UNIT 13 RANDOM NUMBER GENERATION FOR DISCRETE VARIABLES

Structure

13.1 INTRODUCTION

We shall see in subsequent units that for simulation of any system or process, which contains some random components, we require a method for obtaining random numbers. For example, the queuing and inventory models, which we shall discuss in later units, require inter-arrival times, service times, demand sizes, etc. which are random in nature. For this we may need random variables from Exponential, Gamma distributions, etc. Usually a large number of random numbers from standard uniform distribution U (0, 1) are required which are independently and identically distributed (i.i.d.). Basically, they are required for generation of random variables from non-uniform distribution such as Binomial, Poisson, Normal, etc. They are also required for calculating integral of a function by Monte-Carlo technique. We shall use these random numbers to generate random variables from some popular discrete and continuous distributions, which we shall require for simulation purposes.

The concepts of Random numbers and Pseudo random numbers (PRN) are described in Section 13.2. The different methods of generation of random numbers and pseudo random numbers are explained in Section 13.3. The Inverse probability transformation (IPT) method for generating the discrete and continuous is explored in Section 13.4. In Section 13.5 describes the methods of random number generation for discrete variate from discrete

Uniform, Bernoulli, Binomial, Geometric, Negative Binomial and Poisson distribution.

Objectives

After studying this unit, you will be able to

- define the random numbers and Pseudo random numbers;
- explain Lottery method of generation of random numbers;
- explain Middle Square method of generation of Pseudo random numbers (PRN);
- explain Congruential method of generation of Pseudo random numbers (PRN);
- explain Inverse probability transformation (IPT) method for generating random variables; and
- describe the generation of random numbers from some discrete probability distributions such as Discrete Uniform, Bernoulli, Binomial, Poisson, Geometric, Negative Binoimial, etc.

13.2 RANDOM NUMBERS AND PSEUDO RANDOM NUMBERS (PRN)

The methodology of generating random numbers has a long history. The earliest methods were carried out by hands, throwing dice, dealing out well suffuled cards, etc. Many lotteries are still operated in this way. In the early twentieth century many mechanised devices were built to generate numbers more quickly. Later electronic devices were used to generate random numbers. Rand Corporation (1955) generated a million numbers and are available in table form. An easy method to read from Rand Corporation table or some other statistical tables which contain a large number of random numbers is to select a row or a column of one, two or more figures.

For example, Fisher and Yates (1963) Statistical Tables give 7500 two-figure random numbers arranged in six pages. Suppose one wants to select ten random numbers from 00-99, the simplest way of doing this is to select a row, column or diagonal of two figure numbers randomly and read ten numbers from 00-99 as they appear in a column. If one wants ten uniformly distributed numbers U (0, 1) then one can divide the chosen number by 99 (range of numbers between 00-99). For example, first ten numbers of the first column of the Random Number Table contains two digit numbers as:

03 97 16 12 55 16 84 63 33 57

These numbers are selected randomly (giving equal probabilities), without replacement, from two digit numbers 00-99. If one wants to convert them to uniformly distributed U (0, 1) variables then one has to divide them by 99 which give:

0.030 0.980 0.162 0.121 0.555 0.162 0.848 0.636 0.333 0.575

However, such methods are satisfactory when one requires a very small number. But in most simulations a very large number in thousands and millions are required. This requires a lot of memory, time and computers are not very efficient. For this purpose we require a procedure which should be fast and does not need much memory. Now modern computer use some inbuilt methods for generating.

Pseudo Random Numbers (PRN) are not random numbers but they behave like random numbers for all practical purposes. That is why they are called Pseudo Random Numbers. One may define PRN as:

A sequence of PRN (U_i) is a deterministic sequence in the interval [0, 1] having the same statistical properties as a sequence of random numbers.

This means that any statistical test applied to a finite part of sequence (U_i), which aims to detect departure from randomness, would not reject the null hypothesis that the sequence consists of random numbers. Hence, we shall generate a deterministic sequence of PRN and then apply some relevant statistical tests to examine the hypothesis that the sequence is random. In case these tests do not reject this hypothesis, we shall take them as random numbers. But this may be noted here that they are not random numbers in the strict sense.

