# UNIT 4 TRANSPORTATION PROBLEM

## Structure



## 4.1 INTRODUCTION

In Units 2 and 3 of this block, we have discussed linear programming problems and methods of solving them. You have studied the graphical and simplex methods of solving LPPs in Unit 2 and Unit 3, respectively. Transportation problem is also one of the sub-classes of Linear Programming problems in which the objective is to transport various quantities (goods) initially stored at various origins/plants/factories to different destinations/distribution centres/warehouses in such a way that the total transportation cost is minimised. To achieve this objective, we must know the quantity of available supplies and the quantities demanded. Also, we must know the costs of shipping a unit from various origins to various destinations. Solving LPPs by either of the methods discussed in Units 2 and 3 in this block involves a lot of computational work and is quite cumbersome and difficult for solving the transportation problems. So in this unit, we discuss the methods, which are specifically applied for solving transportation problems.

### Objectives

After studying this unit, you should be able to:

- define a transportation problem;
- determine the basic feasible solution of a transportation problem using North-West Corner Rule, Least Cost and Vogel's Approximation methods;
- state the conditions for performing optimality test;
- apply the Stepping Stone and Modified Distribution (MODI) methods of obtaining the optimal solution of a transportation problem; and

- solve the transportation problems for special cases such as unbalanced transportation problem, case of degeneracy, case of alternative solutions, maximisation transportation problem, problems with prohibited routes.

## 4.2 MATHEMATICAL FORMULATION OF THE TRANSPORTATION PROBLEM

Suppose a manufacturer of an item has m origins (plants/factories) situated at places $O_1, O_2, \ldots, O_i, \ldots, O_m$ and suppose there are n destinations (warehouses/distribution centres) at $D_1, D_2, \ldots D_j \ldots D_n$. The aggregate of the capacities of all m origins is assumed to be equal to the aggregate of the requirements of all n destinations. Let $C_{ij}$ be the cost of transporting one unit from origin i to destination j. Let $a_i$ be the capacity/availability of items at origin i and $b_j$, the requirement/demand of the destination j. Then this transportation problem can be expressed in a tabular form as follows:

| Origin | Destinations $D_1$ $D_2$ ... $D_j$ ... $D_n$ | Availability / Capacity |
|---|---|---|
| $O_1$ | $C_{11}$ $C_{12}$ ... $C_{1j}$ ... $C_{1n}$ | $a_1$ |
| $O_2$ | $C_{21}$ $C_{22}$ ... $C_{2j}$ ... $C_{2n}$ | $a_2$ |
| . | . . . . | . |
| . | . . . . | . |
| . | . . . . | . |
| $O_i$ | $C_{i1}$ $C_{i2}$ ... $C_{ij}$ ... $C_{in}$ | $a_i$ |
| . | . . . . | . |
| . | . . . . | . |
| . | . . . . | . |
| $O_m$ | $C_{m1}$ $C_{m2}$ ... $C_{mj}$ ... $C_{mn}$ | $a_m$ |
| Requirement/ Demand | $b_1$ $b_2$ ... $b_j$ ... $b_n$ | Total |

The condition for the existence of a feasible solution to a transportation problem is given as

$$\sum_{j=1}^{m} a_i = \sum_{j=1}^{n} b_i \qquad \ldots(1)$$

Equation (1) tells us that the total requirement equals the total capacity. If it is not so, a dummy origin or destination is created to balance the total capacity and requirement.

Now let $x_{ij}$ be the number of units to be transported from origin i to destination j and $C_{ij}$ the corresponding cost of transportation. Then the total transportation cost is $\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij}$. It means that the product of the number of units allocated to each cell with the transportation cost per unit for that cell is calculated and added for all cells. Our objective is to allocate the units in such a way that the **total transportation cost is least**. The problem can also be stated as a linear programming problem as follows:

We are to minimise the total transportation cost

$$Z = \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}x_{ij} \qquad \ldots(2)$$

subject to the constraints:

$$\sum_{j=1}^{n} x_{1j} = a_1, \quad \sum_{j=1}^{n} x_{2j} = a_2, \ldots, \quad \sum_{j=1}^{n} x_{mj} = a_m$$

$$\sum_{i=1}^{m} x_{i1} = b_1, \quad \sum_{i=1}^{m} x_{i2} = b_2, \ldots, \quad \sum_{i=1}^{m} x_{in} = b_n \qquad \ldots(3)$$

and $\quad x_{ij} \geq 0 \quad$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

We can solve this transportation problem by the Simplex method. However, the solution is very lengthy. Moreover, it is a cumbersome process since a large number of decision variables and artificial variables are involved. Hence, we use an alternative method of solving a transportation problem known as the **transportation method**. It requires far less computational effort as compared to the Simplex method. In the transportation method, we first obtain the initial basic feasible solution and then perform the optimality test as explained in Secs. 4.3 and 4.4.

Solution by transportation method involves making a transportation model in the form of a matrix, finding an initial basic feasible solution, performing optimality test and moving towards an optimal solution.

## 4.3 METHODS OF FINDING INITIAL BASIC FEASIBLE SOLUTION

We can use any one of the following methods to determine the initial basic feasible solution:

i) **North-West Corner Rule**

ii) **Least Cost Method**

iii) **Vogel's Approximation Method**

Vogel's Approximation method generally gives a solution closer to the optimum solution. Hence, it is preferred over other methods.

We first discuss the North West Corner Rule.

### 4.3.1 North-West Corner Rule

The algorithm involved in the North-West Corner Rule (NWC) consists of the following steps.

We start with the north-west corner of the given problem and proceed as follows:

If the quantity needed at First Distribution Centre/Destination ($b_1$) is **less** than the quantity available at First Plant/Origin ($a_1$), we allocate a quantity equal to the requirement at First Distribution Centre to the cell (1, 1). At this stage, Column 1 is exhausted, so we cross it out. Since the requirement $b_1$ is fulfilled, we reduce the availability $a_1$ by $b_1$ and proceed to north-west corner of the resulting matrix, i.e., cell (1, 2).

If the quantity needed at First Distribution Centre/Destination ($b_1$) is **greater** than the quantity available at First Plant/Origin ($a_1$), we allocate a quantity equal to the quantity available at First Plant/Origin ($a_1$) to cell (1, 1). At this stage, Row 1 is exhausted, so we cross it out and proceed to north-west corner of the resulting matrix, i.e., cell (2, 1).

If the quantity needed at First Distribution Centre/Destination ($b_1$) is **equal** to the quantity available at First Plant/ Origin ($a_1$), we allocate a quantity equal to the requirement at First Distribution Centre/Destination (or the quantity available at First Plant/Origin). At this stage, both Column 1 as well as Row 1 is exhausted. We cross them out and proceed to the north-west corner of the resulting matrix, i.e., cell (2, 2).

We continue in this manner, until we reach the south-east corner of the original matrix.

Let us take up an example to further explain NWC rule.

**Example 1:** Apply the North-West Corner rule for finding the basic feasible solution of the following transportation problem:

| Warehouse / Factory | D | E | F | G | Capacity |
|---|---|---|---|---|---|
| A | 42 | 48 | 38 | 37 | 160 |
| B | 40 | 49 | 52 | 51 | 150 |
| C | 39 | 38 | 40 | 43 | 190 |
| Requirement | 80 | 90 | 110 | 220 | 500 |

**Solution:** We start from the North West corner, i.e., the Factory A and Warehouse D. The quantity needed at the First Warehouse (Warehouse D) is 80, which is less than the quantity available (160) at the First Factory A. Therefore, a quantity equal to the requirement at Warehouse D is to be allocated to the cell (A, D). Thus, the requirement of Warehouse D is met by Factory A. So we cross out column 1 and reduce the capacity of Factory A by 80. Then we go to cell (A, E), which is the North-West corner of the resulting matrix.

| Warehouse / Factory | D | E | F | G | Capacity |
|---|---|---|---|---|---|
| A | ~~42~~ 80 | 48 | 38 | 37 | ~~160~~ 80 |
| B | ~~40~~ | 49 | 52 | 51 | 150 |
| C | ~~39~~ | 38 | 40 | 43 | 190 |
| Requirement | ~~80~~ 0 | 90 | 110 | 220 | 500 |

The resulting matrix is

| Warehouse / Factory | E | F | G | Capacity |
|---|---|---|---|---|
| A | 48 | 38 | 37 | 80 |
| B | 49 | 52 | 51 | 150 |
| C | 38 | 40 | 43 | 190 |
| Requirement | 90 | 110 | 220 | |

Now, the quantity needed at the Second Warehouse (Warehouse E) is 90, which is greater than the quantity available (80) at the First Factory A. Therefore, we allocate a quantity equal to the capacity at Factory A, i.e., 80 to the cell (A, E). The requirement of warehouse E is reduced to 10. The capacity of Factory A is exhausted and has to be removed from the matrix. Therefore, we cross out row 1 and proceed to cell (B, E). This is the North-West corner of the resulting matrix.

| Warehouse / Factory | E | F | G | Capacity |
|---|---|---|---|---|
| A | ~~48~~ 80 | ~~38~~ | ~~37~~ | ~~80~~ 0 |
| B | 49 | 52 | 51 | 150 |
| C | 38 | 40 | 43 | 190 |
| Requirement | ~~90~~ 10 | 110 | 220 | |

The resulting matrix is

| Warehouse / Factory | E | F | G | Capacity |
|---|---|---|---|---|
| B | 49 | 52 | 51 | 150 |
| C | 38 | 40 | 43 | 190 |
| Requirement | 10 | 110 | 220 | |

