UNIT 8 DOUBLE SAMPLING PLANS

Structure

8.1 INTRODUCTION

In Unit 7, you have learnt about the single sampling plan and the procedure for implementing it. In a single sampling plan, we take the decision of accepting or rejecting a lot on the basis of a single sample.

Sometimes, situations arise when it is not possible to decide whether to accept or reject the lot on the basis of a single sample. In such situations, we use a sampling plan known as the **double sampling plan**. In this plan, the decision of acceptance or rejection of a lot is taken on the basis of **two samples**. A lot may be accepted immediately if the first sample is good or may be rejected if it is bad. If the first sample is neither good nor bad, the decision is based on the evidence of the first and second sample combined.

In this unit, we explain the concept of the double sampling plan and the procedure for implementing it (Sec. 8.2). In Secs. 8.3 to 8.7, we discuss various features of the double sampling plan viz. the OC curve, producer's risk and consumer's risk, AOQ, ASN and ATI. In the last section of the unit (Sec. 8.8), we explain how to design the double sampling plan. In the next unit, we shall introduce decision theory.

Objectives

After studying this unit, you should be able to:

- describe a double sampling plan and explain how to implement it;
- differentiate between single and double sampling plans;
- compute the probability of accepting or rejecting a lot of incoming quality in a double sampling plan;
- construct the operating characteristic (OC) curve for a double sampling plan;

- compute the producer's risk and consumer's risk for a double sampling plan;
- compute the average sample number (ASN) and the average total inspection (ATI) for a double sampling plan; and
- design double sampling plans.

8.2 DOUBLE SAMPLING PLAN

A sampling plan in which a decision about the acceptance or rejection of a lot is based on two samples that have been inspected is known as a **double sampling plan**.

The double sampling plan is used when a clear decision about acceptance or rejection of a lot cannot be taken on the basis of a single sample. In double sampling plan, generally, the decision of acceptance or rejection of a lot is taken on the basis of two samples. If the first sample is bad, the lot may be rejected on the first sample and a second sample need not be drawn. If the first sample is good, the lot may be accepted on the first sample and a second sample is not needed. But if the first sample is neither good nor bad and there is a doubt about its results, we take a second sample and the decision of acceptance or rejection of a lot is taken on the basis of the evidence obtained from both the first and the second samples.

For example, suppose a buyer purchases cricket balls in lots of 500 from a company. To check the quality of the lots, the buyer and the company decide that the buyer will draw two samples of sizes 10 (first sample) and 20 (second sample) and the acceptance numbers for the plan are 1 and 3. The buyer takes two samples and makes the decision of acceptance or rejection of the lot on the basis of two samples. Since the decision of acceptance or rejection of the lot is taken on the basis of two samples, this is a double sampling plan.

A double sampling plan requires the specification of four quantities which are known as its **parameters**. These parameters are

n_1 – size of the first sample,

c_1 – acceptance number for the first sample,

n_2 – size of the second sample, and

c_2 – acceptance numbers for both samples combined.

Therefore, the parameters of the double sampling plan in the above example are

the size of the first sample (n_1) = 10,

the acceptance number for the first sample (c_1), = 1

the size of the second sample (n_2) = 20, and

the acceptance numbers for both the samples combined (c_2) = 3.

So far you have learnt the definition of the double sampling plan and why it is used. We now describe the procedure for implementing it and its advantages over the single sampling plan.

8.2.1 Implementation of Double Sampling Plan

Suppose, lots of the same size, say N, are received from the supplier or the final assembly line and submitted for inspection one at a time. The procedure for implementing the double sampling plan to arrive at a decision about the lot is described in the following steps:

Step 1: We draw a random sample (first sample) of size n_1 from the lot received from the supplier or the final assembly.

Step 2: We inspect each and every unit of the sample and classify it as defective or non-defective. At the end of the inspection, we count the number of defective units found in the sample. Suppose the number of defective units found in the first sample is d_1.

Step 3: We compare the number of defective units (d_1) found in the first sample with the stated acceptance numbers c_1 and c_2.

Step 4: We take the decision on the basis of the first sample as follows:

Under acceptance sampling plan

If the number of defective units (d_1) in the first sample is less than or equal to the stated acceptance number (c_1) for the first sample, i.e., if $d_1 \leq c_1$, we accept the lot and if $d_1 > c_2$, we reject the lot. But if $c_1 < d_1 \leq c_2$, the first (single) sample is failed.

Under rectifying sampling plan

If $d_1 \leq c_1$, we accept the lot and replace all defective units found in the sample by non-defective units. If $d_1 > c_2$, we accept the lot after inspecting the entire lot and replacing all defective units in the lot by non-defective units. But if $c_1 < d_1 \leq c_2$, the first (single) sample is failed.

Step 5: If $c_1 < d_1 \leq c_2$, we draw a second random sample of size n_2 from the lot.

Step 6: We inspect each and every unit of the second sample and count the number of defective units found in it. Suppose the number of defective units found in the second sample is d_2.

Step 7: We combine the number of defective units (d_1 and d_2) found in both samples and consider $d_1 + d_2$ for taking the decision about the lot on the basis of the second sample as follows:

Under acceptance sampling plan

If $d_1 + d_2 \leq c_2$, we accept the lot and if $d_1 + d_2 > c_2$, we reject the lot.

Under rectifying sampling plan

If $d_1 + d_2 \leq c_2$, we accept the lot and replace all defective units found in the second sample by non-defective units. If $d_1 + d_2 > c_2$, we accept the lot after inspecting the entire lot and replacing all defective units in the lot by non-defective units.

The steps described above are shown in Fig. 8.1 for the acceptance sampling plan.

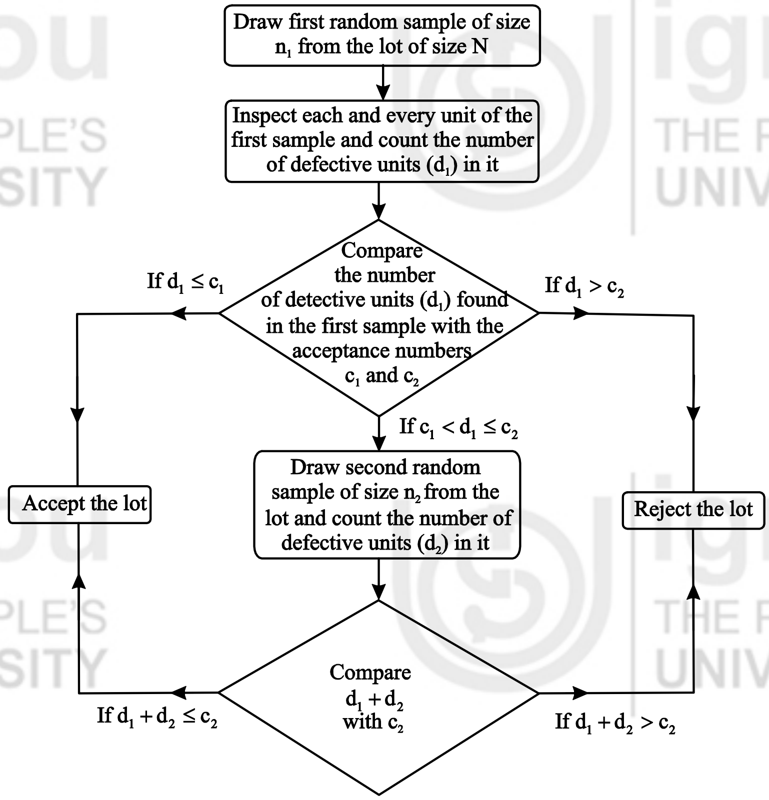

Fig. 8.1: Procedure for implementation of an acceptance double sampling plan.

Let us explain these steps further with the help of an example.

Example 1: Suppose a mobile phone company produces mobile phones in lots of 400 phones each. To check the quality of the lots, the quality inspector of the company uses a double sampling plan with $n_1 = 15$, $c_1 = 1$, $n_2 = 30$, $c_2 = 3$. Explain the procedure for implementing it under acceptance sampling plan.

Solution: For implementing the double sampling plan, the quality inspector of the company randomly draws first sample of 15 mobiles from the lot and classifies each mobile of the first sample as defective or non-defective. At the end of the inspection, he/she counts the number of defective mobiles (d_1) found in the first sample and compares d_1 with the acceptance numbers c_1 and c_2. If $d_1 \le c_1 = 1$, he/she accepts the lot and if $d_1 > c_2 = 3$, he/she rejects the lot. If $c_1 < d_1 \le c_2$, it means that if the number of defective mobiles is 2 or 3, he /she draws the second sample from the lot. He/she counts the number of defective mobiles (d_2) found in the second sample and compares the total number of defective mobiles ($d_1 + d_2$) in both samples with the acceptance number c_2. If $d_1 + d_2 \le c_2 = 3$, he/she accepts the lot and if $d_1 + d_2 > c_2 = 3$, he/she rejects the lot.

