UNIT 15 k-SAMPLE TESTS

Structure

15.1 INTRODUCTION

In Unit 13 of this block, we have studied some one-sample non-parametric tests for testing the hypothesis about the population median and in Unit 14, we have studied some two-sample non-parametric tests for testing the hypothesis about the equality of two population medians. If there is a situation where three or more populations are under study and we are interested to test the equality of their mean or medians. In such situations, generally, we use analysis of variance (ANOVA) technique that will describe in Block 2 of MST-005. But ANOVA procedure requires the assumption that the populations being compared are all normally distributed with equal variances. When the populations under study are not normally distributed then we cannot use the ANOVA procedure. Therefore, we require non-parametric tests that not based on any assumption about the shape (form) of the population distributions. This unit is devoted to discuss some such tests that use in these situations as Kruskal-Wallis test and Friedman test.

This unit is divided into five sections. Section 15.1 is described the need of k-sample non-parametric tests. The Kruskal-Wallis test is discussed in Section 15.2 in details whereas Friedman test is discussed in Section 15.3. Unit ends by providing summary of what we have discussed in this unit in Section 15.4 and solution of exercises in Section 15.5.

Objectives

After studying this unit, you should be able to:

- describe the need of k-sample test;
- perform the Kruskal-Wallis test for testing the hypothesis about equality of more than two population medians; and
- perform the Friedman test.

15.3 KRUSKAL-WALLIS TEST

Kruskal-Wallis test is the non-parametric alternative to the one way analysis of variance or completely randomised design. Also this test represents a generalisation of the two-sample Mann-Whitney U test.

As we have discussed in previous section that analysis of variance (ANOVA) test whether three or more independent samples have been drawn from the populations having the same mean or not. Also it is considered as the extension of two-sample t-test for equality of two means. But ANOVA procedure requires the assumption that the populations being compared are all normally distributed with equal variances. When the populations under study are not normally distributed then we cannot use the ANOVA procedure. Therefore, we require a non-parametric test that not based on any assumption about the shape of the population distribution and use in testing of such situations. Such a test is the Kruskal-Wallis test. This test is a non-parametric alternative to the one-way analysis of variance or completely randomised design. Kruskal-Wallis test also a generalisation of the two-sample Mann-Whitney U test.

Assumptions

This test works under the following assumptions:

(i) k samples are randomly and independently drawn from their respective populations.

(ii) The variable under study is continuous.

(iii) The measurement scale is at least ordinal.

(iv) The populations are identical in shape except for a possible difference in location.

Let us suppose that we have k independent random samples of sizes $n_1, n_2, ..., n_k$ taken from k populations respectively which are shown as below:

Sample (Treatment) Number	Sample Observation			
1	X_{11}	X_{12}	…	X_{1n_1}
2	X_{21}	X_{22}	…	X_{2n_2}
.	.	.	…	.
.	.	.	…	.
.	.	.	…	.
k	X_{k1}	X_{k2}	…	X_{kn_k}

Generally, we are interested to test whether all k populations have same median or not so we can take the null and alternative hypotheses as

H_0 : All k populations have same median

H_1 : All k populations have not same median

A treatment is an object or a factor which is allocated to one or more units to estimate its effect. For example, if some fertilizers are applied on experimental fields to know the effect to these then fertilizers are known as treatments.

In ANOVA generally, we test the effect of treatments so we can also take the null and alternative hypotheses as

H_0 : All k treatments have same effect

H_1 : All k treatments have not same effect

After setting the null and alternative hypotheses, this test involves following steps:

Step 1: First of all, we combine the observations of all samples.

Step 2: After that, ranking all these combined observations from smallest to largest, that is, the rank 1 is given to the smallest of the combined observations, rank 2 is given to the second smallest and so on up to the largest observation. If several values are same (tied), we assign each the average of ranks they would have received if there were no repetition.

Step 3: If the null hypothesis is true, we expect that sum of ranks of all samples to be equal. Therefore, in this step we find the sum of ranks for each sample. Let $R_1, R_2, ..., R_k$ represent the sum of ranks of 1^{st} sample, 2^{nd} sample and so on k^{th} sample respectively.

