UNIT 6 BIVARIATE DISCRETE RANDOM VARIABLES

Structure

6.1 INTRODUCTION

In Unit 5, you have studied one-dimensional random variables and their probability mass functions, density functions and distribution functions. There may also be situations where we have to study two-dimensional random variables in connection with a random experiment. For example, we may be interested in recording the number of boys and girls born in a hospital on a particular day. Here, 'the number of boys' and 'the number of girls' are random variables taking the values 0, 1, 2, … and both these random variables are discrete also.

In this unit, we concentrate on the two-dimensional discrete random variables defining them in Sec. 6.2. The joint, marginal and conditional probability mass functions of two-dimensional random variable are described in Sec. 6.3. The distribution function and the marginal distribution function are discussed in Sec. 6.4.

Objectives

A study of this unit would enable you to:

- define two-dimensional discrete random variable;
- specify the joint probability mass function of two discrete random variables;
- obtain the marginal and conditional distributions for two-dimensional discrete random variable;
- define two-dimensional distribution function;
- define the marginal distribution functions; and
- solve various practical problems on bivariate discrete random variables.

6.2 BIVARIATE DISCRETE RANDOM VARIABLES

In Unit 5, the concept of single-dimensional random variable has been studied in detail. Proceeding in analogy with the one-dimensional case, concept of two-dimensional discrete random variables is discussed in the present unit.

A situation where two-dimensional discrete random variable needs to be studied has already been given in Sec. 6.1 of this unit. To describe such situations mathematically, the study of two random variables is introduced.

Definition: Let X and Y be two discrete random variables defined on the sample space S of a random experiment then the function (X, Y) defined on the same sample space is called a two-dimensional discrete random variable. In others words, (X, Y) is a two-dimensional random variable if the possible values of (X, Y) are finite or countably infinite. Here, each value of X and Y is represented as a point (x, y) in the xy-plane.

As an illustration, let us consider the following example:

Let three balls b_1, b_2, b_3 be placed randomly in three cells. The possible outcomes of placing the three balls in three cells are shown in Table 6.1.

Table 6.1 : Possible Outcomes of Placing the Three Balls in Three Cells

Arrangement Number	Placement of the Balls in		
	Cell 1	Cell 2	Cell 3
1	b_1	b_2	b_3
2	b_1	b_3	b_2
3	b_2	b_1	b_3
4	b_2	b_3	b_1
5	b_3	b_1	b_2
6	b_3	b_2	b_1
7	b_1,b_2	b_3	-
8	b_1,b_2	-	b_3
9	-	b_1,b_2	b_3
10	b_1,b_3	b_2	-
11	b_1,b_3	-	b_2
12	-	b_1,b_3	b_2
13	b_2,b_3	b_1	-
14	b_2,b_3	-	b_1
15	-	b_2,b_3	b_1
16	b_1	b_2,b_3	-
17	b_1	-	b_2,b_3
18	-	b_1	b_2,b_3

19	b_2	b_3,b_1	-
20	b_2	-	b_3,b_1
21	-	b_2	b_3,b_1
22	b_3	b_1,b_2	-
23	b_3	-	b_1,b_2
24	-	b_3	b_1,b_2
25	b_1,b_2,b_3	-	-
26	-	b_1,b_2,b_3	-
27	-	-	b_1,b_2,b_3

Now, let X denote the number of balls in Cell 1 and Y be the number of cells occupied. Notice that X and Y are discrete random variables where X take on the values 0, 1, 2, 3 ($\because$ number of balls in Cell 1 may be 0 or 1 or 2 or 3) and Y take on the values 1, 2, 3 ($\because$ number of occupied cells may be 1 or 2 or 3). The possible values of two-dimensional random variable (X, Y), therefore, are all ordered pairs of the values x and y of X and Y, respectively, i.e. are (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3).

Now, to each possible value $\left(x_i, y_j\right)$ of (X, Y), we can associate a number $p\left(x_i, y_j\right)$ representing $P\left(X = x_i, Y = y_j\right)$ as discussed in the following section of this unit.