8.2.2 Advantages of Double Sampling Plan

A double sampling plan has the following two main advantages over a single sampling plan:

i) The principal advantage of the double sampling plan over the single sample plan is that for the same degree of protection (i.e., the same probability of accepting a lot of a given quality), the double sampling plan may have a smaller average sample number (ASN) than that corresponding to the single sampling plan. The underlying reason is that the size (n_1) of the first sample in the double sampling plan is always smaller than the sample size

(n) of an equivalent single sampling plan. Thus, if a decision is taken on the basis of the first sample, ASN will be lower for the double sampling plan or if a decision is taken after the second sample, the ASN will be reduced.

ii) The double sampling plan has the psychological advantage of giving a lot a second chance. From the viewpoint of the producer/manufacture, it is unfair to reject a lot on the basis of a single sample. The double sampling plan permits the decision to be made on the basis of two samples.

However, double sampling plans are costlier to administer in comparison with the single sampling plans.

Try the following exercise to explain the procedure for implementing a double sampling plan.

E1) A manufacturer of silicon chip produces lots of 1000 chips for shipment. A buyer uses a double sampling plan with $n_1 = 5$, $c_1 = 0$, $n_2 = 20$, $c_2 = 2$ to test the quality of the lots. Explain the procedure for implementing it under acceptance sampling plan.

8.2.3 Difference between Single and Double Sampling Plans

The differences between single and double sampling plans are given below:

1. Single sampling plans are easier to design and operate in comparison with the double sampling plans.
2. The advantage of double sampling plans over single sampling plans seems to be psychological. It looks unreasonable to reject a lot on the basis of one sample and it is more convincing to say that the lot was rejected on the basis of inspection of two samples.
3. Under the double sampling plan, the good lot will generally be accepted and the bad lot will usually be rejected on the basis of the first sample. Thus, in all cases, where a decision to accept or reject is taken on the basis of the first sample, there is a considerable saving in the amount of inspection than required by a comparable single sampling plan. Moreover, when a second sample is taken, it may be possible to reject the lot without completely inspecting the entire second sample (see Sec.8.6).
4. In the double sampling plan, on an average, 25% to 33 % less number of items/units need to be inspected as compared to the single sampling plan.
5. The operating characteristic (OC) curve for a double sampling plan is steeper than a single sampling plan, i.e., the discriminatory power of the double sampling plan is higher than that of the single sampling plan.

So far you have learnt what a double sampling plan is and how it is implemented in industry. You have also learnt the differences between the double and single sampling plans. In Secs. 8.3 to 8.7, we describe various features of the double sampling plan.

8.3 OPERATING CHARACTERISTIC (OC) CURVE

You have learnt in Unit 6 that the operating characteristic (OC) curve displays the discriminatory power of the sampling plan. That is, it shows the probability that a lot submitted with a certain fraction defective will either be accepted or rejected.

You have learnt in Units 6 and 7 that for constructing an OC curve, we require the probabilities of accepting a lot corresponding to different quality levels. Therefore, we now describe how to compute the probability of accepting a lot in a double sampling plan.

You have learnt in Sec. 8.2.1 that in a double sampling plan, the decision of acceptance or rejection of the lot is taken on the basis of two samples. The lot is accepted on the first sample if the number of defective units (d_1) in the first sample is less than the acceptance number c_1. The lot is accepted on the second sample if the number of defective units ($d_1 + d_2$) in both samples is greater than c_1 and less than or equal to the acceptance number c_2. Therefore, if $P_{a1}(p)$ and $P_{a2}(p)$ denote the probabilities of accepting a lot on the first sample and the second sample, respectively, the probability of accepting a lot of quality level p is given by:

$$P_a(p) = P_{a1}(p) + P_{a2}(p) \qquad \ldots (1)$$

If X and Y denote the observed number of defective units in the first and second samples, respectively, we accept the lot on the first sample if $X \le c_1$, i.e., if X = 0 or 1or 2,…, or c_1.

You have studied in Unit 3 of MST-003 that if A and B are mutually exclusive events then $P[A \text{ or } B] = P[A] + P[B]$

Therefore,

$$P_{a1}(p) = P[X \le c_1] = P[X = 0 \text{ or } 1 \text{ or } 2, \ldots, \text{or } c_1]$$

$$= P[X=0] + P[X=1] + \ldots + P[X=c_1] \quad \left(\begin{matrix}\because X = 0, 1, 2, \ldots, c_1 \text{ are} \\ \text{mutually exclusive}\end{matrix}\right)$$

$$= \sum_{x=0}^{c_1} P[X = x] \qquad \ldots (2)$$

We can calculate this probability if we know the distribution of X. You have learnt in Unit 5 that generally, in quality control, a random sample is drawn from a lot of finite size without replacement. So in such situations, the number of defective units (X) in the sample follows a hypergeometric distribution.

In a lot of size N and incoming quality p, the number of defective units is Np and non-defective units is N – Np. Therefore, we can compute the probability of getting exactly x defective units in a sample of size n_1 using the hypergeometric distribution as follows:

$$P[X = x] = \frac{{}^{Np}C_x \; {}^{N-Np}C_{n_1-x}}{{}^{N}C_{n_1}}; \; x = 0, 1, \ldots, \min(Np, n_1) \qquad \ldots (3)$$

Thus, we can obtain the probability of accepting a lot of quality p on the first sample by putting the value of P[X = x] in equation (2) as follows:

$$P_{a1}(p) = \sum_{x=0}^{c_1} P[X = x] = \sum_{x=0}^{c_1} \frac{{}^{Np}C_x \; {}^{N-Np}C_{n_1-x}}{{}^{N}C_{n_1}} \qquad \ldots (4)$$

We accept the lot of quality p on the second sample if $c_1 < X + Y \le c_2$. It means that we accept the lot if

$X = c_1 + 1$ and $Y \le c_2 - X = c_2 - c_1 - 1$

or $X = c_1 + 2$ and $Y \le c_2 - c_1 - 2$

$\vdots$

or $X = c_2$ and $Y \le c_2 - c_2 = 0.$

Therefore, from the addition theorem of probability, we can obtain the probability of accepting a lot of quality p on the second sample as follows:

$$P_{a2}(p) = P\left[X = c_1 + 1\right]P\left[Y \le c_2 - c_1 - 1\right] + P\left[X = c_1 + 2\right]P\left[Y \le c_2 - c_1 - 2\right] + ... + P\left[X = c_2\right]P\left[Y \le 0\right]$$

$$= \sum_{x=c_1+1}^{c_2} \sum_{y=0}^{c_2-x} P\left[X = x\right]P\left[Y = y / X = x\right] \qquad \dots (5)$$

where $P\left[Y = y / X = x\right]$ is the conditional probability of observing y defective units in the second sample under the condition that x defective units have already appeared in the first sample.

Since it is given that x defective units have already appeared in the first sample, the remaining defective units and non-defective units in the lot on the second sample are Np – x and $(N - n_1) - (NP - x)$, respectively. Thus, we can compute the probability $P\left[Y = y / X = x\right]$ by using the hypergeometric distribution as follows:

$$P(Y = y / X = x) = \frac{{}^{Np-x}C_y \; {}^{N-n_1-(Np-x)}C_{n_2-y}}{{}^{N-n_1}C_{n_2}} \qquad \dots (6)$$

On putting the values of $P\left[X = x\right]$ and $P\left[Y = y / X = x\right]$ from equations (4) and (6) in equation (5), we get

$$P_{a2}(p) = \sum_{x=c_1+1}^{c_2} \sum_{y=0}^{c_2-x} \frac{{}^{Np}C_x \; {}^{N-Np}C_{n_1-x}}{{}^{N}C_{n_1}} \frac{{}^{Np-x}C_y \; {}^{N-n_1-(Np-x)}C_{n_2-y}}{{}^{N-n_1}C_{n_2}} \qquad \dots (7)$$

Therefore, the required probability of accepting a lot of quality p in a double sampling plan is given by

$$P_a(p) = \sum_{x=0}^{c_1} \frac{{}^{Np}C_x \; {}^{N-Np}C_{n_1-x}}{{}^{N}C_{n_1}} + \sum_{x=c_1+1}^{c_2} \sum_{y=0}^{c_2-x} \frac{{}^{Np}C_x \; {}^{N-Np}C_{n_1-x}}{{}^{N}C_{n_1}} \frac{{}^{Np-x}C_y \; {}^{N-n_1-(Np-x)}C_{n_2-y}}{{}^{N-n_1}C_{n_2}} \qquad \dots (8)$$

However, we can approximate the hypergeometric distribution to the binomial distribution with parameters n and p as we have explained in Unit 7. It means that if the sample sizes n_1 and n_2 are small compared to the lot size (N), the probability of accepting a lot of quality p (using the binomial approximation) is given by

$$P_a(p) = \sum_{x=0}^{c_1} {}^{n_1}C_x \, p^x \left(1-p\right)^{n_1-x} + \sum_{x=c_1+1}^{c_2} \sum_{y=0}^{c_2-x} {}^{n_1}C_x \, p^x \left(1-p\right)^{n_1-x} \; {}^{n_2}C_y \, p^y \left(1-p\right)^{n_2-y} \qquad \dots (9)$$

However, for rapid calculation, we can use Table I entitled **Cumulative Binomial Probability Distribution** given at the end of this block for calculating this probability.