13.3 RANDOM NUMBER GENERATION

In the past, many methods have been used for generation of random numbers but we shall not discuss all of them here and describe only following three methods:

1. Lottery Method
2. Middle Square Method
3. Linear Congruential Method

13.3.1 Lottery Method

This is the simplest method of generating random numbers from U (0,1). In this method, we have ten cards which are made as homogeneous as possible in shape, size, color, etc. and we assign the numbers 0 to 9 on these cards. Then these cards are put in a rotated drum. If we have to draw random numbers of two digits, then we draw a card from the drum and note the number of the drawn card. This card is replaced in the drum and drum is again rotated. Again, we draw a card and note the number of drawn card and this card is replaced in the drum. So we obtain a random number of two digits. Similarly, we can obtain more digital random numbers.

Another way of generating random numbers is to fix up a spinning arrow on a common clock. When the arrow is spin, the number at which it stops would be noted. The arrow is again spin and the number at which it stops would be noted. In this way we find a random number of two digits. Similarly, we can generate another random number of different numbers of digits. Random numbers can also be generated by tossing a coin or dice etc.

Drawback of Lottery Method

The main drawback of this procedure is that if we want to draw large random numbers of digit three or more, then we will have to draw cards or spin the arrow as a large number of times. So this method is much time consuming.

13.3.2 Middle Square Method

Still on a more sophisticated level computers are used for generating the random numbers. With computer it is typically easier to generate the random numbers by an arithmetic process. The method proposed for use on digital computers to generate random number is "Middle Square Method". In this method, if we want to generate n random numbers of r digits then we take a random number of digits r which is generated from any other method, then square this random number. If there are 2r digits in the square then we take the middle r digits as the next random number. If there are less than 2r digits in the square then we put zeroes in front of it to make 2r digits and then take the middle r digits. For example, if we want to generate a four digit integer random number then take a random number (which is generated by any other method) suppose 8937. To obtain the next number in this sequence we square it. We get 79869969 and take the middle four digits 8699. So that the next random number generated is 8699. The next few random numbers in this sequence are 6726, 2390, 7121, …, etc.

If we have to generate a random number of two digits then take a random number (which is generated by any other method) suppose the number is 13, then square of this is 169. The square of this contain $(2r-1)$ digits, so put a single zero in front of this square to make 2r digits and then take middle two digits as the next random number, so that the next random number generated is 16. The few next random numbers generated in this sequence are 48, 30, 90, …, etc.

Drawbacks of Middle-Square Method

The middle-square method has the following drawbacks:

1. This method tends to degenerate rapidly. A random number may reproduce itself. For example $x_9 = 7600$, $x_{10} = 7600$, $x_{11} = 7600$,…
2. If the number zero is ever generated, all subsequent numbers generated will also have a zero value unless steps are provided to handle this case.
3. A loop many generate i.e. the same sequence of random numbers can repeat. For example $x_{15} = 6100$, $x_{16} = 2100$, $x_{17} = 4100$, $x_{18} = 6100$,

   $x_{19} = 2100$.
4. This method is slow since many multiplications and divisions are required to access the middle digits in a fixed word binary computer.

13.3.3 Linear Congruential Generator (LCG) Method

A sequence of integers z_1, $z_{2, \ldots}$is generated by a recursive formula

$$z_i = (az_{i-1} + c) \bmod m \qquad \ldots (1)$$

where, m, a and c are positive integers.

Equation (1) can also be written as:

$$z_i = (n) \bmod m$$

by replacing $(az_{i-1} + c)$ by n.

Here **"(n) mod m"** means the remainder part of n/m. Generally "(n) mod m" is always less than m.

For example, if n = 4592 and m = 543

Then "(n) mod m" = "(4592) mod 543" = 248.

