UNIT 4 BAYES' THEOREM

Structure

4.1 INTRODUCTION

In the first three units of this block, we have seen as to how probability of different types of events is calculated. We have also discussed the operation of events and evaluation of their probabilities by using addition and multiplication laws. Still, there are situations whose probability evaluations require use of more results.

Probabilities obtained in the earlier units are per-revised or priori probabilities. However, these probabilities can be revised on the basis of some new related information. The revised probabilities are obtained using Bayes' theorem for which knowledge of total probability is also required. So, the present unit first discusses the law of total probability, its applications and then Bayes' theorem and its applications.

Objectives

After completing this unit, you should be able to:

- explain law of total probability;
- know as to how and when to apply the law of total probability;
- learn Bayes' theorem; and
- learn as to how and when to apply Bayes' theorem.

4.2 LAW OF TOTAL PROBABILITY

There are experiments which are conducted in two stages for completion. Such experiments are termed as two-stage experiments. At the first stage, the experiment involves selection of one of the given numbers of possible mutually exclusive events. At the second stage, the experiment involves happening of an event which is a sub-set of at least one of the events of first stage.

As an illustration for a two-stage experiment, let us consider the following example:

Suppose there are two urns – Urn I and Urn II. Suppose Urn I contains 4 white, 6 blue and Urn II contains 4 white, 5 blue balls. One of the urns is selected at random and a ball is drawn. Here, the first stage is the selection of one of the urns and second stage is the drawing of a ball of particular colour.

If we are interested in finding the probability of the event of second stage, then it is obtained using law of total probability, which is stated and proved as under:

Law of Total Probability

Statement: Let S be the sample space and $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events with $P(E_i) \neq 0$; $i = 1, 2, \ldots, n$. Let A be any event which is a sub-set of $E_1 \cup E_2 \cup \ldots \cup E_n$ (i.e. at least one of the events $E_1, E_2, \ldots, E_n$) with $P(A) > 0$, then $P(A) = P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + \ldots + P(E_n)\, P(A|E_n)$

$$= \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)$$

Proof: As A is a sub-set of $E_1 \cup E_2 \cup \ldots \cup E_n$

$\therefore\ A = A \cap (E_1 \cup E_2 \cup \ldots \cup E_n)$ [$\because$ if A is sub-set of B, then $A = A \cap B$]

$\Rightarrow A = (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n)$ [Distributive property of set theory]

$= (E_1 \cap A) \cup (E_2 \cap A) \cup \ldots \cup (E_n \cap A)$

> The above expression can be understood theoretically/logically also as explained under :
>
> A happens in any of the following mutually exclusive ways:
>
> (E_1 happens and then A happens) or (E_2 happens and then A happens) or (E_3 happens and then A happens) or … or (E_n happens and then A happens).
> Now, as meanings of 'and' and 'or' in set theory are '$\cap$' and '$\cup$' respectively
>
> $\therefore\ A = [(E_1 \cap A) \cup (E_2 \cap A) \cup (E_3 \cap A) \cup \ldots \cup (E_n \cap A)]$

$\Rightarrow P(A) = P[(E_1 \cap A) \cup (E_2 \cap A) \cup \ldots \cup (E_n \cap A)]$

$= P(E_1 \cap A) + P(E_2 \cap A) + \ldots + P(E_n \cap A)$

[$\because$ $E_1, E_2, \ldots, E_n$ and hence $E_1 \cap A, E_2 \cap A, \ldots, E_n \cap A$ are mutually exclusive]

$= P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + \ldots + (E_n)\, P(A|E_n)$

[Using multiplication theorem for dependent events]

$$= \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)$$

Hence proved

4.3 APPLICATIONS OF LAW OF TOTAL PROBABILITY

Here, in this section, we are going to take up various situations through examples, where the law of total probability is applicable.

Example 1: There are two bags. First bag contains 5 red, 6 white balls and the second bag contains 3 red, 4 white balls. One bag is selected at random and a ball is drawn from it. What is the probability that it is (i) red, (ii) white.

