UNIT 2 DIFFERENT APPROACHES TO PROBABILITY THEORY

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2.1 INTRODUCTION

In the previous unit, we have defined the classical probability. There are some restrictions in order to use it such as the outcomes must be equally likely and finite. There are many situations where such conditions are not satisfied and hence classical definition cannot be applied. In such a situation, we need some other approaches to compute the probabilities.

Thus, in this unit, we will discuss different approaches to evaluate the probability of a given situation based on past experience or own experience or based on observed data. Actually classical definition is based on the theoretical assumptions and in this unit, our approach to evaluate the probability of an event is different from theoretical assumptions and will put you in a position to answer those questions related to probability where classical definition does not work. The unit discusses the relative frequency (statistical or empirical probability) and the subjective approaches to probability. These approaches, however, share the same basic axioms which provide us with the unified approach to probability known as axiomatic approach. So, the axiomatic approach will also be discussed in the unit.

Objectives

After studying this unit, you should be able to:

- explain the relative frequency approach and statistical(or empirical) probability;
- discuss subjective approach to probability; and
- discuss axiomatic approach to probability.

2.2 RELATIVE FREQUENCY APPROACH AND STATISTICAL PROBABILITY

Classical definition of probability fails if

i) the possible outcomes of the random experiment are not equally likely or/and

ii) the number of exhaustive cases is infinite.

In such cases, we obtain the probability by observing the data. This approach to probability is called the relative frequency approach and it defines the statistical probability. Before defining the statistical probability, let us consider the following example:

Following table gives a distribution of daily salary of some employees:

Salary per day (In Rs)	Below 100	100-150	150-200	200 and above
Employees	**20**	**40**	**50**	**15**

If an individual is selected at random from the above group of employees and we are interested in finding the probability that his/her salary was under Rs. 150, then as the number of employees having salary less than Rs 150 is 20 + 40 = 60 and the total number employees is 20 + 40 + 50 + 15 = 125, therefore the relative frequency that the employee gets salary less than Rs. 150 is

$$\frac{60}{125} = \frac{12}{25}.$$

This relative frequency is nothing but the probability that the individual selected is getting the salary less than Rs. 150.

So, in general, if X is a variable having the values $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n$, respectively. Then

$$\frac{f_1}{\sum_{i=1}^{n} f_i}, \frac{f_2}{\sum_{i=1}^{n} f_i}, \ldots, \frac{f_n}{\sum_{i=1}^{n} f_i}$$

are the relative frequencies of $x_1, x_2, \ldots, x_n$ respectively and hence the probabilities of X taking the values $x_1, x, \ldots, x_n$ respectively.

But, in the above example the probability has been obtained using the similar concept as that of classical probability.

Now, let us consider a situation where a person is administered a sleeping pill and we are interested in finding the probability that the pill puts the person to sleep in 20 minutes. Here, we cannot say that the pill will be equally effective for all persons and hence we cannot apply classical definition here.

To find the required probability in this case, we should either have the past data or in the absence of the past data, we have to undertake an experiment where we administer the pill on a group of persons to check the effect. Let m be the number of persons to whom the pill put to sleep in 20 minutes and n be the total number of persons who were administered the pill.

Then, the relative frequency and hence the probability that a particular person will put to sleep in 20 minutes is $\frac{m}{n}$. But, this measure will serve as probability only if the total number of trials in the experiment is very large.

In the relative frequency approach, as the probability is obtained by repetitive empirical observations, it is known as statistical or empirical probability.

Statistical (or Empirical) Probability

If an event A (say) happens m times in n trials of an experiment which is performed repeatedly under essentially homogeneous and identical conditions (e.g. if we perform an experiment of tossing a coin in a room, then it must be performed in the same room and all other conditions for tossing the coin should also be identical and homogeneous in all the tosses), then the probability of happening A is defined as:

$$P(A) = \lim_{n\to\infty} \frac{m}{n}.$$

As an illustration, we tossed a coin 200 times and observed the number of heads. After each toss, proportion of heads i.e. $\frac{m}{n}$ was obtained, where m is the number of heads and n is the number of tosses as shown in the following table (Table 2.1):

