UNIT 4 STANDARD SAMPLING DISTRIBUTIONS-II

Structure

4.1 INTRODUCTION

In Unit 3, we have discussed chi-square and t-distributions in detail with their properties and applications. As we know that the sampling distributions χ^2, t and F are mostly in use. In this unit, we shall discuss the F-distribution in brief with their properties and applications. As we have said that F-distribution was introduced by Prof. R. A. Fisher and it is the ratio of two independent chi-square variates when divided by its respective degrees of freedom. Later on the F-distribution was also described by the square of t-variate.

As usual after giving a brief introduction of this unit in this Section 4.1, we start our discussion by giving an over view of the F-distribution in Section 4.2. Different properties of F-distribution are explained in Section 4.3. The probability curves of F-distribution for various degrees of freedom $(\nu_1 = 5, \nu_2 = 5)$, $(\nu_1 = 5, \nu_2 = 20)$ and $(\nu_1 = 20, \nu_2 = 5)$ are described along with the rth moment about origin, mean and variance. The distribution of F is used in testing the equality of two variances, equality of several population means, etc. Therefore in Section 4.4 the different applications of F-distribution are explored. As mentioned above, all the standard sampling distribution discussed in this unit and previous unit are very useful and inter-related. Therefore, in Section 4.5 the relations between the t, χ^2 and F -distributions are obtained. In Section 4.6, the method of obtaining the tabulated value of a variate is explained which follows either of t, χ^2 and F-distributions. Unit ends by providing summary of what we have discussed in this unit in Section 4.7 and solution of exercises in Section 4.8.

Objectives

After studying this unit, you should be able to:

- introduce the F-distribution;
- explain the properties of F-distribution;

- describe the probability curve of F-distribution;
- derive the mean and variance of F-distribution;
- explore the applications of F-distribution;
- describe the relations between t, χ^2 and F-distributions; and
- explain the method of obtaining the tabulated value of a variate which follows either of t, χ^2 and F-distributions.

4.2 INTRODUCTION TO F-DISTRIBUTION

As we have said in previous unit that F-distribution was introduced by Prof. R. A. Fisher and defined as the ratio of two independent chi-square variates when divided by their respective degrees of freedom. If we draw a random sample $X_1, X_2, ..., X_{n_1}$ of size n_1 from a normal population with mean μ_1 and variance σ_1^2 and another independent random sample $Y_1, Y_2, ..., Y_{n_2}$ of size n_2 from another normal population with mean μ_2 and variance σ_2^2 respectively then as we have studied in Unit 3 that $\nu_1 S_1^2 / \sigma_1^2$ is distributed as chi-square variate with ν_1 df i.e.

$$\chi_1^2 = \frac{\nu_1 S_1^2}{\sigma_1^2} \sim \chi^2_{(\nu_1)} \qquad \ldots (1)$$

where, $\nu_1 = n_1 - 1,\ \bar{X} = \frac{1}{n_1}\sum_{i=1}^{n_1} X_i$ and $S_1^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}\left(X_i - \bar{X}\right)^2$

Similarly, $\nu_2 S_2^2 / \sigma_2^2$ is distributed as chi-square variate with ν_2 df i.e.

$$\chi_2^2 = \frac{\nu_2 S_2^2}{\sigma_2^2} \sim \chi^2_{(\nu_2)} \qquad \ldots (2)$$

where, $\nu_2 = n_2 - 1,\ \bar{Y} = \frac{1}{n_2}\sum_{i=1}^{n_2} Y_i$ and $S_2^2 = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}\left(Y_i - \bar{Y}\right)^2$

Now, if we take the ratio of the above chi-square variates given in equations (1) and (2), then we get

$$\frac{\chi_1^2}{\chi_2^2} = \frac{\nu_1 S_1^2/\sigma_1^2}{\nu_2 S_2^2/\sigma_2^2}$$

$$\Rightarrow \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{\chi_1^2/\nu_1}{\chi_2^2/\nu_2} \sim F_{(\nu_1, \nu_2)} \qquad \ldots (3)$$

By observing the above form given in equation (3) we reveal that the ratio of two independent chi-square variates when divided by their respective degrees of freedom follows F-distribution where, ν_1 and ν_2 are called the degrees of freedom of F-distribution.

Now, if variances of both the populations are equal i.e. $\sigma_1^2 = \sigma_2^2$ then F-variate is written in the form of ratio of two sample variances i.e.

$$F = \frac{S_1^2}{S_2^2} \sim F_{(\nu_1, \nu_2)} \qquad \ldots (4)$$

By observing the above form of F-variate given in equation (4) we reveal that under the assumption $\sigma_1^2 = \sigma_2^2$, the ratio of two independent random sample variances follows F-distribution with ν_1 and ν_2 df. The probability density function of F-variate is given as

$$f(F) = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \frac{F^{(\nu_1/2)-1}}{\left(1 + \frac{\nu_1}{\nu_2}F\right)^{(\nu_1+\nu_2)/2}}; \qquad 0 < F < \infty \qquad \ldots (5)$$

After describing the F-distribution, we try to calculate the value of F-variate as in the given example.

Example 1: A statistician selects 7 women randomly from the population of women, and 12 men from a population of men. The table given below shows the standard deviation of each sample and population:

Population	Population Standard Deviation	Sample Standard Deviation
Women	40	45
Men	80	75

Compute the value of F-variate.

Solution: Here, we are given that

$$n_1 = 7,\ n_2 = 12,\ \sigma_1 = 40,\ \sigma_2 = 80,\ S_1 = 45,\ S_2 = 75$$

The value of F-variate can be calculated by the formula given below

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

where, S_1^2 & S_2^2 are the sample variances.

Therefore, we have

$$F = \frac{(45)^2/(40)^2}{(75)^2/(80)^2}$$

$$= \frac{1.27}{0.88} = 1.93$$

For the above calculation, the degrees of freedom ν_1 for women's data are $7 - 1 = 6$ and the degrees of freedom ν_2 for men's data are $12 - 1 = 11$.

Now, try to answer the following exercises to see how much you learn about F-distribution.

E1) Write down the pdf of F-distribution in each of the following cases:

(i) (4, 2) degrees of freedom

(ii) (2, 4) degrees of freedom

E2) If random variable X follows the F-distribution whose pdf is given by

$$f(x) = \frac{1}{(1+x)^2}; \qquad 0 < x < \infty$$

Obtain the degrees of freedom of this distribution.

E3) For the purpose of a survey 15 students are selected randomly from class A and 10 students are selected randomly from class B. At the stage of the analysis of the sample data, the following information is available:

Class	Population Standard Deviation	Sample Standard Deviation
Class A	65	60
Class B	45	50

Calculate the value of F-variate.

After introducing the F-distribution, one may be interested in knowing the properties of this distribution. Now, we shall discuss some of the important properties of F-distribution in the next section.

