UNIT 3 SIMPLEX METHOD

Structure

3.1 INTRODUCTION

In Unit 2, you have studied the graphical method of solving linear programming problems and learnt how to express a linear programming problem in canonical and standard forms. You know that the graphical method cannot be used for problems involving more than two decision variables. For more than two decision variables, methods based on the concept of slack variables, i.e., trial and error method and Simplex method, are used. The trial and error method is extremely cumbersome and time-consuming and successive solutions may not culminate into the optimal one.

The Simplex method is an improvement over the trial and error method and it overcomes its inefficiencies. In this unit, we shall first discuss the trial and error method and then the Simplex method of solving the linear programming problems.

In the next unit, you will learn how to obtain an initial basic feasible and an optimum solution of a transportation problem.

Objectives

After studying this unit, you should be able to:

- explain the trial and error method of solving Linear Programming problems;
- apply the Simplex method to solve Linear Programming problems;
- describe the Artificial Variable Technique;
- apply the Big M method to solve Linear Programming problems; and
- explain the concept of degeneracy while solving an LPP by Simplex method.

3.2 TRIAL AND ERROR METHOD OF SOLVING A LINEAR PROGRAMMING PROBLEM

In the trial and error method, we first write the given LPP in the standard form as discussed in Unit 2 of this block. Suppose there are m constraints with n variables (where $n > m$) in the standard form of LPP. Before proceeding further, we state the Extreme Point theorem and Basic theorem.

Extreme Point Theorem: An optimal solution to an LPP occurs at one of the vertices of the feasible region.

Basic Theorem: For a system of m equations in n variables (where n > m), a solution in which at least (n – m) variables have the value zero is called a **basic solution**.

Now, the objective function is optimal at least at one of the basic solutions. Some of the vertices may be infeasible if they have negative coordinates. Such solutions are dropped in view of the non-negativity condition on all variables including slack and surplus variables.

We consider the following LPP to explain the **trial and error method**:

$$\text{Max. } Z = 3x_1 + 2x_2$$

subject to the constraints:

$$x_1 + x_2 \le 4$$

$$x_1 - x_2 \le 2$$

$$x_1 \ge 0, x_2 \ge 0$$

Introducing slack variables s_1 and s_2, we write the standard form of the LPP as:

$$\text{Max. } Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$$

subject to the constraints:

$$x_1 + x_2 + s_1 = 4$$

$$x_1 - x_2 + s_2 = 2$$

$$x_1,\ x_2,\ s_1,\ s_2 \ge 0$$

Here, we have n = 4 variables and m = 2 equations. Thus, n – m = 4 – 2 = 2. So, at least 2 variables have the value zero as a vertex.

According to the Basic theorem, we set n – m = 2 variables equal to zero at a time. Thus, we have the following sets of solutions:

i) $x_1 = 0,\ x_2 = 0$

$\Rightarrow\ s_1 = 4,\ s_2 = 2$

ii) $x_1 = 0,\ s_1 = 0$

$\Rightarrow\ x_2 = 4,\ -x_2 + s_2 = 2$

$\Rightarrow\ x_2 = 4,\ s_2 = 2 + 4 = 6$

iii) $x_1 = 0,\ s_2 = 0$

Þ $x_2 + s_1 = 4,\ x_2 = -2$

$\Rightarrow\ s_1 = 6,\ x_2 = -2$

iv) $x_2 = 0,\ s_1 = 0$

Þ $x_1 = 4,\ x_1 + s_2 = 2$

$\Rightarrow\ x_1 = 4,\ s_2 = -2$

v) $x_2 = 0,\ s_2 = 0$

$\Rightarrow\ x_1 + s_1 = 4,\ x_1 = 2$

$\Rightarrow\ s_1 = 2,\ x_1 = 2$

vi) $s_1 = 0,\ s_2 = 0$

$\Rightarrow\ x_1 + x_2 = 4,\ x_1 - x_2 = 2$

$\Rightarrow\ x_1 = 3,\ x_2 = 1$

Since solutions (iii) and (iv) yield a negative coordinate each, these are infeasible and are dropped.

Since the optimal solution lies at one of the vertices, we find the value of the objective function $Z = 3x_1 + 2x_2$ for each of the above sets of solutions for x_1 and x_2 except (iii) and (iv).

For Set (i), $Z = 3(0) + 2(0) = 0$

For Set (ii), $Z = 3(0) + 2(4) = 8$

For Set (v), $Z = 3(2) + 2(0) = 6$

For Set (vi), $Z = 3(3) + 2(1) = 11$

Thus, solution (vi) is optimal with a profit of 11.

This is how a problem is solved by the trial and error method. But this method becomes cumbersome and time consuming if m and n are large. In such cases we shall have to solve numerous sets of simultaneous equations. Hence, we use another method known as the Simplex method to solve the linear programming problems.

3.3 SIMPLEX METHOD OF SOLVING A LINEAR PROGRAMMING PROBLEM

The Simplex method is an improvement over the trial and error method. It is an iterative optimising technique where we first find an initial basic solution (a vertex) and then proceed to an adjacent vertex and continue moving from adjacent vertex to another adjacent vertex till an optimal solution is attained. In this method, the value of the objective function improves with each solution and the optimum solution is achieved in a finite number of steps.

You can watch the video lecture series of Prof. G. Srinivasan on Simplex method on the following link: http://youtube/qxls3cYg8to?list=PLDFD8C5517267A8D5. In these video lectures, all steps are explained in detail.

Let us explain the steps for solving the LPP by the Simplex method.

Steps for Computation of an Optimum Solution by Simplex Method

Suppose we have to maximise (or minimise) Z, a linear function of n basic variables $X_1, X_2, \ldots, X_n$. The LPP is written as:

$$\text{Maximise } Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n \qquad \ldots (1)$$

subject to the constraints:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \end{array} \qquad \ldots (2)$$

and $x_j \ge 0$ for all j=1, 2, …, n $\qquad \ldots (3)$

where the constants $C_1, C_2, \ldots, C_n$ are the **cost coefficients** of decision variables. Let (a_{ij}) be an m×n real matrix and $\{b_1, b_2, \ldots, b_m\}$ be a set of constants.

The linear function Z given in equation (1) is called the **objective function**. The set of inequalities given in equation (2) is called **constraints** of LPP and the inequalities given in equation (3) are known as **non-negative restrictions** of LPP (which means that all x_j values are non-negative).

