UNIT 8 MATHEMATICAL EXPECTATION

Structure

8.1 INTRODUCTION

In Units 1 to 4 of this course, you have studied probabilities of different events in various situations. Concept of univariable random variable has been introduced in Unit 5 whereas that of bivariate random variable in Units 6 and 7. Before studying the present unit, we advice you to go through the above units.

You have studied the methods of finding mean, variance and other measures in context of frequency distributions in MST-002 (Descriptive Statistics). Here, in this unit we will discuss mean, variance and other measures in context of probability distributions of random variables. Mean or Average value of a random variable taken over all its possible values is called the expected value or the expectation of the random variable. In the present unit, we discuss the expectations of random variables and their properties.

In Secs. 8.2, 8.3 and 8.4, we deal with expectation and its properties. Addition and multiplication laws of expectation have been discussed in Sec. 8.5.

Objectives

After studying this unit, you would be able to:

- find the expected values of random variables;
- establish the properties of expectation;
- obtain various measures for probability distributions; and
- apply laws of addition and multiplication of expectation at appropriate situations.

8.2 EXPECTATION OF A RANDOM VARIABLE

In Unit 1 of MST-002, you have studied that the mean for a frequency distribution of a variable X is defined as

$$\text{Mean} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}.$$

If the frequency distribution of the variable X is given as

$x:$	x_1	x_2	$x_3 \ldots x_n$
$f:$	f_1	f_2	$f_3 \ldots f_n$

The above formula of finding mean may be written as

$$\text{Mean} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{f_1 x_1 + f_2 x_2 + \ldots + f_n x_n}{\sum_{i=1}^{n} f_i}$$

$$= \frac{x_1 f_1}{\sum_{i=1}^{n} f_i} + \frac{x_2 f_2}{\sum_{i=1}^{n} f_i} + \ldots + \frac{x_n f_n}{\sum_{i=1}^{n} f_i}$$

$$= x_1 \left(\frac{f_i}{\sum_{i=1}^{n} f_i} \right) + x_2 \left(\frac{f_2}{\sum_{i=1}^{n} f_i} \right) + \ldots + x_n \left(\frac{f_n}{\sum_{i=1}^{n} f_i} \right)$$

Notice that $\frac{f_1}{\sum_{i=1}^{n} f_i}, \frac{f_2}{\sum_{i=1}^{n} f_i}, \ldots, \frac{f_n}{\sum_{i=1}^{n} f_i}$ are, in fact, the relative frequencies or the proportion of individuals corresponding to the values $x_1, x_2, \ldots, x_n$ respectively of variable X and hence can be replaced by probabilities. [See Unit 2 of this course]

Let us now define a similar measure for the probability distribution of a random variable X which assumes the values say $x_1, x_2, \ldots, x_n$ with their associated probabilities $p_1, p_2, \ldots, p_n$. This measure is known as expected value of X and in the similar way is given as

$x_1(p_1) + x_2(p_2) + \ldots + x_n(p_n) = \sum_{i=1}^{n} x_i p_i$ with only difference is that the role of relative frequencies has now been taken over by the probabilities. The expected value of X is written as E(X).

The above aspect can be viewed in the following way also:

Mean of a frequency distribution of X is $\dfrac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i}$, similarly mean of a probability distribution of r.v. X is $\dfrac{\sum_{i=1}^{n} x_i p_i}{\sum_{i=1}^{n} p_i}$.

Now, as we know that $\sum_{i=1}^{n} p_i = 1$ for a probability distribution, therefore the mean of the probability distribution becomes $\sum_{i=1}^{n} x_i p_i$.

$\therefore$ Expected value of a random variable X is $E(X) = \sum_{i=1}^{n} x_i p_i$.

The above formula for finding the expected value of a random variable X is used only if X is a discrete random variable which takes the values $x_1, x_2, ..., x_n$ with probability mass function $p(x_i) = P[X = x_i], \ i = 1, 2, ..., n.$

But, if X is a continuous random variable having the probability density function $f(x)$, then in place of summation we will use integration and in this case, the expected value of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx,$$

The expectation, as defined above, agrees with the logical/theoretical argument also as is illustrated in the following example.

Suppose, a fair coin is tossed twice, then answer to the question, "How many heads do we expect theoretically/logically in two tosses?" is obviously 1 as the coin is unbiased and hence we will undoubtedly expect one head in two tosses. Expectation actually means "what we get on an average"? Now, let us obtain the expected value of the above question using the formula.

Let X be the number of heads in two tosses of the coin and we are to obtain E(X), i.e. expected number of heads. As X is the number of heads in two tosses of the coin, therefore X can take the values 0, 1, 2 and its probability distribution is given as

$$\begin{array}{llll} X: & 0 & 1 & 2 \\ p(x): & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{array}.$$

[Refer Unit 5 of MST-003]

$$\therefore \ E(X) = \sum_{i=1}^{3} x_i p_i$$

$$= x_1 p_1 + x_2 p_2 + x_3 p_3$$

$$= (0)\left(\frac{1}{4}\right)+(1)\left(\frac{1}{2}\right)+(2)\left(\frac{1}{4}\right)= 0+\frac{1}{2}+\frac{1}{2}=1$$

So, we get the same answer, i.e. 1 using the formula also.

So, expectation of a random variable is nothing but the average (mean) taken over all the possible values of the random variable or it is the value which we get on an average when a random experiment is performed repeatedly.