Solution: Let E_1 be the event that first bag is selected and E_2 be the event that second bag is selected.

$$\therefore\ P(E_1) = P(E_2) = \frac{1}{2}.$$

(i) Let R be the event of getting a red ball from the selected bag.

$$\therefore P(R \mid E_1) = \frac{5}{11}, \text{ and } P(R \mid E_2) = \frac{3}{7}.$$

Thus, the required probability is given by

$$P(R) = P(E_1)P(R \mid E_1) + P(E_2)P(R \mid E_2)$$

$$= \frac{1}{2} \times \frac{5}{11} + \frac{1}{2} \times \frac{3}{7}$$

$$= \frac{5}{22} + \frac{3}{14} = \frac{35+33}{154} = \frac{68}{154} = \frac{34}{77}$$

(ii) Let W be the event of getting a white ball from the selected bag.

$$\therefore P(W \mid E_1) = \frac{6}{11}, \text{ and } P(W \mid E_2) = \frac{4}{7}.$$

Thus, the required probability is given by

$$P(W) = P(E_1)P(W \mid E_1) + P(E_2)P(W \mid E_2)$$

$$= \frac{1}{2} \times \frac{6}{11} + \frac{1}{2} \times \frac{4}{7} = \frac{3}{11} + \frac{2}{7} = \frac{21+22}{77} = \frac{43}{77}.$$

Example 2: A factory produces certain type of output by 3 machines. The respective daily production figures are-machine X : 3000 units, machine Y: 2500 units and machine Z: 4500 units. Past experience shows that 1% of the output produced by machine X is defective. The corresponding fractions of defectives for the other two machines are 1.2 and 2 percent respectively. An item is drawn from the day's production. What is the probability that it is defective?

Solution: Let E_1, E_2 and E_3 be the events that the drawn item is produced by machine X, machine Y and machine Z, respectively. Let A be the event that the drawn item is defective.

As the total number of units produced by all the machines is

3000 + 2500 + 4500 = 10000,

$\therefore$ $P(E_1) = \dfrac{3000}{10000} = \dfrac{3}{10}$, $P(E_2) = \dfrac{2500}{10000} = \dfrac{1}{4}$, $P(E_3) = \dfrac{4500}{10000} = \dfrac{9}{20}$.

$P(A \mid E_1) = \dfrac{1}{100} = 0.01$, $P(A \mid E_2) = \dfrac{1.2}{100} = 0.012$, $P(A \mid E_3) = \dfrac{2}{100} = 0.02$.

Thus, the required probability = Probability that the drawn item is defective

$$= P(A)$$

$$= P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2) + P(E_3)\,P(A|E_3)$$

$$= \frac{3}{10}\times 0.01 + \frac{1}{4}\times 0.012 + \frac{9}{20}\times 0.02$$

$$= \frac{3}{1000} + \frac{3}{1000} + \frac{9}{1000}$$

$$= \frac{15}{1000} = 0.015.$$

Example 3: There are two coins-one unbiased and the other two- headed, otherwise they are identical. One of the coins is taken at random without seeing it and tossed. What is the probability of getting head?

Solution: Let E_1 and E_2 be the events of selecting the unbiased coin and the two-headed coin respectively. Let A be the event of getting head on the tossed coin.

$\therefore$ $P(E_1) = \dfrac{1}{2}$, $P(E_2) = \dfrac{1}{2}$ [$\because$ selection of each of the coin is equally likely]

$P(A \mid E_1) = \dfrac{1}{2}$ [$\because$ if it is unbiased coin, then head and tail are equally likely]

$P(A|E_2) = 1$ [$\because$ if it is two-headed coin, then getting the head is certain]

Thus, the required probability = P(A)

$$= P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2)$$

$$= \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times 1$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Example 4: The probabilities of selection of 3 persons for the post of a principal in a newly started colleage are in the ratio $4:3:2$. The probabilities that they will introduce co-education in the college are 0.2, 0.3 and 0.5, respectively. Find the probability that co-education is introduced in the college.