When p is small and n is large such that np is finite, we know that the binomial distribution approaches the Poisson distribution with parameter $\lambda = np$. Therefore, the probability of accepting a lot of quality p using the Poisson approximation is given by

$$P_a(p) = \sum_{x=0}^{c_1} \frac{e^{-\lambda_1}\lambda_1^x}{\lfloor x} + \sum_{x=c_1+1}^{c_2} \sum_{y=0}^{c_2-x} \frac{e^{-\lambda_1}\lambda_1^x}{\lfloor x} \frac{e^{-\lambda_2}\lambda_2^y}{\lfloor y} \qquad \ldots (10)$$

where $\lambda_1 = n_1 p$ and $\lambda_2 = n_2 p$.

We can use Table II entitled **Cumulative Poisson Probability Distribution** given at the end of this block for calculating this probability.

We now illustrate how to compute the probability of accepting a lot in a double sampling plan.

Example 2: Suppose, in Example 1, the incoming quality of the lot is 0.05. What is the probability of accepting the lot on the first sample? What is the probability of final acceptance?

Solution: It is given that

$$N = 400, n_1 = 15, c_1 = 1, n_2 = 30, c_2 = 3 \text{ and } p = 0.05$$

If $P_{a1}(p)$ and $P_{a2}(p)$ denote the probabilities of accepting the lot on the first sample and the second sample, respectively, we can calculate the probability of accepting the lot of quality level (p) in a double sampling plan as follows:

$$P_a(p) = P_{a1}(p) + P_{a2}(p) \qquad \ldots (i)$$

If X represents the number of defective mobiles in the first sample of size 15, the lot is accepted on the first sample if $X \le c_1 = 1$. Therefore,

$$P_{a1}(p) = P[X \le c_1] = P[X \le 1] \qquad \ldots (ii)$$

Since $N \ge 10n_1$, we can use the binomial distribution and calculate $P_{a1}(p)$ using Table I.

From Table I, for $n = n_1 = 15, x = c_1 = 1$ and $p = 0.05$, we have

$$P_{a1}(p) = P[X \le 1] = \sum_{x=0}^{1} {}^nC_x\, p^x (1-p)^{n-x} = 0.8290 \qquad \ldots (iii)$$

Hence, the probability of accepting the lot on the first sample is 0.8290. It means that about 82.9% of the lot of quality p = 0.05 will be accepted on the first sample.

If Y represents the number of defective mobiles in the second sample, the lot is accepted on the second sample if $c_1 < X + Y \le c_2$, i.e., $1 < X + Y \le 3$. It means that we shall accept the lot if

$X = c_1 + 1 = 1 + 1 = 2$ and $Y \le c_2 - X = 3 - 2 = 1$

or $X = c_1 + 2 = 1 + 2 = 3$ and $Y \le c_2 - X = 3 - 3 = 0$.

Therefore,

$$P_{a2}(p) = P[X=2]P[Y \le 1] + P[X=3]P[Y \le 0] \qquad \dots \text{(iv)}$$

The probabilities $P[X=2]$ and $P[X=1]$ can also be obtained using Table I, which contains the cumulative probabilities of the binomial distribution, using the following expression:

$$P[X=x] = P[X \le x] - P[X \le x-1] \qquad \dots \text{(v)}$$

From Table I, for $n = n_1 = 15$ and $p = 0.05$, we have

$$P[X=2] = P[X \le 2] - P[X \le 1] = 0.9638 - 0.8290 = 0.1348$$

$$P[X=1] = P[X \le 1] - P[X \le 0] = 0.8290 - 0.4633 = 0.3657$$

From Table I, for $n = n_2 = 30$ and $p = 0.05$, we have

$$P[Y \le 1] = 0.5535 \text{ and } P[Y \le 0] = 0.2146$$

On putting these values in equation (iv), we get

$$P_{a2}(p) = P[X=2]P[Y \le 1] + P[X=3]P[Y \le 0]$$

$$= 0.1348 \times 0.5535 + 0.3657 \times 0.2146 = 0.1531$$

Hence, the probability of accepting the lot on the second sample is 0.1531. It means that about 15.31% of the lot of quality p = 0.05 will be accepted on the second sample.

From equation (i), we get the probability of accepting the lot in the double sampling plan as follows:

$$P_a(p) = P_{a1}(p) + P_{a2}(p) = 0.8290 + 0.1531 = 0.9821$$

Thus, overall, 98.21% of the lots will be accepted by this sampling plan.

In a similar way, we can calculate the probability of accepting a lot for different lot qualities.

The construction of the OC curve for a double sampling plan is beyond the scope of this course. If you are interested in constructing the OC curve, you may consider different quality levels such as p = 0.01, 0.02, 0.03… and then calculate the corresponding probabilities of accepting the lot as explained in Example 2. The OC curve may then be constructed by taking the quality level (proportion defective) on the X-axis and the probability of accepting the lot on the Y-axis as explained for the single sampling plan in Unit 7.

You may now like to calculate the probability of accepting a lot for practice. Try the following exercise.

E2) Suppose that in E1, the incoming quality of the lot is 0.03. Calculate the probabilities of accepting the lot on the first sample and on the second sample. What is the probability of accepting the lot?

8.4 PRODUCER’S RISK AND CONSUMER’S RISK

In Units 5 and 7, we have defined the **producer’s risk** as follows:

The probability of rejecting a lot of acceptance quality level (p_1) is known as the producer’s risk.

Thus, the producer's risk for a double sampling plan is given by

$$P_p = P\left[\text{rejecting a lot of acceptance quality level } p_1\right]$$

$$= 1 - P\left[\text{accepting a lot of acceptance quality level } p_1\right]$$

$$= 1 - P_a\left(p_1\right) \quad \text{… (11)}$$

We can compute $P_a\left(p_1\right)$ from equation (8) by replacing the quality level p with p_1 as follows:

$$P_a(p_1) = \sum_{x=0}^{c_1} \frac{{}^{Np_1}C_x \, {}^{N-Np_1}C_{n_1-x}}{{}^{N}C_{n_1}} + \sum_{x=c_1+1}^{c_2} \sum_{y=0}^{c_2-x} \frac{{}^{Np_1}C_x \, {}^{N-Np_1}C_{n_1-x}}{{}^{N}C_{n_1}} \cdot \frac{{}^{Np_1-x}C_y \, {}^{N-n_1-(Np_1-x)}C_{n_2-y}}{{}^{N-n_1}C_{n_2}} \quad \text{... (12)}$$

Therefore, from equation (11), the producer's risk for a double sampling plan is given by

$$P_p = 1 - \sum_{x=0}^{c_1} \frac{{}^{Np_1}C_x \, {}^{N-Np_1}C_{n_1-x}}{{}^{N}C_{n_1}} - \sum_{x=c_1+1}^{c_2} \sum_{y=0}^{c_2-x} \frac{{}^{Np_1}C_x \, {}^{N-Np_1}C_{n_1-x}}{{}^{N}C_{n_1}} \cdot \frac{{}^{Np_1-x}C_y \, {}^{N-n_1-(Np_1-x)}C_{n_2-y}}{{}^{N-n_1}C_{n_2}} \quad \text{... (13)}$$

For rapid calculation of the producer's risk for a double sampling plan, we can also use the approximations as explained in Sec. 8.3. Therefore, if we use the approximation of the hypergeometric distribution to the binomial distribution, the producer's risk is given by

$$P_p = 1 - \sum_{x=0}^{c_1} {}^{n_1}C_x \, p_1^x \left(1-p_1\right)^{n_1-x} - \sum_{x=c_1+1}^{c_2} \sum_{y=0}^{c_2-x} {}^{n_1}C_x \, p_1^x \left(1-p_1\right)^{n_1-x} \, {}^{n_2}C_y \, p_1^y \left(1-p_1\right)^{n_2-y} \quad \text{...(14)}$$

We now explain the **Consumer's Risk** for a double sampling plan:

By definition, the probability of accepting a lot of unsatisfactory quality (LTPD) p_2 is known as the consumer's risk.

