UNIT 3 LAWS OF PROBABILITY

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3.1 INTRODUCTION

In the preceding two units of this block, you have studied various approaches to probability, their direct applications and various types of events in terms of set theory. However, in many situations, we may need to find probability of occurrence of more complex events. Now, we are adequately equipped to develop the laws of probability i.e. the law of addition and the law of multiplication which will help to deal with the probability of occurrence of complex events.

Objectives

After completing this unit, you should be able to discuss:

- addition law of probability;
- conditional probability;
- multiplication law of probability;
- independent events;
- probability of happening at least one of the independent events; and
- problems on addition and multiplicative laws of probability.

3.2 ADDITION LAW

Addition Theorem on Probability for Two Events

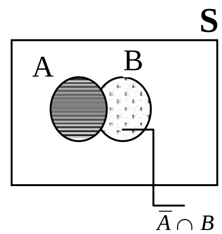
Statement

Let S be the sample space of a random experiment and events A and $B \subseteq S$ then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof: From the Venn diagram, we have

$$A \cup B = A \cup \left(\bar{A} \cap B\right)$$

$$\therefore\ P\left(A \cup B\right) = P\left(A\right) + P\left(\bar{A} \cap B\right) \qquad \left[\begin{array}{l}\text{By axiom (iii)} \because\ A \text{ and } \bar{A} \cap B \\ \text{are mutually disjoint}\end{array}\right]$$



$$= P(A) + P(B) - P\left(A \cap B\right) \left[\begin{array}{l}\text{Refer to result 3 of Sec. 2.6} \\ \text{of Unit 2 of this Course}\end{array}\right]$$

Hence proved

Corollary: If events A and B are mutually exclusive events, then

$P(A \cup B) = P(A) + P(B)$.

[This is known as the addition theorem for mutually exclusive events]

Proof: For any two events A and B, we know that

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Now, if the events A and B are mutually exclusive then

$A \cap B = \phi$

Also, we know that probability of impossible event is zero i.e.

$P(A \cap B) = P(\phi) = 0$.

Hence,

$P(A \cup B) = P(A) + P(B)$

Similarly, for three non-mutually exclusive events A, B and C, we have

$$P\left(A \cup B \cup C\right) = P\left(A\right) + P\left(B\right) + P\left(C\right) - P\left(A \cap B\right) - P\left(A \cap C\right) - P\left(B \cap C\right)$$
$$+ P\left(A \cap B \cap C\right)$$

and for three mutually exclusive events A, B and C, we have

$P\left(A \cup B \cup C\right) = P\left(A\right) + P\left(B\right) + P\left(C\right)$.

The result can similarly be extended for more than 3 events.

Applications of Addition Theorem of Probability

Example 1: From a pack of 52 playing cards, one card is drawn at random. What is the probability that it is a jack of spade or queen of heart?

Solution: Let A and B be the events of drawing a jack of spade and queen of heart, respectively.

$\therefore$ P(A) $= \dfrac{1}{52}$ and P(B) $= \dfrac{1}{52}$ $\left[\begin{array}{l}\because \text{there is one card each of jack} \\ \text{of spade and queen of heart}\end{array}\right]$

Here, a card cannot be both the jack of spade and the queen of heart, hence A and B are mutually exclusive,

$\therefore$ applying the addition theorem for mutually exclusive events,

the required probability = P(A $\cup$ B) = P(A) + P(B)

$$= \frac{1}{52} + \frac{1}{52} = \frac{2}{52} = \frac{1}{26}.$$

Example 2: 25 lottery tickets are marked with first 25 numerals. A ticket is drawn at random. Find the probability that it is a multiple of 5 or 7.

Solution: Let A be the event that the drawn ticket bears a number multiple of 5 and B be the event that it bears a number multiple of 7.

Therefore,

A = {5, 10, 15, 20, 25}

B = {7, 14, 21}

Here, as A $\cap$ B = ϕ,

$\therefore$ A and B are mutually exclusive, and hence,

$$P(A \cup B) = P(A) + P(B) = \frac{5}{25} + \frac{3}{25} = \frac{8}{25}$$

Example 3: Find the probability of getting either a number multiple of 3 or a prime number when a fair die is thrown.