Solution: Let E_1, E_2, E_3 be the events of selection of first, second and third person for the post of a principal respectively. Let A be the event that co-education is introduced.

$\therefore$ $P(E_1) = \dfrac{4}{9}$, $P(E_2) = \dfrac{3}{9}$, $P(E_3) = \dfrac{2}{9}$

$P(A \mid E_1) = 0.2$, $P(A \mid E_2) = 0.3$, $P(A \mid E_3) = 0.5$.

Thus, the required probability = P(A)

$$= P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)\,P(A|E_3)$$

$$= \frac{4}{9}\times 0.2 + \frac{3}{9}\times 0.3 + \frac{2}{9}\times 0.5$$

$$= \frac{0.8}{9} + \frac{0.9}{9} + \frac{1}{9} = \frac{2.7}{9} = 0.3$$

Now, you can try the following exercises.

E1) A person gets a construction job and agrees to undertake it. The completion of the job in time depends on whether there happens to be strike or not in the company. There are 40% chances that there will be a strike. Probability that job is completed in time is 30% if the strike takes place and is 70% if the strike does not take place. What is the probability that the job will be completed in time?

E2) What is the probability that a year selected at random will contains 53 Sundays?

E3) There are two bags, first bag contains 3 red, 5 black balls and the second bag contains 4 red, 5 black balls. One ball is drawn from the first bag and is put into the second bag without noticing its colour. Then two balls are drawn from the second bag. What is the probability that balls are of opposite colours?

4.4 BAYES' THEOREM

In Sec. 4.2 of this unit, we have discussed that if we are interested in finding the probability of the event of second stage, then it is obtained using law of total probability. But if the happening of the event of second stage is given to us and on this basis we find the probability of the events of first stage, then the probability of an event of first stage is the revised (or posterior) probabilities and is obtained using an important theorem known as Bayes' theorem given by Thomas Bayes (died in 1761, at the age of 59), a British Mathematician, published after his death in 1763. This theorem is also known as 'Inverse probability theorem', because here moving from first stage to second stage, we again find the probabilities (revised) of the events of first stage i.e. we move inversely. Thus, using this theorem, probabilities can be revised on the basis of having some related new information.

As an illustration, let us consider the same example as taken in Sec. 4.2 of this unit. In this example, if we are given that the drawn ball is of particular colour and it is asked to find, on this basis, the probability that Urn I or Urn II was selected, then these are the revised (posterior) probabilities and are obtained using Bayes' theorem, which is stated and proved as under:

Statement: Let S be the sample space and $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events with $P(E_i) \neq 0$; i = 1, 2, .., n. Let A be any event which is a sub-set of $E_1 \cup E_2 \cup \ldots \cup E_n$ (i.e. at least one of the events $E_1, E_2, \ldots, E_n$) with $P(A) > 0$ [Notice that up to this line the statement is same as that of law of total probability], then

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{P(A)}, i = 1, 2, \ldots, n$$

where $P(A) = P(E_1)\ P(A|E_1) + P(E_2)\ P(A|E_2) + \ldots + P(E_n)\ P(A|E_n)$.

Proof: First you have to prove that

$P(A) = P(E_1)\ P(A|E_1) + P(E_2)\ P(A|E_2) + \ldots + P(E_n)\ P(A|E_n)$.

which is nothing but the law of total probability that has already been proved in Sec. 4.2. After proving this, proceed as under:

$$P(E_i \mid A) = \frac{P(E_i \cap A)}{P(A)} \qquad \left[\begin{array}{l}\text{Applying the formula of conditional} \\ \text{probability i.e. } P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}\end{array}\right]$$

$$= \frac{P(E_i)P(A \mid E_i)}{P(A)} \qquad \left[\begin{array}{l}\text{Applying multiplication theorem for dependent} \\ \text{events i.e. } P(A \cap B) = P(A)P(B \mid A)\end{array}\right]$$

