UNIT 2   LINEAR PROGRAMMING PROBLEMS

Structure

2.1   INTRODUCTION

In Unit 1, you have learnt about the origin, scope, uses and limitations of Operations Research. We have also discussed the concept of optimisation and explained the basic feasible solution of linear programming problem.

In this unit, we discuss linear programming problems and explain how they are formulated mathematically in Secs. 2.2 and 2.3, respectively. We also define the objective function and explain how its optimum value is obtained graphically in Sec. 2.4. The objective function may be profit, cost, revenue, production capacity, etc. Linear programming can be applied to a variety of problems such as production, transportation, advertising and problems in public and private organisations, e.g., business, industry, hospitals, libraries as also in education. In order to solve linear programming problems, we need to convert them into a canonical or standard form, as explained in Sec. 2.5.

In the next unit, you will learn how to solve an LPP using trial and error method and the Simplex method. We shall also discuss the artificial variable technique and the Big-M method.

Objectives

After studying this unit, you should be able to:

- describe a linear programming problem and its mathematical formulation;

- discuss the applications and limitations of linear programming problems;

- formulate the linear programming problems;

- explain how linear programming problems are solved graphically; and

- express the linear programming problems to their canonical and standard form.

2.2 LINEAR PROGRAMMING PROBLEM (LPP)

In Unit 1, you have learnt about the concept of optimisation. Usually, the requirements of any agency, industry or country far exceed their limited resources of land, workforce, capital, organisational facilities, etc. Since the resources at their disposal are limited, the problem is to use them in such a way as to obtain the maximum production or profit, or to minimise the cost of production. You have learnt that such problems are called **optimisation problems**. A problem wherein the objective is to allot the limited available resources to the jobs in such a way as to optimise the overall effectiveness, minimising the total cost or maximising the total profit, is called mathematical programming. Mathematical programming in which constraints are expressed as linear equalities/inequalities is called **linear programming**.

Linear programming deals with the optimisation of the total effectiveness expressed as a linear function of **decision variables**, known as the objective function, subject to a set of linear equalities/inequalities known as **constraints**. Decision variables are the variables in terms of which the problem is defined.

Every organisation, big or small, has at its disposal four Ms, i.e., Men, Machines, Money and Materials. The supply of each of these four Ms is limited. If the supply of these resources is unlimited, the need for management tools such as linear programming will not arise at all. Since it is limited, there is a need to find the best allocation of resources in order to optimise the objective function. Linear programming can be used if the following conditions are satisfied:

1. Objective function, to be optimised, should be well defined and can be expressed as a linear function of the decision variables.

2. Resources leading to constraints should be finite and can be expressed as linear equalities or inequalities in terms of variables.

3. There must be alternative and finite courses of action.

4. Decision variables should be non-negative.

Applications

Linear programming methods are used in various fields including business and industry by almost all their departments such as production, marketing, finance, personnel. For example, using linear programming methods, we can:

i)   maximise profits or the number of effective exposures or returns or revenue; or

ii) minimise costs or times of assembling the parts or the number of personnel for a job or the transportation cost or the travelling distance of a salesman.

There are many more applications. However, there are some limitations as well.

Limitations

1. Linear programming method cannot be applied if relationships are not linear.

2. It may give non-integral values even for those decision variables which have only integral values.

3. Constraints in the linear programming methods are written assuming all parameters are known. However, in real problems, sometimes these are not known and hence the full set of constraints cannot be written.

2.3 MATHEMATICAL FORMULATION OF LPP

The mathematical formulation of linear programming problem (LPP) is described in the following steps:

1. Identify the decision variables of the problem.

2. Express the objective function, which is to be optimised, i.e., maximised or minimised, as a linear function of the decision variables.

3. Identify the limited available resources, i.e., the constraints and express them as linear inequalities or equalities in terms of decision variables.

4. Since negative values of the decision variables do not have any valid physical interpretation, introduce non-negative restrictions.

Let us take an example to illustrate these steps.

Example 1: A small scale industry manufactures two products P and Q which are processed in a machine shop and assembly shop. Product P requires 2 hours of work in a machine shop and 4 hours of work in the assembly shop to manufacture while product Q requires 3 hours of work in machine shop and 2 hours of work in assembly shop. In one day, the industry cannot use more than 16 hours of machine shop and 22 hours of assembly shop. It earns a profit of `3 per unit of product P and `4 per unit of product Q. Give the mathematical formulation of the problem so as to maximise profit.

Solution: Let x and y be the number of units of product P and Q, which are to be produced. Here, x and y are the **decision variables**. Suppose Z is the profit function.

Since one unit of product P and one unit of product Q gives the profit of `3 and `4, respectively, the objective function is

$$\text{Maximise } Z = 3x + 4y$$

The requirement and availability in hours of each of the shops for manufacturing the products are tabulated as follows:

	Machine Shop	**Assembly Shop**	**Profit**
Product P	2 hours	4 hours	`3 per unit
Product Q	3 hours	2 hours	`4 per unit
Available hours per day	16 hours	22 hours	

Total hours of machine shop required for both types of product $= 2x + 3y$

Total hours of assembly shop required for both types of product $= 4x + 2y$

Hence, the constraints as per the limited available resources are:

$$2x + 3y \le 16$$
and $\quad\quad 4x + 2y \le 22$

Since the number of units produced for both P and Q cannot be negative, the non-negative restrictions are:

$$x \ge 0, y \ge 0$$

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Thus, the mathematical formulation of the given problem is

> Maximise $Z = 3x + 4y$
>
> subject to the constraints
>
> $$2x + 3y \le 16$$
>
> $$4x + 2y \le 22$$
>
> and non-negative restrictions
>
> $$x \ge 0, \quad y \ge 0$$

Now, you should try to formulate the following problem mathematically.

g is the symbol of gram.

E1) A company produces two types of items P and Q that require gold and silver. Each unit of type P requires 4g silver and 1g gold while that of Q requires 1g silver and 3g gold. The company can produce 8g silver and 9g gold. Suppose each unit of type P brings a profit of `44 and that of type Q, `55. Give the mathematical formulation for the problem to determine the number of units of each type that the company should produce to maximise the profit.

Before studying the next section, match your solution with the solution given in Sec. 2.7.

2.4 GRAPHICAL SOLUTION OF LINEAR PROGRAMMING PROBLEMS

In Sec. 2.3, you have learnt the mathematical formulation of a linear programming problem (LPP). In this section, we discuss how to solve this linear programming problem graphically using the method of graphs of the inequalities.

The graphical method is used to solve linear programming problems having two decision variables. For solving LPPs involving more than two decision variables, we use another method called the Simplex method. We shall discuss it in Unit 3. But you need to learn the graphical method to acquire the necessary grounding for learning the Simplex method.

