UNIT 16 APPLICATIONS OF SIMULATION

Structure

16.1 INTRODUCTION

In Unit 15, we have discussed the preliminaries of starting simulation and general problems that can be solved by simulation. In this unit, we shall consider some important applications of simulations. In this unit, we shall be discussing the applications of simulation in the area of Queuing theory and Inventory problems. Some other problems are also tackled with simulation i.e Air traffic problem control, Bank teller scheduling, Fire station location and Computer network problems. We shall discuss advantages and disadvantages of solving problem by simulation.

Some of the important applications i.e A waiting time model (Queuing Model) and simulation of a very simple inventory problem are explored in Section 16.2. Tests of randomness i.e. Chi-square goodness of fit test, Kolmogrov-Smirnov goodness of fit test and Runs up and down test (Test of independence) are discussed in Section 16.3. The advantages and disadvantages of simulation technique are described in Section 16.2.

Objectives

After studying this unit, you would be able to:

- describe certain applications of simulation with examples i.e. in inventory prblems, in queening theory, etc;
- describe advantages and disadvantages of simulation;
- tests the randomness of the generated sequence i.e. Chi-square goodness of fit test, Kolomogrov-Smirnov goodness of fit test and Runs up and down test of independence; and
- explain the methods of testing the randomness of the generated sequence.

16.2 SOME APPLICATIONS OF SIMULATION

Simulation is becoming a very popular tool for the analysis of a wide variety of the problems. In the last few years it has been used in all major manufacturing process design. Some problems that can be tackled by simulation are:

1. Capacity and feasibility
2. Comparing alternatives
3. Troubleshooting fine tuning

Some problems that can be tackled in other areas are:

- Air traffic control (delays in landing);
- Bank teller scheduling (customer waiting times);
- Location of fire stations (response times);
- Computer network (delays); etc.

Variables given within brackets are performance measures.

Now-a-days a lot of statistical inference methods are also based on Monte-Carlo simulation techniques such as Jack-Knife, Bootstrap, Markov Chain Monte-Carlo (MCMC) methods.

In the following sections, we shall give examples of some important applications.

16.2.1 A Waiting Time Model (Queuing Model)

These models can be used in a wide variety of situations in the real world, where entities (or customers) enter a system for some service. The time between two customers arrival is taken as random variable. The time of serving of each customer is also taken as random variable. Examples of queuing theory are many as follows:

1. Patients arriving at an emergency room and waiting for doctor;
2. Job on a computer centre waiting for CPU time;
3. Airplanes entering an airport's airspace waiting for an available runway; etc.

One may like to use these models to study

(i) What is the average amount of time a customers or job spends for waiting for service?

(ii) What is the probability that the number of customers or job in the system will exceed some fixed level?

(iii)Total number of customers that have been served in an interval [0, t].

If the arriving customer finds the server free, his service starts immediately, and he departs from the system after completion of his service. If the arriving customer finds the server busy, he waits till his turn to be served. Customers are served on first-come-first-out basis (FCFO).

Let S_i denote the service time of the i^{th} customer who arrives at time t_i and let

$$I_i = t_i - t_{i-1}, \quad i \geq 1$$

denote the inter-arrival time between arrival of the i^{th} and $(i-1)^{th}$ customers. Assume that S_i ($i \geq 1$) and I_i ($i \geq 1$), each are independent random variables and follow exponential distribution.

Let μ and λ be the mean service rate (customers/unit of time) and mean arrival rates (customers/unit of time), respectively.

Then by inverse probability transformation (IPT) method

$$F(X) = P[X \leq x] = u$$

$$\Rightarrow 1 - e^{-I\lambda} = u$$

$$\Rightarrow e^{-I\lambda} = 1 - u \approx u$$

$$\Rightarrow -I\lambda = \log u$$

$$\Rightarrow \quad I = (-\log u) / \lambda$$

Similarly,

$$S = -(\log u) / \mu$$

The parameter

$$\rho = \lambda / \mu$$

is known as **traffic intensity** and measures the congestion of the system.

If $\rho < 1$, then the system is to be stable.

Note that the total time spent in the system of the n^{th} customer, W_n, from arrival till departure, may be simply expressed as

$$W_n = W_{n-1} - I_n + S_n \qquad \text{if } W_{n-1} > I_n,\ n = 1, 2, \ldots$$

$$= S_n \qquad \text{if } W_{n-1} < I_n$$

starting with $W_1 = S_1$

For general random variables S_i and I_i and single server (GI/G/I) queuing model, it is very difficult to find mean value of W_i, i.e. $E(W_i)$, analytically and simulation may be used. In order to estimate E (W) we run the queuing system N times, each time starting from $W_1 = S_1$.