Therefore, the consumer's risk for a double sampling plan is given by

$$P_c = P\left[\text{accepting a lot of quality (LTPD) } p_2\right] = P_a\left(p_2\right)$$

We can compute the consumer's risk for a double sampling plan from equation (8) by replacing the quality level p with the quality LTPD (p_2) as follows:

$$P_c = P_a(p_2) = \sum_{x=0}^{c_1} \frac{{}^{Np_2}C_x \, {}^{N-Np_2}C_{n_1-x}}{{}^{N}C_{n_1}} + \sum_{x=c_1+1}^{c_2} \sum_{y=0}^{c_2-x} \frac{{}^{Np_2}C_x \, {}^{N-Np_2}C_{n_1-x}}{{}^{N}C_{n_1}} \cdot \frac{{}^{Np_2-x}C_y \, {}^{N-n_2-(Np_2-x)}C_{n_2-y}}{{}^{N-n_1}C_{n_2}} \quad \text{... (15)}$$

If we approximate the hypergeometric distribution to the binomial distribution, the consumer's risk is given by

$$P_c = P_a(p_2) = \sum_{x=0}^{c_1} {}^{n_1}C_x\, p_2^x (1-p_2)^{n_1-x} + \sum_{x=c_1+1}^{c_2} \sum_{y=0}^{c_2-x} {}^{n_1}C_x\, p_2^x (1-p_2)^{n_1-x}\; {}^{n_2}C_y\, p_2^y (1-p_2)^{n_2-y} \qquad \text{... (16)}$$

Let us consider an example from a real life situation to explain these concepts.

Example 3: Suppose a shirt manufacturing company supplies shirts in lots of size 500 to the buyer. A double sampling plan with $n_1 = 10$, $c_1 = 0$, $n_2 = 25$, $c_2 = 1$ is being used for the lot inspection. The company and the buyer's quality control inspector decide that AQL = 0.04 and LTPD = 0.10. Compute the producer's risk and consumer's risk for this sampling plan.

Solution: It is given that

$$N = 500, n_1 = 10, c_1 = 0, n_2 = 25, c_2 = 1,$$

$$AQL(p_1) = 0.04 \text{ and } LTPD(p_2) = 0.10$$

We know that the producer's risk for a double sampling plan is given by

$$P_p = P[\text{rejecting a lot of acceptance quality level } p_1]$$

$$= 1 - P[\text{accepting a lot of acceptance quality level } p_1]$$

$$= 1 - P_a(p_1) \qquad \text{... (i)}$$

Therefore, for calculating the producer's risk, we first calculate the probability of accepting the lot of quality $p = p_1 = AQL = 0.04$ as we have explained in Sec. 8.3.

In a double sampling plan, the lot may be accepted either on the first sample or on the second sample. So if $P_{a1}(p)$ and $P_{a2}(p)$ denote the probabilities of accepting the lot on the first sample and on the second sample, respectively, we can calculate the probability of accepting the lot of quality level p, as follows:

$$P_a(p) = P_{a1}(p) + P_{a2}(p) \qquad \text{... (ii)}$$

If X represents the number of defective shirts in the first sample, the lot will be accepted on the first sample if $X \le c_1 = 0$. Therefore,

$$P_{a1}(p) = P[X \le c] = P[X \le 0] \qquad \text{... (iii)}$$

Since $N \ge 10n$, we can use the binomial distribution and calculate the probability using Table I.

From Table I, for $n = n_1 = 10, x = c_1 = 0$ and $p = p_1 = 0.04$, we have

$$P_{a1}(p_1) = P[X \le 0] = \sum_{x=0}^{0} {}^{n_1}C_x\, p_1^x (1-p_1)^{n_1-x} = 0.6648 \qquad \text{... (iv)}$$

If Y represents the number of defective shirts in the second sample, the lot will be accepted on the second sample if $c_1 < X + Y \le c_2$. It means that we shall accept the lot if $X = c_1 + 1 = 0 + 1 = 1$ and $Y \le c_2 - X = 1 - 1 = 0$.

Therefore,

$$P_{a2}(p_1) = P[X=1]P[Y \le 0] \qquad \ldots (v)$$

We also know that $P[X = x] = P[X \le x] - P[X \le x-1]$

From Table I, for $n = n_1 = 10$ and $p = p_1 = 0.04$, we have

$$P[X=1] = P[X \le 1] - P[X \le 0] = 0.9418 - 0.6648 = 0.2770$$

From Table I, for $n = n_2 = 25, x = y = 0$ and $p = p_1 = 0.04$, we have

$$P[Y \le 0] = 0.3604$$

On putting these values in equation (v), we get

$$P_{a2}(p_1) = P[X=1]P[Y \le 0]$$

$$= 0.2770 \times 0.3604 = 0.0998$$

Hence, from equation (ii), we get the probability of accepting the lot in this double sampling plan as follows:

$$P_a(p_1) = P_{a1}(p_1) + P_{a2}(p_1) = 0.6648 + 0.0998 = 0.7646$$

Therefore, we calculate the producer's risk for this plan using equation (i) as follows:

$$P_p = 1 - P_a(p_1) = 1 - 0.7646 = 0.2354$$

It means that if there are several lots of the same quality p = 0.04, about 23.54% out of these will be rejected. This is obviously a risk for the manufacturing company because it was agreed upon by both that lots of quality 0.04 will be accepted whereas the quality inspector is rejecting 23.54% of them.

Similarly, we can calculate the consumer's risk as follows:

We know the consumer's risk for a double sampling plan is given by

$$P_c = P[\text{accepting a lot of quality (LTPD) } p_2] = P_a(p_2) \qquad \ldots (vi)$$

We first calculate the probability of accepting the lot of quality $p = p_2$ = LTPD = 0.10 using equations (ii), (iii) and (v).

From Table I, for $n = n_1 = 10, x = c_1 = 0$ and $p = p_2 = 0.10$, we have

$$P_{a1}(p_2) = P[X \le 0] = \sum_{x=0}^{0} {}^{n_1}C_x\, p_2^x (1-p_2)^{n-x} = 0.3487 \qquad \ldots (vii)$$

From Table I, for $n = n_1 = 10$ and $p = p_2 = 0.10$, we have

$$P[X=1] = P[X \le 1] - P[X \le 0] = 0.7361 - 0.3487 = 0.3874$$

From Table I, for $n = n_2 = 25, x = y = 0$ and $p = p_2 = 0.10$, we have

$$P[Y \le 0] = 0.0718$$

On putting these values in equation (v), we get

$$P_{a2}(p_2) = P[X=1]P[Y \le 0]$$

$$= 0.3874 \times 0.0718 = 0.0278 \qquad \ldots (viii)$$

On putting the values of $P_{a1}(p_2)$ and $P_{a2}(p_2)$ in equation (ii), we get

$$P_a(p_2) = 0.3487 + 0.0278 = 0.3765$$

Hence, from equation (vi), we get the consumer's risk for this plan as follows:

$$P_c = P_a(p_2) = 0.3765$$

It means that if there are several lots of the same quality p = 0.10, about 37.65% out of these will be accepted by the quality inspector even though this quality is unsatisfactory. This is obviously the buyer's risk.

For practice, you can also compute the producer's risk and consumer's risk in the following exercise.

E3) Suppose that in E2, the acceptance quality level (AQL) and lot tolerance percent defective (LTPD) are 0.05 and 0.14, respectively. Calculate the producer's risk and consumer's risk for this plan.

8.5 AVERAGE OUTGOING QUALITY (AOQ)

In Unit 6, you have learnt that the average outgoing quality (AOQ) is defined as follows:

The expected quality of the lots after the application of sampling inspection is called the **average outgoing quality (AOQ)**. It is given by:

$$AOQ = \frac{\text{Number of defective units in the lot after the inspection}}{\text{Total number of units in the lot}} \quad \ldots (17)$$

You know that the concept of AOQ is particularly useful in the rectifying sampling plan wherein the rejected lots are inspected 100% and all defective units are replaced by non-defective units. So in a double sample plan, we can obtain the formula for average outgoing quality by considering the following situations:

i) If the lot of size N is accepted on the first sample of size n_1, $(N – n_1)$ units remain un-inspected. If the incoming quality of the lot is p, we expect that $p(N – n_1)$ defective units are left in the lot after the inspection on the first sample. The probability that the lot will be accepted on the first sample is P_{a1}. Therefore, the expected number of defective units per lot in the outgoing stage is $p(N - n_1)P_{a1}$.

ii) If the lot is rejected on the first sample, all units of the lot go for 100% inspection and all defective units found in the lot are replaced by non-defective units. So there is no defective unit in the outgoing stage. However, the probability that the lot will be rejected on the first sample is $(1 - P_{a1})$. Therefore, the expected number of defective units per lot at the outgoing stage is $0 \times (1 - P_{a1}) = 0$.

iii) If the lot is accepted on the second sample of size n_2, $(N – n_1 – n_2)$ units remain un-inspected. If the incoming quality of the lot is p, we expect that $p(N – n_1 – n_2)$ defective units are left in the lot after the inspection on the second sample. The probability that the lot will be accepted on the second sample is P_{a2}. Therefore, the expected number of defective units per lot in the outgoing stage is $p(N-n_1-n_2)P_{a2}$.

iv) If the lot is rejected on the second sample, all units of the lot go for 100% inspection and all defective units found in the lot are replaced by non-defective units. So there is no defective unit in the outgoing stage. However, the probability that the lot will be rejected on the second sample is $(1-P_{a2})$. Therefore, the expected number of defective units per lot at the outgoing stage is $0\times(1-P_{a2})=0$.