4.3 PROPERTIES OF F-DISTRIBUTION

The F-distribution has wide properties in Statistics. Some of them are as follow:

1. The probability curve of F-distribution is positively skewed curve. The curve becomes highly positive skewed when ν_2 is smaller than ν_1.
2. F-distribution curve extends on abscissa from 0 to ∞.
3. F-distribution is a uni-modal distribution, that is, it has single mode.
4. The square of t-variate with ν df follows F-distribution with 1 and ν degrees of freedom.
5. The mean of F-distribution with (ν_1, ν_2) df is $\frac{\nu_2}{\nu_2 - 2}$ for $\nu_2 > 2$.
6. The variance of F-distribution with (ν_1, ν_2) df is

$$\frac{2\nu_2^2\left(\nu_1 + \nu_2 - 2\right)}{\nu_1\left(\nu_2 - 2\right)^2\left(\nu_2 - 4\right)} \text{ for } \nu_2 > 4.$$

7. If we interchange the degrees of freedom ν_1 and ν_2 then there exists a very useful relation as

$$F_{(\nu_1, \nu_2),(1-\alpha)} = \frac{1}{F_{(\nu_2, \nu_1),\alpha}}$$

Here, we have deliberated some of the important properties of the F-distribution and some of them have discussed such as probability curve, mean and variance of F-distribution in Sub-sections 4.3.1 and 4.3.2 respectively.

4.3.1 Probability Curve of F-distribution

In Section 4.2, we have defined the F-variate as the ratio of two sample variances when population variances are equal and follows F-distribution with ν_1 and ν_2 df i.e.

$$F = \frac{S_1^2}{S_2^2} \sim F_{(\nu_1, \nu_2)} \qquad \ldots (6)$$

where, S_1^2 and S_2^2 are the sample variances of two normal populations. As you have studied that the F-variate varies from 0 to ∞, therefore, it is always positive so probability curve of F-distribution wholly lies in the first quadrant. These ν_1 and ν_2 are the degrees of freedom and are called the parameters of F-distribution. Hence, the shape of probability curve F-distribution depends on ν_1 and ν_2. The probability curves of F-distribution for various pairs of degrees of freedom (ν_1, ν_2) i.e. (5, 5), (5, 20) and (20, 5) are shown in Fig. 4.1 given below:

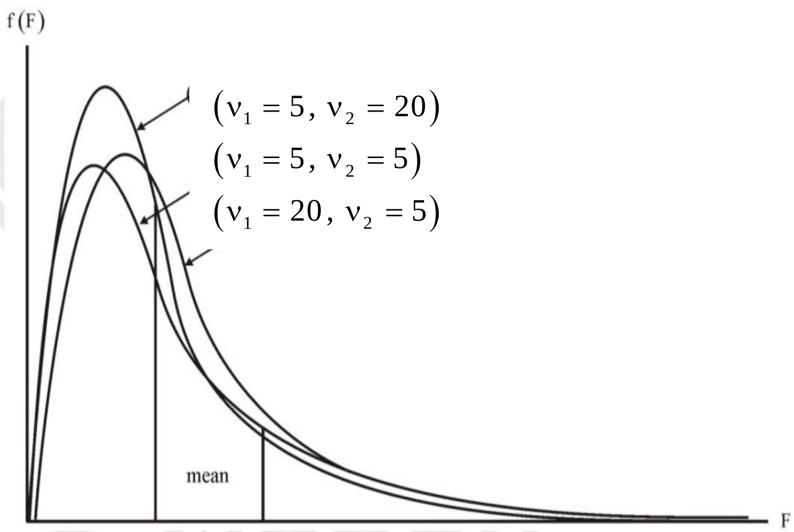

Fig. 4.1: Probability curves of F-distribution for (5, 5), (5, 20) and (20, 5) degrees of freedom.

After looking on the probability curves of F-distribution drawn above, we can observe that the probability curve of the F-distribution is uni-model curve. And by increasing in the first degrees of freedom from $\nu_1 = 5$ to $\nu_1 = 20$ the mean of the distribution (shown by vertical line) does not change but probability curve shifted from the tail to the centre of the distribution whereas increasing in the second degrees of freedom from $\nu_2 = 5$ to $\nu_2 = 20$ the mean of the distribution (shown by vertical line) decrease and the probability curve shifted from the tail to the centre of the distribution. One can also get an idea about the skewness of the F-distribution, we have observed from the probability curve of the F-distribution that it is positive skewed curve and it becomes very highly positive skewed if ν_2 becomes small.

4.3.2 Mean and Variance of F-distribution

In previous sub-section, the probability curve of F-distribution is discussed with some of its features. Now in this sub-section, we shall describe the mean and variance of the F-distribution. The mean and variance of F-distribution is derived with the help of the moment about origin. As we have discussed in Unit 3 of MST-002 that the first order moment about origin is known as mean and central second order moment of is known as variance of the distribution. The r[th] order moment about origin of F-distribution is obtained as

$$\mu_r' = E\left(F^r\right) = \int_0^\infty F^r f\left(F\right) dF$$

$$= \frac{\left(\nu_1/\nu_2\right)^{\nu_1/2}}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \int_0^\infty F^r \frac{F^{(\nu_1/2)-1}}{\left(1+\frac{\nu_1}{\nu_2}F\right)^{(\nu_1+\nu_2)/2}} dF \qquad \ldots (7)$$

If we take $\frac{\nu_1}{\nu_2}F = x \Rightarrow F = \frac{\nu_2}{\nu_1}x$ then $dF = \frac{\nu_2}{\nu_1}dx$. Also when $F = 0 \Rightarrow x = 0$ and when $F \to \infty \Rightarrow x \to \infty$.

Now, by putting these values in above equation (7), we get

$$\mu_r' = \frac{\left(\nu_1/\nu_2\right)^{\nu_1/2}}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \int_0^\infty \frac{\left\{\left(\frac{\nu_2}{\nu_1}\right)x\right\}^{\frac{\nu_1}{2}+r-1}}{\left(1+x\right)^{(\nu_1+\nu_2)/2}}\left(\frac{\nu_2}{\nu_1}\right)dx$$

$$= \frac{\left(\frac{\nu_2}{\nu_1}\right)^r}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \int_0^\infty \frac{x^{(\nu_1/2)+r-1}}{\left(1+x\right)^{(\nu_1/2)+r+\{(\nu_2/2)-r\}}} dx$$

$$= \frac{\left(\frac{\nu_2}{\nu_1}\right)^r}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} B\left(\frac{\nu_1}{2}+r,\frac{\nu_2}{2}-r\right) \qquad \left[\begin{array}{l}\because \text{For } a>0 \;\&\; b>0 \\ \int_0^\infty \frac{x^{a-1}}{(1+x)^{a+b}}dx \\ \qquad = B\left(a,b\right)\end{array}\right]$$

Thus,

$$\mu_r' = \left(\frac{\nu_2}{\nu_1}\right)^r \frac{\Gamma\left(\frac{\nu_1}{2}+r\right)\Gamma\left(\frac{\nu_2}{2}-r\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \text{ for } \nu_2 > 2r \quad \left[B\left(a,b\right) = \frac{\Gamma\left(a+b\right)}{\Gamma a\,\Gamma b}\right] \qquad \ldots (8)$$

Now, if we put r = 1 in the above expression given in equation (8), we get the value of first order moment about origin which is known as mean i.e.