Then we obtain a sequence of service times $\{S_{ik}, i \geq 1, k = 1, 2, \ldots, N\}$ and sequence of inter-arrival times $[I_{ik}, i \geq 1, k = 1, 2, \ldots, N]$ and calculate W_{ik} and estimate $E(W_i)$ by the sample mean

$$\overline{W} = \sum_{k=1}^{N} \frac{W_{ik}}{N}$$

If the distributions of I and S are considered to be exponential with rates λ and μ respectively, then the system is known as M/M/1 queuing system and analytical solution $E(W_i)$ is available as:

$$E(W_i) = 1/(\mu - \lambda)$$

The average number of customers in the system (waiting or being served) is $\rho / (1 - \rho)$. This may be verified by simulation of M/M/1 process.

16.2.2 Simulation of a Very Simple Inventory Problem: An Example

A newspaper vendor buys daily newspaper at Rs. 2.30 and sells them at Rs. 3.50 each. Newspapers are purchased at the beginning of the day (before the news paper boy knows what the demand of the papers will be), and any paper left unsold at the end of the day are thrown out.

Suppose the probability distribution of the random demand of newspapers is given as below. The news boy wishes to know how many papers he should buy each day in order to maximize his profit. The demand per day and their probabilities are given below:

Demand (D)	20	21	22	23	24	25	26	27	28	29	30
Probabilities	0.05	0.05	0.10	0.10	0.10	0.15	0.15	0.10	0.10	0.05	0.05

Suppose we begin with an inventory of arbitrary number, say, 25 papers purchased each day. We then evaluate the profit resulting from a policy of purchasing 25 papers at the beginning of each day. To do this, we fill out a table describing the actual sale and profits for each simulated day. The demand per day for 30 days can be simulated by the method of IPT and a simulation is given in the following table:

Day	1	2	3	4	5	6	7	8	9	10
Demand (D)	28	20	26	26	23	22	24	24	29	21
Profit (P)	30.0	12.5	30.0	30.0	23.0	19.5	26.5	26.5	30.0	16.0
Day	11	12	13	14	15	16	17	18	19	20
Demand (D)	24	24	28	28	26	27	26	26	24	21
Profit (P)	26.5	26.5	30.0	30.0	30.0	30.0	30.0	30.0	26.5	16.0
Day	21	22	23	24	25	26	27	28	29	30
Demand (D)	26	23	25	25	25	25	27	24	22	23
Profit (P)	30.0	23.0	30.0	30.0	30.0	30.0	30.0	26.5	19.5	23.0

$$\text{Sale} = D \qquad \text{if } D \leq 25$$

$$= 25 \qquad D > 25$$

$$\text{Profit} = \text{Sale} \times 3.50 - 25 - 2.30$$

Average Profit = Rs. 26.38

In the above example, we have given the profitability of ordering 25 papers. In this way one can calculate the profitability of ordering any other number and make a choice of ordering that number which gives the maximum profit.

E1) Simulate a M/M/1 process with $\lambda = 0.6$ and $\mu = 1.0$ and find out average waiting time W_i by taking N = 10.

E2) The following data give the arrival times and service times that each customer will require for the first 13 customers at a single server system:

Arrival Times:	12	31	63	95	99	154	198	221	304	346	411	455	537
Service Times:	40	32	55	48	18	50	47	18	28	54	40	72	12

a) Determine the waiting times of 13 customers.

b) Determine the average waiting time.

16.3 TEST FOR RANDOMNESS

In Section 13.3, we have seen that pseudo random numbers (PRN) generated are completely deterministic. In this section, we shall describe some statistical tests to see that how close they are to independent random numbers from U (0, 1) distribution. There are two quite different kind of tests as follows:

1. Theoretical tests
2. Empirical tests

Theoretical tests are based on parameters a, c and m of equation in a global manner without actually generating U_i's at all. This is outside the scope of this course and we shall not describe them here. We shall discuss empirical test in details in further sections.

In the following sections we shall consider two types of tests. The first type of tests i.e. Chi-square and Kolmogrov-Smirnov, are tests of goodness of fit**.** For given numbers they test whether they are from the assumed distribution e.g. Normal, Exponential, Poisson, etc. or not.

The second type of test i.e. Runs Up-and-Down test, is a test of independence. This tests whether the consecutive numbers which appear in a sequence are independent or not (i.e. consecutive numbers do not have any trend).