6.3 JOINT, MARGINAL AND CONDITIONAL PROBABILITY MASS FUNCTIONS

Let us again consider the example discussed in Sec. 6.2. In this example, we have obtained all possible values of (X, Y), where X is the number of balls in Cell 1 and Y be the number of occupied cells. Now, let us associate numbers $p\left(x_i, y_j\right)$ representing $P\left[X = x_i, Y = y_j\right]$ as follows:

p(0, 1) = P[X = 0, Y = 1] = P[no ball in Cell 1 and 1 cell occupied] = P[Arrangement numbers 26, 27] = $\frac{2}{27}$
p(0, 2) = P[X = 0, Y = 2] = P[no ball in Cell 1 and 2 cells occupied] = P[Arrangement numbers 9, 12, 15, 18, 21, 24] = $\frac{6}{27}$
p(0, 3) = P[X = 0, Y = 3] = P[no ball in Cell 1 and 3 cells occupied] = P[Impossible event] = 0
p(1, 1) = P[X = 1, Y = 1] = P[one ball in Cell 1 and 1 cell occupied] = P[Impossible event] = 0

p(1, 2) = P[X = 1, Y = 2] = P[one ball in Cell 1 and 2 cells occupied]   = P[Arrangement numbers 16, 17, 19, 20, 22, 23] $= \frac{6}{27}$
p(1, 3) = P[X = 1, Y = 3] = P[one ball in Cell 1 and 3 cells occupied]   = P[Arrangement numbers1 to 6] = $\frac{6}{27}$
p(2, 1) = P[X = 2, Y = 1] = P[two balls in Cell 1 and 1 cell occupied]   = P[Impossible event] = 0
p(2, 2) = P[X = 2, Y = 2] = P[two balls in Cell 1 and 2 cells occupied]   = P[Arrangement numbers 7, 8, 10, 11, 13, 14] $= \frac{6}{27}$
p(2, 3) = P[X = 2, Y = 3] = P[two balls in Cell 1 and 3 cells occupied]   = P[Impossible event] = 0
p(3, 1) = P[X = 3, Y = 1] = P[three balls in Cell 1 and 1 cell occupied]   = P[Arrangement number 25] = $\frac{1}{27}$
p(3, 2) = P[X = 3, Y = 2] = P[three balls in Cell 1 and 2 cells occupied]   = P[Impossible event] = 0
p(3, 3) = P[X = 3, Y = 3] = P[three balls in Cell 1 and 3 cells occupied]   = P[Impossible event] = 0

The values of (X, Y) together with the number associated as above constitute what is known as joint probability distribution of (X, Y) which can be written in the tabular form also as shown below:

X \ Y	1	2	3	Total
0	2/27	6/27	0	8/27
1	0	6/27	6/27	12/27
2	0	6/27	0	6/27
3	1/27	0	0	1/27
Total	3/27	18/27	6/27	1

We are now in a position to define joint, marginal and conditional probability mass functions.

Joint Probability Mass Function

Let (X, Y) be a two-dimensional discrete random variable. With each possible outcome $\left(x_i, y_j\right)$, we associate a number $p\left(x_i, y_j\right)$ representing $P\left[X = x_i, Y = y_j\right]$ or $P\left[X = x_i \cap Y = y_j\right]$ and satisfying following conditions:

(i) $p\left(x_i, y_j\right) \ge 0$

(ii) $\sum_i \sum_j p\left(x_i, y_j\right) = 1$

The function p defined for all $\left(x_i, y_j\right)$ is in analogy with one-dimensional case and called the **joint probability mass function of X and Y.** It is usually represented in the form of the table as shown in the example discussed above.

Marginal Probability Function

Let (X, Y) be a discrete two-dimensional random variable which take up finite or countably infinite values $\left(x_i, y_j\right)$. For each such two-dimensional random variable (X, Y), we may be interested in the probability distribution of X or the probability distribution of Y, individually.