4.5 APPLICATIONS OF BAYES' THEOREM

Example 5: Let us consider the problem given in Example 1 of Sec 4.3 of this unit after replacing the question asked therein (i.e. the last sentence) by the following question:

If it is found to be red, what is the probability of?

i) selecting the first bag

ii) selecting the second bag

Solution: First, we have to give the solution exactly as given for Example 1 of Sec. 4.3 of this unit. After that, we are to proceed as follows:

i) Probability of selecting the first bag given that the ball drawn is red

$$= P(E_1 \mid R)$$

$$= \frac{P(E_1)P(R \mid E_1)}{P(R)} \qquad \left[\begin{array}{l}\text{Applying Bayes'} \\ \text{theorem}\end{array}\right]$$

$$= \frac{\frac{1}{2} \times \frac{5}{11}}{\frac{34}{77}} = \frac{5}{22} \times \frac{77}{34} = \frac{35}{68}$$

ii) Probability of selecting the second bag given that the ball drawn is red

$$P(E_2|R) = \frac{P(E_2)P(R \mid E_2)}{P(R)} = \frac{\frac{1}{2} \times \frac{3}{7}}{\frac{34}{77}} = \frac{3}{14} \times \frac{77}{34} = \frac{33}{68}$$

Example 6: Consider the problem given in Example 2 of Sec. 4.3 of this unit after replacing the question asked therein (i.e. the last sentence) by the following question:

If the drawn item is found to be defective, what is the probability that it has been produced by machine Y?

Solution: Proceed exactly in the manner the Example 2 of Sec. 4.3 of this unit has been solved and then as under:

Probability that the drawn item has been produced by machine Y given that it is defective

$$= P(E_2 \mid A)$$

$$= \frac{P(E_2)P(A \mid E_2)}{P(A)} \quad \begin{bmatrix}\text{Applying Bayes'} \\ \text{theorem}\end{bmatrix}$$

$$= \frac{\frac{1}{4} \times 0.012}{0.015} = \frac{0.003}{0.015} = \frac{1}{5}.$$

Example 7: Consider the problem given in Example 3 of Sec. 4.3 of this unit after replacing the question asked therein (i.e. the last sentence) by the following question:

If head turns up, what is the probability that

i) it is the two-headed coin

ii) it is the unbiased coin.

Solution: First give the solution of Example 3 of Sec. 4.3 of this unit and then proceed as under:

Now, the probability that the tossed coin is two-headed given that head turned up

$$= P(E_2 \mid A)$$

$$= \frac{P(E_2)P(A \mid E_2)}{P(A)} \quad \begin{bmatrix}\text{Applying Bayes'} \\ \text{theorem}\end{bmatrix}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{3}{4}} = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}.$$

ii) The probability that the tossed coin is unbiased given that head turned up

$$= P(E_1 \mid A)$$

$$= \frac{P(E_1)P(A \mid E_1)}{P(A)} \quad \begin{bmatrix}\text{Applying Bayes'} \\ \text{theorem}\end{bmatrix}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{3}{4}} = \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$$

Example 8: Consider the statement given in Example 4 of Sec. 4.3 of this unit after replacing the question asked therein (i.e. the last sentence) by the following question:

If the co-education is introduced by the candidate selected for the post of principal, what is the probability that first candidate was selected.

Solution: First give the solution of Example 4 of Sec. 4.3 of this unit and then proceed as under:

The required probability $= P(E_1 | A)$

$$= \frac{P(E_1)P(A|E_1)}{P(A)} \quad \begin{bmatrix}\text{Applying Bayes'} \\ \text{theorem}\end{bmatrix}$$

$$= \frac{\frac{4}{9} \times 0.2}{0.3} = \frac{4}{9} \times \frac{0.2}{0.3} = \frac{8}{27}.$$

Now, you can try the following exercises.

E4) A bag contains 4 red and 5 white balls. Another bag contains 2 red and 3 white balls. A ball is drawn from the first bag and is transferred to the second bag. A ball is then drawn from the second bag and is found to be red, what is the probability that red ball was transferred from first to second bag?