Step 4: To test the null hypothesis, the Kruskal-Wallis test statistic is defined as:

$$H = \frac{12}{n(n+1)}\left(\sum_{i=1}^{k}\frac{R_i^2}{n_i}\right) - 3(n+1)$$

where, $n = n_1 + n_2 + ... + n_k$

If tie occurs, an adjustment in Kruskal-Wallis test statistic has to be made. The adjustment factor is

$$C = 1 - \frac{1}{(n^3 - n)}\sum_{i=1}^{r}\left(t_i^3 - t_i\right)$$

where, r is the number of groups of different tied ranks, and t_i is the number of tied values within i^{th} group that are tied at a particular value. Therefore, adjusted Kruskal-Wallis test statistic is

$$H_C = H / C$$

Step 5: Obtain critical value of Kruskal-Wallis test statistic at given level of significance under the condition, that null hypothesis is true. **Table VIII** in the Appendix at the end of this block provides critical values for three samples (k = 3) and each sample has five or fewer observations ($n_i \le 5$) at α level of significance.

Step 6: **Decision rule:**

To take the decision about the null hypothesis, the calculated value of Kruskal-Wallis test statistic (computed in Step 4) is compared with the critical (tabulated) value (obtain in Step 5) at given level of significance(α) under the condition that null hypothesis is true.

If calculated value of test statistic is greater than critical (tabulated) value at α level of significance then we reject the null hypothesis at α level of significance otherwise we do not reject the null hypothesis H_0.

For large samples

For large number of samples (k > 3) or for large sample sizes ($n_i > 5$), the distribution of the test statistic under the null hypothesis is well approximated by the chi-square distribution with (k –1) degrees of freedom. So when number of samples or observations per sample are such that we cannot use **Table VIII** then we use χ^2-table (given in the Appendix of the Block 1 of this course) for critical value of the test statistic.

Now, it is time to do some examples based on above test.

Example 1: Three different feeds are to be compared to determine that they have same distribution of weight gains on experimental animals (such as pig). Suppose 12 animals are divided at random into three groups of four animals and each group at different feed. The following results are obtained:

Feed	Weight Gains			
1	104	110	106	102
2	112	117	115	114
3	120	126	121	128

Test that the median weight gains due to three feeds are same by Kruskal-Wallis test at 5% level of significance.

Solution: Here, we want to test that the median weight gains due to three feeds are same. Therefore, our claim is "the median weight gains due to three feeds are same" and its complement is "the median weight gains due to three feeds are not same". So we can take claim as the null hypothesis and complement as the alternative hypothesis. Thus,

H_1 : The median weight gains due to three feeds are same

H_1 : The median weight gains due to three feeds are not same

Here, the distributions of weight gains due to different feeds are not given so the assumption of normality for one-way ANOVA is not fulfilled. Therefore, we go for Kruskal-Wallis test.

To perform the Kruskal-Wallis test, the test statistic is given by

$$H = \frac{12}{n(n+1)} \sum_{i=1}^{k} \left(\frac{R_i^2}{n_i} \right) - 3(n+1) \qquad \ldots (1)$$

where, R_i is the sum of ranks of i^{th} group.

Therefore, first we rank all the observations of three groups and then find the sum of ranks (R_i) of each group.

Calculation for R_i:

Weight Gains					
Feed I	**Rank**	**Feed II**	**Rank**	**Feed III**	**Rank**
104	2	112	5	120	9
110	4	117	8	126	11
106	3	115	7	121	10
102	1	114	6	128	12
Total	$R_1 = 10$		$R_2 = 26$		$R_3 = 42$

Putting the values in equation (1), we have

$$H = \frac{12}{12(12+1)} \left[\frac{(10)^2}{4} + \frac{(26)^2}{4} + \frac{(42)^2}{4} \right] - 3(12+1)$$

$$= \frac{12}{12 \times 13} \left[\frac{100}{4} + \frac{676}{4} + \frac{1764}{4} \right] - 39$$

$$= 48.85 - 39 = 9.85$$

The critical (tabulated) value corresponding to k = 3, $n_1 = 4$, $n_2 = 4$, $n_3 = 4$ at 5% level of significance is 5.692.

Since calculated value of test statistic H (= 9.85) is greater than critical value (= 5.692) so we reject the null hypothesis i.e. we reject the claim at 5% level of significance.

Thus, we conclude that samples provide us sufficient evidence against the claim so the median weight gains due to feeds are not same.

