UNIT 10 DECISION MAKING PROCESS

Structure

10.1 INTRODUCTION

In Unit 9, we have introduced the components of decision making. You have learnt about courses of action, states of nature, payoff values, payoff matrix and opportunity loss table. We have also introduced various types of decision making environments, namely:

- Decision making under certainty;
- Decision making under uncertainty; and
- Decision making under risk.

The first two environments have been discussed in Unit 9 and we discuss the third environment in Sec. 10.2 of this unit. Note that the situations discussed in Unit 9 and Sec. 10.2 are the ones wherein the decision maker has to make only one decision. However, in many real life situations, the decision maker has to make a sequence of decisions. This means that he/she has to select an optimum course of action more than once, because the decision taken by the decision maker at a given stage generally leads to the next stage. Such multistage decision making problems are solved using decision tree analysis and roll-back technique and we discuss them in Sec. 10.3.

In the next unit, we shall introduce game theory and solve two-person-zero-sum games with saddle point.

Objectives

After studying this unit, you should be able to:

- select an optimum course of action by applying the expected monetary value (EMV) and expected opportunity loss (EOL) criteria;
- draw decision tree diagrams for multistage decision making problems; and
- solve multistage decision making problems by applying the roll-back technique.

10.2 DECISION MAKING UNDER RISK

In the environment, ‘decision making under certainty’ discussed in Unit 9, we had only one state of nature and so there was no question of decision making for different states of nature. In the environment of ‘decision making under uncertainty’ there were more than one states of nature but we had no data regarding the occurrence of the states of nature in terms of probabilities. For the environment of uncertainty, we have discussed five criteria to select an optimum course of action.

We now discuss decision making under risk. In this environment, we have additional information about the occurrence of the states of nature: we have the past data containing information about the occurrence of different states of nature. We know either:

1) The directly available probabilities of occurrence of different states of nature

   or

2) The frequency data for different states of nature which can be converted into probabilities using the relative frequency approach of probability (we have explained relative frequency approach of probability in Unit 2 of MST-003)

   or

3) Subjective probabilities on the basis of experience of individuals (we have explained subjective probabilities in Unit 2 of MST-003).

In all cases, we have with us the probabilities of occurrence of different states of nature in the environment of decision making under risk. The following criteria are used to select an optimum course of action in this environment:

(i) Expected Monetary Value (EMV) Criterion,

(ii) Expected Opportunity Loss (EOL) Criterion.

Let us now explain these criteria.

10.2.1 Expected Monetary Value (EMV) Criterion

In this criterion, we first form the payoff table or payoff matrix if it is not already given. Then for each course of action we find the expected value by multiplying the payoff value for each course of action one at a time with the probabilities of the corresponding state of nature. The resulting values are called the **expected monetary values** (EMVs). Next we select the maximum of the EMVs in the case of profit or gain, and the minimum of the EMVs in the case of loss or cost. In the case of profit or gain the course of action corresponding to the maximum expected monetary value is the optimum course of action according to this criterion. And in the case of loss or cost, the course of action corresponding to the minimum expected monetary value is the optimum course of action according to this criterion.

We follow the steps given below for the calculations of this criterion:

Step 1: If the payoff table or payoff matrix is already given, Step 1 is not needed. Otherwise, we first define the courses of action and states of nature and then obtain the payoff table or payoff matrix for the given situation. We also add one more column to the table indicating the probabilities of different states of nature.

Step 2: To obtain the expected monetary value (EMV) for each course of action, we multiply the payoff value of each course of action with the probability of the corresponding state of nature and then add the results. For example, let $x_{1j}, x_{2j}, ..., x_{mj}$ be the payoff values for the j^{th} course of action corresponding to m states of nature $N_1, N_2, ..., N_m$ and let $p_1, p_2, ..., p_m$ be the corresponding probabilities of these m states of nature, respectively. Then the expected monetary value (EMV) for the j^{th} course of action is given as:

$$\text{EMV for the } j^{th} \text{ course of action} = p_1 x_{1j} + p_2 x_{2j} + \ldots + p_m x_{mj} = \sum_{i=1}^{m} p_i x_{ij} \ldots (1)$$

Step 3: We select the maximum expected monetary value from among the expected monetary values obtained in Step 2 if payoff values represent profit or gain. We select the minimum EMV if the values represent loss or cost.

Step 4: Under this criterion, the course of action corresponding to the maximum (or minimum) EMV selected in Step 3 will be the optimum course of action.

Let us consider an example to explain the steps involved in this criterion.

Example 1: A vendor buys newspapers at the rate of Rs 3 per newspaper and sells at the rate of Rs 4 per newspaper. Assume that a newspaper which is not sold on the same day goes to scrap and pays him Rs 0.50 as regret value. The information for the past 200 days about the sale of the newspapers is shown in Table 10.1.

Table 10.1: Frequency Table representing Demand of Newpapers over past 200 Days

Number of Newspapers Demanded	200	204	206	208	Total
Number of Days	40	100	40	20	200

On the basis of this information, how many newspapers should be bought by the vendor so that his profit is maximum?

Solution:

Step 1: On the basis of the given information, it is clear that the vendor should buy either 200, 204, 206 or 208 newspapers per day. Since the number of newspapers he will buy is under his control, purchases of newspapers form the courses of action. If we denote these courses of action by A_1, A_2, A_3, A_4, respectively, then we have

$$A_1 = 200,\ A_2 = 204,\ A_3 = 206,\ A_4 = 208$$

But the future demand of newspapers on any day is not under his control. So the demands of newspapers form the states of nature. If we denote these states of nature by N_1, N_2, N_3, N_4, respectively, then we have

$$N_1 = 200,\ N_2 = 204,\ N_3 = 206,\ N_4 = 208$$

Now, the frequencies corresponding to these demands can be used to calculate probabilities (using the relative frequency approach of probability discussed in Unit 2 of MST-003) and for the states of nature N_1, N_2, N_3, N_4, these are given as:

$$\frac{40}{200} = 0.2,\ \frac{100}{200} = 0.5,\ \frac{40}{200} = 0.2,\ \frac{20}{200} = 0.1, \text{respectively.}$$

Let us now calculate the payoff values.

The cost of a newspaper = Rs 3

The selling price of a newspaper = Rs 4

Profit gained by the vendor on selling one newspaper = Rs (4 – 3)

= Rs 1

Loss to the vendor on an unsold newspaper = Rs (3 – 0.5) = Rs 2.5

$\therefore$ conditional profit = (profit on a sold newspaper) $\times$ (Number of newspapers sold) $-$ (Loss on an unsold newspaper) $\times$ (Number of newspapers unsold)

$= 1 \times$ (Number of newspapers sold) $-(2.5)\times$ (Number of newspapers unsold)

For example, for A_1 = 200, N_1 = 200,

conditional profit $= 1\times 200 - 2.5\times 0 = 200$

For A_1 = 200, N_2 = 204,

conditional profit $= 1\times 200 - 2.5\times 0 = 200$

Similarly, other calculations can be done as shown in Table 10.2 given below:

Table 10.2: Payoff Table for the Vendor

States of Nature	Courses of Action				
	Prob.	**200 (A_1)**	**204 (A_2)**	**206 (A_3)**	**208 (A_4)**
200 (N_1)	0.2	200	$1\times 200-2.5\times 4 = 190$	$1\times 200-2.5\times 6 = 185$	$1\times 200-2.5\times 8 = 180$
204 (N_2)	0.5	200	$1\times 204-2.5\times 0 = 204$	$1\times 204-2.5\times 2 = 199$	$1\times 204-2.5\times 4 = 194$
206 (N_3)	0.2	200	204	206	$1\times 206-2.5\times 2 = 201$
208 (N_4)	0.1	200	204	206	208

Step 2: In this step, we have to calculate the expected monetary value for each course of action. Expected monetary values (EMVs) for different courses of action are given by:

$$\text{EMV for } A_1 = 0.2\times 200 + 0.5\times 200 + 0.2\times 200 + 0.1\times 200$$
$$= (0.2+0.5+0.2+0.1)\times 200 = 1\times 200 = 200$$

$$\text{EMV for } A_2 = 0.2\times 190 + 0.5\times 204 + 0.2\times 204 + 0.1\times 204$$
$$= 38 + 102 + 40.8 + 20.4 = 201.2$$

$$\text{EMV for } A_3 = 0.2\times 185 + 0.5\times 199 + 0.2\times 206 + 0.1\times 206$$
$$= 37 + 99.5 + 41.2 + 20.6 = 198.3$$

$$\text{EMV for } A_4 = 0.2\times 180 + 0.5\times 194 + 0.2\times 201 + 0.1\times 208$$
$$= 36 + 97 + 40.2 + 20.8 = 194$$

Step 3: $\max\{200, 201.2, 198.3, 194\} = 201.2$

Step 4: Max EMV corresponds to the course of action A_2. Hence, under this criterion, A_2 is the optimum course of action.