$$\text{Mean} = \mu_1' = \left(\frac{\nu_2}{\nu_1}\right).\frac{\Gamma\left(\frac{\nu_1}{2}+1\right)\Gamma\left(\frac{\nu_2}{2}-1\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \text{ for } \nu_2 > 2$$

$$= \left(\frac{\nu_2}{\nu_1}\right).\frac{\frac{\nu_1}{2}\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}-1\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\left(\frac{\nu_2}{2}-1\right)\Gamma\left(\frac{\nu_2}{2}-1\right)} \qquad \left(\because \Gamma r = \left(r-1\right)\Gamma\left(r-1\right)\right)$$

$$= \frac{\nu_2}{\nu_2-2} \quad \text{for} \quad \nu_2 > 2 \qquad \ldots (9)$$

Similarly, if we put r = 2 in the formula of μ_r' given in equation (8) then we get the value of second order moment about origin i.e.

$$\mu_2' = \left(\frac{\nu_2}{\nu_1}\right)^2 . \frac{\Gamma\left(\frac{\nu_1}{2}+2\right)\Gamma\left(\frac{\nu_2}{2}-2\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \quad \text{for } \nu_2 > 4$$

$$= \left(\frac{\nu_2}{\nu_1}\right)^2 \frac{\left(\frac{\nu_1}{2}+1\right)\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}-2\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\left(\frac{\nu_2}{2}-1\right)\left(\frac{\nu_2}{2}-2\right)\Gamma\left(\frac{\nu_2}{2}-2\right)} \quad \text{for } \nu_2 > 4$$

$$= \frac{\nu_2^2(\nu_1+2)}{\nu_1(\nu_2-2)(\nu_2-4)} \quad \text{for } \nu_2 > 4 \qquad \ldots (10)$$

Now, we obtain the value of variance by putting the value of first order and second order moments about origin in the formula given below

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

$$= \frac{\nu_2^2(\nu_1+2)}{\nu_1(\nu_2-2)(\nu_2-4)} - \frac{\nu_2^2}{(\nu_2-2)^2}$$

$$= \frac{\nu_2^2\left[(\nu_1+2)(\nu_2-2)-\nu_1(\nu_2-4)\right]}{\nu_1(\nu_2-2)^2(\nu_2-4)}$$

$$= \frac{\nu_2^2(\nu_1\nu_2-2\nu_1+2\nu_2-4-\nu_1\nu_2+4\nu_1)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$$

$$= \frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)} \quad \text{for } \nu_2 > 4 \qquad \ldots (11)$$

From the above derivation of mean and variance, we can conclude that mean of F-distribution is independent of ν_1.

Similarly, on putting r = 3 and 4 in the formula given in equation (8) one may get μ_3' and μ_4' which are the third and fourth order moments about origin respectively.

Now, you can try some exercises to see how much you learn about the properties of F-distribution.

E4) Write any five properties of F-distribution.

E5) What are the mean and variance of F-distribution with $\nu_1 = 5$ and $\nu_2 = 10$ degrees of freedom?

4.4 APPLICATIONS OF F-DISTRIBUTION

After discussing the main properties of F-distribution in previous section, we are now discussing some of the important applications of F-distribution.

Though F-distribution has many applications but some of them are listed here.

The F-distribution has the following applications:

1. F-distribution is used to test the hypothesis about equality of the variances of two normal populations.
2. F-distribution is used to test the hypothesis about multiple correlation coefficients.
3. F-distribution is used to test the hypothesis about correlation ratio.
4. F-distribution is used to test the equality of means of k-populations, when one characteristic of the population is considered i.e. F-distribution is used in one-way analysis of variance.
5. F-distribution is used to test the equality of k-population means for two characteristics at a time i.e. F-distribution is used in two-way analysis of variance.

The first application listed above will be discussed in Unit 12 of this course, fourth and fifth applications will be discussed in Block-2 of MST-005, second application will be discussed in detail in specialisation courses and the remaining third application is beyond the scope of this course.

Now, try the following exercise.

E6) Write four applications of F-distribution.

4.5 RELATION BETWEEN t, χ^2 AND F-DISTRIBUTIONS

After describing the exact sampling distributions χ^2, t & F, we shall explain relationship between t & F-distributions and χ^2 & F-distributions. These relations can be described in terms of the following two statements:

Statement 1: If a variate t follows Student's t-distribution with ν df, then square of t follows F-distribution with (1, ν) df i.e. if $t \sim t_{(\nu)}$ then $t^2 \sim F(1, \nu)$.

Statement 2: If F-variate follows F-distribution with (ν_1, ν_2) df and if

$\nu_2 \to \infty$ then $\chi^2 = \nu_1 F$ follows a χ^2-distribution with ν_1 df.

The proofs of these statements are beyond the scope of this course.

Now, try to answer the following exercises.

E7) If a variate t follows Student's t-distribution with 4 df, then what is the distribution of square of t. Also write the pdf of that distribution.

4.6 TABULATED VALUES OF t, χ^2 AND F

As we have seen in Unit14 of MST-003 that the area or probability of normal distribution can be calculated with the help of tables. Similarly, we can also find the area or probability of a variate which follows either of t, χ^2 and F-distributions. In this section, we shall discuss the method of obtaining the tabulated values of t, χ^2 and F-variates.

4.6.1 Tabulated Values of t-Variate

A variate which follows the t-distribution is described with the help of degrees of freedom (ν). **Table-II** in the Appendix contains the tabulated (critical) values of t-variate for different degrees of freedom (ν) such that the area under the probability curve of t-distribution to its right tail (upper tail) is equal to α as shown in Fig. 4.2. The critical value of t-variate is represented by $t_{(\nu),\alpha}$ where, ν represents the df and α area under the right tail or level of significance.

For level of significance, please see Unit 9 of this course.

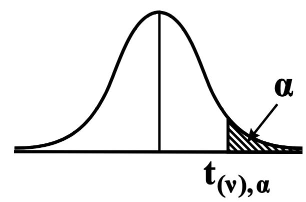

Fig. 4.2

In this table, the first column of left hand side contains values of degrees of freedom (ν) and the column heading represents the values of α on the right hand side (tail) of the probability curve of the t-distribution and the entries in the body of the table are the values of t-variate for each particular value of ν and α. We discuss the process of finding the values of t-variate by taking an example as:

If we want to find out the value of t-variate for which the area on the right tail is 0.05 and the degrees of freedom is 8. Then we start with the first column of the t-table and downward headed 'ν' until entry 8 is reached. Then proceed right to the column headed α = 0.05. For your convenient, part of the t-table is shown in Table 4.1 given below:

Table 4.1: t-table for Tabulated (Critical) Values of t-variate

One- tail α =	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**
ν =1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355

Thus, we get the required value of t-variate as $t_{(\nu)} = t_{(\nu),\alpha} = t_{(8),0.05} = 1.860$. The value of t-variate equal to 1.860 means, the probability that t-variate would exceed 1.860 is 0.05 as shown in Fig. 4.3.

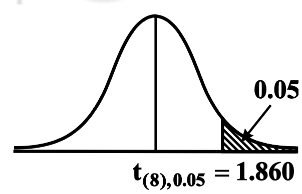

Fig. 4.3

Since t-distribution is symmetrical about t = 0 line, therefore, it is sufficient to tabulate the values of t-variate for right tail only. Therefore, the value of t-variate corresponding to a left hand tail for α is $-t_{(\nu),\alpha}$ as shown in Fig. 4.4.Thus, in above example, if we want to find the value of t-variate such that left area is 0.05 for 8 df then due to symmetry of t-distribution the value of t-variate is −1.860.