Example 2: An analyst in the publishing industry wants to test whether the price of newspaper advertisement of a given size is about the same in four large newspaper groups. Random sample of seven newspapers from each group is selected and the price (in thousand rupees) of an ad is recorded which are given below:

Group A:	65	62	45	70	48	57	50
Group B:	81	64	72	55	90	68	75
Group C:	42	58	48	59	61	60	64
Group D:	85	92	69	82	94	64	73

Do you believe that there is significant difference in the price of an ad across the four groups by Kruskal-Wallis test at 1% level of significance?

Solution: Here, we want to test that price of an ad across the four groups is different. Therefore, our claim is "the price of an ad across the four groups is not same" and its complement is "the price of an ad across the four groups is same". Therefore, we can take complement as the null hypothesis and claim as the alternative hypothesis. Thus,

H_0: The price of an ad across the four newspapers is same

H_1: The price of an ad across the four newspapers is not same

Here the assumption of normality is not fulfilled, so we cannot use one-way ANOVA. Therefore, we go for Kruskal-Wallis test.

To perform the Kruskal-Wallis test, the test statistic is given by

$$H = \frac{12}{n(n+1)}\sum_{i=1}^{k}\left(\frac{R_i^2}{n_i}\right) - 3(n+1) \quad \ldots (2)$$

where, R_i is the sum of ranks of i^{th} group.

Therefore, first we rank all the observations of three groups. For convenient, we arrange the data of each group in ascending order and then find the sum of ranks (R_i) of each group.

Calculation for R_i:

Group A	**Rank**	**Group B**	**Rank**	**Group C**	**Rank**	**Group D**	**Rank**
45	2	55	6	42	1	64	14
48	3.5	64	14	48	3.5	69	18
50	5	68	17	58	8	73	21
57	7	72	20	59	9	82	24
62	12	75	22	60	10	85	25
65	16	81	23	61	11	92	27
70	19	90	26	64	14	94	28
Total	R_1 = 64.5		R_2 =128		R_3 =56.5		R_4 =157

Putting the values in equation (2), we have

$$H = \frac{12}{28(28+1)}\left(\frac{(64.5)^2}{7}+\frac{(128)^2}{7}+\frac{(56.5)^2}{7}+\frac{(157)^2}{7}\right) - 3(28+1)$$

$$= \frac{12}{28\times 29}\left(\frac{4160.25+16384+3192.25+24649}{7}\right) - 87$$

$$= \frac{12}{28\times 29}\times 6912.21 - 87 = 102.15 - 87 = 15.15$$

From the above calculations, we see that there are tied ranks therefore, we have

Number of groups of tied ranks = r = 2 (3.4 and 14)

Number of times group 3.4 of tied ranks occurs = $t_1(3.4) = 2$

Number of times group 14 of tied ranks occurs = $t_2(14) = 3$

Since ties occur, therefore, the correction factor is

$$C = 1 - \frac{1}{(n^3 - n)}\sum_{i=1}^{r}\left(t_i^3 - t_i\right) = 1 - \frac{1}{(28)^3 - 28}\left[\left(2^3 - 2\right) - \left(3^3 - 3\right)\right]$$

$$= 1 - \frac{1}{21924} \times 30 = 1 - 0.0013 = 0.9987$$

Therefore, adjusted Kruskal-Wallis test statistic is

$$H_C = H/C = 15.15/0.9987 = 15.17$$

Since all sample sizes $n_1 = 7$, $n_2 = 7$, $n_3 = 7$ exceed 5 therefore, the test statistic H_c is approximated by chi square with $k - 1 = 4 - 1 = 3$ degrees of freedom. Thus, we compare calculated test statistic H_c with critical value given in χ^2-table.

The critical (tabulated) value of χ^2 with 3 degrees of freedom at 1% level of significance is 11.34.

Since calculated value of test statistic H_c (= 15.17) is greater than critical value (= 11.34) so we reject the null hypothesis and support the alternative hypothesis i.e. we support the claim at 1% level of significance.

Thus, we conclude that the samples fail to provide us sufficient evidence against the claim so we may assume that the price of an ad across the four newspapers is different.

Now, you can try the following exercises in same manner.

E1) Suppose dairy farmer wishes to compare the effect of four different diets on the amount of milk produced by cattle. He randomly selects 12 cows and assigns four cows randomly to each diet for eight week. The following table gives the rank assigned to each cows.