Some Empirical Tests

16.3.1 Chi-Square Goodness of Fit Test

Let $x_1, x_2, \ldots, x_n$ be a sample from a distribution F(x) which is unknown. We wish to test the null hypothesis

$$H_0: F(x) = F_0(x), \qquad \text{for all } x$$

where, $F_0(x)$ is completely specified, against the alternative

$$H_1: F(x) \neq F_0(x), \qquad \text{for some } x$$

Assume that n observations have been grouped into k mutually exclusive categories. Denote n_j and np_{j0} be the observed and expected number for the j^{th} category, j = 1, 2, …, k. Here p_{j0} is the probability of an observation lying in the j^{th} class and is calculated from $F_0(x)$.

$$\chi^2 = \sum_{j=1}^{k} \frac{(n_j - np_{j0})^2}{np_{j0}},$$

where, $\sum_{j=1}^{k} n_j = n$

which should be small when H_0 is true and large when H_0 is false. The cut-off point between large and small is decided by distribution of statistic χ^2 which has approximately a Chi-square distribution on (k - 1) degrees of freedom when H_0 is true and n is large.

Under H_0 we have

$$P(\chi^2 > \chi^2_\alpha) = \alpha$$

and then we find χ^2_α from table of Chi-square distribution.

We reject H_0 if $\chi^2 > \chi^2_\alpha$.The α should be taken = 0.10 or 0.05.

When testing for uniform distribution, we divide the [0,1] interval into k non-overlapping classes of equal length 1/k.

Now, under H_0: $np_{j0} = n/k$

The test statistic is

$$\chi^2 = k\sum_{j=1}^{k} \frac{(n_j - n/k)^2}{n},$$

Under null hypothesis χ^2 has a Chi-square distribution on (k - 1) degrees of freedom. For good approximation of the distribution one should have n > 50,000 and $k = 4n^2/5$.

For a good approximation to Chi-square distribution the categories should be such that the expected frequencies in each class, np_{j0}, should be greater than 5. In case this is not so the consecutive classes should be merged to assure that the expected frequencies are greater than 5.

16.3.2 Kolmogrov-Smirnov Goodness of Fit Test

Let $x_1, x_2, ..., x_n$ denoete a random sample from unknown continuous distribution function F(x).The empirical distribution derived from sample, denoted by $F_n(x)$ is:

$$F_n(x) = (\text{Number of } x_i\text{'s} \le x)/n$$

If null hypothesis

$$H_0: F(x) = F_0(x) \quad \text{for all } x$$

is correct then one would expect that deviation $| F_0(x) - F_n(x) |$ should be small for all values of x. The asymptotic distribution of D_n

$$D_n = \sup_{\substack{x \\ \approx}} |F_0(x) - F_n(x)|$$

has been studied by Kolmogrov-Smirnov and the critical values, d_n, have been tabulated

$$P\,(D_n > d_{1-\alpha}) = \alpha$$

for various values of n.

In case $D_n > d_{1-\alpha}$ then we reject the null hypothesis at a significance level α. When $F_0(x)$ is uniform distribution U (0, 1) then D_n

$$D_n = \sup_{\substack{x \\ \approx}} |x - F_n(x)|$$

and one can test the goodness of random numbers generated.

Example 1: Twenty uniform U (0, 1) numbers (x) generated by a generator are given as below:

x	0.867	0.778	0.741	0.480	0.441	0.095	0.096	0.442	0.273	0.140
Rank (r_i)	17	15	14	11	9	2	3	10	6	4
x	0.968	0.786	0.019	0.330	0.408	0.681	0.144	0.978	0.889	0.579
Rank (r_i)	19	16	1	7	8	13	5	20	18	12

Demonstrate Kolmogrov-Smirnov test on these numbers.

Solution: We have the calculations in following table:

x	0.867	0.778	0.741	0.480	0.441	0.095	0.096	0.442	0.273	0.140
Rank (r_i)	17	15	14	11	9	2	3	10	6	4
$F_n(x)$ = r/n	0.850	0.750	0.700	0.550	0.450	0.100	0.150	0.500	0.300	0.200
$\|x-F_n(x)\|$	0.017	0.028	0.041	0.070	0.009	0.005	0.054	0.058	0.027	0.060
x	0.968	0.786	0.019	0.330	0.408	0.681	0.144	0.978	0.889	0.579
Rank (r_i)	19	16	1	7	8	13	5	20	18	12
$F_n(x)$ = r/n	0.950	0.800	0.050	0.350	0.400	0.650	0.250	1.000	0.900	0.600
$\|x-F_n(x)\|$	0.018	0.014	0.031	0.020	0.008	0.031	0.106*	0.022	0.003	0.021

We also have $F_0(x) = x$, so

$$D_n = \sup_{\underset{\approx}{x}} |x - F_n(x)|$$

$$D_n = 0.106$$

$d_{0.95} = 0.29$ for n = 20 (from Kolmogrov-Smirnov Table)

$$D_n = 0.106 < 0.29 \text{ (the tabulated value)}$$

Thus, null hypothesis is not rejected, i.e. there is no significant departure from uniform distribution.