10.2.2 Expected Opportunity Loss (EOL) Criterion

We have already explained how to obtain the opportunity loss table in Sec. 9.4.4 of Unit 9. This criterion suggests the course of action which minimizes our expected opportunity loss. The steps involved in the procedure of this criterion are the same as in the expected monetary value (EMV) criterion except that instead of dealing with payoff values, here we deal with opportunity loss values. We follow the steps explained below:

Step 1: If the payoff table or payoff matrix is already given, then Step 1 is not needed. Otherwise, we first define the courses of action, states of nature and then obtain the payoff table. We also add one more column indicating the probabilities of different states of nature.

Step 2: We obtain the opportunity loss values or regret values or conditional opportunity loss values for **each** state of nature by subtracting

- all payoff values corresponding to each state of nature from their respective maximum payoff values in case of profit or gain.

  or

- the minimum payoff value corresponding to each state of nature from all other payoff values of the states of nature in case of cost or loss.

The calculation has been explained in Tables 9.4 and 9.5, respectively, in Unit 9.

Step 3: Next, we obtain the expected opportunity loss values for each course of action by finding the sum of the products of the opportunity loss values of the course of action with the probabilities of the corresponding states of nature as explained in Example 2 given below.

Step 4: Finally, we select the minimum from among the expected opportunity loss values calculated in Step 3. The course of action corresponding to the minimum expected opportunity loss value will be the optimum course of action.

Example 2: A company has decided to buy an equipment for its powerhouse station located in a remote area. But this equipment contains an expensive part, which is subject to random failure. The failure data of the same part on the basis of the experience of other users is given in Table 10.3.

Table 10.3: Probability Distribution of the Random Variable, Number of Failures

Number of Failures	0	1	2	3 and more
Probability of Failure	0.70	0.20	0.10	0.00

If the company purchases spares of this part at the time of purchasing the equipment, it costs Rs 6000 per unit. If it is ordered after the failure of the part during its operation, the total cost including the cost of down time of the equipment is Rs 30000. Assume that there is no scrap value of the part. On the basis of this information, what should the suggestion of a decision maker to the company be in each of the following cases?

Down time cost includes all those costs and losses that occur during the entire period in which the system remains in non-operating mode.

(i) What is the optimal number of spares the company should buy at the time of purchasing the equipment using the EMV criterion?

(ii) What is the optimal course of action using the EOL criterion?

Solution: We first define the states of nature and courses of action. Then we shall obtain the payoff matrix. The purchases of spare parts of the equipment, (when the equipment is purchased or after the failure of the parts) are under the control of the company and so form the courses of action. If we denote these courses of action by A_1, A_2 and A_3, then we can write:

A_1 : No spare part was purchased at the time of purchasing the equipment

A_2 : One spare part was purchased at the time of purchasing the equipment

A_3 : Two spare parts were purchased at the time of purchasing the equipment

But the number of spares required by the company depends on the number of failures, which is a random event and not under the control of the company. So these numbers correspond to the states of nature. Since the probability of 3 or more failures is zero from Table 10.3, there are only three states of nature N_1, N_2, N_3 as given below:

N_1 : No failure occurs

N_2 : One failure occurs

N_3 : Two failures occur

Let x_{ij} denotes the payoff value corresponding to the i^{th} state of nature N_i and j^{th} course of action A_j. We first carry out the calculations shown in Table 10.4.

Table 10.4: Calculation(s) of Total Cost for Different Combinations of Courses of Action and States of Nature

States of Nature	Courses of Action	Cost when Spare(s) is/are Purchased at the Time when Equipment is Purchased	Cost when Company Orders after Occurrence of Failure	Total Cost
0	0	0	0	0
0	1	6000	0	6000
0	2	12000	0	12000
1	0	0	30000	30000
1	1	6000	0	6000
1	2	12000	0	12000
2	0	0	60000	60000
2	1	6000	30000	36000
2	2	12000	0	12000

Then we obtain the payoff table (Table 10.5) using the 9 payoff values given in the last column of Table 10.4.

Table 10.5: Payoff Table for the Data of Example 2

States of Nature	Probability	Courses of Action		
		A_1	A_2	A_3
N_1	0.70	0	6000	12000
N_2	0.20	30000	6000	12000
N_3	0.10	60000	36000	12000

(i) We now find the optimal number of spares that should be purchased by the company using the EMV criterion.

Step 1: We have already obtained the payoff table, which also includes a column indicating probabilities of different states of nature.

Step 2: Next, we calculate the expected monetary value (EMV) for each course of action.

$$\text{EMV for } A_1 = 0.70\times 0 + 0.20\times 30000 + 0.10\times 60000$$
$$= 0 + 6000 + 6000 = 12000$$

$$\text{EMV for } A_2 = 0.70\times 6000 + 0.20\times 6000 + 0.10\times 36000$$
$$= 4200 + 1200 + 3600 = 9000$$

$$\text{EMV for } A_3 = 0.70\times 12000 + 0.20\times 12000 + 0.10\times 12000$$
$$= 8400 + 2400 + 1200 = 12000$$

Step 3: Here payoff values represent a cost to the company. So, in this step, instead of the maximum EMV we select the minimum EMV from among the values obtained in Step 2.

$$\therefore \min\{12000, 9000, 12000\} = 9000$$

Step 4: Minimum EMV as obtained in Step 3 corresponds to the course of action A_2. Hence, the suggestion of the decision maker to the company under EMV criterion is that the company should buy one spare part of the equipment to minimise the cost.

(ii) Now, we use an EOL criterion to find the optimum course of action.

Step 1: We have already obtained the payoff table, which includes a column indicating the probabilities of different states of nature.

Step 2: In this step, we obtain the opportunity loss values (or regret values or conditional opportunity loss values) for each state of nature. In this case, the payoff values represent costs. So, the opportunity loss values are obtained by subtracting the minimum payoff value corresponding to each state of nature from all other payoff values of the states of nature as shown in Table 10.6.

Table 10.6: Opportunity Loss Table for the Company

States of Nature	Prob.	Courses of Action		
		A_1	A_2	A_3
N_1	0.70	0 – 0 = 0	6000 – 0 = 6000	12000 – 0 = 12000
N_2	0.20	30000 – 6000 = 24000	6000 – 6000 = 0	12000 – 6000 = 6000
N_3	0.10	60000 – 12000 = 48000	36000 – 12000 = 24000	12000 – 12000 = 0

Step 3: We now obtain the expected opportunity loss (EOL) value for each course of action as follows:

$$\text{EOL for } A_1 = 0.70\times 0 + 0.20\times 24000 + 0.10\times 48000$$
$$= 0 + 4800 + 4800 = 9600$$
$$\text{EOL for } A_2 = 0.70\times 6000 + 0.20\times 0 + 0.10\times 24000$$
$$= 4200 + 0 + 2400 = 6600$$
$$\text{EOL for } A_3 = 0.70\times 12000 + 0.20\times 6000 + 0.10\times 0$$
$$= 8400 + 1200 + 0 = 9600$$

Step 4: We select the minimum value from among the EOL values obtained in Step 3:

$$\min\{9600, 6600, 9600\} = 6600$$

This corresponds to the course of action A_2. Hence, the suggestion of the decision maker to the company under EOL criterion is that the company should buy one spare part of the equipment to minimise the cost.

