UNIT 7 BIVARIATE CONTINUOUS RANDOM VARIABLES

Structure

7.1 INTRODUCTION

In Unit 6, we have defined the bivariate discrete random variable (X, Y), where X and Y both are discrete random variables. It may also happen that one of the random variables is discrete and the other is continuous. However, in most applications we deal only with the cases where either both random variables are discrete or both are continuous. The cases where both random variables are discrete have already been discussed in Unit 6. Here, in this unit, we are going to discuss the cases where both random variables are continuous.

In Unit 6, you have studied the joint, marginal and conditional probability functions and distribution functions in context of bivariate discrete random variables. Similar functions, but in context of bivariate continuous random variables, are discussed in this unit.

Bivariate continuous random variable is defined in Sec. 7.2. Joint and marginal density functions are described in Sec. 7.3. Sec. 7.4 deals with the conditional distribution and density functions. Independence of two continuous random variables is dealt with in Sec. 7.5. Some practical problems on two-dimensional continuous random variables are taken up in Sec. 7.6.

Objectives

A study of this unit would enable you to:

- define two-dimensional continuous random variable;
- specify the joint and marginal probability density functions of two continuous random variables;
- obtain the conditional density and distribution functions for two-dimensional continuous random variable;
- check the independence of two continuous random variables; and
- solve various practical problems on bivariate continuous random variables.

7.2 BIVARIATE CONTINUOUS RANDOM VARIABLES

Definition: If X and Y are continuous random variables defined on the sample space S of a random experiment, then (X, Y) defined on the same sample space S is called bivariate continuous random variable if (X, Y) assigns a point in xy-plane defined on the sample space S. Notice that it (unlike discrete random variable) assumes values in some non-countable set. Some examples of bivariate continuous random variable are:

1. A gun is aimed at a certain point (say origin of the coordinate system). Because of the random factors, suppose the actual hit point is any point (X, Y) in a circle of radius unity about the origin.

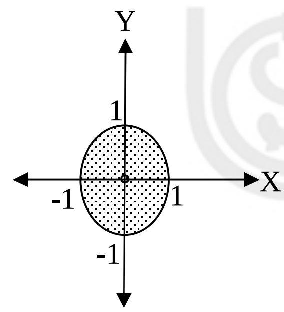

Fig.7.1: Actual Hit Point when a Gun is Aimed at a Certain Point

Then (X, Y) assumes all the values in the circle $\{(x, y): x^2 + y^2 \le 1\}$ i.e. (X, Y) assumes all values corresponding to each and every point in the circular region as shown in Fig.7.1. Here, (X, Y) is bivariate continuous random variable.

2. (X, Y) assuming all values in the rectangle $\{(x, y): a \le x \le b, c \le y \le d\}$ is a bivariate continuous random variable.

   Here, (X, Y) assumes all values corresponding to each and every point in the rectangular region as shown in Fig.7.2.

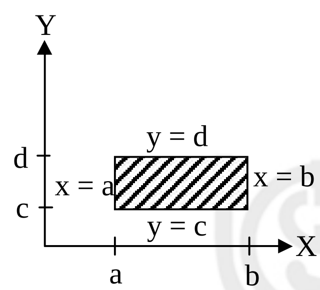

Fig.7.2: (X, Y) Assuming All Values in the Rectangle $\{(x,y): a \le x \le b, c \le y \le d\}$

3. In a statistical survey, let X denotes the daily number of hours a child watches television and Y denotes the number of hours he/she spends on the studies. Here, (X, Y) is a two-dimensional continuous random variable.

7.3 JOINT AND MARGINAL DISTRIBUTION AND DENSITY FUNCTIONS

Two-Dimensional Continuous Distribution Function

The distribution function of a two-dimensional continuous random variable (X, Y) is a real-valued function and is defined as

$F(x, y) = P[X \le x, Y \le y]$ for all real x and y.