16.3.3 Runs Up and Down Test (Test of Independence)

One important departure from randomness is the propensity for the occurrence of long monotonic sub-sequences. Run tests are designed to detect non random behaviour of this type. A run is a monotonic subsequence for assessing the randomness 'runs up', 'runs down' and 'runs up and down' may be considered. If it is too large or small than the number of runs expected in a sequence of independent numbers, then it indicates its departure from independence. Consider an example of a sequence of nine numbers:

1,	5,	4,	1,	3,	1,	3,	4,	7
+	–	–	+	–	+	+	+	+

+ sign is given when the next number is larger than the previous one and

– sign, when the next number is smaller than the previous one. This sequence contains three runs up (+) and two runs down (-) thus making a total, T, of five runs (T = 5).

Under independence, this statistic T is approximately normally distributed with mean and variance

$$E(T) = (2n-1)/3,$$

$$V(T) = (16n - 29)/90$$

for n greater than 25. Test statistic Z:

$$Z = \frac{T - \frac{(2n-1)}{3}}{\left[\frac{(16n-29)}{90}\right]^{1/2}}$$

has a standard normal distribution under null hypothesis and test can be applied.

One potential disadvantage of empirical test is that they are only local in nature i.e. only that segment of a cycle (of LCG for example) which was actually used, is tested for randomness, therefore we cannot say how the generator might perform in other segments of the cycle. On the other hand, this local nature of empirical tests can be an advantage, since it might allow us to examine the actual random numbers that will be used in simulation.

Example 2: A random number generator produces the following U(0,1) random numbers:

0.34	0.50	0.04	0.75	0.76	0.61	0.66	0.32	0.48	0.94
+	-	+	+	-	+	-	+	+	+
0.19	0.18	0.49	0.39	0.66	0.48	0.21	0.07	0.88	0.80
-	-	+	-	+	-	-	-	+	-
0.31	0.06	0.88	0.27	0.31	0.64	0.86	0.93	0.57	0.82
-	-	+	-	+	+	+	+	-	+
0.76	0.68	0.61	0.49	0.13	0.92	0.03	0.21	0.22	0.17
-	-	-	-	-	+	-	+	+	-

Apply Chi-square goodness of fit test to test that random numbers come from uniform distribution and successive numbers are independent.

Solution: We construct the following frequency table**:**

Class	**Frequency (n_j)**
0.0-0.2	8
0.2-0.4	9
0.4-0.6	6
0.6-0.8	9
0.8-1.0	8

Chi-square Goodness of Fit Test

Here one is interested in testing whether these numbers can be considered to be coming from a U(0,1) population.

Under uniform distribution $p_1 = p_2 = p_3 = p_4 = p_5 = 1/5$, $n = 40$,

$$\chi^2 = \sum_{j=1}^{5} \frac{(n_j - 8)^2}{8}$$

$$= (0^2 + 1^2 + 2^2 + 1^2 + 0^2) / 8$$

$$= 6 / 8 = 0.75$$

Therefore, $0.75 < \chi^2_{4,\, 0.05} = 9.488$

Hence, the null hypothesis is accepted i.e. the hypothesis of Uniform distribution is not rejected.

Run Test

Here one is interested in testing whether the consecutive numbers can be considered as independent.

Total number of runs (up and down) T = 24 and n = 40.

$E(T) = 79/3 = 26.333,$

$V(T) = 6.789,$

$$Z = \frac{(24 - 26.333)}{\sqrt{6.789}}$$

$$= \frac{(24 - 26.333)}{2.605} = -0.89$$

We have $\alpha/2 = 0.025$.

From table $Z_{\alpha/2} = Z_{0.025} = 1.96$

$|Z| = 0.89 < Z_{0.025} = 1.96$

Hence we do not reject the null hypothesis of randomness. Thus, we conclude that successive observations are independent.

E3) Following U (0, 1) were generated by a generator. Apply Chi-square goodness of fit test to test the fit of the distribution.

0.151	0.669	0.053	0.475	0.290	0.235	0.966	0.004	0.100
0.919	0.497	0.729	0.912	0.822	0.591	0.717	0.964	0.439
0.319	0.205	0.349	0.898	0.819	0.370	0.272	0.751	0.543
0.409	0.813	0.345	0.364	0.347	0.179	0.313	0.394	0.872
0.883	0.332	0.268	0.474					

E4) The following table gives the grouped data for number of items demanded per day. They were generated by Poisson distribution algorithm with mean $\lambda = 6$. By using Chi-square goodness of fit statistic test the hypothesis that the generated data fit the Poisson distribution with $\lambda = 6$.