E5) An insurance company insured 1000 scooter drivers, 3000 car drivers and 6000 truck drivers. The probabilities that scooter, car and truck drivers meet an accident are 0.02, 0.04, 0.25 respectively. One of the insured persons meets with an accident. What is the probability that he is a
(i) car driver

(ii) truck driver

E6) By examining the chest X-ray, the probability that T.B is detected when a person is actually suffering from T.B. is 0.99. The probability that the doctor diagnoses incorrectly that a person has T.B. on the basis of X-ray is 0.002. In a certain city, one in 1000 persons suffers from T.B. A person is selected at random and is diagnosed to have T.B., what is the chance that he actually has T.B.?

E7) A person speaks truth 3 out of 4 times. A die is thrown. She reports that there is five. What is the chance there was five?

4.6 SUMMARY

In this unit, we have covered the following points:

1) There are experiments which are conducted in two stages for completion. Such experiments are termed as two-stage experiments. At the first stage, the experiment involves selection of one of the given number of possible mutually exclusive events. At the second stage, the experiment involves happening of an event which is a sub-set of at least one of the events of first stage. If we are interested in finding the probability of the event of second stage, then it is obtained using **law of total probability**, but if the

happening of the event of the second stage is given to us and on this basis we find the probability of an event of the first stage, then this probability of the event of the first stage is the revised (or posterior) probability and is obtained by using **Bayes' theorem.**

2) **Law of total probability** Let S be the sample space and $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events with $P(E_i) \neq 0$; i = 1, 2, …, n. Let A be any event which is a sub-set of $E_1 \cup E_2 \cup \ldots \cup E_n$ (i.e. at least one of the events $E_1, E_2, \ldots, E_n$) with P(A) > 0, then

$$P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2) + \ldots + P(E_n)\,P(A|E_n)$$

$$= \sum_{i=1}^{n} P(E_i)\,P(A \mid E_i).$$

3) **Bayes' theorem** is an extension of law of total probability and is stated as:

Let S be the sample space. Let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events with $P(E_i) \neq 0$; i = 1, 2, …, n. Let A be any event which is a sub-set of $E_1 \cup E_2 \cup \ldots \cup E_n$ (i.e. at least one of the events $E_1, E_2, \ldots, E_n$) with P(A) > 0 [Notice that up to this line the statement is as same as that of law of total probability], then

$$P(E_i \mid A) = \frac{P(E_i)\,P(A \mid E_i)}{P(A)}, \; i = 1, 2, \ldots, n$$

where $P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2) + \ldots + P(E_n)\,P(A|E_n)$.

4.7 SOLUTIONS/ANSWERS

E1) Let E_1 be the event that strike takes place and E_2 be the event that strike does not take place.

Let A be the event that the job will be completed in time.

$$\therefore P(E_1) = 40\% = \frac{40}{100}, \; P(E_2) = 1 - P(E_1) = 1 - \frac{40}{100} = \frac{60}{100},$$

$$P(A \mid E_1) = 30\% = \frac{30}{100}, \; P(A \mid E_2) = 70\% = \frac{70}{100}$$

Thus, the required probability is

$$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$$

$$= \frac{40}{100} \times \frac{30}{100} + \frac{60}{100} \times \frac{70}{100}$$

$$= \frac{1200}{10000} + \frac{4200}{10000} = \frac{12}{100} + \frac{42}{100} = \frac{54}{100} = 54\%$$

E2) Such probability in case of a leap year and in case of a non-leap year has already been obtained in Unit 1 of this course. But here we do not know as to whether it is a leap year or non-leap year, hence we will have to do it using law of total probability considering both the possibilities i.e. we shall proceed as follows:

Let E_1 be the event that the year selected is a leap year and E_2 be the event that it is a non-leap year. Let A be the event that the selected year contains 53 Sundays.