Remark 1: Sometimes summations and integrals as considered in the above definitions may not be convergent and hence expectations in such cases do not exist. But we will deal only those summations (series) and integrals which are convergent as the topic regarding checking the convergence of series or integrals is out of the scope of this course. You need not to bother as to whether the series or integral is convergent or not, i.e. as to whether the expectation exists or not as we are dealing with only those expectations which exist.

Example 1: If it rains, a rain coat dealer can earn Rs 500 per day. If it is a dry day, he can lose Rs 100 per day. What is his expectation, if the probability of rain is 0.4?

Solution: Let X be the amount earned on a day by the dealer. Therefore, X can take the values Rs 500, $-$ Rs 100 ($\because$ loss of Rs 100 is equivalent to negative of the earning of Rs100).

$\therefore$ Probability distribution of X is given as

	Rainy Day	Dry day
$X(\text{in Rs.})$:	500	-100
$p(x)$:	0.4	0.6

Hence, the expectation of the amount earned by him is

$$E(X)=\sum_{i=1}^{2} x_i p_i = x_1p_1 + x_2p_2$$

$$= (500)(0.4)+(-100)(0.6) = 200-60 = 140$$

Thus, his expectation is Rs 140, i.e. on an overage he earns Rs 140 per day.

Example 2: A player tosses two unbiased coins. He wins Rs 5 if 2 heads appear, Rs 2 if one head appears and Rs1 if no head appears. Find the expected value of the amount won by him.

Solution: In tossing two unbiased coins, the sample space, is

$S = \{HH, HT, TH, TT\}$.

$$\therefore\ P[2\,\text{heads}]=\frac{1}{4},\ \ P(\text{one head})=\frac{2}{4},\ \ P(\text{no head})=\frac{1}{4}.$$

Let X be the amount in rupees won by him

$\therefore$ X can take the values 5, 2 and 1 with

$P[X=5] = P(2\text{heads}) = \frac{1}{4}$,

$P[X=2] = P[1\text{Head}] = \frac{2}{4}$, and

$P[X=1] = P[\text{no Head}] = \frac{1}{4}$.

$\therefore$ Probability distribution of X is

$X:$	5	2	1
$p(x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

Expected value of X is given as

$$E(X) = \sum_{i=1}^{3} x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3$$

$$= 5\left(\frac{1}{4}\right) + 2\left(\frac{2}{4}\right) + 1\left(\frac{1}{4}\right) = \frac{5}{4} + \frac{4}{4} + \frac{1}{4} = \frac{10}{4} = 2.5.$$

Thus, the expected value of amount won by him is Rs 2.5.

Example 3: Find the expectation of the number on an unbiased die when thrown.

Solution: Let X be a random variable representing the number on a die when thrown.

$\therefore$ X can take the values 1, 2, 3, 4, 5, 6 with

$$P[X=1] = P[X=2] = P[X=3] = P[X=4] = P[X=5] = P[X=6] = \frac{1}{6}.$$

Thus, the probability distribution of X is given by

$X:$	1	2	3	4	5	6
$p(x):$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Hence, the expectation of number on the die when thrown is

$$E(X) = \sum_{i=1}^{6} x_i p_i = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6} = \frac{21}{6} = \frac{7}{2}$$

Example 4: Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the expected value for the number of aces.

Solution: Let A_1, A_2 be the events of getting ace in first and second draws, respectively. Let X be the number of aces drawn. Thus, X can take the values 0, 1, 2 with

$$P[X=0] = P[\text{no ace}] = P\left[\bar{A}_1 \cap \bar{A}_2\right]$$

$$= P\left[\bar{A}_1\right] P\left[\bar{A}_2\right] \quad \left[\begin{array}{l}\because \text{cards are drawn with replacement} \\ \text{and hence the events are independent}\end{array}\right]$$

$$= \frac{48}{52} \times \frac{48}{52} = \frac{12}{13} \times \frac{12}{13} = \frac{144}{169},$$

$$P[X = 1] = [\text{one Ace and one other card}]$$

$$= P\left[\left(A_1 \cap \bar{A}_2\right) \cup \left(\bar{A}_1 \cap A_2\right)\right]$$

$$= P\left[A_1 \cap \bar{A}_2\right] + P\left[\bar{A}_1 \cap A_2\right] \quad \left[\begin{array}{l}\text{By Addition theorem of probability}\\ \text{for mutually exclusive events}\end{array}\right]$$

$$= P[A_1]P\left[\bar{A}_2\right] + P\left[\bar{A}_1\right]P[A_2] \quad \left[\begin{array}{l}\text{By multiplication theorem of}\\ \text{probability for independent events}\end{array}\right]$$

$$= \frac{4}{52} \times \frac{48}{52} + \frac{48}{52} \times \frac{4}{52} = \frac{1}{13} \times \frac{12}{13} + \frac{12}{13} \times \frac{1}{13} = \frac{24}{169}, \text{ and}$$

$$P[X = 2] = P[\text{both aces}] = P[A_1 \cap A_2]$$

$$= P[A_1]P[A_2] = \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}.$$

Hence, the probability distribution of random variable X is

$X:$	0	1	2
$p(x):$	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

$\therefore$ The expected value of X is given by

$$E(X) = \sum_{i=1}^{3} x_i p_i = 0 \times \frac{144}{169} + 1 \times \frac{24}{169} + 2 \times \frac{1}{169} = \frac{26}{169} = \frac{2}{13}$$

Example 5: For a continuous distribution whose probability density function is given by:

$$f(x) = \frac{3x}{4}(2 - x),\ 0 \le x \le 2, \text{ find the expected value of X.}$$

Solution: Expected value of a continuous random variable X is given by

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^2 x \frac{3x}{4}(2 - x)\,dx = \frac{3}{4}\int_0^2 x^2(2 - x)\,dx$$

$$= \frac{3}{4}\int_0^2 \left(2x^2 - x^3\right)dx = \frac{3}{4}\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[2\frac{(2)^3}{3} - \frac{(2)^4}{4} - 0\right]$$

$$= \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{3}{4} \times \frac{16}{12} = 1$$

Now, you can try the following exercises.