Diet	Rank				Total R_i
A	2	5	4	9	20
B	7	1	8	3	19
C	6	10	12	11	39

Test that the diets are equally productive by Kruskal-Wallis test at 1% level of significance.

E2) A manufacturer tried three alternative manufacturing processes and recorded the number of defective articles in each process as follows:

Number of Defectives per Batch		
Process I	**Process II**	**Process III**
12	10	13
5	15	11
2	2	6
6	7	16
0	8	14
4	6	20

Assuming that same number of articles is produced in each batch. Can you conclude that there is no difference in the three processes in terms of number of defective per batch by Kruskal-Wallis test at the 1% level of significance?

15.4 FRIEDMAN TEST

In the previous section, we have studied Kruskal-Wallis test which is a non-parametric version of one-way ANOVA or completed randomised design and also an extension of Mann-Whitney U test. The Friedman test is a non-parametric version of two-way ANOVA or randomised block design with one item per call that will be discussed in Block 2 of MST-005. This test is also based on ranks and may be viewed as an extension of the Wilcoxon signed-rank test for two samples.

Kruskal-Wallis test is a non-parametric version of one way ANOVA or completely randomised design whereas the Friedman test is a non-parametric version of the randomised block design or two way ANOVA with one item per call.

Assumptions

This test work under the following assumptions:

(i) The observations are independent within and between samples.

(ii) The treatments are randomly assigned to experimental units within each block.

(iii) The measurement scale is at least ordinal.

(iv) The variable under study is continuous.

Suppose, we have k mutually independent random samples of equal size n, that is $X_{i1}, X_{i2}, ..., X_{in}$ where, i = 1, 2, …, k. In other words, we can say that there are n treatments and k blocks of each of size n and each treatment occurs once in each block. The data for k blocks and n treatments can be presented in the following two-way table:

Block (Sample)	Treatment					
	1	2	…	j	…	n
1	X_{11}	X_{12}	…	X_{1j}	…	X_{1n}
2	X_{21}	X_{22}	…	X_{2j}	…	X_{2n}
.	.	.	…	.	…	.
.	.	.	…	.	…	.
.	.	.	…	.	…	.
i	X_{i1}	X_{i2}	…	X_{ij}	…	X_{in}
.	.	.	…	.	…	.
.	.	.	…	.	…	.
.	.	.	…	.	…	.
k	X_{k1}	X_{k2}	…	X_{kj}	…	X_{kn}

Generally, we are interested to test whether all treatments have same effect or not so we can take the null and alternative hypotheses as

H_0 : All treatments have same effect

H_1 : All treatments have same not effect

After setting null and alternative hypotheses, this test involves following steps:

Step 1: To perform the Friedman test, we rank the observations within each block (sample) independently from smallest to largest. The rank 1 is given to the smallest observation of a block, rank 2 is given to the second smallest and so on up to the largest observation of the block (sample). If several values are same (tied) we assign each the average of ranks they would have received if there were no repetition. Then we replace each observation of the block (sample) by corresponding rank. This process is repeated for each block (sample).

In Kruskal-walls test, we rank all observations which are combined from all samples whereas in Friedman test we rank the observations within each sample or block.

Step 2: After that, we sum all the ranks for each treatment. Let R_1 be the sum of all ranks for 1st treatment, R_2 be the sum of all ranks for 2nd treatment and so on, R_n be the sum of all ranks for nth treatment.

Step 3: If null hypothesis is true, that is, all treatments have same effects then we expect that the sum of the ranks for each treatment would be approximately equal. The sum of squares of differences among the sums of the ranks will be indicative of the difference in the treatment effects. These sum of squares are measured by the Friedman and give in the simple form which is known as Friedman test statistic (F) and given by

$$F = \frac{12}{kn(n+1)}\sum_{j=1}^{n} R_j^2 - 3k(n+1)$$

The distribution of Friedman test statistic F is approximated by chi-square distribution with (n – 1) degrees of freedom. In order to the reasonably good approximation, we require that either number of blocks (k) or number of treatments (n) exceed 5.