Let us again consider the example of the random placement of three balls in there cells wherein X and Y are the discrete random variables representing "the number of balls in Cell 1" and "the number of occupied cells", respectively. Let us consider Table 6.1 showing the joint distribution of (X, Y). From this table, let us take up the row totals and column totals. The row totals in the table represent the probability distribution of X and the column totals represent the probability distribution of Y, individually. That is,

$$P[X = 0] = \frac{2}{27} + \frac{6}{27} + 0 = \frac{8}{27}$$

$$P[X = 1] = 0 + \frac{6}{27} + \frac{6}{27} = \frac{12}{27}$$

$$P[X = 2] = 0 + \frac{6}{27} + 0 = \frac{6}{27}$$

$$P[X = 3] = \frac{1}{27} + 0 + 0 = \frac{1}{27} \text{ and}$$

$$P[Y = 1] = \frac{2}{27} + 0 + 0 + \frac{1}{27} = \frac{3}{27}$$

$$P[Y = 2] = \frac{6}{27} + \frac{6}{27} + \frac{6}{27} + 0 = \frac{18}{27}$$

$$P[Y = 3] = 0 + \frac{6}{27} + 0 + 0 = \frac{6}{27}$$

These distributions of X and Y, individually, are called the **marginal probability distributions** of X and Y, respectively.

So, if (X, Y) is a discrete two-dimensional random variable which take up the values $\left(x_i, y_j\right)$, then the probability distribution of X is determined as follows:

$$p\left(x_i\right) = P\left[X = x_i\right]$$

$$= P\left[\left(X = x_i \cap Y = y_1\right) \text{or } \left(X = x_i \cap Y = y_2\right) \text{or} \ldots\right]$$

$$= P\left[X = x_i \cap Y = y_1\right] + P\left[X = x_i \cap Y = y_2\right] + P\left[X = x_i \cap Y = y_3\right] + \ldots$$

$$= \sum_j P\left[X = x_i \cap Y = y_j\right]$$

$$= \sum_j p\left(x_i, y_j\right) \left[\begin{array}{l} \because\ p\left(x_i, y_j\right), \text{the joint probability mass} \\ \text{function, is } P\left[X = x_i \cap Y = y_j\right] \end{array}\right]$$

which is known as the marginal probability mass function of X. Similarly, the probability distribution of Y is

$$p\left(y_j\right) = P\left[Y = y_j\right]$$

$$= P\left[X = x_1 \cap Y = y_j\right] + P\left[X = x_2 \cap Y = y_j\right] + \ldots$$

$$= \sum_i P\left[X = x_i \cap Y = y_j\right]$$

$$= \sum_i p\left(x_i, y_j\right)$$

and is known as the marginal probability mass function of Y.

Conditional Probability Mass Function

Let (X, Y) be a discrete two-dimensional random variable. Then the conditional probability mass function of X, given $Y = y$ is defined as

$$p\left(x \mid y\right) = P[X = x \mid Y = y]$$

$$= \frac{P\left[X = x \cap Y = y\right]}{P\left[Y = y\right]}, \text{ provided } P[Y = y] \neq 0$$

$$\left[\because P\left[A \mid B\right] = \frac{P\left[A \cap B\right]}{P\left[B\right]}, P\left(B\right) \neq 0\right]$$

Similarly, the conditional probability mass function of Y, given X = x, is defined as

$$p\left(y \mid x\right) = P\left[Y = y \mid X = x\right] = \frac{P\left[Y = y \cap X = x\right]}{P\left[X = x\right]}$$