E1) You toss a fair coin. If the outcome is head, you win Rs 100; if the outcome is tail, you win nothing. What is the expected amount won by you?

E2) A fair coin is tossed until a tail appears. What is the expectation of number of tosses?

E3) The distribution of a continuous random variable X is defined by

$$f(x) = \begin{cases} x^3 & , \ 0 < x \le 1 \\ (2-x)^3 & , \ 1 < x \le 2 \\ 0 & , \ \text{elsewhere} \end{cases}$$

Obtain the expected value of X.

Let us now discuss some properties of expectation in the next section.

8.3 PROPERTIES OF EXPECTATION OF ONE-DIMENSIONAL RANDOM VARIABLE

Properties of mathematical expectation of a random variable X are:

1. E(k) = k, where k is a constant
2. E(kX) = kE(X), k being a constant.
3. E(aX + b) = aE(X) + b, where a and b are constants

Proof:

Discrete case:

Let X be a discrete r.v. which takes the values $x_1, x_2, x_3, \ldots$ with respective probabilities $p_1, p_2, p_3, \ldots$

1. $E(k) = \sum_i k\,p_i$ [By definition of the expectation]

$= k\sum_i p_i$

$= k(1) = k$ $\left[\because \text{sum of probabilities of all the possible value of r.v. is 1}\right]$

2. $E(kX) = \sum_i (kx_i)\,p_i$ [By def.]

$= k\sum_i x_i\,p_i$

$= k\,E(X)$

3. $E(aX + b) = \sum_i (ax_i + b)\,p_i$ [By def.]

$= \sum_i (ax_ip_i + bp_i) = \sum_i ax_ip_i + \sum_i bp_i = a\sum_i x_ip_i + b\sum_i p_i$

$= aE(X) + b(1) = aE(X) + b$

Continuous Case:

Let X be continuous random variable having f(x) as its probability density function. Thus,

1. $E(k) = \int_{-\infty}^{\infty} kf(x)dx$ [By def.]

$= k\int_{-\infty}^{\infty} f(x)dx$

$= k(1) = k$ $\left[\begin{array}{l}\because \text{integral of the p.d.f. over} \\ \text{the entire range is 1}\end{array}\right]$

2. $E(kX) = \int_{-\infty}^{\infty} (kx)f(x)dx$ [By def.]

$= k\int_{-\infty}^{\infty} xf(x)dx = kE(X)$

3. $E(aX+b) = \int_{-\infty}^{\infty} (ax+b)f(x)dx = \int_{-\infty}^{\infty} (ax)f(x)dx + \int_{-\infty}^{\infty} b\,f(x)dx$

$= a\int_{-\infty}^{\infty} x\,f(x)dx + b\int_{-\infty}^{\infty} f(x)dx = aE(X) + b(1) = aE(X) + b$

Example 6: Given the following probability distribution:

X	–2	–1	0	1	2
$p(x)$	0.15	0.30	0	0.30	0.25

Find i) E(X)

ii) E(2X + 3)

iii) $E(X^2)$

iv) E(4X – 5)

Solution

i) $E(X) = \sum_{i=1}^{5} x_i p_i = x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5$

$= (-2)(0.15) + (-1)(0.30) + (0)(0) + (1)(0.30) + (2)(0.25)$

$= -0.3 - 0.3 + 0 + 0.3 + 0.5 = 0.2$

ii) E(2X + 3) = $2E(X) + 3$ [Using property 3 of this section]

= 2(0.2) + 3 [Using solution (i) of the question]

= 0.4 + 3 = 3.4

iii) $E(X^2) = \sum_{i=1}^{5} x_i^2 p_i$ [By def.]

$$= x_1^2 p_1 + x_2^2 p_2 + x_3^2 p_3 + x_4^2 p_4 + x_5^2 p_5$$

$$= (-2)^2 (0.15) + (-1)^2 (0.30) + (0)^2 (0) + (1)^2 (0.30) + (2)^2 (0.25)$$

$$= (4)(0.15) + (1)(0.30) + (0) + (1)(0.30) + (4)(0.25)$$

$$= 0.6 + 0.3 + 0 + 0.3 + 1 = 2.2$$

iv) $E(4X - 5) = E\left[4X + (-5)\right]$

$$= 4E(X) + (-5) \qquad \text{[Using property 3 of this section]}$$

$$= 4(0.2) - 5$$

$$= 0.8 - 5 = -4.2$$

Here is an exercise for you.

E4) If X is a random variable with mean 'μ' and standard deviation 'σ', then what is the expectation of $Z = \frac{X - \mu}{\sigma}$?

[**Note:** Here Z so defined is called standard random variate.]

Let us now express the moments and other measures for a random variable in terms of expectations in the following section.