If tie occurs within blocks, an adjustment in Friedman F-statistic has to be made. The adjustment factor is

$$C = 1 - \frac{1}{k\left(n^2 - 1\right)}\sum_{i=1}^{k}\left(t_i^3 - t_i\right)$$

where t_i is the number of tied observations in the ith block (sample). Therefore, adjusted Friedman statistic is

$$F_C = F / C$$

Step 4: Obtain critical value of Friedman test statistic at given level of significance under the condition that null hypothesis is true. **Table IX** in the Appendix at the end of this block provides critical values for 3 to 6 (k = 3 to 6) samples and each sample has n = 20 or fewer observations at α level of significance.

Step 5: **Decision rule:**

To take the decision about the null hypothesis, the calculated value of Friedman test statistic (computed in Step 3) is compared with the critical (tabulated) value (obtain in Step 4) at a given level of significance(α) under the condition that null hypothesis is true.

If calculated value of test statistic is greater than critical (tabulated) value at α level of significance then we reject the null hypothesis at α level of significance otherwise we do not reject the null hypothesis H_0.

For large samples

For large number of treatments or for large sample sizes the distribution of the test statistic under the null hypothesis is well approximated by the chi-square distribution with (n –1) degrees of freedom. So when number of samples and observations per sample are such that we cannot use **Table IX** then we use **χ^2-table** (given in the Appendix of the Block 1 of this course) for critical values of the test statistic.

Now, it is time to do some examples based on Friedman test.

Example 3: The table given below shows reaction time (in second) data from 6 objects (treatments) each of which was tested under three conditions I, II and III:

Condition \ Object	A	B	C	D	E	F
I	356	590	665	450	540	500
II	401	564	667	560	570	525
III	455	570	650	575	560	550

Apply the Friedman test to test, whether reaction time of each object is same or not at 5% level of significance?

Solution: Here, we want to test that reaction time of each object is same. Therefore, our claim is "reaction time of each object is same" and its complement is "reaction time of each object is not same". So we can take claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$H_0:$ Reaction time of each object is same

$H_1:$ Reaction time of each object is not same

Here, two factors (object and condition) are under study so we can use two-way ANOVA if the distributions of both factors follow normal distributions. But it is not the case so we go for Friedman test.

For testing the null hypothesis, we have Friedman test statistic as

$$F = \frac{12}{kn(n+1)}\sum_{j=1}^{n} R_j^2 - 3k(n+1) \qquad \ldots (3)$$

To perform the Friedman test, we rank the observations within each condition (sample or block) and then calculate the sum of ranks of each object (treatment) as:

Condition	**Object**					
	A	**B**	**C**	**D**	**E**	**F**
I	1	5	6	2	4	3
II	1	4	6	3	5	2
III	1	4	6	5	3	2
Sum	3	13	18	10	12	7

Here, k = 3 and n = 6, therefore, by putting the values in equation (3), we have

$$F = \frac{12}{3\times 6\times(6+1)}\left[3^2 + 13^2 + 18^2 + 10^2 + 12^2 + 7^2\right] - 3\times 3\times(6+1)$$

$$= \frac{2}{21}\times 795 - 63 = 75.71 - 63 = 12.71$$

The critical (tabulated) value of test statistic F corresponding k = 3 and n = 6 at 5% level of significance is 7.0.

Since calculated value of test statistic F (= 12.71) is greater than critical value (= 7.0) so we reject the null hypothesis i.e. we reject the claim at 5% level of significance.

Thus, we conclude that the samples provide us sufficient evidence against the claim so reaction time of each object is not same.

Example 4: An experiment is conducted to investigate the toxic effect of three chemicals A, B and C on the skin of rats. Three adjacent 1-inch squares are marked on the backs of eight rats and each of the three chemicals is applied to each rat. The squares of skin are the scored from 0 to 10 depending on the degree of imitation. The data are given in the following table:

Rat \ Chemical	A	B	C
1	6	5	3
2	9	8	4
3	6	9	6
4	5	8	6
5	7	8	9
6	5	7	6
7	6	7	5
8	6	5	7

Test there is sufficient evidence that effect of all chemicals is same by the Freidman test at 1% level of significance.

Solution: Here, we want to test that effect of all chemicals is same. So our claim is "effect of all chemicals is same" and its complement is "effect of all chemicals is not same". So we can take claim as the null hypothesis and complement as the alternative hypothesis. Thus,

H_0 : Effect of all chemicals is same

H_1 : Effect of all chemicals is not same

Here, two factors (chemical and rat) are under study, so we can use two-way ANOVA if the distributions of both factors follow normal distributions. But it is not the case, so we go for Friedman test.