It is obvious that all numbers z_i will satisfy;

$$0 \le z_i \le m - 1$$

Uniformly distributed PRN from U (0, 1) are obtained as u_i's:

$$u_i = z_i/m$$

Whenever, z_i takes the same value as taken earlier in the sequence then the sequence is repeated again. The length of such a cycle is called **period (p)**. It is clear that $p \le m$. For $p = m$ the LCG is said to be of **full period**.

In actual simulations thousands of random numbers are required and it is desirable to have LCGs of large period, so that the same sequence is not repeated again.

13.3.4 Choice of Linear Congruential Generator's

From the above discussion, it is clear that m should be as large as possible depending on the computer word size. When c = 0 the LCG is called **multiplicative** generator. For a 32-bit computer it has been shown that

$$m = 2^{31} - 1, \quad a = 7^5 = 16807, \quad c = 0$$

results in a good multiplicative generator:

$$z_i = (16807\, z_{i-1} + 0) \bmod (2^{31} - 1) \qquad \ldots (2)$$

Starting with initial arbitrary seed value z_0, one can generate the required sequence of u_i's.

For example, some uniform PRN from LCG given in (2) with

$$z_0 = 441, \qquad m = 2^{31} - 1$$

$$u_1 = z_1/m = 0.0034, \qquad u_2 = z_2/m = 0.8063$$

E1) Using the following LCG

$$x_i = (1573 x_{i-1} + 19) \bmod (10^3)$$

obtain four x_i's with $x_0 = 89$.

E2) Using PRN generated in E1), obtain four U (0, 1) random variables.

13.4 INVERSE PROBABILTY TRANSFORMATION (IPT) METHOD

13.4.1 IPT Method for Discrete Random Variable

Suppose a discrete random variable X takes values at discrete points

$x_1 < x_2 < x_3 < \ldots.$

$$F(x) = P\,(X \le x) = \sum_{x_i \le x} p(x_i) \qquad \ldots (3)$$

The algorithm for generating a random variable from F(x) is

1 Generate a uniform random variable U (0, 1) using Section 13.3.

2 Determine the smallest integer i for which $u_i \le F(x_i)$ (using Equation (3)).

3 Take $x = x_i$ which is the desired variable.

Repeat (1)-(3), starting with a new u, each time to obtain more variables.

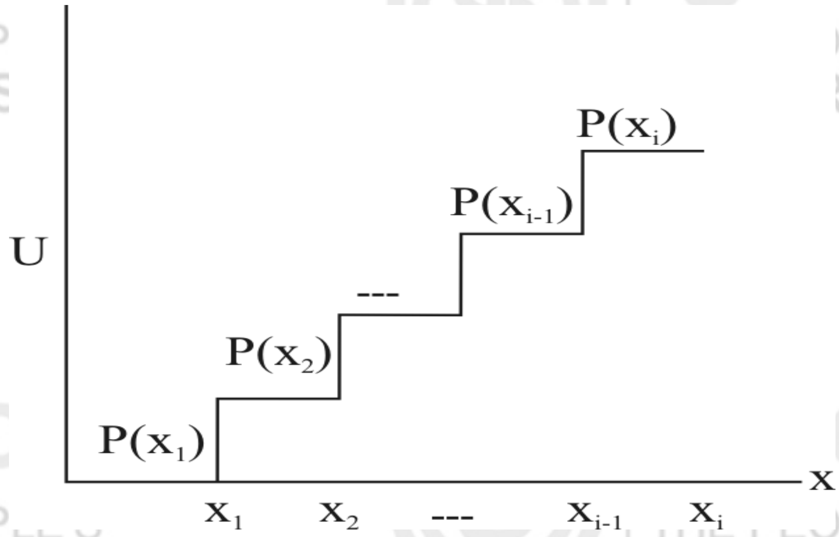

Fig. 1

$$P(X = x_i) = F(x_i) - F(x_{i-1}) = p(x_i) \qquad \text{(See Fig. 1).}$$

Hence, x_i generated in this way has the desired probability $p(x_i)$.

13.4.2 IPT Method for Continuous Random Variable

If F(x) is the cumulative distribution functions of a continuous random variable X,

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\, du$$

where, f(u) is the probability density function (p.d.f.) of the random variable U.