Table 2.1: Table Showing Number of Tosses and Proportion of Heads

n (Number of Tosses)	m (Number of Heads)	Proportion of Heads i.e.P(H)=m/n
1	1	1
2	2	1
3	2	0.666667
4	3	0.75
5	4	0.8
6	4	0.666667
7	4	0.571429
8	5	0.625
9	6	0.666667
10	6	0.6
15	10	0.666667
20	12	0.6
25	14	0.56
30	16	0.533333
35	18	0.514286

40	22	0.55
45	25	0.555556
50	29	0.58
60	33	0.55
70	41	0.585714
80	46	0.575
90	52	0.577778
100	53	0.53
120	66	0.55
140	72	0.514286
160	82	0.5125
180	92	0.511111
200	105	0.525

Then a graph was plotted taking number of tosses (n) on x-axis and proportion of heads $\left(\frac{m}{n}\right)$ on y-axis as shown in Fig. 2.1.

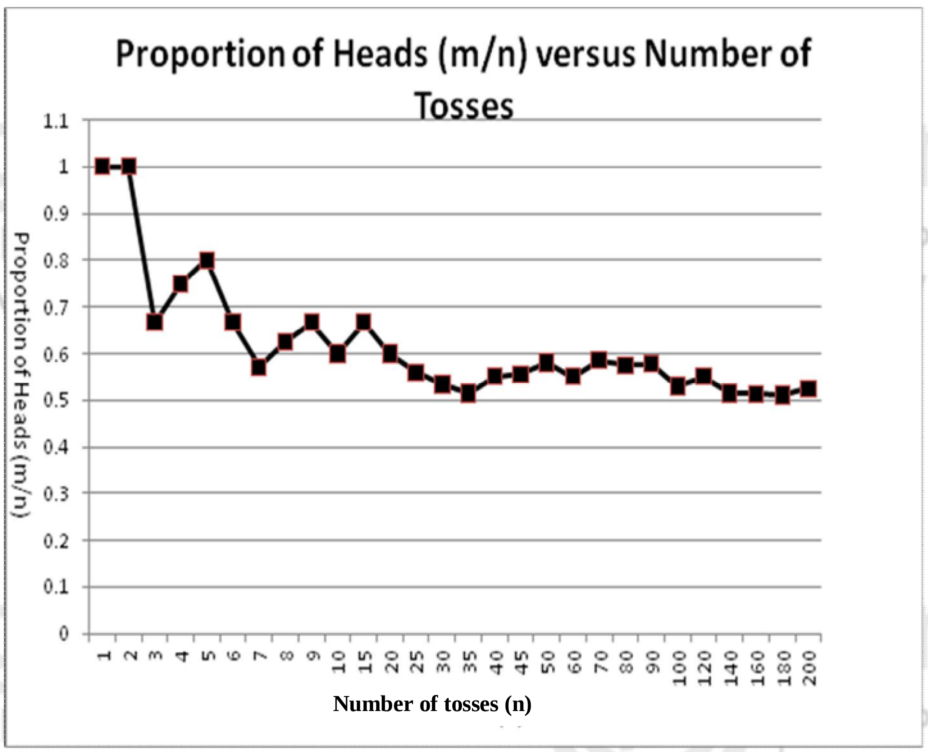


Fig. 2.1: **Proportion of Heads versus Number of Tosses**

The Graph reveals that as we go on increasing n,

$\frac{m}{n}$ tends to $\frac{1}{2}$

i.e. $\lim_{n\to\infty} \frac{m}{n} = \frac{1}{2}$

Hence, by the statistical (or empirical) definition of probability, the probability of getting head is

$$\lim_{n\to\infty} \frac{m}{n} = \frac{1}{2}.$$

Statistical probability has the following limitations:

(i) The experimental condition may get altered if it is repeated a large number of times.

(ii) $\lim_{n\to\infty} \frac{m}{n}$ may not have a unique value, however large n may be.

2.3 PROBLEMS BASED ON RELATIVE FREQUENCY

Example 1: The following data relate to 100 couples

Age of wife / Age of Husband	10-20	20-30	30-40	40-50	50-60
15-25	6	3	0	0	0
25-35	3	16	10	0	0
35-45	0	10	15	7	0
45-55	0	0	7	10	4
55-65	0	0	0	4	5

(i) Find the probability of a couple selected at random has a “age of wife” in the interval 20-50.