For testing the null hypothesis, we have Friedman test statistic as

$$F = \frac{12}{kn(n+1)}\sum_{j=1}^{n} R_j^2 - 3k(n+1) \qquad \ldots (4)$$

To perform the Friedman test, we rank the observations within each rat (block) and then calculate the sum of ranks of each chemical (treatment) as:

Rat \ Chemical	A	B	C
1	3	2	1
2	3	2	1
3	1.5	3	1.5
4	1	3	2
5	1	2	3
6	1	3	2
7	2	3	1
8	2	1	3
Total	14.5	19	14.5

Here, k = 8 and n = 3, therefore, by putting the values in equation (4), we have

$$F = \frac{12}{8 \times 3 \times (3+1)}\left[(14.5)^2 + (19)^2 + (14.5)^2\right] - 3 \times 8 \times (3+1)$$

$$= \frac{1}{8}[210.25 + 361 + 210.25] - 96$$

$$= 97.69 - 96 = 1.69$$

Since tie occurs within 3rd rat (block) therefore, an adjustment in Friedman F statistic has to be made. The adjustment factor is

$$C = 1 - \frac{1}{k(n^2 - 1)}\sum_{i=1}^{k}(t_i^3 - t_i)$$

where, t_i is the number of tied observations in the i^{th} rat (sample or block).

Here, $t_1 = 2$ therefore,

$$C = 1 - \frac{(2^3 - 2)}{8(3^2 - 1)} = 1 - 0.094 = 0.91$$

Adjusted Friedman statistic is

$$F_C = F / C = 1.69 / 0.91 = 1.86$$

Since number of rats (samples or blocks) k = 8 > 6 so the distribution of the test statistic under the null hypothesis is well approximated by the chi-square distribution with n –1 = 2 – 1 = 1 degree of freedom. Thus, we compare calculated test statistic F_c with critical value given in χ^2-table.

The critical (tabulated) value of χ^2 with 1 degree of freedom at 1% level of significance is 6.63.

Since calculated value of test statistic F_c (= 1.86) is less than critical value (= 6.63) so we do not reject the null hypothesis i.e. we support the claim at 1% level of significance.

Thus, we conclude that samples do not provide us sufficient evidence against the claim so we may assume that effect of all chemicals on rats is same.

Now, you can try the following exercises.

E3) An experiment was conducted using randomised block design with four treatments and six blocks. The ranks of the measurements within each block are shown in the following table:

Block / Treatment	1	2	3	4	5	6
A	3	3	2	3	2	3
B	1	1	1	2	1	1
C	4	4	3	4	4	4
D	2	2	4	1	3	2

Use the Friedman test for a randomised block design to test that effect of all treatments is same at 5% level of significance.

E4) A random sample of 10 consumers asked to rank their preferences of four new smells that a perfume manufacturer wants to introduce to the market in the coming fall. The data are as follows where best is ranked by 1, second best by 2 and worst by 4.

Respondent	Smell I	Smell II	Smell III	Smell IV
1	1	2	4	3
2	1	3	4	2
3	1	3	2	4
4	2	1	3	4
5	1	3	4	2
6	1	3	2	4
7	2	1	4	3
8	1	3	4	2
9	1	3	2	4
10	1	4	3	2

Applying Freidman test, do you believe that all four smells are equally liked at 1% level of significance?

We now end this unit by giving a summary of what we have covered in it.

15.4 SUMMARY

In this unit, we have discussed following points:

1. Need of k-sample non-parametric tests.
2. The Kruskal-Wallis test for testing the hypothesis about equality of more than two population medians and is known as the non-parametric version of one-way ANOVA or completely randomised design.
3. Friedman test which is known as the non-parametric version of for randomised block design.

15.5 SOLUTIONS / ANSWERS

E1) Here, we want to test that diets are equally productive of milk. So our claim is "the diets are equally productive of milk" and its complement is "diets are not equally productive of milk". So we can take claim as the null hypothesis and complement as the alternative hypothesis. Thus,

H_0: The diets are equally productive of milk

H_1: The diets are not equally productive of milk

Here, the assumption of normality is not fulfilled and the data are given in the form of ranks so we cannot use one-way ANOVA. Therefore, we go for Kruskal-Wallis test.