We convert the LPP into standard form by adding slack variables $s_1, s_2, \ldots, s_m$ (refer to Sec. 2.5 of Unit 2):

$$\text{Maximise } Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n + 0s_1 + 0s_2 + \ldots + 0s_m \qquad \ldots (4)$$

subject to the given constraints:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + s_2 = b_2 \\ . \qquad . \qquad . \qquad . \\ . \qquad . \qquad . \qquad . \\ . \qquad . \qquad . \qquad . \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + s_m = b_m \end{array} \qquad \ldots (5)$$

and $x_j \geq 0$ and $s_i \geq 0$ for all i =1, 2, ..., m and j=1, 2, …, n … (6)

The initial simplex table is formed as follows:

Table 1: Initial Simplex Table

		$C_j \rightarrow$	C_1	C_2	$..C_r$	..0	0...	
Basic variables	Profit/unit (C_B)	Quantity	X_1	X_2	$..X_r$	$..S_1$	$S_2..$	Replacement Ratios
s_1 s_2 . . .	0 0 . . .	b_1 b_2 . . .	a_{11} a_{21} . . .	a_{12} a_{22} . . .	$..a_{1r}$ $..a_{2r}$ . . .	1 0 . . .	0.. 1.. . . .	
	Z	$z_j \rightarrow$ $c_j - z_j \rightarrow$	C_BX_1	C_BX_2	$..C_BX_r$	C_BS_1	C_BS_2	

Note that in the initial simplex table (Table 1), the slack variables are kept in the column of basic variables. Once we obtain the values of $c_j - z_j$, the nature of the solution could be any one of the following:

1. If all $c_j - z_j \leq 0$, the solution under test is an **optimal** solution.

2. If at least one value of $c_j - z_j$ is positive and corresponding to the most positive $c_j - z_j$, all elements of the column X_j are negative or zero, the solution under test is an **unbounded** solution.

3. Suppose at least one value of $c_j - z_j$ is positive. Suppose the most positive value is, say, $c_2 - z_2$ and at least one entry in the column of X_2 is positive. Then the solution under test is **not optimal**. The most positive entry could be any of ($c_j - z_j$).

   Then we proceed as follows to obtain the optimal solution:

   - Let X_r be the variable which corresponds to the most positive value of $c_j - z_j$. This variable is called the **incoming variable**.
   - We divide the values of the Quantity (Qty) column by the corresponding positive values in the column of X_r. These ratios are called **Replacement Ratios** (RR). Note that we do not consider the negative values in the column of X_r for calculating RR. Then we select the minimum RR. The basic variable corresponding to this value of the RR is called the **outgoing variable**. It is called outgoing variable because it is removed (goes out) from the next simplex table. The element at the intersection of the row corresponding to the outgoing variable and the column corresponding to the incoming variable is called the **key element** or **leading element** or **pivotal element**.
   - We convert the key element to unity by dividing all entries in the row by the key element itself.
   - In the next step, we would like that the values of all other elements in the column corresponding to key element are zero. For this we carry out suitable operations on each row using the row containing the key element.
   - We repeat the procedure until either an optimal solution is obtained or there is an indication of unbounded solution.

For understanding the Simplex method, you should carefully go through the solution of the following example:

Example 1: Maximise $Z = 3x_1 + 2x_2$

subject to the constraints:

$$x_1 + x_2 \le 4$$

$$x_1 - x_2 \le 2$$

$$x_1 \ge 0, \; x_2 \ge 0$$

Solution: Let us re-write the given LPP in the standard form as follows:

$$\text{Max. } Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$$

subject to the constraints:

$$x_1 + x_2 + s_1 = 4$$

$$x_1 - x_2 + s_1 = 2$$

$$x_1, \; x_2, \; s_1, \; s_2 \ge 0.$$

Note that in this problem, a_{11}=1, a_{12}=1, a_{13}=1, a_{14}=0, a_{21}=1, a_{22}= –1, a_{23}=0, a_{24}=1, b_1=4 and b_2=2.

The variables s_1 and s_2 are the slack variables.

1. We now explain **how to construct the initial simplex table** in the following steps:

i) First of all, we write the basic variables s_1 and s_2 in the first column and the variables x_1, x_2, s_1, s_2 in the second row as shown in Table 2. Then we write the values of coefficients a_{ij} (read margin remark). We also write the cost coefficients (c_j) of the objective function in the first row above the corresponding basic variables at the top of the table.

Table 2: Initial Simplex Table

		$c_j \rightarrow$	3	2	0	0
Basic Variables	Profit/ Unit	Qty	x_1	x_2	s_1	s_2
s_1			1	1	1	0
s_2			1	–1	0	1
	Z =	$z_j \rightarrow$ $c_j - z_j \rightarrow$				

ii) Note that the columns corresponding to the basic variables in initial simplex tables in the Simplex method form an identity matrix. For example, in Table 2, the basic variables are s_1 and s_2 and their coefficients in the constraints are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which form an identity matrix.

iii) In the column with the caption "Profit/Unit", we write the coefficients c_j of the basic variables in the objective function. These coefficients have the value zero for basic variables s_1 and s_2. In the column with the caption "Qty (Quantity)", we write the values of the basic variables obtained from the constraints by setting $x_1 = 0$, $x_2 = 0$, i.e., $s_1 = 4$, $s_2 = 2$. Note that these are just the constants in the right hand side of the constraint equations. So, these constants are written in "Qty" column. Thus, we get the complete initial simplex table (Table 3).

Table 3: Complete Initial Simplex Table

		$c_j \rightarrow$	3	2	0	0
Basic Variables	Profit/ Unit	Qty	x_1	x_2	s_1	s_2
s_1	0	4	1	1	1	0
s_2	0	2	1	–1	0	1
	Z =	$z_j \rightarrow$ $c_j - z_j \rightarrow$				

iv) The value of Z in column of Table 3 is obtained from the equation

$$Z = \sum c_j b_j$$

Since $c_1 = 0$, $b_1 = 4$, $c_2 = 0$ and $b_2 = 2$ in the initial simplex table, we get

$$Z = 0 \times 4 + 0 \times 2 = 0$$

The values of z_j's are obtained from the equation

$$z_j = \sum_{i=1}^{m} c_i a_{ij}$$

where m is the number of rows. In this example m = 2 and

$\therefore \quad z_1 = c_1 a_{11} + c_2 a_{21} = 0 \times 1 + 0 \times 1 = 0$

Similarly,

$z_2 = 0 \times 1 + 0 \times (-1) = 0$

$z_3 = 0 \times 1 + 0 \times 0 = 0$

$z_4 = 0 \times 0 + 0 \times 1 = 0$

Thus, z_j's are 0, 0, 0, 0.

v) Next we obtain $c_j - z_j$, known as **net-evaluations**, on subtracting z_j from c_j . These are 3 – 0, 2 – 0, 0 – 0, 0 – 0, or 3, 2, 0, 0.

