UNIT 12 TWO-PERSON ZERO-SUM GAMES WITHOUT SADDLE POINT

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12.1 INTRODUCTION

In Unit 11, you have learnt the definitions of key terms used in game theory. We have also discussed the solution of two-person zero-sum games with saddle point using the maximin-minimax principle. In this unit, we discuss some methods of solving two-person zero-sum games without saddle point. In Sec. 12.2, we describe the algebraic method for solving 2×2 two-person zero-sum games without a saddle point. However, there are rectangular games of sizes other than 2×2 and to obtain the solution for such games, we need to know the dominance rules. These rules help in reducing the size of the games and are described in Sec. 12.3. However, using the dominance ruels, we cannot reduce all rectangular games into games of sizes 2×2. So, we need methods to solve games of sizes other than 2×2. The graphical method discussed in Sec. 12.4 is one such method, which reduces the games of size $2 \times n$ and $m \times 2$ to size 2×2. Once the game is reduced to size 2×2, we can solve it by using the algebraic method discussed in Sec. 12.2.

In Sec. 12.5, we solve a popular 3×3 two-person zero-sum game known as Stone-Paper-Scissors game by extending the technique of solving 2×2 two-person zero-sum games discussed in Sec. 12.2. A flow chart showing the approach followed in this unit to solve the two-person zero-sum games is given in Sec. 12.6.

Objectives

After studying this unit, you should be able to:

- solve 2×2 two-person zero-sum games by applying the algebraic method;
- apply dominance rules to reduce the size of a game;
- solve $2 \times n$ and $m \times 2$ two-person zero-sum games by applying the graphical method; and
- solve 3×3 two-person zero-sum games.

12.2 SOLUTION OF 2×2 TWO-PERSON ZERO-SUM GAMES WITHOUT SADDLE POINT

In Unit 11, you have learnt how to solve two-person zero-sum games with saddle point using the maximin-minimax principle. You have also learnt that the existence of a saddle point depends on the equality of maximin value and minimax value, where

$$\text{Maximin value} = \text{maximum among row minima} = \max_i \min_j \{a_{ij}\} \quad \ldots (1)$$

$$\text{Minimax value} = \text{minimum among column maxima} = \min_j \max_i \{a_{ij}\} \quad \ldots (2)$$

and a_{ij} is the $(i, j)^{th}$ entry of the payoff matrix of the row player.

The operations for obtaining the maximin and minimax values from equations (1) and (2) are different. So, the values obtained from equations (1) and (2) may or may not be equal. If they are equal, the game has a saddle point and can be solved using the maximin-minimax principle, which gives a pure strategy as a solution. But if they are not equal, the game does not have a saddle point. So we need methods to solve the games without saddle point. One such method for solving 2×2 two-person zero-sum games without saddle point is discussed in this section. But before describing the method, we state a theorem required for solving such games.

Fundamental Theorem of Rectangular Games or the Minimax Theorem

It states that for every finite two-person zero-sum game:

(1) There exists a number v, called the value of the game,

(2) There exists at least one mixed strategy for player I such that the gain of player $I \geq v$, irrespective of the strategy adopted by player II, and

(3) There exists at least one mixed strategy for player II such that the loss of player $II \leq v$, irrespective of the strategy adopted by player I.

Note 1: Finite two-person zero-sum game means that the courses of action available to both players should be finite in number.

This theorem simply states that there always exists a solution for every finite two-person zero-sum game in terms of mixed strategies.

The solution of 2×2 two-person zero-sum games in terms of mixed strategies is provided by the **algebraic method**. We first give a statement of the method and then its proof.

Statement: Every 2×2 two-person zero-sum game without a saddle point having the following payoff matrix for player A,

		Player B	
		B_1	**B_2**
Player A	**A_1**	a	b
	A_2	c	d

has a solution in terms of mixed strategies (p_1, p_2), (q_1, q_2) for player A and player B, respectively. The value of the game is v, and p_1, p_2, q_1, q_2 and v are given by

$$p_1 = \frac{d-c}{(a-b)+(d-c)} \quad \ldots (3a)$$

$$p_2 = \frac{a-b}{(a-b)+(d-c)} \quad \ldots (3b)$$

$$q_1 = \frac{d-b}{(a-b)+(d-c)} \quad ...(3c)$$

$$q_2 = \frac{a-c}{(a-b)+(d-c)} \quad ...(3d)$$

$$v = \frac{ad-bc}{(a-b)+(d-c)} \quad ...(3e)$$

such that

$$p_1, p_2, q_1, q_2 \geq 0, \; p_1 + p_2 = 1 \text{ and } q_1 + q_2 = 1 \quad ...(3f)$$

Proof: We are given that the payoff matrix for player A has no saddle point. So there is no solution in terms of pure optimal strategies. But from the fundamental theorem of game theory, there exists a solution in terms of mixed strategies. Let (p_1, p_2) and (q_1, q_2) be the optimal mixed strategies for player A and player B, respectively, and v be the value of the game. We now rewrite the payoff matrix indicating the above probabilities as follows:

			Player B	
			q_1	q_2
			B_1	B_2
Player A	p_1	A_1	a	b
	p_2	A_2	c	d

The expected gain of player A is

$ap_1 + cp_2$ when player B employs strategy B_1

$bp_1 + dp_2$ when player B employs strategy B_2

Similarly, the expected loss of player B is

$aq_1 + bq_2$ when player A employs strategy A_1

$cq_1 + dq_2$ when player A employs strategy A_2

Since v is the value of the game and (p_1, p_2) a mixed optimal strategy for player A, we have

$$\underset{\text{I}}{ap_1 + cp_2} = \underset{\text{II}}{v} = \underset{\text{III}}{bp_1 + dp_2}$$

$$\Rightarrow ap_1 + cp_2 = bp_1 + dp_2$$

$$\Rightarrow ap_1 + c(1-p_1) = bp_1 + d(1-p_1) \quad [\because p_1 + p_2 = 1]$$

$$\Rightarrow (a-c-b+d)p_1 = d-c \Rightarrow p_1 = \frac{d-c}{(a-b)+(d-c)}$$

$$\therefore p_2 = 1 - p_1 = 1 - \frac{d-c}{(a-b)+(d-c)} = \frac{(a-b)+(d-c)-d+c}{(a-b)+(d-c)} = \frac{a-b}{(a-b)+(d-c)}$$

$$\text{The expected gain of player A} = v = ap_1 + cp_2 = \frac{a(d-c)}{(a-b)+(d-c)} + \frac{c(a-b)}{(a-b)+(d-c)}$$

$$= \frac{ad-ac+ca-bc}{(a-b)+(d-c)} = \frac{ad-bc}{(a-b)+(d-c)}$$

Similarly, since v is the value of the game and (q_1, q_2) is a mixed optimal strategy for player B, we have

$$aq_1 + bq_2 = v = cq_1 + dq_2$$
$$\quad \text{I} \qquad \text{II} \qquad \text{III}$$

$$\Rightarrow aq_1 + bq_2 = cq_1 + dq_2 \Rightarrow aq_1 + b(1-q_1) = cq_1 + d(1-q_1)$$

$$\Rightarrow (a-b-c+d)q_1 = d-b \Rightarrow q_1 = \frac{d-b}{(a-b)+(d-c)}$$

$$\therefore q_2 = 1-q_1 = 1-\frac{d-b}{(a-b)+(d-c)} = \frac{(a-b)+(d-c)-d+b}{(a-b)+(d-c)} = \frac{a-c}{(a-b)+(d-c)}$$

The expected gain of player B $= v = aq_1 + bq_2 = \dfrac{a(d-b)}{(a-b)+(d-c)} + \dfrac{b(a-c)}{(a-b)+(d-c)}$

$$= \frac{ad-ab+ab-bc}{(a-b)+(d-c)} = \frac{ad-bc}{(a-b)+(d-c)}$$

Thus, we have proved that

$$p_1 = \frac{d-c}{(a-b)+(d-c)},\ p_2 = \frac{a-b}{(a-b)+(d-c)}$$

$$q_1 = \frac{d-b}{(a-b)+(d-c)},\ q_2 = \frac{a-c}{(a-b)+(d-c)} \text{ and } v = \frac{ad-bc}{(a-b)+(d-c)}$$

Now, it only remains to prove that $p_1, p_2, q_1, q_2 \geq 0$.

Remember that we are given that the payoff matrix of player A has no saddle point.

Therefore, either $a \geq b$ or $a \leq b$.

First, suppose that $a \geq b$. This implies that b is the minimum element of the first row

$\Rightarrow b < d$.

Otherwise, b will also become the maximum element of the second column and so a saddle point.

Now, $b < d \Rightarrow d$ is the maximum element of the second column.

$\Rightarrow d > c$.
Otherwise, d will also become the minimum element of the second row and so a saddle point.

Again, d > c $\Rightarrow$c is the minimum element of the second row.

$\Rightarrow c < a$.

Otherwise, c will also become the maximum element of the first column and so a saddle point.

Finally, $c < a \Rightarrow a$ is the maximum element of the first column.

$\Rightarrow a > b$.

Otherwise, a will also become the minimum element of the first row and so a saddle point.

Thus, we obtain the result that if $a \geq b$ then b < d, d > c, c < a and a > b.

Due to symmetry, if $a \leq b$ then $b > d, d < c, c > a,$ and $a < b$.

Thus, in both cases, either both (a – b) and (d – c) are simultaneously positive or both are simultaneously negative. Therefore, $p_1 \geq 0,\ p_2 \geq 0$.

Similarly, we can show that $q_1 \geq 0,\ q_2 \geq 0$.

Hence proved.

Note 2: For numerical problems you have to substitute the values of a, b, c, d directly in equations (3a to e) for p_1, p_2, q_1, q_2 and v to obtain the solution.

Let us solve the following example using the algebraic method.

Example 1: Solve the two-person zero-sum game having the following payoff matrix for player A:

		Player B	
		$\mathbf{B_1}$	$\mathbf{B_2}$
Player A	$\mathbf{A_1}$	3	2
	$\mathbf{A_2}$	1	5

Solution: We first check whether a saddle point exists or not.

		Player B		**Row Minima**	**Maximin Value**
		$\mathbf{B_1}$	$\mathbf{B_2}$		
Player A	$\mathbf{A_1}$	3	2	2	$\max\{2, 1\} = 2$
	$\mathbf{A_2}$	1	5	1	
Column Maxima		3	5		
Minimax Value		$\min\{3, 5\} = 3$			

Since the maximin value (=2) $\neq$ minimax value (=3), there is no saddle point. Hence, we have to solve the game in terms of mixed strategies. Let (p_1, p_2) and (q_1, q_2) be the optimal mixed strategies for player A and player B, respectively and v be the value of the game. From equations (3a to e), we have

$$p_1 = \frac{d-c}{(a-b)+(d-c)},\ p_2 = \frac{a-b}{(a-b)+(d-c)},\ q_1 = \frac{d-b}{(a-b)+(d-c)}$$

$$q_2 = \frac{a-c}{(a-b)+(d-c)} \quad \text{and} \quad v = \frac{ad-bc}{(a-b)+(d-c)}$$

In this game, a = 3, b = 2, c = 1 and d = 5.

$$\therefore p_1 = \frac{5-1}{(3-2)+(5-1)} = \frac{4}{5},\ p_2 = \frac{3-2}{(3-2)+(5-1)} = \frac{1}{5},\ q_1 = \frac{5-2}{(3-2)+(5-1)} = \frac{3}{5}$$

$$q_2 = \frac{3-1}{(3-2)+(5-1)} = \frac{2}{5} \quad \text{and} \quad v = \frac{3\times5-2\times1}{(3-2)+(5-1)} = \frac{13}{5}$$

Hence, the solution of the game is as follows:

The optimal strategy for player A is (4/5, 1/5), the optimal strategy for player B is (3/5, 2/5) and the value of the game is 13/5.