To perform the Kruskal-Wallis test, the test statistic is given by

$$H = \frac{12}{n(n+1)} \sum_{i=1}^{k} \left(\frac{R_i^2}{n_i} \right) - 3(n+1) \qquad \ldots (5)$$

where, R_i is the sum of ranks of i^{th} feed.

Since data are given in the form of ranks so first we find the sum of rank (R_i) of each feed.

Calculation for R_i:

Feed A	Feed B	Feed C
2	7	6
5	1	10
4	8	12
9	3	11
$R_1 = 20$	$R_1 = 19$	$R_1 = 39$

Putting the values in equation (5), we have

$$H = \frac{12}{12(12+1)}\left[\frac{(20)^2}{4} + \frac{(19)^2}{4} + \frac{(39)^2}{4}\right] - 3(12+1)$$

$$= \frac{1}{13}\left(\frac{400}{4} + \frac{361}{4} + \frac{1521}{4}\right) - 39 \quad = 43.88 - 39 = 4.88$$

The critical (tabulated) value corresponding k = 3, $n_1 = 4$, $n_2 = 4$, $n_3 = 4$ at 1% level of significance is 7.654.

Since calculated value of test statistic H (= 4.88) is less than critical value (= 7.654) so we do not reject the null hypothesis i.e. we support the claim at 1% level of significance.

Thus, we conclude that the samples do not provide us sufficient evidence against the claim so we may assume that the diets are equally productive of milk.

E2) Here, we want to test that there is no difference in the three processes in terms of number of defective articles per batch, that is, three manufacturing processes are identical. Therefore, our claim is “three manufacturing processes are identical” and its complement is “three manufacturing processes are not identical”. So we can take claim as the null hypothesis and complement as the alternative hypothesis. Thus,

H_0: The three manufacturing processes are identical

H_1: The three manufacturing processes are not identical

Here, the assumption of normality is not fulfilled so we cannot use one-way ANOVA. Therefore, we go for Kruskal-Wallis test.

To perform the Kruskal-Wallis test, the test statistic is given by

$$H = \frac{12}{n(n+1)}\sum_{i=1}^{k}\left(\frac{R_i^2}{n_i}\right) - 3(n+1) \qquad \ldots (6)$$

where, R_i is the sum of ranks of i^{th} processes.

Therefore, we first rank all the number of defectives articals manufactured by three processes and then find the sum of ranks (R_i) of each process.

Calculation for R_i:

Process I	Rank	Process II	Rank	Process III	Rank
12	13	10	11	13	14
5	5	15	16	11	12
2	2.5	2	2.5	6	7
6	7	7	9	16	17
0	1	8	10	14	15
4	4	6	7	20	18
Total	$R_1 = 32.5$		$R_2 = 55.5$		$R_3 = 83$

Putting the values in equation (6), we have

$$H = \frac{12}{18(18+1)}\left(\frac{(32.5)^2}{6}+\frac{(55.5)^2}{6}+\frac{(83)^2}{6}\right)-3(18+1)$$

$$= \frac{12}{18\times19}\left(\frac{1056.25+3080.25+6889}{6}\right)-57$$

$$= \frac{12}{18\times19}\times\frac{11025.5}{6}-57 = 64.48-57 = 7.48$$

From the above calculations, we see that there are tied ranks therefore, we have

Number of groups of tied ranks = r = 2 (2.5 and 7)

Number of times group 2.5 of tied ranks occurs = $t_1(2.5) = 2$

Number of times group 7 of tied ranks occurs = $t_2(7) = 3$

Since ties occur so the correction factor is given by

$$C = 1-\frac{1}{(n^3-n)}\sum_{i=1}^{r}\left(t_i^3-t_i\right)$$

$$= 1-\frac{1}{(18)^3-18}\left[\left(2^3-2\right)+\left(3^2-3\right)\right]$$

$$= 1-\frac{1}{5814}\times30 = 0.995$$

Therefore, adjusted Kruskal-Wallis test statistic is

$$H_C = H/C = 7.48/0.995 = 7.52$$

Since all sample sizes exceed 5, therefore, the test statistic H_c is approximated by chi square with k−1 = 3−1 = 2 degrees of freedom. Thus, we compare calculated test statistic H_c with critical value given in **χ^2-table** with 2 degrees of freedom and 1% level of significance.