Solution: When a die is thrown, then the sample space is

S = {1, 2, 3, 4, 5, 6}

Let A be the event of getting a number multiple of 3 and B be the event of getting a prime number,

$\therefore$ A = {3, 6}, B = {2, 3, 5}, A $\cap$ B = {3}

Here as A $\cap$ B is not empty set,

$\therefore$ A and B are non-mutually exclusive and hence,

the required probability = P(A $\cup$ B)

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{2}{6} + \frac{3}{6} - \frac{1}{6}$$

$$= \frac{2+3-1}{6} = \frac{4}{6} = \frac{2}{3}.$$

Example 4: There are 40 pages in a book. A page is opened at random. Find the probability that the number of this opened page is a multiple of 3 or 5.

Solution Let A be the event that the number of the opened page is a multiple of 3 and B be the event that it is a multiple of 5.

$\therefore$ A = {3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39},

B = {5, 10, 15, 20, 25, 30, 35, 40}, and

$A \cap B$ = {15, 30}

As A and B are non-mutually exclusive,

$\therefore$ the required probability $= P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{40} + \frac{8}{40} - \frac{2}{40} = \frac{19}{40}.$$

Example 5: A Card is drawn from a pack of 52 playing cards, find the proabability that the drawn card is an ace or a red colour card.

Solution: Let A be the event that the drawn card is a card of ace and B be the event that it is red colour card.

Now as there are four cards of ace and 26 red colour cards in a pack of 52 playing cards. Also, 2 cards in the pack are ace cards of red colour.

$$\therefore P(A) = \frac{4}{52},\ P(B) = \frac{26}{52} \text{ , and } P(A \cap B) = \frac{2}{52}$$

$\therefore$ the required probability $= P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}.$$

Now, you can try the following exercises.

E1) A card is drawn from a pack of 52 playing cards. Find the probability that it is either a king or a red card.

E2) Two dice are thrown together. Find the probability that the sum of the numbers turned up is either 6 or 8.

3.3 CONDITIONAL PROBABILITY AND MULTIPLICATIVE LAW

Conditional Probability

We have discussed earlier that P(A) represents the probability of happening event A for which the number of exhaustive cases is the number of elements in the sample space S. P(A) dealt earlier was the unconditional probability. Here, we are going to deal with conditional probability.

Let us start with taking the following example:

Suppose a card is drawn at random from a pack of 52 playing cards. Let A be the event of drawing a black colour face card. Then A = $\{J_s, Q_s, K_s, J_c, Q_c, K_c\}$ and hence

P(A) = 6/52 = 3/26.

Let B be the event of drawing a card of spade i.e.

$B = \{1_s, 2_s, 3_s, 4_s, 5_s, 6_s, 7_s, 8_s, 9_s, 10_s, J_s, Q_s, K_s\}$.

If after a card is drawn from the pack of cards, we are given the information that card of spade has been drawn i.e., B has happened, then the probability of event A will no more be $\frac{3}{26}$, because here in this case, we have the information that the card drawn is of spade (i.e. from amongst 13 cards) and hence there are 13 exhaustive cases and not 52. From amongst these 13 cards of spade, there are 3 black colour face cards and hence probability of having black colour face card given that it is a card of spade i.e. $P(A|B) = 3/13$, which is the conditional probability of A given that B has already happened.

Note: Here, the symbol '|' used in P(A|B) should be read as 'given' and not 'upon'. P(A|B) is the conditional probability of happening A given that B has already happened i.e. here A happens depending on the condition of B.

So, the conditional probability P(A|B) is also the probability of happening A but here the information is given that the event B has already happened. P(A|B) refers to the sample space B and not S.

Remark 1: P(A|B) is meaningful only when $P(B) \neq 0$ i.e. when the event B is not an impossible event.

Multiplication Law of Probability

Statement: For two events A and B,

$$P(A \cap B) = P(A)\, P(B|A), \quad P(A) > 0 \qquad \dots (1)$$

$$= P(B)\, P(A|B), \quad P(B) > 0, \qquad \dots (2)$$

where P(B|A) is the conditional probability of B given that A has already happened and P(A|B) is the conditional probability of A given that B has already happened.

