UNIT 1 INTRODUCTION TO PROBABILITY

Structure

1.1 INTRODUCTION

In our daily lives, we face many situations when we are unable to forecast the future with complete certainty. That is, in many decisions, the uncertainty is faced. Need to cope up with the uncertainty leads to the study and use of the probability theory. The first attempt to give quantitative measure of probability was made by Galileo (1564-1642), an Italian mathematician, when he was answering the following question on the request of his patron, the Grand Duke of Tuscany, who wanted to improve his performance at the gambling tables: "With three dice a total of 9 and 10 can each be produced by six different combinations, and yet experience shows that the number 10 is oftener thrown than the number 9?" To the mind of his patron the cases were (1, 2, 6), (1, 3, 5), (1, 4, 4), (2, 2, 5), (2, 3, 4), (3, 3, 3) for 9 and (1, 3, 6), (1, 4, 5), (2, 2, 6), (2, 3, 5), (2, 4, 4), (3, 3, 4) for 10 and hence he was thinking that why they do not occur equally frequently i.e. why there chances are not the same? Galileo makes a careful analysis of all the cases which can occur, and he showed that out of the 216 possible cases 27 are favourable to the appearance of the number 10 since permutations of (1, 3, 6) are (1, 3, 6), (1, 6, 3), (3, 1, 6), (3, 6, 1), (6, 1, 3), (6, 3, 1) i.e. number of permutations of (1, 3, 6) is 6; similarly, the number of permutations of (1, 4, 5), (2, 2, 6), (2, 3, 5), (2, 4, 4), (3, 3, 4) is 6, 3, 6, 3, 3 respectively and hence the total number of cases come out to be $6 + 6 + 3 + 6 + 3 + 3 = 27$ whereas the number of favourable cases for getting a total of 9 on three dice are $6 + 6 + 3 + 3 + 6 + 1 = 25$. Hence, this was the reason for 10 appearing oftener thrown than 9. But the first foundation was laid by the two mathematicians Pascal (1623-62) and Fermat (1601-65) due to a gambler's dispute in 1654 which led to the creation of a mathematical theory of probability by them. Later, important contributions were made by various researchers including Huyghens (1629 - 1695), Jacob Bernoulli (1654-1705), Laplace (1749-1827), Abraham De-Moivre (1667-1754), and Markov (1856-1922). Thomas Bayes (died in 1761, at the age of 59) gave an important technical result known as Bayes' theorem, published after his death in 1763, using which probabilities can be revised on the basis of some new information. Thereafter, the probability, an important branch of Statistics, is being used worldwide.

We will start this unit with very elementary ideas. In other words, we are assuming that reader knows nothing about probability. We will go step by step clearing the basic ideas which are required to understand the probability. In this unit, we will first present the various terms which are used in the definition of probability and then we will give the classical definition of probability and simple problems on it.

Objectives

After studing this unit, you should be able to:

- define and give examples of random experiment and trial;
- define and give examples of sample space, sample point and event;
- explain mutually exclusive, equally likely, exhaustive and favourable cases and why they are different in nature and how much these terms are important to define probability;
- explain the classical definition of probability;
- solve simple problems based on the classical definition of probability; and
- distinguish between odds in favour and odds against the happening of an event.

Random Experiment

An experiment in which all the possible outcomes are known in advance but we cannot predict as to which of them will occur when we perform the experiment, e.g. Experiment of tossing a coin is random experiment as the possible outcomes head and tail are known in advance but which one will turn up is not known.

Similarly, 'Throwing a die' and 'Drawing a card from a well shuffled pack of 52 playing cards 'are the examples of random experiment.

Trial

Performing an experiment is called trial, e.g.

(i) Tossing a coin is a trial.

(ii) Throwing a die is a trial.

1.2 SAMPLE SPACE, SAMPLE POINT AND EVENT

Sample Space

Set of all possible outcomes of a random experiment is known as sample space and is usually denoted by S, and the total number of elements in the sample space is known as size of the sample space and is denoted by n(S), e.g.

(i) If we toss a coin then the sample space is

$S = \{H, T\}$, where H and T denote head and tail respectively and $n(S) = 2$.

(ii) If a die is thrown, then the sample space is

$$S = \{1, 2, 3, 4, 5, 6\} \text{ and } n(S) = 6. \left[\begin{array}{l}\because \text{die has six faces} \\ \text{numbered } 1, 2, 3, 4, 5, 6\end{array}\right]$$

(iii) If a coin and a die are thrown simultaneously, then the sample space is S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6} and n(S) = 12.

where H1 denotes that the coin shows head and die shows 1 etc.

Note: Unless stated the coin means an unbiased coin (i.e. the coin which favours neither head nor tail).

(iv) If a coin is tossed twice or two coins are tossed simultaneously then the sample space is

S = {HH, HT, TH, TT},

where HH means both the coins show head, HT means the first coin shows head and the second shows tail, etc. Here, n(S) = 4.