Thus, the expected number of defective units per lot after sampling inspection is $p(N-n_1)P_{a1}+0+p(N-n_1-n_2)P_{a2}+0=p\left[(N-n_1)(P_{a1}+P_{a2})-n_2P_{a2}\right]$.

Hence, from equation (17), the AOQ for a double sampling plan is given by

$$AOQ=\frac{\text{Number of defective units in the lot after the inspection}}{\text{Total number of units in the lot}}$$

$$=\frac{p\left[(N-n_1)(P_{a1}+P_{a2})-n_2P_{a2}\right]}{N}$$

or $$AOQ=p\left[\left(1-\frac{n_1}{N}\right)(P_{a1}+P_{a2})-\frac{n_2}{N}P_{a2}\right] \quad \ldots (18)$$

If the sample sizes n_1 and n_2 are very small in proportion to the lot size N, i.e., $n_1/N \to 0$ and $n_2/N \to 0$, equation (18) for AOQ becomes

$$AOQ=p(P_{a1}+P_{a2}) \quad \ldots (19)$$

Let us consider an example to illustrate how to calculate the AOQ for a double sampling plan.

Example 4: Suppose that in Example 2, the rejected lots are screened and all defective mobiles are replaced by non-defective ones. Calculate the average outgoing quality (AOQ) for this plan.

Solution: It is given that

$$N=400, n_1=15, c_1=1, n_2=30, c_2=3 \text{ and } p=0.05$$

Since $N>10n_1$ and $N>10n_2$, we can calculate the average outgoing quality (AOQ) for the double sampling plan using equation (18).

For calculating AOQ, we have to calculate the probabilities of accepting the lot on the first sample and on the second sample corresponding to p = 0.05.

We have already calculated these probabilities in Example 2. So we directly use the results:

$$P_{a1}=0.8290 \text{ and } P_{a2}=0.1531$$

Substituting the values of N, n_1, n_2, P_{a1}, P_{a2} and p in equation (18), we get

$$AOQ=p\left[\left(1-\frac{n_1}{N}\right)(P_{a1}+P_{a2})-\frac{n_2}{N}P_{a2}\right]$$

$$=0.05\left[\left(1-\frac{15}{400}\right)(0.8290+0.1531)-\frac{30}{400}\times 0.1531\right]$$

$$= 0.05\left[0.9625 \times 0.9821 - 0.0115\right] = 0.0467$$

In the same way, you can calculate the AOQ for different lot qualities.

The construction of the AOQ curve for a double sampling plan is beyond the scope of this course. If you are interested in constructing the AOQ curve, you may consider different quality levels such as p = 0.01, 0.02, 0.03… and then calculate the corresponding probabilities of accepting the lot. The AOQ can then be constructed as explained in Units 6 and 7.

You may now like to calculate the average outgoing quality (AOQ) in the following exercise.

E4) Suppose that in E2, the rejected lots are screened and all defective silicon chips are replaced by non-defective ones. Calculate the average outgoing quatity (AOQ) for this plan.

So far you have learnt about the OC curve, producer's risk, consumer's risk and AOQ for double sampling plan. We now explain ASN for a double sampling plan.

8.6 AVERAGE SAMPLE NUMBER (ASN)

You have learnt in Unit 6 that the average sample number (ASN) is the expected number of sample units per lot which is required to arrive at a decision about the acceptance or rejection of the lot under the acceptance sampling plan.

In acceptance double sampling plan, the number of units inspected to arrive at a decision of acceptance or rejection of the lot depends upon whether the decision is taken only on the first sample or on the second sample as well. Therefore, we have two situations:

i) If the decision is taken on the first sample of size n_1, the number of units inspected is n_1. However, the probability of taking decision of acceptance or rejection of the lot on the first sample is P_I.

ii) If the decision is taken on the second sample of size n_2, the number of units inspected is $(n_1 + n_2)$. The probability that a second sample is necessary is $(1 - P_I)$.

Therefore, the ASN for a double sampling plan can be obtained as follows:

$$\text{ASN} = \text{Expected number of units inspected per lot}$$

$$= \sum (\text{inspected number of units} \times \text{probability of taking decision})$$

$$\text{ASN} = n_1 \times P_I + (n_1 + n_2) \times (1 - P_I) = n_1 P_I + (n_1 + n_2)(1 - P_I) \qquad \ldots (20)$$

where P_I is the probability of making a decision about acceptance or rejection of the lot on the first sample and can be calculated as follows:

$$P_I = \text{P[lot is accepted on the first sample]} + \text{P[lot is rejected on the first sample]} \qquad \ldots (21)$$

Let us take up an example to illustrate this concept.

Example 5: Calculate the average sample number (ASN) for Example 2.

Solution: It is given that

$$N = 400, n_1 = 15, c_1 = 1, n_2 = 30, c_2 = 3 \text{ and } p = 0.05$$

For computing the ASN, we can use equation (20):

$$ASN = n_1 P_I + (n_1 + n_2)(1 - P_I)$$

where P_I is the probability of making a decision about acceptance or rejection of the lot on the first sample. We calculate it using equation (21).

So we have to find the probability P_I first. In this double sampling plan, the lot is accepted on the first sample if $X \le c_1 = 1$, i.e., if $X \le 1$. Therefore,

$$P[\text{lot is accepted on the first sample}] = P\left[X \le c_1\right] = P\left[X \le 1\right] = P_{a1}\left(p\right)$$

We have already calculated this probability in Example 2. So we directly use the results. Therefore,

$$P_{a1}\left(p\right) = P\left[X \le c_1\right] = P\left[X \le 1\right] = 0.8290 \qquad \ldots \text{(i)}$$

The lot will be rejected on the first sample if $X > c_2$, i.e., $X > 3$. Therefore,

We know that $P\left[X > A\right] = 1 - P\left[X \le A\right]$

$$P[\text{lot is rejected on the first sample}] = P\left[X > 3\right] = 1 - P\left[X \le 3\right] \qquad \ldots \text{(ii)}$$

From Table I, for $n = n_1 = 15, x = 3$ and $p = 0.05$, we have

$$P\left[X \le 3\right] = \sum_{x=0}^{3} {}^{n_1}C_x \, p^x \left(1-p\right)^{n_1 - x} = 0.9945$$

Therefore, from equation (ii), we have

$$P[\text{lot is rejected on the first sample}] = 1 - P\left[X \le 3\right]$$

$$= 1 - 0.9945 = 0.0055 \qquad \ldots \text{(iii)}$$

Thus, on putting the values of equations (ii) and (iii) in equation (21), we get

$$P_I = 0.8290 + 0.0055 = 0.8345$$

Hence, on putting the values of n_1, n_2, and P_I in equation (20), we get the ASN for the plan as follows:

$$ASN = n_1 P_I + (n_1 + n_2)(1 - P_I)$$

$$= 15 \times 0.8345 + \left(15 + 30\right)\left(1 - 0.8345\right) = 19.965 \approx 20$$

In Sec. 8.2.2, you have learnt the advantages of the double sampling plan over the single sampling plan: The average sample number (ASN) for a double sampling plan is expected to be less than that for an equivalent single sampling plan (i.e, the same probability of accepting a lot of a given quality). If we plot the ASN curves for equivalent double and single sampling plans for different values of quality level p, we obtain the curves shown in Fig. 8.2.

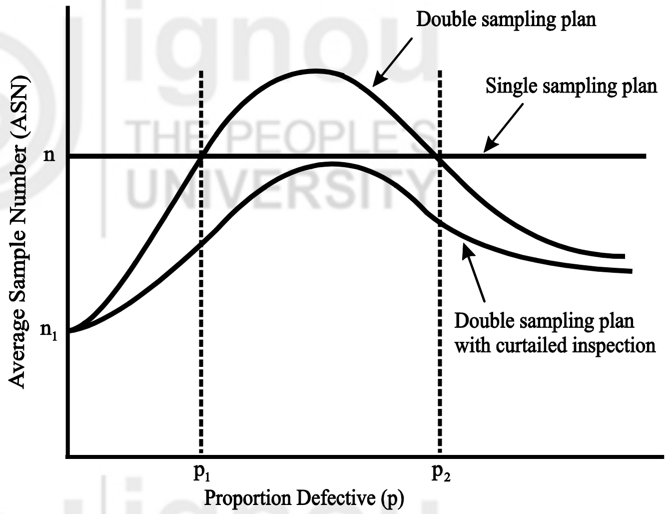

Fig. 8.2: The ASN curves for single and double sampling plans.