The graphical method of solving a linear programming problem comprises the following steps:

1. First of all, the graphs are plotted for the equalities corresponding to the given inequalities for constraints as well as restrictions. That is, we first draw the straight lines. For example, suppose one of the given inequalities is $2x + 3y \le 6$. Then, we first plot the graph for the equation $2x + 3y = 6$, which is a straight line. For this, we take any two points on it as follows and join them:

x	0	3
y	2	0

For example, for $x = 0$, $y = 6/3 = 2$ and for $y = 0$, $x = 6/2 = 3$. So we get the straight line shown in Fig. 2.1.
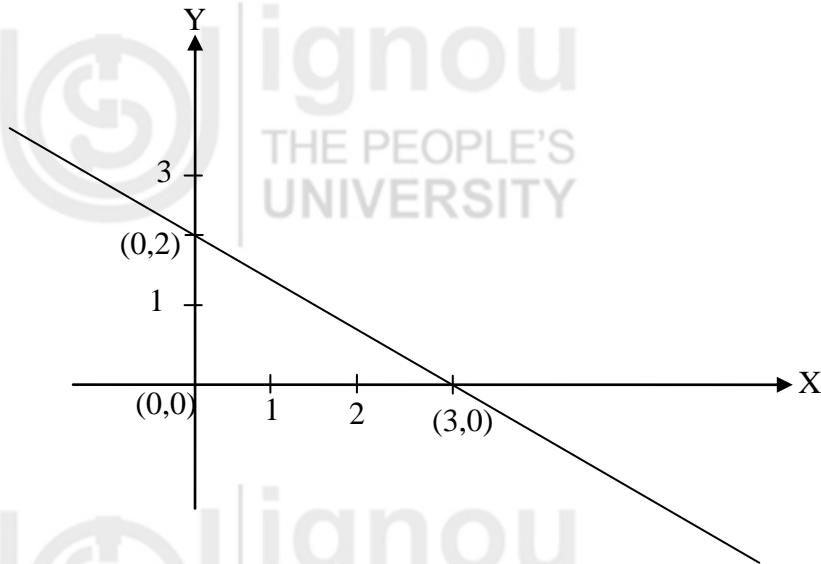
Fig. 2.1

2. Then we determine the region corresponding to each inequality. Let us
   consider the inequality $2x + 3y \leq 6$ again. We can find the region on the
   graph satisfied by this inequality by substituting $x = 0$ and $y = 0$ in it. We get

   $$2(0) + 3(0) \leq 6 \Rightarrow 0 \leq 6$$

   which is correct. So it is the region containing the point $(0, 0)$. Hence, the
   half plane shown in Fig. 2.2 by arrows starting from the line towards the
   point $(0, 0)$ is the graph of the given inequality.
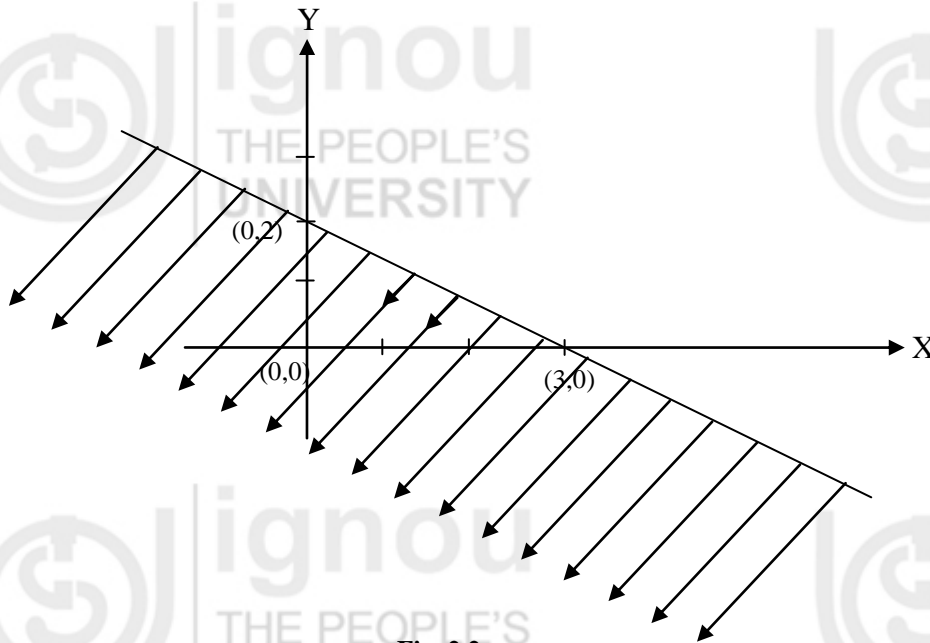
Fig. 2.2

Had the given inequality been $2x + 3y \geq 6$, then we would have drawn
the arrows on the opposite side of the line. This is because on putting
$x = 0, y = 0$ in the inequality, we get

$$2(0) + 3(0) \geq 6 \Rightarrow 0 \geq 6$$

which is not correct.

Thus the point $(0, 0)$ does not satisfy the inequality and hence does not lie
in this region. The graph for the inequality $2x + 3y \geq 6$ would, therefore,
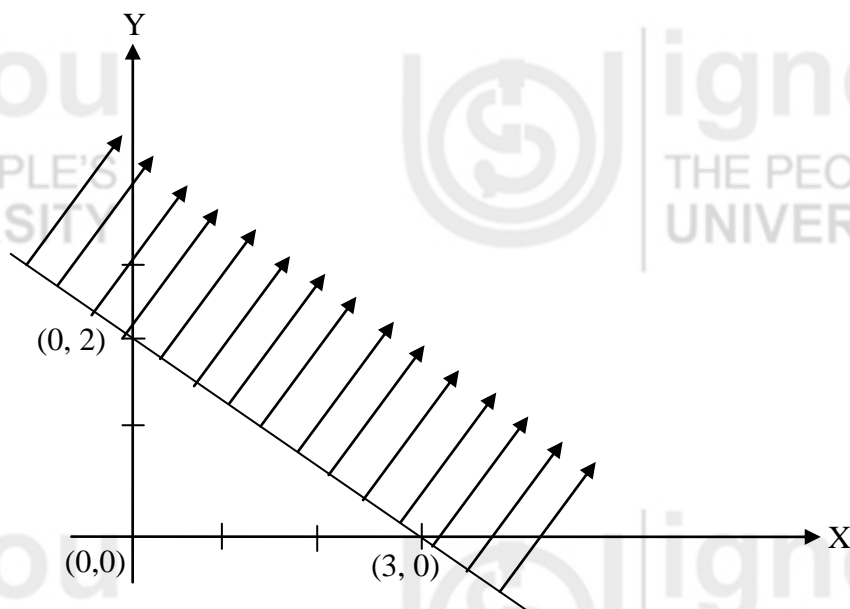be as shown in Fig. 2.3.

Fig. 2.3

In this example, we have used the point (0, 0) to determine which half plane corresponds to the given inequality. However, you can take any other point. But using (0, 0) is far easier. If the right hand side of the given inequality is zero, using the point (0, 0) in it is meaningless. For example, suppose the given inequality is $2x-3y \geq 0$. The plot of $2x-3y = 0$ is given in Fig. 2.4. It is a straight line passing through the origin. Using the point (0, 0) in the inequality $2x - 3y \geq 0$, we get $2(0) - 3(0) \geq 0$, i.e., $0 \geq 0$. So we cannot decide which half plane is the region of the given inequality. Therefore, in this case, we use any other point, say (2, 0). On putting x=2, y=0 in the given inequality, we get

$$2(2) - 3(0) \geq 0 \qquad \Rightarrow \qquad 4 \geq 0$$

which is true. Therefore, the half plane containing (2, 0) is the required region as shown in Fig. 2.4.