Now, before giving you some exercises to solve for practice, we define two more terms commonly used in decision making, namely, EPPI and EVPI.

10.2.3 Expected Profit with Perfect Information (EPPI) and Expected Value of Perfect Information (EVPI)

We first define EPPI.

Expected Profit with Perfect Information (EPPI)

Expected profit with perfect information (EPPI) is the expected profit that we could make if we had perfect information about the occurrence of a particular state of nature. That is, it represents the expected profit under the environment of certainty and is defined as:

$$\text{EPPI} = \sum_i P(N_i) \times \begin{pmatrix} \text{payoff value of optimum course} \\ \text{of action for the state of nature } N_i \end{pmatrix} \qquad \text{... (2)}$$

Thus, EPPI is the maximum attainable value of EMV under perfect information about the occurrence of the states of nature.

We explain how to calculate EPPI in Example 3, after defining the expected value of perfect information (EVPI).

Expected Value of Perfect Information (EVPI)

The expected value of perfect information (EVPI) represents the excess of the amount earned under the environment of certainty over the expected monetary value (EMV) under the environment of risk. In other words, it represents the value of the maximum amount to be paid to get perfect information. If EMV* represents the expected monetary value under the environment of risk (i.e., probabilities are associated with states of nature) then EVPI is defined as

$$\text{EVPI} = \text{EPPI} - \text{EMV}^* \qquad \text{… (3)}$$

The following example will help you understand the calculations of both EPPI and EVPI.

Example 3: Using the information given in Example 2, obtain the expected profit with perfect information (EPPI) and the expected value of perfect information (EVPI).

Solution: First of all, we have to calculate the expected profit with perfect information (EPPI). By definition,

$$\text{EPPI} = \sum_i P(N_i) \times \begin{pmatrix} \text{payoff value of optimum course} \\ \text{of action for the state of nature } N_i \end{pmatrix}$$

The calculations are shown in Table 10.7:

Table 10.7: Calculations of Expected Profit with Perfect Information (EPPI)

States of Nature	Prob.	Optimum Course of Action	Payoff Value for Optimum Course of Action	Weighted Opportunity Loss
(1)	**(2)**	**(3)**	**(4)**	**(2) × (4)**
$\mathbf{N_1}$	0.70	A_1	0	$0.70\times0=0$
$\mathbf{N_2}$	0.20	A_2	6000	$0.20\times6000=1200$
$\mathbf{N_3}$	0.10	A_3	12000	$0.10\times12000=1200$

$$\therefore \text{EPPI} = -(0+1200+1200) \qquad \begin{bmatrix} \text{The negative sign is taken because} \\ \text{the payoff values represent costs.} \end{bmatrix}$$

$$= -2400$$

We now calculate the expected monetary value (EMV*) under the environment of risk.

In the solution of part (i) of Example 2, we have already obtained the expected monetary value (EMV) under the environment of risk as Rs 9000. But in Example 2, payoff values represent costs. So if we want to convert the EMV into profit, we have to put a negative sign with it.

Therefore, if EMV* represents the expected monetary value in the environment of risk, we have

EMV* = – Rs 9000

Now, we know that the expected value of perfect information (EVPI) is given by

EVPI = EPPI – EMV*

$\therefore$ EVPI $= -2400 - (-9000) = \text{Rs}\, 6600$

You may like to try the following exercises on decision making.

E1) A flower seller buys flowers at the rate of Rs 8 per dozen and sells them at the rate of Rs 15 per dozen. He has the following data on the demand for past 100 days:

Table 10.8: Frequency Distribution of Demand of Flowers for Past 100 Days

Demand (in Dozens)	50	51	52	53	Total
Number of Days	10	40	40	10	100

The seller has to give orders to the supplier on the previous day for the next day's sale. Also, the seller donates all the unsold flowers to an orphanage. On the basis of this information, as a decision maker, what will you, suggest to the flower seller about the order to the supplier on the previous day for meeting the next day's demand? Assume that he purchases and sells flowers only in dozens.

E2) Identify the optimum course of action under the expected opportunity loss (EOL) criterion for the data of E1.

E3) Using the data of E1, obtain the expected profit with perfect information (EPPI) and the expected value of perfect information (EVPI).

10.3 DECISION TREE ANALYSIS

So far we have studied decision criteria under different types of environment to solve decision making problems. But in all those criteria, we were identifying an optimum course of action among the available courses of action at a given point in time. That is, we were dealing with the types of situations in which the decision maker has to take the decision at a single stage. But there may exist situations wherein the decision maker has to make a sequence of decisions. That is, he/she has to select an optimum course of action more than once, because generally one decision taken by the decision maker leads to the next. **Decision tree analysis** is the most useful technique for solving such complex problems. You are familiar with the basic shape of a tree shown in Fig. 10.1. It looks like a sequence of branches and sub branches.

Fig. 10.1: Shape of a tree.

If different branches and sub branches of a tree are associated with courses of action, the states of nature, probabilities of the states of nature and payoff values of a given decision making problem, then the tree so obtained is known as a **decision tree**. But, generally we keep the direction of moving the decision trees from left to right, unlike the natural tree which moves in the vertical direction. A typical representation of the decision tree is shown in Fig. 10.2. Study Fig. 10.2 carefully for a better understanding of the decision tree and the terms given below:

Decision Point: A point in the pictorial representation of a decision tree having courses of action as immediate sub branches is known as a **decision point**. The symbol '□' (square) is used to represent a decision point as shown in Fig. 10.2. In other words, a point from which the branches representing the courses of action come out is known as a decision point.

Chance Point: A point in the pictorial presentation of decision tree having states of nature as immediate sub branches is known as a **chance point**, **chance node** or **event point**. The symbol '□' (circle) is used to represent an event point as shown in Fig. 10.2. In other words, a point from which the branches representing the states of nature come out is known as the **chance point** or **chance node**.

Root Node: We have already mentioned that, generally, we keep the direction of moving the decision tree from left to right and so the leftmost node is known as the **root node**. Root node is represented by the same symbol '□' (square) as the one used for the decision point.

Decision and Chance Branches: We know that any branch of a decision tree either represents a course of action or a state of nature. So the branches which represent the courses of action are known as **decision branches** and the branches which represent the states of nature are known as **chance branches**. Also, probabilities corresponding to different states of nature are written along the corresponding chance branches.

End Points: The points in the pictorial representation of a decision tree which are neither followed by any decision branch nor by any chance branch are known as its **end points.**

Terminal Branches: Any branch which ends at an **end point** is known as the **terminal branch**.

You should learn all the terms explained above with the help of Fig. 10. 2, before studying further.

