UNIT 2 CONTROL CHARTS FOR VARIABLES

Structure

2.1 INTRODUCTION

In Unit 1, you have learnt that the **control chart** is the most important tool for process control. With the help of control chart, we quickly detect the occurrence of assignable causes of variation and can take corrective action to eliminate them.

Several control charts may be designed for different situations. Each chart has its own field of applications and its own advantages and disadvantages. However, all control charts have some characteristics in common and are interpreted in almost the same manner. Generally, quality characteristics are of two types (variables and attributes). So control charts are broadly classified into two categories:

1. Control charts for variables (Control charts for measurable characteristics)

2. Control charts for attributes (Control charts for non-measurable characteristics)

In this unit, you study the construction and interpretation of control chart for variables. We introduce the general technique of constructing a control chart in Sec. 2.2. We also discuss control charts for variables, i.e., control chart for mean and control chart for variability in Secs. 2.3 to 2.6. Finally, we introduce the process capability analysis, which is used to check whether the process is able to meet the specifications or not in Sec. 2.7. The control charts for attributes are taken up in Units 3 and 4.

Objectives

After studying this unit, you should be able to:

- explain different types of control charts for variables;

- decide which control chart for variables to use in a given situation;

- describe the procedure for constructing a control chart;

- construct and interpret the control chart for process mean $(\overline{X}$-chart);

- construct and interpret the control chart for process variability (R-chart and S-chart);

- define the concepts of specification limits, natural tolerance limits and process capability.

2.2 CONTROL CHART TECHNIQUE

There are two types of control charts: one for measurable characteristics known as **control charts for variables** and second for non-measurable characteristics known as **control charts for attributes**. The technique for drawing a control chart is the same for all types.

The main steps are as follows:

1. Select the Quality Characteristic

In any manufacturing plant (small or big) of a product, there exist a large number of quality characteristics in the product. A single product/item/unit usually has several quality characteristics such as weight, length, width, strength, thickness, etc. It is, therefore, impossible to construct a control chart for each characteristic because **it is drawn for controlling a single quality characteristic**. So it is necessary to make a judicious selection of quality characteristic.

Joseph Moses Juran (December 24, 1904-February 28, 2008) was a Romanian-born American engineer and management consultant. He is principally remembered for quality and quality management.

For example, a cricket ball has many quality characteristics such as weight, diameter, surface, elasticity, etc. A manufacturer of cricket ball would like to find out which of these quality characteristics, should be given priority for quality control. While selecting a quality characteristic, generally, we give higher priority to the one that causes more non-conforming (defective) items and increases cost. For this, we use **Pareto diagram**. It is a bar diagram of the quality characteristics of a product and was used for the first time by Joseph Moses Juran for quality control. For drawing a Pareto diagram, the frequency of non-conformities (defects) of each quality characteristic such as weight, diameter, surface, etc. is noted. In Pareto diagram, quality characteristics are taken along the X-axis and the percentage of defects occurring in the quality is taken along the Y-axis. For example, consider the data for quality characteristics for 200 cricket balls given in Table 1.

Table 1: Quality Characteristics Data for a Sample of 200 Cricket Balls

S. No.	Quality Characteristic	Frequency of Defects	Percentage of Defects
1	Weight	90	45
2	Diameter	70	35
3	Rough Surface	20	10
4	Elasticity	10	5
5	Other	10	5
Total		200	100