If we compare the ASN curves (shown in Fig. 8.2) for a double sampling plan with an equivalent single sampling plan, we observe that when the lots are of very good quality, they are usually accepted on the first sample. However, if the lots are of very bad quality, they are usually rejected on the first sample. In such cases, the ASN for the double sampling plan is smaller than the ASN for the equivalent single sampling plan because the size of the first sample in the double sampling plan is generally smaller than the size of the sample in the single sampling plan. However, if the lots are of intermediate quality (neither good nor bad), the second sample is usually required for making a decision about the acceptance or rejection of the lot in double sampling plan. In this case, the ASN for the double sampling plan is greater than the ASN for the single sampling plan.

In practice, inspection of the second sample is usually terminated and the lot is rejected as soon as the number of observed defective units in the combined samples (first and second) exceeds the acceptance number for the second sample. This termination is known as **curtailment**.

Thus, if we use the curtailed inspection for the second sample, the ASN for the double sampling plan is lower than ASN for the single sampling plan for the intermediate quality.

You can now calculate the ASN for the following exercise.

E5) Calculate the average sample number (ASN) for E2.

8.7 AVERAGE TOTAL INSPECTION (ATI)

Another important feature of the rectifying sampling plan is the average total inspection (ATI). Under rectifying sampling plan, the rejected lots are 100% inspected. The ATI is defined as follows:

The expected number of units inspected per lot under the rectifying sampling plan is called the average total inspection (ATI).

So in the rectifying double sampling plan, the number of units to be inspected will depend on the three situations given below:

i) If the lot of size N is accepted on the first sample of size n_1, the number of units inspected is n_1 and the probability of accepting the lot on the first sample is $P_{a1}(p)$.

ii) If the lot is accepted on the second sample of size n_2, the number of units inspected is $(n_1+ n_2)$ and the probability of accepting the lot on the second sample is $P_{a2}(p)$.

iii) If the lot is rejected on the first or second sample, the entire lot of size (N) is inspected and the probability of rejecting the lot is $1 – P_a(p)$.

Therefore, we can compute the ATI for a double sampling plan as follows:

$$\text{ATI} = \text{Expected number of units inspected per lot}$$

$$= \sum(\text{inspected number of units} \times \text{probability of taking decision})$$

$$\text{ATI} = n_1P_{a1}(p) + (n_1 + n_2)P_{a2}(p) + N[1 - P_a(p)] \qquad \dots (22)$$

Let us take up an example.

Example 6: Suppose that in Example 2, the rejected lots are screened and all defective mobiles are replaced by non-defective ones. Calculate the average total inspection (ATI) for this plan.

Solution: It is given that

$$N = 400, n_1 = 15, c_1 = 1, n_2 = 30, c_2 = 3 \text{ and } p = 0.05$$

We can calculate the ATI for the double sampling plan using equation (22).

For calculating ATI, we have to calculate the probabilities of accepting the lot on the first sample and the second sample corresponding to p = 0.05.

We have already calculated these probabilities in Example 2. So we directly use the results as follows:

$$P_{a1}(p) = 0.8290, P_{a2}(p) = 0.1531 \text{ and } P_a(p) = 0.9821$$

Substituting the values of N, n_1, n_2, P_{a1} and P_{a2} in equation (22), we get

$$\text{ATI} = n_1P_{a1}(p) + (n_1 + n_2)P_{a2}(p) + N[1 - P_a(p)]$$

$$= 15 \times 0.8290 + (15 + 30) \times 0.1531 + 400 \times (1 - 0.9821)$$

$$= 12.4350 + 6.8895 + 7.16 = 26.4845 \approx 27$$

In the same way, you can calculate the ATI for different submitted lot qualities.

E6) Calculate the average total inspection (ATI) for E3.

We now explain how to design double sampling plans.

8.8 DESIGN OF DOUBLE SAMPLING PLANS

The design of a double sampling plan implies the determination of the parameters of the plan, i.e., the first sample size $\mathbf{n_1}$, the second sample size $\mathbf{n_2}$ and the acceptance numbers $\mathbf{c_1}$ and $\mathbf{c_2}$. These numbers have to be decided in advance before applying the double sampling plan technique. There are several approaches for determining the parameters n_1, n_2, c_1 and c_2. Here we discuss an approach for designing the double sampling plan in which the producer's risk with its corresponding acceptance quality level (AQL) and the consumer's risk with its corresponding lot tolerance percent defective (LTPD) are specified.

We have to design a double sampling plan, which satisfies both the producer's and the consumer's risk, such that the lots of AQL are to be rejected no more than 100α% of the time and lots of LTPD are to be accepted no more than 100β% of the time.

For designing the plan in such situations, we use a pair of tables known as Grubbs Tables (Tables IV and V given at the end of this block). Tables IV and V are based on a relationship imposed on the parameters $\mathbf{n_1}$ and $\mathbf{n_2}$, that $\mathbf{n_1 = n_2}$ and $\mathbf{n_1 = 2n_2}$, respectively. Both tables are based on a producer's risk (α) of 0.05 and a consumer's risk (β) of 0.10. Table IV is used when $\mathbf{n_1 = n_2}$ and Table V is used when $\mathbf{n_1 = 2n_2}$. In this approach, we first find the operating ratio R as follows:

$$R = \frac{p_2}{p_1} \qquad \ldots (23)$$

The values of R for double sampling plan are also listed in Table IV or V. We choose a value of R which is exactly equal to its desired value. Generally, the tabulated value of R is not equal to the desired value of R. In such situations, we take the value closest to the calculated value of R. Then we look up the corresponding values of c_1, c_2 and n_1p from the appropriate table. There are two columns for n_1p, one for $\alpha = 0.05$ and another for $\beta = 0.10$.

The following two approaches are used for deciding the value of n_1:

1. **Satisfy Producer's Risk Stipulation Exactly and come close to Consumer's Risk**

   According to this approach, we look up the value of n_1p for $\alpha = 0.05$. Then we obtain the value of n_1 by dividing n_1p by $p = p_1 = \text{AQL}$ as follows:

$$n_1 = \frac{n_1 p}{p_1} \qquad \ldots (24)$$

2. **Satisfy Consumer's Risk Stipulation Exactly and come close to Producer's Risk**

   According to this approach, we look up the value of n_1p for $\beta = 0.10$. Then we obtain the value of n_1 by dividing n_1p by $p = p_2 = \text{LTPD}$ as follows:

$$n_1 = \frac{n_1 p}{p_2} \qquad \ldots (25)$$

If the computed value of n_1 is a fraction, it is rounded off to the next integer.

We then obtain the value of n_2 by taking $n_1 = n_2$ or $n_1 = 2n_2$ as the case may be.

Let us consider an example to illustrate how to design a specific double sampling plan.

Example 7: Suppose a tyre supplier ships tyres in lots of size 500 to the buyer. The supplier and the quality control inspector of the buyer decide the acceptance quality level (AQL) to be 2% and the lot tolerance percent defective (LTPD) to be 8%. Design a double sampling plan which ensures that lots of quality 2% will be rejected about 5% of the time and lots of quality 8% will be accepted about 10% of the time. The sample sizes are equal.

Solution: It is given that

$$AQL = p_1 = 2\% = 0.02 \text{ and } \alpha = 5\% = 0.05$$

$$LTPD = p_2 = 8\% = 0.08 \text{ and } \beta = 10\% = 0.10$$

$$n_1 = n_2$$

To design the desired sampling plan, we first calculate the operating ratio (R) from equation (23) as follows:

$$R = \frac{p_2}{p_1} = \frac{0.08}{0.02} = 4.0$$

Since sample sizes are equal, we use Table IV to look up the value of R. We see from Table IV that the desired value of R = 4.0 lies closest to 3.88. Then we look up the corresponding value of c_1 and c_2. From Table IV, we have c_1 = 2 and c_2 = 5. We have to find the value of the first sample size (n_1).

We first find the plan which satisfies the desired producer's risk exactly.

For this we look up the value of n_1p corresponding to $c_1 = 2$, $c_2 = 5$ and $\alpha = 0.05$ in Table IV.

From Table IV, we have $n_1p = 1.43$. Therefore, from equation (24), we have

$$n_1 = \frac{n_1p}{p_1} = \frac{1.43}{0.02} = 71.5 \simeq 72$$

Hence, the required double sampling plan is

$$n_1 = 72,\ n_2 = 72, c_1 = 2 \text{ and } c_2 = 5$$

We now find the plan which satisfies the desired consumer's risk exactly.

From Table IV, we have $n_1p = 5.55$ corresponding to $c_1 = 2$, $c_2 = 5$ and $\beta = 0.10$. Therefore, from equation (25), we have

$$n_1 = \frac{n_1p}{p_2} = \frac{5.55}{0.10} = 55.5 \simeq 56$$

Hence, the required double sampling plan is

$$n_1 = 56,\ n_2 = 56, c_1 = 2 \text{ and } c_2 = 5$$

You may like to design a sampling plan yourself. Try the following exercise.

