UNIT 14 AREA PROPERTY OF NORMAL DISTRIBUTION

Structure

14.1 INTRODUCTION

In Unit 13, you have studied normal distribution and its chief characteristics. Some characteristics including moments, mode, median, mean deviation about mean have been established too in Unit 13. The area property of normal distribution has just been touched in the preceding unit. Area property is very important property and has lot of applications and hence it needs to be studied in detail. Hence, in the Unit 14 this property with its diversified applications has been discussed in detail. Fitting of normal distribution to the observed data and computation of expected frequencies have also been discussed in one of the sections i.e. Sec. 14.3 of this unit.

Objectives

After studing this unit, you would be able to:

- describe the importance of area property of normal distribution;
- explain use of the area property to solve many practical life problems; and
- fit a normal distribution to the observed data and compute the expected frequencies using area property.

14.2 AREA PROPERTY OF NORMAL DISTRIBUTION

Let X be a normal variate having the mean μ and variance σ^2.

Suppose we are interested in finding $P[\mu < X < x_1]$ [See Fig.14.1]

Now, $P[\mu < X < x_1] = \int_{\mu}^{x_1} f(x)\,dx = \int_{\mu}^{x_1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

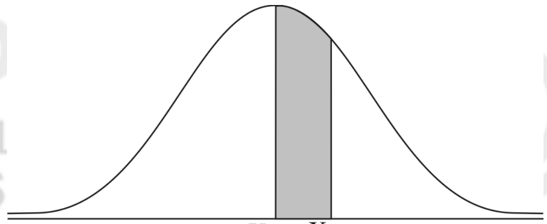

Fig. 14.1: P[μ < X < x_1]

Put $\frac{x-\mu}{\sigma} = z \Rightarrow x - \mu = \sigma z$

$$\frac{dx}{\sigma} = dz$$

Also, when X = μ, Z = 0

and when $X = x_1,\ Z = \frac{x_1-\mu}{\sigma} = z_1 \text{(say)}$

$$\therefore\ P\left[\mu < X < x_1\right] = P\left[0 < Z < z_1\right] = \int_0^{z_1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} \sigma dz$$

$$= \int_0^{z_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dz = \int_0^{z_1} \phi(z) dz$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ is the probability density function of standard normal variate and the definite integral $\int_0^{z_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$ i.e. $\int_0^{z_1} \phi(z) dz$ represents the area under standard normal curve between the ordinates at Z = 0 and Z = z_1. (Fig. 14.2).

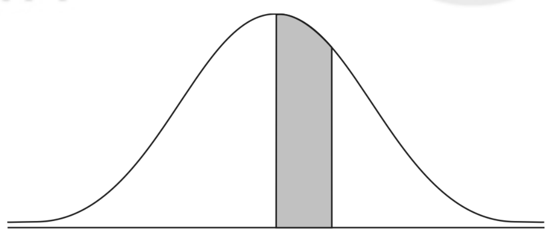

Fig. 14.2: P[0 < Z < z_1]

You need not to evaluate the integral to find the area. Table is available to find such area for different values of z_1.

Here, we have transformed the integral from

$$\int_{\mu}^{x_1} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \text{ to } \int_0^{z_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

i.e. we have transformed normal variate 'X' to standard normal variate (S.N.V.) $Z = \frac{X - \mu}{\sigma}$.

This is because, the computation of

$\int_{\mu}^{x_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$ requires construction of separate tables for different values of μ and σ as the normal variate X may have any values of mean and standard deviation and hence different tables are required for different μ and σ. So, infinitely many tables are required to be constructed which is impossible. But beauty of standard normal variate is that its mean is always '0' and standard deviation is always '1' as shown in Unit 13. So, whatever the values of mean and standard deviation of a normal variate be, the mean and standard deviation on transforming it to the standard normal variate are always '0' and '1' respectively and hence only one table is required.

In particular,

$$P[\mu - \sigma < X < \mu + \sigma] = \int_{\mu-\sigma}^{\mu+\sigma} f(x)\, dx \qquad \text{[See Fig.14.3]}$$

$$\Rightarrow P[-1 < Z < 1] = \int_{-1}^{1} \phi(z)\, dz \quad \left[\begin{array}{l} \because Z = \frac{X - \mu}{\sigma} \text{ when } X = \mu - \sigma, Z = \frac{\mu - \sigma - \mu}{\sigma} = -1 \\ \text{when } X = \mu + \sigma, Z = \frac{\mu + \sigma - \mu}{\sigma} = \frac{\sigma}{\sigma} = 1 \end{array}\right]$$

$$= 2\int_{0}^{1} \phi(z)\, dz \qquad \text{[By Symmetry]}$$

$$= 2 \times 0.34135 \quad \left[\begin{array}{l}\text{From the table given in the} \\ \text{Appendix at the end of the unit}\end{array}\right]$$

$$= 0.6827$$

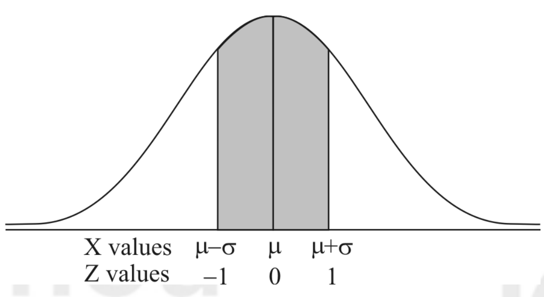

Fig. 14.3: Area within the Range $\mu \pm \sigma$

Similarly,

$$P[\mu - 2\sigma < X < \mu + 2\sigma] = \int_{\mu-2\sigma}^{\mu+2\sigma} f(x)\, dx \qquad \text{[See Fig.14.4]}$$

$$\Rightarrow \; P[-2 < Z < 2] = \int_{-2}^{2} \phi(z)\,dx \;\; = 2\int_{0}^{2} \phi(z)\,dx \quad \left[\begin{array}{l} \because \text{for } Z = \dfrac{X-\mu}{\sigma}, \text{we have} \\ Z = -2 \text{ when} X = \mu - 2\sigma \\ \text{and } Z = 2 \text{ when } X = \mu + 2\sigma \end{array}\right]$$

$$= 2 \times 0.4772 \quad \left[\begin{array}{l} \text{From the table given in the} \\ \text{Appendix at the end of the unit} \end{array}\right]$$

$$= 0.9544$$

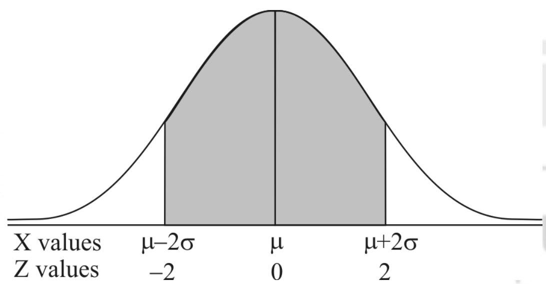

Fig. 14.4: Area within the Range $\mu \pm 2\sigma$

and

$P[\mu - 3\sigma < X < \mu + 3\sigma] = P[-3 < Z < 3]$ = 2.P [0 < Z < 3] [See Fig. 14.5]

$= 2 \times 0.49865 = 0.9973$

$\therefore$ P[X lies within the range $\mu \pm 3\sigma$] = 0.9973

$\Rightarrow$ P[X lies outside the range $\mu \pm 3\sigma$] = 1 – 0.9973 = 0.0027

which is very small and hence usually we expect a normal variate to lie within the range from – 3 to 3, though, theoretically it ranges from – ∞ to ∞.

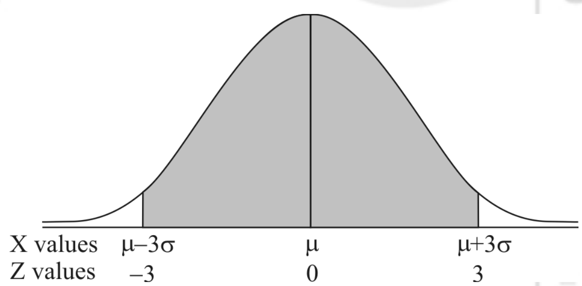

Fig. 14.5: Area within the Range $\mu \pm 3\sigma$

From the above discussion, we conclude that while solving numerical problems, we need to transform the given normal variate into standard normal variate because tables for the area under every normal curve, being infinitely many, cannot be made available whereas the standard normal curve is one and hence table for area under this curve can be made available and this is given in the Appendix at the end of this unit.