Now, you can try the following exercises for solving a couple of games.

E1) Solve the two-person zero-sum game having the following payoff matrix for player A:

		Players B	
		$\mathbf{B_1}$	$\mathbf{B_2}$
Player A	$\mathbf{A_1}$	–2	–1
	$\mathbf{A_2}$	4	–3

E2) Solve the coin matching game explained in Sec. 11.2 of Unit 11.

12.3 DOMINANCE RULES FOR REDUCING THE SIZE OF THE GAME

In Sec. 12.2, we have derived and applied the algebraic method to solve 2×2 two-person zero-sum games without saddle point. So this method cannot be applied directly to $m\times n$ two-person zero-sum games without a saddle point if either m or n or both are greater than 2. But some of these games can be reduced to 2×2 games by applying the dominance rules, if applicable. In this section, we state these rules and explain how to apply them. But before you learn the dominance rules, you should understand inferior and superior strategies, which we explain below.

Inferior and Superior Strategies: Let $a=(a_1, a_2, ..., a_n)$ and $b=(b_1, b_2, ..., b_n)$ be two n-tuples. We say that n-tuple b **dominates** n-tuple a if

$a_i \le b_i$ for all $i, 1 \le i \le n$

If $a_i < b_i$ for all $i, 1 \le i \le n$ then we say that n-tuple b **strictly dominates** n-tuple a. Suppose that $a_{ij}, \left(i=1, 2, ..., m; j=1, 2, ..., n\right)$ represents the payoff value of payoff matrix of player A corresponding to the i^{th} strategy of row player A and j^{th} strategy of column player B. Then we say that the r^{th} strategy of row player A is **inferior** to his/her s^{th} strategy (or s^{th} strategy is **superior** to r^{th} strategy for row player A) if

$a_{rj} \le a_{sj}$ for all $j, 1 \le j \le n$

that is, all the elements of the r^{th} row are less than or equal to the corresponding elements of the s^{th} row.

Remember that in the payoff matrix of row player A, payoff values represent gains of player A while for column player B, they represent losses. So, from the point of view of player B, we say that the r^{th} strategy of column player B is **inferior** to his/her s^{th} strategy (or s^{th} strategy is **superior** to r^{th} strategy for column player B) if

$$a_{ir} \ge a_{is} \text{ for all } i, \text{or } 1 \le i \le m \quad \left[\begin{array}{l}\because \text{payoff values represent} \\ \text{losses for player B}\end{array}\right]$$

that is, if all the elements of the r^{th} column are greater than or equal to the corresponding elements of the s^{th} column.

So, in general, we define the dominant strategy as follows:

Dominant Strategy: A strategy is said to be the dominant strategy for a player if it is better than another strategy available to that player irrespective of the strategy adopted by the other player.

We now state the dominance rules to be used for reducing the size of the payoff matrix.

Rule 1

If all elements in a row (say, the i^{th} row) of a payoff matrix for row player are **less than or equal to** the corresponding elements of the other row (say j^{th} row), then the row player A will never employ his/her i^{th} strategy because it is dominated by his/her j^{th} strategy. So we can delete the i^{th} row from the payoff matrix. Remember that payoff values are gains for row player.

Rule 2

If all the elements in a column (say, the r^{th} column) of a payoff matrix for row player are **greater than or equal to** the corresponding elements of the other column (say, the s^{th} column), then the column player B will never employ his/her r^{th} strategy because it is dominated by his/her s^{th} strategy. So we can delete the r^{th} column from the payoff matrix. Remember that payoff values are losses for column player.

Note 3: It is important to note that in the case of row player A, the inferior row is deleted but in the case of column player B, the superior column is deleted. This is due to the fact that payoff values are gains for row player A but losses for column player B.

Rule 3

A strategy can also be deleted if it is dominated by average (or any convex combination) of two or more other pure strategies.

Note 4: A combination $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$ of $x_1, x_2, ..., x_n$ is said to be a convex combination if $\alpha_i \geq 0$ for all i, $1 \leq i \leq n$ and

$\alpha_1 + \alpha_2 + ... + \alpha_n = 1$.

If you want to know more about convex combination, refer to Unit 1 of the Course MSTE-002.

Let us apply these rules to a couple of examples.

Example 2: Solve the two-person zero-sum game having the following payoff matrix for player A:

		Player B		
		B_1	B_2	B_3
Player A	A_1	–2	–4	6
	A_2	5	4	1
	A_3	3	2	8

Solution: We first check whether a saddle point exists or not.

		Player B			Row Minima	Maximin Value
		B_1	B_2	B_3		
Player A	A_1	–2	–4	6	–4	$\max\{-4, 1, 2\} = 2$
	A_2	5	4	1	1	
	A_3	3	2	8	2	
Column Maxima		5	4	8		
Minimax Value		$\min\{5, 4, 8\} = 4$				

Since maximin value $\neq$ minimax value, there is no saddle point. Hence, we shall obtain the solution in terms of mixed strategies. Let us first reduce the payoff matrix by applying the dominance rules.

Step 1: We note that each element of the first row is less than the corresponding element of the third row. This implies that the first row is dominated by the third row. So we can eliminate the first row from the given payoff matrix. Thus, we write the reduced payoff matrix as:

		Player B		
		B_1	B_2	B_3
Player A	A_2	5	4	1
	A_3	3	2	8

Step 2: Each element of the first column is greater than the corresponding element of the second column. This implies that the first column is dominated by the second column. So we can eliminate the first column and write the reduced payoff matrix as:

		Player B	
		B_2	B_3
Player A	A_2	4	1
	A_3	2	8

In the reduced payoff matrix given above neither any row dominates any other row nor any column dominates any other column. So this matrix cannot be reduced any further using dominance rules. Since the reduced payoff matrix is of order 2×2, we can apply the algebraic method for 2×2 two-person zero-sum game.

Let (p_1, p_2, p_3) and (q_1, q_2, q_3) be the mixed strategies for players A and B, respectively, for the original payoff matrix. Therefore, from the reduced payoff matrix, (p_2, p_3) and (q_2, q_3) are the mixed strategies for the players A and B, respectively. If v is the value of the game, then from equations (3 a to e), we have

$$p_2 = \frac{d-c}{(a-b)+(d-c)},\ p_3 = \frac{a-b}{(a-b)+(d-c)},\ q_2 = \frac{d-b}{(a-b)+(d-c)}$$

$$q_3 = \frac{a-c}{(a-b)+(d-c)} \quad \text{and} \quad v = \frac{ad-bc}{(a-b)+(d-c)}$$

In this case a = 4, b = 1, c = 2, d = 8.

$$\therefore p_2 = \frac{8-2}{(4-1)+(8-2)} = \frac{6}{9} = \frac{2}{3},\ p_3 = \frac{4-1}{(4-1)+(8-2)} = \frac{3}{9} = \frac{1}{3}$$

$$q_2 = \frac{8-1}{(4-1)+(8-2)} = \frac{7}{9},\ q_3 = \frac{4-2}{(4-1)+(8-2)} = \frac{2}{9}$$

$$v = \frac{4\times 8 - 1\times 2}{(4-1)+(8-2)} = \frac{30}{9} = \frac{10}{3}$$

Hence, the solution of the game is:

The optimal strategy for player A is (0, 2/3, 1/3), the optimal strategy for player B is (0, 7/9, 2/9) and the value of the game is 10/3.

Example 3: Solve the two-person zero-sum game having the following payoff matrix for player A:

		Player B				
		B_1	B_2	B_3	B_4	B_5
Player A	A_1	6	1	3	2	3
	A_2	5	7	–5	1	4
	A_3	8	4	–1	2	2
	A_4	3	3	–2	2	5

Solution: We first check whether a saddle point exists or not.

		Player B					Row Minima	Maximin Value
		B_1	B_2	B_3	B_4	B_5		
Player A	A_1	6	1	3	2	3	1	$\max\{1, -5, -1, -2\} = 1$
	A_2	5	7	–5	1	4	–5	
	A_3	8	4	–1	2	2	–1	
	A_4	3	3	–2	2	5	–2	
Column Maxima		8	7	3	2	5		
Minimax Value		$\min\{8, 7, 3, 2, 5\} = 2$						

Since maximin value $\neq$ minimax value, the saddle point does not exist. Hence, we have to obtain the solution in terms of mixed strategies. Let us first reduce the payoff matrix by applying the dominance rules.

Step 1: We note that each element of the first and fifth columns is greater than or equal to the corresponding elements of the fourth column. This implies that the fourth column dominates both first and fifth columns. So we can eliminate the first and fifth columns from the given payoff matrix and write the reduced payoff matrix as:

		Player B		
		B_2	B_3	B_4
Player A	A_1	1	3	2
	A_2	7	–5	1
	A_3	4	–1	2
	A_4	3	–2	2

Step 2: Each element of the fourth row is less than or equal to the corresponding element of the third row. This implies that the fourth row is dominated by the third row. So we can eliminate the fourth row from the reduced payoff matrix. The reduced payoff matrix after this step is:

		Player B		
		B_2	B_3	B_4
Player A	A_1	1	3	2
	A_2	7	–5	1
	A_3	4	–1	2

Step 3: In this reduced payoff matrix, neither any row is directly dominated by any other row nor any column is directly dominated by any other column. But we observe that the average of the first and second columns of the payoff matrix dominates the third column. So, we can eliminate the third column from this reduced payoff matrix and write the reduced payoff matrix as:

		Player B	
		B_2	B_3
Player A	A_1	1	3
	A_2	7	–5
	A_3	4	–1

Step 4: Again, in this reduced payoff matrix neither any row is directly dominated by any other row nor any column is directly dominated by any other column. But we observe that the average of the first and second rows of this reduced payoff matrix dominates the third row. So we can eliminate the third row from this reduced payoff matrix. The reduced payoff matrix after this step is:

		Player B	
		B_2	B_3
Player A	A_1	1	3
	A_2	7	–5

In this reduced payoff matrix, neither any row dominates any other row nor any column dominates any other column. So it cannot be reduced further using dominance rules. Since the reduced payoff matrix is of order 2×2, we can apply the algebraic method for 2×2 two-person zero-sum game. Let (p_1, p_2, p_3, p_4) and $(q_1, q_2, q_3, q_4, q_5)$ be the mixed strategies for players A and B, respectively, for the original payoff matrix. Thus, (p_1, p_2) and (q_2, q_3) are the mixed strategies from the reduced payoff matrix for players A and B, respectively. If v is the value of the game, then from equations (3 a to e) we have

$$p_1 = \frac{d-c}{(a-b)+(d-c)},\ p_2 = \frac{a-b}{(a-b)+(d-c)},\ \ q_2 = \frac{d-b}{(a-b)+(d-c)}$$

$$q_3 = \frac{a-c}{(a-b)+(d-c)} \quad \text{and} \quad v = \frac{ad-bc}{(a-b)+(d-c)}$$

In this case a = 1, b = 3, c = 7, d = –5.

$$\therefore p_1 = \frac{-5-7}{(1-3)+(-5-7)} = \frac{-12}{-14} = \frac{6}{7},\ p_2 = \frac{1-3}{(1-3)+(-5-7)} = \frac{-2}{-14} = \frac{1}{7}$$

$$q_2 = \frac{-5-3}{(1-3)+(-5-7)} = \frac{-8}{-14} = \frac{4}{7}, q_3 = \frac{1-7}{(1-3)+(-5-7)} = \frac{-6}{-14} = \frac{3}{7}$$

$$v = \frac{1\times(-5)-3\times 7}{(1-3)+(-5-7)} = \frac{-26}{-14} = \frac{13}{7}$$

Hence, the solution of the game is:

The optimal strategy for player A is (6/7, 1/7, 0, 0), the optimal strategy for player B is (0, 4/7, 3/7, 0, 0) and the value of the game is 13/7.