(ii) What is the probability that the age of wife is in the interval 20-40 and the age of husband is in the interval 35-45 if a couple selected at random?

Solution: (i) Required probability is given by

$$= \frac{(3+16+10+0+0)+(0+10+15+7+0)+(0+0+7+10+4)}{100}$$

$$= \frac{82}{100} = 0.82$$

(ii) Required probabilty $= \frac{10+15}{100} = \frac{25}{100} = 0.25$

Example 2: A class has 15 students whose ages are 14, 17, 15, 21, 19, 20, 16, 18, 20, 17, 14, 17, 16, 19 and 20 years respectively. One student is chosen at random and the age of the selected student is recorded. What is the probability that

(i) the age of the selected student is divisible by 3,

(ii) the age of the selected student is more than 16, and

(iii) the selected student is eligible to pole the vote.

Solution:

Age X	Frequency f	Relative frequency
14	2	2/15
15	1	1/15
16	2	2/15
17	3	3/15
18	1	1/15
19	2	2/15
20	3	3/15
21	1	1/15

(i) The age divisible by 3 is 15 or 18 or 21.

$$\therefore \text{ Required Probability} = \frac{1+1+1}{15} = \frac{3}{15} = \frac{1}{5}$$

(ii) Age more than 16 means, age may be 17, 18, 19, 20, 21

$$\therefore \text{ Required Probability} = \frac{3+1+2+3+1}{15} = \frac{10}{15} = \frac{2}{3}$$

(iii) In order to poll the vote, age must be ≥ 18 years. Thus, we are to obtain the probability that the selected student has age 18 or 19 or 20 or 21.

$$\therefore \text{ Required Probability} = \frac{1+2+3+1}{15} = \frac{7}{15}$$

Example 3: A tyre manufacturing company kept a record of the distance covered before a tyre needed to be replaced. The following table shows the results of 2000 cases.

Distance (in km)	Less than 4000	4001-10000	100001-20000	20001-40000	More than 40000
Frequency	20	100	200	1500	180

If a person buys a tyre of this company then find the probability that before the need of its replacement, it has covered

(i) at least a distance of 4001 km.

(ii) at most a distance 20000 km

(iii) more than a distance 20000 km

(iv) a distance between 10000 to 40000

Solution: The record is based on 2000 cases,

$\therefore$ Exhaustive cases in each case = 2000

(i) Out of 2000 cases, the number of cases in which tyre covered at least 4001 km

= 100 + 200 + 1500 + 180 = 1980

$$\therefore \text{Required Probability} = \frac{1980}{2000} = \frac{198}{200} = \frac{99}{100}$$

(ii) Number of cases in which distance covered by tyres of this company is at most 20000 km = 20 + 100 + 200 = 320

$$\therefore \text{Required Probability} = \frac{320}{2000} = \frac{32}{200} = \frac{4}{25}$$

(iii) Number of cases in which tyres of this company covers a distance of more than 20000 = 1500 + 180 = 1680

$$\therefore \text{Required Probability} = \frac{1680}{2000} = \frac{168}{200} = \frac{21}{25}$$

(iv) Number of cases in which tyres of this company covered a distance between 10000 to 40000 = 200 + 1500 = 1700

$$\therefore \text{Required Probability} = \frac{1700}{2000} = \frac{17}{20}$$

Now, you can try the following exercises.

E 1) An insurance company selected 5000 drivers from a city at random in order to find a relationship between age and accidents. The following table shows the results related to these 5000 drivers.

Age of driver (in years)	Accidents in one year				
Class interval	0	1	2	3	4 or more
18-25	600	260	185	90	70
25-40	900	240	160	85	65
40-50	1000	195	150	70	50
50 and above	500	170	120	60	30

If a driver from the city is selected at random, find the probability of the following events:

(i) Age lying between 18-25 and meet 2 accidents

(ii) Age between 25-50 and meet at least 3 accidents

(iii) Age more than 40 years and meet at most one accident

(iv) Having one accident in the year

(v) Having no accident in the year.