Proof: Let n be the number of exhaustive cases corresponding to the sample space S and m_1, m_2, m_3 be the number of favourable cases for events A, B and $A \cap B$ respectively.

$$\therefore\ P(A) = \frac{m_1}{n},\ P(B) = \frac{m_2}{n},\ P(A \cap B) = \frac{m_3}{n}$$

Now, as B|A represents the event of happening B given that A has already happened and hence it refers to the sample space A ($\because$ we have with us the information that A has already happened) and thus the number of exhaustive cases for B|A is m_1 (i.e. the number of cases favourable to "A relative to sample space S"). The number of cases favourable to B|A is the number of those elements of B which are in A i.e. the number of favourable cases to B|A is the number of favourable cases to "$A \cap B$ relative to S". So, the number of favourable cases to B|A is m_3

$$\therefore\ P(B\,|\,A) = \frac{m_3}{m_1}$$

$$= \frac{m_3/n}{m_1/n} \qquad \begin{bmatrix} \text{Dividing the numerator} \\ \text{and denominator by n} \end{bmatrix}$$

$$= \frac{P(A \cap B)}{P(A)}, \; P(A) \neq 0$$

$\Rightarrow P(A \cap B) = P(A)\, P(B|A), \; P(A) \neq 0$

Similarly, you can prove yourself that

$P(A \cap B) = P(B)\, P(A|B), \; P(B) \neq 0.$

This result can be extended for 3 or more events e.g. for three events A, B and C, we have

$P(A \cap B \cap C) = P(A)\, P(B|A)\, P(C|A \cap B),$

where $P(C|A \cap B)$ represents the probability of happening C given that A and B both have already happened.

3.4 INDEPENDENT EVENTS

Before defining the independent events, let us again consider the concept of conditional probability taking the following example:

Suppose, we draw a card from a pack of 52 playing cards, then probability of drawing a card of spade is 13/52. Now, if we do not replace the card back and draw the next card. Then, the probability of drawing the second card 'a card of spade' if it being given that the first card was spade would be 12/51 and it is the conditional probability. Now, if the first card had been replaced back then this conditional probability would have been 13/52. So, if sampling is done without replacement, the probability of second draw and that of subsequent draws made following the same way is affected but if it is done with replacement, then the probability of second draw and subsequent draws made following the same way remains unaltered.

So, if in the above example, if the next draw is made with replacement, then the happening or non-happening of any draw is not affected by the preceding draws. Let us now define independent events.

Independent Events

Events are said to be independent if happening or non-happening of any one event is not affected by the happening or non-happening of other events. For example, if a coin is tossed certain number of times, then happening of head in any trial is not affected by any other trial i.e. all the trials are independent.

Two events A and B are independent if and only if $P(B|A) = P(B)$ i.e. there is no relevance of giving any information. Here, if A has already happened, even then it does not alter the probability of B. e.g. Let A be the event of getting head in the 4^{th} toss of a coin and B be the event of getting head in the 5^{th} toss of the coin. Then the probability of getting head in the 5^{th} toss is $\frac{1}{2}$, irrespective of the case whether we know or don't know the outcome of 4^{th} toss, i.e. $P(B|A) = P(B)$.

Multiplicative Law for Independent Events:

If A and B are independent events, then

$P(A \cap B) = P(A)\, P(B).$

This is because if A and B are independent then $P(B|A) = P(A)$ and hence the equation (1) discussed in Sec. 3.3 of this unit becomes $P(A \cap B) = P(A)\, P(B)$.

Similarly, if A, B and C are three independent events, then

$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$.

The result can be extended for more than three events also.

Remark 2: Mutually exclusive events can never be independent.

Proof: Let A and B be two mutually exclusive events with positive probabilities (i.e. $P(A) > 0$, and $P(B) > 0$)

$\therefore\ P(A \cap B) = 0\ [\because A \cap B = \phi]$

Also, by multiplication law of probability, we have

$P(A \cap B) = P(A)\, P(B|A),\ P(A) \neq 0.$

$\therefore\ 0 = P(A)\, P(B|A).$

Now as $P(A) \neq 0,\ \therefore\ P(B|A) = 0$

But $P(B) \neq 0 \quad [\because P(B) > 0]$

$\therefore\ P(B|A) \neq P(B),$

Hence A and B are not independent.

Result: If events A and B are independent then prove that

(i) A and $\overline{B}$ are independent

(ii) $\overline{A}$ and B are independent

(iii) $\overline{A}$ and $\overline{B}$ are independent

Proof:

(i) We know that

$$P(A \cap \overline{B}) = P(A) - P(A \cap B) \qquad \text{[Already proved in Unit 2 of this course]}$$

$$= P(A) - P(A)P(B) \qquad [\because \text{events A and B are independent.}]$$

$$= P(A)(1 - P(B))$$

$$= P(A)P(\overline{B})$$

$\Rightarrow$ Events A and $\overline{B}$ are independent

(ii) We know that

$$P(\overline{A} \cap B) = P(B) - P(A \cap B) \qquad \text{[Already proved in Unit 2 of this course]}$$

$$= P(B) - P(A)P(B) \qquad [\because \text{ events A and B are independent.}]$$

$$= P(B)(1 - P(A))$$

$$= P(B)P(\overline{A})$$

$$= P(\overline{A})P(B)$$

$\Rightarrow$ Events $\overline{A}$ and B are independent

(iii) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$ [By De-Morgan's law]