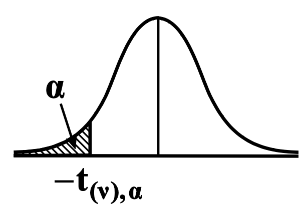

Fig. 4.4

If we want to find the values of t-variate when the total area on both tails is α then by symmetry the half area i.e. α/2 lies in both tails as shown in Fig. 4.5. In such situation, we find the value of t-variate corresponding to α/2 area and there are two values of t-variate as $\pm t_{(\nu),\alpha/2}$. For example, if we want to find out the values of t-variate for which the area on the both tails is 0.05 and the degrees of freedom is 8. Then by symmetry, we find the value of variate t corresponding to α/2 = 0.05/2 = 0.025 and 8 df. Therefore, from the t-table we get $t_{(\nu),\alpha/2} = t_{(8),0.025} = 2.571$. So required values of t-variate are $\pm t_{(\nu),\alpha/2} = \pm t_{(8),0.025} = \pm 2.571$.

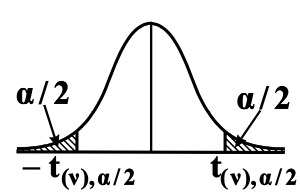

Fig. 4.5

Example 2: Find the value of t-variate in each case for which the area on the right tail (α) and the degrees of freedom (ν) are given in next page:

(i) $\alpha = 0.01$ and $\nu = 12$

(ii) $\alpha = 0.10$ and $\nu = 15$

Solution:

(i) Here, we want to find the value of t-variate for

$\alpha = 0.01$ and $\nu = 12$

Therefore, we start with the first column of t-table given in the Appendix and downward headed ν until entry 12 is reached. Then proceed right to the column headed $\alpha = 0.01$. So we get the required value of t-variate as $t_{(\nu),\, \alpha} = t_{(12),\, 0.01} = 2.681$.

(ii) Here, we are given that

$\alpha = 0.10$ and $\nu = 15$

By proceeding same way as above, from t-table, we get the required value of t-variate as $t_{(\nu),\, \alpha} = t_{(15),\, 0.10} = 1.341$. For your convenient, part of the t-table is shown in Table 4.2 given below:

Table 4.2: t-Table for Tabulated (Critical) Values of t-variate

One- tail α =	0.10	0.05	0.025	0.01	0.005
ν = 10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921

Example 3: Find the values of t-variate in each part for which the area on the both tails and the degrees of freedom are

(i) $\alpha = 0.01$ and $\nu = 12$

(ii) $\alpha = 0.10$ and $\nu = 29$

Solution:

(i) Here, we want to find the value of t-variate for two-tails and for

$\alpha = 0.01$ and $\nu = 12$

Since total area on the both tails is 0.01 therefore, by symmetry the area on the right tail is 0.01/2 = 0.005. Thus, we start with the first column of t-table and downward headed ν until entry 12 is reached. Then we proceed right to the column headed $\alpha = 0.005$. We get $t_{(\nu),\, \alpha/2} = t_{(12),\, 0.005} = 3.055$. So the required values of t-variate are $\pm\, t_{(\nu),\, \alpha/2} = \pm\, t_{(12),\, 0.005} = \pm\, 3.055$.

(ii) Here, we are given that

$\alpha = 0.10$ and $\nu = 29$

By proceeding same way as above, from the t-table, we get the required values of t-variate as $\pm\, t_{(\nu),\, \alpha/2} = \pm\, t_{(29),\, 0.05} = \pm\, 1.699$.

Method of Finding the Values of t-variate for Degrees of Freedom which are not listed in the Table

The t-table given in the Appendix does not list values for every possible degree of freedom. Therefore, it becomes necessary to know how to find values of t-variate for degrees of freedom not listed in the table.

We discuss the process of finding the values which are not listed in the table with the help of an example as:

Suppose we want to find out the value of t-variate for 34 degrees of freedom such that the area on the right side is equal to 0.05. The t-table in the Appendix does not list a tabulated value for this degree of freedom, so we need to interpolate it.

First of all, we read the tabulated values of t-variate that are just greater and just smaller than the degrees of freedom for our interest. Thus, from the t-table, we get the values of t-variate for 40 and 30 df and area $\alpha = 0.05$ as

$$t_{(40),\,0.05} = 1.684 \qquad t_{(30),\,0.05} = 1.697$$

Note that larger the degrees of freedom smaller the tabulated value of t-variate.

Now, we calculate how much the t-value changes for each degree of freedom between these two tabulated values. Here, there is a difference of 10(40 – 30) degrees of freedom and t-value change of 0.013(1.697 – 1.684).

Thus, to determine how much it changes for each degree of freedom, divide the difference between the t-values by the difference between the degrees of freedom as

$$\frac{0.013}{10} = 0.0013$$

Since, we have to obtain the value for 34 degrees of freedom, this is either 4 more than 30 or 6 less than 40. Therefore, we can interpolate from either values. To get from 30 to 34 degrees of freedom there is a difference of 4 (34 – 30). So multiply this difference by the amount by which the t-value change per degree of freedom i.e. 0.0013. This result as

$$4 \times 0.0013 = 0.0052$$

Since larger the degrees of freedom smaller the tabulated value of t-variate therefore, subtracting this value 0.0052 from $t_{(30,\,0.05)} = 1.697$ to get the required value. Thus,

$$t_{(34),\,0.05} = 1.697 - 0.0052 = 1.6918$$

Now, if we interpolate from 40 degrees of freedom then the difference 6(40 – 34) multiply by 0.0013 and adding this to 1.684, we get

$$t_{(34),\,0.05} = 1.684 + 6 \times 0.0013 = 1.6918$$

Thus, we get the same value.

Example 4: Find the value of t-variate in each part for which the area on the right tail (α) and the degrees of freedom (ν) are given below:

(i) $\alpha = 0.01$ and $\nu = 42$ (ii) $\alpha = 0.01$ and $\nu = 35$

Solution:

(i) Here, we want to find the value of t-variate for

$$\alpha = 0.01 \text{ and } \nu = 42$$

Since t-table does not have the tabulated value for 42 degrees of freedom so we need to interpolate it. For this, we find the tabulated values of t-variate that are just greater and just less than the degree of freedom 42. Thus, we get

$$t_{(60),\,0.01} = 2.390 \quad t_{(40),\,0.01} = 2.423$$

There is a difference of 20 degrees of freedom between these two and a difference 0.033(2.423-2.390) in the t-values.

Thus, each degree of freedom has an approximate change in the value of t-variate is

$$\frac{0.033}{20} = 0.00165$$

To get 42 degrees of freedom, multiply 0.000165 by 2 (42−40), we get

$$0.00165 \times 2 = 0.003$$

Since larger the degrees of freedom smaller the tabulated value of t-variate, therefore, subtracting this from 2.423, we get required values as

$$t_{(42),\,0.01} = 2.423 - 0.003 = 2.420$$

(ii) Similarly, we find the tabulated values of t-variate that are just greater and just less than the degree of freedom 35. Thus, we have

$$t_{(40),\,0.01} = 2.423 \quad t_{(30),\,0.01} = 2.457$$

There is a difference of 10 degrees of freedom between these two and a difference 0.034(2.457−2.423) in the t-values.