Notice that the above function is in analogy with one-dimensional continuous random variable case as studied in Unit 5 of the course.

Remark 1: $F(x, y)$ can also be written as $F_{X,Y}(x, y)$.

Joint Probability Density Function

Let (X, Y) be a continuous random variable assuming all values in some region R of the xy-plane. Then, a function $f(x, y)$ such that

$$F(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x, y)\,dydx$$

is defined to be a joint probability density function.

As in the one-dimensional case, a joint probability density function has the following properties.

i) $f(x, y) \ge 0$

ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dydx = 1$

Remark 2:

As in the one-dimensional case, $f(x, y)$ does not represent the probability of anything. However, for positive δx and δy sufficiently small, $f(x, y)\delta x\delta y$ is approximately equal to

$P[x \le X \le x + \delta x, y \le Y \le y + \delta y]$.

In the one-dimensional case, you have studied that for positive δx sufficiently small $f(x)\delta x$ is approximately equal to $P[x \le X \le x + \delta x]$. So, the two-dimensional case is in analogy with the one-dimensional case.

Remark 3:

In analogy with the one-dimensional case [See Sec. 5.4 of Unit 5 of this course],

$f(x, y)$ can be written as $\lim\limits_{\substack{\delta x \to 0 \\ \delta y \to 0}} \dfrac{P[x \le X \le x + \delta x, y \le Y \le y + \delta y]}{\delta x\delta y}$

and is equal to

$\dfrac{\partial^2}{\partial x \partial y}(F(x, y))$, i.e. second order partial derivative with respect to x and y.

[See Sec. 5.5 of Unit 5 where $f(x)=\frac{d}{dx}(F(x))$]

Note: $\frac{\partial^2}{\partial x \partial y}(F(x,y))$ means first differentiate $F(x,y)$ partially w.r.t. y and then the resulting function w.r.t. x. When we differentiate a function partially w.r.t. one variable, then the other variable is treated as constant

For example, Let $F(x,y)=xy^3+x^2y$

If we differentiate it partially w.r.t. y, we have

$$\frac{\partial}{\partial y}(F(x,y)) = x(3y^2)+x^2.1 \qquad [\because \text{ here, x is treated as constant.}]$$

If we now differentiate this resulting expression w.r.t. x, we have

$$\frac{\partial^2}{\partial x \partial y}(F(x,y)) = 3y^2+2x \qquad [\because \text{ here, y is treated as constant.}]$$

Marginal Continuous Distribution Function

Let (X, Y) be a two-dimensional continuous random variable having $f(x,y)$ as its joint probability density function. Now, the marginal distribution function of the continuous random variable X is defined as

$$F(x)=P[X\le x]$$

$$=P[X\le x, y<\infty] \qquad [\because \text{ for } X \le x, \text{ Y can take any real value}]$$

$$=\int_{-\infty}^{x}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]dx,$$

and the marginal distribution function of the continuous random variable Y is defined as

$$F(y)=P[Y\le y]$$

$$=P[Y\le y, X<\infty] \qquad [\because \text{ for } Y \le y, \text{ X can take any real value}]$$

$$=\int_{-\infty}^{y}\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right]dy$$

Marginal Probability Density Functions

Let (X, Y) be a two-dimensional continuous random variable having $F(x,y)$ and $f(x,y)$ as its distribution function and joint probability density function, respectively. Let $F(x)$ and $F(y)$ be the marginal distribution functions of X and Y, respectively. Then, the marginal probability density function of X is given as

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy,$$

or, it may also be obtained as $\frac{d}{dx}\left(F(x)\right)$,

and the marginal probability density function of Y is given as

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

or

$$= \frac{d}{dy}\left(F(y)\right)$$

7.4 CONDITIONAL DISTRIBUTION AND DENSITY FUNCTIONS

Conditional Probability Density Function

Let (X, Y) be a two-dimensional continuous random variable having the joint probability density function f(x, y). The conditional probability density function of Y given X = x is defined as

$f(y \mid x) = \frac{f(x,y)}{f(x)}$, where f(x) > 0 is the marginal density of X.