$= 1 - P(A \cup B)$ $\left[\because P(E) + P(\overline{E}) = 1\right]$

$= 1 - [P(A) + P(B) - P(A \cap B)]$ [Using addition law on probability]

$= 1 - [P(A) + P(B) - P(A)P(B)]$ $[\because$ events A and B are independent. $]$

$= 1 - P(A) - P(B) + P(A)P(B)$

$= (1 - P(A)) - P(B)(1 - P(A))$

$= (1 - P(A))(1 - P(B))$

$= P(\overline{A})P(\overline{B})$

$\Rightarrow$ Events $\overline{A}$ and $\overline{B}$ are independent

Now let us take up some examples on conditional probability, multiplicative law and independent events:

Example 6: A die is rolled. If the outcome is a number greater than 3, what is the probability that it is a prime number?

Solution: The sample space of the experiment is

S = {1, 2, 3, 4, 5, 6}

Let A be event that the outcome is a number greater than 3 and B be the event that it is a prime number.

$\therefore$ A = {4, 5, 6}, B = {2, 3, 5} and hence $A \cap B$ = {5}.

$\Rightarrow P(A) = 3/6,\ P(B) = 3/6,\ P(A \cap B) = 1/6.$

Now, the required probability = P(B|A)

$$= \frac{P(A \cap B)}{P(A)} \qquad \left[\begin{array}{l}\text{Refer the multiplication}\\ \text{law given by (1) in}\\ \text{Sec. 3.3 of the Unit}\end{array}\right]$$

$$= \frac{1/6}{3/6} = \frac{1}{3}$$

Example 7: A couple has 2 children. What is the probability that both the children are boys, if it is known that?

(i) younger child is a boy

(ii) older child is a boy

(iii) at least one of them is boy

Solution: Let B_i, G_i denote that i^{th} birth is of boy and girl respectively, i =1, 2.

Then for a couple having two children, the sample space is

$S = \{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$

Let A be the event that both children are boys then

$A = \{B_1B_2\}$

(i) Let B be the event of getting younger child as boy i.e.

$B = \{B_1B_2, G_1B_2\}$. Hence $A \cap B = \{B_1B_2\}$

$$\therefore \text{required probability } P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2}$$

(ii) Let C be the event of getting older child as boy, then C = $\{B_1B_2, B_1G_2\}$

and hence $A \cap C = \{B_1B_2\}$.

$$\therefore \text{ required probability } P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{1/4}{2/4} = \frac{1}{2}.$$

(iii) Let D be the event of getting at least one of the children as boy, then

$D = \{B_1B_2, B_1G_2, G_1B_2\}$ and hence

$A \cap D = \{B_1B_2\}$.

$$\therefore \text{required probability } P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Example 8: An urn contains 4 red and 7 blue balls. Two balls are drawn one by one without replacement. Find the probability of getting 2 red balls.

Solution: Let A be the event that first ball drawn is red and B be the event that the second ball drawn is red.

$$\therefore P(A) = 4/11 \text{ and } P(B|A) = 3/10 \left[\begin{array}{l}\because \text{it is given that one red ball} \\ \text{has already been drawn}\end{array}\right]$$

$\therefore$ The required probability = P(A and B)

$$= P(A)\,P(B|A)$$

$$= \left(\frac{4}{11}\right)\left(\frac{3}{10}\right) = \frac{6}{55}$$

Example 9: Three cards are drawn one by one without replacement from a well shuffled pack of 52 playing cards. What is the probability that first card is jack, second is queen and the third is again a jack.

Solution: Define the following events

E_1 be the event of getting a jack in the first draw,

E_2 be the event of getting a queen in second draw, and

E_3 be the event of getting a jack in third draw,

$\therefore$ Required probability = $P(E_1 \cap E_2 \cap E_3)$

$$= P(E_1)P(E_2|E_1)P(E_3 \mid E_1 \cap E_2)$$

$$\left[\begin{array}{l}\because \text{cards are drawn without replacement and} \\ \text{hence the events are not independent}\end{array}\right]$$

$$= \frac{4}{52} \times \frac{4}{51} \times \frac{3}{50} = \frac{1}{13} \times \frac{2}{17} \times \frac{1}{25} = \frac{2}{5525}.$$

Example 10: (i) If A and B are independent events with

$P(A \cup B) = 0.8$ and $P(B) = 0.4$ then find $P(A)$.