Thus, the table takes the following form (Table 4):

Table 4: Simplex Table

		$c_j \rightarrow$	3	2	0	0
Basic Variables	Profit/Unit	Qty	x_1	x_2	s_1	s_2
s_1	0	4	1	1	1	0
s_2	0	2	1	–1	0	1
	Z = 0	$z_j \rightarrow$ $c_j - z_j \rightarrow$	0 3	0 2	0 0	0 0

vi) Now, we select the most positive value of $c_j - z_j$, which is 3 in this case. This corresponds to the variable x_1 which becomes the **incoming variable**. We shall enter it as a basic variable in the next simplex table. So, one of the variables s_1, s_2 will now be the outgoing variable which would be replaced by x_1. To find out which one of these variables (s_1 or s_2) is the outgoing variable, we determine the replacement ratio (RR). Recall that RR for any row is obtained by dividing the value of the Qty column by the corresponding value of the Incoming variable for that particular row. For example, consider the first row in Table 4. The value of Qty in the first row is 4 and the value of the incoming variable x_1 in this row is 1.

Therefore, RR for the first row is $\frac{4}{1}$. For the second row, Qty is 2 and the value of x_1 is 1. Hence, RR is $\frac{2}{1}$. RR is shown in the last column of the next simplex table (Table 5).

The outgoing variable is the variable for which RR minimum. In Table 5, RR in the second row is minimum. It corresponds to s_2 and hence s_2 is the outgoing variable.

The element which lies at the intersection of the column of incoming variable (x_1) and the row of outgoing variable (s_2) is called the key element. Here it is 1 and is enclosed by the rectangle as shown in Table 5.

Table 5: Simplex Table **Incoming variable**

	$c_j \rightarrow$		3	2	0	0	
Basic Variables	Profit/ Unit	Qty	x_1	x_2	s_1	s_2	R.R.
s_1	0	4	1	1	1	0	4/1 = 4
(outgoing variable) s_2	0	2	$\boxed{1}$	–1	0	1	2/1 = 2
	Z = 0	$z_j \rightarrow$ $c_j - z_j \rightarrow$	0 3	0 2	0 0	0 0	

Key (or Pivotal or Leading) element

The objective function Z = 0 at $x_1 = 0$ and $x_2 = 0$ (since x_1 and x_2 do not appear in the column of basic variables, these are non-basic variables. The values of non-basic variables are taken as zero). This is the initial solution, which we shall improve.

2. Now, we form the next simplex table to find the adjacent vertex, i.e., the improved solution. The steps for forming this table are explained below:

   i) The initial simplex table (Table 3) has revealed that x_1 is an incoming variable which will enter in place of s_2, the outgoing variable. The cost coefficient of s_2 will also be replaced by the cost coefficient of x_1 in "Profit/Unit" column. In this case its value is 3. Therefore, now the simplex table takes the form of Table 6:

Table 6: Simplex Table

	$c_j \rightarrow$		3	2	0	0
Basic Variables	Profit/Unit	Qty	x_1	x_2	s_1	s_2
s_1	0					
x_1	3					

   Other entries will be filled up as explained below:

   ii) Since x_1 has entered as a basic variable, the coefficients of x_1 along with s_1 should form an identity matrix; i.e., the column corresponding to x_1 should be $\begin{pmatrix}0\\1\end{pmatrix}$. Thus, we have to make the key element unity and the other element zero. Note that it is already unity in this case (see Table 5). Had it been any number other than unity, we would have divided the row containing leading element by the leading element itself, excluding the elements of the column "profit/unit".

   So, the Table takes the form:

Table 7: Simplex Table

		$c_j \rightarrow$	3	2	0	0
Basic Variables	Profit/Unit	Qty	x_1	x_2	s_1	s_2
s_1	0	4	1	1	1	0
x_1	3	2	1	–1	0	1

iii) Now, we have to make the other element in the column of key element (x_1) zero. In this case, its value is 1 (a_{11} =1). For this, we multiply the row of the key element (excluding profit/unit) by negative of the element a_{11} (in this case) and add it to the first row. This row operation is shown below:

First row of Table 7 →	4	1	1	1	0
Second row of Table 7 → (on multiplying by –1)	–2	–1	1	0	–1
Sum of the rows	2	0	2	1	–1

Thus, the sum of the rows is the new first row (excluding profit/unit), which replaces the first row of Table 7. We get Table 8 as follows:

Table 8: Simplex Table

		c_j→	3	2	0	0
Basic Variables	Profit/Unit	Qty	x_1	x_2	s_1	s_2
s_1	0	2	0	2	1	–1
x_1	3	2	1	–1	0	1

Note that now the matrix for the basic variables s_1 and x_1 in Table 8 is the identity matrix.

iv) Next we calculate z_j and $c_j–z_j$ again as explained in Steps 1(iii) and 1(iv) of this section. The resulting simplex table is given below:

Table 9: Simplex Table

		c_j→	3	2	0	0	
Basic Variables	Profit/ Unit	Qty	x_1	x_2	s_1	s_2	RR
←s_1	0	2	0	$\boxed{2}$	1	–1	2/2=1←
x_1	3	2	1	–1	0	1	–
	Z = 0×2 +3×2= 6	z_j →	0×0+3×1 =3	0×2+ 3×(–1) = –3	0×1+ 3×0 =0	0×(–1) +3×1=3	
		$c_j–z_j$ →	0	5 ↑	0	–3	

Here Z = 6 at x_1 = 2 (see the value in the Qty column) and x_2 = 0 (x_2 being non-basic variable).

In Table 9, the incoming variable is x_2 corresponding to the most positive value of $c_j–z_j$ = 5. The key element is 2. We find the RR by dividing the elements in the Qty column by the corresponding elements in the column of the incoming variable x_2 and ignore the negative or zero values of the incoming variable. We have to divide by only the positive values. So, in this case, there will be only one RR and that will be considered as minimum RR. This implies that s_1 is the outgoing variable. If none of the elements in the column of incoming variable is positive, then the given LPP has an unbounded solution and we will stop there.