We know that out of 4 consecutive years, 1 is a leap year and 3 are non-leap years, therefore,

$$P(E_1)=\frac{1}{4},\ P(E_2)=\frac{3}{4}.$$

A leap years consists of 366 days, i.e. 52 complete weeks and 2 over days. These 2 over days may be "Sunday and Monday", "Monday and Tuesday", "Tuesday and Wednesday" ," Wednesday and Thursday", "Thursday and Friday", " Friday and Saturday" and " Saturday and Sunday". Out of these 7 cases, Sunday is included in 2 cases and hence the probability that a leap year will consist of 53 Sundays is

$$P(A\,|\,E_1)=\frac{2}{7}.$$

A non-leap year consists of 365 days, i.e. 52 complete weeks and 1 over day. This over day may be one of the seven days of the week and hence the probability that a non-leap year will consist of 53 sundays is

$$P(A\,|\,E_2)=\frac{1}{7}.$$

$\therefore$ By the total law of probability, the probability that a year selected at random will consists of 53 Sundays is

$$P(A)=P(E_1)P(A\,|\,E_1)+P(E_2)P(A\,|\,E_2)$$

$$=\frac{1}{4}\times\frac{2}{7}+\frac{3}{4}\times\frac{1}{7}=\frac{2}{28}+\frac{3}{28}=\frac{5}{28}.$$

E3) We are given that first of all one ball is transferred from bag I to bag II and then two balls are drawn from bag II. This job can be done in two mutually exclusive ways

(a) A red ball is transferred from bag I to bag II and then two balls are drawn from bag II

(b) A black ball is transferred from bag I to bag II and then two balls are drawn from bag II

We define the following events:

E_1 be the event of getting a red ball from bag I

E_2 be the event of getting a black ball from bag I

A be the event of getting two balls of opposite colours from the bag II

$$\therefore P(E_1)=\frac{{}^3C_1}{{}^8C_1}=\frac{3}{8}$$

$$P(E_2)=\frac{{}^5C_1}{{}^8C_1}=\frac{5}{8}$$

$$P(A|E_1) = \frac{{}^5C_1 \times {}^5C_1}{{}^{10}C_2} = \frac{5\times 5}{45} = \frac{5}{9}.$$

$$P(A|E_2) = \frac{{}^4C_1 \times {}^6C_1}{{}^{10}C_2} = \frac{4\times 6}{45} = \frac{8}{15}$$

Hence, the required probability is given by

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2)$$

$$= \frac{3}{8}\times\frac{5}{9} + \frac{5}{8}\times\frac{8}{15}$$

$$= \frac{5}{24} + \frac{1}{3} = \frac{5+8}{24} = \frac{13}{24}$$

E4) Let E_1 be the event that a red ball is drawn from the first bag and E_2 be the event that the drawn ball from the first bag is white. Let A be the event of drawing a red ball from the second bag after transferring the ball drawn from first bag into it.

$$\therefore\ P(E_1) = \frac{4}{9}, P(E_2) = \frac{5}{9}$$

$$P(A|E_1) = \frac{3}{6}, P(A|E_2) = \frac{2}{6}$$

$\therefore$ By law of total probability,

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2)$$

$$= \frac{4}{9}\times\frac{3}{6} + \frac{5}{9}\times\frac{2}{6} = \frac{11}{27}.$$

Thus, the required probability = $P(E_1|A)$

$$= \frac{P(E_1)P(A|E_1)}{P(A)} \quad \left[\begin{array}{l}\text{Applying Bayes'}\\ \text{theorem}\end{array}\right]$$

$$= \frac{\frac{4}{9}\times\frac{3}{6}}{\frac{11}{27}} = \frac{6}{11}$$

E5) Total number of drivers = 1000 + 3000 + 6000 = 10000.