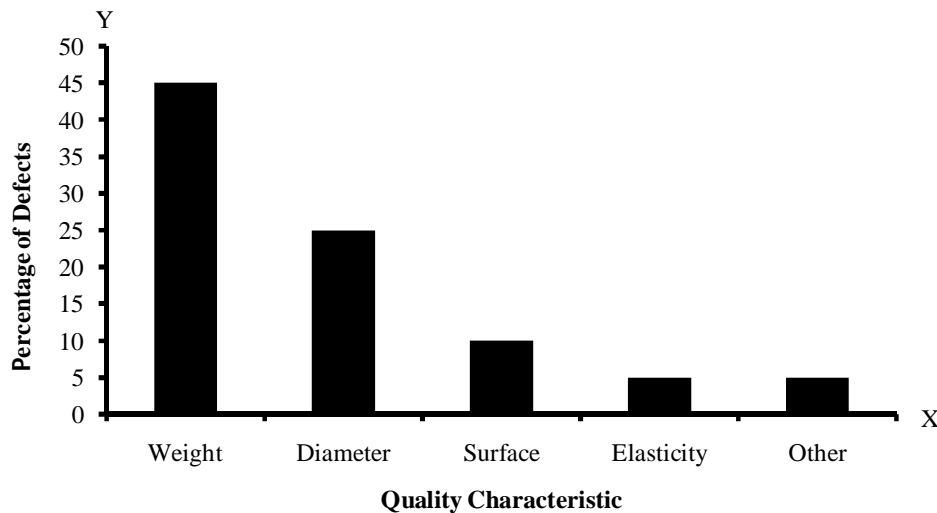The Pareto diagram drawn from the above data is as follows:

Fig. 2.1: Pareto diagram for data of Table 1.

From the Pareto diagram shown in Fig. 2.1, we observe that the weight and diameter are serious problems accounting for most of the non-conforming (defective) cricket balls produced. Rough surface occurs **less frequently**, elasticity and other problems are relatively **rare**. So for this problem, we should select quality characteristics, weight and diameter of the balls for quality control.

2. **Select the Type of Control Chart**

The choice of control chart depends upon the measurement, quality characteristic and the cost involved. For selection of control chart, we first select the category of the chart to be used as variables or attributes. If the characteristic to be controlled is **measureable** such as weight, length, diameter, etc., we use the **control chart for variables**. In the example of cricket balls, we have selected weight and diameter for control and both are measurable. So we will use control charts for variables for both quality characteristics.

If a characteristic to be controlled is **non-measureable**, e.g., colour, surface roughness, etc., we use the **control chart for attributes**. If many characteristics need control, it is too costly to use control charts for variables because these charts are used for controlling only one characteristic at a time. Therefore, each characteristic has its variables chart. In such situations, a single chart for attributes can be used in place of all the charts for variables.

After selecting the category of the chart to be used (variables or attributes), we select the actual chart to be used. For this, we need to know which chart can be used for a given situation and also the purpose of the control chart. We shall explain this aspect while describes each control chart.

3. **Selection of Rational Subgroups**

We know that control charts for quality characteristics are drawn with the help of sample data. Now, the question may arise "how are the data collected for analysis and controlling processes"? For this Walter A. Shewhart developed and introduced the concept of **rational subgroups** for control charts. He suggested that **the differences between rational**

subgroups are an indication of process changes (assignable causes) while **differences within rational subgroup are an indication of inherent variability, i.e., chance causes**. So the subgroups are selected in such a way that we are able to differentiate between assignable causes and chance causes. If assignable causes are present in the process, they will show up as differences between the subgroups rather than differences within a subgroup.

A rational subgroup is a small set of items/units that are produced under similar conditions within a relatively short time, i.e., the variation within the subgroup is only due to chance causes.

Generally, in industries, the word **'sample'** is used for a subgroup. So we use words, **sample and subgroup**, in this course.

4. Size of Subgroup (Sample Size)

To provide maximum homogeneity within a subgroup/sample, the size of the sample should be as small as possible. However, a sample size of four or five units/items is quite common in the industry. It is also valid on statistical grounds. As we know from sampling distribution theory, the distribution of the sample mean $\bar{X}$ is nearly normal for samples of four or more, even though the samples are taken from a non-normal population. This fact is helpful in interpretation of control chart limits. When a sample of size 5 is used there is ease in the computation of the average.

When we have to make the control chart more sensitive, samples of size 10 or 20 are used. This is because the standard error of a statistic is inversely proportion to sample size, i.e., as the sample size increases, the standard error decreases. Therefore, 3σ limits (upper and lower control limits) will lie closer to the centre line. However, if the items produced are destroyed under inspection or are expensive, a small sample of size 2 or 3 is used.