Similarly, the conditional probability density function of X given Y = y is defined to be

$f(x \mid y) = \frac{f(x,y)}{f(y)}$, where f(y) > 0 is the marginal density of Y.

As $f(y \mid x)$ and $f(x \mid y)$, though conditional yet, are the probability density functions, hence possess the properties of a probability density function. Properties of $f(y \mid x)$ are:

i) $f(y \mid x)$ is clearly ≥ 0

ii) $\int_{-\infty}^{\infty} f(y \mid x)\,dy = \int_{-\infty}^{\infty} \frac{f(x,y)}{f(x)}\,dy$

$$= \frac{1}{f(x)}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]$$

$$= \frac{1}{f(x)}\left[f(x)\right] \qquad \left[\because \int_{-\infty}^{\infty} f(x,y)\,dy \text{ is the marginal probability density function of X}\right]$$

$$= 1$$

Similarly, $f(x|y)$ satisfies

i) $f(x|y) \geq 0$ and

ii) $\int_{-\infty}^{\infty} f(x|y)dx = 1$

Conditional Continuous Distribution Function

For a two-dimensional continuous random variable (X, Y), the conditional distribution function of Y given X = x is defined as

$$F(y|x) = P[Y \leq y | X = x]$$

$$= \int_{-\infty}^{y} f(y|x)dy \text{, for all x such that } f(x) > 0;$$

and the conditional distribution function of X given Y = y is defined as

$$F(x|y) = P[X \leq x | Y = y]$$

$$= \int_{-\infty}^{x} f(x|y)dx \text{, for all y such that f(y) > 0.}$$

7.5 STOCHASTIC INDEPENDENCE OF TWO CONTINUOUS RANDOM VARIABLES

You have already studied in Unit 3 of this course that independence of events is closely related to conditional probability, i.e. if events A and B are independent, then $P[A|B] = P[A]$, i.e. conditional probability of A is equal to the unconditional probability of A. Likewise independence of random variables is closely related to conditional distributions of random variables, i.e. two random variables X and Y with joint probability function $f(x, y)$ and marginal probability functions $f(x)$ and $f(y)$ respectively are said to be stochastically independent if and only if

i) $f(y|x) = f(y)$

ii) $f(x|y) = f(x)$.

Now, as defined in Sec. 7.4, we have

$$f(y|x) = \frac{f(x, y)}{f(x)}$$

$$\Rightarrow f(x, y) = f(x)f(y|x) \qquad \text{[On cross-multiplying]}$$

So, if X and Y are independent, then

$$f(x, y) = f(x)f(y) \qquad [\because \ f(y|x) = f(y)]$$

Remark 4: The random variables, if independent, are actually stochastically independent but often the word "stochastically" is omitted.

Definition: Two random variables are said to be (stochastically) independent if and only if their joint probability density function is the product of their marginal density functions.

Let us now take up some problems on the topics covered so far in this unit.

7.6 PROBLEMS ON TWO-DIMENSIONAL CONTINUOUS RANDOM VARIABLES

Example 1: Let X and Y be two random variables. Then for

$$f(x,y)=\begin{cases}k(2x+y), & 0<x<1,\ 0<y<2\\ 0, & \text{elsewhere}\end{cases}$$

to be a joint density function, what must be the value of k ?