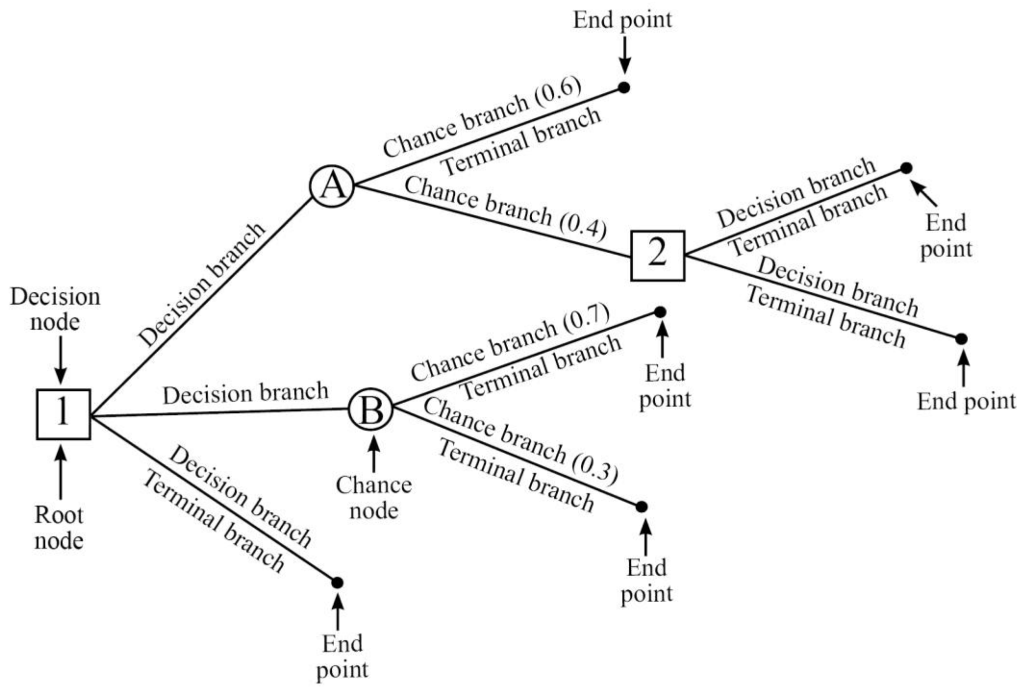

Fig. 10.2: Pictorial representation of a typical decision tree.

To solve multistage decision making problems using decision tree analysis, we apply the roll-back technique, which is explained below:

In the situations where the decision maker has to make a sequence of decisions in multiple stages, the ultimate consequence of the decision taken at the first stage depends on the output of all subsequent decisions that will be taken in future as a result of this decision. Also, the output of the last result is of primary concern for the decision maker. That is why, in decision tree analysis, we start evaluating the output of the decisions from the last decision and move in the backward direction until we reach the root node. Since we work in the backward direction, this technique is known as the **roll-back technique**.

Let us consider some examples to explain the application of the roll-back technique in numerical problems.

Example 4: Mr. Singh had to decide whether or not to drill a tubewell at his farm. In his village, only 40% of the tubewells were successful at 60 feet of depth. Some farmers who did not get water at 60 feet drilled up to 150 feet, but only 30% struck water at 150 feet. The cost of drilling is Rs 300 per foot. Mr. Singh estimated that he would have to pay Rs 20000 for the next 5 years, if he continued to buy water from his neighbour instead of drilling the tubewell, which would have a life of 5 years. Also, if he struck water, the total cost of drawing water for 5 years from his own tubewell would be Rs 3000. If this problem is given to a decision maker, what should his/her suggestion be to Mr. Singh? Assume that all amounts are calculated in terms of the present value.

Solution: So far you have learnt that for solving a decision making problem, we first have to state/identify the courses of action and states of nature. Recall the discussion about the courses of action and states of nature from Sec. 9.4 of Unit 9. The courses of action are under the control of the decision maker. But the states of nature are future outcomes and beyond the control of the decision maker. For this situation, you may like to think for a while as to which of the following situations is under the control of the decision maker and which one is beyond his/her control?

- Deciding whether to drill the tubewell or not.

- Status of water struck at a given depth.

We hope that you have got the correct answer. Deciding whether to drill the tubewell or not is under the control of the decision maker and so forms the courses of action. Thus, the courses of action and states of nature in this situation are as under:

Courses of Action

A_1: Not to drill any tubewell and continue buying water from the neighbour.

A_2: To drill the tubewell up to a depth of 60 feet.

A_3: To drill the tubewell up to a depth of 150 feet, if water is not struck water at 60 feet.

A_4: Not to drill the tubewell up to 150 feet, if water is not struck at 60 feet.

States of Nature

N_1: Water is struck at 60 feet.

N_2: Water is not struck at 60 feet.

N_3: Water is struck at 150 feet.

N_4: Water is not struck at 150 feet.

Further, it is given that

the cost of drilling = Rs 300 per foot

Therefore, the cost of drilling up to 60 feet $= \text{Rs}\,60 \times 300 = \text{Rs}\,18000,$ and

the cost of drilling further 90 feet $(= 150 - 60) = \text{Rs}\,90 \times 300 = \text{Rs}\,27000$

If Mr. Singh does not drill the tubewell, he will have to pay Rs 20000 to his neighbour, as water charges for the next 5 years. But if water is struck, then the cost of watering from his own tubewell for the next five years is Rs 3000.

Now, all these amounts represent costs. So to convert them into profit we have to put a negative sign before each amount. Further, it is a multistage decision making problem. So we have to solve it using the roll-back technique as explained below:

Roll-back Technique: We know that to solve the problem using roll-back technique, we have to draw the decision tree for the problem at hand. As explained earlier in this section, the decision tree is a pictorial representation of the sequence of the decisions. Here, Mr. Singh has to first decide whether he should drill the tubewell or not. This is represented by two branches coming out from the root node 1(see Fig. 10.3). Similarly, other decisions that lead from these branches are also shown in the decision tree in Fig. 10.3.

Let us now carry out the calculations. We know that in the roll-back technique, we start the calculations from right and move towards the left as explained below:

$$\begin{aligned}\text{Expected monetary value (EMV) at chance node B} &= 0.3 \times (-3000) \\ &\quad + 0.70 \times (-20000) \\ &= -900 - 14000 = -\text{Rs}\,14900\end{aligned}$$

But if Mr. Singh decides to drill upto 150 ft, then he has to pay Rs 27000 as drill charges for 90 ft (=150 – 60). Therefore,

total EMV at node $\text{B} = \text{Rs}(-14900 - 27000) = -\text{Rs}\,41900$

Since node 2 is a decision node, the EMV at node 2 $= \max\{-41900, -20,000\}$

$$= -\text{Rs}\,20000$$

Since node A is a chance node, the EMV at node A $= 0.4 \times (-3000) + 0.6 \times (-20000)$

$$= -\text{Rs}(1200 + 12000) = -\text{Rs}\,13200$$

But if Mr. Singh decides to drill up to 60 ft, then he has to pay Rs 18000 as drill charges. Therefore,

total EMV at node A $= \text{Rs}(-13200 - 18000) = -\text{Rs}\,31200$

Finally, node 1 is a decision node, therefore EMV at node 1 $= \max\{-31200, -20,000\}$

$$= -\text{Rs}\,20000$$

The decision tree representing this information is shown in Fig. 10.3.

Now, study node 1 in Fig. 10.3. The EMV at node 1 corresponds to the course of action A_1.

Hence, the suggestion of the decision maker to Mr. Singh should be:

Mr. Singh should not go for drilling a tubewell at all. He should continue to buy water from his neighbour.

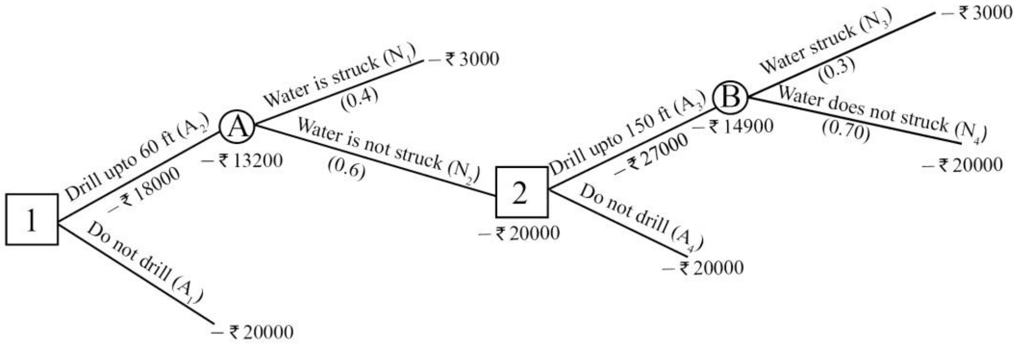

Fig. 10.3: Decision tree for Example 4.