Let us again consider the example as already discussed in this section. Suppose, we are interested in finding the conditional probability mass function of X given Y = 2. Then the conditional probabilities are found separately for each value of X given Y = 2. That is, we proceed as follows:

$$P\left[X=0\,|\,Y=2\right]=\frac{P\left[X=0\cap Y=2\right]}{P\left[Y=2\right]}=\frac{\frac{6}{27}}{\frac{18}{27}}=\frac{1}{3}$$

$$P\left[X=1\,|\,Y=2\right]=\frac{P\left[X=1\cap Y=2\right]}{P\left[Y=2\right]}=\frac{\frac{6}{27}}{\frac{18}{27}}=\frac{1}{3}$$

$$P\left[X=2\,|\,Y=2\right]=\frac{P\left[X=2\cap Y=2\right]}{P\left[Y=2\right]}=\frac{\frac{6}{27}}{\frac{18}{27}}=\frac{1}{3}$$

$$P\left[X=3\,|\,Y=2\right]=\frac{P\left[X=3\cap Y=2\right]}{P\left[Y=2\right]}=\frac{0}{\frac{18}{27}}=0$$

[Note that values of numerator and denominator in the above expressions have already been obtained while discussing the joint and marginal probability mass functions in this section of the unit.]

Independence of Random Variables

Two discrete random variables X and Y are said to be independent if and only if

$$P\left[X=x_i\cap Y=y_j\right]=P\left[X=x_i\right]P\left[Y=y_j\right]$$

[$\because$ two events A and B are independent if and only if P(A $\cap$ B) = P(A) P(B)]

Example 1: The following table represents the joint probability distribution of the discrete random variable (X, Y):

X \ Y	1	2
1	0.1	0.2
2	0.1	0.3
3	0.2	0.1

Find **:**

i) The marginal distributions.

ii) The conditional distribution of X given Y = 1.

iii) P[(X + Y) < 4].

Solution:

i) To find the marginal distributions, we have to find the marginal totals, i.e. row totals and column totals as shown in the following table:

Y / X	1	2	$p(x)$ (Totals)
1	0.1	0.2	0.3
2	0.1	0.3	0.4
3	0.2	0.1	0.3
$p(y)$ (Totals)	0.4	0.6	1

Thus, the marginal probability distribution of X is

X	1	2	3
p(x)	0.3	0.4	0.3

and the marginal probability distribution of Y is

Y	1	2
P(y)	0.4	0.6

ii) As $P[X=1|Y=1]=\dfrac{P[X=1,Y=1]}{P[Y=1]}=\dfrac{0.1}{0.4}=\dfrac{1}{4}$,

$$P[X=2|Y=1]=\frac{P[X=2,Y=1]}{P[Y=1]}=\frac{0.1}{0.4}=\frac{1}{4} \text{ and}$$

$$P[X=3|Y=1]=\frac{P[X=3\cap Y=1]}{P[Y=1]}=\frac{0.2}{0.4}=\frac{1}{2},$$

$\therefore$ The conditional distribution of X given Y = 1 is

X	1	2	3
$P[X=x\,|\,Y=1]$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

(iii) As the values of (X, Y) which satisfy X + Y < 4 are (1, 1), (1, 2) and (2, 1) only.

$$\therefore P[(X+Y)<4]=P[X=1,Y=1]+P[X=1,Y=2]+P[X=2,Y=1]$$

$$=0.1+0.2+0.1=0.4$$

Example 2: Two discrete random variables X and Y have

P[X = 0, Y = 0] $=\dfrac{2}{9}$, P[X = 0, Y = 1] = $\dfrac{1}{9}$, $P[X=1,Y=0]=\dfrac{1}{9}$, and

$P[X=1, Y=1]=\frac{5}{9}$. Examine whether X and Y are independent?

Solution: Writing the given distribution in tabular form as follows:

X \ Y	0	1	p(x)
0	2/9	1/9	3/9
1	1/9	5/9	6/9
p(y)	3/9	6/9	1

$\therefore\ P[X=0]=\frac{3}{9},\ P[X=1]=\frac{6}{9},$

$P[Y=0]=\frac{3}{9},\ P[Y=1]=\frac{6}{9}$

Now $P[X=0]P[Y=0]=\frac{3}{9}\times\frac{3}{9}=\frac{1}{9}$

But $P[X=0, Y=0]=\frac{2}{9}$

$\therefore\ P[X=0, Y=0]\neq P[X=0]P[Y=0]$

Hence X and Y are not independent

[**Note**: If $P[X=x, Y=y]=P[X=x]P[Y=y]$ for each possible value of X and Y, only then X and Y are independent.]