Now, the quantity needed at the Second Warehouse (Warehouse E) is 10, which is less than the quantity available at the Second Factory B, which is 150. Therefore, the quantity 10 equal to the requirement at Warehouse E is allocated to the cell (B, E). Hence, the requirement of Warehouse E is met and we cross out column 1. We reduce the capacity of Factory B by 10 and proceed to cell (B, F), which is the new North-West corner of the resulting matrix.

| Warehouse / Factory | E | F | G | Capacity |
|---|---|---|---|---|
| B | ~~49~~ 10 | 52 | 51 | ~~150~~ 140 |
| C | ~~38~~ | 40 | 43 | 190 |
| Requirement | ~~10~~ 0 | 110 | 220 | |

The resulting matrix is

| Warehouse / Factory | F | G | Capacity |
|---|---|---|---|
| B | ~~52~~ 110 | 51 | ~~140~~ 30 |
| C | ~~40~~ | 43 | 190 |
| Requirement | ~~110~~ 0 | 220 | |

Again, the quantity needed at the Third Warehouse (Warehouse F) is 110. It is less than the quantity available at the Second Factory (Factory B), which is 140. Therefore, a quantity equal to the requirement at Warehouse F is allocated to the cell (B, F). Since, the requirement of warehouse F is met, we cross out Column 1 and reduce the capacity of Factory B by 110. Then we proceed to cell (B, G), which is the new North-West corner of the resulting matrix given below:

| Warehouse / Factory | G | Capacity |
|---|---|---|
| B | 51 | 30 |
| C | 43 | 190 |
| Requirement | 220 | |

Now, the quantity needed at the Fourth Warehouse (Warehouse G) is 220, which is greater than the quantity available at the Second Factory (Factory B). Therefore, we allocate the quantity equal to the capacity at Factory B to the cell (B, G) so that the capacity of Factory B is exhausted and the requirement of Warehouse G is reduced to 190. Hence, we cross out Row 1 and proceed to cell (C, G).

| Factory \ Warehouse | G | Capacity |
|---|---|---|
| B | ~~51~~ 30 | ~~30~~ 0 |
| C | 43 | 190 |
| Requirement | ~~220~~ 190 | |

The resulting matrix is

| Factory \ Warehouse | G | Capacity |
|---|---|---|
| C | 43 | 190 |
| Requirement | 190 | |

Thus, the allocations given using North-West corner rule are as shown in the following matrix along with the cost per unit of transportation:

| Factory \ Warehouse | D | E | F | G | Capacity |
|---|---|---|---|---|---|
| A | **42** 80 | **48** 80 | 38 | 37 | 160 |
| B | 40 | **49** 10 | **52** 110 | **51** 30 | 150 |
| C | 39 | 38 | 40 | **43** 190 | 190 |
| Requirement | 80 | 90 | 110 | 220 | 500 |

Thus, the total transportation cost for these allocations

$$= 42\times80 + 48 \times 80+49\times10 + 52 \times 110 + 51 \times 30 + 43 \times 190$$

$$= 3360 + 3840 + 490 + 5720 + 1530 + 8170 = 23110$$

Now, you should try to solve the following exercises to practice the NWC rule.

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**E1)** Find the basic feasible solution of the following problem using North-West Corner Rule:

| Origin / Distribution Centre | 1 | 2 | 3 | 4 | 5 | 6 | Availability |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 6 | 9 | 2 | 7 | 8 | 10 |
| 2 | 3 | 5 | 4 | 8 | 10 | 0 | 12 |
| 3 | 2 | 6 | 9 | 8 | 4 | 13 | 4 |
| 4 | 4 | 4 | 5 | 9 | 3 | 6 | 18 |
| 5 | 9 | 8 | 7 | 3 | 2 | 14 | 20 |
| Requirements | 8 | 8 | 16 | 3 | 8 | 21 | |

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Before studying the next section, match your answer with the answer given in Sec. 4.7.

### 4.3.2 Least Cost Method

The least cost method is also known as the Matrix Minimum method or Inspection method. It starts by making the first allocation to the cell for which the shipping cost (or transportation cost) per unit is lowest. The row or column for which the capacity is exhausted or requirement is satisfied is removed from the transportation table. The process is repeated with the reduced matrix till all the requirements are satisfied. If there is a tie for the lowest cost cell while making any allocation, the choice may be made for a row or a column by which maximum requirement is exhausted. If there is a tie in making this allocation as well, then we can arbitrarily choose a cell for allocation.

Let us explain this method with the help of an example.

**Example 2:** Apply the Least Cost method for finding the Basic Feasible solution of the transportation problem of Example 1.

**Solution:** Here, the least cost is 37 in the cell (A, G). The requirement of the Warehouse G is 220 and the capacity of Factory A is 160. Hence, the maximum number of units that can be allocated to this cell is 160. Thus, Factory A is exhausted and the row has to be removed from the next matrix. The requirement of Warehouse G is reduced by 160.

| Factory \ Warehouse | D | E | F | G | Capacity |
|---|---|---|---|---|---|
| A | ~~42~~ | ~~48~~ | ~~38~~ | ~~37~~ 160 | ~~160~~ 0 |
| B | 40 | 49 | 52 | 51 | 150 |
| C | 39 | 38 | 40 | 43 | 190 |
| Requirement | 80 | 90 | 110 | ~~220~~ 60 | 500 |

The reduced matrix, therefore, is

| Factory \ Warehouse | D | E | F | G | Capacity |
|---|---|---|---|---|---|
| B | 40 | 49 | 52 | 51 | 150 |
| C | 39 | 38 | 40 | 43 | 190 |
| Requirement | 80 | 90 | 110 | 60 | |

Now, the least cost is 38, which is in the cell (C,E). The requirement of the Warehouse E is 90 and the capacity of Factory C is 190. Hence, the maximum number of units that can be allocated to this cell is 90. Thus, Warehouse E is exhausted and has to be removed for the next matrix. So we cross out this column. Moreover, we reduce the capacity of factory C by 90.

| Factory \ Warehouse | D | E | F | G | Capacity |
|---|---|---|---|---|---|
| B | 40 | ~~49~~ | 52 | 51 | 150 |
| C | 39 | ~~38~~ 90 | 40 | 43 | ~~190~~ 100 |
| Requirement | 80 | ~~90~~ 0 | 110 | 60 | |

The reduced matrix, therefore, is

| Factory \ Warehouse | D | F | G | Capacity |
|---|---|---|---|---|
| B | 40 | 52 | 51 | 150 |
| C | 39 | 40 | 43 | 100 |
| Requirement | 80 | 110 | 60 | |

The least cost in this matrix is 39, which is in the cell (C, D). The requirement of Warehouse D is 80 and the capacity of Factory C is 100. Hence, the maximum number of units that can be allocated to this cell is 80. Thus, the requirement of Warehouse D is exhausted and this column is removed for the next matrix. The capacity of Factory C is also reduced by 80.

| Warehouse<br>Factory | D | F | G | Capacity |
|---|---|---|---|---|
| B | ~~40~~ | 52 | 51 | 150 |
| C | ~~39~~ 80 | 40 | 43 | ~~100~~ 20 |
| Requirement | ~~80~~ 0 | 110 | 60 | |

Thus, the reduced matrix is

| Warehouse<br>Factory | F | G | Capacity |
|---|---|---|---|
| B | 52 | 51 | 150 |
| C | 40 | 43 | 20 |
| Requirement | 110 | 60 | |

The least cost in this matrix is 40 which is in the cell (C, F). The requirement of Warehouse F is 110 and the capacity of Factory C is 20. Hence, the maximum number of units that can be allocated to this cell is 20. Thus, Factory C is exhausted and we remove the row for the next matrix. The requirement of Warehouse F is reduced by 20. It is now 90 in the reduced matrix.

| Warehouse<br>Factory | F | G | Capacity |
|---|---|---|---|
| B | 52 20 | 51 | 150 |
| C | ~~40~~ | ~~43~~ | ~~20~~ 0 |
| Requirement | ~~110~~ 90 | 60 | |

The reduced matrix is

| Warehouse<br>Factory | F | G | Capacity |
|---|---|---|---|
| B | 52 | ~~51~~ 60 | ~~150~~ 90 |
| Requirement | 90 | ~~60~~ 0 | |

The least cost is 51 in the cell (B, G) and the requirement of warehouse G is 60 units. So we allocate 60 units to cell (B, G) and the remaining 90 units to the cell (B, F). Thus, the allocations given using Least Cost method are as shown in the following matrix along with the cost per unit of transportation:

| Warehouse<br>Factory | D | E | F | G | Capacity |
|---|---|---|---|---|---|
| A | 42 | 48 | 38 | 37 160 | 160 |
| B | 40 | 49 | 52 90 | 51 60 | 150 |
| C | 39 80 | 38 90 | 40 20 | 43 | 190 |
| Requirement | 80 | 90 | 110 | 220 | 500 |

Thus, the total transportation cost $= 37\times160 + 52\times90 + 51\times60 + 39\times80 + 38\times90 + 40\times20$

$$= 5920 + 4680 + 3060 + 3120 + 3420 + 800 = 21000$$

Note that this method has reduced the total transportation cost in comparison to the NWC rule.

Now, you should try to solve the following exercise and apply the Least Cost method.