(v) If a coin is tossed thrice or three coins are tossed simultaneously, then the sample space is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT} and n(S) = 8.

(vi) If a coin is tossed 4 times or four coins are tossed simultaneously then the sample space is

S = {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT} and n(S) = 16.

(vii) If a die is thrown twice or a pair of dice is thrown simultaneously, then sample space is

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (5, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

here, e.g., (1, 4) means the first die shows 1 and the second die shows 4.

Here, n(S) = 36.

(viii) If a family contains two children then the sample space is

S = {B_1B_2, B_1G_2, G_1B_2, G_1G_2}

where B_i denotes that i^{th} birth is of boy, i = 1, 2, and

G_i denotes that i^{th} birth is of girl, i = 1, 2.

This sample space can also be written as

S = {BB, BG, GB, GG}

(ix) If a bag contains 3 red and 4 black balls and

(a) One ball is drawn from the bag, then the sample space is

$\{R_1, R_2, B_1, B_2, B_3, B_4\}$, where R_1, R_2, R_3 denote three red balls and B_1, B_2, B_3, B_4

denote four black balls in the bag.

(b) Two balls are drawn one by one without replacement from the bag, then the sample space is

$$S = \{R_1R_2, R_1R_3, R_1B_1, R_1B_2, R_1B_3, R_1B_4, R_2R_1, R_2R_3, R_2B_1, R_2B_2, R_2B_3, R_2B_4, R_3R_1, R_3R_2, R_3B_1, R_3B_2, R_3B_3, R_3B_4, B_1R_1, B_1R_2, B_1R_3, B_1B_2, B_1B_3, B_1B_4, B_2R_1, B_2R_2, B_2R_3, B_2B_1, B_2B_3, B_2B_4, B_3R_1, B_3R_2, B_3R_3, B_3B_1, B_3B_2, B_3B_4, B_4R_1, B_4R_2, B_4R_3, B_4B_1, B_4B_2, B_4B_3\}$$

Note: It is very simple to write the above sample space – first write all other balls with R_1, then with R_2, then with R_3 and so on.

Remark 1: If a random experiment with x possible outcomes is performed n times, then the total number of elements in the sample is x^n i.e. $n(S) = x^n$, e.g. if a coin is tossed twice, then n(S) = 2^2 =4; if a die is thrown thrice, then n(S) = 6^3 = 216.

Now you can try the following exercise.

E1) Write the sample space if we draw a card from a pack of 52 playing cards.

Sample Point

Each outcome of an experiment is visualised as a sample point in the sample space. e.g.

(i) If a coin is tossed then getting head or tail is a sample point.

(ii) If a die is thrown twice, then getting (1, 1) or (1, 2) or (1, 3) or…or (6, 6) is a sample point.

Event

Set of one or more possible outcomes of an experiment constitutes what is known as event. Thus, an event can be defined as a subset of the sample space, e.g.

i) In a die throwing experiment, event of getting a number less than 5 is the set {1, 2, 3, 4},

which refers to the combination of 4 outcomes and is a sub-set of the sample space

= {1, 2, 3, 4, 5, 6}.

ii) If a card is drawn form a well-shuffled pack of playing cards, then the event of getting a card of a spade suit is

$$\{1_S, 2_S, 3_S, ..., 9_S, 10_S, J_S, Q_S, K_S\}$$

where suffix S under each character in the set denotes that the card is of spade and J, Q and K represent jack, queen and king respectively.

1.3 EXHAUSTIVE CASES, FAVOURABLE CASES, MUTUALLY EXCLUSIVE CASES AND EQUALLY LIKELY CASES

Exhaustive Cases

The total number of possible outcomes in a random experiment is called the exhaustive cases. In other words, the number of elements in the sample space is known as number of exhaustive cases, e.g.

(i) If we toss a coin, then the number of exhaustive cases is 2 and the sample space in this case is {H, T}.

(ii) If we throw a die then number of exhaustive cases is 6 and the sample space in this case is {1, 2, 3, 4, 5, 6}

Favourable Cases

The cases which favour to the happening of an event are called favourable cases. e.g.

(i) For the event of drawing a card of spade from a pack of 52 cards, the number of favourable cases is 13.

(ii) For the event of getting an even number in throwing a die, the number of favourable cases is 3 and the event in this case is {2, 4, 6}.

Mutually Exclusive Cases

Cases are said to be mutually exclusive if the happening of any one of them prevents the happening of all others in a single experiment, e.g.

(i) In a coin tossing experiment head and tail are mutually exclusive as there cannot be simultaneous occurrence of head and tail.

