k}$$

Demonstration:

First we determine $P(A_i)$

$$P(A_i) = 1 - P(\bar{A}_i) = 1 - P(I \;\; never \;\; get \;\; the \;\; i - card \;\; on \;\; k \;\; purchases)$$

$$= 1 - P(\bigcap_{i=1}^{k} \; I \;\; don't \;\; get \;\; the \;\; i-card \;\; on \;\; the \;\; k \;\; purchase)$$

$$= 1 - \prod_{i=1}^{k} P(I \;\; don't \;\; get \;\; the \;\; i-card \;\; on \;\; the \;\; k \;\; purchase)$$

$$= 1 - \prod_{i=1}^{k} (1 - P_i) = 1 - (1 - P_i)^k$$

Second we determine $P(A_i \cap A_j)$

$$P(A_i \cap A_j) = 1 - P(\overline{A_i \cap A_j}) = 1 - P(\bar{A}_i \cup \bar{A}_j)$$

$$= 1 - (P(\bar{A}_i) + P(\bar{A}_j) - P(\bar{A}_i \cap \bar{A}_j))$$

$$= 1 - ((1 - P_i)^k + (1 - P_j)^k - P(\bar{A}_i \cap \bar{A}_j))$$

To finish we have to determine $P(\bar{A}_i \cap \bar{A}_j)$

$$P(\bar{A}_i \cap \bar{A}_j) = P(I \;\; never \;\; get \;\; the \;\; i-card \;\; and \;\; j-card \;\; on \;\; k \;\; purchases)$$

$$= P(\bigcap_{i=1}^{k} \; I \;\; don't \;\; get \;\; the \;\; i-card \;\; and \;\; j-card \;\; on \;\; the \;\; k \;\; purchase)$$

$$= \prod_{i=1}^{k} (1 - P_i - P_j) = (1 - P_i - P_j)^k$$

So we have determinated that:

$$P(A_i \cap A_j) = 1 - (1 - P_i)^k + (1 - P_j)^k - (1 - P_i - P_j)^k$$

Demonstrated.