Example 1: If X ~ N (45, 16) and Z is the standard normal variable (S.N.V.) i.e $Z = \dfrac{X-\mu}{\sigma}$ then find Z scores corresponding to the following values of X.

(i) X = 45 (ii) X = 53 (iii) X = 41 (iv) X = 47

Solution: We are given X ~ N (45, 16)

$\therefore$ In usual notations, we have

$\mu = 45,\ \sigma^2 = 16 \Rightarrow \sigma = \pm\sqrt{16} \Rightarrow \sigma = 4\ \ [\because \sigma > 0 \text{ always}]$

Now $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-45}{4}$

(i) When X = 45, $Z = \dfrac{45-45}{4} = \dfrac{0}{4} = 0$

(ii) When X = 53, $Z = \dfrac{53-45}{4} = \dfrac{8}{4} = 2$

(iii) When X = 41, $Z = \dfrac{41-45}{4} = \dfrac{-4}{4} = -1$

(iv) When X = 47, $Z = \dfrac{47-45}{4} = \dfrac{2}{4} = 0.5$

Example 2: If the r.v. X is normally distributed with mean 80 and standard deviation 5, then find

(i) $P[X > 95]$, (ii) $P[X < 72]$, (iii) $P[60.5 < X < 90]$,

(iv) $P[85 < X < 97]$, and (v) $P[64 < X < 76]$

Solution: Here we are given that X is normally distributed with mean 80 and standard deviation (S.D.) 5.

i.e. $\text{Mean} = \mu = 80$ and $\text{variance} = \sigma^2 = (\text{S.D.})^2 = 25.$

If Z is the S.N.V., then $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-80}{5}$

Now

(i) $X = 95,\ Z = \dfrac{95-80}{5} = \dfrac{15}{5} = 3$

$\therefore P[X > 95] = P[Z > 3]$ [See Fig.14.6]

$= 0.5 - P[0 < Z < 3]$

= 0.5 – 0.4987 [Using table area under normal curve]

= 0.0013

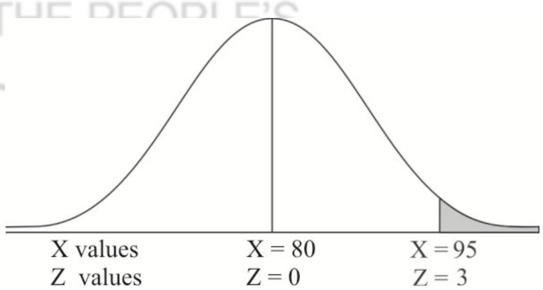

Fig. 14.6: Area to the Right of X= 95

(ii) X = 72, $Z = \dfrac{72-80}{5} = \dfrac{-8}{5} = -1.6$

$\therefore P[X < 72] = P[Z < -1.6]$ [See Fig.14.7]

$= P[Z > 1.6]$ $\left[\begin{array}{l}\because \text{normal curve is symmetrical} \\ \text{about the line } Z = 0\end{array}\right]$

$= 0.5 - P[0 < Z < 1.6]$

= 0.5 – 0.4452 [Using table area under normal curve]

= 0.0548

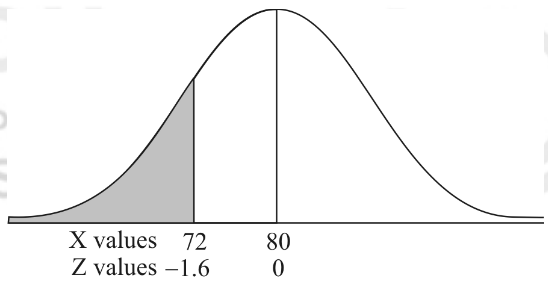

Fig. 14.7: Area to the Left of X = 72

(iii) X = 60.5, $Z = \dfrac{60.5-80}{5} = \dfrac{-19.5}{5} = -3.9$

X = 90, $Z = \dfrac{90-80}{5} = \dfrac{10}{5} = 2$

$\therefore P[60.5 < X < 90] = P[-3.9 < X < 2]$ [See Fig.14.8]

$= P[-3.9 < X < 0] + P[0 < Z < 2]$

$= P[0 < X < 3.9] + P[0 < Z < 2]$ $\left[\begin{array}{l}\because \text{normal curve is} \\ \text{symmetrical about} \\ \text{the line } Z = 0\end{array}\right]$

= 0.5000+ 0.4772 $\left[\begin{array}{l}\text{Using table area} \\ \text{under normal curve}\end{array}\right]$

= 0.9772

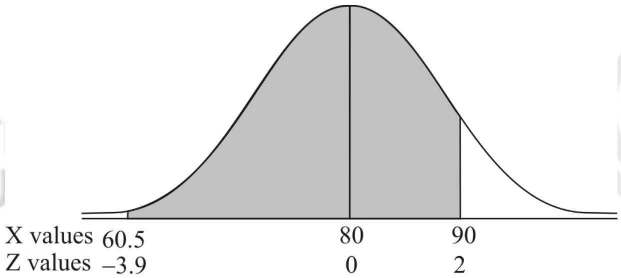

Fig. 14.8: Area between X = 60.5 and X = 90

(iv) $X = 85, \quad Z = \frac{85-80}{5} = \frac{5}{5} = 1$

$X = 97, Z = \frac{97-80}{5} = \frac{17}{5} = 3.4$

$\therefore P[85 < X < 97] = P[1 < Z < 3.4]$ [See Fig.14.9]

$= P[0 < Z < 3.4] - P[0 < Z < 1]$

= 0.4997 – 0.3413 [Using table area under normal curve]

= 0.1584

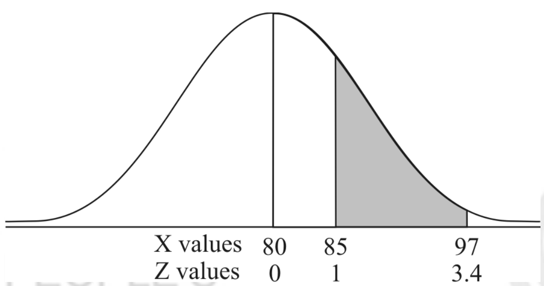

Fig. 1 4.9: Area between X = 85 and X= 97

(v) $X = 64, \quad Z = \frac{64-80}{5} = \frac{-16}{5} = -3.2$

$X = 76, \quad Z = \frac{76-80}{5} = \frac{-4}{5} = -0.8$

$P[64 < X < 76] = P[-3.2 < Z < -0.8]$ [See Fig.14.10]

$= P[0.8 < Z < 3.2] \quad \left[\begin{array}{l} \because \text{normal curve is symmetrical} \\ \text{about the line } Z = 0 \end{array}\right]$

$= P[0 < Z < 3.2] - P[0 < Z < 0.8]$

$= 0.4993 - 0.2881 \quad \left[\begin{array}{l} \text{Using table area} \\ \text{under normal curve} \end{array}\right]$

= 0.2112

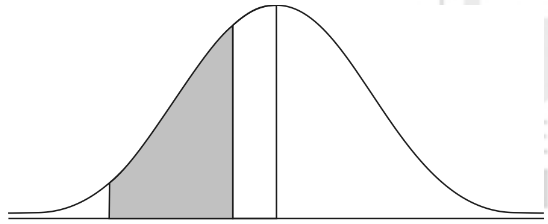

Fig. 14.10: Area between X = 64 and X= 76

Example 3: In a university the mean weight of 1000 male students is 60 kg and standard deviation is 16 kg.

(a) Find the number of male students having their weights
   i) less than 55 kg
   ii) more than 70 kg
   iii) between 45 kg and 65 kg

(b) What is the lowest weight of the 100 heaviest male students?

(Assuming that the weights are normally distributed)

Solution: Let X be a normal variate, "The weights of the male students of the university". Here, we are given that μ = 60 kg, σ = 16 kg, therefore,

X ~ N(60, 256).

We know that if $X \sim N(\mu, \sigma^2)$, then the standard normal variate is given by $Z = \frac{X-\mu}{\sigma}$.

Hence, for the given information, $Z = \frac{X-60}{16}$

(a) i) For X = 55, $Z = \frac{55-60}{16} = -0.3125 \simeq -0.31$.