You may like to try the following exercise for practice.

E3) Solve the two-person zero-sum game having the following payoff matrix for player A:

		Player B			
		B_1	B_2	B_3	B_4
Player A	A_1	4	3	2	1
	A_2	6	4	5	0
	A_3	1	2	0	3

Note 5: You have seen that dominance rules (if applicable) reduce the size of the game. You can apply these rules to any game whether it has a saddle point or not. So we can also apply these rules to the games discussed in Unit 11. If we do so,the size of the game reduces to 1×1. And if the size of the game reduces to 1×1, it implies that the game has been solved. For example, let us apply these rules to Example 1 of Unit 11.

Step 1: We note that each element of the first row is greater than or equal to the corresponding element of the second row. This implies that the first row dominates the second row. So we can eliminate the second row from the given payoff matrix and write the reduced payoff matrix as:

		Player B		
		B_1	B_2	B_3
Player A	A_1	5	7	4
	A_3	6	1	3

Step 2: Each element of the first column is greater than or equal to the corresponding element of the third column. This implies that the third column dominates the first column. So we can eliminate the first column from the reduced payoff matrix. The reduced payoff matrix after this step is:

		Player B	
		B_2	B_3
Player A	A_1	7	4
	A_3	1	3

Step 3: Each element of the first row is greater than or equal to the corresponding element of the second row. This implies that the first row dominates the second row. So we can eliminate the second row from the reduced payoff matrix. The reduced payoff matrix after this step is:

		Player B	
		B_2	B_3
Player A	A_1	7	4

Step 4: The single element 7 of the first column is greater than the corresponding single element of the second column. This implies that the second column dominates the first column. So we can eliminate the first column from the reduced payoff matrix. The reduced payoff matrix after this step is:

		Player B
		B_3
Player A	A_1	4

Thus payoff matrix reduces to order 1×1. Hence the solution of the game is:

The optimal strategy for player A is (1, 0, 0), the optimal strategy for player B is (0, 0, 1) and the value of the game is 4. You can match the solution obtained here with the one obtained in Example 1 of Unit 11.

12.4 GRAPHICAL METHOD FOR REDUCING THE $2\times n$ AND $m\times 2$ GAMES TO 2×2 GAMES

In Sec. 12.3, you have learnt the dominance rules for reducing the size of the game if the payoff matrix has some inferior strategies. But many-a-times we have to deal either directly with $2\times n$ and $m\times 2$ games or they reduce to one of these forms after applying dominance rules. Therefore, it becomes important to learn how to solve $2\times n$ and $m\times 2$ games. In this section, we discuss one such method known as the **graphical method**, which reduces the $2\times n$ and $m\times 2$ games into 2×2 games. Once a game is reduced to size 2×2, we can solve it using the algebraic method discussed in Sec. 12.2. So, let us first discuss the graphical method for $2\times n$ games.

12.4.1 Graphical Method for Reducing $2\times n$ Games to 2×2 Games

In $2\times n$ games, the row player A has two moves and the column player B may have any finite number of moves (say, n). Let the payoff matrix for player A for a $2\times n$ game be:

		Player B					
		B_1	B_2	…	B_j	…	B_n
Player A	A_1	a_{11}	a_{12}	…	a_{1j}	…	a_{1n}
	A_2	a_{21}	a_{22}	…	a_{2j}	…	a_{2n}

If the game has a saddle point, it can be solved using the maximin-minimax principle discussed in Unit 11, which gives pure strategies as a solution. If the game does not have a saddle point, its solution will be in terms of mixed strategies. Let (p_1, p_2) and $(q_1, q_2, ..., q_j, ..., q_n)$ be the optimal mixed strategies for player A and player B, respectively. Let us rewrite the payoff matrix with these probabilities.

			Player B					
			q_1	q_2	…	q_j	…	q_n
			B_1	B_2	…	B_j	…	B_n
Player A	p_1	A_1	a_{11}	a_{12}	…	a_{1j}	…	a_{1n}
	p_2	A_2	a_{21}	a_{22}	…	a_{2j}	…	a_{2n}

Now, if $e_1, e_2, ..., e_n$ be the expected payoffs for player A corresponding to different strategies of player B, then these are given by:

$e_1 = a_{11}p_1 + a_{21}p_2$ if player B employs strategy B_1

$e_2 = a_{12}p_1 + a_{22}p_2$ if player B employs strategy B_2

$\vdots$

$e_n = a_{1n}p_1 + a_{2n}p_2$ if player B employs strategy B_n

But $p_1 + p_2 = 1 \Rightarrow p_2 = 1 - p_1$. So writing the above expected payoffs for player A in terms of p_1, we get

$e_1 = (a_{11} - a_{21})p_1 + a_{21}$ if player B employs strategy B_1

$e_2 = (a_{12} - a_{22})p_1 + a_{22}$ if player B employs strategy B_2

$\vdots$

$e_n = (a_{1n} - a_{2n})p_1 + a_{2n}$ if player B employs strategy B_n

The above expected payoffs for player A are linear equations in p_1. We know from Sec. 2.5 of Unit 2 of MST-001 that the graph of a linear function given by

$$y = ax + b$$

is a straight line (here we have the expected payoff in place of y and p_1 in place of x). Hence, the n expected payoffs of row player A corresponding to different strategies employed by column player B represent n straight lines. Let us now explain how to plot these lines on a graph paper. First of all, we take values of p_1 along the horizontal axis. However, the values of p_1 vary from 0 to 1

$(0 \le p_1 \le 1)$ because p_1 represents probability. Therefore, we restrict the horizontal axis at 1 unit, starting from $p_1 = 0$ and ending at $p_1 = 1$ (see Fig. 12.1a).

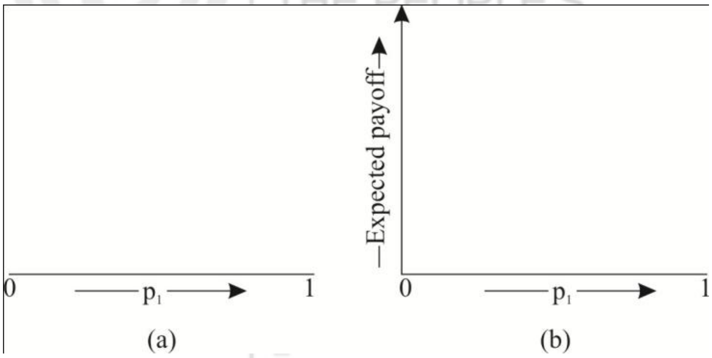

Fig. 12.1: Horizontal and vertical axes for the graph of player A.

We take the expected payoff along the vertical axis (see Fig. 12.1b).

Now, consider the first line:

$$e_1 = (a_{11} - a_{21})p_1 + a_{21}$$

Actually, two points are sufficient to draw the graph of a straight line (read the margin remark). So we obtain two points corresponding to $p_1 = 0$ and $p_1 = 1$, respectively.

Recall from Sec. 2.5 of Unit 2 of MST-001, how to draw the graph of a straight line. We simply obtain any three points from the equation of the straight line, plot them on the graph paper and finally join the points. In that unit, we took three points just to keep an automatic check on our calculations. Because if the three points do not fall on the line, it means that we have made some mistake in obtaining them. We do not get this advantage if we work with two points.

At $p_1 = 0,\ e_1 = a_{21}$

At $p_1 = 1,\ e_1 = (a_{11} - a_{21})(1) + a_{21} = a_{11}$

Thus, the two points on the first line are $(0,\ a_{21})$ and $(1,\ a_{11})$.

Similarly, two points on the second line are $(0,\ a_{22}),\ (1,\ a_{12})$.

.
.
.

Two points on the n^{th} line are $(0,\ a_{2n}),\ (1,\ a_{1n})$.

Let us plot these lines: we first draw two vertical lines: the first along $p_1 = 0$ and the second along $p_1 = 1$ (see Fig. 12.2a).

We observe that the second coordinates of all points having first coordinate 0 are the same as the payoff values in the second row. Further, the second coordinate of the points having first coordinate 1 are the same as payoff values in the first row. Thus, for numerical problems we need not obtain two points for each line. We can simply take payoff values of the second row on the vertical line along $p_1 = 0$ and payoff values of the first row on the vertical line along $p_1 = 1$(see Fig. 12.2a). Here we have made one assumption that $a_{21} < a_{22} < ... < a_{2n}$ and $a_{12} < a_{11} < a_{13} < a_{14} < ... < a_{1n}$. The lines can be plotted by simply joining the points a_{21} to a_{11}, a_{22} to a_{12},..., a_{2n} to a_{1n} (see Fig. 12.2b).

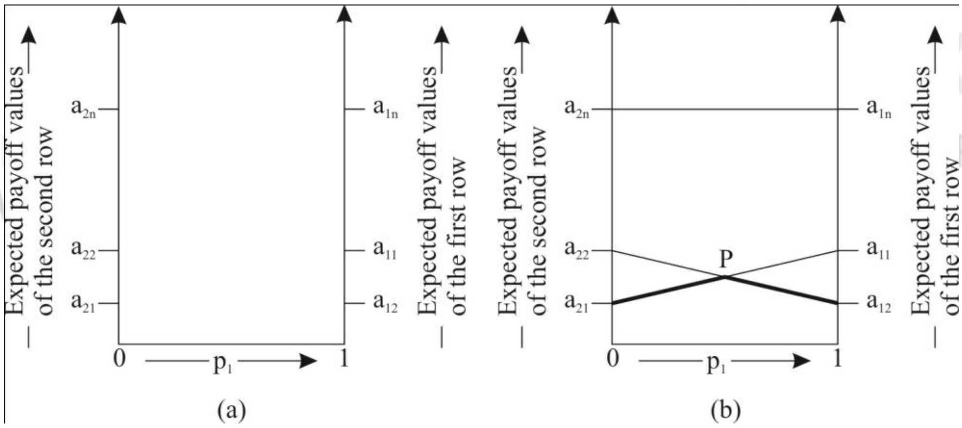

Fig. 12.2: Representation of expected payoff values and expected payoff line segments.