Demand (X) :	≤3	4	5	6	7	8-9	≥10
Frequency (n_j):	12	10	12	18	10	20	5

E5) The following table gives the frequency distribution of 100 variables generated from N (0, 1) distribution. Using Chi- square goodness of fit test statistics test whether fit is satisfactory.

Class Interval	Frequency
≤ (-2.5)	02
(-2.5) - (-1.5)	04
(-1.5) - (-1.0)	08
(-1.0) - (-0.5)	18
(-0.5) - 0	19
0 - (0.5)	12
(0.5) - (1.0)	14
(1.0) - (1.5)	14
(1.5) - (2.0)	05
(2.0) - (2.5)	02
(2.5) - (3.0)	02

E6) Times between successive crashes of a computer system were generated for a 6-month period and are given in increasing order as follows (time in hours):

1 10 20 30 40 52 63 70 80 90 100 102 130 140 190 210 266 310 530 590 640 1340

The parameter $\alpha = 0.00435$, Mean $= 1/\alpha = 230$ hrs.

Use Kolmogrov-Smirnov test to examine the goodness of fit of Exponential distribution.

16.4 ADVANTAGES AND DISADVANTAGES OF SIMULATION

By using simulation techniques it is possible to study and experiment with the complex internal interactions of a given system whether it is a firm, an industry or economy.

It is possible to study the effect of certain changes which may lead to improvement of the system.

Simulation can be used to experiment with new situation about which we have little or no information. This is a very powerful tool for decision making. In some situations it may be used to verify analytical solutions and obtain solutions of problems which are difficult to solve analytically.

Simulation is a very useful technique for those situations where analytical techniques are inadequate. It has a number of disadvantages too. It is an imprecise technique which is subject to random variations rather than exact results. It gives results for the parameter values actually used in simulation. It may not be advisable to generalise those results which have not been covered in the simulation. Usefulness of the simulation depends on the adequacy of the model it represents. In case it is not correct representation of the system then conclusions drawn may not be valid for the system under study.

16.5 SUMMARY

In this unit, we have discussed:

1. Areas where simulation can be applied to solve certain problems which are very difficult to solve otherwise;
2. The queuing and inventory problems in particular which have many applications of simulation;
3. The advantages and disadvantages of simulation; and
4. Some empirical tests of randomness of a sequence of random numbers generated.

16.6 SOLUTIONS /ANSWERS

E1) Ten U (0, 1) variates given in the following table are:

U	0.34	0.5	0.04	0.75	0.76	0.61	0.66	0.32	0.48	0.94
I = (-logU)/λ	1.80	1.15	5.36	0.48	0.46	0.82	0.69	1.90	1.22	0.10
U	0.19	0.18	0.49	0.39	0.66	0.48	0.21	0.07	0.88	0.87
S= (-logU)/μ	1.66	1.71	0.71	0.94	0.41	0.73	1.56	2.66	0.13	0.14

Using the formula

$$W_n = W_{n-1} - I_n + S_n \quad \text{if } W_{n-1} > I_n,\ n = 1, 2, \ldots$$
$$= S_n \quad \text{if } W_{n-1} < I_n$$
$$W_1 = S_1$$

gives,

W: 1.66 2.22 0.71 1.17 1.12 1.03 1.90
2.66 1.57 1.61

Average, $\overline{W} = 1.565$

Theoretical value = 1/(μ− λ) = 1/0.4 = 2.5

E2) We have $I_i = t_i - t_{i-1}$

$$W_n = W_{n-1} - I_n + S_n \quad \text{if } W_{n-1} > I_n,\ n = 1, 2, \ldots$$
$$= S_n \quad \text{if } W_{n-1} < I_n$$
$$W_1 = S_1$$

I_i	12	19	32	32	4	55	44	23	83	42	65	44	82
S_i	40	32	55	48	18	50	47	18	28	54	40	72	12
W_i	40	53	76	92	106	101	104	99	44	56	40	72	12

Therefore, $\overline{W} = 68.85$

E3) Chi-square goodness of fit test:

Class Interval	Class Frequency (n_j)	np_{j0}
0-0.2	5	8
0.2-0.4	14	8
0.4-0.6	7	8
0.6-0.8	4	8
0.8-1.0	10	8

Expected frequency = 40/5 = 8 = np_{j0}

Then $$\chi^2 = \sum_{j=1}^{5} \frac{(n_j - np_{j0})^2}{np_{j0}} = \sum_{j=1}^{5} \frac{(n_j - 8)^2}{8}$$

$$= (3^2 + 6^2 + 1^2 + 4^2 + 2^2)/8$$

$$= 8.25 < \chi^2_{4,0.05} = 9.488$$

Hence, the null hypothesis is not rejected, i.e. there is no significant departure from the Uniform distribution.