(ii) If A and B are independent events with

$P(A) = 0.2$, $P(B) = 0.5$ then find $P(A \cup B)$.

(iii) If A and B are independent events and

$P(A) = 0.4$ and $P(B) = 0.3$, then find $P(A|B)$ and $P(B|A)$.

(iv) If A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.2$, then find

$$P(\overline{A} \cap B),\ P(A \cap \overline{B}),\ P(\overline{A} \cap \overline{B})$$

Solution:

(i) We are given

$$P(A \cup B) = 0.8,\ P(B) = 0.4.$$

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{[By Addition theorem of probability]}$$

$$= P(A) + P(B) - P(A)P(B) \quad [\because \text{events A and B are independent}]$$

$$\Rightarrow 0.8 = P(A) + 0.4 - 0.4P(A)$$

$$\Rightarrow 0.4 = (1 - 0.4)\, P(B)$$

$$= 0.6\, P(B)$$

$$\Rightarrow P(B) = \frac{0.4}{0.6} = \frac{4}{6} = \frac{2}{3}.$$

(ii) We are given that $P(A) = 0.2$, $P(B) = 0.5$.

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{[Addition theorem of probability]}$$

$$= P(A) + P(B) - P(A)P(B) \quad [\because \text{events A and B are independent}]$$

$$= 0.2 + 0.5 - 0.2 \times 0.5$$

$$= 0.7 - 0.10 = 0.6$$

(iii) We are given that $P(A) = 0.4$, $P(B) = 0.3$.

Now, as A and B are independent events,

$\therefore P(A \cap B) = P(A)P(B)$

$= 0.4 \times 0.3 = 0.12$

And hence from conditional probability, we have

$$P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{0.12}{0.3} = \frac{12}{30} = \frac{2}{5} = 0.4,$$

and $$P(B | A) = \frac{P(A \cap B)}{P(A)} = \frac{0.12}{0.4} = \frac{12}{40} = \frac{3}{10} = 0.3.$$

(iv) We are given that P(A) = 0.4, P(B) = 0.2.

We know that if two events A and B are independent then

$\bar{A}$ and B; A and $\bar{B}$; $\bar{A}$ and $\bar{B}$ are also independent events.

$\therefore P(\bar{A} \cap B) = P(\bar{A})P(B)$ [Using the concept of independent events]

$= (1 - P(A))(P(B))$

$= (1 - 0.4)(0.2)$

$= (0.6)(0.2)$

$= 0.12$

$P(A \cap \bar{B}) = P(A)\, P(\bar{B})$ [$\because$ A and $\bar{B}$ are independent]

$= (0.4)(1 - 0.2) = 0.32$

$$P\left(\bar{A} \cap \bar{B}\right) = P\left(\bar{A}\right)P\left(\bar{B}\right) = \left(1-0.4\right)\left(1-0.2\right) = \left(0.6\right)\left(0.8\right) = 0.48.$$

Example 11: Three unbiased coins are tossed simultaneously. In which of the following cases are the events A and B independent?

(i) A be the event of getting exactly one head

B be the event of getting exactly one tail

(ii) A be the event that first coin shows head

B be the event that third coin shows tail

(iii) A be the event that shows exactly two tails

B be the event that third coin shows head

Solution: When three unbiased coins are tossed simultaneously, then the sample space is given by

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

(i) A = {HTT, THT, TTH}

B = {HHT, HTH, THH}

$A \cap B = \{\} = \phi$

$$\therefore P(A) = \frac{3}{8}, P(B) = \frac{3}{8}, P(A \cap B) = \frac{0}{8} = 0$$

Hence, $P(A)\, P(B) = \frac{3}{8} \times \frac{3}{8} = \frac{9}{64}$

$\Rightarrow P(A \cap B) \neq P(A)P(B)$

$\Rightarrow$ Events A and B are not independent.

(ii) A = {HHH, HHT, HTH, HTT}

B = {HHT, HTT, THT, TTT}

$A \cap B = \{HHT, HTT\}$

$$\therefore P(A) = \frac{4}{8} = \frac{1}{2},\ P(B) = \frac{4}{8} = \frac{1}{2},\ P(A \cap B) = \frac{2}{8} = \frac{1}{4}$$

$$P(A)P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(A \cap B)$$

$\Rightarrow$ Events A and B are independent.