8.4 MOMENTS AND OTHER MEASURES IN TERMS OF EXPECTATIONS

Moments

The moments for frequency distribution have already been studied by you in Unit 3 of MST-002. Here, we deal with moments for probability distributions. The r^{th} order moment about any point 'A' (say) of variable X already defined in Unit 3 of MST-002 is given by:

$$\mu_r' = \frac{\sum_{i=1}^{n} f_i (x_i - A)^r}{\sum_{i=1}^{n} f_i}$$

So, the r^{th} order moment about any point 'A' of a random variable X having probability mass function $P\left[X = x_i\right] = p(x_i) = p_i$ is defined as

$$\mu_r' = \frac{\sum_{i=1}^{n} p_i (x_i - A)^r}{\sum_{i=1}^{n} p_i}$$

[Replacing frequencies by probabilities as discussed in Sec. 8.2 of this unit.]

$$= \sum_{i=1}^{n} p_i (x_i - A)^r \qquad \left[\because \sum_{i=1}^{n} p_i = 1\right]$$

The above formula is valid if X is a discrete random variable. But, if X is a continuous random variable having probability density function f(x), then

r^{th} order moment about A is defined as $\mu_r' = \int_{-\infty}^{\infty} (x-A)^r f(x)\,dx.$

So, r^{th} order moment about any point 'A' of a random variable X is defined as

$$\mu_r' = \begin{cases} \sum_i p_i (x_i - A)^r, & \text{if X is a discrete r.v.} \\ \int_{-\infty}^{\infty} (x-A)^r f(x)\,dx, & \text{if X is a continous r.v} \end{cases}$$

$$= E(X - A)^r$$

Similarly, r^{th} order moment about mean (μ) i.e. r^{th} order central moment is defined as

$$\mu_r = \begin{cases} \sum_i p_i (x_i - \mu)^r, & \text{if X is a discrete r.v.} \\ \int_{-\infty}^{\infty} (x-\mu)^r f(x)\,dx, & \text{if X is a continous r.v} \end{cases}$$

$$= E(X-\mu)^r = E\left[X - E(X)\right]^r$$

Variance

Variance of a random variable X is second order central moment and is defined as

$$\mu_2 = V(X) = E[X-\mu]^2 = E\left[X - E(X)\right]^2$$

Also, we know that

$$V(X) = \mu_2' - (\mu_1')^2$$

where μ_1', μ_2' be the moments about origin.

$\therefore$ We have $V(X) = E(X^2) - \left[E(X)\right]^2$

$$\left[\because \mu_1' = E[X-0]^1 = E(X), \text{ and } \mu_2' = E[X-0]^2 = E(X^2)\right]$$

Theorem 8.1: If X is a random variable, then $V(aX+b) = a^2 V(X)$, where a and b are constants.

Proof: $V(aX+b) = E\left[(aX+b) - E(aX+b)\right]^2$ [By def. of variance]

$$= E\left[aX + b - \left(aE(X)+b\right)\right]^2 \quad \text{[Using property 3 of Sec. 8.3]}$$

$$= E\left[aX + b - aE(X) - b\right]^2$$

$$= E\left[a\left\{X - E(X)\right\}\right]^2$$

$$= E\left[a^2\left(X - E(X)\right)^2\right]$$

$$= a^2E\left[X - E(X)\right]^2 \qquad \text{[Using property 2 of section 8.3]}$$

$$= a^2V(X) \qquad \text{[By definition of Variance]}$$

Cor. (i) $V(aX) = a^2V(X)$

(ii) V(b) = 0

(iii) V(X + b) = V(X)

Proof: (i) This result is obtained on putting b = 0 in the above theorem.

(ii) This result is obtained on putting a = 0 in the above theorem.

(iii) This result is obtained on putting a = 1 in the above theorem.

Covariance

For a bivariate frequency distribution, you have already studied in Unit 6 of MST-002 that covariance between two variables X and Y is defined as

$$\text{Cov}(X, Y) = \frac{\sum_i f_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i f_i}$$

$\therefore$ For a bivariate probability distribution, Cov (X, Y) is defined as

$$\text{Cov}(X, Y) = \begin{cases} \sum_i p_{ij}(x_i - \bar{x})(y_i - \bar{y}), & \text{if } (X, Y) \text{ is two-dimensional discrete r.v.} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - \bar{x})(y - \bar{y}) f(x, y)\,dydx, & \text{if } (X, Y) \text{ is two dimensional continuous r.v.} \end{cases}$$

where $p_{ij} = P\left[X = x_i, Y = y_j\right]$

$$= E(X - \bar{X})(Y - \bar{Y}) \qquad \text{[By definition of expectation]}$$

$$= E\left[(X - E(X))(Y - E(Y))\right] \qquad \left[\begin{array}{l} \because E(X) = \text{Mean of X i.e. } \bar{X}, \\ E(Y) = \text{Mean of Y i.e. } \bar{Y} \end{array}\right]$$

On simplifying,

Cov(X, Y) = E(XY) – E(X) E(Y).

Now, if X and Y are independent random variables then, by multiplication theorem,

E(XY) = E(X)E(Y) and hence in this case Cov(X, Y) = 0.

Remark 2:

i) If X and Y are independent random variables, then

V(X + Y) = V(X) +V(Y).

Proof: $V(X+Y) = E\left[(X+Y) - E(X+Y)\right]^2$

$$= E\left[X + Y - E(X) - E(Y)\right]^2$$

$$= E\left[\{X - E(X)\} + \{Y - E(Y)\}\right]^2$$

$$= E\left[\{X - E(X)\}^2 + \{Y - E(Y)\}^2 + 2\{X - E(X)\}\{Y - E(Y)\}\right]$$

$$= E\left[X - E(X)\right]^2 + E\left[Y - E(Y)\right]^2 + 2E\left[(X - E(X))(Y - E(Y))\right]$$

$$= V(X) + V(Y) + 2\text{Cov}(X, Y)$$

$$= V(X) + V(Y) + 0 \qquad [\because \text{ X and Y are independent}]$$

$$= V(X) + V(Y)$$

ii) If X and Y are independent random variables, then

$V(X - Y) = V(X) + V(Y).$

Proof: This can be proved in the similar manner as done is Remark 2(i) above.

iii) If X and Y are independent random variables, then

$V(aX + bY) = a^2V(X) + b^2V(Y).$

Proof: Prove this result yourself proceeding in the similar fashion as in proof of Remark 2(i).