Example 5: An investor has a certain amount of money to invest for five years and only three portfolios X, Y and Z are available to him for investing his money. The estimated profits per rupee on the basis of past experience are shown in Table 10.4. These are subject to the economic condition of the economy in future.

Table 10.9: Estimated Profits per Rupee Subject to theFuture Condition of Economy

Economic condition	Probability	Estimated Profit per INR on an Average basis during the Period of Next Five Years		
		In Case of Portfolio X	In Case of Portfolio Y	In Case of Portfolio Z
Economy Grows	0.5	2.5	3	2
Economy Remains Stable	0.2	1.5	2	1.5
Economy Declines	0.3	–0.5	–1.5	–1

Form a decision tree for the given situation and suggest an optimum portfolio to the investor.

Solution: We first define the courses of action and states of nature.

Courses of Action

A_1 : Invest in portfolio X

A_2 : Invest in portfolio Y

A_3 : Invest in portfolio Z

States of Nature

N_1 : Economy grows

N_2 : Economy remains stable

N_3 : Economy declines

Since all amounts represent profit, we shall keep their signs unchanged.

These types of problems (i.e., single stage decision making problems) have already been solved in the previous section without using the decision tree and roll-back technique. But here we want to solve it using the roll-back technique as explained below:

We know that under the roll-back technique, we start the calculations from right and move towards left as follows:

Since the nodes A, B, C are chance nodes, the expected monetary values (EMV) at these nodes are given as:

$$\text{EMV at node A} = 0.5\times 2.5 + 0.2\times 1.5 + 0.3\times(-0.5) = 1.25 + 0.3 - 0.15 = \text{Rs}1.40$$

$$\text{EMV at node B} = 0.5\times 3.0 + 0.2\times 2.0 + 0.3\times(-1.5) = 1.5 + 0.40 - 0.45 = \text{Rs}1.45$$

$$\text{EMV at node C} = 0.5\times 2 + 0.2\times 1.5 + 0.3\times(-1) = 1 + 0.30 - 0.30 = \text{Rs}1.0$$

$$\text{Since node 1 is the decision node, the EMV at node1} = \max\{1.40, 1.45, 1\} = \text{Rs}1.45$$

Hence, portfolio B is the optimum portfolio for the investor as per the information given in this case.

The decision tree representing the entire information is shown in Fig 10.4.

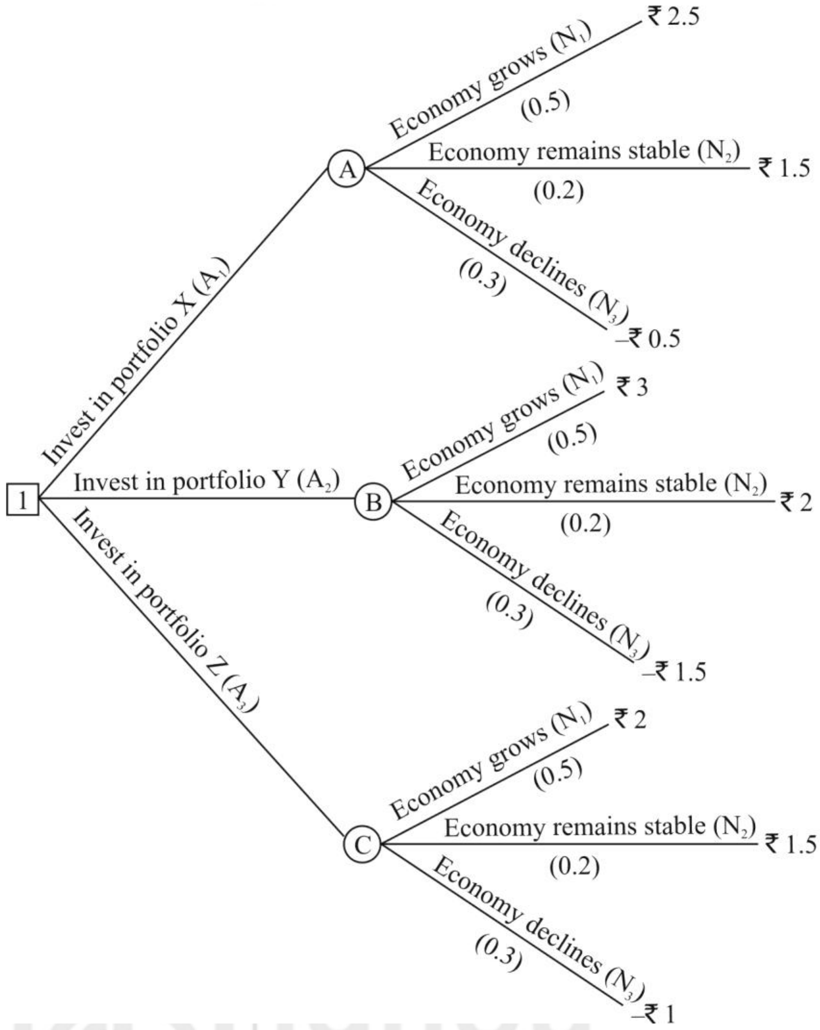

Fig. 10.4: Decision tree for the investor.

You may like to try the following exercises for practice.

E4) Company X is planning to launch a new product of direct-to-home (DTH) cables which can be introduced initially either in eastern Madhya Pradesh (MP) or in the entire state of MP. If the product is introduced only in eastern MP, the investment outlay will be Rs 80 lakh for 5 years. After one year, the company can evaluate the project to determine whether it should cover the entire state of MP. For such an expansion, it will have to incur an additional investment of Rs 40 lakh for 4 years. To introduce the product in the entire state of MP right at the beginning would involve an outlay of Rs 140 lakh for five years. The product, in any case, will have a life of 5 years after which it will have zero net value.

If the product is introduced only in eastern MP, demand would be high, moderate or low with the probabilities of 0.5, 0.3 and 0.2, respectively, with annual cash inflow of Rs 30 lakh, Rs 25 lakh and Rs 20 lakh, respectively.

If the product is introduced in the entire state of MP right in the beginning, the demand would be high, moderate or low with the probabilities of 0.4, 0.5 and 0.1, respectively, with annual cash inflows of Rs 50 lakh, 40 lakh and Rs 35 lakh, respectively.

Based on the observed demand in eastern MP, if the product is introduced in the entire state of MP, the following probabilities would exist for high, moderate and low demands:

Table 10.10: Probabilities for High, Moderate and Low Demands if the Product is Introduced in the Entire State of MP as per Observed Demand in Eastern MP

Eastern MP	Entire MP		
	High Demand	Moderate Demand	Low Demand
High Demand	0.7	0.2	0.1
Moderate Demand	0.6	0.2	0.2
Low Demand	0.3	0.2	0.5

On the basis of this information, form a decision tree and using the roll-back technique, obtain the optimal path that should be followed by the company. Assume that all amounts are calculated in terms of the present value.

E5) Suppose one of your friends has a certain amount of money to invest for a period of one year and he/she wants to opt for one of the following three choices available to him/her:

- To lend the money to his neighbour at the interest rate of 12% p.a.
- To invest in real estate.
- To invest in gold.