Equally Likely Cases

Cases are said to be equally likely if we do not have any reason to expect one in preference to others. If there is some reason to expect one in preference to others, then the cases will not be equally likely, For example,

(i) Head and tail are equally likely in an experiment of tossing an unbiased coin. This is because if someone is expecting say head, he/she does not have any reason as to why he/she is expecting it.

(ii) All the six faces in an experiment of throwing an unbiased die are equally likely.

You will become more familiar with the concept of "equally likely cases" from the following examples, where the non-equally likely cases have been taken into consideration:

(i) Cases of "passing" and "not passing" a candidate in a test are not equally likely. This is because a candidate has some reason(s) to expect "passing" or "not passing" the test. If he/she prepares well for the test, he/she will pass the test and if he/she does not prepare for the test, he/she will not pass. So, here the cases are not equally likely.

(ii) Cases of "falling a ceiling fan" and "not falling" are not equally likely. This is because, we can give some reason(s) for not falling if the bolts and other parts are in good condition.

1.5 CLASSICAL OR MATHEMATICAL PROBABILITY

Let there be 'n' exhaustive cases in a random experiment which are mutually exclusive as well as equally likely. Let 'm' out of them be favourable for the happening of an event A (say), then the probability of happening event A (denoted by P (A)) is defined as

$$P(A) = \frac{\text{Number of favourable cases for event A}}{\text{Number of exhaustive cases}} = \frac{m}{n} \qquad \dots (1)$$

Probability of non-happening of the event A is denoted by $P\left(\overline{A}\right)$ and is defined as

$$P(\overline{A}) = \frac{\text{Number of favourable cases for event } \overline{A}}{\text{Number of exhaustive cases}} = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

So, $\quad P\left(A\right) + P\left(\overline{A}\right) = 1$

Therefore, we conclude that, the sum of the probabilities of happening an event and that of its complementary event is 1.

Let us now prove that $\mathbf{0 \le P(A) \le 1}$

Proof: We know that

$0 \le$ Number of favourable cases $\le$ No of exhaustive cases

[$\because$ Number of favourable cases can never be negative and can at the most be equal to the number of exhaustive cases.]

$\Rightarrow 0 \le m \le n$

Dividing both sides by n, we get

$$\Rightarrow \frac{0}{n} = \frac{m}{n} = \frac{n}{n}$$

$\Rightarrow 0 \le P(A) \le 1$

Remark 2: Probability of an impossible event is always zero and that of certain event is 1, e.g. probability of getting 7 when we throw a die is zero as getting 7 here is an impossible event and probability of getting either of the six faces is 1 as it is a certain event.

Classical definition of probability fails if

(i) The cases are not equally likely, e.g. probability of a candidate passing a test is not defined.

$$\left[\begin{array}{l}\text{Passing or failing in a test} \\ \text{are not equally likely cases.}\end{array}\right]$$

(ii) The number of exhaustive cases is indefinitely large, e.g.

probability of drawing an integer say 2 from the set of integers i.e.

$\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ by classical definition probability, is $\frac{1}{\infty} = 0.$

But, in actual, it is not so, happening of 2 is not impossible, i.e. there are some chances of drawing 2. Hence, classical definition is failed here also.

Before we give some examples on classical definition of probability, let us take up some examples which define the events as subsets of sample space.

Example 1: If a fair die is thrown once, what is the event of?

(i) getting an even number

(ii) getting a prime number

(iii) getting a number multiple of 3

(iv) getting an odd prime

(v) getting an even prime

(vi) getting a number greater than 4

(vii) getting a number multiple of 2 and 3

Solution: When a die is thrown, then the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

(i) Let E_1 be the event of getting an even number,

$\therefore E_1 = \{2, 4, 6\}$

(ii) Let E_2 be the event of getting a prime number,

$\therefore E_2 = \{2, 3, 5\}$

(iii) Let E_3 be the event of getting a number multiple of 3

$\therefore E_3 = \{3, 6\}$

(iv) Let E_4 be the event of getting an odd prime,

$\therefore E_4 = \{3, 5\}$

(v) Let E_5 be the event of getting an even prime,

$\therefore E_5 = \{2\}$

(vi) Let E_6 be the event of getting a number greater than 4,

$\therefore E_6 = \{5, 6\}$

(vii) Let E_7 be the event of getting a number multiple of 2 and 3,

$\therefore E_7 = \{6\}$

Example 2: If a pair of a fair dice is thrown, what is the event of

(i) getting a doublet

(ii) getting sum as 11

(iii) getting sum less than 5

(iv) getting sum greater than 16

(v) getting 3 on the first die

(vi) getting a number multiple of 3 on second die

(vii) getting a number multiple of 2 on first die and a multiple of 3 on second die.

Solution: When two dice are thrown, then the sample space is already given in (vii) of Sec.1.3.

(i) Let E_1 be the event of getting a doublet.

$\therefore E_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

(ii) Let E_2 be the event of getting sum 11.