Mean Deviation about Mean

Mean deviation about mean in context of frequency distribution is

$$\frac{\sum_{i=1}^{n} f_i \,|\, x_i - \bar{x} \,|}{\sum_{i=1}^{n} f_i}, \text{ and}$$

therefore, mean deviation about mean in context of probability distribution is

$$\frac{\sum_{i=1}^{n} p_i \,|\, x - \text{mean} \,|}{\sum_{i=1}^{n} p_i} = \sum_{i=1}^{n} p_i \,|\, x - \text{mean} \,|$$

$\therefore$ by definition of expectation, we have

M.D. about mean = E|X – Mean|

= E|X – E(X)|

$$= \begin{cases} \sum p_i \,|\, x - \text{Mean} \,|\, & \text{for discrete r.v} \\ \int_{-\infty}^{\infty} |\, x - \text{Mean} \,|\, f(x)\,dx & \text{for continuous r.v} \end{cases}$$

Note: Other measures as defined for frequency distributions in MST-002 can be defined for probability distributions also and hence can be expressed in terms of the expectations in the manner as the moments; variance and covariance have been defined in this section of the Unit.

Example 7: Considering the probability distribution given in Example 6, obtain

i) V(X)

ii) V(2X + 3).

Solution:

(i) $V(X) = E(X^2) - [E(X)]^2$

$= 2.2 - (0.2)^2$ $\left[\begin{array}{l}\text{The values have already been obtained} \\ \text{in the solution of Example 6}\end{array}\right]$

$= 2.2 - 0.04 = 2.16$

(ii) $V(2X+3) = (2)^2 V(X)$ [Using the result of Theorem 8.1]

$= 4V(X) = 4(2.16) = 8.64$

Example 8: If X and Y are independent random variables with variances 2 and 3 respectively, find the variance of 3X + 4Y.

Solution: $V(3X + 4Y) = (3)^2 V(X) + (4)^2 V(Y)$ [By Remark 3 of Section 8.4]

$= 9(2) + 16(3) = 18 + 48 = 66$

Here are two exercises for you:

E5) If X is a random variable with mean μ and standard deviation σ, then find the variance of standard random variable $Z = \dfrac{X-\mu}{\sigma}$.

E6) Suppose that X is a random variable for which E(X) = 10 and V(X) = 25. Find the positive values of a and b such that Y = aX – b has expectation 0 and variance 1.

8.5 ADDITION AND MULTIPLICATION THEOREMS OF EXPECTATION

Now, we are going to deal with the properties of expectation in case of two-dimensional random variable. Two important properties, i.e. addition and multiplication laws of expectation are discussed in the present section.

Addition Theorem of Expectation

Theorem 8.2: If X and Y are random variables, then $E(X+Y) = E(X) + E(Y)$

Proof:

Discrete case:

Let (X, Y) be a discrete two-dimensional random variable which takes up the values (x_i, y_j) with the joint probability mass function $p_{ij} = P\left[X = x_i \cap Y = y_j\right]$.

Then, the probability distribution of X is given by

$$p_i = p(x_i) = P[X = x_i]$$

$$= P[X = x_i \cap Y = y_1] + P[X = x_i \cap Y = y_2] + \ldots \begin{bmatrix} \because \text{event } X = x_i \text{ can happen with} \\ Y = y_1 \text{ or } Y = y_2 \text{ or } Y = y_3 \text{ or} \ldots \end{bmatrix}$$

$$= p_{i1} + p_{i2} + p_{i3} + \ldots$$

$$= \sum_j p_{ij}$$

Similarly, the probability distribution of Y is given by

$$p_j' = p(y_j) = P\left[Y = y_j\right] = \sum_i p_{ij}$$

$$\therefore \; E(X) = \sum_i x_i p_i, \; E(Y) = \sum_j y_j p_j' \text{ and } E(X+Y) = \sum_i \sum_j (x_i + y_j) p_{ij}$$

Now $E(X+Y) = \sum_i \sum_j (x_i + y_j) p_{ij}$

$$= \sum_i \sum_j x_i p_{ij} + \sum_i \sum_j y_j p_{ij}$$

$$= \sum_i x_i \sum_j p_{ij} + \sum_j y_j \sum_i p_{ij}$$

[$\because$ in the first term of the right hand side, x_i is free from j and hence can be taken outside the summation over j; and in second term of the right hand side, y_j is free from i and hence can be taken outside the summation over i.]

$$\therefore \; E(X+Y) = \sum_i x_i p_i + \sum_j y_j p_j' = E(X) + E(Y)$$

Continuous Case:

Let (X, Y) be a bivariate continuous random variable with probability density function $f(x, y)$. Let $f(x)$ and $f(y)$ be the marginal probability density functions of random variables X and Y respectively.