Let E_1, E_2, E_3 be the events that the selected insured person is a scooter driver, car driver, truck driver respectively. Let A be the event that a driver meets with an accident.

$$\therefore\ P(E_1) = \frac{1000}{10000} = \frac{1}{10},\ P(E_2) = \frac{2000}{10000} = \frac{1}{5},\ P(E_3) = \frac{6000}{10000} = \frac{6}{10} = \frac{3}{5}.$$

$$P(A|E_1) = 0.02 = \frac{2}{100},\ P(A|E_2) = 0.04 = \frac{4}{100},\ P(A|E_3) = 0.25 = \frac{25}{100}$$

$\therefore$ By total probability theorem, we have

$$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + P(E_3)P(A \mid E_3)$$

$$= \frac{1}{10} \times \frac{2}{100} + \frac{1}{5} \times \frac{4}{100} + \frac{3}{5} \times \frac{25}{100}$$

$$= \frac{1}{500} + \frac{4}{500} + \frac{75}{500} = \frac{80}{500} = \frac{4}{25}$$

i) The required probability $= P\left(E_2 \mid A\right)$

$$= \frac{P\left(E_2\right)P\left(A \mid E_2\right)}{P\left(A\right)} \quad \left[\begin{array}{l}\text{Applying Bayes'} \\ \text{theorem}\end{array}\right]$$

$$= \frac{\frac{1}{5} \times \frac{4}{100}}{\frac{4}{25}} = \frac{4}{500} \times \frac{25}{4} = \frac{1}{20}.$$

ii) The required probability $= P\left(E_3 \mid A\right)$

$$= \frac{P\left(E_3\right)P\left(A \mid E_3\right)}{P\left(A\right)} \quad \left[\begin{array}{l}\text{Applying Bayes'} \\ \text{theorem}\end{array}\right]$$

$$= \frac{\frac{3}{5} \times \frac{25}{100}}{\frac{4}{25}} = \frac{75}{500} \times \frac{25}{4} = \frac{75}{80} = \frac{15}{16}.$$

E6) Let E_1 be the event that a person selected from the population of the city is suffering from the T.B. and E_2 be the event that he/she is not suffering from the T.B. Let A be the event that the selected person is diagnosed to have T.B. Therefore, according to given,

$$P(E_1) = \frac{1}{1000} = 0.001, \quad P(E_2) = 1 - 0.001 = 0.999 = \frac{999}{1000},$$

$$P(A \mid E_1) = 0.99 = \frac{99}{100}, P(A \mid E_2) = 0.002 = \frac{2}{1000}.$$

$\therefore$ By total probabilities theorem, we have

$$P(A) = P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2)$$

$$= \frac{1}{1000} \times \frac{99}{100} + \frac{999}{1000} \times \frac{2}{1000}$$

$$= \frac{99}{100000} + \frac{1998}{1000000} = \frac{990 + 1998}{1000000} = \frac{2988}{1000000}$$

Thus, the required probability = $P(E_1|A)$

$$= \frac{P(E_1)P(A|E_1)}{P(A)} \quad \begin{bmatrix} \text{Applying Bayes'} \\ \text{theorem} \end{bmatrix}$$

$$= \frac{\frac{1}{1000} \times \frac{99}{100}}{\frac{2988}{1000000}} = \frac{990}{2988} = \frac{55}{166}.$$

E7) Let E_1 be the event that the person speaks truth, E_2 be the event that she tells a lie and A be the event that she reports a five.

$$\therefore\ P(E_1) = \frac{3}{4},\ P(E_2) = \frac{1}{4},\ P(A|E_1) = \frac{1}{6},\ P(A|E_2) = \frac{5}{6}.$$

By law of total probability, we have

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2)$$

$$= \frac{3}{4} \times \frac{1}{6} + \frac{1}{4} \times \frac{5}{6}$$

$$= \frac{3}{24} + \frac{5}{24} = \frac{8}{24} = \frac{1}{3}.$$

Thus, the required probability = $P(E_1|A)$

$$= \frac{P(E_1)P(A|E_1)}{P(A)} \quad \begin{bmatrix} \text{Applying Bayes'} \\ \text{theorem} \end{bmatrix}$$

$$= \frac{\frac{3}{4} \times \frac{1}{6}}{\frac{1}{3}} = \frac{3}{4} \times \frac{1}{6} \times \frac{3}{1} = \frac{3}{8}.$$