Here are two exercises for you.

E1) The joint probability distribution of a pair of random variables is given by the following table:

X \ Y	1	2	3
1	1/12	0	1/18
2	1/6	1/9	1/4
3	0	1/5	2/15

i) Evaluate marginal distribution of X.

ii) Evaluate conditional distribution of Y given X = 2

iii) Obtain $P[X+Y<5]$.

E2) For the following joint probability distribution of (X, Y),

X \ Y	1	2	3
1	$1/20$	$1/10$	$1/10$
2	$1/20$	$1/10$	$1/10$
3	$1/10$	$1/10$	$1/20$
4	$1/10$	$1/10$	$1/20$

i) find the probability that Y = 2 given that X = 4,

ii) find the probability that Y = 2, and

iii) examine if the two events X = 4 and Y = 2 are independent.

6.4 JOINT AND MARGINAL DISTRIBUTION FUNCTIONS FOR DISCRETE RANDOM VARIABLES

Two-Dimensional Joint Distribution Function

In analogy with the distribution function $F(x) = P[X \le x]$ of one-dimensional random variable X discussed in Unit 5 of this course, the distribution function of the two-dimensional random variable (X, Y) for all real x and y is defined as

$$F(x, y) = P[X \le x, Y \le y]$$

Marginal Distribution Functions

Let (X, Y) be a two-dimensional discrete random variable having $F(x, y)$ as its distribution function. Now the **marginal distribution function of X** is defined as

$$\begin{aligned} F(x) &= P[X \le x] \\ &= P[X \le x, Y = y_1] + P[X \le x, Y = y_2] + \ldots \\ &= \sum_j P[X \le x, Y = y_j] \end{aligned}$$

Similarly, the **marginal distribution function of Y** is defined as

$$\begin{aligned} F(y) &= P[Y \le y] \\ &= P[X = x_1, Y \le y] + P[X = x_2, Y \le y] + \ldots \\ &= \sum_i P[X = x_i, Y \le y] \end{aligned}$$

Example 3: Considering the probability distribution function given in Example 1, find

i) F(2, 2), F(3,2)

ii) $F_X(3)$

iii) $F_Y(1)$

Solution:

i) $F(2,2) = P[X \le 2, Y \le 2]$

$= P[X = 2, Y \le 2] + P[X = 1, Y \le 2]$

$= P[X = 2, Y = 2] + P[X = 2, Y = 1] + P[X = 1, Y = 2]$

$+ P[X = 1, Y = 1]$

= 0.3 + 0.1 + 0.2 + 0.1 = 0.7

$F(3,2) = P[X \le 3, Y \le 2]$

$= P[X \le 2, Y \le 2] + P[X = 3, Y \le 2]$

$= 0.7 + P[X = 3, Y \le 2]$ $\left[\begin{array}{l}\because \text{first term on R.H.S. has been} \\ \text{obtained in part (i) of this example}\end{array}\right]$

$= 0.7 + P[X = 3, Y = 2] + P[X = 3, Y = 1] = 0.7 + 0.1 + 0.2 = 1$

ii) $F_X(3) = P[X \le 3]$

$= P[X \le 3, Y = 1] + P[X \le 3, Y = 2]$

$= P[X = 3, Y = 1] + P[X = 2, Y = 1] + P[X = 1, Y = 1]$

$+ P[X = 3, Y = 2] + P[X = 2, Y = 2] + P[X = 1, Y = 2]$

= 0.2 + 0.1 + 0.1 +0.1+ 0.3 + 0.2 = 1

iii) $F_Y(1) = P[Y \le 1]$

$= P[X = 1, Y \le 1] + P[X = 2, Y \le 1] + P[X = 3, Y \le 1]$

$= P[X = 1, Y = 1] + P[X = 2, Y = 1] + P[X = 3, Y = 1]$

= 0.1 + 0.1 + 0.2 = 0.4

Example 4: Find the joint and marginal distribution functions for the joint probability distribution given in Example 2.