**E2)** Find the basic feasible solution of the following problem using the Least Cost method:

| D. Centre \ Origin | 1 | 2 | 3 | 4 | 5 | 6 | Availability |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 6 | 9 | 2 | 7 | 8 | 10 |
| 2 | 3 | 5 | 4 | 8 | 10 | 0 | 12 |
| 3 | 2 | 6 | 9 | 8 | 4 | 13 | 4 |
| 4 | 4 | 4 | 5 | 9 | 3 | 6 | 18 |
| 5 | 9 | 8 | 7 | 3 | 2 | 14 | 20 |
| Requirement | 8 | 8 | 16 | 3 | 8 | 21 | |

Before studying the next section, match your answer with the answer given in Sec. 4.7.

### 4.3.3 Vogel's Approximation Method (VAM)

We describe the procedure for finding the initial basic feasible solution by Vogel's Approximation method in the following steps:

1. For each row of the transportation table, we identify the least and second least costs. We find the difference between these two and display it to the right of that row in a new column formed by extending the table on the right. Likewise, we find such differences for each column and display it below that column in a new row formed for the purpose by extending the table at the bottom. The new column and row formed by extending the table at the right and bottom are labelled as **penalty column** and **penalty row**, respectively. If two cells in a row (or column) contain the same least costs then the difference is taken as zero.

2. From amongst these difference values displayed in the penalty column and the penalty row, we select the largest value (largest difference). We, allocate the maximum possible units to the least cost cell in the selected column or row. If a tie occurs amongst the largest differences, the choice may be made for that row or column, which has the least cost. In case there is a tie in such least cost as well, choice may be made from that row or column by which maximum requirements are exhausted. The cell so chosen is allocated the units and the corresponding exhausted row (or column) is removed (or ignored) from further consideration.

3. Next, we compute the column and row differences for the reduced transportation table and repeat the procedure until all column and row totals are exhausted.

This method is also called the **Penalty method**. Let us illustrate it with the help of an example.

**Example 3:** Apply the Vogel's Approximation Method for finding the Basic Feasible Solution for the transportation problem of Example 1.

**Solution:** In the first row, the least and the second least costs are 37 and 38 and their difference is 1. We write 1 in a new column created on the right. It is labelled **Penalty**. Similarly, the differences between the least and the second least costs in the second and third row, respectively, are 49−40=9 and 39−38=1. So we write the values (differences), i.e., 9 and 1 in the penalty column.

Next we find the differences of the least and second least elements of each of the columns D, E, F and G. These are 40−39=1, 48−38=10, 40−38=2 and 43−37=6 , respectively. We write them in a newly created **penalty row** at the bottom of the table.

We now select the largest of these differences in the penalty row and column, which is 10 in this case. This value (10) corresponds to the second column (Column E) and the least cost in this column is 38. Hence the allocation of 90 units (the maximum requirement of Warehouse E) is to be made in the cell (C, E) from Factory C. Since the column corresponding to E is exhausted, it is removed for the next reduced matrix and the capacity of C is reduced by 90.

| Factory \ Warehouse | D | E | F | G | Capacity | Penalty |
|---|---|---|---|---|---|---|
| A | 42 | ~~48~~ | 38 | 37 | 160 | 1 |
| B | 40 | ~~49~~ | 52 | 51 | 150 | 9 |
| C | 39 | ~~38~~ 90 | 40 | 43 | ~~190~~ 100 | 1 |
| Requirement | 80 | ~~90~~ 0 | 110 | 220 | 500 | |
| Penalty | 1 | 10 | 2 | 6 | | |

The reduced matrix, therefore, is

| Factory \ Warehouse | D | F | G | Capacity | Penalty |
|---|---|---|---|---|---|
| A | 42 | 38 | 37 | 160 | 1 |
| B | 40 | 52 | 51 | 150 | 11 |
| C | 39 | 40 | 43 | 100 | 1 |
| Requirement | 80 | 110 | 220 | 410 | |
| Penalty | 1 | 2 | 6 | | |

We now take the differences between the least and second least cost for each row and column of the reduced matrix. In the first row, the least and the second least costs are 37 and 38 and their difference is 1. We write it in the newly created penalty column. Similarly, we write the second difference element 51−40 = 11 and third difference element 40−39 = 1 in the second and third row of this column. Likewise, the differences of the smallest and second smallest elements of each of the columns D, F and G are 40−39 = 1, 40−38 = 2 and 43−37 = 6, respectively. We write these in a newly created penalty row at the bottom of the table.

Now, we select the largest of these differences in the penalty row and column, which is 11 in this case. This value (11) corresponds to Row B. Since the least cost in row B is 40, we allocate 80 units (the maximum requirement of Warehouse D) to the cell (B, D). Thus, the requirement of

Warehouse D is exhausted and we can remove it. We also reduce the capacity of Factory B by 80 in the next reduced matrix.

| Factory \ Warehouse | D | F | G | Capacity |
|---|---|---|---|---|
| A | ~~42~~ | 38 | 37 | 160 |
| B | 80 ~~40~~ | 52 | 51 | ~~150~~ 70 |
| C | ~~39~~ | 40 | 43 | 100 |
| Requirement | ~~80~~ 0 | 110 | 220 | 330 |

The reduced matrix is

| Factory \ Warehouse | F | G | Capacity | Penalty |
|---|---|---|---|---|
| A | 38 | 37 | 160 | 1 |
| B | 52 | 51 | 70 | 1 |
| C | 40 | 43 | 100 | 3 |
| Requirement | 110 | 220 | 330 | |
| Penalty | 2 | 6 | | |

Again, in the first row, the least and the second least costs are 37 and 38 and their difference is 1. We write it to the right of this row in the newly created penalty column. Similarly, the second and third elements in the second and third rows of this column are 52−51 = 1 and 43−40 = 3, respectively. Likewise, the differences of the least and second least elements of each of the columns F and G are 40−38 = 2 and 43−37 = 6, respectively. We write these in a newly created penalty row at the bottom of the table. Now, we select the largest of these differences, which is 6 in this case. It corresponds to Column G and the least cost in this column is 37. Hence, we allocate 160 units (the maximum capacity of Factory A) to the cell (A, G). Since Row A is exhausted, it is removed for the next reduced matrix. We also reduce the requirement of Warehouse G by 160 units.

| Factory \ Warehouse | F | G | Capacity |
|---|---|---|---|
| A | ~~38~~ | ~~37~~ 160 | ~~160~~ 0 |
| B | 52 | 51 | 70 |
| C | 40 | 43 | 100 |
| Requirement | 110 | ~~220~~ 60 | 170 |

The reduced matrix is

| Factory \ Warehouse | F | G | Capacity | Penalty |
|---|---|---|---|---|
| B | 52 | 51 | 70 | 1 |
| C | 40 | 43 | 100 | 3 |
| Requirement | 110 | 60 | 170 | |
| Penalty | 12 | 8 | | |

Once again, the difference of the least and the second least costs in the first row is 52−51=1. We write it in the newly created penalty column. Similarly, for the second row, the difference is 43−40=3. Likewise, the differences of the least and second least elements of each of the columns F and G are 52−40=12 and 51−43=8, respectively. We write them in the newly created penalty row at the bottom of the table. The largest of these differences is 12 in this case. It corresponds to Column F and the least cost in this column is 40. Hence, we allocate 100 units from the row (the maximum capacity of Factory C) to the cell (C, F). Since Row C is exhausted, it is removed and the requirement of Warehouse F is reduced to 10 for the next reduced matrix.

| Factory \ Warehouse | F | G | Capacity |
|---|---|---|---|
| B | 52 | 51 | 70 |
| C | ~~40~~ 100 | ~~43~~ | ~~100~~ 0 |
| Requirement | ~~110~~ 10 | 60 | 70 |

The reduced matrix is

| Factory \ Warehouse | F | G | Capacity |
|---|---|---|---|
| B | 52 | 60 ~~51~~ | ~~70~~ 10 |
| Requirement | 10 | 0 ~~60~~ | 10 |

At the end, of the 70 units available in Factory B, we allocate 60 units to the lower cost (51), i.e., to the cell (B, G) and the remaining 10 units to the cell (B, F).

The entire procedure of allocating units by Vogel's Approximation Method can be done in a single table also, as shown below:

| Factory \ Warehouse | D | E | F | G | Capacity | $Diff_1$ | $Diff_2$ | $Diff_3$ | $Diff_4$ |
|---|---|---|---|---|---|---|---|---|---|
| A | 42 | 48 | 38 | **37** 160 | 160 | 1 | 1 | 1 | - |
| B | **40** 80 | 49 | **52** 10 | **51** 60 | 150 | 9 | 11* | 1 | 1 |
| C | 39 | **38** 90 | **40** 100 | 43 | 190 | 1 | 1 | 3 | 3 |
| Requirement | 80 | 90 | 110 | 220 | 500 | | | | |
| $Diff_1$ | 1 | 10* | 2 | 6 | | | | | |
| $Diff_2$ | 1 | - | 2 | 6 | | | | | |
| $Diff_3$ | - | - | 2 | 6* | | | | | |
| $Diff_4$ | - | - | 12* | 8 | | | | | |

Thus, the total transportation cost

$$= 40\times80 + 38\times90 + 52\times10 + 40\times100 + 37\times160 + 51\times60$$

$$= 3200 + 3420 + 520 + 4000 + 5920 + 3060$$

$$= 20120$$

This is the lowest total transportation cost among the three methods.

Now, you should apply Vogal's Approximation method to solve the following exercise.