E4) Following table gives the probability p_j for each class (obtained from Poisson table) with $\lambda = 6$:

p_{j0}	:	0.1512	0.1339	0.1606	0.1606	0.1377	0.1721	0.0839
np_{jo}	:	13.15	11.65	13.97	13.97	11.98	14.97	7.30
$n_j - np_{j0}$	:	-1.15	-1.65	-1.97	4.03	-1.98	5.03	-2.30

Therefore, $$\chi^2 = \sum_{j=1}^{5} \frac{(n_j - np_{j0})^2}{np_{j0}} = 4.517$$

Tabulated value of Chi-Square on 6 degrees of freedom for $\alpha = 0.05$, is $\chi^2_{6,\,0.05} = 12.6$.

Thus calculated $\chi^2 < \chi^2_{6,\,0.05}$. Hence we do not reject the null hypothesis i.e. the data generated give satisfactory fit.

E5) Seeing the Normal tables we have obtained probability of each class given as p_{j0}

Class Interval	p_{j0}	np_{j0}		Observed Frequency (n_j)		$n_j - np_{j0}$
≤ 25	0.0062	00.62		02		
(-2.5, -1.5)	0.0606	06.06	6.68	04	06	0.68
(-1.5, -1.0)	0.0918	09.18		08		1.18
(-1.0, -0.5)	0.1499	14.99		18		-3.01
(-0.5, 0.0)	0.1915	19.15		19		0.15
(0.0, 0.5)	0.1915	19.15		12		7.15
(0.5, 1.0)	0.1499	14.99		14		0.99
(1.0, 1.5)	0.0918	9.18		14		-4.82
(1.5, 2.0)	0.0441	4.41		05		
(2.0, 2.5)	0.0165	1.65	7.23	02	09	-1.77
(2.5, 3.0)	0.0117	1.17		02		

We have merged two classes in the top of the table and three in the bottom of the table so as to get better approximation to the Chi-square distribution. The calculated value of the test statistic is

$$\chi^2 = \sum_{j=1}^{8} \frac{\left(n_j - np_{j0}\right)^2}{np_{j0}} = 6.52$$

Tabulated value of Chi-Square on 7 degrees of freedom for $\alpha = 0.05$ is

$$\chi^2_{7,\,0.05} = 14.1$$

Thus $\chi^2 < 14.1$. Hence, we do not reject the null hypothesis and conclude that the fit is satisfactory.

E6) For the Exponential distribution

$$P(X \le x) = F_0(x) = 1 - e^{-\alpha x} = 1 - e^{-0.00435x}$$

The following table gives the required calculations $F_n(x) = x/22$

S.No.	1	2	3	4	5	6	7	8	9	10	11
x	1	10	20	30	40	52	63	70	80	90	100
$F_n(x)$	0.045	0.091	0.136	0.182	0.227	0.273	0.318	0.364	0.409	0.454	0.500
$F_0(x)$	0.009	0.051	0.032	0.122	0.160	0.202	0.240	0.262	0.294	0.324	0.353
$\|F_n(x)-F_0(x)\|$	0.036	0.040	0.104	0.060	0.067	0.071	0.078	0.102	0.115	0.130	0.147
S.No.	12	13	14	15	16	17	18	19	20	21	22
x	102	130	140	190	210	266	310	530	590	640	1340
$F_n(x)$	0.545	0.591	0.636	0.682	0.727	0.773	0.818	0.864	0.909	0.954	1.000
$F_0(x)$	0.358	0.432	0.456	0.562	0.599	0.686	0.740	0.900	0.923	0.938	0.997
$\|F_n(x)-F_0(x)\|$	0.187*	0.159	0.180	0.120	0.128	0.087	0.078	0.036	0.014	0.016	0.003

$$D_n = \sup_{\substack{x \\ \approx}} \left|F_0(x) - F_n(x)\right| = 0.187,$$

and $\quad d_{22,0.05} = 0.28$ (from table)

D_n observed is smaller than tabulated value of 0.28 at $\alpha = 0.05$. Hence, we do not reject the null hypothesis. The conclusion is that it gives a satisfactory fit.