On the basis of past experience, he/she has the following information about the returns from real estate and gold:

Real Estate: Return may vary in percentage depending on the state of the market as given below:

Table 10.11: Probability of Earning in Percentage for an Investment in Real Estate

Earning in Percentage	Probability
40	0.2
30	0.4
10	0.3
5	0.1

If he/she invests in gold, the chances of earning 15% or 9% return on the investment are 0.6 and 0.4, respectively. On the basis of this information, suggest an optimum option to your friend.

Let us now summarise the main points that we have discussed in this unit.

10.4 SUMMARY

1) We find the **expected monetary values** for each course of action by multiplying the payoff value for each course of action with the probability of the corresponding state of nature and taking the sum of these products.

2) **Expected opportunity loss** criterion suggests the course of action that minimises our expected opportunity loss.

3) If different branches and sub branches of a tree are associated with courses of action, states of nature, probabilities of the states of nature and payoff values of a given decision making problem, then the tree so obtained is known as a **decision tree**.

4) **Decision Point:** A point in the pictorial representation of a decision tree having courses of action as immediate sub branches is known as a decision point.

5) **Chance Point:** A point in the pictorial representation of decision tree having states of nature as immediate sub branches is known as chance point, chance node or event point.

6) **Root Node:** Generally, we keep the direction of moving the decision tree from left to right and so the leftmost node is known as the root node.

7) **Decision and Chance Branches:** The branches that represent courses of action are known as **decision branches** and the branches that represent states of nature are known as **chance branches**.

8) **End Points:** The points in the pictorial representation of a decision which are neither followed by any decision branch nor by any chance branch are known as end points of the decision tree.

10.5 SOLUTIONS/ANSWERS

E1) The steps involved in the EMV criterion are explained below:

Step 1: We first define the courses of action, states of nature and obtain a payoff matrix. The quantity of flowers the seller will order to the supplier on the previous day for the next day's sale is under the control of the seller. So the flowers he will buy (in dozens) form the courses of action. If $A_1, A_2, A_3, A_4,$ denote the courses of action, then

$$A_1 = 50,\ A_2 = 51,\ A_3 = 52,\ A_4 = 53$$

But the flowers he will sell (in dozens) on the next day is not under the control of the seller. Hence, the next day's demand form the states of nature. If N_1, N_2, N_3, N_4 denote the states of nature, then

$$N_1 = 50,\ N_2 = 51,\ N_3 = 52,\ N_4 = 53$$

The probabilities for each state of nature are obtained by dividing the respective frequencies by the sum of all frequencies (using the relative frequency approach). Hence, we obtain 0.10, 0.40, 0.40, and 0.10, respectively, as probabilities for these states of nature.

Cost price of 1 dozen flowers = Rs 8

Selling price of 1 dozen flowers = Rs 15

Profit on selling 1 dozen flowers = Rs (15 – 8) = Rs 7

Loss per dozen on unsold flowers = Rs 8

The payoff values can be obtained as

$$\begin{aligned}\text{Payoff value} &= 7\times(\text{Number of flowers sold in dozens})\\ &\quad -8\times(\text{Number of unsold flowers in dozens})\end{aligned}$$

Calculation of payoff values for different combinations of courses of action and states of nature are shown in Table 10.12.

Table 10.12: Payoff Table for the Flower Seller

States of Nature	Prob.	Courses of Action			
		$A_1(50)$	$A_2(51)$	$A_3(52)$	$A_4(53)$
$N_1(50)$	0.10	$7\times50-8\times0$ $=350$	$7\times50-8\times1$ $=342$	$7\times50-8\times2$ $=334$	$7\times50-8\times3$ $=326$
$N_2(51)$	0.40	$7\times50-8\times0$ $=350$	$7\times51-8\times0$ $=357$	$7\times51-8\times1$ $=349$	$7\times51-8\times2$ $=341$
$N_3(52)$	0.40	$7\times50-8\times0$ $=350$	$7\times51-8\times0$ $=357$	$7\times52-8\times0$ $=364$	$7\times52-8\times1$ $=356$
$N_4(53)$	0.10	$7\times50-8\times0$ $=350$	$7\times51-8\times0$ $=357$	$7\times52-8\times0$ $=364$	$7\times53-8\times0$ $=371$

Step 2: We now obtain the expected monetary value (EMV) for each course of action as follows:

$$\text{EMV for } A_1 = 0.10\times350+0.40\times350+0.40\times350+0.10\times350$$
$$=(0.10+0.40+0.40+0.10)\times350 = 1\times350 = 350$$

$$\text{EMV for } A_2 = 0.10\times342+0.40\times357+0.40\times357+0.15\times357$$
$$=34.2+(0.40+0.40+0.10)\times357 = 34.2+0.9\times357$$
$$=34.2+321.3=355.5$$

$$\text{EMV for } A_3 = 0.10\times334+0.40\times349+0.40\times364+0.10\times364$$
$$=33.4+139.6+145.6+36.4=355$$

$$\text{EMV for } A_4 = 0.10\times326+0.40\times341+0.40\times356+0.10\times371$$
$$=32.6+136.4+142.4+37.1=348.5$$

Step 3: $\text{Max}\{350,\ 355.5,\ 355,\ 348.5\} = 355.5$

Step 4: The maximum EMV corresponds to the course of action A_2. Hence, under this criterion, A_2 is the optimum course of action.

E 2) To identify an optimum course of action using the expected opportunity loss (EOL) criterion, we follow the steps given below:

Step 1: We first define the courses of action, states of nature and obtain the payoff table with one additional column of probabilities. But we have already done these calculations while solving E1.

Step 2: We now obtain the opportunity loss matrix. The payoff values in the payoff table obtained in Step 1 represent profit of the flower seller. So to obtain the opportunity loss table, we first calculate the maximum payoff value (Max PV) for each state of nature as follows:

$$\text{Max PV for state of nature } N_1 = \max\{350,342,334,326\} = 350$$
$$\text{Max PV for state of nature } N_2 = \max\{350,357,349,341\} = 357$$
$$\text{Max PV for state of nature } N_3 = \max\{350,357,364,356\} = 364$$
$$\text{Max PV for state of nature } N_4 = \max\{350,357,364,371\} = 371$$

Then we subtract all payoff values corresponding to different courses of action for the state of nature N_1 from 350. We do the same calculation for the states of nature N_2, N_3 and N_4 by subtracting the payoffs from 357, 364 and 371, respectively, as shown in Table 10.13.

Table 10.13: Opportunity Loss Table for the Flower Seller

States of Nature	Prob.	Courses of Action			
		$A_1(50)$	$A_2(51)$	$A_3(52)$	$A_4(53)$
$N_1(50)$	0.10	$350-350=0$	$350-342=8$	$350-334=16$	$350-326=24$
$N_2(51)$	0.40	$357-350=7$	$357-357=0$	$357-349=8$	$357-341=16$
$N_3(52)$	0.40	$364-350=14$	$364-357=7$	$364-364=0$	$364-356=8$
$N_4(53)$	0.10	$371-350=21$	$371-357=14$	$371-364=7$	$371-371=0$

Step 3: We now obtain the expected opportunity loss (EOL) values for each course of action as follows:

$$\text{EOL for } A_1 = 0.10\times 0+0.40\times 7+0.40\times 14+0.10\times 21$$
$$=0+2.8+5.6+2.1=10.5$$

$$\text{EOL for } A_2 = 0.10\times 8+0.40\times 0+0.40\times 7+0.10\times 14$$
$$=0.8+0+2.8+1.4=5$$

$$\text{EOL for } A_3 = 0.10\times 16+0.40\times 8+0.40\times 0+0.10\times 7$$
$$=1.6+3.2+0+0.7=5.5$$

$$\text{EOL for } A_4 = 0.10\times 24+0.40\times 16+0.40\times 8+0.10\times 0$$
$$=2.4+6.4+3.2+0=12$$

Step 4: We select the minimum from among the EOL values obtained in Step 3:

$$\text{Min}\{10.5,\ 5,\ 5.5,\ 12\}=5$$

This corresponds to the course of action A_2. Hence, the optimum course of action under the EOL criterion is A_2.