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**E3)** Obtain the basic feasible solution of the following problem using Vogal's Approximation method:

| Origin / Distribution Centre | 1 | 2 | 3 | 4 | 5 | 6 | Availability |
|---|---|---|---|---|---|---|---|
| 1 | 4 | 6 | 9 | 2 | 7 | 8 | 10 |
| 2 | 3 | 5 | 4 | 8 | 10 | 0 | 12 |
| 3 | 2 | 6 | 9 | 8 | 4 | 13 | 4 |
| 4 | 4 | 4 | 5 | 9 | 3 | 6 | 18 |
| 5 | 9 | 8 | 7 | 3 | 2 | 14 | 20 |
| Requirement | 8 | 8 | 16 | 3 | 8 | 21 | |

Before studying the next section, match your answer with the answer given in Sec. 4.7.

## 4.4 METHODS OF FINDING OPTIMAL SOLUTION

Once the initial basic feasible solution is determined, the optimality test is performed to find whether the obtained feasible solution is optimal or not. The optimality test is performed by applying one of the following methods:

i) **Stepping Stone Method**

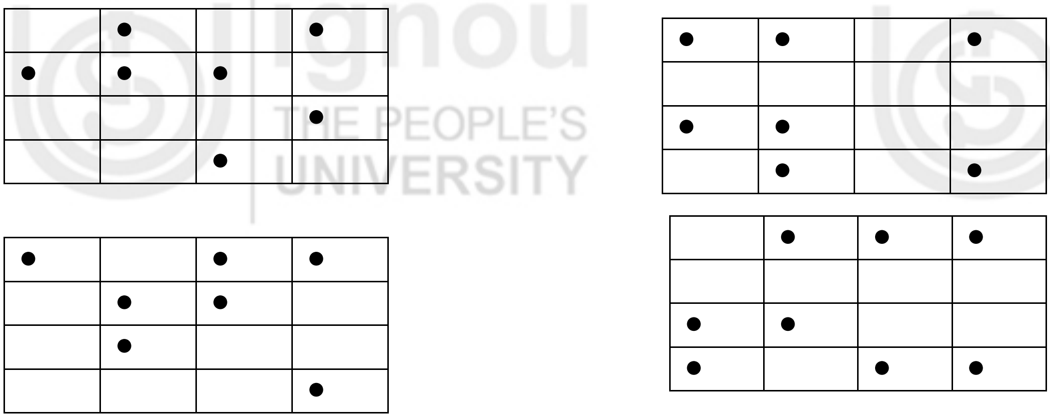
ii) **Modified Distribution (MODI) Method**

These methods reveal whether the initial basic feasible solution is optimal or not. They also improve the solution until the optimal solution is obtained. The Stepping Stone method is applied to a problem of small dimension as its application to a problem of large dimension is quite tedious and cumbersome. The MODI method is usually preferred over the Stepping Stone method.

Before discussing the methods of performing optimality test and finding optimal solutions, we explain the concept of independent allocations. We also state the conditions for performing optimality test.

**Independent and Non-Independent Allocations**

If it is not possible to form any closed loop through the allocations under consideration then the allocations are said to be **independent**. Here, formation of closed loop means that it is possible to travel from any allocation, back to itself by a series of horizontal and vertical jumps from one occupied cell (i.e., the cell containing allocation) to another, without a direct reversal of route (see Fig. 4.1a) . If such a loop can be formed using some or all of the allocations under consideration, the allocations are known as **non-independent**.



**(a)** **(b)**

**Fig. 4.1: Example of a) Independent Allocations; b) Non-Independent Allocations.**

**Note:** Every loop has an even number of cells.

**Conditions for Performing Optimality Test**

An optimality test can be applied to that feasible solution which satisfies the following conditions:

1. It contains exactly m + n−1 allocations where m and n represent the number of rows and columns, respectively, of the transportation table.
2. These allocations are independent.

Let us now discuss the methods of performing optimality test and hence finding optimal solutions for each of these methods.

## 4.4.1 Stepping Stone Method

In the Stepping Stone method for obtaining the optimal solution of a transportation problem, you should follow the steps given below:

1. Determine an initial basic feasible solution.
2. Evaluate all unoccupied cells for the effect of transferring one unit from an occupied cell to the unoccupied cell as follows:
   a) Select an unoccupied cell to be evaluated.
   b) Starting from this cell, form a closed path (or loop) through at least three occupied cells. The direction of movement is immaterial because the result will be the same in both directions. Note that except for the evaluated cell, all cells at the corners of the loop have to be occupied.
   c) At each corner of the closed path, assign plus (+) and minus (−) sign alternatively, beginning with the plus sign for the unoccupied cell to be evaluated.
   d) Compute the net change in cost with respect to the costs associated with each cell traced in the closed path.
   e) Repeat steps 2(a) to 2(d) until the net change in cost has been calculated for all occupied cells.
3. If all net changes are positive or zero, an optimal solution has been arrived at. Otherwise go to step 4.
4. If some net changes are negative, select the unoccupied cell having the most negative net change. If two negative values are equal, select the one that results in moving more units into the selected unoccupied cell with the minimum cost.
5. Assign as many units as possible to this unoccupied cell.
6. Go to Step 2 and repeat the procedure until all unoccupied cells are evaluated and the value of net change, i.e., net evaluation is positive or zero.

Let us now take up an example of transportation problem and apply the Stepping Stone method for finding the optimal solution.

**Example 4:** A company is spending `1000 on transportation of its units from three plants to four distribution centres. The availability of unit per plant and requirement of units per distribution centre, with unit cost of transportation are given as follows:

| D. Centres / Plants | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Availablity |
|---|---|---|---|---|---|
| $P_1$ | 19 | 30 | 50 | 12 | 7 |
| $P_2$ | 70 | 30 | 40 | 60 | 10 |
| $P_3$ | 40 | 10 | 60 | 20 | 18 |
| Requirement | 5 | 8 | 7 | 15 | |

What is the maximum possible saving by optimum distribution? Use the Stepping Stone method to solve the problem.

**Solution:** Applying the Vogel's Approximation Method to the problem, the initial feasible solution is given as follows:

| D Cs / Plants | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Av. | $diff_1$ | $diff_2$ | $diff_3$ | $diff_4$ |
|---|---|---|---|---|---|---|---|---|---|
| $P_1$ | 19 5 | 30 | 50 | 12 2 | 7 | 7 | 18 | 38 | 38 |
| $P_2$ | 70 | 30 | 40 7 | 60 3 | 10 | 10 | 10 | 20 | 20 |
| $P_3$ | 40 | 10 8 | 60 | 20 10 | 18 | 10 | 10 | 40* | - |
| Requirements | 5 | 8 | 7 | 15 | | | | | |
| $diff_1$ | 21* | 20 | 10 | 8 | | | | | |
| $diff_2$ | - | 20* | 10 | 8 | | | | | |
| $diff_3$ | - | - | 10 | 8 | | | | | |
| $diff_4$ | - | - | 10 | 48* | | | | | |

In the above table, the encircled values are the allocations. The total transportation cost associated with this initial basic feasible solution is given as:

Total Transportation Cost $=19\times5+12\times2+40\times7+60\times3+10\times8+20\times10$

$= 95+24+280+180+80+200 =$ `859

This solution has 3+4–1 = 6 occupied cells. These are independent as it is not possible to form any closed loop through these allocations (see Fig. 4.2).

| | | | |
|---|---|---|---|
| • | | | • |
| | | • | • |
| | • | | • |

**Fig. 4.2: The allocations are Independent as a closed loop cannot be formed in this case.**

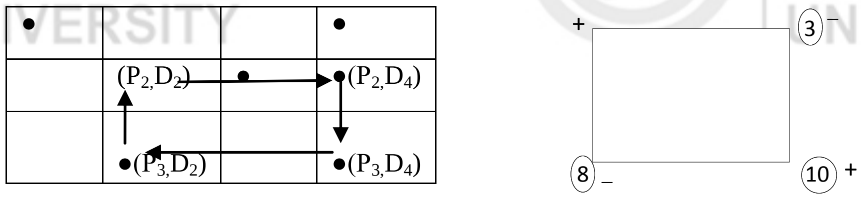
Hence, the optimality test can be performed. Let us apply the optimality test using the Stepping Stone method.

### Optimality Test using the Stepping Stone Method

We evaluate the effect of allocating one unit to each of the unoccupied cells making closed paths. Note that the unoccupied cells are ($P_1$, $D_2$), ($P_1$, $D_3$), ($P_2$, $D_1$), ($P_2$, $D_2$), ($P_3$, $D_1$) and ($P_3$, $D_3$). We have to make closed paths so that each path contains at least three occupied cells. We also have to evaluate the net change in cost for each and every unoccupied cell. Then we have to select the one unoccupied cell, which has the most negative opportunity cost and allocate as many units as possible to reduce the total transportation cost. The computations are shown below:

| Unoccupied Cell | Closed Path | Net Change in Cost (`) |
|---|---|---|
| $(P_1, D_2)$ | $(P_1, D_2) \rightarrow (P_1, D_4) \rightarrow (P_3, D_4) \rightarrow (P_3, D_2)$ | 30–12+20–10 = 28 |
| $(P_1, D_3)$ | $(P_1, D_3) \rightarrow (P_1, D_4) \rightarrow (P_2, D_4) \rightarrow (P_2, D_3)$ | 50–12+60–40 = 58 |
| $(P_2, D_1)$ | $(P_2, D_1) \rightarrow (P_1, D_1) \rightarrow (P_1, D_4) \rightarrow (P_2, D_4)$ | 70–19+12–60 = 3 |
| $(P_2, D_2)$ | $(P_2, D_2) \rightarrow (P_3, D_2) \rightarrow (P_3, D_4) \rightarrow (P_2, D_4)$ | 30–10+20–60 = –20 |
| $(P_3, D_1)$ | $(P_3, D_1) \rightarrow (P_1, D_1) \rightarrow (P_1, D_4) \rightarrow (P_3, D_4)$ | 40–19+12–20 = 13 |
| $(P_3, D_3)$ | $(P_3, D_3) \rightarrow (P_2, D_3) \rightarrow (P_2, D_4) \rightarrow (P_3, D_4)$ | 60–40+60–20 = 60 |

The cell $(P_2, D_2)$ has the most negative opportunity cost (net change in cost). Therefore, transportation cost can be reduced by making allocation to this unoccupied cell. This means that if one unit is shifted to this unoccupied cell through the loop shown in the fourth row of the table, then `20 can be saved (see Fig. 4.3a). Hence, we shall shift as many units as possible to the cell $(P_2, D_2)$ through this loop. The maximum number of units that can be allocated to $(P_2, D_2)$ through this loop is 3.