F(x) is strictly increasing function of x. The algorithm is

1 Generate a u~U (0, 1), using Section 13.3.
2 Take F(x) = u and find that x for which F(x) = u. This can be written as $x = F^{-1}(u)$, which is always defined.

X is the desired random variable with distribution F(x).

Proof: We have

$$P(X \le x) = P(F^{-1}(u) \le x) = P(u \le F(x)) = F(x)$$

Which is the desired result (see Fig. 2)

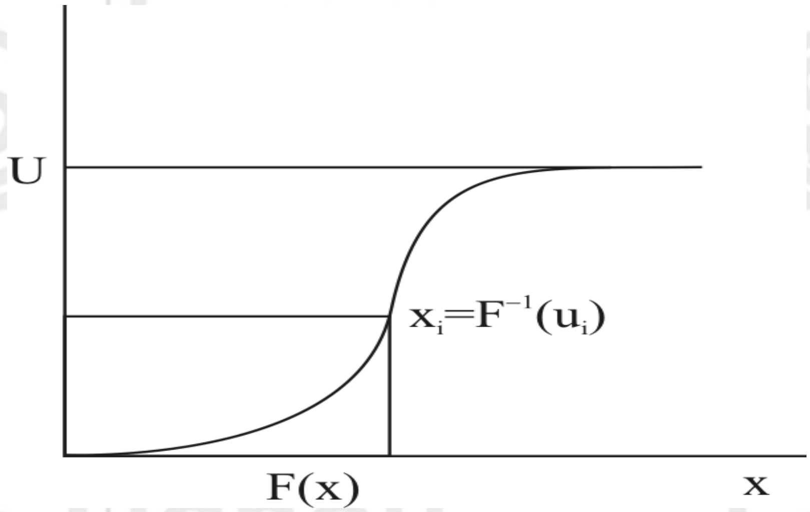

Fig. 2

13.5 RANDOM NUMBER GENERATION FOR DISCRETE VARIABLES

In this section, we shall describe methods for generation of random variables from some important distributions using the PRN described in Section 13.3.

13.5.1 Discrete Uniform Random Variable

Suppose we wish to generate a discrete uniform random variable X whose probability mass function P(x) is given by

$$P(X = i) = 1/N \qquad i = 1, 2, 3, \ldots, N$$

Using Inverse Probabilty Transformation (IPT) method, the algorithm is:

1 Select a uniform random variable u~U (0, 1).

2 Multiply u by N and take x = [Nu] +1

where, [Nu] is the integer part of Nu. X is the desired uniform random variable.

Repeat (1)-(2) to generate more random variables with new u's.

For example, suppose u_1, u_2, u_3 and u_4 are all independent U (0, 1) variables and take values:

$$u_1 = 0.0535, \quad u_2 = 0.5292, \quad u_3 = 0.1189, \quad u_4 = 0.3829$$

Then, we can use them to generate discrete uniform random variables given N = 50. Using the algorithm, we obtain $x_i = [Nu_i] + 1$ and the values are:

$$x_1 = 3, \qquad x_2 = 27, \qquad x_3 = 6, \qquad x_4 = 20$$

13.5.2 Bernoulli Random Variable

Probability mass function of Bernoulli random variable B (1, p) is given by

$$P(X = 1) = p$$

$$P(X = 0) = 1 - p$$

i.e. X takes only two values 1 and 0 with probability p and (1 - p) respectively.

Using IPT method the algorithm is given by

1 Generate a $u \sim U(0, 1)$.
2 If $u \leq p$ take $x = 1$
   $u > p$ take $x = 0$

Repeat (1)-(2) to generate more random variables with new u's.

For example, if we wish to generate three independent Bernoulli random variables for $p = 0.3$.