$\therefore E_2 = \{(5, 6), (6, 5)\}$

(iii) Let E_3 be the event of getting sum less than 5 i.e. sum can be 2 or 3 or 4

$\therefore E_3 = \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2)\}$

(iv) Let E_4 be the event of getting sum greater than 16.

$\therefore E_4 = \{\ \}$ i.e. E_4 is a null event.

(v) Let E_5 be the event of getting 3 on the first die i.e. 3 on first die and second die may have any number

$\therefore E_5 = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}$

(vi) Let E_6 be the event of getting a number multiple of 3 on second die i.e. first die may have any number and the second has 3 or 6.

$$\therefore E_6 = \{(1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3), (1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6)\}$$

(vii) Let E_7 be the event of getting a multiple of 2 on the first die and a multiple of 3 on the second die i.e. 2 or 4 or 6 on first die and 3 or 6 on the second.

$\therefore E_7 = \{(2, 3), (4, 3), (6, 3), (2, 6), (4, 6), (6, 6)\}$

Now, you can try the following exercise.

E2) If a die and a coin are tossed simultaneously, write the event of getting

(i) head and prime number

(ii) tail and an even number

(iii) head and multiple

1.6 SIMPLE PROBLEMS ON PROBABILITY

Now let us give some examples so that you become familiar as to how and when the classical definition of probability is used.

Example 3: A bag contains 4 red, 5 black and 2 green balls. One ball is drawn from the bag. Find the probability that?

(i) It is a red ball
(ii) It is not black
(iii) It is green or black

Solution: Let R_1, R_2, R_3, R_4 denote 4 red balls in the bag. Similarly B_1, B_2, B_3, B_4, B_5 denote 5 black balls and G_1, G_2 denote two green balls in the bag. Then the sample space for drawing a ball is given by

$\{R_1, R_2, R_3, R_4, B_1, B_2, B_3, B_4, B_5, G_1, G_2\}$

(i) Let A be the event of getting a red ball, then A = $\{R_1, R_2, R_3, R_4\}$

$$\therefore\ P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{4}{11}$$

(ii) Let B be the event that drawn ball is not black, then
B = $\{R_1, R_2, R_3, R_4, G_1, G_2\}$

$$\therefore\ P(B) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{6}{11}$$

(iii) Let C be the event that drawn ball is green or black, then
C = $\{B_1, B_2, B_3, B_4, B_5, G_1, G_2\}$.

$$\therefore\ P(C) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{7}{11}$$

Example 4: Three unbiased coins are tossed simultaneously. Find the probability of getting

(i) at least two heads

(ii) at most two heads

(iii) all heads

(iv) exactly one head

(v) exactly one tail

Solution: The sample space in this case is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

(i) Let E_1 be the event of getting at least 2 heads, then
E_1 = {HHT, HTH, THH, HHH}

$$\therefore\ P(E_1) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{4}{8} = \frac{1}{2}$$

(ii) Let E_2 be the event of getting at most 2 heads then

E_2 = {TTT, TTH, THT, HTT, HHT, HTH, THH}

$$\therefore\ P(E_2) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{7}{8}$$

(iii) Let E_3 be the event of getting all heads, then

E_3 = {HHH}

$$\therefore\ P(E_3) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{1}{8}$$

(iv) Let E_4 be the event of getting exactly one head then

E_4 = {HTT, THT, TTH}

$$\therefore\ P(E_4) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{3}{8}$$

(v) Let E_5 be the event of getting exactly one tail, then

E_5 = {HHT, HTH, THH}

$$\therefore\ P(E_5) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{3}{8}$$

Example 5: A fair die is thrown. Find the probability of getting

(i) a prime number

(ii) an even number

(iii) a number multiple of 2 or 3

(iv) a number multiple of 2 and 3

(v) a number greater than 4

Solution: The sample space in this case is

S = {1, 2, 3, 4, 5, 6}

(i) Let E_1 be the event of getting a prime number, then

E_1 = {2, 3, 5}.

$$\therefore\ P(E_1) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{3}{6} = \frac{1}{2}$$

(ii) Let E_2 be the event of getting an even number, then

$E_2 = \{2, 4, 6\}$

$$\therefore \quad P(E_2) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{3}{6} = \frac{1}{2}$$

(iii) Let E_3 event of getting a multiple of 2 or 3, then

$E_3 = \{2, 3, 4, 6\}$

$$\therefore \quad P(E_3) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{4}{6} = \frac{2}{3}$$

(iv) Let E_4 event of getting a number multiple of 2 and 3, then

$E_4 = \{6\}$

$$\therefore \quad P(E_4) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{1}{6}$$

(v) Let E_5 be the event of getting a number greater than 4, then

$E_5 = \{5, 6\}$

$$\therefore \quad P(E_5) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{2}{6} = \frac{1}{3}$$

Example 6: In an experiment of throwing two fair dice, find the probability of getting

(i) a doublet
(ii) sum 7
(iii) sum greater than 8
(iv) 3 on first die and a multiple of 2 on second die
(v) prime number on the first die and odd prime on the second die.