3. For the next simplex table, x_2 will enter in place of s_1 as a basic variable and accordingly we shall write the cost coefficient of x_2 in the LPP as the value for the column of profit/unit, i.e., 2. The element 2 enclosed in a rectangle in Table 9 is the key element. So we shall divide the row containing the key element by the key element itself, i.e., by 2, excluding the values in the column of profit/unit. Thus, we get Table 10:

Table 10: Simplex Table

		$c_j\rightarrow$	3	2	0	0
Basic Variables	Profit/Unit	Qty	x_1	x_2	s_1	s_2
x_2	2	1	0	1	1/2	–1/2
x_1	3	2	1	–1	0	1

Now the coefficients of x_2 and x_1 have to form an identity matrix, i.e., the column corresponding to x_2 should be $\begin{pmatrix}1\\0\end{pmatrix}$. We have already made the key element unity. Now, to make the second element, (i.e., –1) in its column as zero, we just add the first row corresponding to x_2, to the second row of Table 10 excluding the values in the column of profit/unit as follows:

First row of Table 10	$\rightarrow$	1	0	1	1/2	–1/2
Second row of Table 10 (on multiplying by 1)	$\rightarrow$	2	1	–1	0	1
Sum of the rows		3	1	0	1/2	1/2

We write the sum above as the second row, excluding profit/unit as shown in Table 11. Then we can obtain Z, z_j, and $c_j–z_j$ as explained in Steps 1(iii) and (iv) and also in Step 2(iii). The complete resulting simplex table is given below:

Table 11: Simplex Table

		$c_j\rightarrow$	3	2	0	0	RR
Basic Variables	Profit/ Unit	Qty	x_1	x_2	s_1	s_2	
x_2	2	1	0	1	1/2	–1/2	
x_1	3	3	1	0	1/2	1/2	
	$Z = 2\times1 +3\times3 = 11$	$z_j\rightarrow$	$2\times0+3\times1=3$	$2\times1+3\times0= 2$	$2\times1/2+3\times1/2=5/2$	$2\times(–1/2)+3\times(1/2)=1/2$	
		$c_j–z_j\rightarrow$	0	0 $\uparrow$	–5/2	–1/2	

Now, in Table 11 none of the net-evaluations, i.e., the values of $c_j – z_j$ are positive. Therefore, the optimum solution is attained for $x_1 = 3$ and $x_2 = 1$, the values of x_1 and x_2 in the Qty column of Table 11.

At this stage, it is necessary to check whether any of the non-basic variables (other than those appearing in the first column, i.e., the column with caption "Basic Variables", i.e., x_1 and x_2 in Table 11) has value 0 in the net-evaluation row. If "yes", then the LPP has multiple optimum solutions. If "not", then we stop here concluding that the LPP has the unique solution. In this example, it is given by

Max. Z = 11 at $x_1 = 3$ and $x_2 = 1$ (see the values in the Qty column).

We discuss the case of multiple optimum solutions in the next example.

Example 2: Max. Z = 6x + 3y

subject to the constraints:

$$2x + 5y \leq 120$$
$$2x + y \leq 40$$
$$x \geq 0, y \geq 0$$

Solution: Rewriting the given LPP in the standard form, we have

$$\text{Max.} \quad Z = 6x + 3y + 0s_1 + 0s_2$$

subject to the constraints:

$$2x + 5y + s_1 = 120$$
$$2x + y + s_2 = 40$$
$$x, y, s_1, s_2 \geq 0$$

We form the initial simplex table (Table 1) as explained in Example 1:

Table 1: Initial Simplex Table

		$c_j \rightarrow$	6	3	0	0	
Basic Variables	Profit/Unit	Qty	x	y	s_1	s_2	RR
s_1	0	120	2	5	1	0	120/2=60
$\leftarrow s_2$	0	40	$\boxed{2}$	1	0	1	40/2 = 20
	Z = 0	$z_j \rightarrow$ $c_j - z_j \rightarrow$	0 6 ↑	0 3	0 0	0 0	

Note from Table 1 that $c_j - z_j = 6$ is maximum for x and RR is minimum for s_2. Therefore, x is the incoming variable and s_2 is the outgoing variable. Then in the second simplex table, x will enter in place of s_2 as a basic variable. Profit/unit will be written accordingly. The key element is 2 and enclosed by the rectangle in Table 1. To make the key element unity, we divide the second row in Table 1 by 2 excluding the values of the column of profit/unit. So, we get Table 2 as follows:

Table 2: Simplex Table

		$c_j \rightarrow$	6	3	0	0
Basic Variables	Profit/Unit	Qty	x	y	s_1	s_2
s_1	0	120	2	5	1	0
x	6	20	1	1/2	0	1/2

Now we have to make the other element in column of x zero so that the coefficients of s_1 and x form the identity matrix. So we multiply the second row of Table 2 (excluding the elements in the column of profit/unit) by the negative of the element (a_{ij}) in the row of s_1 and column of x in Table 1, i.e., by −2 and then add it to the first row of the Table 2 as follows:

Second row of Table 2 (after multiplying by −2)	→	−40	−2	−1	0	−1
First row of the Table 2	→	120	2	5	1	0
Sum of the rows	→	80	0	4	1	−1

We also calculate Z, z_j, and $c_j - z_j$. So the resulting completed simplex table is as follows:

Table 3: Simplex Table

	$c_j \rightarrow$		6	3	0	0	RR
Basic Variables	Profit/Unit	Qty	x	y	s_1	s_2	
s_1	0	80	0	4	1	−1	
x	6	20	1	1/2	0	1/2	
	Z = 120	$z_j \rightarrow$ $c_j - z_j \rightarrow$	6 0	3 0	0 0	3 −3	

As none of the net-evaluations is positive, the optimum solution is attained. The optimum solution is

Max. Z = 120, when x = 20 (see the value in the Qty column),

y = 0 (since y is non-basic variable)

But we also note that the non-basic variable y has value zero in its net-evaluation row in Table 3. Therefore, the given LPP has multiple optimal solutions. To find another optimal solution, let us find another vertex at which Max. Z = 120.

So, another simplex table has to be formed. Here, instead of selecting the column corresponding to the most positive element in the net-evaluation row, we select the column of non-basic variable which has zero in the net-evaluation row, i.e., the column of y shown in Table 4.

Table 4: Simplex Table

	$c_j \rightarrow$		6	3	0	0	RR
Basic Variables	Profit/ Unit	Qty	x	y	s_1	s_2	
← s_1	0	80	0	$\boxed{4}$	1	−1	80/4=20←
x	6	20	1	1/2	0	1/2	20/(1/2)=40
	Z = 120	$z_j \rightarrow$ $c_j - z_j \rightarrow$	6 0	3 0	0 0	3 −3	
				↑			

Note that in Table 4, the key element is 4 and the minimum RR corresponds to s_1. So s_1 is the outgoing variable and y is the incoming variable. We form the next simplex table (Table 5) following the steps explained for forming Table 3. The resulting simplex table (Table 5) is given a follows:

Table 5: Simplex Table

	$c_j \rightarrow$		6	3	0	0	RR
Basic Variables	Profit/Unit	Qty	x	y	s_1	s_2	
y	3	20	0	1	1/4	–1/4	
x	6	10	1	0	–1/8	5/8	
	Z = 120	$z_j \rightarrow$ $c_j - z_j \rightarrow$	6 0	3 0	0 0	3 –3	

You should verify all entries of Table 5 before studying further. Note that for forming Table 5, we have first divided the first row by the value of key element to make the key element unity. Then we obtain the second row of Table 5 as follows:

First row of the Table 5 (after multiplying by –1/2)	$\rightarrow$	–10	0	–1/2	–1/8	1/8
Second row of the Table 4	$\rightarrow$	20	1	1/2	0	1/2
Sum of the rows	$\rightarrow$	10	1	0	–1/8	5/8

From the above simplex table (Table 5), we find that Max. Z = 120 at (10, 20) also (see the values of x = 10, y = 20 in the Qty column).