Solution: As $f(x,y)$ is the joint probability density function,

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx=1$$

$$\Rightarrow \int_0^1\int_0^2 f(x,y)\,dy\,dx=1 \qquad [\because 0<x<1,\ 0<y<2\ ]$$

$$\Rightarrow \int_0^1\int_0^2 k(2x+y)\,dy\,dx=1$$

$$\Rightarrow k\int_0^1\left[\int_0^2(2x+y)\,dy\right]dx=1$$

$$\Rightarrow k\int_0^1\left[2xy+\frac{y^2}{2}\right]_0^2 dx=1$$

[Firstly the integral has been done w.r.t. y treating x as constant.]

$$\Rightarrow k\int_0^1\left[2x(2)+\frac{(2)^2}{2}-0\right]dx=1$$

$$\Rightarrow k\int_0^1(4x+2)\,dx=1$$

$$\Rightarrow k\left[\frac{4x^2}{2}+2x\right]_0^1=1$$

$$\Rightarrow k\left[\frac{4}{2}+2-0\right]=1 \Rightarrow 4k=1\Rightarrow k=\frac{1}{4}$$

Example 2: Let the joint density function of a two-dimensional random variable (X, Y) be:

$$f(x,y) = \begin{cases} x+y & \text{for } 0 \le x < 1 \text{ and } 0 \le y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the conditional density function of Y given X.

Solution: The conditional density function of Y given X is $f(y|x) = \frac{f(x,y)}{f(x)}$,

where $f(x,y)$ is the joint density function, which is given; and $f(x)$ is the marginal density function which, by definition, is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{0}^{1} f(x,y)\,dy \qquad [\because 0 \le y < 1]$$

$$= \int_{0}^{1} (x+y)\,dy$$

$$= \left[xy + \frac{y^2}{2}\right]_0^1.$$

$$= \left[x(1) + \frac{(1)^2}{2} - 0\right] = x + \frac{1}{2},\ 0 \le x < 1.$$

$\therefore$ the conditional density function of Y given X is

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{x+y}{x+\frac{1}{2}},\ \text{for } 0 \le x < 1 \text{ and } 0 \le y < 1.$$

Example 3: Two-dimensional random variable (X, Y) have the joint density

$$f(x,y) = \begin{cases} 8xy, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

i) Find $P[X < \frac{1}{2} \cap Y < \frac{1}{4}]$.

ii) Find the marginal and conditional distributions.

iii) Are X and Y independent?

Solution:

i) $$P\left[X < \frac{1}{2} \cap Y < \frac{1}{4}\right] = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{4}} f(x,y)\,dy\,dx = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{4}} 8xy\,dy\,dx = \int_0^{\frac{1}{2}} 8x\left[\frac{y^2}{2}\right]_0^{\frac{1}{4}} dx$$

$$= \int_0^{\frac{1}{2}} 8x\left[\frac{1}{16(2)}\right]dx = \int_0^{\frac{1}{2}} \frac{x}{4}dx = \frac{1}{4}\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{4}\left[\frac{1}{8}\right] = \frac{1}{32}.$$

ii) Marginal density function of X is

$$f(x) = \int_x^1 f(x, y)\,dy \qquad [\because 0 < x < y < 1]$$

$$= \int_x^1 8xy\,dy = 8x\left[\frac{y^2}{2}\right]_x^1$$

$$= 8x\left[\frac{1}{2} - \frac{x^2}{2}\right] = 4x\left(1 - x^2\right) \text{ for } 0 < x < 1$$

Marginal density function of Y is

$$f(y) = \int_0^y f(x, y)\,dx \qquad [\because 0 < x < y]$$

$$= \int_0^y 8xy\,dx$$

$$= 8y\left[\frac{x^2}{2}\right]_0^y = \frac{8y^3}{2} = 4y^3 \quad \text{for} \quad 0 < y < 1$$

Conditional density function of X given Y(0 < Y < 1) is

$$f(x \mid y) = \frac{f(x, y)}{f(y)}$$

$$= \frac{8xy}{4y^3} = \frac{2x}{y^2},\ 0 < x < y$$

Conditional density function of Y given X(0 < X < 1) is

$$f(y \mid x) = \frac{f(x, y)}{f(x)}$$

$$= \frac{8xy}{4x\left(1 - x^2\right)} = \frac{2y}{\left(1 - x^2\right)},\ x < y < 1$$

iii) $f(x, y) = 8xy,$

But $f(x)f(y) = 4x\left(1 - x^2\right)4y^3$

$$= 16x\left(1 - x^2\right)y^3$$

$$\therefore f(x, y) \neq f(x) f(y)$$

Hence, X and Y are not independent random variables.