$\therefore\ E(X)=\int_{-\infty}^{\infty} x\ f(x)dx,\ E(Y)=\int_{-\infty}^{\infty} y\ f(y)dy,$

and $E(X+Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x,y)\,dy\,dx$.

$$\text{Now, } E(X+Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x,y)\,dy\,dx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,f(x,y)\,dy\,dx+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\,f(x,y)\,dy\,dx$$

$$=\int_{-\infty}^{\infty} x\left(\int_{-\infty}^{\infty} f(x,y)\,dy\right)dx+\int_{-\infty}^{\infty} y\left(\int_{-\infty}^{\infty} f(x,y)\,dx\right)dy$$

[$\because$ in the first term of R.H.S., x is free from the integral w.r.t. y and hence can be taken outside this integral. Similarly, in the second term of R.H.S, y is free from the integral w.r.t. x and hence can be taken outside this integral.]

$$=\int_{-\infty}^{\infty} x\,f(x)\,dx+\int_{-\infty}^{\infty} y\,f(y)\,dy\quad\left[\begin{array}{l}\text{Refer to the definition of marginal density}\\ \text{function given in Unit 7 of this course}\end{array}\right]$$

$$=E(X)+E(Y)$$

Remark 3: The result can be similarly extended for more than two random variables.

Multiplication Theorem of Expectation

Theorem 8.3: If X and Y are independent random variables, then

E(XY) = E(X) E(Y)

Proof:

Discrete Case:

Let (X, Y) be a two-dimensional discrete random variable which takes up the values (x_i, y_j) with the joint probability mass function

$p_{ij}=P\left[X=x_i\cap Y=y_j\right]$. Let p_i and $p_j^{'}$ be the marginal probability mass functions of X and Y respectively.

$\therefore\ E(X)=\sum_i x_i p_i,\ E(Y)=\sum_j y_j p_j^{'},$ and

$$E(XY)=\sum_i\sum_j (x_i y_j)p_{ij}$$

But as X and Y are independent,

$\therefore\ p_{ij}=P\left[X=x_i\cap Y=y_j\right]$

$$= P[X = x_i]P[Y = y_j] \quad \begin{bmatrix} \because \text{ if events A and B are independent,} \\ \text{then } P(A \cap B) = P(A)P(B) \end{bmatrix}$$

$$= p_i p_j^{'}$$

Hence, $E(XY) = \sum_i \sum_j (x_i y_j) p_i p_j^{'}$

$$= \sum_i \sum_j x_i y_j p_i p_j^{'}$$

$$= \sum_i \sum_j (x_i p_i y_j p_j^{'})$$

$$= \sum_i x_i p_i \sum_j y_j p_j^{'} \quad \begin{bmatrix} \because \ x_i p_i \text{ is free from j and hence can be} \\ \text{taken outside the summation over j} \end{bmatrix}$$

$$= E(X)\, E(Y)$$

Continuous Case:

Let (X, Y) be a bivariate continuous random variable with probability density function f(x, y). Let f(x) and f(y) be the marginal probability density function of random variables X and Y respectively.

$$\therefore \ E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx, \ E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy,$$

and $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dy\, dx$.

Now $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dy\, dx$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x)\, f(y)\, dy\, dx \quad \begin{bmatrix} \because \text{ X and Y are independent, } f(x,y)=f(x)f(y) \\ \text{(see Unit 7 of this course)} \end{bmatrix}$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} (x\, f(x))(y f(y))\right) dy\, dx$$

$$= \left(\int_{-\infty}^{\infty} x\, f(x)\, dx\right)\left(\int_{-\infty}^{\infty} y\, f(y)\, dy\right)$$

$$= E(X)E(Y)$$

Remark 4: The result can be similarly extended for more than two random variables.

Example 8: Two unbiased dice are thrown. Find the expected value of the sum of number of points on them.

Solution: Let X be the number obtained on the first die and Y be the number obtained on the second die, then

$E(X)=\frac{7}{2}$ and $E(Y)=\frac{7}{2}$ [See Example 3 given in Section 8.2]

$\therefore$ The required expected value $= E(X + Y)$

$$= E(X) + E(Y) \quad \begin{bmatrix}\text{Using addition theorem}\\ \text{of expectation}\end{bmatrix}$$

$$= \frac{7}{2}+\frac{7}{2} = 7$$

Remark 5: This example can also be done considering one random variable only as follows:

Let X be the random variable denoting “the sum of numbers of points on the dice”, then the probability distribution in this case is

$X:$	2	3	4	5	6	7	8	9	10	11	12
$p(x):$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

and hence $E(X) = 2\times\frac{1}{36}+3\times\frac{2}{36}+...+12\times\frac{1}{36} = 7$

Example 9: Two cards are drawn one by one with replacement from 8 cards numbered from 1 to 8. Find the expectation of the product of the numbers on the drawn cards.

Solution: Let X be the number on the first card and Y be the number on the second card. Then probability distribution of X is

X	1	2	3	4	5	6	7	8
$p(x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

and the probability distribution of Y is

Y	1	2	3	4	5	6	7	8
$p(y)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

$$\therefore \quad E(X)=E(Y)=1\times\frac{1}{8}+2\times\frac{1}{8}+...+8\times\frac{1}{8}$$

$$=\frac{1}{8}(1+2+3+4+5+6+7+8)=\frac{1}{8}(36) = \frac{9}{2}$$

Thus, the required expected value is

$E(XY)=E(X)E(Y)$ [Using multiplication theorem of expectation]

$$=\frac{9}{2}\times\frac{9}{2}=\frac{81}{4}.$$

Expectation of Linear Combination of Random Variables

Theorem 8.4: Let $X_1, X_2, ..., X_n$ be any n random variables and if $a_1, a_2, ..., a_n$ are any n constants, then

$$E\left(a_1X_1 + a_2X_2 + ... + a_nX_n\right) = a_1E\left(X_1\right) + a_2E\left(X_2\right) + ... + a_nE\left(X_n\right)$$

[**Note :** Here $a_1X_1 + a_2X_2 + ... + a_nX_n$ is a linear combination of $X_1, X_2, ... , X_n$]

Proof: Using the addition theorem of expectation, we have

$$E\left(a_1X_1 + a_2X_2 + ... + a_nX_n\right) = E\left(a_1X_1\right) + E\left(a_2X_2\right) + ... + E\left(a_nX_n\right)$$

$$= a_1E\left(X_1\right) + a_2E\left(X_2\right) + ... + a_nE\left(X_n\right).$$

[Using second property of Section 8.3 of the unit]

Now, you can try the following exercises.