Therefore,

$P[X < 55] = P[Z < -0.31] = P[Z > 0.31]$ [See Fig. 14.11]

$= 0.5 - P[0 < Z < 0.31]$ $\left[\begin{array}{l}\because \text{area on both} \\ \text{sides of } Z = 0 \text{ is } 0.5\end{array}\right]$

$= 0.5 - 0.1217$ $\left[\begin{array}{l}\text{Using table area} \\ \text{under normal curve}\end{array}\right]$

$= 0.3783$

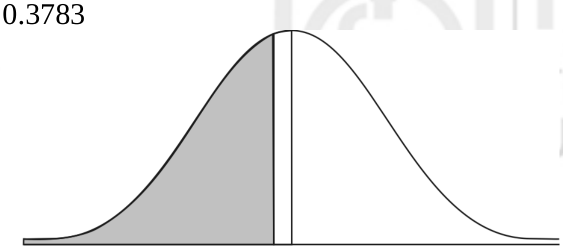

Fig. 14.11: Area Representing Students having Less than 55 kg weight

Number of male students having weight less than 55 kg $= N \times P(X < 55)$

$= 1000 \times 0.3783$

$= 378$

ii) For X = 70, $Z = \dfrac{70-60}{16} = 0.625 \approx 0.63$

$P[X>70] = P\,[Z > 0.63]$ [See Fig. 14.12]

$= 0.5 - P\,[0 < Z < 0.63]$ $\left[\begin{array}{l}\because \text{area on both} \\ \text{sides of } Z = 0 \text{ is } 0.5\end{array}\right]$

$= 0.5 - 0.2357$ $\left[\begin{array}{l}\text{Using table area} \\ \text{under normal curve}\end{array}\right]$

$= 0.2643$

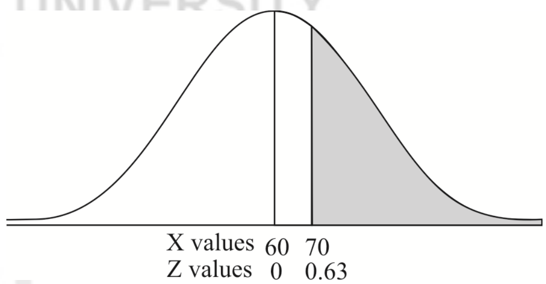

Fig. 14.12: Area Representing Students having More than 70 kg weight

Number of male students having weight more than 70 kg $= N \times P[X > 70]$

$= 1000 \times 0.2643$

$= 264$

iii) For X = 45, $Z = \dfrac{45-60}{16} = -0.9375 \approx -0.94$

For X = 65, $Z = \dfrac{65-60}{16} = 0.3125 \approx 0.31$

$P\left[45 < X < 65\right] = P\left[-0.94 < Z < 0.31\right]$ [See Fig. 14.13]

$= P[-\ 0.94 < Z < 0] + P[0 < Z < 0.31]$

$= P[0 < Z < 0.94] + P[0 < Z < 0.31]$

$= 0.3264 + 0.1217 = 0.\ 4481$

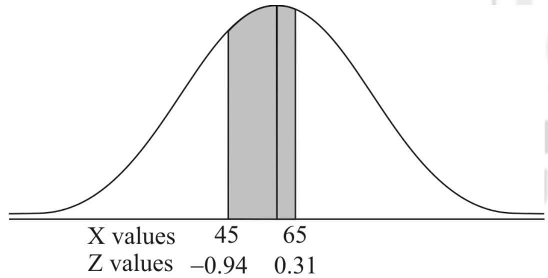

Fig. 14.13: Area Representing Students having Weight between 45 kg and 65 kg

$\therefore$ Number of male students having weight between 45 kg & 65 kg

$= P[45 < X < 65]$

$= 1000 \times 0.4481 = 448$

b) Let x_1 be the lowest weight amongst 100 heaviest students.

Now, for $X = x_1$, $Z = \dfrac{x_1 - 60}{16} = z_1$ (say).

$P[X \ge x_1] = \dfrac{100}{1000} = 0.1$ [See Fig.14.14]

$\Rightarrow P[Z \ge z_1] = 0.1$

$\Rightarrow P\left[0 \le Z \le z_1\right] = 0.5 - 0.1 - 0.4.$

$\Rightarrow z_1 = 1.28$ [From Table]

$\Rightarrow x_1 = 60+16 \times 1.28 = 60+20.48 = 80.48.$

Therefore, the lowest weight of 100 heaviest male students is 80.48 kg.

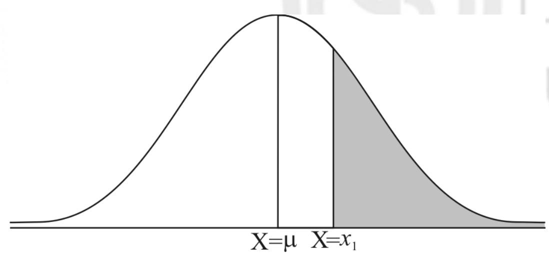

Fig. 14.14: Area Representing the 100 Heaviest Male Students

Example 4: In a normal distribution 10% of the items are over 125 and 35% are under 60. Find the mean and standard deviation of the distribution.

Solution:

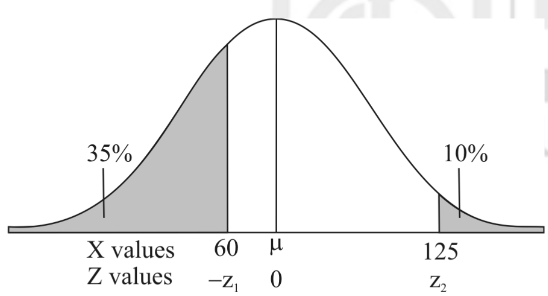

Fig. 14.15: Area Representing the Items under 60 and over 125

Let $X \sim N(\mu, \sigma^2)$, where μ and σ^2 are unknown and are to be obtained.

Here we are given

P[X > 125] = 0.1 and P[X < 60] = 0.35. [See Fig. 14.15]

We know that if X ~ N(μ, σ^2), then $Z = \frac{X-\mu}{\sigma}$.

For X = 60, $Z = \frac{60-\mu}{\sigma} = -z_1 \text{(say)}$ ... (1) $\left[\begin{array}{l}-\text{ve sign is taken because} \\ P[Z<0] = P[Z>0] = 0.5\end{array}\right]$

For X = 125, $Z = \frac{125-\mu}{\sigma} = z_2 \text{(say)}$ ...(2)

Now $P[X < 60] = P[Z < -z_1] = 0.35$

$\Rightarrow P[Z > z_1] = 0.35$ [By symmetry of normal curve]

$\Rightarrow 0.5 - P[0 < Z < z_1] = 0.35$

$\Rightarrow P[0 < Z < z_1] = 0.15$

$\Rightarrow z_1 = 0.39$ $\left[\begin{array}{l}\text{From the table areas} \\ \text{under normal curve}\end{array}\right]$

and $P[X > 125] = P[Z > z_2] = 0.10$

$\Rightarrow 0.5 - P[0 < Z < z_2] = 0.10$

$\Rightarrow P[0 < Z < z_2] = 0.40$

$\Rightarrow z_2 = 1.28$ [From the table]

Putting the values of z_1 and z_2 in Equations (1) and (2), we get

$$\frac{60-\mu}{\sigma} = -0.39 \quad \dots (3)$$

$$\frac{125-\mu}{\sigma} = 1.28 \quad \dots (4)$$

(4) – (3) gives

$$\frac{125-\mu-60+\mu}{\sigma} = 1.28 + 0.39$$

$$\frac{65}{\sigma} = 1.67 \Rightarrow \sigma = \frac{65}{1.67} = 38.92$$

From Eq. (4), $\mu = 125 - 1.28\sigma \Rightarrow \mu = 125 - 1.28 \times 38.92 = 75.18$

Hence $\mu = \text{mean} = 75.18; \sigma = \text{S.D.} = 38.92$

Example 5: Find the quartile deviation of the normal distribution having mean μ and variance σ^2.

Solution: Let X ~ N(μ, σ^2). Let Q_1 and Q_3 are the first and third quartiles. Now as Q_1, Q_2 and Q_3 divide the distribution into four equal parts, therefore, areas under the normal curve to the left of Q_1 , between Q_1 and Q_2 (Median), between Q_2 and Q_3 and to the right of Q_3 all are equal to 25 percent of the total area. This has been shown in Fig. 14.16.