So, we have two vertices at which the maximum value of Z is the same, i.e., 120. So, the other solutions of the LPP are obtained as follows:

First ordinate of other solutions

= t × (First ordinate of first vertex)
+ (1 – t) × (First ordinate of second vertex)

Second ordinate of other solutions

= t × (second ordinate of first vertex)
+ (1 – t) × (second ordinate of second vertex)

So, the other solutions are given as

First ordinate = t×10 + (1 – t)×20 = 10t + 20 – 20t = 20 – 10t

Second ordinate = t×20 + (1 – t)×0 = 20 t

The other solutions are (20 – 10t, 20t), $0 \le t \le 1$.

Now, you should try to solve the following exercises.

E1) Maximise $Z = 2x_1 + 4x_2$

subject to the constraints:

$$x_1 + 2x_2 \le 5$$

$$x_1 + x_2 \le 4$$

$$x_1 \ge 0, \ \ x_2 \ge 0$$

E2) Solve the following LPP by the Simplex method:

Maximise $Z = 100x_1 + 60x_2 + 40x_3$

subject to the constraints:

$$x_1 + x_2 + x_3 \le 100$$

$$10x_1 + 4x_2 + 5x_3 \le 500$$
$$x_1 + x_2 + 3x_3 \le 150$$
$$x_1,\ x_2,\ x_3 \ge 0$$

Before studying next section, match your answers with the solutions given in Sec. 2.7.

3.4 ARTIFICIAL VARIABLE TECHNIQUE AND THE BIG M METHOD

After converting the given LPP into standard form, sometimes we observe that some or all the variables whose coefficients in the constraints should form the identity matrix are missing in the initial simplex table. Such a situation is usually observed if the constraints equation(s) is (are) of the type "$\ge$". In such a situation, we introduce new type of variable(s) known as artificial variable(s). These are added to act as basic variables. They have no physical meaning and are used only to initiate the solution so that the simplex procedure may be adopted as usual till the optimal solution is obtained. The artificial variables are eliminated from the simplex table as and when they become non-basic variables. This technique is called the **artificial variable technique** for solving LPP. LPPs involving artificial variables are solved by either of the following methods:

i) Big-M or Penalty Method

ii) Two-phase Technique

In this unit, we discuss only the Big-M method for solving an LPP.

Big-M Method

In this method, the objective function coefficients impose a huge and hence unacceptable penalty. In case of maximisation, the objective function is modified by adding $-MA_1$, where M is arbitrarily large and A_1 is an artificial variable. If there are two artificial variables A_1 and A_2, then $-MA_1 - MA_2$ is added to the objective function. Similar treatment is done for more artificial variables. The logic behind taking the coefficient as $-M$ is that we should never get the net-evaluation positive in the column of the artificial variable, i.e., the artificial variable should not enter again as a basic variable. M is very big and hence adding $-MA_1$ is the penalty to the objective function. Hence this method is called the **penalty method**. Though $-M$ is a big penalty, it does not affect the objective function. This is because the value of the artificial variable should come out to be zero so that $-MA_1$ becomes zero. If the artificial variable remains as a basic variable till the final simplex table, then its value in the Qty column should be zero for the solution of LPP to exist. Otherwise, if in the final simplex table, an artificial variable appears as a basic variable and is non-zero, the LPP does not possess any feasible solution.

Example 3: Maximise $Z = x_1 + 2x_2$

subject to the constraints:

$$x_1 - x_2 \ge 3$$
$$2x_1 + x_2 \le 10$$
$$x_1 \ge 0,\ x_2 \ge 0$$

Solution: The standard form of the LPP is

$$\text{Max } Z = x_1 + 2x_2 + 0s_1 + 0s_2$$

subject to the constraints:

$$x_1 - x_2 - s_1 = 3$$
$$2x_1 + x_2 + s_2 = 10$$
$$x_1,\ x_2, s_1,\ s_2 \geq 0$$

Now, let us try to form simplex table as follows:

Table 1: Simplex Table

		$c_j \rightarrow$	1	2	0	0
Basic Variables	Profit/Unit	Qty	x	x_2	s_1	s_2
			1	–1	–1	0
			2	1	0	1

We cannot form the initial simplex table with the variables x_1, x_2, s_1, s_2 as there is only one variable s_2 that has a column of identity matrix. The column $\begin{pmatrix}1\\0\end{pmatrix}$ is missing. Therefore, one more variable, i.e., artificial variable A_1 (say) needs to be introduced in the first constraint so as to get $\begin{pmatrix}1\\0\end{pmatrix}$. Its coefficient in the objective function will be taken as –M.

Thus, the objective function is

$$\text{Max. } Z = x_1 + 2x_2 + 0s_1 + 0s_2 - MA_1$$

subject to the constraints

$$x_1 - x_2 - s_1 + A_1 = 3$$
$$2x_1 + x_2 + s_2 = 10$$
$$x_1,\ x_2, s_1,\ s_2,\ A_1 \geq 0$$

The initial simplex table, therefore, is as follows:

Table 2: Initial Simplex Table

		$c_j \rightarrow$	1	2	0	0	–M	
Basic Variables	Profit/Unit	Qty	x_1	x_2	s_1	s_2	A_1	R.R.
$\leftarrow A_1$	–M	3	$\boxed{1}$	–1	–1	0	1	3/1 = 3$\leftarrow$
s_2	0	10	2	1	0	1	0	10/2 = 5
	Z = –3M	$z_j \rightarrow$ $c_j - z_j \rightarrow$	–M 1+M	M 2–M	M –M	0 0	–M 0	
			$\uparrow$					

Here, since M is big, 1 + M is most positive and x_1 is the incoming variable. The least replacement ratio is $\left(\frac{3}{1}\right)$ which corresponds to A_1 and hence it is the outgoing variable. Thus, the resulting simplex table is as follows:

Table 3: Simplex Table

	$c_j \rightarrow$		1	2	0	0	–M	
Basic Variables	Profit/unit	Qty	x_1	x_2	s_1	s_2	A_1	R.R.
x_1	1	3	1	–1	–1	0	×	–
$\leftarrow s_2$	0	4	0	$\boxed{3}$	2	1	×	4/3←
	Z = 3	$z_j \rightarrow$ $c_j - z_j \rightarrow$	1 0	–1 3 ↑	–1 1	0 0	×	

We obtain the second row (excluding profit/unit) in Table 3 as follows:

First row of the Second Simplex table (Table 3) (after multiplying by –2)	→	–6	–2	2	2	0	×
Second row of First Simplex table (Table 2)	→	10	2	1	0	1	×
Second row of the Third Simplex table	→	4	0	3	2	1	×

Once the artificial variable is removed from the basic variables, there is no need to do any computational work for it.