E7) Two cards are drawn one by one with replacement from ten cards numbered 1 to 10. Find the expectation of the sum of points on two cards.

E8) Find the expectation of the product of number of points on two dice.

Now before ending this unit, let's summarize what we have covered in it.

8.6 SUMMARY

The following main points have been covered in this unit:

1) Expected value of a random variable X is defined as

$$E\left(X\right) = \sum_{i=1}^{n} x_i p_i \text{ , if X is a discrete random variable}$$

$$= \int_{-\infty}^{\infty} xf\left(x\right)dx \text{ , if X is a continuous random variable.}$$

2) Important properties of expectation are:

i) E(k) = k, where k is a constant.

ii) E(kX) = kE(X), k being a constant.

iii) E(aX + b) = aE(X) + b, where a and b are constants

iv) Addition theorem of Expectation is stated as:

If X and Y are random variables, then $E\left(X+Y\right) = E\left(X\right) + E\left(Y\right)$.

v) Multiplication theorem of Expectation is stated as:

If X and Y are independent random variables, then

E(XY) = E(X)E(Y).

vi) If $X_1, X_2, ..., X_n$ be any n random variables and if $a_1, a_2, ..., a_n$ are any n constants, then

$$E(a_1X_1 + a_2X_2 + ... + a_nX_n) = a_1E(X_1) + a_2E(X_2) + ... + a_nE(X_n).$$

3) Moments and other measures in terms of expectation are given as:

i) r^{th} order moment about any point is given as

$$\mu_r' = \begin{cases} \sum_i p_i (x_i - A)^r, & \text{if X is a discrete r.v.} \\ \int_{-\infty}^{\infty} (x-A)^r f(x)\,dx, & \text{if X is a continous r.v} \end{cases}$$

$$= E(X - A)^r$$

ii) Variance of a random variable X is given as

$$V(X) = E[X-\mu]^2 = E[X - E(X)]^2$$

iii) $$\text{Cov(X,Y)} = \begin{cases} \sum_i p_i (x_i - \bar{x})(y_i - \bar{y}), & \text{if (X, Y) is discrete r.v.} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})(y-\bar{y}) f(x,y)\,dy dx, & \text{if (X, Y) is continuous r.v.} \end{cases}$$

$$= E\left[(X - E(X))(Y - E(Y))\right]$$

$$= E(XY) - E(X)\,E(Y).$$

iv) M.D. about mean $= E|X - E(X)|$

$$= \begin{cases} \sum p_i \,|\,x - \text{Mean}\,| & \text{for discrete r.v} \\ \int_{-\infty}^{\infty} |\,x - \text{Mean}\,| f(x)\,dx & \text{for continuous r.v} \end{cases}$$

If you want to see what our solutions to the exercises in the unit are, we have given them in the following section.

8.7 SOLUTIONS/ANSWERS

E1) Let X be the amount (in rupees) won by you.

$\therefore$ X can take the values 100, 0 with P[X = 100] = P[Head] = $\frac{1}{2}$, and

P[X = 0] = P[Tail] = $\frac{1}{2}$.

$\therefore$ probability distribution of X is

$X:$	100	0
$p(x)$	$\frac{1}{2}$	$\frac{1}{2}$

and hence the expected amount won by you is

$$E(X) = 100 \times \frac{1}{2} + 0 \times \frac{1}{2} = 50.$$

E2) Let X be the number of tosses till tail turns up.

$\therefore$ X can take values 1, 2, 3, 4… with

P[X = 1] = P[Tail in the first toss] = $\frac{1}{2}$

P[X = 2] = P[Head in the first and tail in the second toss] = $\frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^2$,

P[X = 3] = P[HHT] = $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^3$, and so on.

$\therefore$ Probability distribution of X is

X :	1	2	3	4	5...
$p(x)$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^5$ ...

and hence

$$E(X) = 1 \times \frac{1}{2} + 2 \times \left(\frac{1}{2}\right)^2 + 3 \times \left(\frac{1}{2}\right)^3 + 4 \times \left(\frac{1}{2}\right)^4 + \ldots \qquad \ldots (1)$$

Multiplying both sides by $\frac{1}{2}$, we get

$$\frac{1}{2}E(X) = \left(\frac{1}{2}\right)^2 + 2 \times \left(\frac{1}{2}\right)^3 + 3 \times \left(\frac{1}{2}\right)^4 + 4 \times \left(\frac{1}{2}\right)^5 + \ldots$$

$$\Rightarrow \frac{1}{2}E(X) = \left(\frac{1}{2}\right)^2 + 2 \times \left(\frac{1}{2}\right)^3 + 3 \times \left(\frac{1}{2}\right)^4 + \ldots \qquad \ldots (2)$$

[Shifting the position one step towards right so that we get the terms having same power at the same positions as that in (1)]