Note that these lines represent expected payoffs of row player A. So the lowest boundary of these lines will give the minimum expected payoff as a function of p_1. Pay attention to the lowest boundary formed by the lower line segments of expected payoff lines in Fig. 12.2b. It is known as the **lower envelope** and indicated by **bold line segments.** But, obviously player A would like to maximise his/her expected payoffs (gains). This can be achieved by selecting the highest point on the lower envelope in Fig. 12.2b, it is indicated by the point P. If there are only two straight lines passing through the highest point, then we identify the two strategies corresponding to these lines and eliminate all other strategies of player B from the payoff matrix. This reduces the game to a 2×2 two-person zero-sum game and can be solved using the algebraic method discussed in Sec 12.2.

But if there are more than two straight lines passing through the highest point, then there will be alternative solutions. To obtain one solution, we select any two straight lines from among the lines passing through the highest point and identify the strategies corresponding to these two lines. This will again reduce the game into 2×2 two-person zero-sum game and can be solved using the algebraic method discussed in Sec. 12.2. In brief, we can summarise the above procedure in the following steps:

Step 1: Draw a horizontal line of length one unit which extends from $p_1 = 0$ to $p_1 = 1$.

Step 2: Draw two vertical lines passing through the two points $p_1 = 0$ and $p_1 = 1$ on the horizontal axis.

Step 3: Represent the payoff values of the second row on the vertical line $p_1 = 0$ and payoff values of the first row on the vertical line $p_1 = 1$.

Step 4: Join a_{21} to a_{11}, a_{22} to $a_{12}, ..., a_{2n}$ to a_{1n}. The n straight lines thus plotted will represent the expected payoffs of row player A.

Step 5: Identify the lowest boundary of the expected payoff lines as obtained in Step 4, called the lower envelope. Draw bold lines along the line segments of the boundary.

Step 6: Identify the highest point on the lower envelope.

Step 7: Identify the strategies corresponding to the expected payoff lines, which pass through the highest point. If there are only two such strategies, then eliminate all other strategies. This reduces the game to 2×2 two-person zero-sum game. Solve it using the algebraic method. If there are more than two such strategies, select any two among them and eliminate all other strategies. This will again reduce the game into 2×2 two-person zero-sum game and can be solved using the algebraic method.

Let us consider an example to explain the graphical method for solving $2\times n$ games.

Example 4: Solve the two-person zero-sum game having the following payoff matrix for player A:

		Player B		
		B_1	B_2	B_3
Player A	A_1	1	2	7
	A_2	6	4	2

Solution: We first check if a saddle point exists or not.

		Player B			Row Minima	Maximin Value
		B_1	B_2	B_3		
Player A	A_1	1	2	7	1	$\max\{1, 2\} = 2$
	A_2	6	4	2	2	
Column Maxima		6	4	7		
Minimax Value		$\min\{6, 4, 7\} = 4$				

Since maximin value $\neq$ minimax value, there is no saddle point. Hence, we have to obtain the solution in terms of mixed strategies. Moreover, this game cannot be reduced by using dominance rules, because it has neither any inferior row nor any inferior column. But this game is of the type $2\times n$ where n = 3. Hence, we can apply the graphical method for $2\times n$ games which will reduce the game into a 2×2 two-person zero-sum game. Let (p_1, p_2) and (q_1, q_2, q_3) be the optimal mixed strategies for players A and B, respectively. Then the expected payoffs of player A corresponding to the moves B_1, B_2, B_3 of the player B are given by

$$e_1 = 1p_1 + 6p_2 = p_1 + 6(1-p_1) = -5p_1 + 6 \qquad \ldots \text{(i)}$$

$$e_2 = 2p_1 + 4p_2 = 2p_1 + 4(1-p_1) = -2p_1 + 4 \qquad \ldots \text{(ii)}$$

$$e_3 = 7p_1 + 2p_2 = 7p_1 + 2(1-p_1) = 5p_1 + 2 \qquad \ldots \text{(iii)}$$

We now follow the steps given below to apply the graphical method for this 2×3 game. Refer to Fig. 12.3.

Step 1: We draw a horizontal line of length 1 unit extending from the point $R(p_1 = 0)$ to point $S(p_1 = 1)$.

Step 2: We draw two vertical lines $p_1 = 0$ and $p_1 = 1$ passing through the points R and S, respectively, on the horizontal axis.

Step 3: We represent the payoff values of the second row (6, 4, 2) along the vertical line $p_1 = 0$ and payoff values of the first row (1, 2, 7) along the vertical line $p_1 = 1$.

Step 4: We plot the straight lines given by (i), (ii) and (iii) by joining the payoff values of two rows with their corresponding values. Here we plot the lines by joining 6 to 1, 4 to 2 and 2 to 7 as shown in Fig. 12.3.

Step 5: We draw bold lines along the line segments, which form the lowest boundary of expected payoff lines given by (i), (ii) and (iii) to obtain the lower envelope (see Fig. 12.3).

Step 6: We identify the highest point on the lower envelope. In Fig. 12.3, the highest point is P, which is the point of intersection of expected payoff lines given by (ii) and (iii). Thus, the given game reduces to:

			Player B	
			q_2	q_3
			B_2	B_3
Player A	p_1	A_1	2	7
	p_2	A_2	4	2

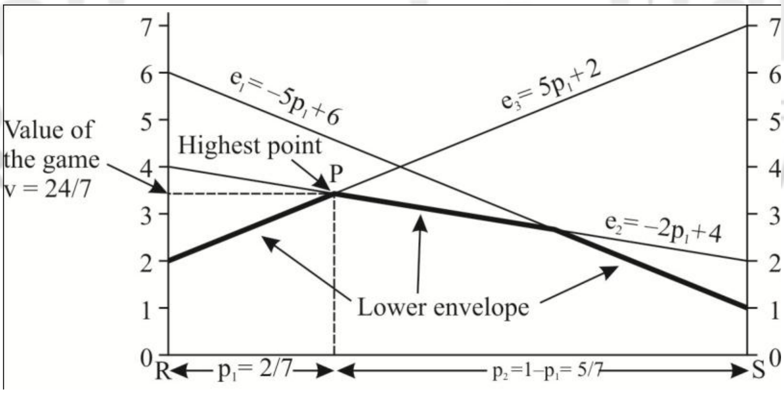

Fig. 12.3: Graph for player A.

We know that for 2×2 two-person zero-sum games p_1, p_2, q_2, q_3 and v are given from equations (3 a to e) as:

$$p_1 = \frac{d-c}{(a-b)+(d-c)},\ p_2 = \frac{a-b}{(a-b)+(d-c)},\ q_2 = \frac{d-b}{(a-b)+(d-c)}$$

$$q_3 = \frac{a-c}{(a-b)+(d-c)} \text{ and } v = \frac{ad-bc}{(a-b)+(d-c)}$$

In this case, a = 2, b = 7, c = 4, d = 2.

$$\therefore p_1 = \frac{2-4}{(2-7)+(2-4)} = \frac{-2}{-7} = \frac{2}{7},\ p_2 = \frac{2-7}{(2-7)+(2-4)} = \frac{-5}{-7} = \frac{5}{7}$$

$$q_2 = \frac{2-7}{(2-7)+(2-4)} = \frac{-5}{-7} = \frac{5}{7},\ q_3 = \frac{2-4}{(2-7)+(2-4)} = \frac{-2}{-7} = \frac{2}{7}$$

$$v = \frac{2\times2-7\times4}{(2-7)+(2-4)} = \frac{-24}{-7} = \frac{24}{7}$$

Hence, the optimal strategies for original game for players A and B are (2/7, 5/7) and (0, 5/7, 2/7), respectively, and the value of the game is 24/7.

We now discuss the graphical method for $m\times2$ games.

12.4.2 Graphical Method for Reducing $m\times2$ Games to 2×2 Games

In an $m\times2$ game, the column player B has two moves and the row player A may have any finite number of moves (say, m). Let the payoff matrix for player A for an $m\times2$ game be:

		Player B	
		B_1	B_2
Player A	A_1	a_{11}	a_{12}
	A_2	a_{21}	a_{22}
	$\vdots$	$\vdots$	$\vdots$
	A_i	a_{i1}	a_{i2}
	$\vdots$	$\vdots$	$\vdots$
	A_m	a_{m1}	a_{m2}

If the game has a saddle point, it can be solved using the maximin-minimax principle discussed in Unit 11, which gives pure strategies as a solution. Suppose the game does not have a saddle point, which means that the solution of the game will be in terms of mixed strategies. Let $(p_1, p_2, ..., p_m)$ and (q_1, q_2) be the optimal mixed strategies for player A and player B, respectively. Let us rewrite the payoff matrix with these probabilities as follows:

			Player B	
			q_1	q_2
			B_1	B_2
Player A	p_1	A_1	a_{11}	a_{12}
	p_2	A_2	a_{21}	a_{22}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	p_i	A_i	a_{i1}	a_{i2}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	p_m	A_m	a_{m1}	a_{m2}

If $e_1, e_2, ..., e_m$ are the expected payoffs for player B corresponding to different strategies of player A, then these are given by:

$$
\begin{aligned}
e_1 &= a_{11}q_1 + a_{12}q_2 && \text{if player A employs strategy } A_1 \\
e_2 &= a_{21}q_1 + a_{22}q_2 && \text{if player A employs strategy } A_2 \\
&\vdots && \qquad\vdots \\
e_m &= a_{m1}q_1 + a_{m2}q_2 && \text{if player A employs strategy } A_m
\end{aligned}
$$

But $q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$.

So we can write the expected payoffs for player B in terms of q_1 as follows:

$$
\begin{aligned}
e_1 &= (a_{11} - a_{12})q_1 + a_{12} && \text{if player A employs strategy } A_1 \\
e_2 &= (a_{21} - a_{22})q_1 + a_{22} && \text{if player A employs strategy } A_2 \\
&\vdots && \qquad\vdots \\
e_m &= (a_{m1} - a_{m2})q_1 + a_{m2} && \text{if player A employs strategy } A_m
\end{aligned}
$$

The expected payoffs for player B given above are linear equations in q_1 and hence represent straight lines. We can plot these lines on a graph paper as explained in Sec. 12.4.1. We take the values of q_1 along the horizontal axis from $q_1 = 0$ to $q_1 = 1$ $(\text{i.e., } 0 \le q_1 \le 1)$. So the horizontal axis is restricted at 1 unit starting from $q_1 = 0$ and ending at $q_1 = 1$ (see Fig. 12.4). We take expected payoffs along the vertical axis.