And the three independent U (0, 1) are obtained as

$$u_1 = 0.928, \quad u_2 = 0.535, \quad u_3 = 0.259$$

Using the algorithm we obtain

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 1.$$

13.5.3 Binomial Random Variable

Probability mass function of Binomial random variable is given by

$$P(X = x) = {}^nC_x\, p^x\, q^{n-x}; \qquad x = 0, 1, 2, \ldots, n.$$

Sum of n independently distributed Bernoulli random variables B (1, p) has a Binomial distribution B (n, p). Therefore, one can generate Binomial random variable by summing n independent Bernoulli random variables, described in Sub-section 13.5.2, algorithm is:

1. Generate n independent B (1, p) described in Sub-section 13.5.2 as $x_1, x_2, \ldots, x_n$.
2. Obtain $x = x_1 + x_2 + \ldots + x_n$

Repeat (1)-(2) with new B (1, p) variables to generate more Binomial random variables.

For example, using four uniform random variables generated in E2), we can generate a binomial random variable B (n, p), with $n = 4$ and $p = 0.3$ as follows:

Take $x_i = 1$, if $u_i < 0.3$

$= 0$, otherwise

Therefore, for

$$u_1 = 0.016 \quad u_2 = 0.187, \quad u_3 = 0.170, \quad u_4 = 0.429$$

we get

$$x_1 = 1, \quad x_2 = 1, \quad x_3 = 1, \quad x_4 = 0$$

x_i's are independent Bernoulli random variable B (1, 0.3). Sum of n independent Bernoulli random variables has a B (n, 0.3). Hence the value of a Binomial random variable X with $n = 4$ and $p = 0.3$ is given by

$$x = x_1 + x_2 + x_3 + x_4 = 3$$

E3) Using five uniform random variables $u_1 = 0.316$, $u_2 = 0.087$, $u_3 = 0.270$, $u_4 = 0.129$, $u_5 = 0.249$, generate a Binomial random variable B (n, p), with n = 5 and p = 0.2.

E4) Using Bernoulli random variables $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$ and $x_5 = 0$ with p = 0.2, generate a Binomial variable with n = 5 and p = 0.2.

13.5.4 Geometric Random Variable

Geometric distribution gives the probability of first success in the n^{th} trial, when the trials are independent and probability of success is p in each trial. The probability is given by

$$P(X = n) = (1-p)^{n-1} p = p\, q^{n-1}, \quad n = 1, 2, 3, \ldots$$

$$P(X \le t) = p\,(1 + q^1 + q^2 + \ldots + q^{t-1}) = p\,(1-q^t)/(1-q)$$

Obtain u~U (0, 1) and using IPT, put u = P (X ≤ t). Solving for t gives the following algorithm:

1 Select a u~ U(0,1);

2 t = log (1-u)/log (q), or equivalently t = log(u)/log (q);

where q =1 - p

3 Take x = [t]+1;

where, [t] is the rounded up value of t and log is the simple logarithm at the base e.

Generate more x's by repeating (1)-(3) with new u's.

For example, if four uniform random variables from U (0, 1) are given as:

$u_1 = 0.39$, $u_2 = 0.89$, $u_3 = 0.23$, $u_4 = 0.76$

and if we wish to obtain x_1, x_2, x_3, x_4 which are following geometric distribution with p = 0.3.

So, q = 1 - 0.3 = 0.7

Therefore, $t_i = \log(u_i)/\log(q)$ then

$t_1 = 2.64$, $t_2 = 0.33$, $t_3 = 4.12$, $t_4 = 0.76$

$x_1 = 3$, $x_2 = 1$, $x_3 = 5$, $x_4 = 1$

13.5.5 Negative Binomial Random Variable

The distribution of number of failures until k successes is known as Negative Binomial (NB) distribution. It has two parameters p and k, where p is the probability of success in a trial.

$$P(X = x) = {}^{k+x-1}C_{k-1}\, p^k q^x; \qquad x = 0, 1, 2, \ldots$$

The sum of k independent Geometric variables, each with parameter p, has a NB distribution. Therefore, one can generate Negative Binomial random variable by summing k independent Geometric random variables, described in Sub-section 13.5.4, algorithm is:

1. Generate k independent Geometric random variables with parameter p described in Sub-section 13.5.4 as $x_1, x_2, \ldots, x_k$;
2. Obtain $x = x_1 + x_2 + \ldots + x_n$

Repeat (1)-(2) with new Geometric random variables to generate more Negative Binomial random variables.