Now, subtracting (2) from (1), we have

$$E(X) - \frac{1}{2}E(X) = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \ldots$$

$$\Rightarrow \frac{1}{2}E(X) = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$$

$$\Rightarrow E(X) = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$$

(Which is an infinite G.P. with first term a = 1 and common ratio $r = \frac{1}{2}$)

$$= \frac{1}{1-\frac{1}{2}} \qquad \left[\because S_\infty = \frac{a}{1-r} \text{ (see Unit 3 of course MST - 001)}\right]$$

$$= \frac{1}{\frac{1}{2}} = 2.$$

E3) $E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$= \int_{-\infty}^{0} x\,f(x)\,dx + \int_{0}^{1} x\,f(x)\,dx + \int_{1}^{2} x\,f(x)\,dx + \int_{2}^{\infty} x\,f(x)\,dx$$

$$= \int_{-\infty}^{0} x(0)\,dx + \int_{0}^{1} x\left(x^3\right)dx + \int_{1}^{2} x(2-x)^3\,dx + \int_{2}^{\infty} x(0)\,dx$$

$$= 0 + \int_{0}^{1} x^4 dx + \int_{1}^{2} x\left[8 - x^3 - 6x(2-x)\right]dx + 0$$

$$= \int_{0}^{1} x^4 dx + \int_{1}^{2}\left(8x - x^4 - 12x^2 + 6x^3\right)dx$$

$$= \left[\frac{x^5}{5}\right]_0^1 + \left[8\frac{x^2}{2} - \frac{x^5}{5} - 12\frac{x^3}{3} + 6\frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{5} + \left[\left\{\frac{8(2)^2}{2} - \frac{(2)^5}{5} - \frac{12(2)^3}{3} + \frac{6(2)^4}{4}\right\} - \left\{\frac{8(1)^2}{2} - \frac{(1)^5}{5} - \frac{12(1)^3}{3} + \frac{6(1)^4}{4}\right\}\right]$$

$$= \frac{1}{5} + \left[\left\{16 - \frac{32}{5} - 32 + 24\right\} - \left\{4 - \frac{1}{5} - 4 + \frac{3}{2}\right\}\right]$$

$$= \frac{1}{5} + \left[\frac{8}{5} - \frac{13}{10}\right] = \frac{1}{5} + \frac{3}{10} = \frac{1}{2}.$$

E4) As X is a random variable with mean μ,

$\therefore$ E(X) = μ ... (1)

$$\left[\begin{array}{l}\because \text{expectation is nothing but simply the average taken over all} \\ \text{the possible values of random variable as defined in Sec. 8.2}\end{array}\right]$$

Now, $E(Z) = E\left(\frac{X-\mu}{\sigma}\right)$

$= E\left[\frac{1}{\sigma}(X-\mu)\right]$

$= \frac{1}{\sigma}E[X-\mu]$ [Using Property 2 of Sec. 8.3]

$= \frac{1}{\sigma}[E(X)-\mu]$ [Using Property 3 of Sec. 8.3]

$= \frac{1}{\sigma}[\mu-\mu]$ [Using (1)]

$= 0$

Note: Mean of standard random variable is zero.

E5) Variance of standard random variable $Z = \frac{X-\mu}{\sigma}$ is given as

$V(Z) = V\left(\frac{X-\mu}{\sigma}\right) = V\left(\frac{X}{\sigma}-\frac{\mu}{\sigma}\right)$

$= V\left[\frac{1}{\sigma}X + \left(-\frac{\mu}{\sigma}\right)\right]$

$= \left(\frac{1}{\sigma}\right)^2 V(X)$ [Using the result of the Theorem 8.1 of Sec. 8.5 of this unit]

$= \frac{1}{\sigma^2}V(X)$

$= \frac{1}{\sigma^2}(\sigma^2) = 1$ [$\because$ it is given that the standard deviation of X is and hence its variance is σ^2]

Note: The mean of standard random variate is '0' [See **(E4)**] and its variance is 1.

E6) Given that $E(Y) = 0 \Rightarrow E(aX - b) = 0 \Rightarrow a\,E(X) - b = 0$

$\Rightarrow a(10) - b = 0$

$\Rightarrow 10a - b = 0$ ... (1)

Also as V(Y) = 1,

hence V(aX – b) = 1

$\Rightarrow a^2V(X) = 1 \Rightarrow a^2(25) = 1 \Rightarrow a^2 = \frac{1}{25}$

$\Rightarrow a = \frac{1}{5}$ [$\because$ a is positive]

$\therefore$ From (1), we have

$10\left(\frac{1}{5}\right) - b = 0 \Rightarrow 2 - b = 0 \Rightarrow b = 2$

Hence, $a = \frac{1}{5}$, b = 2.

E7) Let X be the number on the first card and Y be the number on the second card. Then probability distribution of X is:

X	1	2	3	4	5	6	7	8	9	10
$p(x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

and the probability distribution of Y is

X	1	2	3	4	5	6	7	8	9	10
$p(x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

$$\therefore\ E(X) = E(Y) = 1\times\frac{1}{10}+2\times\frac{1}{10}+...+10\times\frac{1}{10}$$

$$= \frac{1}{10}\left[1+2+3+4+5+6+7+8+9+10\right] = \frac{1}{10}(55) = 5.5$$

and hence the required expected value is

$$E(X+Y) = E(X)+E(Y) = 5.5 + 5.5 = 11$$

E8) Let X be the number obtained on the first die and Y be the number obtained on the second die.

Then $E(X) = E(Y) = \frac{7}{2}$. [See Example 3 given in Section 8.2]

Hence, the required expected value is

$$E(XY) = E(X)E(Y) \qquad \text{[Using multiplication theorem of expectation]}$$

$$= \frac{7}{2}\times\frac{7}{2} = \frac{49}{4}.$$