Solution: The sample space has already been given in (vii) of Sec. 1.3.

Here, the sample space contains 36 elements i.e. number of exhaustive cases is 36.

(i) Let E_1 be the event of getting a doublet (i.e. same number on both dice), then

$E_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$.

$$\therefore \quad P(E_1) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{6}{36} = \frac{1}{6}$$

(ii) Let E_2 be the event of getting sum 7, then

$E_2 = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$

$$\therefore \quad P(E_2) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{6}{36} = \frac{1}{6}$$

(iii) Let E_3 be the event of getting sum greater than 8, then

E_3 = {(3, 6), (6, 3), (4, 5), (5, 4), (4, 6),

(6, 4), (5, 5), (5, 6), (6, 5), (6, 6)}

$$\therefore \quad P(E_3) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{10}{36} = \frac{5}{18}$$

(iv) Let E_4 be the event of getting 3 on first die and multiple of 2 on second die, then

E_4 = {(3, 2), (3, 4), (3, 6)}

$$\therefore \quad P(E_4) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{3}{36} = \frac{1}{12}$$

(v) Let E_5 be the event of getting prime number on first die and odd prime on second die, then

E_5 = {(2, 3), (2, 5), (3, 3), (3, 5), (5, 3), (5, 5)}

$$\therefore \quad P(E_5) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{6}{36} = \frac{1}{6}$$

Example 7: Out of 52 well shuffled playing cards, one card is drawn at random. Find the probability of getting

(i) a red card

(ii) a face card

(iii) a card of spade

(iv) a card other than club

(v) a king

Solution: Here, the number of exhaustive cases is 52 and a pack of playing cards contains 13 cards of each suit (spade, club, diamond, heart).

(i) Let A be the event of getting a red card. We know that there are 26 red cards,

$$\therefore \quad P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{26}{52} = \frac{1}{2}$$

(ii) Let B be the event of getting a face card. We know that there are 12 face cards (jack, queen and king in each suit),

$$\therefore \quad P(B) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{12}{52} = \frac{3}{13}$$

(iii) Let C be the event of getting a card of spade
We know that there are 13 cards of spade

$$\therefore\ P(C) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{13}{52} = \frac{1}{4}$$

(iv) Let D be the event of getting a card other than club.
As there are 39 cards other than that of club,.

$$\therefore\ P(D) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{39}{52} = \frac{3}{4}$$

(v) Let E be the event of getting a king.
We know that there are 4 kings,

$$\therefore\ P(E) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{4}{52} = \frac{1}{13}$$

Example 8: In a family, there are two children. Write the sample space and find the probability that

(i) the elder child is a girl

(ii) younger child is a girl

(iii) both are girls

(iv) both are of opposite sex

Solution: Let G_i denotes that i^{th} birth is of girl (i = 1, 2) and B_i denotes that i^{th} birth is of boy, (i =1, 2).

$$\therefore\ S = \{G_1G_2, G_1B_2, B_1G_2, B_1B_2\}$$

(i) Let A be the event that elder child is a girl

$$\therefore\ A = \{G_1G_2, G_1B_2\}$$

$$\text{and } P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{2}{4} = \frac{1}{2}$$

(ii) Let B be the event that younger child is a girl

$$\therefore\ B = \{G_1G_2, B_1G_2\}$$

$$\text{and } P(B) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{2}{4} = \frac{1}{2}$$

(iii) Let C be the event that both the children are girls

$$\therefore\ C = \{G_1G_2\}$$

$$\text{and } P(C) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{1}{4}$$

(iv) Let D be the event that both children are of opposite sex

$\therefore$ D = $\{G_1B_2, B_1G_2\}$

and $P(D) = \dfrac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \dfrac{2}{4} = \dfrac{1}{2}$

Example 9: Find the probability of getting 53 sundays in a randomly selected non-leap year.

Solution: We know that there are 365 days in a non-leap year.

$$\frac{365}{7} = 52\frac{1}{7} \text{ weeks}$$

i.e. one non-leap year = (52 complete weeks + one over day). This over day may be one of the days

Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday

So, the number of exhaustive cases = 7

Let A be the event of getting 53 Sundays

There will be 53 Sundays in a non leap year if and only if the over day is Sunday.

$\therefore$ Number of favourable cases for event A= 1

$$\therefore \quad P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{1}{7}$$

Example 10: A single letter selected at random from the word 'STATISTICS'. What is the probability that it is a vowel?