Solution: For the joint distribution function, we have to find $F(x, y) = P[X \le x, Y \le y]$ for each x and y, i.e. we are to find F(0,0), F(0,1), F(1,0), F(1,1).

$$F(0,0) = P[X \le 0, Y \le 0] = P[X = 0, Y = 0] = \frac{2}{9}$$

$$F(0,1) = P[X \le 0, Y \le 1] = P[X = 0, Y = 0] + P[X = 0, Y = 1]$$

$$= \frac{2}{9} + \frac{1}{9} = \frac{3}{9}$$

$$F(1,0) = P[X \le 1, Y \le 0] = P[X = 1, Y = 0] + P[X = 0, Y = 0]$$

$$= \frac{1}{9} + \frac{2}{9} = \frac{3}{9}$$

$$F(1,1) = P[X \le 1, Y \le 1] = P[X = 1, Y = 1] + P[X = 1, Y = 0]$$

$$+ P[X = 0, Y = 1] + P[X = 0, Y = 0]$$

$$= \frac{5}{9} + \frac{1}{9} + \frac{1}{9} + \frac{2}{9} = 1$$

Above distribution function $F(x, y)$ can be shown in the tabular form as follows:

	$Y \le 0$	$Y \le 1$
$X \le 0$	$2/9$	$3/9$
$X \le 1$	$3/9$	1

Marginal distribution function of X is obtained on finding $F(x) = P[X \le x]$ for each x, i.e. we have to obtain $F_X(0)$, $F_X(1)$.

$$F_X(0) = P[X \le 0] = P[X = 0]$$

$$= P[X = 0, Y = 0] + P[X = 0, Y = 1]$$

$$= \frac{2}{9} + \frac{1}{9} = \frac{3}{9}$$

$$F_X(1) = P[X \le 1] = P[X \le 1, Y = 0] + P[X \le 1, Y = 1]$$

$$= P[X = 1, Y = 0] + P[X = 0, Y = 0]$$

$$+ P[X = 1, Y = 1] + P[X = 0, Y = 1]$$

$$= \frac{1}{9} + \frac{2}{9} + \frac{5}{9} + \frac{1}{9} = 1$$

$\therefore$ marginal distribution function of X is given as

X	F(x)
≤ 0	$3/9$
≤ 1	1

Similarly, marginal distribution function of Y can be obtained. [Do it yourself]

Here is an exercise for you.

E3) Obtain the joint and marginal distribution functions for the joint probability distribution given in **E 1).**

Now before ending this unit, let us summarizes what we have covered in it.

6.5 SUMMARY

In this unit we have covered the following main points:

1) If X and Y be two discrete random variables defined on the sample space S of a random experiment then the function (X, Y) defined on the same sample space is called a **two-dimensional discrete random variable**. In others words, (X, Y) is a two-dimensional random variable if the possible values of (X, Y) are finite or countably infinite.

2) A number $p(x_i, y_j)$ associated with each possible outcome (x_i, y_j) of a two-dimensional discrete random variable (X, Y) is called the **joint probability mass function of X and Y** if it satisfies the following conditions:

   (i) $p(x_i, y_j) \geq 0$

   (ii) $\sum_i \sum_j p(x_i, y_j) = 1$

3) If (X,Y) is a discrete two-dimensional random variable which take up the values (x_i, y_j), then the probability distribution of X given by $p(x_i) = \sum_j p(x_i, y_j)$ is known as the **marginal probability mass function of X** and the probability distribution of Y given by $p(y_j) = \sum_i p(x_i, y_j)$ is known as the **marginal probability mass function of Y**.