Thus, each degree of freedom has an approximate change in the value of t-variate is

$$\frac{0.034}{10} = 0.0034$$

To get 35 degrees of freedom, multiplying 0.0034 by 5 (35−30), we have

$$0.0034 \times 5 = 0.017$$

Since larger the degrees of freedom smaller the tabulated value of t-variate, therefore, subtracting this from 2.457, we get required value as

$$t_{(35),\,0.01} = 2.457 - 0.017 = 2.440$$

Now, you can try the following exercise.

E8) Find the values of t-variate for which the area and the degrees of freedom are

(i) $\alpha = 0.01$ (one-right tail) and $\nu = 11$ (ii) $\alpha = 0.05$ (one-left tail) and $\nu = 16$

(iii) $\alpha = 0.05$ (two-tails) and $\nu = 19$ (iv) $\alpha = 0.05$ (two-tails) and $\nu = 45$

4.6.2 Tabulated Value of χ^2-Variate

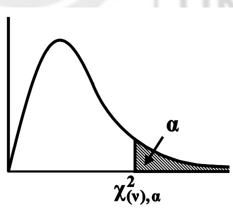

Fig. 4.6

As similar to t-table, the **Table-III** in the Appendix contains the tabulated (critical) values of chi-square variate for different degrees of freedom (ν) such that the area under the probability curve of χ^2-distribution to its right (upper) tail is equal to α as shown in Fig. 4.6.

In this table, the column headings indicates the area on the upper portion (right tail) of the probability curve of chi-square distribution and the first column on the left hand side indicates the values of degrees of freedom (ν). Also the entry in the body of the table indicates the value of chi-square variate. Therefore, $\chi^2_{(\nu),\alpha}$ is the value such that the area under the probability curve of chi-square distribution with ν df to its right tail is equal to α i.e. the probability that any value of χ^2-variate is greater than or equal to $\chi^2_{(\nu),\alpha}$ is α. We discuss the process of finding the values of χ^2-variate by taking an example as:

If we want to find out the value of chi-square variate with 7 df for which the area on the right tail is 0.01 then we start with first column i.e. degrees of freedom (ν) and proceed downward until 7 is reached then proceed right to the column headed α = 0.01 which gives the required value of χ^2 as shown in the Table 4.3 given below and is also indicate in Fig. 4.7 as

$$\chi^2_\alpha = \chi^2_{(\nu),\alpha} = \chi^2_{(7),0.01} = 18.48.$$

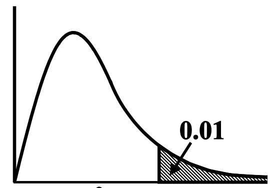

Fig. 4.7

Table 4.3: Chi-square Table

α =	**0.995**	**0.99**	**0.975**	**0.95**	**0.90**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**
ν =1	---	---	---	---	0.02	2.71	3.84	5.02	6.63	7.88
2	0.01	0.02	0.05	0.10	0.21	4.61	5.99	7.38	9.21	10.60
3	0.07	0.11	0.22	0.35	0.58	6.25	7.81	9.35	11.34	12.84
4	0.21	0.30	0.48	0.71	1.06	7.78	9.49	11.14	13.28	14.86
5	0.41	0.55	0.83	1.15	1.61	9.24	11.07	12.83	15.09	16.75
6	0.68	0.87	1.24	1.64	2.20	10.64	12.59	14.45	16.81	18.55
7	0.99	1.24	1.69	2.17	2.83	12.02	14.07	16.01	18.48	20.28

The chi-square distribution is a skewed (not symmetrical) distribution. Therefore, it is necessary to tabulate values of χ^2 for $\alpha > 0.50$ also.

Note 1: Values of chi-square variate for degrees of freedom which are not listed in the table are obtained in the similar manner as discussed in case of t-variate. This is explained in part (iv) of the Example 5 given below.

Now, let us do one example based on the above discussion.

Example 5: Find the values of chi-square variate for which the area and degrees of freedom are

(i) α = 0.05(one-tail) and ν = 2

(ii) α = 0.99(one-tail) and ν = 15

(iii) α = 0.05(two-tails) and ν = 6

(iv) α = 0.01(one-tail) and ν = 64

Solution:

(i) Here, we want to find the value of χ^2-variate for one-tail and for

$$\alpha = 0.05 \text{ and } \nu = 2$$

Thus, we first start with the first column of χ^2-table given in the Appendix and downward headed ν until entry 2 is reached. Then proceed right to the column headed α = 0.05. We get the required value of χ^2-variate as $\chi^2_{(\nu),\alpha} = \chi^2_{(2),0.05} = 5.99$.

(ii) Here, we want to find the value of χ^2-variate for one-tail and for

$\alpha = 0.99$ and $\nu = 15$

By proceeding same way as above, from the χ^2-table, we get the required value of χ^2-variate as $\chi^2_{(\nu),\alpha} = \chi^2_{(15),0.99} = 5.23.$

(iii) Here, we want to find the value of χ^2-variate for two-tails and for

$\alpha = 0.05$ and $\nu = 6$

Since total area on the both tails is 0.05 therefore, half area lies on both tails as 0.05/2 = 0.025. In this case, χ^2-variate has two values one on right-tail as $\chi^2_{\alpha/2} = \chi^2_{(\nu),\alpha/2} = \chi^2_{(6),0.025}$ and one on left-tail as $\chi^2_{(1-\alpha/2)} = \chi^2_{(\nu),(1-\alpha/2)} = \chi^2_{(6),0.975}$. So by proceeding same way as above we get required values of χ^2-variate as $\chi^2_{(6),0.025} = 14.45$ and $\chi^2_{(6),0.975} = 1.24$ as shown in Fig.4.8.

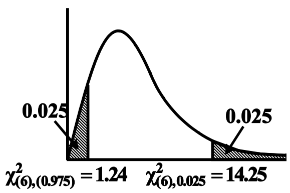

Fig. 4.8

(iv) Here, we want to find the value of χ^2-variate for one-tail and for

$\alpha = 0.01$ and $\nu = 64$

Since χ^2-table does not have the tabulated value for 64 degrees of freedom so we need to interpolate it. For this we find the tabulated values of χ^2-variate that are just greater and just less than the degree of freedom 64 with $\alpha = 0.01$. Thus, we have

$$\chi^2_{(70),\,0.01} = 100.42 \qquad \chi^2_{(60),\,0.01} = 88.38$$

There is a difference of 10 degrees of freedom between these two and a difference 12.04(100.42−88.38) in the χ^2-value. Also note that larger the degrees of freedom larger the tabulated value of χ^2-variate.