E 2) Past experience of 200 consecutive days speaks that weather forecasts of a station is 120 times correct. A day is selected at random of the year, find the probability that

(i) weather forecast on this day is correct

(ii) weather forecast on this day is false

E 3) Throw a die 200 times and find the probability of getting the odd number using statistical definition of probability.

2.4 SUBJECTIVE APPROACH TO PROBABILITY

In this approach, we try to assess the probability from our own experiences. This approach is applicable in the situations where the events do not occur at all or occur only once or cannot be performed repeatedly under the same conditions. Subjective probability is based on one's judgment, wisdom, intuition and expertise. It is interpreted as a measure of degree of belief or as the quantified judgment of a particular individual. For example, a teacher may express his /her confidence that the probability for a particular student getting first position in a test is 0.99 and that for a particular student getting failed in the test is 0.05. It is based on his personal belief.

You may notice here that since the assessment is purely subjective one, it will vary from person to person, depending on one's perception of the situation and past experience. Even when two persons have the same knowledge about the past, their assessment of probabilities may differ according to their personal prejudices and biases.

2.5 AXIOMATIC APPROACH TO PROBABILITY

All the approaches i.e. classical approach, relative frequency approach (Statistical/Empirical probability) and subjective approach share the same basic axioms. These axioms are fundamental to the probability and provide us with unified approach to probability i.e. axiomatic approach to probability. It defines the probability function as follows:

Let S be a sample space for a random experiment and A be an event which is subset of S, then P(A) is called probability function if it satisfies the following axioms

(i) $P(A)$ is real and $P(A) \geq 0$

(ii) P (S) = 1

(iii) If $A_1, A_2, \ldots$ is any finite or infinite sequence of disjoint events in S, then

$$P(A_1 \text{ or } A_2 \text{ or} \ldots \text{or } A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

Now, let us give some results using probability function. But before taking up these results, we discuss some statements with their meanings in terms of set theory. If A and B are two events, then in terms of set theory, we write

i) 'At least one of the events A or B occurs' as $A \cup B$

ii) 'Both the events A and B occurs' as $A \cap B$

iii) 'Neither A nor B occurs' as $\bar{A} \cap \bar{B}$

iv) 'Event A occurs and B does not occur' as $A \cap \bar{B}$

v) 'Exactly one of the events A or B occurs' as $(\bar{A} \cap B) \cup (A \cap \bar{B})$

vi) 'Not more than one of the events A or B occurs' as

$$\left(A \cap \bar{B}\right) \cup \left(\bar{A} \cap B\right) \cup \left(\bar{A} \cap \bar{B}\right).$$

Similarly, you can write the meanings in terms of set theory for such statement in case of three or more events e.g. in case of three events A, B and C, happening of at least one of the events is written as $A \cup B \cup C$.

2.6 SOME RESULTS USING PROBABILITY FUNCTION

1 Prove that probability of the impossible event is zero

Proof: Let S be the sample space and ϕ be the set of impossible event.

$\therefore \; S \cup \phi = S$

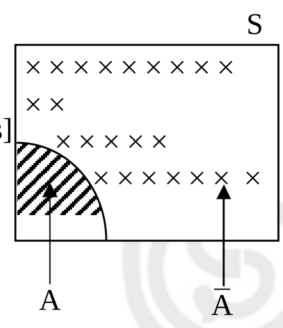
$\Rightarrow P(S \cup \phi) = P(S)$

$\Rightarrow P(S) + P(\phi) = P(S)$ [By axiom (iii)]

$\Rightarrow 1 + P(\phi) = 1$ [By axiom (ii)]

$\Rightarrow P(\phi) = 0$

2 Probability of non-happening of an event A i.e. complementary event $\bar{A}$ of A is given by $P(\bar{A}) = 1 - P(A)$



Proof: If S is the sample space then

$A \cup \bar{A} = S$ [$\because$ A and $\bar{A}$ are mutually disjoint events]

$\Rightarrow P(A \cup \bar{A}) = P(S)$

$\Rightarrow P(A) + P(\bar{A}) = P(S)$ [Using axiom (iii)]