$$2x - 3y = 0 \Rightarrow 3y = 2x \Rightarrow y = \frac{2x}{3}$$
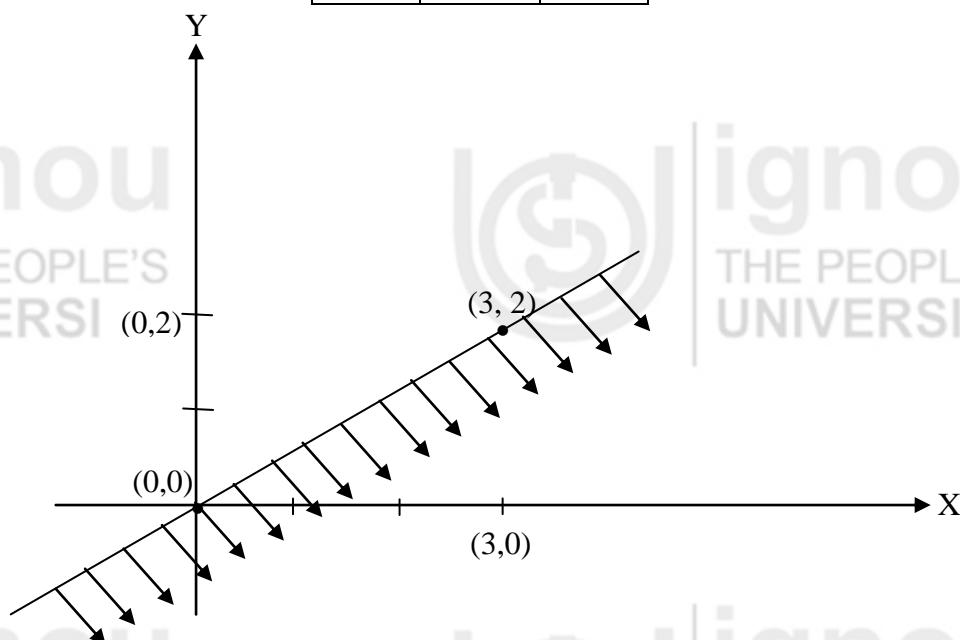
x	0	3
y	0	2

Fig. 2.4

3. After determining the regions for each inequality, we find their common region, i.e., the region obtained on superimposing their regions of inequality. This is the region where all the given inequalities and the non-negative restrictions are satisfied. This common region is known as the **feasible region** or the **solution set** or the **polygonal convex set** (Recall that you have learnt about the convex set in Sec. 1.6 of Unit 1).

4. Next we determine each of the corner points (vertices) of the polygon obtained in step 3. This is done either by plotting the graphs on graph paper or by solving the two equations of the lines intersecting at that point.

5. a) The objective function at each corner point is evaluated. If the feasible region is bounded, then

   i) the maximum of the values obtained for the objective function at the corner points is the optimum value when the objective function is of the maximisation form. The point corresponding to this maximum value gives the required values of the decision variables.

   ii) the minimum of the values obtained for the objective function at the corner points is the optimum value when the objective function is of the minimisation form. The point corresponding to this minimum value gives the required values of the decision variables.

   b) The feasible region may not be bounded. Then either there are additional hidden conditions which can be used to bound the region or there is no solution to the problem.

   c) If the same optimum value exists at two of the vertices, then there are multiple solutions to the problem. Suppose these two points are (x_1, y_1) and (x_2, y_2). Then other solutions are given by the points as follows:

   [First ordinate of first point $\times$ t + First ordinate of second point $\times$ (1–t), Second ordinate of first point $\times$ t + Second ordinate of second point $\times$ (1–t)]

   For the points (x_1, y_1) and (x_2, y_2), the other solutions exist at the points: $[(x_1 \times t + x_2 \times (1 - t), y_1 \times t + y_2 \times (1 - t)]$

   where t is any real number lying between 0 and 1.

For example, let the objective function be $Z = 3x - y$ and let A (2, 1) and B(3, 4) be the points which give the same optimum value of the objective function, i.e., $Z = 5$. Then other solutions which give the same value of the objective function are:

$$(2 \times t + 3 \times (1 - t), 1 \times t + 4 \times (1 - t))$$

or $\quad (2t + 3 - 3t, t + 4 - 4t)$

or $\quad (3 - t, 4 - 3t), \ 0 \le t \le 1$

Here t = 0 gives the point (3, 4), which is point B and t = 1 gives the point (2, 1), which is point A. The real values of t between 0 and 1 give other points which give the same optimum solution. One such point other than A and B is

$$\left(3 - \frac{1}{2}, 4 - 3 \times \frac{1}{2}\right), \quad \text{for} \quad t = \frac{1}{2}, \text{i.e.,} \left(\frac{5}{2}, \frac{5}{2}\right)$$

You can verify that Z = 5 at the point $\left(\frac{5}{2}, \frac{5}{2}\right)$. We illustrate this method in

Example 2.

$$Z = 3x - y$$
$$= 3\left(\frac{5}{2}\right) - \frac{5}{2}$$
$$= \frac{15}{2} - \frac{5}{2} = 5$$

Example 2: A company produces two types of items P and Q that require gold and silver. Each unit of type P requires 4g silver and 1g gold while that of type Q requires 1g silver and 3g gold. The company can produce 8g silver and 9g gold. If each unit of type P brings a profit of `44 and that of type Q `55, determine the number of units of each type that the company should produce to maximise the profit. What is the maximum profit?

Solution: Let x be the number of units of type P to be produced and y be the number of units of type Q to be produced. It is given that:

	Silver	Gold	Profit
Type P	4g	1g	`44 per unit
Type Q	1g	3g	`55 per unit
Available (at the most)	8g	9g	

Let Z be the profit function. The mathematical formulation of the given problem is

$$\text{Max. } Z = 44x + 55y$$

Max. stands for Maximisation

subject to the constraints:

$$4x + y \leq 8,$$
$$x + 3y \leq 9,$$
$$x \geq 0, y \geq 0.$$

First of all, we plot the graphs for the following equations:

$$4x + y = 8, \quad x + 3y = 9, \quad x = 0, y = 0.$$

Since these equations are of straight lines, only two points are sufficient to plot the graphs (see Fig. 2.5). For the line $4x + y = 8$, we take the following two points:

x	0	2
y	8	0

Similarly, for the line $x + 3y = 9$, we take

x	0	9
y	3	0

Now, plotting the above lines, we get Fig. 2.5.

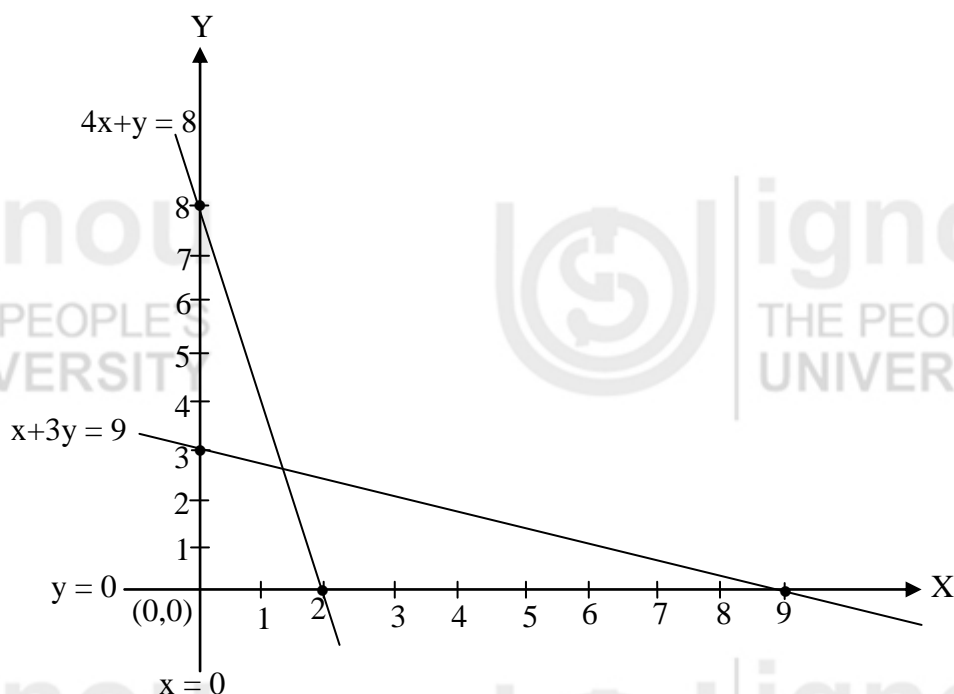**Fig. 2.5**

Note that x = 0 is the y-axis and y = 0 is the x-axis.