E7) A computer manufacturer purchases a computer part from a supplier in lots of 2000. The supplier and the quality control inspector of the company decide the acceptance quality level (AQL) to be 1.5% and the lot tolerance percent defective (LTPD) to be 10%. Assuming that the lot of the second sample is twice that of the first sample, design a double sampling plan which ensures that lots of quality 1.5% will be rejected about 5% of the time and lots of quality 10% will be accepted about 10% of the time.

We now end this unit by giving a summary of what we have covered in it.

8.9 SUMMARY

1. A sampling plan in which a decision about the acceptance or rejection of a lot is based on the inspection of two samples is known as a **double sampling plan**. There are four parameters of a double sampling plan:

n_1 – size of the first sample,

c_1 – acceptance number for the first sample,

n_2 – size of the second sample, and

c_2 – acceptance numbers for both samples combined.

2. The probability of accepting a lot of quality p under a double sampling plan is given by

$$P_a(p) = \sum_{x=0}^{c_1} {}^{n_1}C_x\, p^x (1-p)^{n_1-x}$$
$$+ \sum_{x=c_1+1}^{c_2} \sum_{y=0}^{c_2-x} {}^{n_1}C_x\, p^x (1-p)^{n_1-x}\, {}^{n_2}C_y\, p^y (1-p)^{n_2-y}$$

3. The AOQ for a double sampling plan is

$$AOQ = p\left[\left(1-\frac{n_1}{N}\right)(P_{a1}+P_{a2}) - \frac{n_2}{N}P_{a2}\right]$$

4. The ASN for a double sampling plan is

$$ASN = n_1 P_I + (n_1+n_2)(1-P_I)$$

where P_I is the probability of making a decision about the acceptance or rejection of the lot on the first sample and is given by

$$P_I = P[\text{lot is accepted on the first sample}] + P[\text{lot is rejected on the first sample}]$$

5. The ATI for a double sampling plan is

$$ATI = n_1 P_{a1} + (n_1+n_2)P_{a2} + N[1-P_a]$$

8.10 SOLUTIONS/ANSWERS

E1) For implementing the double sampling plan, the buyer randomly draws the first sample of 5 chips from the lot and classifies each chip of the first sample as defective or non-defective. At the end of the inspection, he/she counts the number of defective chips (d_1) found in the first sample and compares the number of defective chips (d_1) with the acceptance numbers c_1 and c_2. If $d_1 \le c_1 = 0$, he/she accepts the lot and if $d_1 > c_2 = 2$, he/she rejects the lot. If $c_1 < d_1 \le c_2$, it means that if the number of defective chips is 1, he /she draws the second sample of size 20 from the lot. He/she counts the number of defective chips (d_2) found in the second sample and compares the total number of defective chips ($d_1 + d_2$) in both samples with the acceptance numbers c_2. If $d_1 + d_2 \le c_2 = 2$, he/she accepts the lot and if $d_1 + d_2 > c_2 = 2$, he/she rejects the lot.

E2) It is given that

$$N = 1000, n_1 = 5, c_1 = 0, n_2 = 20, c_2 = 2 \text{ and } p = 0.03$$

If $P_{a1}(p)$ and $P_{a2}(p)$ denote the probabilities of accepting the lot on the first sample and the second sample, respectively, we can calculate the probability of accepting the lot of quality level p in a double sampling plan as follows:

$$P_a(p) = P_{a1}(p) + P_{a2}(p) \qquad \ldots \text{(i)}$$

If X represents the number of defective chips in the first sample of size n_1, the lot is accepted on the first sample if $X \le c_1 = 0$. Therefore,

$$P_{a1}(p) = P[X \le c_1] = P[X \le 0] \qquad \dots \text{(ii)}$$

Since $N \ge 10n_1$, we can use the binomial distribution and calculate $P_{a1}(p)$ using Table I.

From Table I, for $n = n_1 = 5, x = c_1 = 0$ and $p = p_1 = 0.03$, we have

$$P_{a1}(p) = P[X \le 0] = \sum_{x=0}^{0} {}^{n_1}C_x\, p^x (1-p)^{n_1-x} = 0.8587 \qquad \dots \text{(iii)}$$

Hence, the probability of accepting the lot on the first sample is 0.8587. It means that about 85.87% of the lot of quality p = 0.03 will be accepted on the first sample.

If Y represents the number of defective chips in the second sample, the lot is accepted on the second sample if $c_1 < X + Y \le c_2$. It means that we shall accept the lot if

$X = c_1 + 1 = 0 + 1 = 1$ and $Y \le c_2 - X = 2 - 1 = 1$ or

$X = c_1 + 2 = 0 + 2 = 2$ and $Y \le c_2 - X = 2 - 2 = 0$.

Therefore,

$$P_{a2}(p) = P[X = 1]P[Y \le 1] + P[X = 2]P[Y \le 0] \qquad \dots \text{(iv)}$$

We also know that $P[X = x] = P[X \le x] - P[X \le x - 1]$

From Table I, for $n = n_1 = 5$ and $p = 0.03$, we have

$$P[X = 1] = P[X \le 1] - P[X \le 0] = 0.9915 - 0.8587 = 0.1328$$

$$P[X = 2] = P[X \le 2] - P[X \le 1] = 0.9997 - 0.9915 = 0.0082$$

From Table I, for $n = n_2 = 20$ and $p = 0.03$, we have

$$P[Y \le 1] = 0.8802 \text{ and } P[Y \le 0] = 0.5438$$

On putting these values in equation (iv), we get

$$P_{a2}(p) = P[X = 1]P[Y \le 1] + P[X = 2]P[Y \le 0]$$

$$= 0.1328 \times 0.8802 + 0.008 \times 0.5438 = 0.1213$$

Hence, the probability of accepting the lot on the second sample is 0.1213. It means that about 12.13% of the lot of quality p = 0.03 will be accepted on the second sample.

Thus, from equation (i), we get the probability of accepting a lot in a double sampling plan as follows:

$$P_a(p) = P_{a1}(p) + P_{a2}(p) = 0.8587 + 0.1213 = 0.9800$$

Thus, overall, 98% of the lots will be accepted by this sampling plan.

E3) It is given that

$$N = 1000, n_1 = 5, c_1 = 0, n_2 = 20, c_2 = 2,$$

$$AQL = p_1 = 0.05 \text{ and } LTPD = p_2 = 0.14$$

For calculating the producer's risk, we can use equation (11):

$$P_p = P\left[\text{rejecting a lot of acceptance quality level } p_1\right]$$

$$= 1 - P_a\left(p_1\right)$$

Therefore, for calculating the producer's risk, we first calculate the probability of accepting the lot of quality $p = p_1 = AQL = 0.05$.

In a double sampling plan, the lot may be accepted either on the first sample or on the second sample. So if $P_{a1}(p)$ and $P_{a2}(p)$ denote the probabilities of accepting the lot on the first sample and the second sample, respectively, we can calculate the probability of acceptance of the lot of quality level p as follows:

$$P_a(p) = P_{a1}(p) + P_{a2}(p) \qquad \ldots \text{(i)}$$

If X represents the number of defective chips in the first sample, the lot will be accepted on the first sample if $X \le c_1$, i.e., $X \le 0$. Therefore,

$$P_{a1}\left(p\right) = P\left[X \le c\right] = P\left[X \le 0\right] \qquad \ldots \text{(ii)}$$

Since $N \ge 10n_1$, we can use the binomial distribution and calculate the probability using Table I.

From Table I, for $n = n_1 = 5, x = c_1 = 0$ and $p = p_1 = 0.05$, we have

$$P_{a1}\left(p_1\right) = P\left[X \le 0\right] = \sum_{x=0}^{0} {}^{n_1}C_x\, p_1^x \left(1 - p_1\right)^{n_1 - x} = 0.7738 \qquad \ldots \text{(iii)}$$

If Y represents the number of defective chips in the second sample, the lot will be accepted on the second sample if $c_1 < X + Y \le c_2$. It means that we shall accept the lot if

$$X = c_1 + 1 = 0 + 1 = 1 \text{ and } Y \le c_2 - X = 2 - 1 = 1$$

$$\text{or } X = c_1 + 2 = 0 + 2 = 2 \text{ and } Y \le c_2 - X = 2 - 2 = 0.$$

Therefore,

$$P_{a2}\left(p_2\right) = P\left[X = 1\right]P\left[Y \le 1\right] + P\left[X = 2\right]P\left[Y \le 0\right] \qquad \ldots \text{(iv)}$$

We also know that $P\left[X = x\right] = P\left[X \le x\right] - P\left[X \le x - 1\right]$

From Table I, for $n = n_1 = 5$ and $p = p_1 = 0.05$, we have

$$P\left[X = 1\right] = P\left[X \le 1\right] - P\left[X \le 0\right] = 0.9774 - 0.7738 = 0.2036$$

$$P\left[X = 2\right] = P\left[X \le 2\right] - P\left[X \le 1\right] = 0.9988 - 0.9774 = 0.0214$$