TABLE 1: Random Number Table

03339	19233	50911	14209	39594	68368	97742	36252	27671	55091
97971	19968	31709	40197	16313	80020	01588	21654	50328	04577
16779	47712	33846	84716	49870	59670	46946	71716	50623	38681
12675	95993	08790	13241	71260	16558	83316	68482	10294	45137
55804	72742	16237	72550	10570	31470	92612	94917	48822	79794
16835	56263	53062	71543	67632	30337	28739	17582	40924	32434
84544	14327	07580	48813	30161	10746	96470	60680	63507	14435
63230	41243	90765	08867	08033	05038	10908	00633	21740	55450
33564	93563	10770	10595	71323	84243	09402	62877	49762	56151
57461	55618	40570	72906	30794	49144	65239	21788	38288	29180
91645	42451	83776	99246	45548	02457	74804	49536	89815	74285
78305	63797	26995	23146	56071	97081	22376	09819	56855	97424
97888	55122	65545	02904	40042	70653	24483	31258	96475	77668
67286	09001	09718	67231	54033	24185	52097	78713	95910	84400
53610	59459	89945	72102	66595	02198	26968	88467	46939	52318
52965	76189	68892	64541	02225	09603	59304	38179	75920	80486
25336	39735	25594	50557	96257	59700	27715	42432	27652	88151
73078	44371	77616	49296	55882	71507	30168	31876	28283	53424
31797	52244	38354	47800	48454	43304	14256	74281	82279	28882
47772	22798	36910	39986	34033	39868	24009	97123	59151	27583
54153	70832	37575	31898	39212	63993	05419	77565	73150	98537
93745	99871	37129	55032	94444	17884	27082	23502	06136	89476
81686	51330	58828	74199	87214	13727	80539	95037	73536	16862
79788	02193	33250	05865	53018	62394	56997	41534	01953	13763
92112	61235	68760	61201	02189	09424	24156	10368	26257	89107
87542	28171	45150	75523	66790	63963	13903	68498	02891	25219
37535	48342	48943	07719	20407	33748	93650	39356	01011	22099
95957	96668	69380	49091	90182	13205	71802	35482	27973	46814
34642	85350	53361	63940	79546	89956	96836	91313	80712	73572
50413	31008	09231	46516	61672	79954	01291	72278	55658	
84893									
53312	73768	59931	55182	43761	59424	79775	17772	41552	45236
16302	64092	76045	28958	21182	30050	96256	85737	86962	27067
96357	98654	01909	58799	87374	53184	87233	55275	59572	56476
38529	89095	89538	15600	33687	86353	61917	63876	52367	79032
45939	05014	06099	76041	57638	55342	41269	96173	94872	35605
02300	23739	68485	98567	77035	91533	62500	31548	09511	80252
59750	14131	24973	05962	83215	25950	43867	75213	21500	17758
21285	53607	82657	22053	88931	84439	94747	77982	61932	21928
93703	60164	19090	63030	88931	84439	94747	77982	61932	21928
15576	76654	19775	77518	43259	82790	08193	63007	68824	75315
12752	33321	69796	03625	37328	75200	77262	99004	96705	15540
89038	53455	93322	25069	88186	45026	31020	52540	10838	72490
62411	56968	08379	40159	27419	12024	99694	68668	73039	87682
45853	68103	38927	77105	65241	70387	01634	59665	30512	66161
84558	24272	84355	00116	68344	92805	52618	51584	75901	53021
45272	58388	69131	61075	80192	45959	76992	19210	27126	45525
68015	99001	11832	39832	80462	70468	89929	55695	77524	20675
13263	92240	89559	66545	06433	38634	36645	22350	81169	97417
66309	31446	97705	46996	69059	33771	95004	89037	38054	80853
56348	05291	38713	82303	26293	61319	45285	72784	50043	44438

TABLE 1 (Continued)