Thus, each degree of freedom has an approximate change in the value of χ^2-variate as

$$\frac{12.04}{10} = 1.204$$

To get the value of χ^2-variate for 64 degrees of freedom, multiplying 1.204 by 4 (64−60), we get

$$1.204 \times 4 = 4.816$$

Since larger the degrees of freedom larger the tabulated value of χ^2-variate so adding 4.816 in 88.38, we get required value as

$$\chi^2_{(64),0.01} = 88.38 + 4.816 = 93.196$$

Now, you can try the following exercise.

E9) Find the value of χ^2-variate for which the area on the right tail and the degrees of freedom are

(i) $\alpha = 0.01$ and $\nu = 11$

(ii) $\alpha = 0.90$ and $\nu = 19$

(iii) $\alpha = 0.05$ and $\nu = 44$

4.6.3 Tabulated Value of F-Variate

F-distribution is described with the help of two degrees of freedom, one for numerator and other for denominator. For each combination of degrees of freedom, F-variate has different tabulated value. **Table-IV** in the Appendix contains the tabulated (critical) values of F-variate for various degrees of freedom such that the area under the probability curve of F-distribution to its right (upper) tail is equal to $\alpha = 0.10, 0.05, 0.025$ and 0.01. A general case is shown in Fig. 4.9.

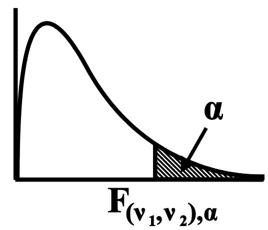

Fig. 4.9

The first row of F-table indicates the values of degrees of freedom (ν_1) for numerator and first column on the left hand side indicates the values of degrees of freedom (ν_2) for denominator. We discuss the process of finding the tabulated values of F-variate by taking an example as:

Suppose we want to find the value of F-variate for which the area on the right tail is 0.05 and the degrees of freedom for numerator and denominator are $\nu_1 = 10$ and $\nu_2 = 15$ respectively. Since $\alpha = 0.05$ so we select the F-table for $\alpha = 0.05$ and start with the first column of this table and proceed down to this column headed ν_2 until entry 15 is reached then proceed right to the column headed $\nu_1 = 10$ as shown in the Table 4.4. In this way, we get the required value of F as $F_{(\nu_1,\nu_2),\alpha} = F_{(10,15),0.05} = 2.54$ which is also shown in Fig. 4.10.

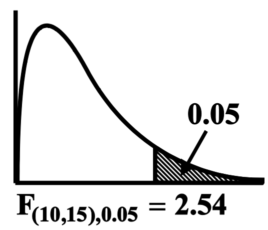

Fig. 4.10

Table 4.4: F-Table for α = 0.05

Degrees of freedom for numerator (ν_2)	Degrees of freedom for numerator(ν_1)											
	1	2	3	4	5	6	7	8	9	10	11	12
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42

Note 2: The value of F-variate for ν_1 degrees of freedom for numerator and ν_2 for denominator is not same for ν_2 degrees of freedom for numerator and ν_1 for denominator, that is,

$$F_{(\nu_1,\nu_2)} \neq F_{(\nu_2,\nu_1)}$$

The F-distribution is also a skewed (not symmetrical) distribution. Therefore, it becomes necessary to tabulate values of F-distribution for $\alpha > 0.50$ also. The tabulated values of F-distribution for $\alpha > 0.50$ can be calculated by the relation as

$$F_{(\nu_1,\nu_2),(1-\alpha)} = \frac{1}{F_{(\nu_2,\nu_1),\alpha}} \quad \text{[Recall property listed in Section 4.3]}$$

Note 3: Values of F-variate for degrees of freedom which are not listed in the table are obtained in the similar manner as discussed in case of t-variate. This is explained in part (iv) of the Example 6 given below.

Now, let us do one example based on the above discussion.

Example 6: Find the values of F-variate in each part for which the area (α) and the degrees of freedom(ν_1, ν_2) are

(i) $\alpha = 0.01$(one-tail) and ($\nu_1 = 5$, $\nu_2 = 8$)

(ii) $\alpha = 0.95$(one-tail) and ($\nu_1 = 12$, $\nu_2 = 16$)

(iii) $\alpha = 0.05$ (two-tails)and ($\nu_1 = 4$, $\nu_2 = 10$)

(iv) $\alpha = 0.01$(one-tail) and ($\nu_1 = 10$, $\nu_2 = 32$)

Solution:

(i) Here, we want to find the value of F-variate for

$$\alpha = 0.01 \text{ and } (\nu_1 = 5,\ \nu_2 = 8)$$

Thus, we select the F-table for $\alpha = 0.01$ and start with the first column of this table and downward headed ν_2 until entry 8 is reached. Then proceed right to the column headed $\nu_1 = 5$. We get the required value of F-variate as 6.63.

(ii) Here, we want to find the value of F-variate for

$$\alpha = 0.95 \text{ and } (\nu_1 = 12,\ \nu_2 = 16)$$

Since we have F-tables for $\alpha = 0.10$, 0.05, 0.025 and 0.01only therefore, for $\alpha = 0.95$, we can use the relation

$$F_{(\nu_1,\nu_2),(1-\alpha)} = \frac{1}{F_{(\nu_2,\nu_1),\alpha}}$$

From the F-table for $\alpha = 0.05$, we have

$$F_{(\nu_2,\nu_1),\alpha} = F_{(16,12),0.05} = 2.60$$

Thus,

$$F_{(\nu_1,\nu_2),(1-\alpha)} = F_{(12,16),0.95} = \frac{1}{F_{(16,12),0.05}} = \frac{1}{2.60} = 0.3846$$

(iii) Here, we want to find the value of F-variate for two-tails and for

$$\alpha = 0.05 \text{ and } \nu_1 = 4,\ \nu_2 = 10$$

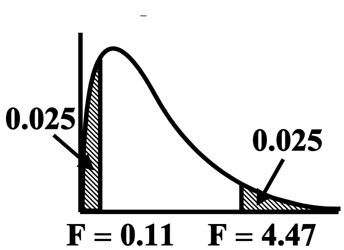

Fig.4.11

Since total area on the both tails is 0.05 therefore, half area lies on both tails as 0.05/2 = 0.025. In this case, F-variate has two values one on right-tail as $F_{\alpha/2} = F_{(\nu_1,\nu_2),\alpha/2} = F_{(4,10),0.025}$ and one on left-tail as $F_{(1-\alpha/2)} = F_{(\nu_1,\nu_2),(1-\alpha/2)} = F_{(4,10),0.975}$. So by proceeding same way as above we get required values of F-variate as $F_{(4,10),0.025} = 4.47$ and $F_{(4,10),0.975} = \frac{1}{F_{(10,4),0.025}} = \frac{1}{8.84} = 0.11$ as shown in Fig.4.11.

(iv) Here, we want to find the value of F-variate for

$$\alpha = 0.01 \text{ and } (\nu_1 = 10,\ \nu_2 = 32)$$

Since F-table for $\alpha = 0.01$ does not have the tabulated value for these degrees of freedom so we need to interpolate it. For this we find the tabulated values of F that are just greater and just less than the degree of freedom 32. Thus we have

$$F_{(10,\ 40)\ ,\ 0.01} = 2.80 \qquad F_{(10,\ 30)\ ,\ 0.01} = 2.98$$

There is a difference of 10 degrees of freedom between these two and a difference 0.18(2.98-2.80) in the F-values. Also note that larger the degrees of freedom smaller the tabulated value of F-variate.