Solution: Here, as the total number of letters in the word 'STATISTICS' is n = 10, and the number of vowels in the word is m = 3 (vowels are a, i, i),

$$\therefore \text{ The required probability} = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{m}{n} = \frac{3}{10}$$

Example 11: Three horses A, B and C are in a race. A is twice as likely to win as B and B is twice as likely to win as C. What are the respective probabilities of their winning?

Solution: Let p be the probability that A wins the race.

$\therefore$ Probability that B wins the race = twice the probability of A's winning

= 2(p) = 2 p

and probability of C's winning = twice the probability of B's winning

= 2(2p) = 4p

Now, as the sum of the probability of happening an event and that of its complementary event(s) is 1. Here, the complementary of A is the happening of B or C]

$$\therefore \qquad p + 2p + 4p = 1, \text{ and hence } p = \frac{1}{7}.$$

Therefore, the respective chances of winning A, B and C are $\frac{1}{7}, \frac{2}{7}$ and $\frac{4}{7}$.

Now, let us take up some problems on probability which are based on permutation/combination which you have already studied in Unit 4 of Course MST-001.

Example 12: Out of 52 well shuffled playing cards, two cards are drawn at random. Find the probability of getting.

(i) One red and one black

(ii) Both cards of the same suit

(iii) One jack and other king

(iv) One red and the other of club

Solution: Out of 52 playing cards, two cards can be drawn in ${}^{52}C_2$ ways i.e.

$$\frac{52\times51}{2!} = 26\times51 \text{ ways}$$

(i) Let A be the event of getting one red and one black card, then the number of favourable cases for the event A are ${}^{26}C_1 \times {}^{26}C_1$ [As one red card out of 26 red cards can be drawn in ${}^{26}C_1$ ways and one black card out of 26 black cards can be drawn in ${}^{26}C_1$ ways.]

$$\therefore P(A) = \frac{{}^{26}C_1 \times {}^{26}C_1}{{}^{52}C_2} = \frac{26\times26}{26\times51} = \frac{26}{51}$$

(ii) Let B be the event of getting both the cards of the same suit and i.e. two cards of spade or two cards of club or 2 cards of diamond or 2 cards of heart.

$\therefore$ Number of favourable cases for event B $= {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2$

$$= 4\times{}^{13}C_2 = 4\times\frac{13\times12}{2!} = 2\times13\times12$$

$$\therefore P(B) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{2\times13\times12}{26\times51} = \frac{4}{17}$$

(iii) Let C be the event of getting a jack and a king.

$$\therefore P(C) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{{}^{4}C_1 \times {}^{4}C_1}{{}^{52}C_2} = \frac{4\times4}{26\times51} = \frac{8}{663}$$

(iv) Let D be the event of getting one red and one card of club.

$$\therefore P(D) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{{}^{26}C_1 \times {}^{13}C_1}{{}^{52}C_2} = \frac{26\times13}{26\times51} = \frac{13}{51}$$

Example 13: If the letters of the word STATISTICS are arranged randomly then find the probability that all the three T's are together.

Solution: Let E be the event that selected word contains 3 T's together.

There are 10 letters in the word STATISTICS. If we consider three T's as a single letter $\boxed{\text{TTT}}$, then we have 8 letters i.e.1 '$\boxed{\text{TTT}}$'; 3 'S' ; 1 'A'; 2 'I' and 1 'C'

Number of possible arrangements with three T's coming together $= \frac{8!}{2!.3!}$

Number of favourable cases for event E $= \frac{8!}{2!.3!}$ and

Number of exhaustive cases = Total number of permutations of 10 letters in the word STATISTICS

$$= \frac{10!}{2!.3!.3!}$$

$\left[\because \text{out of 10 letters, 3 are T's, 2 are I's and 3 are S's}\right]$

$$P(A) = \frac{\frac{8!}{2!.3!}}{\frac{10!}{2!.3!.3!}} = \frac{8!.3!}{10!} = \frac{8!\times 6}{10\times 9\times 8!} = \frac{6}{10\times 9} = \frac{1}{15}$$

Example 14: In a lottery, one has to choose six numbers at random out of the numbers from 1 to 30. He/ she will get the prize only if all the six chosen numbers matched with the six numbers already decided by the lottery committee. Find the probability of wining the prize.

Solution: Out of 30 numbers 6 can be drawn in

$$^{30}C_6 = \frac{30\times 29\times 28\times 27\times 26\times 25}{6!} = \frac{30\times 29\times 28\times 27\times 26\times 25}{720} = 593775 \text{ ways}$$

$\therefore$ Number of exhaustive cases = 593775

Out of these 593775 ways, there is only one way to win the prize (i.e. choose those six numbers that are already fixed by committee).

Here, the number of favourable cases is 1.

Hence, $P(\text{wining the prize}) = \frac{\text{Favourable cases}}{\text{Exhaustive cases}} = \frac{1}{593775}$

Now, you can try the following exercises.

E3) If two coins are tossed then find the probability of getting.