For plotting the graph of the inequality $4x + y \leq 8$ we put (0, 0) in it. We get $0 \leq 8$, which is true. Therefore, starting from the line $4x + y \leq 8$, we shall shade towards origin. Similarly, for the graph $x + 3y \leq 9$, we shall shade towards origin. For the graph $x \geq 0$, we shall shade towards right side of x = 0 and for the graph $y \geq 0$, the region above y = 0 will be shaded.

Thus, the regions for the inequalities are shown in Fig. 2.6 by arrows:
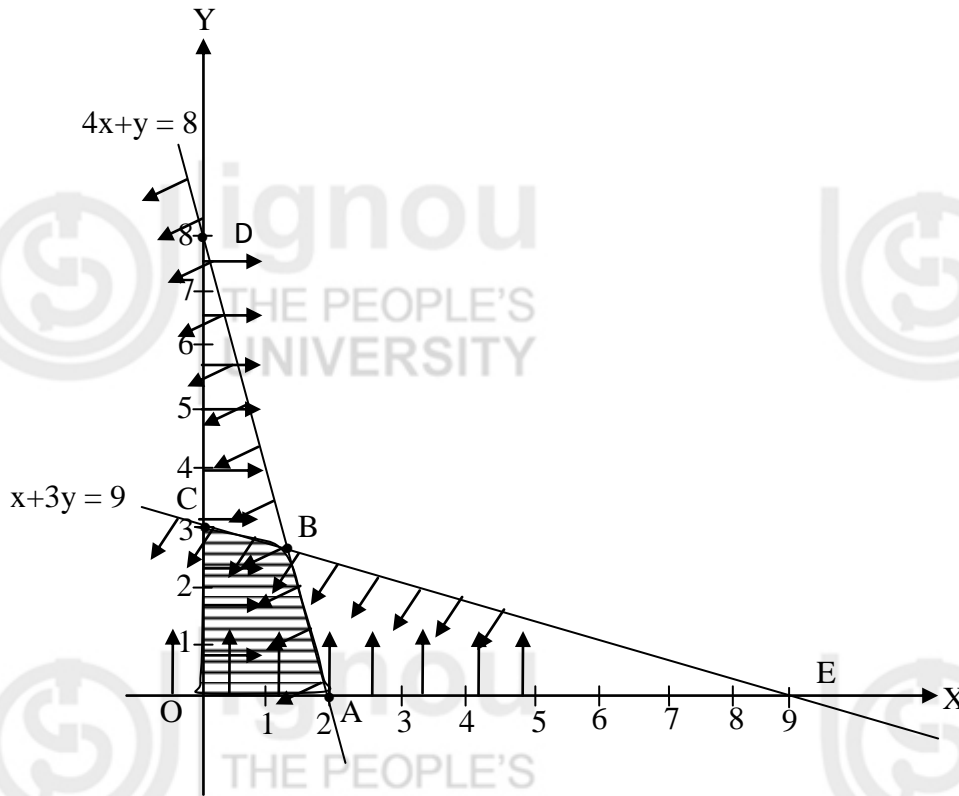

Fig. 2.6

The most common shaded portion is the region inside and on the polygon OABC shown in Fig. 2.6. You can see from Fig. 2.6 that the coordinates of O, A and C are (0,0), (2,0) and (0,3), respectively.

The coordinates of the point B are obtained by solving the equations $4x + y = 8$ and $x + 3y = 9$ as it is the point of intersection of the two lines represented by them. The solution of $4x + y = 8$ and $x + 3y = 9$ is given as

$$x = \frac{15}{11} \quad \text{and} \quad y = \frac{28}{11}$$

So the vertices of OABC are O (0, 0), A (2, 0), $B\left(\frac{15}{11}, \frac{28}{11}\right)$ and C(0, 3).

We now obtain the values of $Z = 44x + 55y$ at each of the vertices of OABC as follows:

At O (0, 0), Z = 44 (0) + 55 (0) = 0

At A (2, 0), Z = 44 (2) + 55 (0) = 88

At $B\left(\frac{15}{11}, \frac{28}{11}\right)$, $Z = 44\left(\frac{15}{11}\right) + 55\left(\frac{28}{11}\right)$

$= 60 + 140 = 200$

At C (0, 3), Z = 44(0) + 55(3) = 165

$4x + y = 8$ (i)

$x + 3y = 9$ (ii)

We multiply equation (i) by 3:

$12x + 3y = 24$ (iii)

We subtract equation (ii) from equation (iii):

$11x = 15$

So we get $x = \frac{15}{11}$

Putting $x = \frac{15}{11}$ in equation (ii),

we get $\frac{15}{11} + 3y = 9 \ \Rightarrow \ y = \frac{28}{11}$

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Thus, the value of Z is maximum at $B\left(\dfrac{15}{11}, \dfrac{28}{11}\right)$

and the optimum solution is

$$\text{Max. } Z = 200 \text{ when } x = \dfrac{15}{11} \text{ and } y = \dfrac{28}{11}.$$

Now, you should try to solve the following exercises.

E2) Maximise $Z = 6X + 3Y$

subject to the constraints

$$2X + 5Y \le 120$$
$$4X + 2Y \le 80$$
$$X \ge 0, Y \ge 0$$

E3) Maximise $z = 3x_1 + 2x_2$

subject to the constraints

$$x_1 - x_2 \le 1$$
$$x_1 + x_2 \ge 3$$
$$x_1 \ge 0, x_2 \ge 0$$

E4) Maximise $Z = x_1 + x_2$

subject to the constraints

$$x_1 + x_2 \le 1$$
$$-3x_1 + x_2 \ge 3$$
$$x_1 \ge 0, x_2 \ge 0$$

E5) A company manufactures two products X and Y, each of which requires three types of processing. The length of time for processing each unit and the profit per unit are given in the following table:

	Product X (hr/unit)	Product Y (hr/unit)	Available capacity per day (hr)
Process I	12	12	840
Process II	3	6	300
Process III	8	4	480
Profit per unit (`)	5	7	

How many units of each product should the company manufacture per day in order to maximise profit?

E6) A company produces soft drinks and has a contract requiring that a minimum of 80 units of chemical A and 60 units of chemical B go into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. The supplier X_1 has a mix of 4 units of A and 2 units of B that costs `10, and the supplier $X_2$ has a mix of 1 unit of A and 1 unit of B that costs `4. How many mixes from the company X_1 and company X_2 should the company purchase to honour contract requirement and yet minimise cost?

Before studying the next section, match your solution with the solution given in Sec. 2.7.