Let us consider the first line:

$$e_1 = (a_{11} - a_{12})q_1 + a_{12}$$

We know that a straight line can be plotted easily by joining two points on it. So, we take the points $q_1 = 0$ and $q_1 = 1$ on the line:

At $q_1 = 0,\ e_1 = a_{12}$

At $q_1 = 1,\ e_1 = (a_{11} - a_{12})(1) + a_{12} = a_{11}$

Thus, two points on the first line are $(0,\ a_{12})$ and $(1,\ a_{11})$.

Similarly, two points on the second line are $(0,\ a_{22})$ and $(1,\ a_{21})$.

.
.
.

Two points on the m^{th} line are $(0,\ a_{m2})$ and $(1,\ a_{m1})$.

Let us plot these lines. For convenience, we draw two vertical lines: the first along $q_1 = 0$ and the second along $q_1 = 1$ as shown in Fig 12.4. We observe that the second coordinates of all points having first coordinate 0 are the same as the payoff values in second column. Moreover, the second coordinates of the points having first coordinate 1 are the same as the payoff values in the first column. Thus, for numerical problems we need not obtain two points for each line. We can simply take payoff values of the second column along the vertical line $q_1 = 0$ and payoff values of the first column along the vertical line $q_1 = 1$. The lines can be plotted by simply joining a_{11} to a_{12}, a_{21} to $a_{22}, ..., a_{m1}$ to a_{m2}.

Note that these lines represent expected payoffs (losses) of column player B. So the uppermost boundary of these lines will give the maximum expected payoffs (losses) as a function of q_1. The uppermost boundary formed by the uppermost line segments of expected payoff lines is known as the **upper envelope** and indicated by bold line segments.

Now, player A would like to minimise his/her expected payoffs (losses). This can be achieved by selecting the lowest point on the upper envelope. If only two straight lines pass through the lowest point, then we identify the strategies corresponding to these lines and eliminate all other strategies of player A from the payoff matrix. This reduces the game to a 2×2 two-person zero-sum game. So it can be solved using the algebraic method.

However, if more than two straight lines pass through the lowest point, then there will be alternative solutions. To obtain one solution, we select any two straight lines from among the lines passing through the lowest point and identify the strategies corresponding to these two lines. This will again reduce the game to a 2×2 two-person zero-sum game, which can be solved using the algebraic method. In brief, we can summarise the above procedure in the following steps.

Step 1: Draw a horizontal line of length one unit, which extends from the point $R(q_1 = 0)$ to the point $S(q_1 = 1)$.

Step 2: Draw two vertical lines passing through the points R and S, respectively, on the horizontal axis.

Step 3: Represent the payoff values of the second column on the vertical line along $q_1 = 0$ and payoff values of the first column on the vertical line along $q_1 = 1$.

Step 4: Join a_{11} to a_{12}, a_{21} to $a_{22}, \ldots, a_{m1}$ to a_{m2}. The m straight lines thus plotted will represent the expected payoffs (losses) of column player B.

Step 5: Identify the uppermost boundary of expected payoff lines obtained in Step 4, called the upper envelope. Draw bold lines along the line segments of the boundary.

Step 6: Identify the lowest point on the upper envelope.

Step 7: Identify the strategies corresponding to expected payoff lines, which pass through the lowest point. If there are only two such strategies, eliminate all other strategies. This will reduce the $m \times 2$ game to a 2×2 two-person zero-sum game. We can solve it using the algebraic method. But if there are more than two such strategies, select any two from them and eliminate all other strategies. This will again reduce the game to a 2×2 two-person zero-sum game, which can be solved using the algebraic method.

Let us consider an example to explain the graphical method for $m \times 2$ games.

Example 5: Solve the two-person zero-sum game having the following payoff matrix for player A:

		Player B	
		B_1	**B_2**
Player A	**A_1**	5	–2
	A_2	–3	6
	A_3	–2	3

Solution: We first check if a saddle point exists or not.

		Player B		Row Minima	Maximin Value
		B_1	**B_2**		
Player A	**A_1**	5	–2	–2	$\max\{-2, -3, -2\} = -2$
	A_2	–3	6	–3	
	A_3	–2	3	–2	
Column Maxima		5	6		
Minimax Value		$\min\{5, 6\} = 5$			

Since maximin value $\neq$ minimax value, there is no saddle point. Hence, we have to obtain the solution in terms of mixed strategies. Also, we cannot reduce this game using dominance rules, because there is neither any inferior row nor any inferior column. But this game is of the type $m \times 2$ where m = 3. Hence, we can apply the graphical method for $m \times 2$ games, which will reduce the game to a 2×2 two-person zero-sum game. Let (p_1, p_2, p_3) and (q_1, q_2) be the optimal mixed strategies for players A and B, respectively. The expected payoffs of player B corresponding to the moves A_1, A_2, A_3 of the player A are given by

$$e_1 = 5q_1 - 2q_2 = 5q_1 - 2(1 - q_1) = 7q_1 - 2 \qquad \ldots \text{(i)}$$

$$e_2 = -3q_1 + 6q_2 = -3q_1 + 6(1 - q_1) = -9q_1 + 6 \qquad \ldots \text{(ii)}$$

$$e_3 = -2q_1 + 3q_2 = -2q_1 + 3(1 - q_1) = -5q_1 + 3 \qquad \ldots \text{(iii)}$$

We now follow the steps given below to apply the graphical method for $m \times 2$ games.

Step 1: We draw a horizontal line, 1 unit in length, extending from the point $R(q_1 = 0)$ to the point $S(q_1 = 1)$. (see Fig. 12.4).

Step 2: We draw two vertical lines, $q_1 = 0$ and $q_1 = 1$, passing through the two points R and S on the horizontal axis.

Step 3: We represent the payoff values of the second column (i.e., –2, 6, 3) along the vertical line $q_1 = 0$ and payoff values of the first column (i.e., 5, –3, –2) along the vertical line $q_1 = 1$ (see Fig. 12.4).

Step 4: We plot the straight lines given by (i), (ii) and (iii) by joining the payoff values of the two columns with their corresponding values. Here we plot the line by joining 5 to –2, –3 to 6 and –2 to 3 as shown in Fig. 12.4.

Step 5: We draw bold lines along the line segments, which are the uppermost boundary of expected payoff lines given by (i), (ii) and (iii), to obtain the upper envelope.

Step 6: We identify the lowest point on the upper envelope. Here the lowest point is P, which is the point of intersection of expected payoff lines given by (i) and (ii). Thus, the given game reduces to:

			Player B	
			q_1	q_2
			B_1	B_2
Player A	p_1	A_1	5	–2
	p_2	A_2	–3	6

We know that for 2×2 two-person zero-sum games, p_1, p_2, q_1, q_2 and the value of the game v are given from equations (3a to e) as:

$$p_1 = \frac{d-c}{(a-b)+(d-c)},\ p_2 = \frac{a-b}{(a-b)+(d-c)},\ q_1 = \frac{d-b}{(a-b)+(d-c)}$$

$$q_2 = \frac{a-c}{(a-b)+(d-c)} \text{ and } v = \frac{ad-bc}{(a-b)+(d-c)}$$

In this case a = 5, b = –2, c = –3, d = 6.

$$\therefore p_1 = \frac{6-(-3)}{(5-(-2))+(6-(-3))} = \frac{9}{16},\ p_2 = \frac{5-(-2)}{(5-(-2))+(6-(-3))} = \frac{7}{16}$$

$$q_1 = \frac{6-(-2)}{(5-(-2))+(6-(-3))} = \frac{8}{16} = \frac{1}{2},\ q_2 = \frac{5-(-3)}{(5-(-2))+(6-(-3))} = \frac{8}{16} = \frac{1}{2}$$

$$\text{and } v = \frac{5\times 6-(-2)(-3)}{(5-(-2))+(6-(-3))} = \frac{24}{16} = \frac{3}{2}$$

Hence, the optimal strategies for the original game for players A and B are $\left(\frac{9}{16}, \frac{7}{16}, 0\right)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$, respectively, and the value of the game is $\frac{3}{2}$.

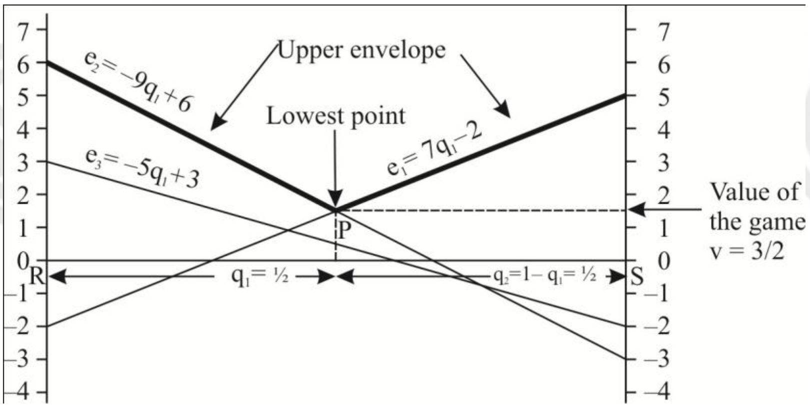

Fig. 12.4: Graph for player B.

You may try the following exercises for applying the graphical methods to solve $2\times n$ and $m\times 2$ games.

E4) Solve the two-person zero-sum game having the following payoff matrix for player A:

		Player B				
		B_1	B_2	B_3	B_4	B_5
Player A	A_1	3	4	5	–2	2
	A_2	1	6	–3	3	7

E5) Solve the two-person zero-sum game having the following payoff matrix for player A:

		Player B	
		B_1	B_2
Player A	A_1	8	–2
	A_2	5	1
	A_3	2	3
	A_4	4	7

12.5 SOLUTION OF 3×3 TWO-PERSON ZERO-SUM GAME

In Sec. 12.2, we have derived a general result for solving 2×2 two-person zero-sum games. In this section, we are not going to derive such a result for 3×3 two-person zero-sum games. We shall simply solve a 3×3 two-person zero-sum game to give you an idea how such games can be solved. Let us explain this idea with the help of a very popular game known as Stone-Paper-Scissors game.

Stone-Paper-Scissors Game

Let us call the two players playing the game as player A and player B. The rules of the game are as follows:

- In each turn, both players will simultaneously have to produce by a gesture of their hands, either a stone, a paper or a pair of scissors (see Fig. 12.5).
- Stone beats scissors (since a stone can crush scissors), scissors beat paper (since scissors can cut paper) and paper beats stone (since a stone can be covered by paper).

- If both players produce the same gesture of their hands, it is a tie.
- In each turn, the loser has to pay Rs 1 to the winner, and in case of a tie both get rupee zero as payoff from each other.

What are the optimal strategies for players A and B?

Solution: The above information can be represented by a payoff matrix as given below. Note that the first entry of each of the nine cells represents the payoff value of row player A and the second entry represents the payoff value of column player B. You should also study Fig. 12.5 to understand the game better.