(iii) A= {HTT, THT, TTH}

B = {HHH, HTH, THH, TTH}

$A \cap B = \{TTH\}$

$$P(A) = \frac{3}{8},\ P(B) = \frac{4}{8} = \frac{1}{2},\ P(A \cap B) = \frac{1}{8}.$$

Hence, $P(A)P(B) = \frac{3}{8} \times \frac{1}{2} = \frac{3}{16} \neq P(A \cap B)$

$\Rightarrow$ Events A and B are not independent.

Example 12: Two cards are drawn from a pack of cards in succession with replacement of first card. Find the probability that both are the cards of 'heart'.

Solution: Let A be the event that the first card drawn is a heart card and B be the event that second card is a heart card.

As the cards are drawn with replacement,

$\therefore$ A and B are independent and hence the required probability

$$= P(A \cap B) = P(A)P(B) = \left(\frac{13}{52}\right)\left(\frac{13}{52}\right) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}.$$

Example 13: A class consists of 10 boys and 40 girls. 5 of the students are rich and 15 students are brilliant. Find the probability of selecting a brilliant rich boy.

Solution: Let A be the event that the selected student is brilliant, B be the event that he/she is rich and C be the event that the student is boy.

$\therefore P(A) = \frac{15}{50}, P(B) = \frac{5}{50}, P(C) = \frac{10}{50}$ and hence

the required probability = $P(A \cap B \cap C)$

$$= P(A)P(B)P(C) \quad [\because A, B \text{ and } C \text{ are independent}]$$

$$= \left(\frac{15}{50}\right)\left(\frac{5}{50}\right)\left(\frac{10}{50}\right) = \left(\frac{3}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{5}\right) = \frac{3}{500}$$

Here are some exercises for you.

E3) A card is drawn from a well-shuffled pack of cards. If the card drawn is a face card, what is the probability that it is a king?

E4) Two cards are drawn one by one without replacement from a well shuffled pack of 52 cards. What is the probability that both the cards are red?

E5) A bag contains 10 good and 4 defective items, two items are drawn one by one without replacement. What is the probability that first drawn item is defective and the second one is good?

E6) The odds in favour of passing driving test by a person X are $3:5$ and odds in favour of passing the same test by another person Y are $3:2$. What is the probability that both will pass the test?

3.5 PROBABILITY OF HAPPENING AT LEAST ONE OF THE INDEPENDENT EVENTS

If A and B be two independent events, then probability of happening at least one of the events is

$$P(A \cup B) = 1 - P(\overline{A \cup B})$$

$$= 1 - P(\bar{A} \cap \bar{B}) \qquad \text{[By DeMorgan's Law]}$$

$$= 1 - P(\bar{A})P(\bar{B}) \qquad \left[\because \text{ A and B and hence } \bar{A} \text{ and } \bar{B} \text{ are independent.}\right]$$

Similarly if we have n independent event $A_1, A_2, \ldots, A_n$, then probability of happening at least one of the events is

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - \left[P(\bar{A}_1)P(\bar{A}_2) \ldots P(\bar{A}_n)\right]$$

I.e. probability of happening at least one of the independent events

= 1 – probability of happening none of the events.

Example 14: A person is known to hit the target in 4 out of 5 shots whereas another person is known to hit 2 out of 3 shots. Find the probability that the target being hit when they both try.

Solution: Let A be the event that first person hits the target and B be the event that second person hits the target.

$$\therefore P(A) = \frac{4}{5}, \quad P(B) = \frac{2}{3}$$

Now, as both the persons try independently,

$\therefore$ the required probability = probability that the target is hit

= probability that at least one of the persons hits the target

$= P(A \cup B)$

$= 1 - P(\bar{A})P(\bar{B})$

$$= 1 - \left(1 - \frac{4}{5}\right)\left(1 - \frac{2}{3}\right)$$

$$= 1 - \left(\frac{1}{5}\right)\left(\frac{1}{3}\right) = 1 - \frac{1}{15} = \frac{14}{15}.$$

E 7) A problem in statistics is given to three students A, B and C whose chances of solving it are 0.3, 0.5 and 0.6 respectively. What is the probability that the problem is solved?

3.6 PROBLEMS USING BOTH ADDITION AND MULTIPLICATIVE LAWS

Here we give some examples which are based on both the addition and multiplication laws.

Example15: Husband and wife appear in an interview for two vacancies for the same post. The probabilities of husband's and wife's selections are $\frac{2}{5}$ and $\frac{1}{5}$ respectively. Find the probability that

(i) Exactly one of them is selected

(ii) At least one of them is selected

(iii) None is selected.