Now, you can try some exercises.

E1) Let X and Y be two random variables. Then for

$$f(x, y) = \begin{cases} kxy & \text{for } 0 < x < 4 \text{ and } 1 < y < 5 \\ 0, & \text{otherwise} \end{cases}$$

to be a joint density function, what must be the value of k?

E2) If the joint p.d.f. of a two-dimensional random variable (X, Y) is given by

$$f(x, y) = \begin{cases} 2 & \text{for } 0 < x < 1 \text{ and } 0 < y < x \\ 0, & \text{otherwise,} \end{cases}$$

Then,

i) Find the marginal density functions of X and Y.

ii) Find the conditional density functions.

iii) Check for independence of X and Y.

E3) If (X, Y) be two-dimensional random variable having joint density function.

$$f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y); & 0 < x < 2, 2 < y < 4 \\ 0, & \text{elsewhere} \end{cases}$$

Find (i) $P[X < 1, Y < 3]$ (ii) $P[X < 1 | Y < 3]$

Now before ending this unit, let's summarize what we have covered in it.

7.7 SUMMARY

In this unit, we have covered the following main points:

1) If X and Y are continuous random variables defined on the sample space S of a random experiment, then (X, Y) defined on the same sample space S is called **bivariate continuous random variable** if (X, Y) assigns a point in xy-plane defined on the sample space S.

2) The distribution function of a two-dimensional continuous random variable (X, Y) is a real-valued function and is defined as

   $F(x, y) = P[X \leq x, Y \leq y]$ for all real x and y.

3) A function $f(x, y)$ is called **joint probability density function** if it is such that

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y)\, dy\, dx$$

and satisfies

i) $f(x,y) \geq 0$

ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dydx = 1.$

4) The **marginal distribution function** of the continuous random variable X is defined as

$$F(x) = P[X \leq x] = \int_{-\infty}^{x}\left[\int_{-\infty}^{\infty} f(x,y)\,dy\right]dx,$$

and that of continuous random variable Y is defined as

$$F(y) = P[Y \leq y] = \int_{-\infty}^{y}\left[\int_{-\infty}^{\infty} f(x,y)\,dx\right]dy.$$

5) The **marginal probability density function** of X is given as

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{d}{dx}(F(x)),$$

and that of Y is given as

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{d}{dx}(F(y)).$$

6) The **conditional probability density function** of Y given X = x is defined as

$$f(y\,|\,x) = \frac{f(x,y)}{f(x)},$$

and that of X given Y = y is defined as

$$f(x\,|\,y) = \frac{f(x,y)}{f(y)}.$$

7) The **conditional distribution function** of Y given X = x is defined as

$$F(y\,|\,x) = \int_{-\infty}^{y} f(y\,|\,x)\,dy, \text{ for all x such that } f(x) > 0;$$

and that of X given Y = y is defined as

$$F(x\,|\,y) = \int_{-\infty}^{x} f(x\,|\,y)\,dx, \text{ for all y such that } f(y) > 0.$$

8) Two random variables are said to be **(stochastically) independent** if and only if their joint probability density function is the product of their marginal density functions.