Thus, we get the resulting simplex table.

Table 4: Simplex Table

	$c_j \rightarrow$		1	2	0	0	–M	
Basic Variables	Profit/unit	Qty	x_1	x_2	s_1	s_2	A_1	
x_1	1	13/3	1	0	–1/3	1/3	×	
x_2	2	4/3	0	1	2/3	1/3	×	
	Z = 21/3 = 7	$z_j \rightarrow$ $c_j - z_j \rightarrow$	1 0	2 0	1 – 1	1 –1	× ×	

The first row (excluding profit/unit) of Table 4 is obtained as follows:

Second row of Third Simplex table (Table 4) (after multiplying by 1)	→	4/3	0	1	2/3	1/3	×
First row of Second Simplex table (Table 3)	→	3	1	–1	–1	0	×
Sum of the rows	→	13/3	1	0	–1/3	1/3	×

Since none of the net-evaluations is positive, the optimum solution is attained and is given by

$$\text{Max. } Z = 7 \quad \text{at} \quad x_1 = \frac{13}{3} \quad \text{and} \quad x_2 = \frac{4}{3}.$$

Now, you should try to solve the following exercises.

E3) Minimise $Z = 4x_1 + 2x_2$

subject to the constraints:

$$3x_1 + x_2 \geq 27$$
$$x_1 + x_2 \geq 21$$
$$x_1 + 2x_2 \geq 30$$
$$x_1,\ x_2 \geq 0$$

E4) Maximise $Z = x_1 + 2x_2$

subject to the constraints:

$$x_1 + x_2 \leq 4$$
$$x_1 + x_2 \geq 6$$
$$x_1,\ x_2 \geq 0$$

E5) Maximise $Z = 10x_1 + 4x_2$

subject to the constraints:

$$-x_1 + x_2 \leq 2$$
$$x_1 + x_2 \geq 4$$
$$x_1,\ x_2 \geq 0$$

Before going to the next section, match your solutions with the solutions given in Sec. 3.7.

3.5 DEGENERACY

Degeneracy is the process of obtaining a degenerate basic feasible solution in an LPP. Degeneracy may occur at the initial stage while forming the initial simplex table if the values of some basic variables are zero. Degeneracy may also occur at any iteration stage if in any of the simplex tables there is a tie in the minimum replacement ratios. In case of tie, if we select any of the rows arbitrarily, it may be possible that the subsequent iterations may not produce improvements in the value of the objective function. This concept is known as cycling. Then we have to simply go back and select another row.

The situation of cycling may be avoided if in case of tie, the row is not selected arbitrarily but is selected adopting the following procedure:

Compute the non-negative ratios of the first column of the identity matrix (and not that of Qty column) to the entries of the entering variable. Then choose the minimum of the values occurring at the places of a tie. If there is a tie again, we compute the ratios of the second column of the identity matrix to the entries of the entering variable. We continue the process till the ratios do not break the tie.

Example 4: Maximise $Z = 3x_1 + 9x_2$

subject to the constraints:

$$x_1 + 4x_2 \leq 8$$
$$x_1 + 2x_2 \leq 4$$
$$x_1 \geq 0,\ x_2 \geq 0$$

Solution: The standard form of the LPP is

$$\text{Maximise } Z = 3x_1 + 9x_2 + 0s_1 + 0s_2$$

subject to the constraints:

$$x_1 + 4x_2 + s_1 = 8$$

$$x_1 + 2x_2 + s_2 = 4$$

$$x_1,\ x_2, s_1,\ s_2 \geq 0$$

Therefore, the initial simplex table is as follows:

Table 1: Initial Simplex Table

	$c_j \rightarrow$		3	9	0	0	
Basic Variables	Profit/unit	Qty	x_1	x_2	s_1	s_2	R.R.
s_1	0	8	1	4	1	0	8/4=2
s_2	0	4	1	2	0	1	4/2=2
	Z = 0	$z_j \rightarrow$	0	0	0	0	
		$c_j - z_j \rightarrow$	3	9 ↑	0	0	

In Table 1, there is tie in the minimum replacement ratio as it is 2 in each of the two rows. Now, we should not select any row arbitrarily. We will proceed as follows:

We divide the non-negative ratios of the first column of identity matrix, i.e., $\begin{pmatrix}1\\0\end{pmatrix}$ by the entries of the entering variable, i.e., $\begin{pmatrix}4\\2\end{pmatrix}$. So we have replacement ratios as $\begin{pmatrix}1/4\\0/2\end{pmatrix}$, i.e., $\begin{pmatrix}1/4\\0\end{pmatrix}$. So here we have found the minimum RR to be 0 and the tie has been broken. Therefore, we choose the second row. Thus, the completed initial simplex table is as follows:

Table 2: Complete Initial Simplex Table

	$c_j \rightarrow$		3	9	0	0	
Basic Variables	Profit/unit	Qty	x_1	x_2	s_1	s_2	R.R.
s_1	0	8	1	4	1	0	2, ¼
$\leftarrow s_2$	0	4	1	$\boxed{2}$	0	1	2, 0←
	Z = 0	$z_j \rightarrow$	0	0	0	0	
		$c_j - z_j \rightarrow$	3	9 ↑	0	0	

Now, you can proceed yourself to form the next simplex table(s) and the answer you should get is Max. Z = 18 at $x_1 = 0$, $x_2 = 2$.

Let us now summarise the main points which have been covered in this unit.

3.6 SUMMARY

1. In **trial and error method**, the given LPP is first re-written in the standard form. After writing it in standard form, we set n – m = 2 variables equal to zero at a time if there are m constraints in n variables

(where $n > m$). Some of the solutions so obtained may be infeasible if they have negative coordinates. Such solutions are dropped in view of non-negativity condition on all variables, including slack and surplus variables. The optimal solution lies at one of the vertices and hence we find the value of the objective function for each set of the solutions except the dropped ones.