93108	77033	68325	10160	38667	62441	87023	94372	06164	30700
28271	08589	83279	48838	60935	70541	53814	95588	05832	80235
21841	35545	11148	25255	50283	94037	57463	92925	12042	91414
09210	20779	02994	02258	86978	85092	54052	18354	20914	28460
90552	71129	03621	20517	16908	06668	29916	51537	93658	29525
01130	06995	20258	10351	99248	51660	38861	49668	74742	47181
22604	56719	21784	68788	38358	59827	19270	99287	81193	43366
06690	01800	34272	65479	94891	14537	91358	21587	95765	72605
59809	69982	71809	64984	48709	43991	24987	69246	86400	29559
56475	02726	58511	95405	70293	84971	06676	44075	32338	31980
02730	34870	83209	03138	07715	31557	55242	61308	26507	06186
74482	33990	13509	92588	10462	76546	46097	01825	20153	36271
19793	22487	94238	81054	95488	23617	15539	94335	73822	93481
19020	27856	60526	24144	98021	60564	46373	86928	52135	74919
69565	60635	65709	77887	42766	86698	14004	94577	27936	47220
69274	23208	61035	84263	15034	28717	76146	22021	23779	98562
83658	14204	09445	40430	54072	82164	68977	95583	11765	81072
14980	74158	78216	38985	60838	82806	49777	85321	90463	11813
63172	28010	29405	91554	75195	51183	65805	87525	35952	83204
71167	37984	52737	06869	38122	95322	41356	19391	96787	64410
78530	56410	19195	34434	83712	20758	83454	22756	83959	96347
98324	03774	07573	67864	06497	20758	83454	22756	83959	96347
55793	30055	08373	32652	02654	75980	02095	87545	88815	80086
05674	34471	61967	91266	38814	44728	32455	17057	08339	93997
15643	22245	07592	22078	73628	60902	41561	54608	41023	98345
66750	19609	70358	03622	64898	82220	69304	46235	97332	64539
42320	74314	50222	82339	51564	42885	50482	98501	00245	88990
73752	73818	15470	04914	24936	65514	56633	72030	30856	85183
97546	02188	46373	21486	28221	08155	23486	66134	88799	49496
32569	52162	38444	42004	78011	16909	94194	79732	47114	23919
36048	93973	82596	28739	86985	58144	65007	08786	14826	04896
40455	36702	38965	56042	80023	28169	04174	65533	52718	55255
33597	47071	55618	51796	71027	46690	08002	45066	02870	60012
22828	96380	35883	15910	17211	42358	14056	55438	98148	35384
00631	95925	19324	31497	88118	06283	84596	72091	53987	01477
75722	36478	07634	63114	27164	15467	03983	09141	60562	65725
80577	01771	61510	17099	28731	41426	18853	41523	14914	76661
10524	20900	65463	83680	05005	11611	64426	59065	06758	02892
93185	69446	75253	51915	97839	75427	90685	60352	96288	34248
81867	97119	93446	20862	46591	97677	42704	13718	44975	67145
64649	07689	16711	12169	15238	74106	60655	56289	74166	78561
55768	09210	52439	33355	57884	36791	00853	49969	74814	09270
38080	49460	48137	61589	42742	92035	21766	19435	92579	27683
22360	16332	05343	34613	24013	98831	17157	44089	07366	66196
40521	09057	00239	51284	71556	22605	41293	54854	39736	05113
19292	40078	06838	05509	68581	39400	85615	52314	83202	40313
64138	27983	84048	42635	58658	62243	82572	45211	37060	15017

TABLE 2: The χ^2 Table

The first column identifies the specific χ^2 distribution according to its number of degrees of freedom. Other columns give the proportion of the area under the entire curve which falls above the tabled value of χ^2.

Area in the Upper Tail

df	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
1	0.00039	0.00016	0.00098	0.0039	0.016	2.71	3.84	5.02	6.63	7.88
2	0.010	0.020	0.051	0.10	0.21	4.61	5.99	7.38	9.21	10.60
3	0.072	0.11	0.22	0 .35	0.58	6.25	7.81	9.35	11.34	12.84
4	0.21	0.30	0.48	0.71	1.06	7.78	9.49	11.14	13.28	14.86
5	0.41	0.55	0.83	1.15	1.61	9.24	11.07	12.83	15.09	16.75
6	0.68	0.87	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	0.99	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	17.28	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	6.30	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	21.05	23.68	26.12	29.14	31.32
15	4.60	5.23	6.26	7.26	8.55	22.31	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	9.31	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.86	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	34.38	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	36.74	40.11	43.19	46.96	49.64
28	12.46	13.56	15.31	16.93	18.94	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	85.53	90.53	95.02	100.42	104.22
80	51.17	53.54	57.15	60.39	64.28	96.58	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	73.29	107.56	113.14	118.14	124.12	128.30
100	67.33	70.06	74.22	77.93	82.36	118.50	124.34	129.56	135.81	140.17
120	83.85	86.92	91.58	95.70	100.62	140.23	146.57	152.21	158.95	163.64

TABLE 3: Critical values of D in the Kolmogorov-Smimov One Sample Test

Sample size (n)	Level of significance of d = Sup $\lvert F_0(x) - F_n(x) \rvert$ .20	.15	.10	.05	.01
1	.900	.925	.950	.975	.995
2	.684	.726	.776	.842	.929
3	.565	.597	.642	.708	.828
4	.494	.525	.564	.624	.733
5	.446	.474	.510	.565	.669
6	.410	.436	.470	.521	.618
7	.381	.405	.438	.486	.577
8	.358	.381	.411	.457	.543
9	.339	.360	.383	.432	.514
10	.322	.342	.368	.410	.490
11	.307	.326	.352	.391	.468
12	.295	.313	.338	.375	.450
13	.284	.302	.325	.361	.433
14	.274	.292	.314	.349	.418
15	.266	.283	.304	.338	.404
16	.258	.274	.295	.328	.392
17	.250	.266	.286	.318	.381
18	.244	.259	.278	.309	.371
19	.237	.252	.272	.301	.363
20	.231	.246	.264	.294	.356
25	.21	.22	.24	.27	.32
30	.19	.20	.22	.24	.29
35	.18	.19	.21	.23	.27
Over 35	1.07	1.14	1.22	1.36	1.63