For example, if Geometric variables generated in Example given above, for p = 0.3 are

$$x_1 = 3, \qquad x_2 = 1, \qquad x_3 = 5, \qquad x_4 = 1$$

Therefore, $\quad x = x_1 + x_2 + x_3 + x_4 = 3 + 1 + 5 + 1 = 10$

is the desired value of Negative Binomial variable with k = 4 and p = 0.3

E5) Using Geometric random variables $x_1 = 2$, $x_2 = 3$, $x_3 = 1$, $x_4 = 5$ and $x_5 = 0$ with p = 0.2, generate a Negative Binomial variable with k = 5 and p = 0.2.

13.5.6 Poisson Random Variable

The probability mass function of Poisson random variable with mean λ is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots\ldots$$

IPT method is rather slow for generation of Poisson variables. For this its relationship with Exponential random variable is used. If inter-arrival times are independent and have Exponential distribution with mean 1, then the number of arrivals X in an interval $(0, \lambda)$ time has a Poisson distribution with mean λ.

Suppose $y_1, y_2, y_3, \ldots$ are inter-arrival times for the first, second, third, …….. arrivals and suppose λ lies between n^{th} and $(n+1)^{th}$ arrival, i.e.

$$\sum_{i=1}^{n} y_i < \lambda \le \sum_{i=1}^{n+1} y_i$$

then, X = n has a Poisson distribution with mean λ. To generate y_i from Exponential distribution with mean 1, one uses

$$y_i = -\log u_i$$

where, $u_i \sim U(0,1)$ and log is simple logarithm at base e. Algorithm is given by:

1. Generate $u_1, u_2, \ldots$ as independent U(0,1)
2. Find cumulative sums of (n+1) log u_i's and take X = n, when

$$-\sum_{i=1}^{n} \log u_i < \lambda \le -\sum_{i=1}^{n+1} \log u_i$$

For obtaining more Poisson variables repeat (1)-(2) with new sets of u_i's.

For example, if a sequence of independent u_i's is given by

$u_1 = 0.29$, $u_2 = 0.89$, $u_3 = 0.35$, $u_4 = 0.56$, $u_5 = 0.69$

So we generate a random variable X which has a Poisson distribution with

$\lambda = 2.5$ as:

$\log u_1 = -1.24$, $\log u_2 = -0.12$, $\log u_3 = -1.05$, $\log u_4 = -0.58$, and

$\log u_5 = -0.37$

Then the cumulative sums of (n+1) $\log u_i$'s

$$-\sum_{i=1}^{3} \log u_i = 2.41 < 2.5 \le -\sum_{i=1}^{4} \log u_i = 2.99$$

Hence x = 3.

E6) Using uniform random variables generated in E2) generate a Poisson random variable with $\lambda = 4.5$.

E7) Generate a complete cycle for the LCG given below:

$x_i = (5x_{i-1} + 3) \bmod 16$, with $x_0 = 5$

E8) Generate ten uniform random numbers U(0, 1) from the multiplicative LCG given below:

$x_i = (49x_{i-1}) \bmod 61$, with $x_0 = 1$

E9) A man tosses an unbiased coin ten times. Using the first ten random numbers generated in E7) obtain a sequence of heads and tails.

E10) Using the first ten random numbers generated in E9) simulate the number of heads obtained in two games of 5 trials each when the probability of obtaining a head is 0.6.

E11) Using uniform random numbers generated in E9) obtain the numbers of trials required for the first success when probability of success p is 0.3.

E12) Using uniform random numbers generated in E9) generate number of trials required for obtaining exactly two successes.

E13) Using the uniform random numbers given below obtain three Poisson random variables when $\lambda = 2$.

0.696, 0.457, 0.493, 0.784, 0.123, 0.478, 0.487, 0.031, 0.681, 0.258

E14) For $\lambda = 2$ cumulative probability for Poisson distribution are given in the following:

X:	0	1	2	3	4

∑p(x):	0.1353	0.4060	0.6767	0.8570	0.9473
X:	5	6	7	8	9
∑p(x):	0.9834	0.9955	0.9989	0.9998	1.000

Using uniform random variables given in E13) obtain three Poisson variables with $\lambda=2$, using Inverse Probability Transformation (IPT).