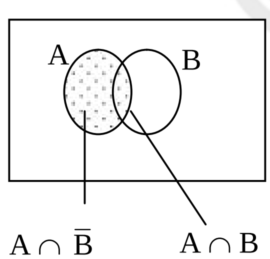
$= 1$ [Using axiom (ii)]

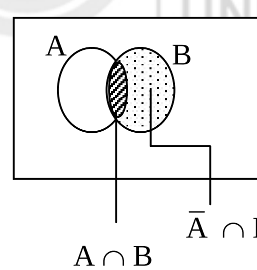
$\Rightarrow P(\bar{A}) = 1 - P(A)$

3. Prove that

(i) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

(ii) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$





Proof

If S is the sample space and A, B $\subset$ S then

(i) $A = (A \cap \bar{B}) \cup (A \cap B)$

$$\Rightarrow P(A) = P((A \cap \bar{B}) \cup (A \cap B))$$

$$= P(A \cap \bar{B}) + P(A \cap B)$$

[Using axiom (iii) as $A \cap \bar{B}$ and $A \cap B$ are mutually disjoint]

$$\Rightarrow P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

(ii) $B = (A \cap B) \cup (\bar{A} \cap B)$

$$P(B) = P((A \cap B) \cup (\bar{A} \cap B))$$

$$= P(A \cap B) + P(\bar{A} \cap B)$$

[Using axiom (iii) as $A \cap B$ and $\bar{A} \cap B$ are mutually disjoint]

$$\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Example 4: A, B and C are three mutually exclusive and exhaustive events associated with a random experiment. Find P(A) given that :

$$P(B) = \frac{3}{4}P(A) \text{ and } P(C) = \frac{1}{3}P(B)$$

Solution:

As A, B and C are mutually exclusive and exhaustive events,

$\therefore A \cup B \cup C = S$

$\Rightarrow P(A \cup B \cup C) = P(S)$

$$\Rightarrow P(A) + P(B) + P(C) = 1 \quad \begin{bmatrix} \because \text{using axiom (iii) as A, B, C} \\ \text{are mutually disjoint events} \end{bmatrix}$$

$$\Rightarrow P(A) + \frac{3}{4}P(A) + \frac{1}{3}P(B) = 1$$

$$\Rightarrow P(A) + \frac{3}{4}P(A) + \frac{1}{3}\left(\frac{3}{4}P(A)\right) = 1$$

$$\Rightarrow \left(1 + \frac{3}{4} + \frac{1}{4}\right)P(A) = 1$$

$\Rightarrow 2P(A) = 1$

$$\Rightarrow P(A) = \frac{1}{2}$$

Examples 5: If two dice are thrown, what is the probability that sum is

a) greater than 9, and

b) neither 10 or 12.

Solution:

a) P[sum > 9] = P[sum = 10 or sum = 11 or sum = 12]

= P[sum =10] + P[sum = 11] + P[sum = 12]

[using axiom (iii)]

$$= \frac{3}{36}+\frac{2}{36}+\frac{1}{36} = \frac{6}{36}=\frac{1}{6}$$

[$\because$ for sum = 10, there are three favourable cases (4, 6), (5, 5) and (6, 4). Similarly for sum =11 and 12, there are two and one favourable cases respectively.]

Let A denotes the event for sum =10 and B denotes the event for sum = 12,

$\therefore$ Required probability = $P\left(\bar{A}\cap\bar{B}\right)=P\left(\overline{A\cup B}\right)$ [Using De- Morgan's law

(see Unit 1 of Course MST-001)]

= 1 –P(A $\cup$ B)

= 1– [P(A) + P(B)] [Using axiom (iii)]

$$= 1-\left[\frac{3}{36}+\frac{1}{36}\right]=1-\frac{4}{36}=1-\frac{1}{9} = \frac{8}{9}$$

Now, you can try the following exercises.

E4) If A, B and C are any three events, write down the expressions in terms of set theory:

a) only A occurs
b) A and B occur but C does not
c) A, B and C all the three occur
d) at least two occur
e) exactly two do not occur
f) none occurs

E5) Fourteen balls are serially numbered and placed in a bag. Find the probability that a ball is drawn bears a number multiple of 3 or 5.