2. The **Simplex method** is an improvement over the trial and error method. It is an iterative optimizing technique where we first find an initial basic solution (a vertex) and then proceed to an adjacent vertex and continue moving from adjacent vertex to another adjacent vertex till an optimal solution is attained. In this method, the value of the objective function improves with each solution and the optimum solution is achieved in a finite number of steps.

3. If at least one of the net evaluations is positive and corresponding to the most positive net evaluation, all the elements of the column are negative or zero, the solution under test is **unbounded solution**.

4. At the stage of final simplex table, it is necessary to check whether any of the non-basic variables (other than those appearing in the first column, i.e., the column with the caption "Basic Variables") has 0 value in the net-evaluation row. If "yes", then the LPP has **multiple optimum solutions.**

5. After converting the given LPP into standard form, sometimes we observe that some or all of the variables whose coefficients in the constraints should form the identity matrix are missing in the initial simplex table. Such a situation is usually observed if the constraints equation(s) is (are) of the type " $\geq$". In such a situation, we introduce new type of variable(s) known as **artificial variable(s)**. These are added to act as basic variables. They have no physical meaning and are used only to initiate the solution so that the simplex procedure may be adopted as usual till the optimal solution is obtained. The artificial variables are eliminated from the simplex table as and when they become non-basic variables.

6. In **Big M Method**, the objective function coefficients impose a huge and hence unacceptable penalty. In case of maximization, the objective function is modified by adding $-MA_1$, where M is arbitrarily large and A_1 is an artificial variable. If there are two artificial variables A_1 and A_2, then $-MA_1 - MA_2$ is added to the objective function. Similar treatment is done for more artificial variables. While solving such a problem, the value of the artificial variable should come out to be zero in the final simplex table. If in the final simplex table, an artificial variable appears as a basic variable and is not at zero level, the LPP does not possess any **feasible solution**.

7. While forming the initial simplex table, if the values of some basic variables are zero or if in any table, there is a tie in the minimum replacement ratio, then **degeneracy** is said to occur in a linear programming problem. In this case, the non-negative ratios of the first column of the identity matrix (and not that of Qty column) to the entries of the entering variable are computed and then the minimum of the values occurring at the places of tie is chosen. If there is a tie again, the ratios of the second column of the identity matrix to the entries of the entering variable are computed. The process is continued till the ratios do not break the tie.

3.7 SOLUTIONS/ANSWERS

E1) Proceeding in the same way as explained in Example 2, we obtain

Max. Z = 10 at the two vertices (0, 5/2) and (3, 1).

Max. Z = 10 at many other points also which are given as

$(3 - 3t, 1 + 3t/2)$, $0 \le t \le 1$.

E2) Standard form of the LPP is

$$\text{Max. } Z = 100x_1 + 60x_2 + 40x_3 + 0s_1 + 0s_2 + 0s_3$$

subject to the constraints:

$$x_1 + x_2 + x_3 + s_1 = 100$$

$$10x_1 + 4x_2 + 5x_3 + s_2 = 600$$

$$x_1 + x_2 + 3x_3 + s_3 = 150$$

$$x_1,\ x_2,\ x_3, s_1,\ s_2,\ s_3 \ge 0$$

The initial simplex table is as follows:

Table 1: Initial Simplex Table

		$c_j \rightarrow$	100	60	40	0	0	0	
Basic Variables	Profit/Unit	Qty	x_1	x_2	x_3	s_1	s_2	s_3	RR
s_1	0	100	1	1	1	1	0	0	100
$\leftarrow s_2$	0	600	$\boxed{10}$	4	5	0	1	0	$60\leftarrow$
s_3	0	150	1	1	3	0	0	1	150
	Z = 0	$z_j \rightarrow$ $c_j - z_j \rightarrow$	0 100	0 60	0 40	0 0	0 0	0 0	
			↑						

The resulting simplex table is

Table 2: Simplex Table

		$c_j \rightarrow$	100	60	40	0	0	0	
Basic Variables	Profit/ Unit	Qty	x_1	x_2	x_3	s_1	s_2	s_3	RR
$\leftarrow s_1$	0	40	0	$\boxed{3/5}$	1/2	1	-1/10	0	$200/3\leftarrow$
x_1	100	60	1	2/5	1/2	0	1/10	0	150
s_3	0	90	0	3/5	5/2	0	-1/10	1	150
	Z = 6000	$z_j \rightarrow$ $c_j - z_j \rightarrow$	100 0	40 20	50 –10	0 0	10 –10	0 0	
				↑					

The resulting simplex table:

Table 3: Simplex Table

		$c_j \rightarrow$	100	60	40	0	0	0	
Basic Variables	Profit/ Unit	Qty	x_1	x_2	x_3	s_1	s_2	s_3	RR
x_2	60	200/3	0	1	5/6	5/3	-1/6	0	
x_1	100	100/3	1	0	1/6	–2/3	1/6	0	
s_3	0	50	0	0	1	–1	0	1	
	Z = 22000/3	$z_j \rightarrow$ $c_j - z_j \rightarrow$	100 0	60 0	200/3 –80/3	100/3 –100/3	20/3 –20/3	0 0	

Explanation to get the second row:

First row of Table 3	$\frac{-80}{3}$	0	$-\frac{2}{5}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{15}$	$-\frac{2}{5}$
Second row of Table 2	60	1	$\frac{2}{5}$	$\frac{1}{2}$	0	$\frac{1}{10}$	0
Sum of the rows	100/3	1	0	1/6	–2/3	1/6	0

Explanation to get the third row:

First row of Table 3	–40	0	$-\frac{3}{5}$	$-\frac{1}{2}$	–1	$-\frac{1}{10}$	0
Third row of Table 3	90	0	$\frac{3}{5}$	$\frac{5}{2}$	0	$-\frac{1}{10}$	1
Sum of the rows	50	0	0	1	–1	0	1

As none of the net-evaluations in the simplex table given in Table 3 is positive, therefore optimal solution is attained and is given by

$$Z = \frac{22000}{3} \quad \text{at} \quad x_1 = \frac{100}{3}, \quad x_2 = \frac{200}{3}, \quad x_3 = 0$$