		Player B		
		Stone (B_1)	**Paper (B_2)**	**Scissors (B_3)**
Player A	**Stone (A_1)**	0, 0	–1,1	1, –1
	Paper (A_2)	1, –1	0, 0	–1, 1
	Scissors (A_3)	–1, 1	1, –1	0, 0

Since it is a two-person zero-sum game, the payoff values of row player are equal in magnitude but opposite in sign to the payoff values of the column player in each play. So we can simply represent the game by the following payoff matrix for the row player (refer to Note 2 in Unit 11 of this course):

		Player B		
		Stone (B_1)	**Paper (B_2)**	**Scissors (B_3)**
Player A	**Stone (A_1)**	0	–1	1
	Paper (A_2)	1	0	–1
	Scissors (A_3)	–1	1	0

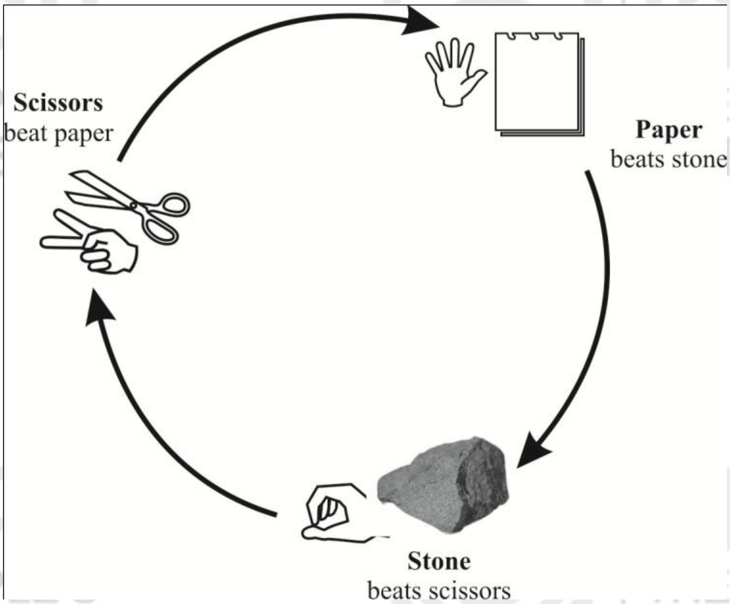

Fig. 12.5: Gestures for the stone-paper-scissors game.

We first check for the saddle point as follows:

		Player B			**Row Minima**	**Maximin Value**
		Stone (B_1)	**Paper (B_2)**	**Scissors (B_3)**		
Player A	**Stone (A_1)**	0	–1	1	–1	$\max\{-1,-1,-1\}=-1$
	Paper (A_2)	1	0	–1	–1	
	Scissors (A_3)	–1	1	0	–1	
Column Maxima		1	1	1		
Minimax Value		$\min\{1,1,1\}=1$				

Since the maximin value $\neq$ minimax value, there is no saddle point. Hence, we have to obtain the solution in terms of mixed strategies. Also, we cannot reduce this game by using dominance rules because there is neither any inferior row nor any inferior column. Further, it is neither a $2\times n$ nor an $m\times 2$ game. So it cannot be reduced to a 2×2 two-person zero-sum game using the graphical method. Hence, we cannot obtain its solution using any of the methods discussed so far. What do we do now? Let us find out.

We first apply the fundamental theorem of game theory (stated in Sec. 12.2) to this game. According to the theorem, since this is a finite two-person zero-sum game, it has a solution in terms of mixed strategies. Let (p_1, p_2, p_3) and (q_1, q_2, q_3) be the optimal mixed strategies for players A and B, respectively, and v be the value of the game. Note that there is one commonality between 2×2 games (discussed in Sec. 12.2) and the 3×3 game. Both are square games. So let us proceed as in Sec. 12.2 and rewrite the payoff matrix with their probabilities:

			Player B		
			q_1	q_2	q_3
			Stone (B_1)	**Paper (B_2)**	**Scissors (B_3)**
Player A	p_1	**Stone (A_1)**	0	–1	1
	p_2	**Paper (A_2)**	1	0	–1
	p_3	**Scissors (A_3)**	–1	1	0

Now, the expected gain of player A is

$0p_1 + 1p_2 + (-1)p_3 = p_2 - p_3$ when player B employs strategy B_1

$(-1)p_1 + 0p_2 + 1p_3 = -p_1 + p_3$ when player B employs strategy B_2

$1p_1 + (-1)p_2 + 0p_3 = p_1 - p_2$ when player B employs strategy B_3

Similarly, the expected loss of player B is

$0q_1 + (-1)q_2 + 1q_3 = -q_2 + q_3$ when player A employs strategy A_1

$1q_1 + 0q_2 + (-1)q_3 = q_1 - q_3$ when player A employs strategy A_2

$(-1)q_1 + 1q_2 + 0q_3 = -q_1 + q_2$ when player A employs strategy A_3

Since v is the value of the game and (p_1, p_2, p_3) is the optimal mixed strategy for player A, we have

$$\underset{\text{I}}{p_2 - p_3} = \underset{\text{II}}{-p_1 + p_3} = \underset{\text{III}}{p_1 - p_2} = \underset{\text{IV}}{v}$$

Now,

$\text{I} \,\&\, \text{II} \Rightarrow p_1 + p_2 - 2p_3 = 0$ … (i)

and $\text{II} \,\&\, \text{III} \Rightarrow -2p_1 + p_2 + p_3 = 0$ … (ii)

Also $p_1 + p_2 + p_3 = 1$ … (iii)

Subtracting (i) from (iii) gives

$$3p_3 = 1 \Rightarrow p_3 = \frac{1}{3}$$

Subtracting (ii) from (iii) gives

$$3p_1 = 1 \Rightarrow p_1 = \frac{1}{3}$$

Putting the values of p_1 and p_3 in (iii), we get

$$\frac{1}{3} + p_2 + \frac{1}{3} = 1 \Rightarrow p_2 = \frac{1}{3}$$

Again $\text{I} \,\&\, \text{IV} \Rightarrow v = p_2 - p_3 \Rightarrow v = \frac{1}{3} - \frac{1}{3} \Rightarrow v = 0$

Also, v is value of the game and (q_1, q_2, q_3) is the optimal mixed strategy for the player B.

$$\therefore \underset{\text{I}}{-q_2 + q_3} = \underset{\text{II}}{q_1 - q_3} = \underset{\text{III}}{-q_1 + q_2} = \underset{\text{IV}}{v}$$

$\text{I} \,\&\, \text{II} \Rightarrow -q_1 - q_2 + 2q_3 = 0$ … (iv)

$\text{II} \,\&\, \text{III} \Rightarrow 2q_1 - q_2 - q_3 = 0$ … (v)

Also $q_1 + q_2 + q_3 = 1$ … (vi)

Adding (iv) and (vi) gives

$$3q_3 = 1 \Rightarrow q_3 = \frac{1}{3}$$

Adding (v) and (vi) gives

$$3q_1 = 1 \Rightarrow q_1 = \frac{1}{3}$$

Putting the values of q_1 and q_3 in (vi), we get

$$\frac{1}{3} + q_2 + \frac{1}{3} = 1 \Rightarrow q_2 = \frac{1}{3}$$

$$\text{I} \,\&\, \text{IV} \Rightarrow v = -q_2 + q_3 \Rightarrow v = -\frac{1}{3} + \frac{1}{3} \Rightarrow v = 0$$

Hence, the optimal strategy for both players A and B is (1/3, 1/3, 1/3) and the value of the game is 0.

12.6 FLOWCHART

In Unit 11, we have introduced some key terms involved in the game theory. You have studied the maximin-minimax principle to solve two-person zero-sum games with saddle point. In Sec. 12.2, we have derived and applied the algebraic method to solve 2×2 two-person zero-sum games. In Sec. 12.3, you have learnt the dominance rules and applied them to reduce the size of the payoff matrix by eliminating inferior row(s) or column(s) or both. In Sec. 12.4, we have discussed the graphical method, which reduces $2 \times n$ and $m \times 2$ games into 2×2 games. We have also solved some games to explain the steps involved in this method. Finally, in Sec. 12.5, you have seen how 3×3 two-person zero-sum games can be solved by extending the technique for 2×2 two-person zero-sum games discussed in Sec 12.2. We have also solved a commonly played game known as Stone-Paper-Scissors game. At this stage, you may wish to know: How do we solve the $m \times n$ games, which are not reducible to any of the games discussed in this unit? The answer is that $m \times n$ two-person zero-sum games can be solved by converting them into linear programming problems. But this discussion is beyond the scope of this course. However, if you are keen to learn this technique, you should first learn how to solve linear programming problems. This is discussed in detail in the course MSTE-002. Then you may refer to any book recommended at the end of this block or any other book available to you. We end the discussion by giving a flow chart in Fig. 12.6, which shows all the information that we have explained in this unit sequentially.

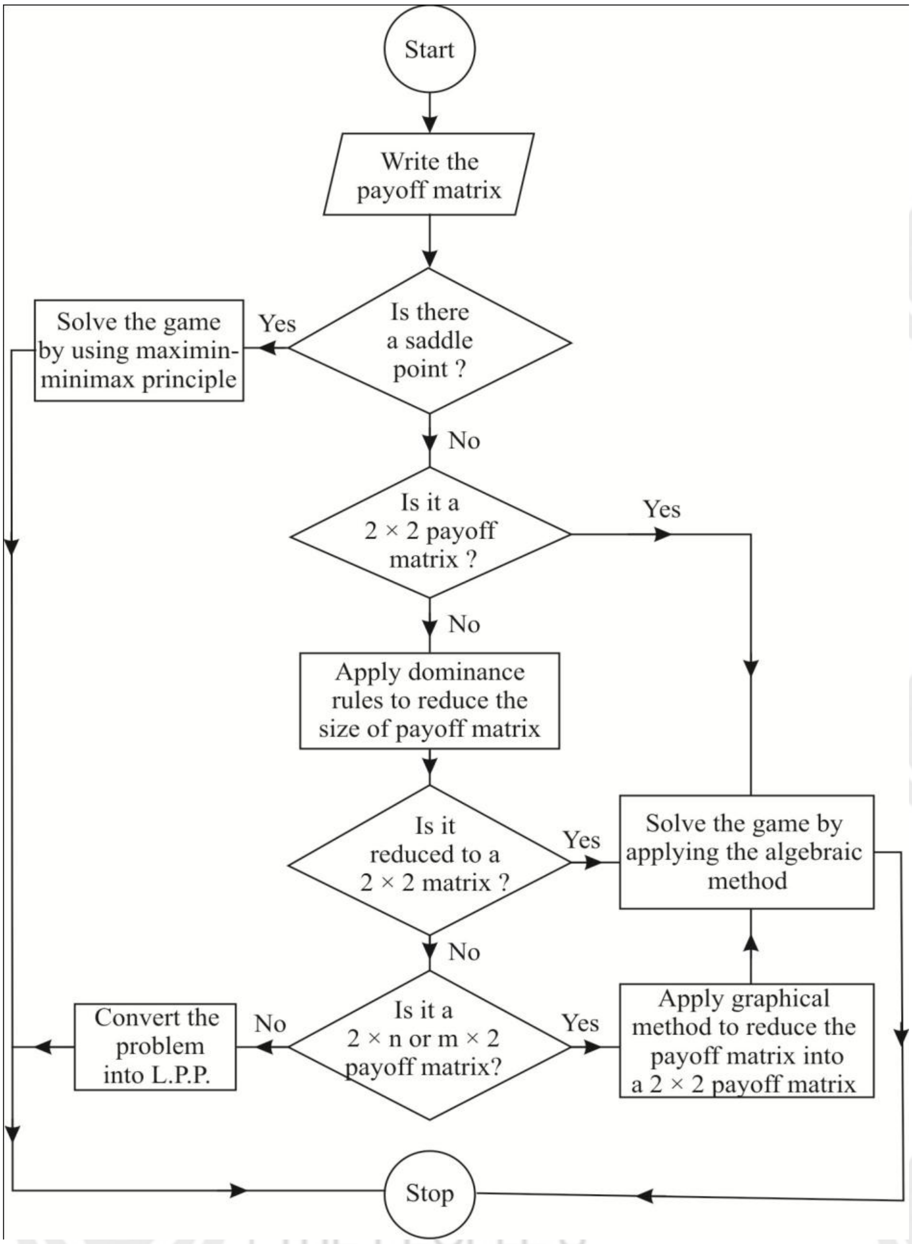

Fig. 12.6: Flowchart of game theory approach.