E3) First of all, we have to calculate expected profit with perfect information (EPPI) and by definition it is given as

$$\text{EPPI}=\sum_i P(N_i)\times\begin{pmatrix}\text{payoff value of optimum course}\\ \text{of action for the state of nature } N_i\end{pmatrix}$$

The calculations are shown in Table 10.14.

Table 10.14: Calculation for Expected Profit with Perfect Information (EPPI)

States of Nature	Prob.	Optimum Course of Action	Payoff Value for Optimum Course of Action	Weighted Opportunity Loss
(1)	(2)	(3)	(4)	(2) × (4)
N_1	0.10	A_1	350	$0.10\times 350=35$
N_2	0.40	A_2	357	$0.40\times 357=142.8$
N_3	0.40	A_3	364	$0.40\times 364=145.6$
N_4	0.10	A_4	371	$0.10\times 371=37.1$

Thus, the expected profit with perfect information (EPPI) is given by

$$\text{EPPI}=35+142.8+145.6+37.1=360.5$$

The expected monetary value (EMV*) under the environment of risk has already been calculated in the solution of E1, i.e.,

$EMV^* = 348.5.$

Thus, the expected value of perfect information (EVPI) is given by

$$EVPI = EPPI - EMV^* = 360.5 - 348.5 = \text{Rs } 12$$

E4) We first define the courses of action and states of nature:

Courses of Action

A_1: Launch the new product in eastern MP

A_2: Launch the new product in entire MP

A_3: Expand the product in entire MP after launching it in eastern MP

A_4: Do not expand the product in entire MP after launching it in eastern MP

States of Nature

N_1: Demand is high

N_2: Demand is moderate

N_3: Demand is low

We now use the roll-back technique to obtain an optimum sequence of courses of action.

We know that under the roll-back technique, we start the calculations from right and move towards left as explained below:

Since node C is a chance node, the expected monetary value (EMV) per annum at node C

$$= \text{Rs}\left[0.7 \times 50 + 0.2 \times 40 + 0.1 \times 35\right]\text{lakh} = \text{Rs } 46.5 \text{ lakh}$$

$$\text{EMV at chance node C for 4 years} = \text{Rs}\,(4 \times 46.5 - 40)\,\text{lakh} = 146\,\text{lakh}$$

Since node 2 is a decision node, EMV at node 2

$$= \text{Rs}\left[\max\{146, 0\} + 30\,\text{lakh cash inflow}\right]\text{lakh} = \text{Rs}\,176\,\text{lakh}$$

$$\text{EMV per annum at chance node D} = \text{Rs}\left[0.6 \times 50 + 0.2 \times 40 + 0.2 \times 35\right]$$

$$= \text{Rs } 45\,\text{lakh}$$

$$\therefore \text{EMV at chance node D for 4 years} = \text{Rs}\left[4 \times 45 - 40\right]\text{lakh} = 140\,\text{lakh}$$

$$\text{EMV at decision node 3} = \text{Rs}\left[\max\{140, 0\} + 25\,\text{lakhs cash inflow}\right]$$

$$= \text{Rs}\left[140 + 25\right]\text{lakh} = \text{Rs}\,165\,\text{lakh}$$

$$\text{EMV per annum at chance node E} = \text{Rs}\left[0.3 \times 50 + 0.2 \times 40 + 0.5 \times 35\right]\text{lakh}$$

$$= \text{Rs } 40.5\,\text{lakh}$$

$$\text{EMV at chance node E for 4 years} = \text{Rs}\left[4 \times 40.5 - 40\right]\text{lakh} = \text{Rs}\,122\,\text{lakh}$$

$$\text{EMV at decision node 4} = \text{Rs}\left[\max\{122, 0\} + 20\,\text{lakh cash inflow}\right]$$

$$= \text{Rs}\left[122 + 20\right]\text{lakh} = 142\,\text{lakh}$$

Since node A is a chance node, EMV

$$\text{at node A} = \text{Rs}\left[0.5 \times 176 + 0.3 \times 165 + 0.2 \times 142 - 80\right]\text{lakh}$$

$= \text{Rs } 85.9 \text{ lakh}$

Since node B is a chance node, EMV per annum

at node $B = \text{Rs}\left[0.4\times 50+0.5\times 40+0.1\times 35\right]\text{lakh} = \text{Rs}\,43.5\,\text{lakh}$

EMV at chance node B for 5 years $= \text{Rs}\left[5\times 43.5 - 140\right]\text{lakh} = \text{Rs}\,77.5\,\text{lakh}$

Finally, since node 1 is a decision node, EMV at node $1 = \max\{85.9, 77.5\}$

$= 85.9\,\text{lakh}$

Hence, the optimum path for the company X is as follows:

- The company should launch the product initially in eastern MP.
- If the demand of the product is high, the company should launch the product in entire MP.
- In the case of moderate or low demand, the company should not launch the product in entire MP.

All the information given above is shown in the decision tree diagram (Fig. 10. 5).

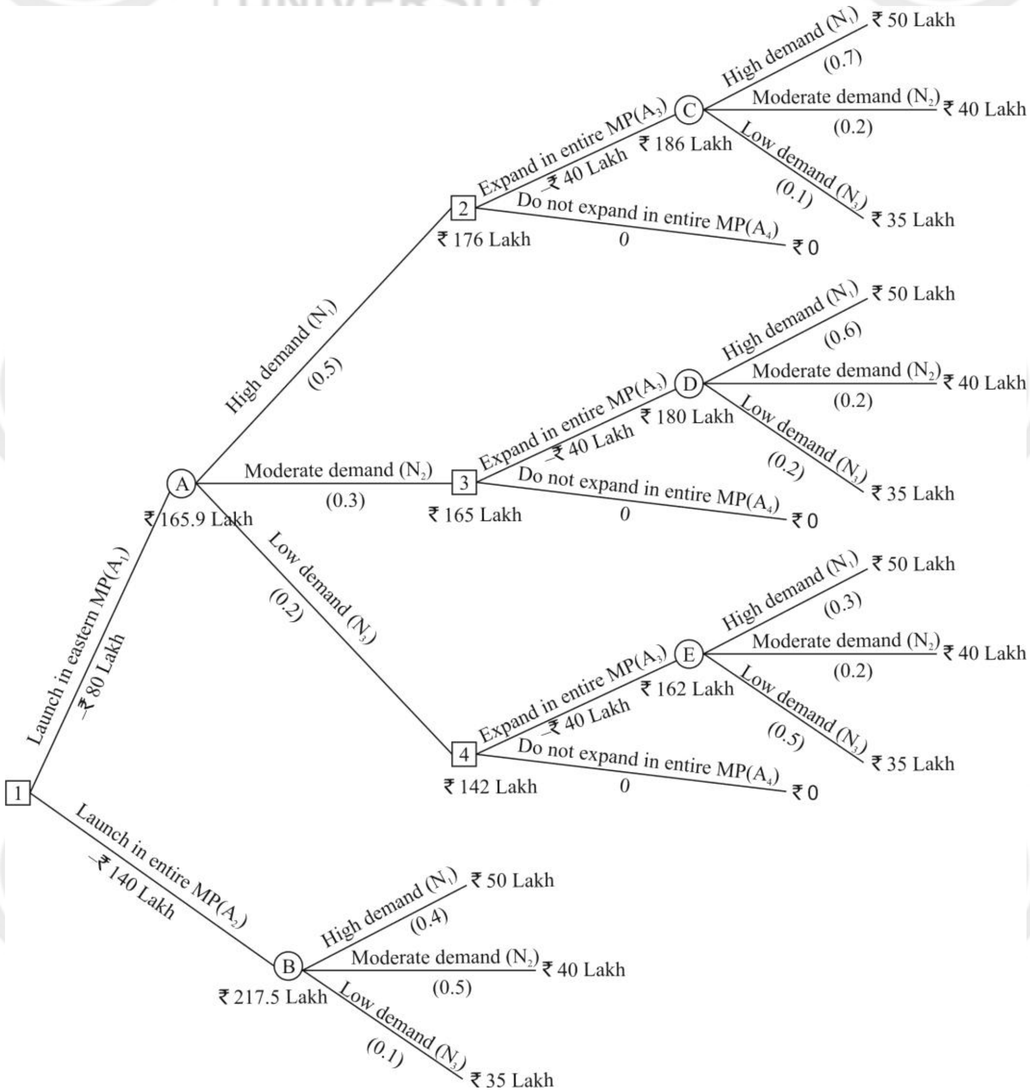

Fig. 10.5: Decision tree for company X.