4) The **conditional probability mass function of X given** $Y = y$ in case of a two-dimensional discrete random variable (X, Y) is defined as

$$p(x \mid y) = P[X = x \mid Y = y]$$

$$= \frac{P[X = x \cap Y = y]}{P[Y = y]}; \text{ and}$$

   the **conditional probability mass function of Y, given X = x** is defined as

$$p(y \mid x) = P[Y = y \mid X = x]$$

$$= \frac{P[Y = y \cap X = x]}{P[X = x]}$$

5) Two discrete random variables X and Y are said to be independent if and only if

$$P[X = x_i \cap Y = y_j] = P[X = x_i]P[Y = y_j]$$

6.6 SOLUTIONS/ANSWERS

E1) Let us compute the marginal totals. Thus, the complete table with marginal totals is given as

Y / X	1	2	3	p(x)
1	$\frac{1}{12}$	0	$\frac{1}{18}$	$\frac{1}{12}+0+\frac{1}{18}=\frac{5}{36}$
2	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{4}$	$\frac{1}{6}+\frac{1}{9}+\frac{1}{4}=\frac{19}{36}$
3	0	$\frac{1}{5}$	$\frac{2}{15}$	$0+\frac{1}{5}+\frac{2}{15}=\frac{1}{3}$
p(y)	$\frac{1}{4}$	$\frac{14}{45}$	$\frac{79}{180}$	1

Therefore,

i) Marginal distribution of X is

X	p(x)
1	5/36
2	19/36
3	1/3

ii) $$P[Y=1\,|\,X=2]=\frac{P[Y=1,X=2]}{P[X=2]}=\frac{1}{6}\times\frac{36}{19}=\frac{6}{19}$$

$$P[Y=2\,|\,X=2]=\frac{P[Y=2,X=2]}{P[X=2]}=\frac{1}{9}\times\frac{36}{19}=\frac{4}{19}$$

$$P[Y=3\,|\,X=2]=\frac{P[Y=3,X=2]}{P[X=2]}=\frac{1}{4}\times\frac{36}{19}=\frac{9}{19}$$

$\therefore$ The conditional distribution of Y given X = 2 is

Y	$P[Y=y\,|\,X=2]$
1	6/19
2	4/19
3	9/19

iii) $P[X+Y<5]$

$$=P[X=1, Y=1]+P[X=1, Y=2]+P[X=1, Y=3]$$

$$+P[X=2, Y=1]+P[X=2, Y=2]+P[X=3, Y=1]$$

$$=\frac{1}{12}+0+\frac{1}{18}+\frac{1}{6}+\frac{1}{9}+0=\frac{15}{36}.$$

E2) First compute the marginal totals, then you will be able to find

i) $P[X=4]=\frac{1}{4}$, and hence

$$P[Y=2 \mid X=4]=\frac{P[Y=2, X=4]}{P[X=4]}=\frac{2}{5}$$

ii) $P[Y=2]=\frac{2}{5}$

iii) $P[X=4, Y=2]=\frac{1}{10}$, $P[X=4]=\frac{1}{4}$, $P[Y=2]=\frac{2}{5}$

$$P[X=4]\,P[Y=2]=\frac{1}{4}\times\frac{2}{5}=\frac{1}{10}$$

$\therefore$ X= 4 and Y= 2 are independent

E3) To obtain joint distribution function $F(x, y)=P[X \leq x, Y \leq y]$, we have to obtain

$F(x, y)$ for each value of X and Y, i.e. we have to obtain

$F(1,1), F(1,2), F(1,3), F(2,1), F(2,2), F(2,3), F(3,1), F(3,2), F(3,3)$.

Then, the distribution function in tabular form is

	$Y \leq 1$	$Y \leq 2$	$Y \leq 3$
$X \leq 1$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{5}{36}$
$X \leq 2$	$\frac{1}{4}$	$\frac{13}{36}$	$\frac{2}{3}$
$X \leq 3$	$\frac{1}{4}$	$\frac{101}{180}$	1

Marginal distribution function of X is given as

X	F(x)
≤ 1	5/36
≤ 2	2/3
≤ 3	1

Marginal distribution function of Y is

Y	F(y)
≤ 1	1/4
≤ 2	101/180
≤ 3	1