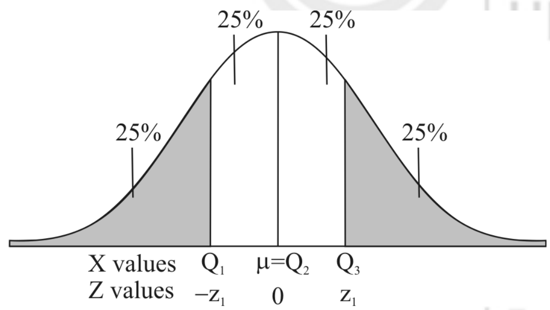

Fig. 14.16: Area to the Left of X = Q_1 and to the Right of X = Q_3

i.e. here, we have

$P[X < Q_1] = 0.25$, $P[Q_1 < X < \mu] = 0.25$, $P[\mu < X < Q_3] = 0.25$ and

$P[X > Q_3] = 0.25$ [See Fig. 14.16]

Now, when $X = Q_1,\ Z = \dfrac{Q_1 - \mu}{\sigma} = -z_1$, (say)

∵ value of Z corresponds to Q_1 which lies to the left of mean which is zero for Z and hence the value to the left of it is negative. Thus, a negative value of Z has been taken here.

$\Rightarrow Q_1 - \mu = -\sigma z_1 \quad \Rightarrow Q_1 = \mu - \sigma z_1$

and when

$X = Q_3,\ Z = \dfrac{Q_3 - \mu}{\sigma} = z_1$

Due to symmetry of normal curve, the values of Z corresponding to Q_1 and Q_3 are equal in magnitude because they are equidistant from mean.

$\Rightarrow Q_3 - \mu = \sigma z_1 \qquad \Rightarrow Q_3 = \mu + \sigma z_1$

Now, as $P[\mu < X < Q_3] = 0.25$, therefore,

$P[0 < Z < z_1] = 0.25$

$\Rightarrow z_1 = 0.67$ [From normal tables]

Now, Q.D. $= \dfrac{Q_3 - Q_1}{2} = \dfrac{(\mu + \sigma z_1) - (\mu - \sigma z_1)}{2} = \sigma z_1 = \sigma(0.67)$ i.e. $\dfrac{2}{3}\sigma$ (approx).

Now, we are sure that you can try the following exercises:

E1) If X ~ N(150, 9) and Z is a S.N.V. i.e $Z = \dfrac{X - \mu}{\sigma}$ then find Z scores corresponding to the following values of X

(i) X = 165 (ii) X = 120

E2) Suppose X ~ N (25, 4) then find

(i) P[X < 22], (ii) P [X > 23], (iii) P[|X – 24| < 3], and (iv) P[|X – 21| > 2]

E3) Suppose X ~ N (30, 16) then find α in each case

(i) $P[X > \alpha] = 0.2492$

(ii) $P[X < \alpha] = 0.0496$

E4) Let the random variable X denote the chest measurements (in cm) of 2000 boys, where X ~ N(85, 36).

a) Then find the number of boys having chests measurement
   i) less than or equal to 87 cm,
   ii) between 86 cm and 90 cm,
   iii) more than 80 cm.

b) What is the lowest value of the chest measurement among the 100 boys having the largest chest measurements?

E5) In a particular branch of a bank, it is noted that the duration/waiting time of the customers for being served by the teller is normally distributed with mean 5.5 minutes and standard deviation 0.6 minutes. Find the probability that a customer has to wait

a) between 4.2 and 4.5 minutes, (b) for less than 5.2 minutes, and (c) more than 6.8 minutes

E6) Suppose that temperature of a particular city in the month of March is normally distributed with mean $24^\circ C$ and standard deviation $6^\circ C$. Find the probability that temperature of the city on a day of the month of March is

(a) less than $20^\circ C$ (b) more than $26^\circ C$ (c) between $23^\circ C$ and $27^\circ C$

14.3 FITTING OF NORMAL CURVE USING AREA PROPERTY

To fit a normal curve to the observed data we first find the mean and variance from the given data. Mean and variance so obtained are μ and σ respectively. Substituting these values of μ and σ^2 in the probability function

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, we get the normal curve fitted to the given data.

Now, the expected frequencies can be computed using either of the following two methods:

1. Area method
2. Method of ordinates

But, here we only deal with the area method. Process of finding the expected frequencies by area method is described in the following steps:

(i) Write the lower limits of each of the given class intervals.

(ii) Find the standard normal variate $Z = \frac{X - \mu}{\sigma}$ corresponding to each lower limit. Suppose the values of the standard normal variate are obtained as z_1, z_2, z_3, …

(iii) Find $P[Z \leq z_1]$, $P[Z \leq z_2]$, $P[Z \leq z_3]$,…i.e. the areas under the normal curve to the left of ordinate at each value of Z obtained in step (ii). Using table given in the Appendix at the end of the unit $Z = z_i$ may be to the right or left of $Z = 0$.

If $Z = z_i$ is to the right of $Z = 0$ as shown in the following figure:

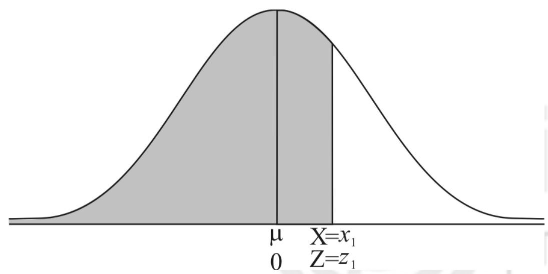

Fig. 14.17: Area to the Left of $Z = Z_i$, when Z_i is to the Right of Z = 0

Then, $P[Z \leq z_i]$ is obtained as

$P[Z \leq z_i] = 0.5 + P[0 \leq Z \leq z_i]$

But, if $Z = z_i$ is to the left of $Z = 0$ (this is the case when z_i is negative) as shown in the following figure:

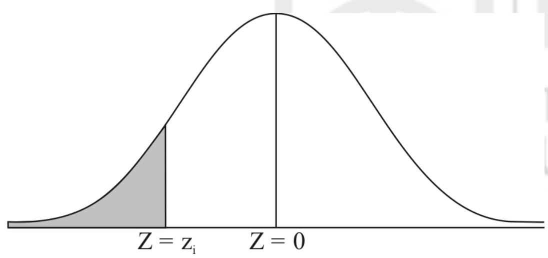

Fig. 14.18: Area to the Left of Z = Z_i, when Z_i is to the left of Z = 0

Then

$P[Z \leq z_i] = 0.5 - P[z_i \leq Z \leq 0]$

$= 0.5 - P[0 \leq Z \leq -z_i]$ [Due to symmetry]

e.g. $z_i = -2$ (say),

Then $P[Z \leq -2] = 0.5 - P[-2 \leq Z \leq 0]$

$= 0.5 - P[0 \leq Z \leq -(-2)]$

$= 0.5 - P[0 \leq Z \leq 2]$

(iv) Obtain the areas for the successive class intervals on subtracting the area corresponding to every lower limit from the area corresponding to the succeeding lower limit.

e.g. suppose 10, 20, 30 are three successive lower limits.

Then areas corresponding to these limits are

$P[X \le 10]$, $P[X \le 20]$, $P[X \le 30]$ respectively.

Now the difference $P[X \le 30] - P[X \le 20]$ gives the area corresponding to the interval 20-30.

(v) Finally, multiply the differences obtained in step (iv) i.e. areas corresponding to the intervals by N (the sum of the observed frequencies), we get the expected frequencies.

Above procedure is explained through the following example.

Example 6: Fit a normal curve by area method to the following data and find the expected frequencies.

X	f
0-10	3
10-20	5
20-30	8
30-40	3
40-50	1

Solution: First we are to find the mean and variance of the given frequency distribution. This you can obtain yourself as you did in Unit 2 of MST-002 and at many other stages. So, this is left an exercise for you.