Solution: Let H be the event that husband is selected and W be the event that wife is selected. Then,

$$P(H) = \frac{2}{5},\ P(W) = \frac{1}{5}$$

$$\therefore P(\overline{H}) = 1 - \frac{2}{5} = \frac{3}{5},\ P(\overline{W}) = 1 - \frac{1}{5} = \frac{4}{5}$$

(i) The required probability $= P\left[(H \cap \overline{W}) \cup (\overline{H} \cap W)\right]$

$$= P(H \cap \overline{W}) + P(\overline{H} \cap W)$$

$\left[\begin{array}{l}\text{By Addition theorem for}\\ \text{mutually exclusive events}\end{array}\right]$

$$= P(H)P(\overline{W}) + P(\overline{H})P(W)$$

$\left[\begin{array}{l}\because \text{selection of husband and}\\ \text{wife are independent}\end{array}\right]$

$$= \frac{2}{5} \times \frac{4}{5} + \frac{3}{5} \times \frac{1}{5} = \frac{8}{25} + \frac{3}{25} = \frac{11}{25}.$$

(ii) The required probability $= P(H \cup W)$

$$= 1 - P\left(\overline{H}\right)P\left(\overline{W}\right)$$

$$\left[\begin{array}{l}\because \text{H and W are independent and hence} \\ \overline{\text{H}} \text{ and } \overline{\text{W}} \text{ are independent events}\end{array}\right]$$

$$= 1 - \frac{3}{5} \times \frac{4}{5} = 1 - \frac{12}{25} = \frac{13}{25}.$$

(iii) The required probability = $P(\overline{H} \cap \overline{W})$

$$= P(\overline{H})P(\overline{W}) \left[\because \overline{H} \text{ and } \overline{W} \text{ are independent}\right]$$

$$= \frac{3}{5} \times \frac{4}{5} = \frac{12}{25}$$

Example 16: A person X speaks the truth in 80% cases and another person Y speaks the truth in 90% cases. Find the probability that they contradict each other in stating the same fact.

Solution: Let A, B be the events that person X and person Y speak truth respectively, then

$$P(A) = \frac{80}{100} = 0.8, \; P(B) = \frac{90}{100} = 0.9.$$

$$\therefore P(\overline{A}) = 1 - 0.8 = 0.2, \; P(\overline{B}) = 1 - 0.9 = 0.1.$$

Thus, the required probability $= P\left[\left(A \cap \overline{B}\right) \cup \left(\overline{A} \cap B\right)\right]$

$$= P(A \cap \overline{B}) + P(\overline{A} \cap B) \left[\begin{array}{l}\text{By addition law for mutually} \\ \text{exclusive events}\end{array}\right]$$

$$= P(A)P(\overline{B}) + P(\overline{A})P(B) \left[\begin{array}{l}\text{By multiplication law for} \\ \text{independent events}\end{array}\right]$$

$$= 0.8 \times 0.1 + 0.2 \times 0.9$$

$$= 0.08 + 0.1 = 0.26 = 26\%.$$

Here is an exercise for you.

E 8) Two cards are drawn from a pack of cards in succession presuming that drawn cards are replaced. What is the probability that both drawn cards are of the same suit?

3.7 SUMMARY

Let us summarize the main points covered in this unit:

1) For two non-mutually exclusive events A and B, the **addition law** of probability is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. If these events are mutually exclusive then, $P(A \cup B) = P(A) + P(B)$.

2) **Conditional probability** for happening of an event say B given that an event A has already happened is given by

$P(B|A) = \dfrac{P(A \cap B)}{P(A)}$, P(A) > 0. The **Multiplicative law** of probability for any two events is stated as $P(A \cap B) = P(A)\,P(B|A)$.

3) Events are said to be **independent** if happening or non-happening of any one event is not affected by the happening or non-happening of other events. Two events A and B are independent if and only if P(B|A) = P(B) and hence **multiplicative law for two independent events** is given by $P(A \cap B) = P(A)\,P(B)$.

4) If events are independent then their complements are also independent. **Probability of happening at least one of the independent events** can be obtained on subtracting from 1 the probability of happening none of the events.

3.8 SOLUTIONS/ANSWERS

E1) Let A be the event of getting a card of king and B be the event of getting a red card.

$\therefore$ the required probability $= P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B) \quad \text{[By addition theorem]}$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} \left[\begin{array}{l}\because \text{there are two cards which} \\ \text{are both king as well as red]}\end{array}\right]$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52}$$

$$= \frac{4+26-2}{52} = \frac{28}{52} = \frac{7}{13}$$

E2) Here, the number of exhaustive cases = 6 × 6 = 36.