(i) At least one head

(ii) head and tail

(iii) At most one head

E4) If three dice are thrown, then find the probability of getting

(i) triplet
(ii) sum 5
(iii) sum at least 17
(iv) prime number on first die and odd prime number on second and third dice.

E5) Find the probability of getting 53 Mondays in a randomly selected leap year.

1.7 CONCEPT OF ODDS IN FAVOUR OF AND AGAINST THE HAPPENING OF AN EVENT

Let n be the number of exhaustive cases in a random experiment which are mutually exclusive and equally likely as well. Let m out of these n cases are favourable to the happening of an event A (say). Thus, the number of cases against A are $n-m$

Then odds in favour of event A are $m:n-m$ (i.e. m ratio $n-m$) and odds against A are $n-m:m$ (i.e. $n-m$ ratio m)

Example 15: If odds in favour of event A are $3:4$, what is the probability of happening A?

Solution: As odds in favour of A are $3:4$,

$\therefore$ $m = 3$ and $n-m = 4$ implies that $n = 7$. Thus,

Probability of happening A i.e. $P(A) = \frac{m}{n} = \frac{3}{7}$.

Example 16: Find the probability of event A if

(i) Odds in favour of event A are 4 : 3

(ii) Odds against event A are 5 : 8

Solution: (i) We know that if odds in favour of A are $m:n$, then

$$P(A)=\frac{m}{m+n} \Rightarrow P(A) = \frac{4}{4+3} = \frac{4}{7}$$

(ii) Here, $n-m=5$ and $m=8$, therefore, $n=5+8=13$.

Now, as we know that if odds against the happening of an event A are $n-m: n$, then

$$P(A)=\frac{m}{n} \Rightarrow P(A) = \frac{8}{13}$$

Example 17 If $P(A) = \frac{3}{5}$ then find

(i) odds in favour of A; (ii) odds against the happening of event A.

Solution: (i) As $P(A) = \frac{3}{5}$,

$\therefore$ odds in favour of A in this case are $3:5-3 = 3:2$

(ii) We know that if $P(A) = \frac{m}{n}$, then odds against the happening of A are $n-m:m$

$\therefore$ In this case odds against the happening of event A are $5-3 : 3 = 2 : 3$

Now, you can try the following exercises.

E6) The odds that a person speaks the truth are 3 : 2. What is the probability that the person speaks truth?

E7) The odds against Manager X setting the wage dispute with the workers are 8 : 6. What is the probability that the manager settles the dispute?

E8) The probability that a student passes a test is $\frac{2}{3}$. What are the odds against passing the test by the student?

E9) Find the probability of the event A if

(i) Odds in favour of the event $\overline{A}$ are $1:4$ (ii) Odds against the event $\overline{A}$ are $7:2$

1.8 SUMMARY

Let us now summarize the main points which have been covered in this unit.

1) An experiment in which all the possible outcomes are known in advance but we cannot predict as to which of them will occur when we perform the experiment is called **random experiment**. Performing an experiment is called **trial**.
2) Set of all possible outcomes of a random experiment is known as **sample space**. Each outcome of an experiment is visualised as a **sample point** and set of one or more possible outcomes constitutes what is known as **event**. The total number of elements in the sample space is called the number of **exhaustive cases** and number of elements in favour of the event is the number of **favourable cases** for the event.
3) Cases are said to be **mutually exclusive** if the happening of any one of them prevents the happening of all others in a single experiment and if we do not have any reason to expect one in preference to others, then they are said to be **equally likely**.
4) **Classical Probability** of happening of an event is the ratio of number of favourable cases to the number of exhaustive cases, provided they are equally likely, mutually exclusive and finite.
5) **Odds in favour of an event** are the number of favourable cases: number of cases against the event, whereas **Odds against the event** are the number of cases against the event : number of cases favourable to the event.

1.9 SOLUTIONS/ANSWERS

E 1) Let suffices C, D, S, H denote that corresponding card is a club, diamond, spade, heart respectively then sample space of drawing a card can be written as

$$\{1_C, 2_C, 3_C, ..., 9_C, 10_C, J_C, Q_C, K_C, 1_D, 2_D, 3_D, ..., 9_D, 10_D, J_D, Q_D, K_D,$$
$$1_S, 2_S, 3_S, ..., 9_S, 10_S, J_S, Q_S, K_S, 1_H, 2_H, 3_H, ..., 9_H, 10_H, J_H, Q_H, K_H\}$$