You will get the mean and variance as

$\mu = 22$ and $\sigma^2 = 111$ respectively

$\Rightarrow \ \sigma = 10.54$

Hence, the equation of the normal curve is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{(10.54)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-22}{10.54}\right)^2}, -\infty < x < \infty$$

Expected frequencies are computed as follows:

Class Interval	Lower Limit	Standard Normal Variate $Z = \frac{X - \mu}{\sigma}$ $= \frac{X - 22}{10.54}$	Area under normal curve to the left of Z $P[X < x]$ $= P[Z \le z]$	Difference between successive areas	Expected frequencies $=20 \times$ col. V
Below 0	$-\infty$	$-\infty$	$P[Z < -\infty]$ $= 0$	0.0183 – 0 = 0.0183	0.366 ≃ 0
0-10	0	–2.09	$P[Z \le -2.09]$ $= 0.0183$	0.1271 – 0.0183 = 0.1088	2.176 ≃ 2
10-20	10	–1.14	$P[z \le -1.14]$ $= 0.1271$	0.4241 – 0.1271 = 0.2970	5.94 ≃ 6
20-30	20	–0.19	$P[Z \le -0.19]$ $= 0.4241$	0.7764 – 0.4241 = 0.3523	7.05 ≃ 7
30-40	30	0.76	$P[Z \le 0.76]$ $= 0.9564$	0.9564 – 0.7764 = 0.1800	3.6 ≃ 4
40-50	40	1.71	$P[Z \le 1.71]$ $= 0.9564$	0.9961 – 0.9564 = 0.0397	0.79 ≃ 1
50 and above	50	2.66	$P[Z \le 2.66]$ $= 0.9961$	–	–

The areas under the normal curve shown in the fourth column of the above tables are obtained as follows:

$$P[Z < -\infty] = 0 \qquad \left[\because \text{there is no value to the left of } -\infty\right]$$

$$\begin{aligned} P[Z \le -2.09] &= 0.5 - P[-2.09 \le Z \le 0] && \text{[See Fig. 14.19]} \\ &= 0.5 - P[0 \le Z \le 2.09] && \text{[Due to symmetry]} \\ &= 0.5 - 0.4817 && \left[\text{From table given at the end of the unit}\right] \\ &= 0.0183 \end{aligned}$$

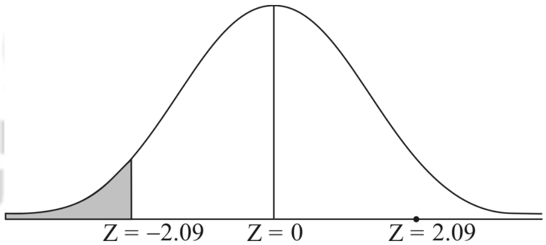

Fig. 14.19: Area to the Left of Z = –2.09

Similarly,

$P[Z \leq -1.14] = 0.5 - 0.3729 = 0.1271$

$P[Z \leq -0.19] = 0.5 - 0.0759 = 0.4241$

Now, $P[Z \leq 0.76] = 0.5 + P[0 \leq Z \leq 0.76]$ [See Fig. 14.20]

$= 0.5 + 0.2764$

$= 0.7764$

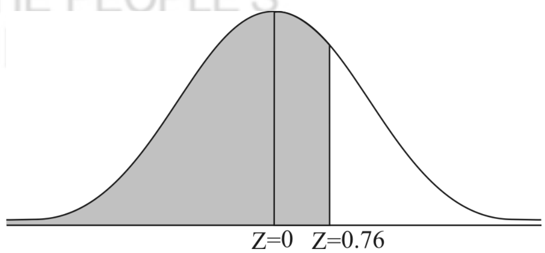

Fig. 14.20: Area to the Left of Z = 0.76

Similarly

$P[Z \leq 1.71] = 0.5 + 0.4564 = 0.9564$

$P[Z \leq 2.66] = 0.5 = 0.4961 = 0.9961$

You can now try the following exercises:

E7) Fit a normal curve to the following distribution and find the expected frequencies by area method.

X	60– 65	65-70	70-75	75-80	80-85
	5	8	12	8	7

E8) The following table gives the frequencies of occurrence of a variate X between certain limits. The distribution is normal. Find the mean and S.D. of X.

X	Less than 40	40-50	50 and more
f	30	33	37

14.4 SUMMARY

The main points covered in this unit are:

1) **Area property and its various applications** has been discussed in detail.
2) **Quartile deviation** has also been obtained using the **area property** in an example.
3) **Fitting of normal distribution** using area property and computation of **expected frequencies** using area method have been explained.

14.5 SOLUTIONS/ANSWERS

E1) We are given X ~ N(150, 9)

$\therefore$ in usual notations, we have

$\mu = 150,\ \sigma^2 = 9 \quad \Rightarrow \sigma = 3$

Now, $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-150}{3}$

(i) When X = 165, $\quad Z = \dfrac{165-150}{3} = \dfrac{15}{3} = 5$

(ii) When X = 120, $\quad Z = \dfrac{120-150}{3} = \dfrac{-30}{3} = -10$

E2) Here X ~ N(25, 4)

$\therefore$ in usual notations, we have

$\text{Mean} = \mu = 25, \quad \text{variance} = \sigma^2 = 4 \Rightarrow \sigma = 2$

If Z is the S.N.V then $Z = \dfrac{X-\mu}{\sigma} = \dfrac{X-25}{2}$

i) $X = 22, \quad Z = \dfrac{22-25}{2} = \dfrac{-3}{2} = -1.5$

$P[X < 22] = P[Z < -1.5]$ [See Fig. 14.21]

$= P[Z > 1.5]$ $\left[\because \text{due to symmetry of normal curve}\right]$

$= 0.5 - P[0 < Z < 1.5]$

$= 0.5 - 0.4332$ $\left[\text{Using table area under normal curve}\right]$

$= 0.0668$

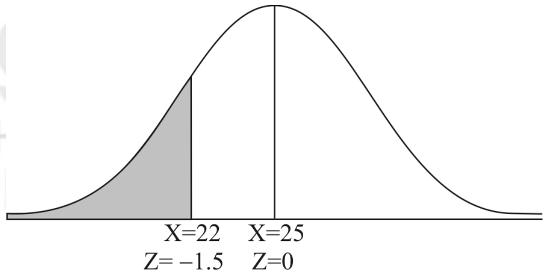

Fig. 14.21: Area to the Left of X = 22

ii) $X = 23, \quad Z = \dfrac{23-25}{2} = \dfrac{-2}{2} = -1$

$P[X > 23] = P[Z > -1]$ [See Fig.14.22]

$= P[Z < 1]$ $\left[\begin{array}{l}\because \text{due to symmetry of} \\ \text{normal curve}\end{array}\right]$

$= 0.5 + P[0 < Z < 1]$

$= 0.5 + 0.3413$ $\left[\begin{array}{l}\text{Using table area} \\ \text{under normal curve}\end{array}\right]$

$= 0.8413$

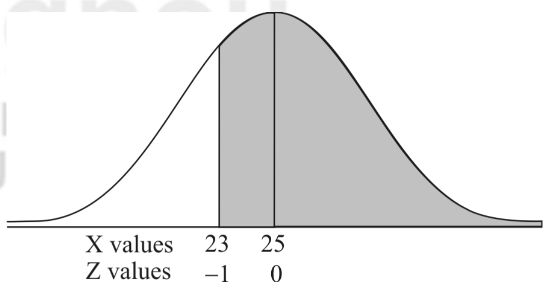

Fig. 14.22: Area to the Right of X = 23

iii) $P[|X-24| < 3] = P[-3 < X - 24 < 3]$ $\left[\begin{array}{l}\because |x-a| < b \\ \Rightarrow -b < x-a < b\end{array}\right]$

$= P[-3+24 < X < 3+24)$

$= P[21 < X < 27]$

$X = 21, \quad Z = \dfrac{21-25}{2} = \dfrac{-4}{2} = -2$

$X = 27, \quad Z = \dfrac{27-25}{2} = \dfrac{2}{2} = 1$

$\therefore P[|X-24| < 3] = P[21 < X < 27]$ [See Fig.14.23]