E15) From n = 10 cumulative probability of Binomial distribution with p = 0.25 i.e. B (10, 0.25) is given as:

X	0	1	2	3	4
∑p(x)	0.0563	0.2440	0.5256	0.7759	0.9219
X	5	6	7	≥8	
∑p(x)	0.9803	0.9965	0.9996	1	

Using uniform random variables given in E13) obtain three Binomial variables from B (10, 0.25) using IPT.

E16) Suppose a population consists of hundred units numbered as 1, 2, …, 100. One wants to select a sample of five units randomly (with equal probabilities 1/100). Using the uniform random numbers given in E13) select a sample of five units.

13.6 SUMMARY

In this unit, we have discussed:

1. A method for generating Pseudo random numbers (PRN);
2. Methods for generating Uniform random variables U (0,1);
3. Method of Inverse probability transformation(IPT) for generating random variables from a given distribution; and
4. Methods for generating Discrete Uniform, Bernoulli, Binomial, Geometric, Netgative Binoimial and Poisson random variates.

13.7 SOLUTIONS /ANSWERS

E1) We have

$x_i = (1573x_{i-1} + 19) \bmod (10^3)$ with $x_0 = 89$

Therefore,

$x_1 = 140016 \bmod (10^3) = 16$

$x_2 = 25187 \bmod (10^3) = 187$

$x_3 = 294170 \bmod (10^3) = 170$

$x_4 = 267429 \bmod (10^3) = 429$

E2) We have

$m = 10^3$ and $u_i = x_i/m$,

then

$u_1 = 0.016$ $u_2 = 0.187$, $u_3 = 0.170$, $u_4 = 0.429$

E3) Take $x_i = 1$, if $u_i < 0.2$

$= 0$, otherwise

Therefore, for

$u_1 = 0.316$, $u_2 = 0.087$, $u_3 = 0.270$, $u_4 = 0.129$, $u_5 = 0.249$

we get

$x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$, $x_5 = 0$

x_i's are independent Bernoulli random variable B (1, 0.2). Sum of n independent Bernoulli random variables has a B (n, 0.2). Hence the value of a Binomial random variable X with n = 5 and p = 0.2 is given by

$x = x_1 + x_2 + x_3 + x_4 + x_5 = 2$

E4) Bernoulli variables for p = 0.2 are given

$x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_5 = 0$

Therefore, $x = x_1 + x_2 + x_3 + x_4 + x_5 = 0 + 1 + 1 + 1 + 0 = 3$

is the desired value of Binomial variable with n = 5 and p = 0.2.

E5) Geometric variables for p = 0.2 are given

$x_1 = 2$, $x_2 = 3$, $x_3 = 1$, $x_4 = 5$, $x_5 = 0$

Therefore, $x = x_1 + x_2 + x_3 + x_4 + x_5 = 2 + 3 + 1 + 5 + 0 = 11$

is the desired value of Negative Binomial variable with k = 5 and p = 0.2.

E6) We have

$u_1 = 0.016$, $u_2 = 0.187$, $u_3 = 0.170$, $u_4 = 0.429$

Therefore,

$\log u_1 = -4.13$, $\log u_2 = -1.68$, $\log u_3 = -1.77$, $\log u_4 = -0.85$

$-\log u_1 = 4.13 \leq 4.5 \leq -\log u_1 - \log u_2 = 5.81$

Hence, Poisson random variable X is given by x =1

E7) A full cycle of random numbers generated from LCG

$x_i = (5x_{i-1} + 3) \bmod 16$, with $x_0 = 5$

is given as follows:

x: 12, 15, 14, 9, 0, 3, 2, 13, 4, 7, 6, 1, 8, 11, 10, 5

E8) The given LCG is

$x_i = (49\, x_{i-1}) \bmod 61$, with $x_0 = 1$

and we also have $u_i = x_i / 61$

x :49	22	41	57	48	34	19	16	52	47
u :0.80	0.36	0.67	0.93	0.79	0.56	0.31	0.26	0.85	0.77