From Table I, for $n = n_2 = 20$ and $p = p_1 = 0.05$, we have

$$P\left[Y \le 0\right] = 0.3585 \text{ and } P\left[Y \le 1\right] = 0.7358$$

On putting these values in equation (iv), we get

$$P_{a2}\left(p_2\right) = P\left[X = 1\right]P\left[Y \le 1\right] + P\left[X = 2\right]P\left[Y \le 0\right]$$

$$= 0.2036 \times 0.7358 + 0.0214 \times 0.3585 = 0.1575$$

Hence, from equation (i), we get the probability of accepting the lot of quality $p = p_1 = 0.05$ in this double sampling plan as follows:

$$P_a(p_1) = P_{a1}(p_1) + P_{a2}(p_1) = 0.7738 + 0.1575 = 0.9313$$

Therefore, we calculate the producer's risk for this plan using equation (11) as follows:

$$P_p = 1 - P_a(p_1) = 1 - 0.9313 = 0.0687$$

Similarly, we can calculate the consumer's risk as follows:

We can calculate the consumer's risk for the double sampling plan using equation (12) which is given by

$$P_c(p) = P[\text{accepting a lot of quality (LTPD) } p_2] = P_a(p_2) \quad \ldots \text{(v)}$$

We first calculate the probability of accepting the lot of quality $p = p_1 = \text{LTPD} = 0.14$ using equations (i), (ii) and (iv).

From Table I, for $n = n_1 = 5, x = c_1 = 0$ and $p = p_2 = 0.14$, we have

$$P_{a1}(p_2) = P[X \le 0] = \sum_{x=0}^{0} {}^{n_1}C_x \, p_2^x (1 - p_2)^{n_1 - x} = 0.4704 \quad \ldots \text{(vi)}$$

From Table I, for $n = n_1 = 5$ and $p = p_2 = 0.14$, we have

$$P[X = 1] = P[X \le 1] - P[X \le 0] = 0.8533 - 0.4704 = 0.3829$$

$$P[X = 2] = P[X \le 2] - P[X \le 1] = 0.9780 - 0.8533 = 0.1247$$

From Table I, for $n = n_2 = 20$ and $p = p_2 = 0.14$, we have

$$P[Y \le 0] = 0.0490 \text{ and } P[Y \le 1] = 0.2084$$

On putting these values in equation (iv), we get

$$P_{a2}(p_2) = P[X = 1]P[Y \le 1] + P[X = 2]P[Y \le 0]$$

$$= 0.3829 \times 0.2084 + 0.1247 \times 0.0490 = 0.0859 \quad \ldots \text{(vii)}$$

On putting the values of $P_{a1}(p_2)$ and $P_{a2}(p_2)$ in equation (i), we get

$$P_a(p_2) = 0.4704 + 0.0859 = 0.5563$$

Hence, from equation (v), we get, the consumer's risk for this plan as follows:

$$P_c = P_a(p_2) = 0.5563$$

E4) It is given that

$$N = 1000, n_1 = 5, c_1 = 0, n_2 = 20, c_2 = 2 \text{ and } p = 0.03$$

It is noted that the sample sizes $n_1 = 5$ and $n_2 = 20$ are very small in proportion to the lot size N = 1000, i.e., $n_1 / N \to 0$ and $n_2 / N \to 0$. So we can calculate the average outgoing quality (AOQ) for the double sampling plan using equation (19).

For calculating AOQ, we have to calculate the probabilities of accepting the lot on the first sample and the second sample corresponding to $p = 0.03$.

We have already calculated these probabilities in E2. So we directly use the results:

$$P_{a1}(p) = 0.8587 \text{ and } P_{a2}(p) = 0.1213$$

Substituting the values of P_{a1}, P_{a2} and p in equation (19), we get

$$AOQ = p(P_{a1} + P_{a2}) = 0.03 \times (0.8587 + 0.1213) = 0.0294$$

E5) It is given that

$$N = 1000, n_1 = 5, c_1 = 0, n_2 = 20, c_2 = 2 \text{ and } p = 0.03$$

For computing ASN for the double sampling plan, we can use equation (20):

$$ASN = n_1 P_I + (n_1 + n_2)(1 - P_I)$$

where P_I is the probability of making a decision about the acceptance or rejection of the lot on the first sample and can be calculated as follows:

$$P_I = P[\text{lot is accepted on the first sample}] + P[\text{lot is rejected on the first sample}] \quad \ldots \text{(i)}$$

So we have to find the probability P_I first. In the double sampling plan, the lot is accepted on the first sample if $X \le c_1$, i.e., if $X \le 0$. Therefore,

$$P[\text{lot is accepted on the first sample}] = P[X \le c_1] = P[X \le 0] = P_{a1}$$

We have already calculated this probability in E2. So we directly use the results. Therefore,

$$P_{a1}(p) = P[X \le 0] = 0.8587 \quad \ldots \text{(ii)}$$

The lot will be rejected on the first sample if $X > c_2$, i.e., if $X > 2$. Therefore,

$$P[\text{lot is rejected on the first sample}] = P[X > 2] = 1 - P[X \le 2] \quad \ldots \text{(iii)}$$

From Table I, for $n = n_1 = 5, x = c_2 = 2$ and $p = 0.03$, we have

$$P[X \le 2] = \sum_{x=0}^{2} {}^{n_1}C_x\, p^x (1-p)^{n_1 - x} = 0.9997$$

Therefore, from equation (iii), we have

$$P[\text{lot is rejected on the first sample}] = 1 - P[X \le 2]$$
$$= 1 - 0.9997 = 0.0003 \quad \ldots \text{(iv)}$$

Thus, on putting the values from equations (ii) and (iv) in equation (i), we get

$$P_I = 0.8587 + 0.0003 = 0.8590$$

Hence, on putting the values of n_1, n_2, and P_I in equation (20), we get the ASN for the plan as follows:

$$ASN = n_1P_I + (n_1 + n_2)(1 - P_I)$$

$$= 5 \times 0.8590 + (5 + 20)(1 - 0.8590) = 7.82 \approx 8$$

E6) It is given that

$$N = 1000, n_1 = 5, c_1 = 0, n_2 = 20, c_2 = 2 \text{ and } p = 0.03$$

We can calculate the ATI for the double sampling plan using equation (22).

For calculating ATI, we have to calculate the probabilities of accepting the lot on the first sample and the second sample corresponding to $p = 0.03$.

We have already calculated these probabilities in E2. So we directly use the results:

$$P_{a1}(p) = 0.8587, P_{a2}(p) = 0.1213 \text{ and } P_a(p) = 0.9800$$

Substituting the values of N, n_1, n_2, P_{a1} and P_{a2} in equation (22), we get

$$ATI = n_1P_{a1}(p) + (n_1 + n_2)P_{a2}(p) + N\left[1 - P_a(p)\right]$$

$$= 5 \times 0.8587 + (5 + 20) \times 0.1213 + 1000 \times (1 - 0.9800)$$

$$= 4.2935 + 3.0325 + 20 = 27.3260 \approx 28$$

E7) It is given that

$$AQL = p_1 = 1.5\% = 0.015 \text{ and } \alpha = 5\% = 0.05$$

$$LTPD = p_2 = 10\% = 0.10 \text{ and } \beta = 10\% = 0.10$$

$$n_2 = 2n_1$$

To design the desired sampling plan, we first calculate the operating ratio (R) from equation (23) as follows:

$$R = \frac{p_2}{p_1} = \frac{0.10}{0.015} = 6.67$$

Since $n_2 = 2n_1$, we use Table V to look up the value of R. We see from Table V that the desired value of R = 6.67 lies closest to 6.48. Then we look up the corresponding values of c_1 and c_2. From Table V, we have $c_1 = 1$ and $c_2 = 3$. We have to find the value of the first sample size (n_1).

We first find the plan which satisfies the desired producer's risk exactly.

For this we look up the value of n_1p corresponding $c_1 = 1$, $c_2 = 3$ and $\alpha = 0.05$ in Table V.

From Table V, we have $n_1p = 0.60$. Therefore, from equation (24), we have

$$n_1 = \frac{n_1p}{p_1} = \frac{0.60}{0.015} = 40$$

Hence, the required double sampling plan is

$$n_1 = 40,\ n_2 = 80, c_1 = 1 \text{ and } c_2 = 3$$

We now find the plan which satisfies the desired consumer's risk exactly.

From Table V, we have $n_1p = 3.89$ corresponding to $c_1 = 1$, $c_2 = 3$ and $\beta = 0.10$. Therefore, from equation (25), we have

$$n_1 = \frac{n_1p}{p_2} = \frac{3.89}{0.10} = 38.9 \approx 39$$

Hence, the required double sampling plan is

$$n_1 = 39,\ n_2 = 78, c_1 = 1 \text{ and } c_2 = 3$$