$= P[-2 < Z < 1]$

$$= P[-2 < Z < 0] + P[0 < Z < 1]$$

$$= P[0 < Z < 2] + P[0 < Z < 1]$$

$$= 0.4772 - 0.3413 = 0.1359$$

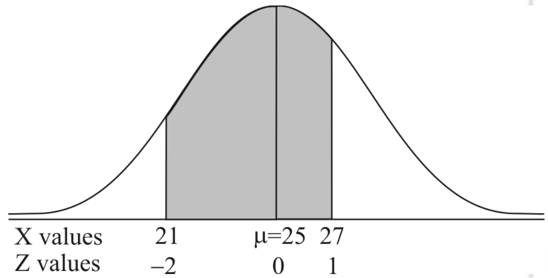

Fig. 14.23: Area between X = 21 and X = 27

iv) $P[|X-21|>2] = P[X-21>2 \text{ or } -(X-21)>2]$

$$\left[\begin{array}{l}\because |x-a|>b \Rightarrow \pm(x-a)>b \\ \Rightarrow x-a>b \text{ or } -(x-a)>b\end{array}\right]$$

$$= P[X>23 \text{ or } -X>2-21]$$

$$= P[X>23 \text{ or } X<19] \quad \left[\begin{array}{l}\because -y>-a \\ \Rightarrow y<a\end{array}\right]$$

For X=19, $Z = \frac{19-25}{2} = \frac{-6}{2} = -3$

For X=23, $Z = \frac{23-25}{2} = \frac{-2}{2} = -1$

$$\therefore P[|X-21|>2] = P[X>23 \text{ or } X<19] \quad [\text{See Fig } 14.24]$$

$$= P[Z>-1 \text{ or } Z<-3]$$

$$= P[Z>-1] + P[Z<-3] \quad \left[\begin{array}{l}\text{By addition theorem for} \\ \text{mutually exclusive events}\end{array}\right]$$

$$= 1 - P[-3<Z<-1]$$

$$= 1 - P[1<Z<3]$$

$$= 1 - [P[0<Z<3] - P[0<Z<1]]$$

$$= 1 - [0.4987 - 0.3413] \quad [\text{From table}]$$

$$= 1 - 0.1574 = 0.8426.$$

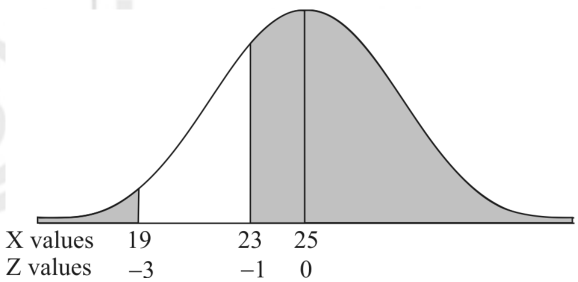

Fig. 14.24: Area between X = 19 and X = 23

E3) Here X ~ N(30, 16)

$\therefore$ in usual notations, we have

Mean = $\mu = 30$, variance $= \sigma^2 = 16 \Rightarrow \sigma = 4$

If Z is S.N.V then $Z = \frac{X-\mu}{\sigma} = \frac{X-30}{4}$

i) $X = \alpha, \quad Z = \frac{\alpha - 30}{4} = z_1 \text{(say)} \qquad \ldots(1)$

Now $P[X > \alpha] = 0.2492 \qquad$ [See Fig.14.25]

$\Rightarrow P[Z > z_1] = 0.2492 \Rightarrow 0.5 - P[0 < Z < z_1] = 0.2492$

$\Rightarrow P[0 < Z < z_1] = 0.2508$

$\Rightarrow z_1 = 0.67 \qquad$ [From the table]

Putting $z_1 = 0.67$ in (1), we get

$\frac{\alpha - 30}{4} = 0.67$

$\alpha - 30 = 2.68$

$\alpha = 30 + 2.68 = 32.68$

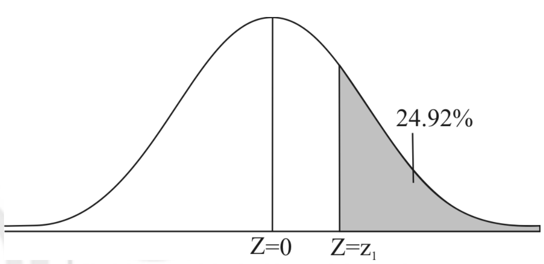

Fig. 14.25: z_1 Corresponding to 24.92 % Area to its Right

ii) For $X = \alpha, \quad Z = \frac{\alpha - 30}{4} = -z_2 \text{(say)} \quad \ldots (2)$

Now $P[X < \alpha] = 0.0496$

$\Rightarrow P[Z < -z_2] = 0.0496 \qquad$ [See Fig.14.26]

$\Rightarrow P[Z > z_2] = 0.0496 \qquad$ [Due to symmetry]

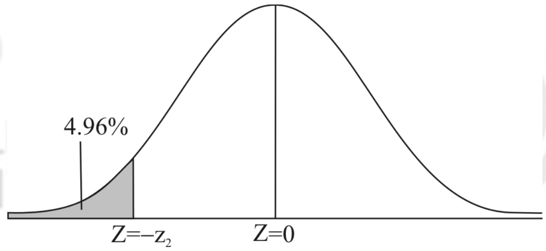

Fig. 14.26: z_2 Corresponding to 4.96 % Area to its Right

$$\Rightarrow 0.5 - P[0 < Z < z_2] = 0.0496$$

$$\Rightarrow P[0 < Z < z_2] = 0.5 - 0.0496 = 0.4504$$

$$\Rightarrow z_2 = 1.65 \qquad \text{[From the table]}$$

Putting $\Rightarrow z_2 = 1.65$ in (2), we get

$$\frac{\alpha - 30}{4} = -1.65$$

$$\Rightarrow \alpha - 30 = -1.65 \times 4$$

$$\Rightarrow \alpha - 30 = -6.60$$

$$\Rightarrow \alpha = 30 - 6.6 = 23.4$$

E4) We are given X ~ N(85, 36), N = 2000

i.e. $\mu = 85\text{cm}, \sigma^2 = 36\text{cm}, N = 2000$

If X ~ N(μ, σ^2) and $Z = \frac{x - \mu}{\sigma}$ then we know that Z ~ N(0, 1)

a) i) For X = 87, $Z = \frac{87 - 85}{6} = \frac{2}{6} \approx 0.33$

Now P[X < 87] = P [Z < 0.33] [See Fig. 14.27]

= 0.5 + P [0 < Z < 0.33]

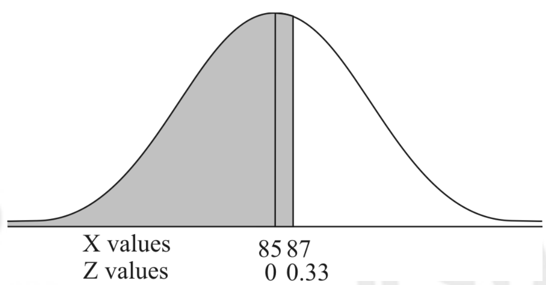

Fig. 14.27: Area to the Left of X = 87 or Z = 0.33

= 0.5 + P [0 < Z < 0.33]

$$= 0.5 + 0.1293 \qquad \begin{bmatrix} \text{From the table of areas} \\ \text{under normal curve} \end{bmatrix}$$

= 0.6293

Therefore, number of boys having chests measurement ≤ 87

$= N.P[X \leq 87]$

$= 2000 \times 0.6293 = 1259$

ii) For $X = 86$, $Z = \frac{86-85}{6} = \frac{1}{6} \approx 0.17$

For $X = 90$, $Z = \frac{90-85}{6} = \frac{5}{6} \approx 0.83$

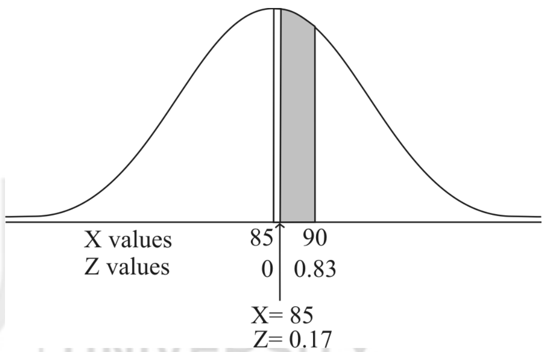

Fig. 14.28: Area between X= 86 and X= 90

$\therefore$ $P[86 < X < 90] = P[0.17 < Z < 0.83]$ [See Fig.14.28]

$= P[0 < Z < 0.83] - P[0 < Z < 0.17]$

$= 0.2967 - 0.0675$ $\left[\begin{array}{l}\text{From the table of areas} \\ \text{under normal curve}\end{array}\right]$