E 2) If a die and a coin are tossed simultaneously then sample space is

{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}

(i) Let A be the event of getting head and prime number, then A = {H2, H3, H5}

(ii) Let B be the event of getting tail and even number, then B = {T2, T4, T6}

(iii) Let C be the event of getting head and multiple of 3, then C = {H3, H6}

E 3) When two coins are tossed simultaneously then sample space is

S = {HH, HT, TH, TT}

(i) Let A be the event of getting at least one head, then

A = {HH, HT, TH} and $P(A) = \frac{3}{4}$

(ii) Let B be the event of getting both head and tail, then

B = {HT, TH} and $P(B) = \frac{2}{4} = \frac{1}{2}$

(iii) Let C be the event of getting at most one head, then

C = {TT, TH, HT} and $P(C) = \frac{3}{4}$

E 4) When 3 dice are thrown, then the sample space is

S = {(1, 1, 1), (1, 1, 2), (1, 1, 3), … , (1, 1, 6),
(1, 2, 1), (1, 2, 2), (1, 2, 3), … , (1, 2, 6),
(1, 3, 1), (1, 3, 2), (1, 3, 3), … , (1, 3, 6),
.
.
.
(6, 6, 1), (6, 6, 2), (6, 6, 3), …, (6, 6, 6)}

Number of elements in the sample space = $6 \times 6 \times 6 = 216$

$$\left[\begin{array}{l}\because \text{We have to fill up 3 positions (.,.,.) and each position can be} \\ \text{filled with 6 options, this can be done in } 6\times6\times6 = 216 \text{ ways}\end{array}\right]$$

(i) Let A be the event of getting triplet i.e. same number on each die.

$$\therefore A = \{(1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6)\}$$

and hence P(A) = $\frac{6}{216} = \frac{1}{36}$

(ii) Let B be the event of getting sum 5

$$\therefore B = \{(1,1,3), (1,3,1), (3,1,1), (1,2,2), (2,1,2), (2,2,1)\}$$

and P (B) = $\frac{6}{216} = \frac{1}{36}$

(iii) Let C be the event of getting sum at least17 i.e. sum 17 or 18

$$\therefore C = \{(5,6,6), (6,5,6), 6,6,5), (6,6,6)\}$$

and hence P (C) = $\frac{4}{216} = \frac{1}{54}$

(iv) Let D be the event of getting prime number on first die and odd prime number on second and third dice.
i.e. first die can show 2 or 3 or 5 and second, third dice can show 3 or 5

$$\therefore D = \{(2,3,3), (2,3,5), (2,5,3), (2,5,5), (3,3,3), (3,3,5), (3,5,3), (3,5,5), (5,3,3), (5,3,5), (5,5,3), (5,5,5)\}$$

and hence P (D) = $\frac{12}{216} = \frac{1}{18}$

E5) We know that there are 366 days in a leap year. i.e. $\frac{365}{7} = 52\frac{2}{7}$ weeks

i.e one leap year = (52 complete weeks + two over days).

These two over days may be

(i) Sunday and Monday

(ii) Monday and Tuesday

(iii) Tuesday and Wednesday

(iv) Wednesday and Thursday

(v) Thursday and Friday

(vi) Friday and Saturday

(vii) Saturday and Sunday

$\therefore$ Number of exhaustive Cases = 7

Let A be the event of getting 53 Mondays

There will be 53 Mondays in a leap year if and only if these two over days are

"Sunday and Monday" or "Monday and Tuesday"

$\therefore$ Number of favourable cases for event A are 2

and $P(A) = \dfrac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \dfrac{2}{7}$

E6) Here, the odds in favour of speaking the truth are 3 : 2,

$\therefore$ here m = 3, n – m = 2 and hence n = 5.

Hence, the probability of speaking the truth $= \dfrac{3}{5}$

E7) As the odds against Manager X setting the wage dispute with the workers are 8 : 6, hence odds in favour of settling the dispute are 6 : 8.

Thus, the probability that the manager settles the dispute $= \dfrac{6}{14} = \dfrac{3}{7}$.

E8) As the probability that student pass a test = 2/3,

$\therefore$ the number of favourable cases = 2 and the number of exhaustive cases = 3, and hence the number of cases against passing the test = 3 – 2 = 1.

Thus, odds against passing the test

= the number cases against the event : the number cases favourable to the event

= 1 : 2

E 9) (i) Odds in favour of the event $\bar{A}$ are given as 1 : 4.

We know that if odds in favour of event E are $m:n$ then P(E) $= \dfrac{m}{m+n}$

$\therefore$ In this case $P(\bar{A}) = \dfrac{1}{1+4} = \dfrac{1}{5}$ and $P(A) = 1 - P(\bar{A}) = 1 - \dfrac{1}{5} = \dfrac{4}{5}$

(ii) Odds against the happening of the event $\bar{A}$ are given as 7 : 2.

We know that if odds against the happening of an event E are $m : n$, then $P(E) = \dfrac{n}{m+n}$.

$\therefore$ In this case $P(\bar{A}) = \dfrac{2}{7+2} = \dfrac{2}{9}$ and hence

$P(A) = 1 - P(\bar{A}) = 1 - \dfrac{2}{9} = \dfrac{7}{9}$.