7.8 SOLUTIONS/ANSWERS

E1) As $f(x, y)$ is the joint probability density function,

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dy\,dx = 1$$

$$\Rightarrow \int_0^4\int_1^5 kxy\,dy\,dx = 1 \;\Rightarrow\; k\int_0^4\left[\int_1^5 xy\,dy\right]dx = 1$$

$$\Rightarrow k\int_0^4\left[x\frac{y^2}{2}\right]_1^5 dx = 1 \;\Rightarrow\; k\int_0^4 12x\,dx = 1$$

$$\Rightarrow 12k\left[\frac{x^2}{2}\right]_0^4 = 1 \;\Rightarrow\; 96\,k = 1$$

$$\Rightarrow k = \frac{1}{96}$$

E2) i) Marginal density function of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x, y)\,dx \;= \int_y^1 2dx$$

[As x is involved in both the given ranges, i.e. $0 < x < 1$ and $0 < y < x$; therefore, here we will combine both these intervals and hence have $0 < y < x < 1$. $\therefore$ x takes the values from y to 1]

$$= [2x]_y^1 = 2 - 2y$$

$$= 2 - 2y$$

$$= 2(1 - y),\; 0 < y < 1$$

Marginal density function of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x, y)\,dy$$

$$= \int_0^x 2dy \qquad [\because\; 0 < \underbrace{y < x}\; < 1]$$

$$= 2[y]_0^x$$

$$= 2x,\; 0 < x < 1.$$

ii) Conditional density function of Y given X(0 < X < 1) is

$$f(y\,|\,x) = \frac{f(x, y)}{f(x)} = \frac{2}{2x} = \frac{1}{x};\; 0 < y < x$$

Conditional density function of X and given Y(0 < Y < 1) is

$$f(x\,|\,y) = \frac{f(x, y)}{f(y)} = \frac{2}{2(1-y)} = \frac{1}{1-y},\; y < x < 1$$

iii) $f(x,y) = 2,$

$$f(x)f(y) = 2(2x)(1-y)$$

As $f(x,y) \neq f(x)f(y),$

$\therefore$ X and Y are not independent.

E3) (i) $P[X<1, Y<3] = \int_{-\infty}^{1}\int_{-\infty}^{3} f(x,y)\,dy\,dx$

$$= \int_0^1\int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= \int_0^1\left[\frac{1}{8}\left(6y - xy - \frac{y^2}{2}\right)\right]_2^3 dx$$

$$= \frac{1}{8}\int_0^1\left[\left\{6(3) - x(3) - \frac{9}{2}\right\} - \{12 - 2x - 2\}\right]dx$$

$$= \frac{1}{8}\int_0^1\left[\left(18 - 3x - \frac{9}{2}\right) - (10 - 2x)\right]dx$$

$$= \frac{1}{8}\int_0^1\left(\frac{7}{2} - x\right)dx \quad = \frac{1}{8}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^1 \quad = \frac{1}{8}\left[\frac{7}{2} - \frac{1}{2}\right] = \frac{3}{8}$$

ii) $P[X<1 \mid Y<3] = \dfrac{P[X<1, Y<3]}{P[Y<3]}$

where $P(Y<3) = \int_0^2\int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$

$$= \frac{1}{8}\int_0^2\left[6y - xy - \frac{y^2}{2}\right]_2^3 dx\,.$$

$$= \frac{1}{8}\int_0^2\left[\left\{18 - 3x - \frac{9}{2}\right\} - \{12 - 2x - 2\}\right]dx$$

$$= \frac{1}{8}\int_0^2\left[\left(18 - 3x - \frac{9}{2}\right) - (10 - 2x)\right]dx$$

$$= \frac{1}{8}\int_0^2\left(\frac{7}{2} - x\right)dx$$

$$= \frac{1}{8}\left[\frac{7}{2}x - \frac{x^2}{2}\right]_0^2$$

$$= \frac{1}{8}\left[7 - \frac{4}{2} - 0\right]$$

$$= \frac{5}{8}$$

$$\therefore P\left[X < 1 \middle| Y < 3\right] = \frac{3/8}{5/8} \quad \begin{bmatrix} \because \text{value of numerator is} \\ \text{already calculated in part(i)} \end{bmatrix}$$

$$= \frac{3}{5}$$