Thus, each degree of freedom has an approximate change in the value of F as

$$\frac{0.18}{10} = 0.018$$

To get 32 degrees of freedom, multiplying 0.018 by 2 (32-30), we get

$$0.018 \times 2 = 0.036$$

Since larger the degrees of freedom smaller the tabulated value of F-variate, therefore, subtracting this from 2.98, we get required values as

$$F_{(10,\,32)} = 2.98 - 0.036 = 2.944$$

Now, try to answer the following exercise.

E10) Find the value of F-variate in each part for which the area on the right side (α) and the degrees of freedom (ν_1, ν_2) are

(i) $\alpha = 0.05$ and ($\nu_1 = 8$, $\nu_2 = 12$)

(ii) $\alpha = 0.99$ and ($\nu_1 = 5$, $\nu_2 = 10$)

(iii) $\alpha = 0.05$ and ($\nu_1 = 23$, $\nu_2 = 5$)

Let us end with a brief look at what we have covered in this unit.

4.7 SUMMARY

In this unit, we have covered the following points:

1. The F-distribution.
2. The properties of F-distribution.
3. The probability curve of F-distribution.
4. Mean and variance of F-distribution.
5. The applications of F-distribution.
6. The relation between t, F and χ^2-distributions.
7. The method of obtaining the tabulated value of a variate which follows either of t, χ^2 and F-distributions.

4.8 SOLUTIONS / ANSWERS

E1) The probability density function of F-distribution with ν_1 and ν_2 df is given as

$$f(F) = \frac{\left(\nu_1/\nu_2\right)^{\nu_1/2}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \frac{F^{(\nu_1/2)-1}}{\left(1 + \frac{\nu_1}{\nu_2}F\right)^{(\nu_1+\nu_2)/2}}; \qquad 0 < F < \infty$$

(i) Therefore, for $\nu_1 = 4$ and $\nu_2 = 2$ df the pdf of F-distribution is given by

$$f(F) = \frac{(4/2)^{4/2}}{B\left(\frac{4}{2},\frac{2}{2}\right)} \frac{F^{(4/2)-1}}{\left(1+\frac{4}{2}F\right)^{(4+2)/2}}$$

$$= \frac{4\overline{|2+1}}{\overline{|2}\,\overline{|1}} \frac{F}{(1+2F)^3} \qquad \left[\because B(a,b) = \frac{\overline{|a}\,\overline{|b}}{\overline{|a+b}}\right]$$

$$= \frac{8F}{(1+2F)^3}; \quad 0 < F < \infty \qquad \left[\begin{array}{l}\because \overline{|3} = \underline{|2} = 2 \\ \& \overline{|2} = \overline{|1} = 1\end{array}\right]$$

(ii) Similarly, for $\nu_1 = 2$ and $\nu_2 = 4$ df the pdf of F-distribution is given by

$$f(F) = \frac{(2/4)^{2/2}}{B\left(\frac{2}{2},\frac{4}{2}\right)} \frac{F^{(2/2)-1}}{\left(1+\frac{2}{4}F\right)^{(2+4)/2}}$$

$$= \frac{\overline{|2+1}}{4\overline{|2}\,\overline{|1}} \frac{1}{\left(1+\frac{F}{2}\right)^3} \qquad \left[\because B(a,b) = \frac{\overline{|a}\,\overline{|b}}{\overline{|a+b}}\right]$$

$$= \frac{1}{2\left(1+\frac{F}{2}\right)^3}; \qquad 0 < F < \infty \quad \left[\begin{array}{l}\because \overline{|3} = \underline{|2} = 2 \\ \& \overline{|2} = \overline{|1} = 1\end{array}\right]$$

Hence, from (i) and (ii) parts, we observed that

$$F_{(\nu_1,\nu_2)} \neq F_{(\nu_2,\nu_1)}$$

E2) If random variable X follows the F-distribution with ν_1 and ν_2 df then probability density function of X is given as

$$f(x) = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)} \frac{x^{(\nu_1/2)-1}}{\left(1+\frac{\nu_1}{\nu_2}x\right)^{(\nu_1+\nu_2)/2}}; \qquad 0 < x < \infty \ldots (12)$$

Here, we are given that

$$f(x) = \frac{1}{(1+x)^2}; \qquad 0 < x < \infty$$

This can be written as

$$f(x) = \frac{(2/2)^{2/2}}{B\left(\frac{2}{2},\frac{2}{2}\right)} \frac{x^{(2/2)-1}}{\left(1+\frac{2}{2}x\right)^{(2+2)/2}}; \quad 0 < x < \infty \left[\because B(1,1) = 1\right]$$

Comparing with equation (12), we get $\nu_1 = 2$ and $\nu_2 = 2$.

E3) Here, we are given that

$$n_1 = 15,\ n_2 = 10,\ \sigma_1 = 65,\ \sigma_2 = 45, S_1 = 60,\ S_2 = 50$$

The value of F-variate can be calculated as follows:

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

where, S_1^2 & S_2^2 are the values of sample variances.

Therefore, we have

$$F = \frac{(60)^2/(65)^2}{(50)^2/(45)^2}$$

$$= \frac{0.85}{1.23} = 0.69$$

E4) Refer Section 4.3.

E5) We know that the mean and variance of F-distribution with ν_1 and ν_2 degrees of freedom are

$$\text{Mean} = \frac{\nu_2}{\nu_2 - 2} \quad \text{for} \quad \nu_2 > 2$$

and

$$\text{Variance} = \frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)} \quad \text{for} \quad \nu_2 > 4.$$

In our case, $\nu_1 = 5$ and $\nu_2 = 10$, therefore,

$$\text{Mean} = \frac{\nu_2}{\nu_2 - 2} = \frac{10}{10-2} = \frac{5}{4}$$

$$\text{Variance} = \frac{2(10)^2(5+10-2)}{5(10-2)^2(10-4)} = \frac{65}{48}$$

E6) Refer Section 4.4.

E7) We know that if a variate t follows Student's t-distribution with ν df, then square of t follows F-distribution with (1, ν) df i.e. if $t \sim t_{(\nu)}$ then $t^2 \sim F_{(1,\nu)}$. In our case, ν = 4, therefore square of t follows F-distribution with (1, 4) df whose pdf is given by

$$f(F) = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} \frac{F^{(\nu_1/2)-1}}{\left(1 + \frac{\nu_1}{\nu_2}F\right)^{(\nu_1+\nu_2)/2}}; \qquad 0 < F < \infty$$

Therefore, for $\nu_1 = 1$ and $\nu_2 = 4$ df the pdf of F-distribution is given by

$$f(F) = \frac{(1/4)^{2/2}}{B\left(\frac{1}{2}, \frac{4}{2}\right)} \frac{F^{(1/2)-1}}{\left(1 + \frac{1}{4}F\right)^{(1+4)/2}}$$

$$= \frac{\Gamma\left(\frac{1}{2}+2\right)}{4\,\Gamma\left(\frac{1}{2}\right)\Gamma 2}\,\frac{1}{\sqrt{F}\left(1+\frac{F}{4}\right)^{5/2}}$$

$$= \frac{\frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}{4\,\Gamma\left(\frac{1}{2}\right)}\,\frac{1}{\sqrt{F}\left(1+\frac{F}{4}\right)^{5/2}} \qquad \left[\because \Gamma 2 = 1\right]$$

$$= \frac{3}{16}\,\frac{1}{\sqrt{F}\left(1+\frac{F}{4}\right)^{5/2}}; \qquad 0 < F < \infty$$

E8)

(i) Here, we want to find the value of t-variate for one-right tail corresponding

$\alpha = 0.01$ and $\nu = 11$

Therefore, we start with the first column of t-table given in the Appendix and downward headed ν until entry 11 is reached. Then proceed right to the column headed $\alpha = 0.01$. We get the required value of t-variate as $t_{(\nu),\,\alpha} = t_{(11),\,0.01} = 2.718$.