2.7 SUMMARY

Let us summarize the main topics covered in this unit.

1) When classical definition fails, we obtain the probability by observing the data. This approach to probability is called the **relative frequency approach** and it defines the statistical probability. If an event A (say) happens m times in n trials of an experiment which is performed repeatedly under essentially homogeneous and identical conditions, then the (**Statistical or Empirical**) probability of happening A is defined as

$$P(A) = \lim_{n\to\infty} \frac{m}{n}.$$

2) **Subjective probability** is based on one's judgment, wisdom, intuition and expertise. It is interpreted as a measure of degree of belief or as the quantified judgment of particular individual.

3) If S be a sample space for a random experiment and A be an event which is subset of S, then P(A) is called probability function if it satisfies the following axioms

(i) P(A) is real and $P(A) \geq 0$

(ii) P (S) = 1

(iii) If $A_1, A_2, \ldots$ is any finite or infinite sequence of disjoint events in S, then

$$P(A_1 \text{ or } A_2 \text{ or} \ldots \text{or } A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

This is the **axiomatic approach to the probability**.

2.8 SOLUTIONS/ANSWERS

E 1) Since the information is based on 5000 drivers,

the number of exhaustive cases is = 5000.

Thus,

(i) the required probability = $\frac{185}{5000} = \frac{37}{1000}$

(ii) the required probability = $\frac{85+65+70+50}{5000} = \frac{270}{5000} = \frac{27}{500}$

(iii) the required probability = $\frac{1000+195+500+170}{5000} = \frac{1865}{5000} = \frac{373}{1000}$

(iv) the required probability = $\frac{260+240+195+170}{5000} = \frac{865}{5000} = \frac{173}{1000}$

(v) the required probability = $\frac{600+900+1000+500}{500} = \frac{3000}{5000} = \frac{3}{5}$

E 2) Since the information is based on the record of 200 days, so the number of exhaustive cases in each case = 200.

(i) Number of favourable cases for correct forecast = 120

$\therefore$ the required probability = $\frac{120}{200} = \frac{12}{20} = \frac{3}{5}$

(iii) Number of favourable outcomes for incorrect forecast = 200 – 120

= 80

$\therefore$ the required probability = $\frac{80}{200} = \frac{2}{5}$

E 3) First throw a die 200 times and note the outcomes. Then construct a table for the number of throws and the number of times the odd number turns up as shown in the following format:

Number of Throws(n)	Number of times the odd number turns up (m)	Proportion (m/n)
1 2 3 . . . 200		

Now, plot the graph taking number of throws (n) on x-axis and the proportion ($\frac{m}{n}$) on y-axis in the manner as shown in Fig. 2.1. Then see to which value the proportion ($\frac{m}{n}$) approaches to as n becoming large. This limiting value of $\frac{m}{n}$ is the required probability.

E 4) a) $A \cap \bar{B} \cap \bar{C}$

b) $A \cap B \cap \bar{C}$

c) $A \cap B \cap C$

d) $\left(A \cap B \cap \bar{C}\right) \cup \left(A \cap \bar{B} \cap C\right) \cup \left(\bar{A} \cap B \cap C\right) \cup \left(A \cap B \cap C\right)$

e) $\left(\bar{A} \cap \bar{B} \cap C\right) \cup \left(\bar{A} \cap B \cap \bar{C}\right) \cup \left(A \cap \bar{B} \cap \bar{C}\right)$

f) $\bar{A} \cap \bar{B} \cap \bar{C}$

E 5) Let A be the event that the drawn ball bears a number multiple of 3 and B be the event that it bears a number multiple of 5, then

A = {3, 6, 9, 12} and B = {5, 10}

$\therefore$ P(A) = $\frac{4}{14} = \frac{2}{7}$ and P(B) = $\frac{2}{14} = \frac{1}{7}$

The required probability = P(A or B)

$$= P(A) + P(B)$$

$$= \frac{2}{7} + \frac{1}{7} = \frac{3}{7}$$

[Using axiom (iii) as A and B are mutually disjoint]