Let A be the event that the sum is 6 and B be the event that the sum is 8.

$\therefore$ A = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}, and

B = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}.

Here, as $A \cap B = \phi$,

$\therefore$ A and B are mutually exclusive and hence

the required probability = $P(A \cup B) = P(A) + P(B)$

$$= \frac{5}{36} + \frac{5}{36} = \frac{10}{36} = \frac{5}{18}$$

E 3) Let A be the event that the card drawn is face card and B be the event that it is a king.

$\therefore$ A = $\{J_s, Q_s, K_s, J_h, Q_h, K_h, J_d, Q_d, K_d, J_c, Q_c, K_c\}$

$B = \{K_s, K_h, K_c, K_d\}$

$\Rightarrow A \cap B = \{K_s, K_h, K_c, K_d\}$

$\Rightarrow P(A) = \frac{12}{52},\ P(B) = \frac{4}{52},$ and $P(A \cap B) = \frac{4}{52}$.

$\therefore$ the required probability = $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{4/52}{12/52} = \frac{4}{12} = \frac{1}{3}$.

E 4) Let A be the event that first card drawn is red and B be the event that second card is red.

$\therefore P(A) = \frac{26}{52}$ and $P(B|A) = \frac{25}{51}$.

Thus, the desired probability = $P(A \cap B)$

$$= P(A)\,P(B|A) \quad \begin{bmatrix}\text{By multiplication} \\ \text{theorem}\end{bmatrix}$$

$$= \left(\frac{26}{52}\right)\left(\frac{25}{51}\right) = \frac{25}{102}.$$

E 5) Let E_1 be the event of getting a defective item in first draw and E_2 be the event of getting a good item in the second draw.

$\therefore$ the required probability = $P(E_1 \cap E_2)$

$$= P(E_1)P(E_2 \mid E_1) \quad \begin{bmatrix}\text{By multiplication} \\ \text{theorem}\end{bmatrix}$$

$$= \frac{4}{14} \times \frac{10}{13} = \frac{20}{91}.$$

E 6) Let A be the event that person X passes the test and B be the event that person Y passes the test.

$\therefore P(A) = \frac{3}{3+5} = \frac{3}{8}$

and $P(B) = \frac{3}{3+2} = \frac{3}{5}$

Now, as both the person take the test independently,

$\therefore$ the required probability = $P(A \cap B)$

$$= P(A)P(B) = \left(\frac{3}{8}\right)\left(\frac{3}{5}\right) = \frac{9}{40}$$

E 7) Let E_1, E_2 and E_3 be the events that the students A, B and C solves the problem respectively.

$\therefore P(E_1) = 0.3,\ P(E_2) = 0.5$ and $P(E_3) = 0.6$.

Now, as the students try independently to solve the problem,

$\therefore$ the probability that the problem will be solved

= Probability that at least one of the students solves the problem

$= P(E_1 \cup E_2 \cup E_3)$

$= 1 - P(\bar{E}_1)P(\bar{E}_2)P(\bar{E}_3)$

$= 1 - (1 - 0.3)(1 - 0.5)(1 - 0.6)$

$= 1 - (0.7)(0.5)(0.4)$

$= 1 - 0.140 = 0.86.$

E 8) Let S_1, C_1, H_1 and D_1 be the events that first card is of spade, club, heart and diamond respectively; and let S_2, C_2, H_2 and D_2 be the events that second card is of spade, club, heart and diamond respectively.

Thus, the required probability

$= P[(S_1 \cap S_2) \text{ or } (C_1 \cap C_2) \text{ or } (H_1 \cap H_2) \text{ or } (D_1 \cap D_2)]$

$= P(S_1 \cap S_2) + P(C_1 \cap C_2) + P(H_1 \cap H_2) + P(D_1 \cap D_2)$

$= P(S_1)P(S_2) + P(C_1)P(C_2) + P(H_1)P(H_2) + P(D_1)P(D_2)$

$\left[\begin{array}{l}\because \text{cards are drawn with replacement and} \\ \text{hence the draws are independent}\end{array}\right]$

$$= \frac{13}{52} \times \frac{13}{52} + \frac{13}{52} \times \frac{13}{52} + \frac{13}{52} \times \frac{13}{52} + \frac{13}{52} \times \frac{13}{52}$$

$$= 4 \times \frac{13}{52} \times \frac{13}{52}$$

$$= \frac{1}{4}$$