E9) We have given LCG

$x_i = (5x_{i-1} + 3) \bmod 16$, with $x_0 = 5$

which gives 10 random numbers as

12, 15, 14, 09, 00, 03, 02, 13, 04, 07

We also have $u_i = x_i /16$

Therefore,

u:	0.75	0.94	0.87	0.56	0.00	0.19	0.12	0.81	0.25	0.44
x :	H	H	H	H	T	T	T	H	T	T

(Taking $u \geq 0.5$ as Head (H) and $u < 0.5$ as Tail (T))

E10) This is the case of Binomial, B (5, 0.6) using the uniform random numbers generated in E9) take H when $U \leq 0.6$ and T when $U > 0.6$, and sum the number of the heads in first five trials and the next five trials. The numbers are:

x: 2, 4

E11) This is the case of Geometric distribution. We have

$t = \log(u)/\log(q)$

and $x = [t] + 1$

$t = \log(u_1)/\log(q)$

$= \log(0.75)/\log(0.7)$

$= -0.2877/-0.3567$

$= 0.80$

Therefore, $x_1 = [0.80] + 1 = 1$

Hence $x_1 = 1$

E12) The random variable required is Negative Binomial with p = 0.3 and k = 2. As in E1), we obtain the next Geometric variable by taking:

$$t = \log(u_2)/\log(q)$$

$$= \log(0.94)/\log(0.7)$$

$$= -0.0619/-0.3567$$

$$= 0.17$$

Therefore $x_2 = [0.17] + 1 = 1$

Hence $x_2 = 1$.

The Negative Binomial variable is the sum of k independent Geometric random variables, and thus

$$x = x_1 + x_2 = 1 + 1 = 2$$

E13) We have

S.No.	1	2	3	4	5	6	7	8	9	10
u_i	0.696	0.457	0.493	0.784	0.123	0.478	0.487	0.031	0.681	0.258
$-\log u_i$	0.36	0.78	0.70	0.24	2.09	0.73	0.71	3.47	0.38	1.35

Therefore,

$$-\sum_{i=1}^{3} \log u_i = 1.85 < 2 \le -\sum_{i=1}^{4} \log u_i = 2.09$$

Hence, $x_1 = 3$.

Starting from u_5

$$-\log u_5 > 2$$

Hence, $x_2 = 0$.

Starting from u_6

$$-\sum_{i=6}^{7} \log u_i = 1.457 < 2 \le -\sum_{i=6}^{8} \log u_i = 4.931$$

Hence, $x_3 = 2$

Thus, three Poisson random variables generated for $\lambda = 2$ are:

3, 0, 2

E14) We have $u_1 = 0.696$, $u_2 = 0.457$, $u_3 = 0.493$

Using IPT, we see that

$P(X \le 2) < u_1 = 0.696 \le P(X \le 3)$, $x_1 = 3$

$P(X \le 1) < u_2 = 0.457 \le P(X \le 2)$, $x_2 = 2$

$P(X \le 1) < u_3 = 0.493 \le P(X \le 2)$, $x_3 = 2$

Hence three random variables from P (2) are:

3, 2, 2.

E15) We have $u_1 = 0.696$, $u_2 = 0.457$, $u_3 = 0.493$.

$$P(X \le 2) < u_1 = 0.696 \le P(X \le 3), \quad x_1 = 3$$

$$P(X \le 1) < u_2 = 0.457 < P(X \le 2), \quad x_2 = 2$$

$$P(X \le 1) < u_3 = 0.493 \le P(X \le 3), \quad x_3 = 2$$

Hence three random variables from B (10, 0.25) are:

$x_1 = 3$, $x_2 = 2$, $x_3 = 2$.

E16) This is the case of discrete uniform random variable with N =100. Using the random numbers in E13) we obtain the following numbers:

$$x_i = [Nu_i] + 1$$

x : 70, 46, 50, 79, 13.