E3) Standard form of the LPP is

$$\text{Max.} -Z = -4x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to the constraints:

$$3x_1 + x_2 - s_1 = 27$$

$$x_1 + x_2 - s_2 = 21$$

$$x_1 + 2x_2 - s_3 = 30$$

$$x_1,\ x_2, s_1,\ s_2,\ s_3 \geq 0$$

But, if we form the initial simplex table using the above equations for constraints, we do not find any basic variable as no column of identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ exists. Hence, we will introduce three artificial variables to initiate the solution. Thus, we have

$$\text{Max.} Z' = -4x_1 - 2x_2 + 0s_1 + 0s_2 + 0s_3 - MA_1 - MA_2 - MA_3\ , \qquad Z' = -Z$$

subject to the constraints:

$$3x_1 + x_2 - s_1 + A_1 = 27$$

$$x_1 + x_2 - s_2 + A_2 = 21$$

$$x_1 + 2x_2 - s_3 + A_3 = 30$$

$$x_1,\ x_2,\ s_1,\ s_2,\ s_3,\ A_1,\ A_2,\ A_3 \ ^3\ 0$$

Thus, the initial simplex table is

Table 1: Initial Simplex Table

	$c_j \rightarrow$		–4	–2	0	0	0	–M	–M	–M	
Basic Variables	Profit/ Unit	Qty	x_1	x_2	s_1	s_2	s_3	A_1	A_2	A_3	RR
$\leftarrow A_1$	–M	27	$\boxed{3}$	1	–1	0	0	1	0	0	27/3=9←
A_2	–M	21	1	1	0	–1	0	0	1	0	21/1=21
A_3	–M	30	1	2	0	0	–1	0	0	1	30/1=30
	Z = –78M	$z_j \rightarrow$ $c_j - z_j \rightarrow$	–5M –4+5M	–4M –2+4M	M –M	M –M	M –M	–M 0	–M 0	–M 0	
			↑								

Now, the second and subsequent simplex tables can be formed and at the end, the answer you get is

Max $Z' = -48$, i.e., Max $-Z = -48$

i.e., Min. $Z = 48$ when $x_1 = 3$, $x_2 = 18$.

E4) Standard form of the linear programming problem is

$$\text{Max. } Z = x_1 + 2x_2 + 0s_1 + 0s_2$$

subject to the constraints:

$$x_1 + x_2 + s_1 = 4$$

$$x_1 + x_2 - s_2 = 6$$

$$x_1,\ x_2, s_1,\ s_2 \geq 0$$

But, with these equations for constraints we have only one basic variable with the column $\begin{pmatrix}1\\0\end{pmatrix}$ and hence we have to introduce an artificial variable in the second equation so as to get the column $\begin{pmatrix}0\\1\end{pmatrix}$.

Thus, we have

$$\text{Max. } Z = x_1 + 2x_2 + 0s_1 + 0s_2 - MA_1$$

subject to the constraints:

$$x_1 + x_2 + s_1 = 4$$

$$x_1 + x_2 - s_2 + A_1 = 6$$

$$x_1,\ x_2, s_1,\ s_2, A_1 \geq 0$$

Hence, the initial simplex table is

Table 1: Initial Simplex Table

	$c_j \rightarrow$		1	2	0	0	–M	
Basic Variables	Profit/ Unit	Qty	x_1	x_2	s_1	s_2	A_1	RR
$\leftarrow s_1$	0	4	1	$\boxed{1}$	1	0	0	4 ←
A_1	–M	6	1	1	0	–1	1	6
	Z =–6M	$z_j \rightarrow$ $c_j - z_j \rightarrow$	–M 1+5M	–M 2+M	0 0	M –M	–M 0	
				↑				

The resulting simplex table, therefore, is

Table 2: Simplex Table

		$c_j \rightarrow$	1	2	0	0	–M	RR
Basic Variables	Profit/Unit	Qty	x_1	x_2	s_1	s_2	A_1	
x_2	2	4	1	1	1	0	0	
A_1	–M	2	0	0	–1	–1	1	
	Z =8–2M	$z_j \rightarrow$ $c_j - z_j \rightarrow$	2 –1	2 0	2+M –M–M	M –M	–M 0	

As none of the net-evaluations is positive, we stop here. We notice that the artificial variable appears as a basic variable and that too not at the zero level as A_1 = 2 and is non-zero. Therefore, the given LPP does not possess any feasible solution.

E5) Standard form of the given LPP is

$$\text{Max.}\, Z = 10x_1 + 4x_2 + 0s_1 + 0s_2$$

subject to the constraints:

$$4x_1 + x_2 - s_1 = 80$$

$$2x_1 + x_2 - s_2 = 60$$

$$x_1,\ x_2, s_1,\ s_2 \geq 0$$

In this problem also we do not get the columns of identity matrix without introducing artificial variables. Introducing artificial variables A_1, A_2 in the two equations, respectively, we have the objective function as

$$\text{Max.}\, Z = 10x_1 + 4x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$$

subject to the constraints:

$$4x_1 + x_2 - s_1 + A_1 = 80$$

$$2x_1 + x_2 - s_2 + A_2 = 60$$

$$x_1,\ x_2, s_1,\ s_2, A_1, A_2 \geq 0$$

Thus, the initial simplex table is

Table 1: Initial Simplex Table

		$c_j \rightarrow$	10	4	0	0	–M	–M	
Basic Variables	Profit/ Unit	Qty	x_1	x_2	s_1	s_2	A_1	A_2	RR
$\leftarrow A_1$	–M	80	$\boxed{4}$	1	–1	0	1	0	20←
A_2	–M	60	2	1	0	–1	0	1	30
	Z = –140M	$z_j \rightarrow$ $c_j - z_j \rightarrow$	–6M 10+6M ↑	– 2M 4+2M	M –M	M –M	–M 0	–M 0	

The resulting simplex table, therefore, is as follows:

Table 2: Simplex Table

	$c_j \rightarrow$		10	4	0	0	–M	–M	
Basic Variables	Profit/ Unit	Qty	x_1	x_2	s_1	s_2	A_1	A_2	RR
$\leftarrow x_1$	10	20	1	1/4	–1/4	0	×	0	–
A_2	–M	20	0	1/2	$\boxed{1/2}$	–1	×	1	40←
	Z = 200–20M	$z_j \rightarrow$ $c_j - z_j \rightarrow$	10 0	5/2–M(1/2) 3/2+(1/2)M	–5/2–(1/2)M 5/2+(1/2)M	M –M	× ×	–M 0	

↑

The resulting simplex table is given below:

Table 3: Simplex Table

	$c_j \rightarrow$		10	4	0	0	–M	–M	RR
Basic Variables	Profit/ Unit	Qty	x_1	x_2	s_1	s_2	A_1	A_2	
$\leftarrow x_1$	10	30	1	1/2	0	–1/2	×	×	–
s_1	0	40	0	1	1	–2	×	×	–
	Z =300	$z_j \rightarrow$ $c_j - z_j \rightarrow$	10 0	5 –1	0 0	–5 5	× ×	× ×	

↑

Note that corresponding to the most positive net-evaluation, i.e., 5, the values in the column of s_2 are $\begin{pmatrix} -1/2 \\ -2 \end{pmatrix}$. Since none of these values is positive, the solution of the given LPP is unbounded.