**(a)** **(b)**

**Fig. 4.3: Closed Loop.**

This is because the shifting can be done only from the corners of the loop and more than 3 units cannot be shifted to $(P_2, D_2)$ as explained below:

The allocations at other corners of the loops are 3, 10 and 8. If we try to shift more than 3 units, say, 4 units from the corner $(P_2, D_4)$, then 4 units will have to be subtracted from the corner $(P_3, D_2)$ so that the total of the column $D_4$ remains unchanged. But this will give 3 – 4= –1 allocations to the cell $(P_2, D_4)$, which is impossible as negative allocations cannot be made.

We obtain the maximum number of units that can be allocated to the cell $(P_2, D_2)$ through the loop as follows:

1. First we assign (+) sign to the unoccupied cell $(P_2, D_2)$ to be evaluated and then (–) and (+) signs alternatively to other corners of the closed loop (moving in one direction) as shown in Fig. 4.3b.
2. Then we take the minimum of the values at the corners that have been assigned the negative sign. In this case, the maximum number of units that can be allocated to the cell $(P_2, D_2)$ through the mentioned loop is the minimum of 3 and 8. It is 3. So we write it as:

$$\min.\begin{cases} \text{the no. of units in } (P_2,\ D_4) \ = \ 3 \\ \text{the no. of units in } (P_3,\ D_2) \ = \ 8 \end{cases} = 3$$

The new table with these changes becomes:

| | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|
| $P_1$ | 19 5 | 30 | 50 | 2 12 |
| $P_2$ | 70 | 3 30 | 7 40 | 60 |
| $P_3$ | 40 | 5 10 | 60 | 13 20 |

In the above table, note that we have also allocated 3 units from ($P_3$, $D_2$) to ($P_3$, $D_4$) so that column $D_4$ remains unchanged. This leaves 5 units in the cell ($P_3$, $D_2$) and there are 13 units in ($P_3$, $D_4$). The total transportation cost associated with this solution is

Total cost = 19 × 5 + 12 × 2 + 30 × 3 + 40 × 7 + 10 × 5 + 20 × 13

= 95 + 24 + 90 + 280 + 50 + 260 = `799

Now, we repeat the optimality test to see if further allocation can be made to reduce the total transportation cost. The computation for the unoccupied cells is as follows:

| **Unoccupied Cell** | **Closed Path** | **Net Change in Cost (`)** |
|---|---|---|
| ($P_1$, $D_2$) | ($P_1$, $D_2$)→ ($P_1$, $D_4$)→ ($P_3$, $D_4$)→ ($P_3$, $D_2$) | 30–12+20–10 = 28 |
| ($P_1$, $D_3$) | ($P_1$, $D_3$)→ ($P_1$, $D_4$)→ ($P_3$, $D_4$)→ ($P_3$, $D_2$) ($P_2$, $D_2$)→ ($P_2$, $D_3$) | 50–12+20–10+30–40= 38 |
| ($P_2$, $D_1$) | ($P_2$, $D_1$)→ ($P_1$, $D_1$)→ ($P_1$, $D_4$)→ ($P_3$, $D_4$) ($P_3$, $D_2$)→ ($P_2$, $D_2$) | 70–19+12–20+10–30 = 23 |
| ($P_2$, $D_4$) | ($P_2$, $D_4$)→ ($P_3$, $D_4$)→ ($P_3$, $D_2$)→ ($P_2$, $D_2$) | 60–20+10–30 = 20 |
| ($P_3$, $D_1$) | ($P_3$, $D_1$)→ ($P_1$, $D_1$)→ ($P_1$, $D_4$)→ ($P_3$, $D_4$) | 40–19+12–20 = 13 |
| ($P_3$, $D_3$) | ($P_3$, $D_3$)→ ($P_2$, $D_3$)→ ($P_2$, $D_2$)→ ($P_3$, $D_2$) | 60–10+30–40 = 40 |

Since all opportunity costs in the unoccupied cells are non-negative, the current solution is an optimal solution with total transportation cost `799. Hence the maximum saving by optimum distribution is `(1000 – 799) = `201.

Now, you should try to solve the following exercise:

**E4)** Perform the optimality test using the Stepping Stone method on the solution of the problem given in Example 1.

Before studying the next method, match your answer with the answer given in Sec. 4.7.

Note that the Stepping Stone method should be applied only to problems of small dimensions. For large dimensions, this method becomes quite tedious. For instance, if a problem involving five factories and seven warehouses has to be solved by the Stepping Stone method, an initial solution may involve 5+7−1=11 occupied cells and hence 5×7−11=24 unoccupied cells. To check whether this solution is an optimal solution or not, we shall have to find the opportunity cost for each of the 24 unoccupied cells by making 24 separate loops: one for each case. Then we will have to proceed as in the above problem. This will be very tedious and cumbersome. So, for problems of large dimensions, we use another method known as the Modified Distribution method (MODI). This method may be conveniently applied to problems of small dimension as well. We now describe the MODI method.

## 4.4.2 Modified Distribution Method (MODI)

The modified distribution method (MODI) is an improved form of the Stepping Stone method for obtaining an optimal solution of a transportation problem. The difference between the two methods is that in the Stepping Stone method, closed loops are drawn for all unoccupied cells for determining their opportunity costs. However, in the MODI method, the opportunity costs of all the unoccupied cells are calculated and the cell with the highest negative opportunity cost is identified without drawing any closed loop. Then only one loop is drawn for the highest negative opportunity cost. The procedure for determining the optimal solution of a transportation problem with the help of the MODI method is as follows:

1. Determine an initial feasible solution using a suitable method.
2. For the current basic feasible solution with exactly (m+n–1) independent allocations, write the cost matrix for only the allocated cells.
3. Using this cost matrix, determine the set of m row numbers $u_i$ (i = 1, 2, …, m) and n column numbers $v_j$ (1, 2, …,n) such that $C_{ij} = u_i + v_j$, taking one of $u_i$ or $v_j$ as zero.
4. Fill the vacant cells using these numbers, i.e., using $u_i + v_j$.
5. Compute the net evaluations $\Delta_{ij} = C_{ij} – (u_i + v_j)$ by subtracting the values so obtained in **Step 4** from the corresponding values of the original cost matrix.
   i) If all $\Delta_{ij} \geq 0$, an optimal solution has been arrived at.
   ii) If some of $\Delta_{ij}$ s are negative, the current solution is not optimal. Then select the cell having the most negative $\Delta_{ij}$ and tick it.
6. Construct a closed path for the unoccupied cell ticked in **Step 5(ii)** using the already allocated cells. At each corner of the closed path, assign plus (+) and minus (–) signs alternatively, beginning with the plus sign for the unoccupied cell. Then θ units are to be allocated to this cell and the same numbers of units are to be added and subtracted alternatively at the corners assigned plus and minus signs, respectively. The value of θ is the maximum number of units that can be allocated to this cell through the loop. It is obtained by equating the allocations at the negative sign corners to zero. In this way, allocations have been improved.
7. Write the cost matrix for only those cells which have improved allocations. Go to **Step 3** and repeat the procedure until all $\Delta_{ij} \geq 0$. Calculate the associated total transportation cost.

**Example 5:** Perform optimality test by applying the MODI method for the problem taken in Example 4.

**Solution:** The basic solution of the problem as already obtained by VAM is

| D Centre / Plants | $D_1$ | $D_2$ | $D_3$ | $D_4$ | Availability |
|---|---|---|---|---|---|
| $P_1$ | 19 (5) | 30 | 50 | 12 (2) | 7 |
| $P_2$ | 70 | 30 | 40 (7) | 60 (3) | 10 |
| $P_3$ | 40 | 10 (8) | 60 | 20 (10) | 18 |
| Requirement | 5 | 8 | 7 | 15 | 35 |

Now we perform optimality test by applying the MODI method. First of all, we write the cost matrix for only allocated cells:

| | | | | |
|---|---|---|---|---|
| 19 | | | 12 | $u_1$ |
| | | 40 | 60 | $u_2$ |
| | 10 | | 20 | $u_3$ |
| $v_1$ | $v_2$ | $v_3$ | $v_4$ | |

Let us denote the row numbers by $u_1$, $u_2$, $u_3$ and column numbers by $v_1$, $v_2$, $v_3$ and $v_4$ such that

$$u_1 + v_1 = 19, \quad u_1 + v_4 = 12, \quad u_2 + v_3 = 40,$$

$$u_2 + v_4 = 60, \quad u_3 + v_2 = 10, \quad u_3 + v_4 = 20.$$

Taking $u_1 = 0$, we have $v_1 = 19$ and $v_4 = 12$ from the first two equations.