2.5 CANONICAL AND STANDARD FORMS OF LPP

After formulating a linear programming problem, our next step is to solve it. You have learnt in Secs. 2.3 and 2.4 that linear programming problems can be presented as problems of maximisation or minimisation with constraints such as $\leq$, = or $\geq$. In order to develop a standard procedure for solving LPPs, we need to convert them into well known forms, namely, the Canonical form and the Standard form. We now discuss the General LPP along with these two forms. The canonical form is especially used in the duality theory and the standard form is used to develop the general procedure for solving any linear programming problem. In order to understand these forms you also need to learn about slack and surplus variables.

2.5.1 General Linear Programming Problem

Let us formulate the general linear programming problem. Let Z be a linear function of n basic variables $X_1, X_2, \ldots, X_n$, which is to be maximised (or minimised). We write the problem as

$$\text{Maximise (or Minimise) } Z = C_1X_1 + C_2X_2 + \ldots + C_nX_n \qquad \ldots (1)$$

where $C_1, C_2, \ldots, C_n$ are known constants termed as **cost coefficients** of basic variables.

Let (a_{ij}) be an m × n real matrix of m×n constants a_{ij}'s and let $\{b_1, b_2, \ldots, b_m\}$ be a set of constants such that

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \; {}^3 \text{ or } \pounds \text{ or } = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \; {}^3 \text{ or } \pounds \text{ or } = b_2$$

$$\begin{matrix} . & . & . & . \\ . & . & . & . \end{matrix} \qquad \ldots (2)$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \; {}^3 \text{ or } \pounds \text{ or } = b_m$$

and $\qquad x_j \geq 0 \quad$ for all j=1, 2, …, n $\qquad \ldots (3)$

The linear function Z in equation (1) is called the **objective function**. The set of inequalities given in (2) is called **constraints** of a general LPP and the set of inequalities given in (3) are known as **non-negative restrictions** of a general LPP.

2.5.2 Slack and Surplus Variables

In general, if in any linear programming problem, we have a constraint of the type

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \; \pounds \; b_1, \quad \text{where } b_1 \; {}^3 \; 0$$

then this inequality can be converted into an equation by **adding** one non-negative variable s_1 to the left hand side. This new variable is called a **slack variable** and the constraints are transformed into the following equation:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 = b_1, \quad \text{where } s_1 \; {}^3 \; 0, b_1 \; {}^3 \; 0$$

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Thus, a **non-negative variable added to the left-hand side** of a less than or equal to $(\leq)$ type of constraint that converts it into an equation is called a **slack variable**. The value of this variable can be interpreted as the amount of unused resource.

Similarly, if in any linear programming problem, we have a constraint of the type

$$a_{11}y_1 + a_{12}y_2 + ... + a_{1n}y_n \geq b_1, \quad b_1 \geq 0$$

then this inequality can be converted into an equation by **subtracting** one non-negative variable s_1 from the left hand side. This new variable is called a **surplus variable** and the constraints are transformed into an equation as

$$a_{11}y_1 + a_{12}y_2 + ... + a_{1n}y_n - s_1 = b_1, \quad \text{where} \ s_1 \geq 0, b_1 \geq 0$$

Thus, a **non-negative variable subtracted from the left-hand side** of a greater than or equal to $(\geq)$ type of constraint that converts it into an equation is called a **surplus variable**. The value of this variable can be interpreted as the amount over and above the required minimum level.

Let us now obtain the canonical form of the LPP.

2.5.3 Canonical Form

The characteristics of the canonical form are explained in the following steps:

i) The objective function should be of maximisation form. If it is given in the minimisation form, it should be converted into maximisation form. Suppose $Z = ax + by$ is the given objective function which is to be minimised. Then, it is equivalent to maximising its negative function, i.e., $-Z$. The reason is that if a function $f(x)$ is minimum at the same point $x = a$ (say), then $-f(x)$ will be maximum at $x = a$ as shown in Fig. 2.7:
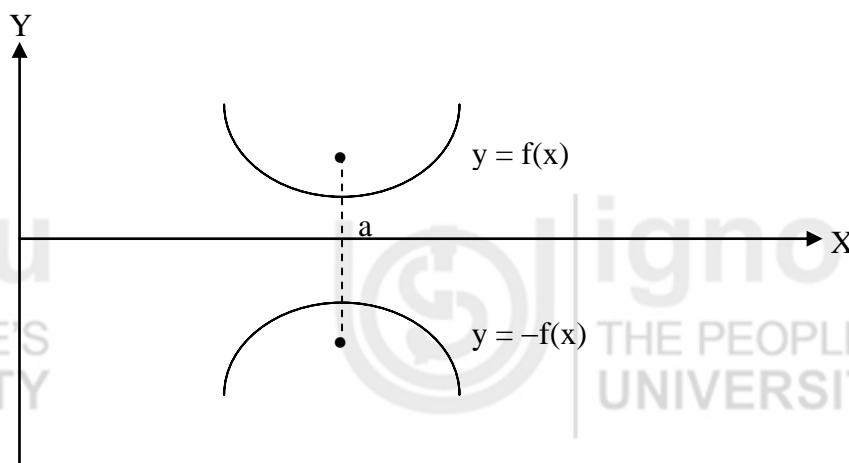

Fig. 2.7

Note: The graph of $y = f(x)$ is the mirror image of the graph of $y = -f(x)$ about the x-axis as shown Fig. 2.7 and $x = a$ is the point where $f(x)$ is minimum and $-f(x)$ is maximum.

ii) All constraints should be of "$\leq$" type, except for non-negative restrictions. Inequality of "$\geq$" type, if any, should be changed to an inequality of the "$\leq$" type by multiplying both sides of the inequality by -1. For example, suppose a given inequality is

$$2x + 3y \geq 5$$

Then it can be written as

$$-2x - 3y \leq -5$$

iii) All the variables should be non-negative. If a given variable is unrestricted in sign (i.e., positive, negative or zero), it can be written as a difference of two non-negative variables. Suppose x is unrestricted in sign, then x can be written as

$$x = x' - x'', \text{ where } x' \geq 0, x'' \geq 0.$$

Let us explain this point further: Suppose $x = -5$. Then it can be written as the difference of two non-negative numbers: $2 - 7$ (say). Here 2 and 7 are non-negative.

Example 3: Rewrite the following linear programming problem in canonical form:

$$\text{Minimise Z} = 2x_1 + x_2 + 4x_3$$

subject to the constraints:

$$-2x_1 + 4x_2 \leq 4$$
$$x_1 + 2x_2 + x_3 \geq 5$$
$$2x_1 + 3x_3 \leq 2$$
$$x_1, x_2 \,^3\, 0 \text{ and } x_3 \text{ is unrestricted in sign}$$

Solution: Here, the objective function is of the minimisation form. We rewrite it in the maximisation form as follows:

$$\text{Minimise Z} = 2x_1 + x_2 + 4x_3$$

Thus, we have to maximise $-Z = -2x_1 - x_2 - 4x_3$. So the problem becomes,

$$\text{Maximise Z}' = -2x_1 - x_2 - 4x_3, \text{ where } Z' = -Z$$

Now, the second constraint is of the type "$\geq$". Hence, to convert it into type "$\leq$", we multiply the inequality by -1 and write

$$-x_1 - 2x_2 - x_3 \leq -5$$

Other constraints are already in the desired form. But x_3 is unrestricted in sign. So we write

$$x_3 = x_3' - x_3'', \quad \text{where } x_3' \geq 0, \ x_3'' \geq 0.$$

The canonical form of the given problem, therefore, is

$$\text{Maximise Z}' = -2x_1 - x_2 - 4(x_3' - x_3''), \quad \text{where } Z' = -Z$$

subject to the constraints:

$$-2x_1 + 4x_2 \qquad\qquad £ \quad 4$$
$$-x_1 - 2x_2 - (x_3' - x_3'') \quad \leq \quad -5$$
$$2x_1 + 3(x_3' - x_3'') \qquad\quad \leq \quad 2$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3' \geq 0, \quad x_3'' \geq 0$$

We now discuss the standard form.