Let us now present a summary of the main points discussed in this unit.

12.7 SUMMARY

1) The solution of a 2×2 two-person zero-sum games in terms of mixed strategies is provided by the **algebraic method**:

$$p_1 = \frac{d-c}{(a-b)+(d-c)}$$

$$p_2 = \frac{a-b}{(a-b)+(d-c)}$$

$$q_1 = \frac{d-b}{(a-b)+(d-c)}$$

$$q_2 = \frac{a-c}{(a-b)+(d-c)}$$

$$v = \frac{ad-bc}{(a-b)+(d-c)}$$

2) **Dominant Strategy:** A strategy is said to be the dominant strategy for a player if it is better than another strategy available to that player irrespective of the strategy adopted by the other player.

3) The dominance rules for reducing the size of the payoff matrix are:

Rule 1

If all the elements in a row (say, the i^{th} row) of a payoff matrix for row player are **less than or equal to** the corresponding elements of the other row (say, the j^{th} row), then the row player A will never employ his/her i^{th} strategy because it is dominated by his/her j^{th} strategy. So we can delete the i^{th} row from the payoff matrix.

Rule 2

If all the elements in a column (say, the r^{th} column) of a payoff matrix for column player are **greater than or equal to** the corresponding elements of the other column (say, the s^{th} column), then the column player B will never employ his/her r^{th} strategy because it is dominated by his/her s^{th} strategy. So we can delete the r^{th} column from the payoff matrix.

Rule 3

A strategy can also be deleted if it is dominated by average (or any convex combination) of two or more other pure strategies.

4) $2 \times n$ and $m \times 2$ games are solved by using the graphical method. The **graphical method** reduces $2 \times n$ and $m \times 2$ games to 2×2 games. Once a game is reduced to size 2×2 it can be solved by the algebraic method.

12.8 SOLUTIONS/ANSWERS

E1) We first check for the saddle point:

		Player B		Row Minima	Maximin Value
		B_1	B_2		
Player A	A_1	–2	–1	–2	$\max\{-2, -3\} = -2$
	A_2	4	–3	–3	
Column Maxima		4	–1		
Minimax Value		$\min\{4, -1\} = -1$			

Since the maximin value (–2) $\neq$ minimax value (–1), there is no saddle point. Hence, we have to obtain the solution in terms of mixed strategies. Let (p_1, p_2) and (q_1, q_2) be the optimal mixed strategies for player A and player B, respectively, and v be the value of the game. Then from equations (3a to e), we have

$$p_1 = \frac{d-c}{(a-b)+(d-c)},\ p_2 = \frac{a-b}{(a-b)+(d-c)},\ q_1 = \frac{d-b}{(a-b)+(d-c)}$$

$$q_2 = \frac{a-c}{(a-b)+(d-c)} \text{ and } v = \frac{ad-bc}{(a-b)+(d-c)}$$

In this case, a = –2, b = –1, c = 4, d = –3.

$$\therefore p_1 = \frac{-3-4}{-2-(-1)+(-3)-4} = \frac{-7}{-8} = \frac{7}{8},\ p_2 = \frac{-2-(-1)}{-2-(-1)+(-3)-4} = \frac{-1}{-8} = \frac{1}{8}$$

$$q_1 = \frac{-3-(-1)}{-2-(-1)+(-3)-4} = \frac{-2}{-8} = \frac{1}{4},\ q_2 = \frac{-2-4}{-2-(-1)+(-3)-4} = \frac{-6}{-8} = \frac{3}{4}$$

$$v = \frac{(-2)(-3)-(-1)\times 4}{-2-(-1)+(-3)-4} = \frac{10}{-8} = -\frac{5}{4}$$

Hence, the solution of the game is:

The optimal strategy for player A is (7/8, 1/8), the optimal strategy for player B is (1/4, 3/4) and the value of the game is $-5/4$.

E2) In Note 2 of Unit 11, we have explained that the coin matching game is a two-person zero-sum game between child X (player I) and child Y (player II). The payoff matrix for player I is given below:

		Child Y (player II)	
		Head	**Tail**
Child X (Player I)	**Head**	1	–1
	Tail	–1	1

We first check for the saddle point.

		Player II		**Row Minima**	**Maximin Value**
		B_1	**B_2**		
Player I	**A_1**	1	–1	–1	$\max\{-1, -1\} = -1$
	A_2	–1	1	–1	
Column Maxima		1	1		
Minimax Value		$\min\{1, 1\} = 1$			

Since maximin value (–1) $\neq$ minimax value (1), there is no saddle point. Hence, we obtain the solution in terms of mixed strategies.

Let (p_1, p_2) and (q_1, q_2) be the optimal mixed strategies for player A and player B, respectively, and v be the value of the game.

From equations (3a to e),

$$p_1 = \frac{d-c}{(a-b)+(d-c)},\ p_2 = \frac{a-b}{(a-b)+(d-c)},\ q_1 = \frac{d-b}{(a-b)+(d-c)}$$

$$q_2 = \frac{a-c}{(a-b)+(d-c)}, \text{ and } v = \frac{ad-bc}{(a-b)+(d-c)}$$

In this case, a = 1, b = –1, c = –1, d = 1

$$\therefore p_1 = \frac{1-(-1)}{1-(-1)+1-(-1)} = \frac{2}{4} = \frac{1}{2},\ p_2 = \frac{1-(-1)}{1-(-1)+1-(-1)} = \frac{2}{4} = \frac{1}{2}$$

$$q_1 = \frac{1-(-1)}{1-(-1)+1-(-1)} = \frac{2}{4} = \frac{1}{2}, q_2 = \frac{1-(-1)}{1-(-1)+1-(-1)} = \frac{2}{4} = \frac{1}{2}$$

$$v = \frac{1\times 1-(-1)\times(-1)}{1-(-1)+1-(-1)} = \frac{0}{4} = 0$$

Hence, the solution of the game is:

The optimal strategy for both child X (player I) and child Y (player II) is (1/2, 1/2) and the value of the game is 0.

E3) We first check for the saddle point.

		Player B				Row Minima	Maximin Value
		B_1	B_2	B_3	B_4		
Player A	A_1	4	3	2	1	1	$\max\{1, 0, 0\} = 1$
	A_2	6	4	5	0	0	
	A_3	1	2	0	3	0	
Column Maxima		6	4	5	3		
Minimax Value		$\min\{6, 4, 5, 3\} = 3$					

Since the maximin value $\neq$ minimax value, there is no saddle point. Hence, we obtain the solution in terms of mixed strategies. Let us first reduce the payoff matrix by applying the dominance rules.

Step 1: We note that each element of the first column is greater than or equal to the corresponding element of the third column. This implies that the third column dominates the first column. So, we eliminate the first column from the given payoff matrix. Thus, the reduced payoff matrix can be written as follows:

		Player B		
		B_2	B_3	B_4
Player A	A_1	3	2	1
	A_2	4	5	0
	A_3	2	0	3

Step 2: In the reduced payoff matrix given above, neither any row is directly dominated by any other row nor any column is directly dominated by any other column. But we observe that the average of the second and the third columns of the reduced payoff matrix dominates the first column. So, we can eliminate the first column from the reduced payoff matrix and obtain a further reduced payoff matrix as:

		Player B	
		B_3	B_4
Player A	A_1	2	1
	A_2	5	0
	A_3	0	3

Step 3: In the reduced payoff matrix given above, neither any row is directly dominated by any other row nor any column is directly dominated by any other column. But we observe that the average of the second and third rows of the reduced payoff

matrix dominates the first row. So, we can eliminate the first row from the reduced payoff matrix and obtain a further reduced payoff matrix as:

		Player B	
		B_3	B_4
Player A	A_2	5	0
	A_3	0	3

Now, neither any row dominates any other row nor any column dominates any other column. So, this reduced payoff matrix cannot be reduced further by using dominance rules. But this reduced payoff matrix is of the order 2×2. So, we can apply the algebraic method for 2×2 two-person zero-sum games. Let (p_1, p_2, p_3), (q_1, q_2, q_3, q_4) be the mixed strategies for players A and B, respectively, for the original payoff matrix. The (2×2) reduced payoff matrix tells us that (p_2, p_3) and (q_3, q_4) are the mixed strategies for players A and B, respectively. If v is the value of the game, then from equations (3a to e), we have

$$p_2 = \frac{d-c}{(a-b)+(d-c)},\ p_3 = \frac{a-b}{(a-b)+(d-c)},\ q_3 = \frac{d-b}{(a-b)+(d-c)}$$

$$q_4 = \frac{a-c}{(a-b)+(d-c)} \text{ and } v = \frac{ad-bc}{(a-b)+(d-c)}$$

In this case, a = 5, b = 0, c = 0, d = 3.

$$\therefore p_2 = \frac{3-0}{(5-0)+(3-0)} = \frac{3}{8},\ p_3 = \frac{5-0}{(5-0)+(3-0)} = \frac{5}{8}$$

$$q_3 = \frac{3-0}{(5-0)+(3-0)} = \frac{3}{8}, q_4 = \frac{5-0}{(5-0)+(3-0)} = \frac{5}{8}$$

$$v = \frac{5\times 3 - 0\times 0}{(5-0)+(3-0)} = \frac{15}{8}$$

Hence, the solution of the game is:

The optimal strategy for player A is (0, 3/8, 5/8), the optimal strategy for player B is (0, 0, 3/8, 5/8) and the value of the game is 15/8.

E4) We first check for the saddle point.