Putting the value of $v_4$ in the fourth equation, we have $u_2 = 60 - 12 = 48$.

Similarly, $u_3 = 20 - 12 = 8$, $v_2 = 10 - 8 = 2$ and $v_3 = 40 - 48 = -8$.

Using these values, we fill the vacant cells of the above table using $C_{ij} = u_i + v_j$ and put dots in the already filled cells so that these cells are not considered again.

| | $v_1 = 19$ | $v_2 = 2$ | $v_3 = -8$ | $v_4 = 12$ |
|---|---|---|---|---|
| $u_1 = 0$ | ● | 2 | –8 | ● |
| $u_2 = 48$ | 67 | 50 | ● | ● |
| $u_3 = 8$ | 27 | ● | 0 | ● |

Now, subtracting these values from the corresponding values of the original cost matrix, we have the net evaluations, i.e., $\Delta_{ij} = C_{ij} - (u_i + v_j)$

| | | | |
|---|---|---|---|
| ● | 30–2=28 | 50– (–8) = 58 | ● |
| 70–67=3 | 30–50= –20 | ● | ● |
| 40–27=13 | ● | 60–0=60 | ● |

Note that the cell ($P_2$, $D_2$) has the most negative opportunity cost (net change in cost). Therefore, the transportation cost can be reduced by making allocation to this unoccupied cell. This means that if one unit is shifted to this unoccupied cell through the closed loop formed, beginning from this cell and using allocated cells, `20 can be saved. We form the loop beginning from this cell, i.e., the cell ($P_2$, $D_2$) (see Fig. 4.4). We shift θ units to this unoccupied cell through the loop and add and subtract θ from the cells at the other corners of the loop which are assigned '+' and '–' signs. So we get Fig. 4.4:


**Fig. 4.4: Closed loop for cell ($P_2$, $D_2$).**

The maximum number of units that can be allocated to the cell ($P_2$, $D_2$) through this loop is given by the minimum of the solution of the equations $3 - \theta = 0$ and $8 - \theta = 0$, i.e.,

$$\theta = \min.\begin{cases} \text{the no. of units in } (P_2,\ D_4) \ = \ 3 \\ \text{the no. of units in } (P_3,\ D_2) \ = \ 8 \end{cases} = 3$$

So with the improved allocations, the table now becomes:

| | | | |
|---|---|---|---|
| 19 5 | 30 | 50 | 12 2 |
| 70 | 30 3 | 40 7 | 60 |
| 40 | 10 5 | 60 | 20 13 |

Thus, the total cost of transportation for this set

$$= 19 \times 5 + 12 \times 2 + 30 \times 3 + 40 \times 7 + 10 \times 5 + 20 \times 13$$

$$= 95 + 24 + 90 + 280 + 50 + 260 = 799$$

Now, let us apply the optimality test to the improved solution. Proceeding in the same way as in the first iteration, first of all, we write the cost matrix for only allocated cells:

| | | | | |
|---|---|---|---|---|
| 19 | | | 12 | $u_1$ |
| | 30 | 40 | | $u_2$ |
| | 10 | | 20 | $u_3$ |
| $v_1$ | $v_2$ | $v_3$ | $v_4$ | |

Let us denote the row numbers by $u_1$, $u_2$, $u_3$ and column numbers by $v_1$, $v_2$, $v_3$ and $v_4$ such that

$$u_1 + v_1 = 19, \quad u_1 + v_4 = 12, \quad u_2 + v_3 = 40,$$

$$u_2 + v_2 = 30, \quad u_3 + v_2 = 10, \quad u_3 + v_4 = 20.$$

Taking $u_1 = 0$, we have $v_1 = 19$, $v_4 = 12$ from the first two equations.

Putting the value of $v_4$ in the sixth equation, we have $u_3 = 20 - 12 = 8$.

Similarly, $u_2 = 30 - 02 = 28$, $v_2 = 10 - 8 = 2$, $v_3 = 40 - 28 = 12$.

Using these values, we fill all the vacant (unoccupied) cells of the table using $c_{ij} = u_i + v_j$ for each unoccupied cell and put dot in already filled cells so that these cells are not considered again.

| | $v_1 = 19$ | $v_2 = 2$ | $v_3 = 12$ | $v_4 = 12$ |
|---|---|---|---|---|
| $u_1 = 0$ | ● | 2 | 12 | ● |
| $u_2 = 28$ | 47 | ● | ● | 40 |
| $u_3 = 8$ | 27 | ● | 20 | ● |

Now, subtracting these values from the corresponding values of the original cost matrix, we have the net evaluations $\Delta_{ij} = C_{ij} - (u_i + v_j)$ as:

| | | | |
|---|---|---|---|
| ● | 28 | 38 | ● |
| 23 | ● | ● | 20 |
| 13 | ● | 40 | ● |

Since none of the net evaluations is negative, this solution is optimal. Thus, the total minimum transportation cost is `799 and the maximum saving

= `(1000 – 799) = `201.

Now, you should try to solve the following exercise.

**E5)** Determine the optimum distribution for the transportation problem given in exercise E1 using the MODI method.

Before studying the next section, match your answer with the answer given in Sec. 4.7.

# 4.5 SOME SPECIAL CASES

We now consider some special cases of the transportation problem such as unbalanced transportation problem, case of degeneracy, case of alternative solution, maximisation transportation problem and problems with prohibited routes.

## 4.5.1 Unbalanced Transportation Problem

The transportation problems wherein the total capacity of all sources and total requirement (demand) of all destinations are not equal is called the **unbalanced transportation problem**. If the total capacity of sources is greater than (less than) the total requirement at destination, we add a dummy destination (source) in the transportation table with zero transportation cost so that the problem becomes balanced. The augmented problem is then solved by the methods explained earlier.

**Example 6:** A company has factories at A, B and C which supply warehouses at D, E, F and G. The monthly factory capacities are 160, 150 and 190 units, respectively. Monthly warehouse requirements are 80, 90, 110 and 160, respectively. Unit shipping costs (in rupees) are as follows:

| | | To | | | |
|---|---|---|---|---|---|
| | | D | E | F | G |
| From | A | 42 | 48 | 38 | 37 |
| | B | 40 | 49 | 52 | 51 |
| | C | 39 | 38 | 40 | 43 |

Determine the optimum distribution for this company to minimise shipping costs.

**Solution:** Here the total capacity of sources (Factories) is 160+150+190=500. It is greater than the total requirement of all the destinations (Warehouses), which is 80+90+110+160=440. Therefore, we add a dummy destination in the transportation table with zero transportation cost and take 60 as its requirement. Thus, the problem becomes balanced, i.e., the total capacity and total requirement are equal. The balanced problem is as follows:

| | | To | | | | | Capacity |
|---|---|---|---|---|---|---|---|
| | | D | E | F | G | H | |
| From | A | 42 | 48 | 38 | 37 | 0 | 160 |
| | B | 40 | 49 | 52 | 51 | 0 | 150 |
| | C | 39 | 38 | 40 | 43 | 0 | 190 |
| Requirement | | 80 | 90 | 110 | 160 | 60 | 500 |

We can solve this problem using VAM to determine the basic feasible solution and then use the MODI method to find the optimal solution. We have solved this problem in Example 3. You may like to try the remaining solution yourself.

The optimum allocation is:

160 to (A, F), 80 to (B, D), 10 to (B, E), 60 to (B, G), 80 to (C, E) and 110 to (C, F).

Note that if the number of allocations in the basic feasible solution is less than m+n−1, we first go through the special case of degeneracy and then apply the test of optimality. Let us explain how the problem of degeneracy is dealt with in the transportation problem.

### 4.5.2 Degeneracy

It may happen sometimes that the number of occupied cells is less than m+n−1. Such a solution is called a **degenerate solution**. We handle such a situation by introducing an infinitesimally small allocation e (say) in the least cost and independent cell. This means that if 'e' is added to or subtracted from any quantity, the quantity remains unaltered, i.e., the value of 'e' is nothing but zero. But we do not write 0 in place of e as it is one of the allocations and will not come into counting if it is written as zero. So we write e and not zero in its place and count it as one of the allocations.