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2.5.4 Standard Form

The characteristics of the standard form are explained in the following steps:

i)   The objective function should be in the maximisation form as already explained in Sec. 2.5.3.

ii)   The right side element of each constraint should be non-negative. If it is negative, we multiply the inequality by -1.

iii) All constraints should be expressed in the form of equations, except for the non-negative restrictions. The inequalities should be converted into equations by augmenting the non-negative variables to the left side of each constraint. For inequalities of the "$\leq$" type, we add the slack variables. If the inequalities are of the "$\geq$" type, we subtract surplus variables.

Example 4: Rewrite the following linear programming problem in the standard form:

$$\text{Minimise } Z = 2x_1 + x_2 + 4x_3$$

subject to the constraints:

$$-2x_1 + 4x_2 \leq 4$$
$$x_1 + 2x_2 + x_3 \geq 5$$
$$2x_1 + 3x_3 \leq -2$$
$$x_1 \geq 0, \quad x_2 \geq 0 \quad x_3 \geq 0.$$

Solution: The objective function should be of maximisation form, i.e., we have to

$$\text{Maximise } Z' = -2x_1 - x_2 - 4x_3, \qquad \text{where } Z' = -Z$$

subject to the constraints:

$$-2x_1 + 4x_2 \leq 4$$
$$x_1 + 2x_2 + x_3 \geq 5$$
$$-2x_1 - 3x_3 \pounds 2 \qquad [\because \text{Right side should be non-negative}]$$
$$x_1 \geq 0, \quad x_2 \geq 0 \quad x_3 \geq 0.$$

Now, the inequalities are to be converted to equations. Note that the first and third inequalities are of the type "less than or equal to ($\leq$)".

Therefore, a slack variable is to be added to the left side of each of these inequalities. The second inequality is of the type "more than or equal to ($\geq$)". So a surplus variable is to be subtracted from the left side of this inequality.

Thus, the standard form of the given LPP is

$$\text{Max. } Z' = -2x_1 - x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3, \qquad \text{where } Z' = -Z$$

subject to the constraints:

$$-2x_1 + 4x_2 + s_1 = 4$$
$$x_1 + 2x_2 + x_3 - s_2 = 5$$
$$-2x_1 - 3x_3 + s_3 = 2$$
$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0, \quad s_3 \geq 0$$

Now, you should try to solve the following exercises.

E7)   Express the following LPP in canonical form:

Minimise $Z = x_1 - 2x_2 + x_3$

subject to the constraints:

$2x_1 + 3x_2 + 4x_3 \geq -4$

$3x_1 + 5x_2 + 2x_3 \geq 7$

$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$

E8)   Express the following LPP in standard form:

Minimise $Z = x_1 - 2x_2 + x_3$

subject to the constraints:

$2x_1 + 3x_2 + 4x_3 \geq -4$

$3x_1 + 5x_2 + 2x_3 \geq 7$

$x_1 \geq 0, \quad x_2 \geq 0 \quad x_3 \geq 0.$

You may like to match your solution with the solution given in Sec. 2.7.

Let us now summarise the main points which have been covered in this unit.

2.6   SUMMARY

1. A problem wherein the objective is to allot the limited available resources to the jobs in such a way as to optimise the overall effectiveness, i.e., minimise the total cost or maximise the total profit, is called **mathematical programming**. Mathematical programming wherein constraints are expressed as linear equalities/inequalities is called **linear programming**. Linear programming deals with the optimisation of the total effectiveness expressed as a linear function of decision variables, known as the **objective function**, subject to a set of linear equalities/inequalities known as **constraints**.

2. Steps involved in the **mathematical formulation** of a linear programming problem (LPP) are: i) Identification of the decision variables of the problem; ii) expressing the objective function as a linear function of the decision variables; iii) identifying the limited available resources to write the constraints as linear inequalities or equalities in terms of decision variables; iv) introducing the non-negative restrictions.

3. **Graphical method** is used to solve linear programming problems having **two decision variables**. The graphical method of solving linear programming programs comprises the following steps: i) The graphs are plotted for the equations corresponding to the given inequalities for constraints as well as restrictions; ii) The region corresponding to each inequality is shaded; iii) After shading the regions for each inequality, the most common shaded portion, i.e., the region obtained on superimposing all the shaded regions is determined. This is the region where all the given inequalities, including non-negative restrictions are

satisfied. This common region is known as the **feasible region** or the **solution set** or the **polygonal convex set**; iv) Each of the corner points (vertices) of the polygon is then determined; v) The objective function at each corner point is evaluated.

4. If the feasible region is bounded, then the **greatest value** obtained for the objective function at the corner points is the **optimum** value when the objective function is of **maximisation** form; and the **least value** obtained for the objective function at the corner points is the **optimum** value when the objective function is of **minimisation** form.

5. The feasible **region may not be bounded**. Then either there are additional hidden conditions which can be used to bound the region or there is no solution to the problem.

6. If the same optimum value occurs at two vertices, then there are **multiple optimal solutions** to the problem.

7. The characteristics of the **canonical form** are: i) Objective function should be of maximisation form. If it is given in minimisation form, it should be converted into maximisation form; ii) All the constraints should be of " $\leq$ " type, except for non-negative restrictions. Inequality of " $\geq$ " type, if any, should be changed to an inequality of the " $\leq$ " type; iii) All variables should be non-negative.

8. The characteristics of **standard form** are: i) The objective function should be of maximisation form; ii) The right side element of each constraint should be non-negative; iii) All constraints should be expressed in the form of equations, except for the non-negative restrictions by augmenting slack or surplus variables.

2.7 SOLUTIONS/ANSWERS

E1) Let x be the number of units of type P to be produced and y be the number of units of type Q to be produced. It is given that:

	Silver	**Gold**	**Profit**
Type P	4g	1g	`44 per unit
Type Q	1g	3g	`55 per unit
Available (at the most)	8g	9g	

Let Z be the profit function.