$= 0.2292$

$\therefore$ number of boys having chests measurement between 86 cm and 90 cm

$= N.P[86 < x < 90]$

$= 2000 \times 0.2292 = 458$

iii) For $X = 80$, $Z = \frac{80-85}{6} = \frac{-5}{6} \approx -0.83$

$P[X > 80] = P[Z > -0.83]$ [See Fig. 14.29]

$= P[Z < 0.83]$

$= 0.5 + P[0 < Z < 0.83]$

$= 0.5 + 0.2967$ $\left[\begin{array}{l}\text{From the table of areas} \\ \text{under normal curve}\end{array}\right]$

$= 0.7967$

$\therefore$ number of boys having chest measurement more than 80 cm

$= N.P[X > 80]$

$= 2000 \times 0.7967$

$= 1593$

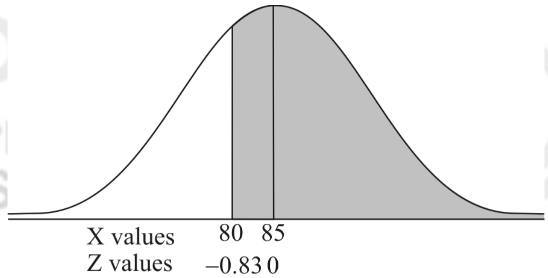

Fig. 14.29: Area to the Right of X = 80 or Z = −0.83

b) Let x_1 be the lowest chest measurement amongst 100 boys having the largest chest measurements.

Now, for $X = x_1$, $Z = \frac{x_1 - 85}{6} = z_1(\text{say})$.

$$P[X \geq x_1] = \frac{100}{2000} = 0.05$$

$$\Rightarrow P\left[Z \geq z_1\right] = 0.05 \qquad \left[\text{See Fig.14.30}\right]$$

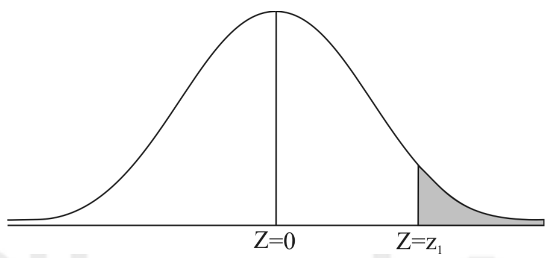

Fig. 14.30: Area Representing the 100 Boys having Largest Chest Measurements

$$\Rightarrow P[0 \leq Z \leq z_1] = 0.5 - 0.05 = 0.45$$

$$\Rightarrow z_1 = 1.64 \qquad \text{[From Table]}$$

$$\Rightarrow x_1 = 85 + 6 \times 1.64 = 85 + 9.84 = 94.84.$$

Therefore, the lowest value of the chest measurement among the 100 boys having the largest chest measurement is 94.84 cm.

E5) We are given

$\mu = 5.5$ minutes, $\sigma = 0.6$ minutes

If $X \sim N(\mu, \sigma^2)$ and $Z = \frac{X - \mu}{\sigma}$ then we know that $Z \sim N(0, 1)$

a) For X = 4.2, $Z = \frac{4.2 - 5.5}{0.6} = \frac{-1.3}{0.6} = \frac{-13}{6} \simeq -2.17$

For X = 4.5, $Z = \frac{4.5 - 5.5}{0.6} = \frac{-1.0}{0.6} = \frac{-10}{6} = \frac{-5}{3} \simeq -1.67$

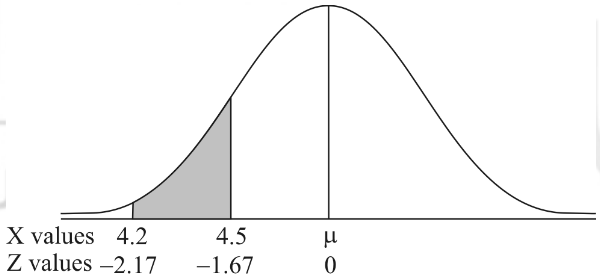

Fig. 14.31: Area Representing Probability of Waiting Time between 4.2 and 4.5 Minutes

$$P[4.2 < x < 4.5] = P\left[-2.17 < Z < -1.67\right] \quad \left[\text{See Fig.14.31}\right]$$

$$= P\ [1.67 < Z < 2.17]$$

$$= P\ [0 < Z < 2.17] - P\ [0 < Z < 1.67]$$

$$= 0.4850 - 0.4525$$

$$= 0.0325$$

Therefore, probability that customer has to wait between 4.2 min and 4.5 min = 0.0325

b) For $X = 5.2,\ Z = \dfrac{5.2-5.5}{0.6} = \dfrac{0.3}{0.6} = \dfrac{-3}{6} = \dfrac{-1}{2} = -0.5$

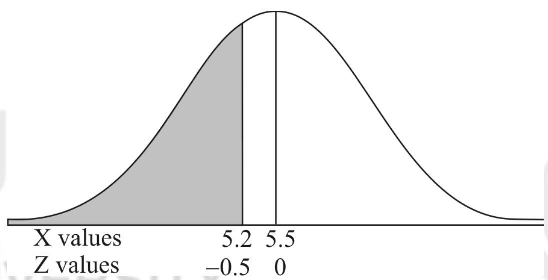

Fig. 14.32: Area Representing Probability of Waiting Time Less than 5.2 Minutes

$$P[X < 5.2] = P\ [Z < -0.5] \qquad [\text{See Fig 14.32}]$$

$$= P\ [Z > 0.5]$$

$$= 0.5 - P\ [0 < Z < 0.5]$$

$$= 0.5 - 0.1915 \qquad \begin{bmatrix}\text{From the table of areas} \\ \text{under normal curve}\end{bmatrix}$$

$$= 0.3085$$

Therefore, probability that customer has to work for less than 5.2 min

$= 0.3085$

c) For $X = 6.8, Z = \dfrac{6.8-5.5}{0.6} = \dfrac{1.3}{0.6} = \dfrac{13}{6} \approx 2.17$

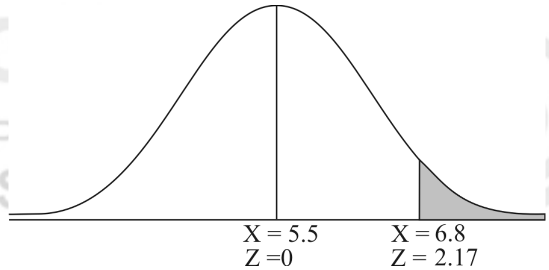

Fig. 14.33: Area Representing Probability of Waiting Time Greater than 6.8 Minutes

$$\begin{aligned} P[X > 6.8] &= P\,[Z > 2.17] \qquad \text{[See Fig 14.33]} \\ &= 0.5 - P\,[0 < Z < 2.17] \\ &= 0.5 - 0.4850 = 0.0150 \end{aligned}$$

Therefore, probability that customer has to wait for more than 6.8 min = 0.0150

E6) Let the random variable X denotes the temperature of the city in the month of March. Then we are given

$X \sim N(\mu, \sigma^2)$, where $\mu = 24^\circ C, \sigma = 6^\circ C$

We know that if $X \sim N(\mu, \sigma^2)$, and $Z = \dfrac{X - \mu}{\sigma}$ then Z ~ N (0, 1)

a) For $X = 20, Z = \dfrac{20-24}{6} = \dfrac{-4}{6} = \dfrac{-2}{3} \approx -0.67$

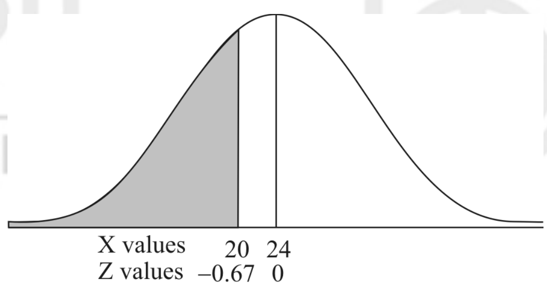

Fig. 14.34: Area Representing Probability of Temperature Less than 20 $^\circ$C

$$\begin{aligned} P[X < 20] &= P[Z < -0.67] \qquad \text{[See Fig. 14.34]} \\ &= P\,[Z > 0.67] \\ &= 0.5 - P[0 < Z < 0.67] \\ &= 0.5 - 0.2486 = 0.2514 \end{aligned}$$