Note: The above calculations can also be expressed in the form of a table, as shown in Table 10.15.

Table 10.15: Calculations Involved in Roll-back Technique

Node	Course of Action	States of Nature	Prob.	Conditional Return (in lakhs)	Expected Monetary Value (EMV) (in lakhs)
2	Expand in Entire MP	High Moderate Low	0.7 0.2 0.1	50 40 35	0.7×50 = 35 (per annum) 0.2×40 = 8 (per annum) 0.1×35 = 3.5 (per annum) Total = 46.5 (per annum) For 4 years = 4×46.5=186 Cost = – 40 EMV = 186 – 40 = 146
	Do not Expand in Entire MP			0	0
3	Expand in Entire MP	High Moderate Low	0.6 0.2 0.2	50 40 35	0.6×50 = 30 (per annum) 0.2×40 = 8 (per annum) 0.2×35 = 7 (per annum) Total = 45 (per annum) For 4 years = $4\times 45 = 180$ Cost = – 40 EMV = 180 – 40 = 140
	Do not Expand in Entire MP			0	0
4	Expand in Entire MP	High Moderate Low	0.3 0.2 0.5	50 40 35	0.3×50 = 15 (per annum) 0.2×40 = 8 (per annum) 0.5×35 =17.5 (per annum) Total = 40.5 (per annum) For 4 years $4\times 40.5 = 162$ Cost = – 40 EMV = 162 – 40 = 122
	Do not Expand in Entire MP			0	0
1	Launch in Eastern M.P.	High Moderate Low	0.5 0.3 0.2	30 +146 = 176 25+140 = 165 20 + 122 = 142	0.5×176 = 88 (for 5 years) 0.3×165 = 49.5 (for 5 years) 0.2×142 = 28.4 (for 5 years) Total = 165.9 (for 5 years) Cost = – 80 EMV = 165.9 – 80 = **85.9**
	Launch in Entire MP	High Moderate Low	0.4 0.5 0.1	50 40 35	0.4×50 = 20 (per annum) 0.5×40 = 20 (per annum) 0.1×35 = 3.5 (per annum) Total = 43.5 (per annum) For 5 years = 5×43.5 = 217.5 Cost = – 140 EMV = 217.5 – 140 = **77.5**

Since node1 is a decision node, the EMV at node1 $= \max\{85.9, 77.5\} = 85.9$.

This is the same as we calculated without using the table. So the optimum path for the company is the same as suggested earlier.

Now, if you feel comfortable in doing the calculations without using a table, you are free to do so. However, if you feel more comfortable in doing the calculations in the format of a table as shown above, you may adopt the same method.

E 5) First of all, we define the courses of action and states of nature for this situation.

Courses of Action:

A_1: Lend the money to his/her neighbour

A_2 : Invest in real estate

A_3 : Invest in gold

States of Nature

In case money is loaned to his/her neighbour, there is only one state of nature: He/she will earn 12% in terms of interest for a period of one year.

In the case of investment in real estate there are four states of nature:

N_1 : Property value increases by 40% after one year of investment

N_2 : Property value increases by 30% after one year of investment

N_3 : Property value increases by 10% after one year of investment

N_4 : Property value increases by 5% after one year of investment

In the case of investment in gold there are two states of nature

N_5 : Rate of gold increases by 15% after one year of investment

N_6 : Rate of gold increases by 9% after one year of investment

Before solving this problem using the roll-back technique, let us assume that the money invested is Rs 100. The outcomes of different states of nature on the basis of an investment of Rs 100 are shown against each terminal branch in Fig. 10.6.

Roll-back Technique: We know that under the roll-back technique, we start the calculations from right and move towards left as explained below:

Node A is a chance node, so EMV at node A = $\text{Rs}1 \times 112 = \text{Rs}112$

Similarly,

$$\begin{aligned}\text{EMV at chance node B} &= \text{Rs}\left(0.2\times140+0.4\times130+0.3\times110+0.1\times105\right)\\ &= \text{Rs}\left(28+52+33+10.5\right)=\text{Rs}123.5\end{aligned}$$

$$\begin{aligned}\text{EMV at chance node C} &= \text{Rs}\left(0.6\times115+0.4\times109\right)\\ &= \text{Rs}\left(69+43.6\right)=\text{Rs}112.6\end{aligned}$$

Now, node 1 is a decision node. So the optimum decision will correspond to the maximum EMV among the EMVs at chance nodes A, B, C.

So $\max\{112, 123.5, 112.6\} = 123.5.$

The decision tree representing this information is shown in Fig. 10.6.

Hence, as a decision maker, you should suggest that your friend should invest in real estate.

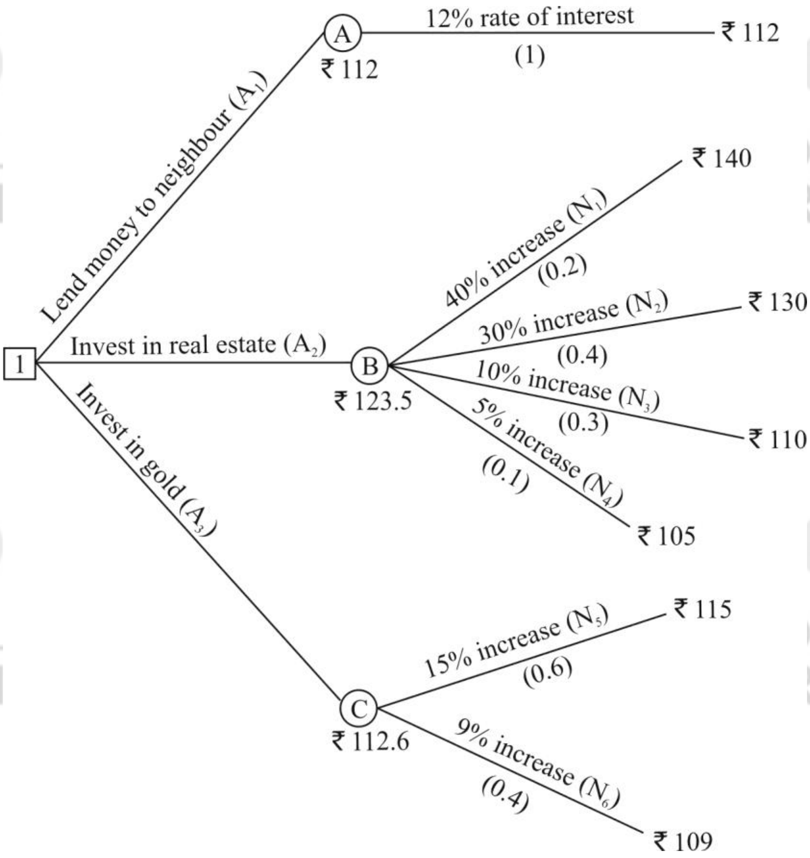

Fig. 10.6: Decision tree for the investment of your friend.