∴ The mathematical formulation of the given problem is

$$\text{Max. } Z = 44x + 55y$$

subject to the constraints:

$$4x + y \leq 8$$
$$x + 3y \leq 9$$
$$x \geq 0, y \geq 0.$$

E2) Max. $Z = 5x + 7y$

subject to the constrains:

$$12x + 12y \leq 840 \Rightarrow x + y \leq 70$$
$$3x + 6y \leq 300 \quad \Rightarrow x + 2y \leq 100$$
$$8x + 4y \leq 480 \quad \Rightarrow 2x + y \leq 120$$
$$x \geq 0 , \ y \geq 0$$

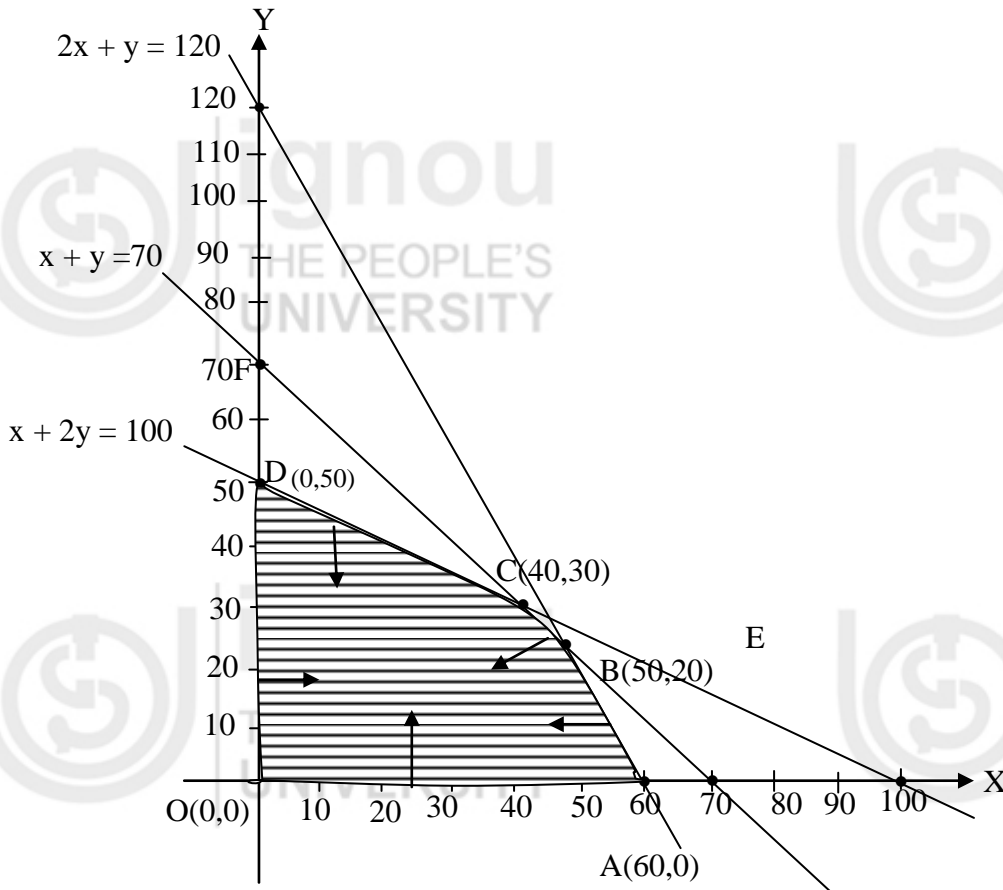**Fig. 2.8**

At O(0, 0), $Z = 5(0) + 7(0) = 0$

At A(60, 0), $Z = 5(60) + 7(0) = 300$

At B(50, 20), $Z = 5(50) + 7(20)$
$$= 250 + 140 = 390$$

At C(40, 30), $Z = 5(40) + 7(30)$
$$= 200 + 210 = 410$$

At D(0, 50), $Z = 5(0) + 7(50) = 350$

The maximum value of Z is 410 at C(40, 30), i.e., at x = 40, y = 30.

E3) Minimise $Z = 10x_1 + 4x_2$

subject to the constraints

$$4x_1 + x_2 \geq 80$$
$$2x_1 + x_2 \geq 60$$
$$x_1 \geq 0, \quad x_2 0$$

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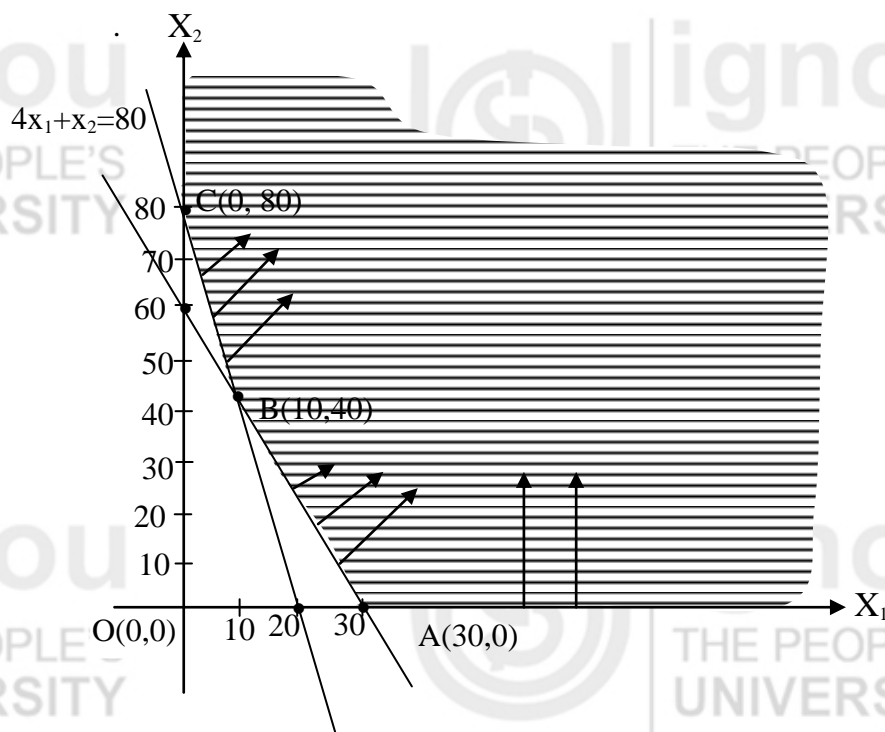
Fig. 2.9

At A(30, 0), $Z = 10(30) + 4(0) = 300$

At B(10, 40), $Z = 10(10) + 4(40)$

$$= 100 + 160 = 260$$

At C (0, 80), $Z = 10(0) + 4(80) = 320$

The minimum value of Z is 260 at B(10, 40), i.e., when $x_1 = 10$ and $x_2 = 40$.

Note: Had the objective function been of maximisation form, the problem would have the unbounded solution. This is because the values of x_1 and x_2 could be increased beyond any limit, which would result in higher and higher value of Z with no upper bound.

E4) At O(0, 0), $Z = 6(0) + 3(0) = 0$

At A(20, 0), $Z = 6(20) + 3(0) = 120$

At B(10, 20), $Z = 6(10) + 3(20) = 60 + 60 = 120$

At C(0, 24), $Z = 6(0) + 3(24) = 0 + 72 = 72$

The maximum value of Z is 120 at A(20, 0) as well as at B(10, 20). Therefore, the given LPP has multiple optimal solutions (Fig. 2.10).