Therefore, probability that temperature of the city is less than 20 $^\circ$C is 0.2514

b) For X= 26, $Z = \dfrac{26-24}{6} = \dfrac{2}{6} = \dfrac{1}{3} \approx 0.33$

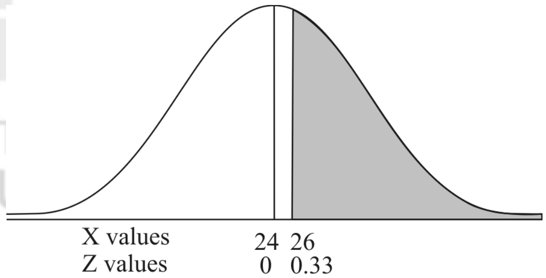

Fig. 14.35: Area Representing Probability of Temperature Greater than 26 $^\circ$C

Since, $P[X > 26] = P[Z > 0.33]$ [See Fig. 14.35]

$$= 0.5 - P[0 < Z < 0.33]$$

$$= 0.5 - 0.1293 \quad \begin{bmatrix} \text{From the table of areas} \\ \text{under normal curve} \end{bmatrix}$$

$$= 0.3707$$

Therefore, probability that temperature of the city is more than 26°C is 0.3707

c) For X= 23, $Z = \dfrac{23-24}{6} = \dfrac{-1}{6} \approx -0.17$

For X= 27, $Z = \dfrac{27-24}{6} = \dfrac{3}{6} = \dfrac{1}{2} = 0.5$

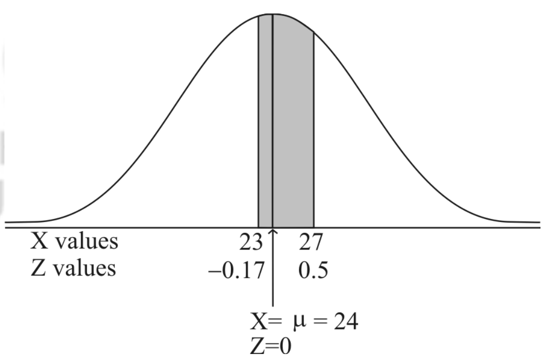

Fig. 14.36: Area Representing Probability of Temperature between 23 $^\circ$C and 27 $^\circ$C

$P[23 < X < 27] = P[-0.17 < Z < 0.5]$ [See Fig. 14.36]

$$= P[-0.17 < Z < 0] + P[0 < Z < 0.5]$$

$$= P[0 < Z < 0.17] + P[0 < Z < 0.5]$$

$$= 0.0675 + 0.1915 \quad \begin{bmatrix} \text{From the table of areas} \\ \text{under Normal Curve} \end{bmatrix}$$

$$= 0.2590$$

Therefore, probability that temperature of the city is between 23°C and 27°C is 0.2590

E7) Mean $(\mu) = 73$, variance$(\sigma^2) = 39.75$

and hence S.D. $(\sigma) = 6.3$

$\therefore$ The equation of the normal curve fitted to the given data is

$$f(x) = \frac{1}{(6.3)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-73}{6.3}\right)^2}, \qquad -\infty < x < \infty$$

Using area method,

The expected frequencies are obtained as follows:

Class interval	Lower limit X	$Z = \frac{X-\mu}{\sigma} = \frac{X-73}{6.3}$	Area under normal curve to the left of z	Difference between successive areas	Expected frequency $40 \times$ col. V
Below 60	$-\infty$	$-\infty$	0	0.0197 – 0 = 0.0197	0.8 ≈ 1
60 – 65	60	–2.06	0.5 – 0.4803 = 0.0197	0.1020 – 0.0197 = 0.0823	3.3 ≈ 3
65 – 70	65	–1.27	0.5 – 0.3980 = 0.1020	0.3156 – 0.1020 = 0.2136	8.5 ≈ 9
70 – 75	70	–0.48	0.5 – 0.1844 = 0.3156	0.6255 – 0.3156 = 0.3099	12.4 ≈ 12
75 – 80	75	+ 0.32	0.5 + 0.1255 = 0.6255	0.8655 – 0.6255 = 0.2400	9.6 ≈ 10
80 – 85	80	1.11	0.5 + 0.3655 = 0.8655	0.9713 – 0.8655 = 1.1058	4.2 ≈ 4
85 and above	85	1.90	0.5 + 0.4713 = 0.9713		

E8) $P[X < 40] = \frac{30}{100} = 0.3,$

$P[40 < X < 50] = \frac{33}{100} = 0.33,$ and

$$P[X > 50] = \frac{37}{100} = 0.37,$$

Now, Let $X \sim N(\mu, \sigma^2)$,

$\therefore$ Standard normal variate is

$$Z = \frac{X - \mu}{\sigma}$$

When X = 40, $Z = \frac{40 - \mu}{\sigma} = -z_1$, (say) $\left[\begin{array}{l}\text{It is taken as} - \text{ve as area to} \\ \text{the left of this value is } 30\% \\ \text{as probability is } 0.3\end{array}\right]$

When X = 50, $Z = \frac{50 - \mu}{\sigma} = z_2$ (say) $\left[\begin{array}{l}\text{It is taken as +ve as} \\ \text{area to the right of this} \\ \text{value is given as } 37\%\end{array}\right]$

Now,

$P[X < 40] = P[Z < -z_1] = 0.3$

$\Rightarrow 0.5 - P[-z_1 < Z < 0] = 0.3$ [See Fig 14.37]

$\Rightarrow 0.5 - P[0 < Z < z_1] = 0.3$ [Due to symmetry]

$\Rightarrow P[0 < Z < z_1] = 0.2$

From table at the end of this unit,

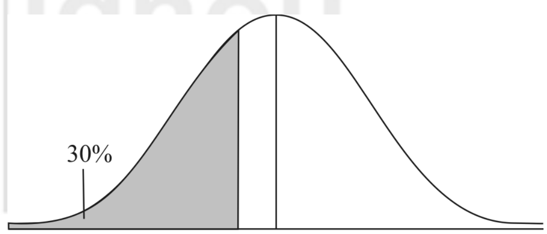

Fig. 14.37: -z_1 Corresponding to the 30% Area to its Left

The value of Z corresponding to probability/area is

$z_1 = 0.525$ $\left[\begin{array}{l}\text{As values of } Z \text{ are } 0.52 \text{ and } 0.53 \\ \text{corresponding to the proabability} \\ 0.1985\end{array}\right]$

$P[X > 50] = 0.37$

$\Rightarrow P[Z > z_2] = 0.37$ [See Fig.14.38]

$\Rightarrow 0.5 - P[0 < Z < z_2] = 0.37$

$\Rightarrow P[0 < Z < z_2] = 0.13$

$z_2 = 0.33$(approx.) [From the table]

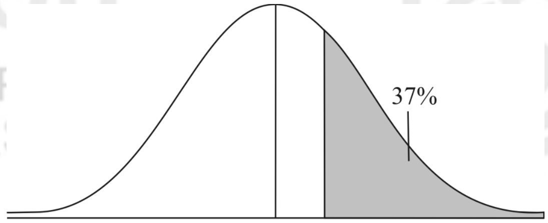

Fig. 14.38: z_2 Corresponding to the 37% Area to its Right

$$\therefore \frac{40-\mu}{\sigma} = -0.525 \text{ and } \frac{50-\mu}{\sigma} = 0.33$$

$$\Rightarrow 40-\mu = 0.525\sigma \text{ and } 50-\mu = 0.33\sigma$$

Solving these equations for μ and σ, we have

$\sigma = 11.7$ and $\mu = 46.14$

APPENDIX

AREAS UNDER NORMAL CURVE

The standard normal probability curve is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right), -\infty < z < \infty$$

The following table gives probability corresponding to the shaded area as shown in the following figure i.e. $P[0 < Z < z]$ for different values of z

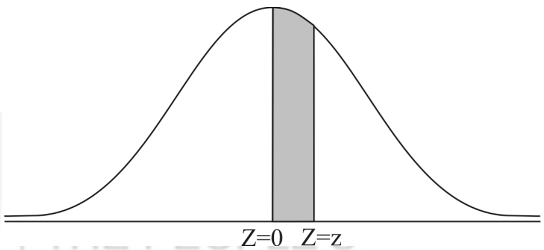

	TABLE OF AREAS									
↓ z →	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0759
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2703	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2005	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3655	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3820
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	4279	.4292	.4306	.4319

1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4678	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4959	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.1960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4879	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4493	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000