The critical (tabulated) value of χ^2 with 2 degrees of freedom at 1% level of significance is 9.21.

Since calculated value of test statistic H_c (= 7.52) is less than critical value (= 9.21) so we do not reject the null hypothesis i.e. we support the claim at 1% level of significance.

Thus, we conclude that the samples fail to provide us sufficient evidence against the claim so we may assume that three manufacturing process are identical.

E3) Here, we want to test that the effect of four treatments is same. Therefore, our claim is “effect of four treatments is same” and its complement is “effect of four treatments is not same”. So we can take claim as the null hypothesis and complement as the alternative hypothesis. Thus,

H_0 : Effect of four treatments is same

H_1 : Effect of four treatments is not same

Here, two factors (treatment and block) are under study so we can use two-way ANOVA if the distributions of both factors follow normal distributions but it is not the case. Also data are given in the form of ranks so we go for Friedman test.

For testing the null hypothesis, we have Friedman test statistic as

$$F = \frac{12}{kn(n+1)}\sum_{j=1}^{n} R_j^2 - 3k(n+1) \quad \dots (7)$$

In this exercise, ranks are given therefore to perform the Friedman test we calculate sum of ranks of each treatment as:

Treatment \ Block	1	2	3	4	5	6	Total
A	3	3	2	3	2	3	16
B	1	1	1	2	1	1	7
C	4	4	3	4	4	4	23
D	2	2	4	1	3	2	14

Here k = 6 and n = 4, therefore, by putting the values in equation (7), we have

$$F = \frac{12}{6\times4\times(4+1)}\left(16^2+7^2+23^2+14^2\right)-3\times6\times\left(4+1\right)$$

$$= \frac{1}{10}\left(256+49+529+196\right)-90 = 13$$

The critical value of test statistic F corresponding k = 6 and n = 4 at 5% level of significance is 10.29.

Since calculated value of test statistic F (= 13) is greater than critical value (= 10.29) so we reject the null hypothesis i.e. we reject the claim at 5% level of significance.

Thus, we conclude that data provide us sufficient evidence against the claim so effect of four treatments is not same.

E4) Here, we want to test that the four smells are equally liked. Therefore, our claim is "four smells are equally liked" and its complement is "four smells are not equally liked". So we can take claim as the null hypothesis and complement as the alternative hypothesis. Thus,

H_0: Four smells are equally liked

H_1: Four smells are not equally liked

Here, two factors (consumer and smell) are under study so we can use two-way ANOVA if the distributions of both factors follow normal distributions but it is not the case. Also data are given in the form of ranks so we go for Friedman test.

For testing the null hypothesis, we have Friedman test statistic as

$$F = \frac{12}{kn(n+1)}\sum_{j=1}^{n} R_j^2 - 3k(n+1) \quad \dots (8)$$

In this exercise, ranks are given therefore to perform the Friedman test we calculate sum of ranks of each smell as:

Respondent	Smell I	Smell II	Smell III	Smell IV
1	1	2	4	3
2	1	3	4	2
3	1	3	2	4
4	2	1	3	4
5	1	3	4	2
6	1	3	2	4
7	2	1	4	3
8	1	3	4	2
9	1	3	2	4
10	1	4	3	2
Total	12	26	32	30

Here k = 10 and n = 4 so by putting the values in equation (8), we have

$$F = \frac{12}{10 \times 4 \times (4+1)}\left(12^2 + 26^2 + 32^2 + 30^2\right) - 3 \times 10 \times \left(4+1\right)$$

$$= \frac{12}{200}\left(144 + 676 + 1024 + 900\right) - 150$$

$$= \frac{12}{200} \times 2744 - 150 = 14.64$$

Since k > 6, therefore, the Friedman test statistic F is approximated by chi square with n − 1 = 4 − 1 = 3 degrees of freedom. Thus, we compare calculated test statistic with critical value given in **χ^2-table** at 3 degrees of freedom and 1% level of significance.

The critical value of χ^2 with 3 degrees of freedom at 1% level of significance is 11.34.

Since calculated value of test statistic (= 14.64) is less than critical value (= 11.34) so we do not reject the null hypothesis i.e. we support the claim at 1% level of significance.

Thus, we conclude that the samples fail to provide us sufficient evidence against the claim so we may assume that four smells are equally liked.