Degeneracy occurs in **Example 6.** We resolve this problem as follows:

Using Vogel Approximation method, we get the basic feasible solution as:

| Warehouse / Factory | D | E | F | G | H | Av. | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|---|---|---|---|---|---|---|---|---|---|---|
| A | 42 | 48 | 38 | 37 160 | 0 e | 160 | 37 | 1 | 1 | 1 |
| B | 40 80 | 49 | 52 10 | 51 | 0 60 | 150 | 40* | 9 | 11* | 1 |
| C | 39 | 38 90 | 40 100 | 43 | 0 | 190 | 38 | 1 | 1 | 3 |
| Requirement | 80 | 90 | 110 | 160 | 60 | 500 | | | | |
| $D_1$ | 1 | 10 | 2 | 6 | 0 | | | | | |
| $D_2$ | 1 | 10* | 2 | 6 | - | | | | | |
| $D_3$ | 1 | - | 2 | 6 | - | | | | | |
| $D_4$ | - | - | 2 | 6* | - | | | | | |

Here the number of allocations is 6 but the number of allocations should be 3+5−1=7 to perform optimality test. Hence it is the case of degeneracy. To resolve such a problem, we introduce an infinitesimally small allocation e (say) in the least cost and independent cell, i.e., the cell (A, H). Another cell (C, H) also has the same least cost as the cell (A, H). But this cell is not independent because a loop (C, H)→ (C, F)→ (B, F)→ (B, H) can be formed beginning with the cell (C, H) as shown in Fig. 4.5. Now the number of allocations is 7, i.e., as many as required for performing the optimality test.

| | | | | |
|---|---|---|---|---|
| 42 | 48 | 38 | • | 0 |
| • | 49 | • (B, F) | 51 | • (B, H) |
| 39 | • | • (C, F) | 43 | 0 (C, H) |

**Fig. 4.5: Closed loop for cell (C, H).**

Thus, applying optimality test by the MODI method, the improved allocations after first iteration will be:

| | | | | |
|---|---|---|---|---|
| 42 | 48 | 38 e | 37 160 | 0 |
| 40 80 | 49 | 52 10 | 51 | 0 60 |
| 39 | 38 90 | 40 100 | 43 | 0 |

Again, applying optimality test, the improved solution is

| | D | E | F | G | H |
|---|---|---|---|---|---|
| A | 42 | 48 | 38 e | 37 160 | 0 |
| B | 40 80 | 49 10 | 52 | 51 | 0 60 |
| C | 39 | 38 80 | 40 110 | 43 | 0 |

Again applying the optimality test, you will see that these allocations give the optimal solution and the total transportation cost = `17050

### 4.5.3 Alternative Optimal Solutions

Such solutions exist if all net-evaluations, i.e., $\Delta_{ij} = C_{ij} - (u_i + v_j) \geq 0$ but one or more of them are equal to zero. To determine the alternative optimal solution, we tick the cell having zero value of net-evaluation and make a closed loop beginning from this cell. The transportation allocation might not be the same, yet the total transportation cost is same.

We illustrate this special case with the help of Example 7.

**Example 7:** Solve the following transportation problem:

| | D | E | F | G | Capacity |
|---|---|---|---|---|---|
| A | 5 | 3 | 6 | 2 | 19 |
| B | 4 | 7 | 9 | 1 | 37 |
| C | 3 | 4 | 7 | 5 | 34 |
| Requirement | 16 | 18 | 31 | 25 | |

**Solution:** Using VAM, the initial basic feasible solution is obtained as follows:

| | D | E | F | G | Capacity | $Diff._1$ | $Diff._2$ | $Diff._3$ | $Diff._4$ |
|---|---|---|---|---|---|---|---|---|---|
| A | 5 | 3 18 | 6 1 | 2 | 19 | 1 | 2 | 2* | 1 |
| B | 4 12 | 7 | 9 | 1 25 | 37 | 3* | 3* | - | - |
| C | 3 4 | 4 | 7 30 | 5 | 34 | 1 | 1 | 1 | 4* |
| Requirement | 16 | 18 | 31 | 25 | 90 | | | | |
| $Diff._1$ | 1 | 1 | 1 | 1 | | | | | |
| $Diff._2$ | 1 | 1 | 1 | - | | | | | |
| $Diff._3$ | 1 | 1 | 1 | - | | | | | |
| $Diff._4$ | 2 | - | 1 | - | | | | | |

Now, performing the optimality test using the MODI method, you can get the following optimal solution:

| | | | |
|---|---|---|---|
| 5 | 3 | 6 19 | 2 |
| 4 12 | 7 | 9 | 1 25 |
| 3 4 | 4 18 | 7 12 | 5 |

But the net evaluations, while performing optimality test for this solution, will be

| | D | E | F | G |
|---|---|---|---|---|
| A | 3 | 0 | ● | 3 |
| B | ● | 2 | 1 | ● |
| C | ● | ● | ● | 5 |

Since one net evaluation is zero, the problem has an alternative optimal solution.

We form the closed loop beginning with the cell containing zero, i.e., the cell (A, E). The loop is (A, E)→ (A, F)→ (C, F)→ (C, E). We perform the optimality test by the MODI method again. The alternative allocations are:

| | | | |
|---|---|---|---|
| 5 | 3 (18) | 6 (1) | 2 |
| 4 (12) | 7 | 9 | 1 (25) |
| 3 (4) | 4 | 7 (30) | 5 |

Here, the first set of allocations gives the transportation cost as

$$6\times19+4\times12+1\times25+3\times4+4\times18+7\times12=114+48+25+12+72+84 = 355$$

The second set of allocations also gives the same transportation cost:

$$3\times18+6\times1+4\times12+1\times25+3\times4+7\times30=54+6+48+25+12+210 = 355$$

Both these different sets of allocations give the same total transportation cost. Hence, any of these alternative solutions may be chosen.

### 4.5.4 Maximisation Transportation Problem

The data given in the transportation problems may be such that we have to maximise it. This may be possible if instead of costs, the profits (or anything else like revenue which needs maximisation) are given in the cells. To handle such a problem, we find the opportunity loss matrix by subtracting the value of each cell from the largest value chosen from amongst all the given cells. When the value of a cell is subtracted from the highest value, it gives the loss of amount caused by not getting the opportunity which would have given the highest value. The matrix so obtained is known as the **opportunity loss matrix**. It is handled in the same way as the minimisation problem.

**Example 8:** A company has 3 factories manufacturing the same product and 5 agencies in different parts of the country. Production costs differ from factory to factory and the sales prices from agency to agency. The shipping cost per unit product from each factory to any agency is known. Given the following data, find the production and distribution schedules most profitable to the company.

| Production cost/unit | Max. Capacity No. of Units | Factory i |
|---|---|---|
| 20 | 150 | 1 |
| 22 | 200 | 2 |
| 18 | 125 | 3 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 5 | 9 | 4 | |
| 2 | 9 | 7 | 8 | 3 | 6 | Shipping costs |
| 3 | 4 | 5 | 3 | 2 | 7 | |
| | 1 | 2 | 3 | 4 | 5 | Agency j |
| | 80 | 100 | 75 | 45 | 125 | Demand to be met |
| | 30 | 32 | 31 | 34 | 29 | Sales price |

**Solution:** We first obtain the profit matrix by finding the excess of sales price over the various costs, i.e.,

Profit/unit = Sales price/unit – Production cost/unit – Shipping cost/unit

Thus, the profit matrix is

| Factory \ Agency | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 30–20–1=9 | 32–20–1=11 | 31–20–5=6 | 34–20–9=5 | 29–20–4=5 |
| 2 | 30–22–9= –1 | 32–22–7=3 | 31–22–8=1 | 34–22–3=9 | 29–22–6=1 |
| 3 | 30–18–4=8 | 32–18–5=9 | 31–18–3=10 | 34–18–2=14 | 29–18–7=4 |

Note that this is the profit matrix and we want to maximise the profit. But the methods which we have discussed deal with minimisation problems. Therefore, we convert this profit matrix into the opportunity loss matrix used for minimisation problems or to minimise the total cost. The maximum value of the profit in this table is 14 and hence the opportunity loss matrix is obtained on subtracting each of the values of the above matrix from 14. The opportunity loss matrix, therefore, is

| | | | | |
|---|---|---|---|---|
| 14 – 9 | 14 – 11 | 14 – 6 | 14 – 5 | 14 – 5 |
| 14 – (–1) | 14 – 3 | 14 – 1 | 14 – 9 | 14 – 1 |
| 14 – 8 | 14 – 9 | 14 – 10 | 14 – 14 | 14 – 4 |

Here, the problem is also unbalanced since demand (425) and capacity (475) are not equal. So, an agency with requirement of 50 units (the difference of demand and capacity) also needs to be introduced to make the problem balanced. We solve the resulting matrix by VAM and then by the MODI method as follows.

If you have attempted E3, you must have obtained the following basic feasible solution using VAM as:

| | | | | | | | | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 5 50 | 3 100 | 8 | 9 | 9 | 0 | 150 | 3 | 2 | 2 | 2* | 4* | - |
| | 15 | 11 | 13 | 5 45 | 13 105 | 0 50 | 200 | 5* | 6* | 2 | 2 | 2 | 2 |
| | 6 30 | 5 | 4 75 | 0 | 10 20 | 0 | 125 | 4 | 4 | 1 | 1 | 4 | 4 |
| | 80 | 100 | 75 | 45 | 125 | 50 | 475 | | | | | | |
| $D_1$ | 1 | 2 | 4 | 5 | 1 | 0 | | | | | | | |
| $D_2$ | 1 | 2 | 4 | 5 | 1 | - | | | | | | | |
| $D_3$ | 1 | 2 | 4* | - | 1 | - | | | | | | | |
| $D_4$ | 1 | 2 | - | - | 1 | - | | | | | | | |
| $D_5$ | 1 | - | - | - | 1 | - | | | | | | | |
| $D_6$ | 9* | - | - | - | 3 | - | | | | | | | |

If you have attempted E5, you can perform the optimality test by using the MODI method. The allocations which give the optimal solution are:

| | | | | | |
|---|---|---|---|---|---|
| 5 (50) | 3 | 8 | 9 | 9 | 0 |
| 15 | 11 (100) | 13 | 5 (25) | 13 | 0 |
| 6 (30) | 5 | 4 (75) | 0 (20) | 10 (125) | 0 (50) |

The maximum profit (on multiplying the allocations with the corresponding profit values)

$$= 5 \times 50 + 11 \times 100 + 5 \times 25 + 10 \times 125 + 0 \times 50 + 6 \times 30 + 4 \times 75 + 0 \times 20$$

$$= 250 + 1100 + 125 + 125 + 0 + 180 + 300 + 0 = 2080$$

### 4.5.5 Prohibited Routes

Sometimes, in a transportation problem, some routes may not be available. There could be several reasons for this such as bad road conditions or strike, etc. To handle such a situation, a very large cost (or a negative profit for maximisation problem) represented by $\infty$ or M is assigned to each such route which is not available. Due to assignment of very large cost, such routes would automatically be eliminated in the final solution.