(ii) Here, we want to find the value of t-variate for one-left tail corresponding

$\alpha = 0.05$ and $\nu = 16$

By proceeding same way as above, from the t-table, we get the value of t-variate as 1.746. So the required values of t-variate for one left-tail is $-\,t_{(\nu),\,\alpha} = -\,t_{(16),\,0.05} = -\,1.746$.

(iii) Here, we want to find the values of t-variate for two-tails corresponding

$\alpha = 0.05$ and $\nu = 19$

Since total area on the both tails is 0.05 therefore, by symmetry, the area on the right tail is $0.05/2 = 0.025$. Thus, we first start with the first column of t-table and downward headed ν until entry 19 is reached. Then proceed right to the column headed 0.025. We get $t_{(\nu),\,\alpha/2} = t_{(19),\,0.025} = 2.093$. So the required values of t-variate are $\pm\, t_{(\nu),\,\alpha/2} = \pm\, t_{(19),\,0.025} = \pm\, 2.093$.

(iv) Here, we want to find the values of t-variate for two-tails corresponding

$\alpha = 0.05$ and $\nu = 45$

Since total area on the both tails is 0.05 therefore, by symmetry the area on the right tail is $0.05/2 = 0.025$ and also t-table does not have the tabulated value for 45 degrees of freedom with $\alpha = 0.025$, so we need to interpolate it. For this, we find the tabulated values of t-variate that are just greater and just less than the degrees of freedom 45 at $\alpha = 0.025$. Thus, we get

$t_{(60)\,,\,0.025} = 2.000 \quad t_{(40)\,,\,0.025} = 2.021$

There is a difference of 20 degrees of freedom between these two and a difference 0.021(2.021−2.000) in the t-values.

Thus, each degree of freedom has an approximate change in the value of t as

$$\frac{0.021}{20} = 0.00105$$

To get 45 degrees of freedom, multiply 0.00105 by 5 (45−40), we get

$$0.00105 \times 5 = 0.00525$$

Now subtracting this from 2.021, we get t-value as

$$t_{(45)\,,\,0.025} = 2.021 - 0.00525 = 2.01575$$

So the required values of t-variate are $\pm\, t_{(\nu),\,\alpha/2} = \pm\, t_{(45),\,0.025} = \pm\, 2.01575$.

E9) (i) Here, we want to find the value of χ^2-variate for

$$\alpha = 0.01 \text{ and } \nu = 11$$

Thus, we start with the first column of χ^2-table given in the Appendix and downward headed ν until entry 11 is reached. Then proceed right to the column headed α = 0.01. We get the required value of χ^2-variate as 24.72.

(ii) Here, we want to find the value of χ^2-variate for

$$\alpha = 0.90 \text{ and } \nu = 19$$

By proceeding same way as above, from the χ^2-table, we get the required value of χ^2 as 11.65.

(iii) Here, we want to find the value of χ^2-variate for

$$\alpha = 0.05 \text{ and } \nu = 44$$

Since χ^2-table does not have the tabulated value for 44 degrees of freedom so we need to interpolate it. For this we find the tabulated values of χ^2-variate that are just greater and just less than the degrees of freedom 44 with α = 0.05. Thus, we have

$$\chi^2_{(50),0.05} = 67.50 \qquad \chi^2_{(40),0.05} = 55.76$$

There is a difference of 10 degrees of freedom between these two and a difference 11.74(67.50 – 55.76) in the χ^2-values. Also note that larger the degrees of freedom larger the tabulated value of χ^2.

Thus, each degree of freedom has an approximate change in the value of χ^2 as

$$\frac{11.74}{10} = 1.174$$

To get 44 degrees of freedom, multiplying 1.174 by 4 (44−40), we get

$$1.174 \times 4 = 4.696$$

Now adding this in 55.76, we get required value as

$$\chi^2_{(44),0.05} = 55.76 + 4.696 = 60.456$$

E10) (i) Here, we want to find the value of F-variate for

$\alpha = 0.05$ and ($\nu_1 = 8$, $\nu_2 = 12$)

Thus, we select the F-table for $\alpha = 0.05$ and start with the first column of this table and downward headed ν_2 until entry 12 is reached. Then proceed right to the column headed $\nu_1 = 8$. We get the required value of F as 2.85.

(ii) Here, we are given that

$\alpha = 0.99$ and ($\nu_1 = 5$, $\nu_2 = 10$)

Since we have F-tables for $\alpha = 0.10, 0.05, 0.025$ and 0.01only therefore, for $\alpha = 0.99$, we can use the relation

$$F_{(\nu_1,\nu_2),(1-\alpha)} = \frac{1}{F_{(\nu_2,\nu_1),\alpha}}$$

From the F-table for $\alpha = 0.01$, we have

$$F_{(10,5),0.01} = 4.47$$

Thus,

$$F_{(\nu_1,\nu_2),(1-\alpha)} = F_{(5,10),0.99} = \frac{1}{F_{(10,5),0.01}} = \frac{1}{10.05} = 0.10$$

(iii) Here, we want to find the value of F-variate for

$\alpha = 0.05$ and ($\nu_1 = 23$, $\nu_2 = 5$)

Since F-table for $\alpha = 0.01$ does not have the tabulated value for these degrees of freedom so we need to interpolate it. For this we find the tabulated values of F that are just greater and just less than the degrees of freedom 23. Thus we have

$F_{(24,\,5)\,,\,0.05} = 4.53$ $F_{(20,\,5)\,,\,0.05} = 4.56$

Here, there is a difference of 4(24–20) degrees of freedom and F value changes by 0.03(4.56 – 4. 53). Thus, to determine how much it approximately changes for each degree of freedom divide the difference between the F values by the difference between the degrees of freedom, we get

$$\frac{0.03}{4} = 0.0075$$

Since, we have 23 degrees of freedom, this is either 3 more than 20 for 1 less than 24. Therefore, we can interpolate from either values. To get from 20 to 23 degrees of freedom there is a difference of 3(23-20). So multiplying this difference by the amount by which the F-value approximate change per degrees of freedom i.e. 0.0075. These results in

$$3 \times 0.0075 = 0.0225$$

Since larger the degrees of freedom smaller the tabulated value of F-variate, therefore, subtracting this value 0.0225 from 4.56 to get required value. Thus,

$F_{(23, 5), 0.05} = 4.56 - 0.0225 = 4.5375$