		Player B					Row Minima	Maximin Value
		B_1	B_2	B_3	B_4	B_5		
Player A	A_1	3	4	5	–2	2	–2	$\max\{-2, -3\} = -2$
	A_2	1	6	–3	3	7	–3	
Column Maxima		3	6	5	3	7		
Minimax Value		$\min\{3, 6, 5, 3, 7\} = 3$						

Since maximin value $\neq$ minimax value, there is no saddle point. Hence, we have to obtain the solution in terms of mixed strategies. But column four dominates the second and fifth columns. So, by dominance rules, both second and fifth columns can be eliminated from this payoff matrix and the reduced payoff matrix can be written as:

		Player B		
		B_1	B_3	B_4
Player A	A_1	3	5	–2
	A_2	1	–3	3

This payoff matrix cannot be reduced further by using dominance rules. But this game is of the type $2\times n$ where n = 3. Hence, we can apply the graphical method for $2\times n$ games, which will reduce the game to a 2×2 two-person zero-sum game. Let (p_1, p_2) and $(q_1, q_2, q_3, q_4, q_5)$ be the optimal mixed strategies for players A and B, respectively, for the original payoff matrix. Then the expected payoffs of player A corresponding to the moves B_1, B_3, B_4 of player B are given by:

$$e_1 = 3p_1 + p_2 = 3p_1 + (1-p_1) = 2p_1 + 1 \qquad \ldots \text{(i)}$$

$$e_3 = 5p_1 - 3p_2 = 5p_1 - 3(1-p_1) = 8p_1 - 3 \qquad \ldots \text{(ii)}$$

$$e_4 = -2p_1 + 3p_2 = -2p_1 + 3(1-p_1) = -5p_1 + 3 \qquad \ldots \text{(iii)}$$

We now follow the steps given below to apply the graphical method for $2\times n$ games as shown in Fig. 12.7:

Step 1: We draw a horizontal line, 1 unit in length, extending from the point $R(p_1 = 0)$ to point $S(p_1 = 1)$.

Step 2: We draw two vertical lines $p_1 = 0$ and $p_1 = 1$ passing through the two points R and S on the horizontal axis.

Step 3: We represent the payoff values of the second row (i.e., 1, –3, 3) along the vertical line $p_1 = 0$ and payoff values of the first row (i.e., 3, 5, –2) along the vertical line $p_1 = 1$.

Step 4: We plot the straight lines given by (i), (ii) and (iii) by joining the payoff values of two rows with their corresponding values. Here we plot the lines by joining 3 to 1, 5 to –3 and –2 to 3 as shown in Fig. 12.7.

Step 5: We draw bold lines along the line segments which form the lowest boundary of expected payoff lines given by (i), (ii), (iii). It is the lower envelope.

Step 6: We identify the highest point P on the lower envelope. Here it is the point of intersection of expected payoff lines given by (ii) and (iii). Thus, the given game reduces to:

			Player B	
			q_3	q_4
			B_3	B_4
Player A	p_1	A_1	5	–2
	p_2	A_2	–3	3

From equations (3a to e), we have

$$p_1 = \frac{d-c}{(a-b)+(d-c)},\ p_2 = \frac{a-b}{(a-b)+(d-c)},\ q_2 = \frac{d-b}{(a-b)+(d-c)}$$

$$q_3 = \frac{a-c}{(a-b)+(d-c)} \quad \text{and} \quad v = \frac{ad-bc}{(a-b)+(d-c)}$$

Here a = 5, b = –2, c = –3, d = 3.

$$\therefore p_1 = \frac{3-(-3)}{(5-(-2))+(3-(-3))} = \frac{6}{13},\ p_2 = \frac{5-(-2)}{(5-(-2))+(3-(-3))} = \frac{7}{13}$$

$$q_2 = \frac{3-(-2)}{(5-(-2))+(3-(-3))} = \frac{5}{13},\ q_3 = \frac{5-(-3)}{(5-(-2))+(3-(-3))} = \frac{8}{13}$$

$$v = \frac{5\times 3-(-2)\times(-3)}{(5-(-2))+(3-(-3))} = \frac{9}{13}$$

Hence, the optimal strategies for original game for the players A and B are $\left(\frac{6}{13}, \frac{7}{13}\right)$ and $\left(0,\ 0, \frac{5}{13}, \frac{8}{13}, 0\right)$, respectively, and the value of the game is $\frac{9}{13}$.

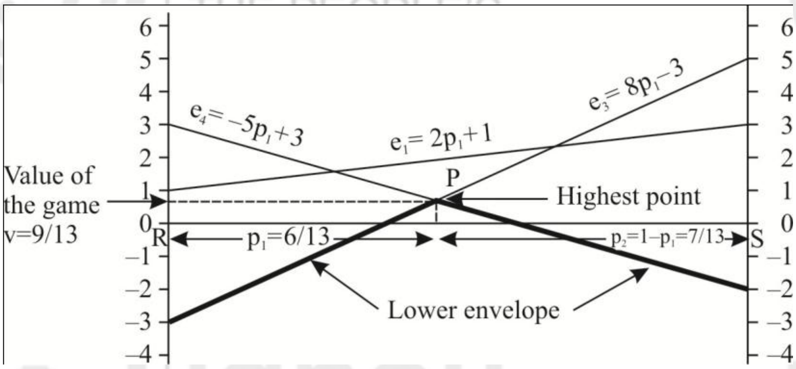

Fig. 12.7: Graph for player A in E4.

E 5) We first check for the saddle point.

		Player B		Row Minima	Maximin Value
		B_1	**B_2**		
Player A	**A_1**	8	–2	–2	$\max\{-2, 1, 2, 4\} = 4$
	A_2	5	1	1	
	A_3	2	3	2	
	A_4	4	7	4	
Column Maxima		8	7		
Minimax Value		$\min\{8, 7\} = 7$			

Since maximin value ≠ minimax value, there is no saddle point. Hence, we have to obtain the solution in terms of mixed strategies. Since the third row is dominated by the fourth row, using dominance rules, we can eliminate the third row from this payoff matrix. The reduced payoff matrix can be written as:

		Player B	
		B_1	**B_2**
Player A	**A_1**	8	–2
	A_2	5	1
	A_4	4	7

This payoff matrix cannot be reduced further by using dominance rules. But this game is of the type $m\times 2$, where m = 3. Hence, we can apply the graphical method for $m\times 2$ games, which will reduce the game to a 2×2

two-person zero-sum game. Let (p_1, p_2, p_3, p_4) and (q_1, q_2) be the optimal mixed strategies for players A and B, respectively, for the original payoff matrix. The expected payoffs of player B corresponding to the moves A_1, A_2, A_4 of player A are given by

$$e_1 = 8q_1 - 2q_2 = 8q_1 - 2(1-q_1) = 10q_1 - 2 \quad \ldots \text{(i)}$$

$$e_2 = 5q_1 + 1q_2 = 5q_1 + 1(1-q_1) = 4q_1 + 1 \quad \ldots \text{(ii)}$$

$$e_4 = 4q_1 + 7q_2 = 4q_1 + 7(1-q_1) = -3q_1 + 7 \quad \ldots \text{(iii)}$$

We follow the steps given below to apply the graphical method for solving $m \times 2$ games as shown in Fig.12.8:

Step 1: We draw a horizontal line, 1 unit in length, extending from the point $R(q_1 = 0)$ to point $S(q_1 = 1)$ (see Fig. 12.8).

Step 2: We draw two vertical lines $q_1 = 0$ and $q_1 = 1$ passing through the two points R and S on the horizontal axis.

Step 3: We represent the payoff values of the second column (i.e.,–2, 1, 7) along the vertical line $q_1 = 0$ and the payoff values of the first column (i.e., 8, 5, 4) along the vertical line $q_1 = 1$ as shown in Fig. 12.8.

Step 4: We plot the straight lines given by (i), (ii) and (iii) by joining the payoff values of two columns with their corresponding values. Here we plot the line by joining 8 to –2, 5 to 1 and 4 to 7 as shown in Fig. 12.8.

Step 5: We draw bold lines along the line segments, which form the uppermost boundary of the expected payoff lines given by (i), (ii), (iii). This is the upper envelope.

Step 6: We identify the lowest point P on the upper envelope. Here it is the point of intersection of expected payoff lines given by (i) and (iii). Thus, the given game reduces to:

			Player B	
			$\mathbf{q_1}$	$\mathbf{q_2}$
			$\mathbf{B_1}$	$\mathbf{B_2}$
Player A	$\mathbf{p_1}$	$\mathbf{A_1}$	8	–2
	$\mathbf{p_4}$	$\mathbf{A_4}$	4	7

From equations (3a to e), we have

$$p_1 = \frac{d-c}{(a-b)+(d-c)}, p_4 = \frac{a-b}{(a-b)+(d-c)}, q_1 = \frac{d-b}{(a-b)+(d-c)}$$

$$q_2 = \frac{a-c}{(a-b)+(d-c)} \text{ and } v = \frac{ad-bc}{(a-b)+(d-c)}$$

In this case, a = 8, b = –2, c = 4, d = 7.

$$\therefore \; p_1 = \frac{7-4}{(8-(-2))+(7-4)} = \frac{3}{13}, \; p_4 = \frac{8-(-2)}{(8-(-2))+(7-4)} = \frac{10}{13}$$

$$q_1 = \frac{7-(-2)}{(8-(-2))+(7-4)} = \frac{9}{13},\ q_2 = \frac{8-4}{(8-(-2))+(7-4)} = \frac{4}{13}$$

$$v = \frac{8\times 7-(-2)\times 4}{(8-(-2))+(7-4)} = \frac{64}{13}$$

Hence, the optimal strategies for original game for players A and B are $\left(\frac{3}{13}, 0, 0, \frac{10}{13}\right)$ and $\left(\frac{9}{13}, \frac{4}{13}\right)$, respectively, and the value of the game is $\frac{64}{13}$.

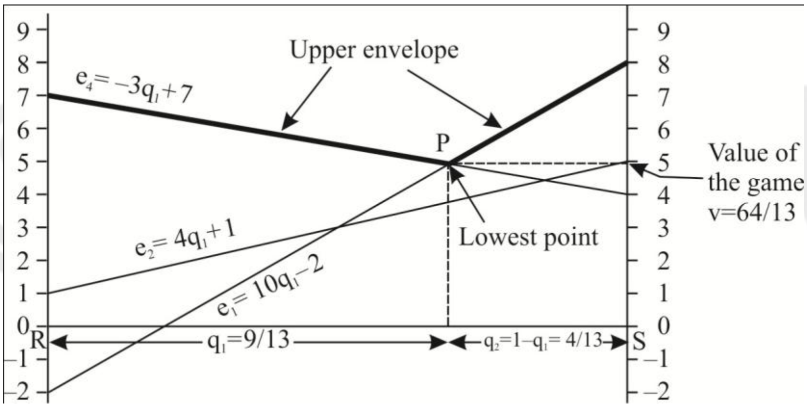

Fig. 12.8: Graph for player B.

FURTHER READING

1) Operations Research by Kanti Swarup, P.K. Gupta and Man Mohan; Sultan Chand & Sons Education Publishers, New Delhi (2006) [Chapters 16 and 17**].**

2) Basic Statistics by B. L. Agarwal; New Age International (P) Limited, Publishers, New Delhi (2009) [Chapter 20].

3) Mathematics Programming Techniques by N. S. Kambo; Affiliated East-West Press Pvt. Ltd., New Delhi (1984) [Chapter 16].