Now, you should try to solve the following exercise.

**E6)** Solve the following transportation problem:

| | | Godowns | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | Stock Available |
| | 1 | 7 | 5 | 7 | 7 | 5 | 3 | 60 |
| Factory | 2 | 9 | 11 | 6 | 11 | - | 5 | 20 |
| | 3 | 11 | 10 | 6 | 2 | 2 | 8 | 90 |
| | 4 | 9 | 10 | 9 | 6 | 9 | 12 | 50 |
| | | 60 | 20 | 40 | 20 | 40 | 40 | |

Before going further, match your answer with the answer given in Sec. 4.7.

Let us now summarise the main points, which have been covered in this unit.

## 4.6 SUMMARY

1. A **transportation problem** is one of the sub-classes of Linear Programming Problems in which the objective is to transport various quantities (initially stored at various origins/plants/factories) to different destinations/distribution centres/warehouses in such a way that the total transportation cost is minimum. These problems can be solved by the Simplex method but this method is cumbersome and hence another method known as **transportation method** is preferred.

   It involves making a transportation model in the form of a matrix, finding initial basic feasible solution, performing optimality test and moving towards optimal solution. Methods of finding initial basic feasible solution are **North-West Corner Rule**, **Least Cost Method** and **Vogel's Approximation Method.** Vogel's approximation method generally gives the solution closest to the optimum solution and hence is preferred over other methods of finding the initial basic feasible solutions. After finding the initial basic feasible solution, optimality test is performed to find whether the obtained feasible solution is optimal or not.

2. **Conditions for applying optimality test** are:

   i) It contains exactly (m + n–1) allocations where m and n represent the number of rows and columns, respectively, of the transportation table.

   ii) These allocations are independent.

   If it is not possible to form any closed loop through the allocations under consideration then the allocations are said to be **independent**. If such a loop can be formed using some or all of the allocations under consideration, the allocations are known as **non-independent**.

3. Optimality test is performed by applying the **Stepping Stone Method** or **Modified Distribution (MODI) Method**.

4. The transportation problems wherein the total capacity of all the sources and total requirement (demand) of all the destinations are not equal is said to be an **unbalanced problem**. If the total capacity of sources is greater than (less than) the total requirement at destination, we add a dummy destination (source) in the transportation table with zero transportation cost so that the problem becomes balanced.

5. It may happen sometimes that the number of occupied cells is less than m + n–1. Such a solution is called a **degenerate solution**. We handle such a situation by introducing an infinitesimally small allocation e (say) in the least cost and independent cell.

6. If all net-evaluations, i.e., $\Delta_{ij=}\ c_{ij} - (u_i + v_j) \geq 0$ but one or more of them are equal to zero then there is an **alternative optimal solution** for the transportation problem.

7. To handle a **Maximisation Transportation Problem,** we find the opportunity loss matrix by subtracting the value of each cell from the largest value chosen from amongst all the given cells.

8. Sometimes, in a transportation problem, some routes may not be available. There could be several reasons for this such as bad road conditions or strike, etc. To handle such a situation, a very large cost (or a negative profit for maximisation problem) represented by ∞ or M is assigned to each of such routes, which are not available.

## 4.7 SOLUTIONS/ANSWERS

**E1)** Using North-West Corner Rule, the allocations are to be made as under:

8 units to cell (1,1), 2 units to cell (1,2), 6 units to cell (2,2), 6 units to cell (2,3), 4 units to cell (3,3), 6 units to cell (4,3), 3 units to cell (4, 4), 8 units to cell (4, 5), 1 unit to cell (4, 6) and 20 units to cell (5,6).

The transportation cost for this solution will be:

$4 \times 8 + 6 \times 2 + 5 \times 6 + 4 \times 6 + 9 \times 4 + 5 \times 6 + 9 \times 3 + 3 \times 8 + 6 \times 1 + 14 \times 20 = 32+12+30+24+36+30+27+24+6+280 = 501$

**E2)** Using Least Cost method, the allocations are to be made as under:

4 units to cell (1,1), 3 units to cell (1,4), 3 units to cell (1,6), 12 units to cell (2,6), 4 units to cell (3,1), 8 units to cell (4,2), 10 units to cell (4,3), 6 units to cell (5, 3), 8 units to cell (5,5) and 6 units to cell (5,6).

The transportation cost for this solution will be:

$4 \times 4 + 2 \times 3 + 8 \times 3 + 0 \times 12 + 2 \times 4 + 4 \times 8 + 5 \times 10 + 7 \times 6 + 2 \times 8 + 14 \times 6 = 16+6+24+0+8+32+50+42+16+84 = 278$

**E3)** Using Vogel's Approximation method, the allocations are to be made as under:

| D. Centre / Factory | 1 | 2 | 3 | 4 | 5 | 6 | Capacity | $D_1$ | $D_2$ | $D_3$ | $D_4$ | $D_5$ | $D_6$ | $D_7$ | $D_8$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 4 | 6 | 9 | 2 3 | 7 | 8 3 | 10 | 2 | 2 | 2* | 2 | 2 | 4* | 1 | 1 |
| 2 | 3 | 5 | 4 | 8 | 10 | 0 12 | 12 | 3 | - | - | - | - | - | - | - |
| 3 | 2 4 | 6 | 9 | 8 | 4 | 13 | 4 | 2 | 2* | - | - | - | - | - | - |
| 4 | 4 | 4 8 | 5 4 | 9 | 3 | 6 6 | 18 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 9 | 8 | 7 12 | 3 | 2 8 | 14 | 20 | 1 | 1 | 1 | 5* | 1 | 2 | 7* | - |
| Requirement | 8 | 8 | 16 | 3 | 8 | 21 | 64 | | | | | | | | |
| $D_1$ | 1 | 1 | 1 | 1 | 1 | 6* | | | | | | | | | |
| $D_2$ | 2 | 2 | 2 | 1 | 1 | 2 | | | | | | | | | |
| $D_3$ | 0 | 2 | 2 | 1 | 1 | 2 | | | | | | | | | |
| $D_4$ | 0 | 2 | 2 | - | 1 | 2 | | | | | | | | | |
| $D_5$ | 0 | 2* | 2 | - | - | 2 | | | | | | | | | |
| $D_6$ | 0 | - | 2 | - | - | 2 | | | | | | | | | |
| $D_7$ | - | - | 2 | - | - | 2 | | | | | | | | | |
| $D_8$ | - | - | 4* | - | - | 2 | | | | | | | | | |

In the above table $D_i$ (i=1, 2, 3, 4, 5, 6, 7, 8) means the differences of the least and next to least costs. Thus, the total transportation cost

$= 16 + 6 + 24 + 0 + 8 + 32 + 20 + 36 + 84 + 16 = 242$

**E4)** The allocations which give optimal solutions are given in the following table:

| | | | |
|---|---|---|---|
| 19 5 | 30 | 50 | 12 2 |
| 70 | 30 3 | 40 7 | 60 |
| 40 | 10 5 | 60 | 20 13 |

Thus, the total cost of transportation for this set

$= 19 \times 5 + 12 \times 2 + 30 \times 3 + 40 \times 7 + 10 \times 5 + 20 \times 13$

$= 95 + 24 + 90 + 280 + 50 + 260 = 799$

and the maximum saving = `(1000 – 799) = `201

**E5)** After obtaining the basic feasible solution by VAM as in E3, we apply the MODI method and write the cost matrix for only the allocated cells as follows:

| | | | | | |
|---|---|---|---|---|---|
| 4 | | | 2 | | 8 |
| | | | | | 0 |
| 2 | | | | | |
| | 4 | 5 | | | 6 |
| | | 7 | | 2 | |

Defining row and column numbers as described in the solution by MODI method for Example 1 and then filling the vacant cells, we get

| | $v_1 = 4$ | $v_2 = 6$ | $v_3 = 7$ | $v_4 = 2$ | $v_5 = 2$ | $v_6 = 8$ |
|---|---|---|---|---|---|---|
| $u_1 = 0$ | | 6 | 7 | | 2 | |
| $u_2 = -8$ | –4 | –2 | –1 | –6 | –6 | |
| $u_3 = -2$ | | 4 | 5 | 0 | 0 | 6 |
| $u_4 = -2$ | 2 | | | 0 | 0 | |
| $u_5 = 0$ | 4 | 6 | | 2 | | 8 |

Now subtracting these from the corresponding costs of original cost matrix i.e. $\Delta_{ij} = c_{ij} - (u_i + v_j)$, we get

| | | | | | |
|---|---|---|---|---|---|
| | 0 | 2 | | 5 | |
| 7 | 7 | 5 | 14 | 16 | |
| | 2 | 4 | 8 | 4 | 7 |
| 2 | | | 9 | 3 | |
| 5 | 2 | | 1 | | 6 |

Since all $\Delta_{ij} \geq 0$, this solution is the optimal solution. Thus, the optimal solution = 242

**E6)** In this problem, the route from Factory 2 to Godown 5 is prohibited and hence to handle this situation, a very large cost represented by $\infty$ or M is assigned to this cell. Due to assignment of very large cost, this route would automatically be eliminated in the final solution as the routes with maximum values are never selected while solving transportation problems. Total transportation cost to be obtained by you will be `1120.