Other points at which Z gives the same maximum value are given as

$$(20 \times t + 10 \times (1 - t), 0 \times t + 20 \times (1 - t))$$

or $(20t + 10 - 10t, 0 + 20 - 20t)$

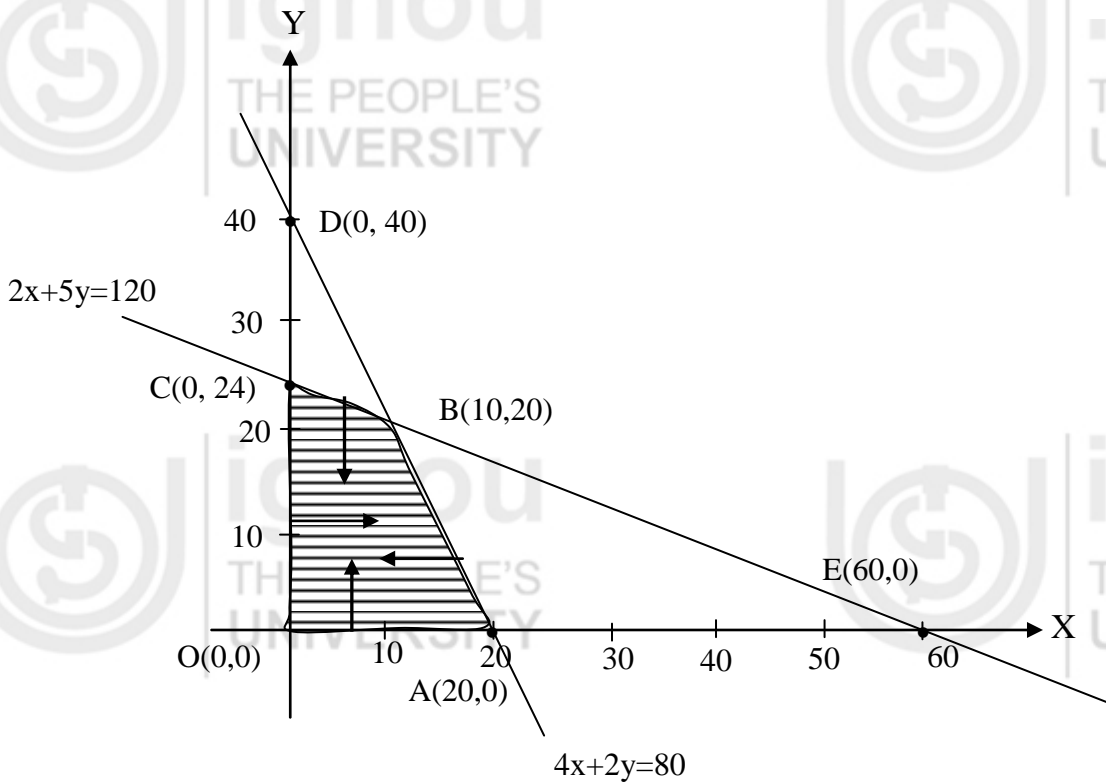or $(10 + 10t, 20 - 20t), 0 \le t \le 1.$

Fig. 2.10

E5) Here, we have to maximise the objective function. But the values of decision variables can be increased infinitely without violating the feasibility condition and hence the value of the objective function can be increased infinitely. Thus, the solution in this case is unbounded (Fig.2.11).
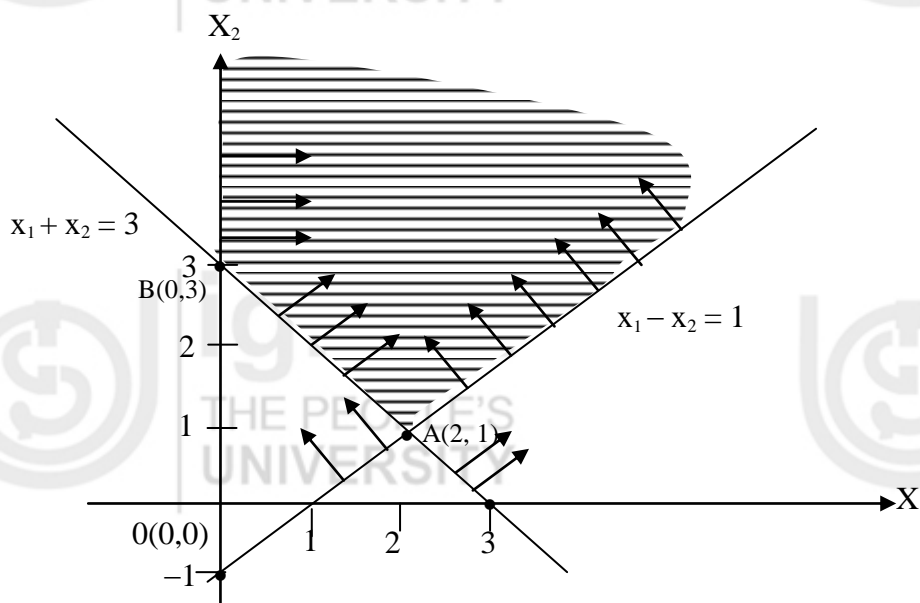

Fig. 2.11

Note: Had it been a minimisation problem, then it would have the optimum solution at one of the points A $(2, 1)$ and B $(3, 0)$ where it has the minimum value.

$$\text{At } A(2, 1), Z = 3(2) + 2(1) = 8$$

$$\text{At } B(3, 0), Z = 3(3) + 2(0) = 9$$

The minimum value is at A(2, 1), i.e., when $x_1 = 2$ and $x_2 = 1$.

E6) We find no common region for all the four graphs and hence the given LPP has no solution (Fig. 2.12).
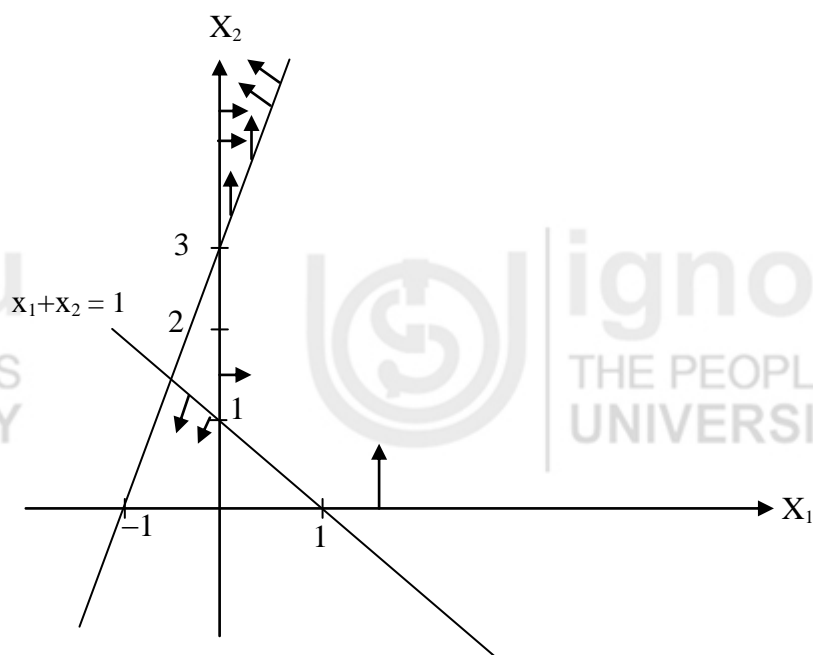

Fig. 2.12

E7) Maximise $Z' = -x_1 + 2x_2 - x_3$, where $Z' = -Z$

subject to the constraints

$$-2x_1 - 3x_2 - 4x_3 \leq 4$$

$$-3x_1 - 5x_2 - 2x_3 \leq -7$$

$$x_1 \geq 0, \quad x_2 \geq 0 \quad x_3 \geq 0.$$

E8) Maximise $Z' = -x_1 + 2x_2 - x_3 + 0s_1 + 0s_2$, where $Z' = -Z$

subject to the constraints

$$-2x_1 - 3x_2 - 4x_3 + s_1 = 4$$

$$3x_1 + 5x_2 + 2x_3 - s_2 = 7$$

$$x_1 \geq 0, \quad x_2 \geq 0 \quad x_3 \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0.$$