5. Frequency of Subgroups (Number of Samples)

There is no hard and fast rule for the number of subgroups/samples, but it should be such that it is enough to detect trend, pattern of level shifts and process changes. To decide the number of samples, we need to keep in mind that the cost of taking samples must be balanced with the data obtained.

There are two ways of taking samples:

i) Taking larger samples at less frequent intervals, or

ii) Taking smaller samples at more frequent intervals.

It is best to take samples more frequently in the beginning of the process. As the processes are brought into control, frequency of sampling can be reduced. For example, in a firm, the analyst may take a sample every hour at the beginning of the process. If the process remains under statistical control for two or three days, the frequency of sampling may be reduced to every 2 or 3 hour. If the process continuously remains under statistical control for a few weeks, the frequency may further reduced. However, if difficulties are encountered in keeping the process under statistical control, samples can be taken more frequently for inspection.

6. Design the Forms for Data Collection

After taking the decision about the quality characteristic, subgroup/sample size and number of subgroups, we design the **data recording form**. The form for data recording should be designed in accordance with the control chart to be used. A typical data recording form for the $\overline{X}$-chart is shown below:

Name of the Chart:					Plot No.:		Pin:		
Part Name:				Operation:			SPEC:		
Operator:		Machine:			Gauge:		Unit of Measurement:		
Subgroup Number	Date	Time	Measurements				Mean	SD	Comment
			X_1	X_2	X_3	X_4			
1	17/5	10:00	52	52	50	51	51.25	0.83	
2		12:00	50	53	52	53	52.00	1.22	
3		02:00	64	60	63	61	62.00	1.58	New temporary operator
4		04:00	51	52	50	53	51.50	1.12	
5	18/5	10:00	50	52	51	50	50.75	0.83	
6		12:00	50	50	52	51	50.75	0.83	
7		02:00	52	51	53	50	51.50	1.12	
8		04:00	52	53	52	55	53.00	1.22	
9	19/5	10:00	42	41	40	43	41.50	1.12	Bad material
10		12:00	52	52	54	52	52.50	0.87	
11		02:00	52	53	54	49	52.00	1.87	
12		04:00	52	51	54	51	52.00	1.22	
13	20/5	10:00	51	51	52	52	51.50	0.50	
14		12:00	50	50	51	52	50.75	0.83	
15		02:00	50	51	53	51	51.25	1.09	
16		04:00	52	50	49	53	51.00	1.58	
17	21/5	10:00	52	51	54	51	52.00	1.22	
18		12:00	51	51	50	52	51.00	0.71	
19		02:00	52	50	53	45	50.00	3.08	
20		04:00	50	49	50	48	49.25	0.83	
Total							1027.50	23.68	

Generally, a data recording form is divided into two parts. The top part of the form contains information about part name, operation machine, gauge used, unit of measurement and specifications. The remaining segment of the form contains information about subgroup/sample number, the date and time when the sample was selected and raw values of the observations. A column for comment is used to incorporate remark about the process.

7. **Determination of Trial Centre Line and Control Limits**

You have learnt in Unit 1 that the quality characteristic can be described by a probability distribution. In most cases, it follows a normal distribution or can be approximated by a normal distribution. You have learnt in Sec. 14.2

of Unit 14 of MST-003 (Probability Theory) that the probability of a normally distributed random variable (X) that lies between $\mu - 3\sigma$ and $\mu + 3\sigma$ is 0.9973 where μ and σ are the mean and standard deviation of random variable X. That is,

$$P[\mu - 3\sigma \leq X \leq \mu + 3\sigma] = 0.9973$$

The probability that the random variable X lies outside the limits $\mu \pm 3\sigma$ is $1 - 0.9973 = 0.0027$, that is, it is very small. It means that if we consider 100 samples, then most probably 0.27 items fall outside $\mu \pm 3\sigma$ limits. Therefore, if an observation falls outside 3σ limits, it is logical to suspect that something might have gone wrong. For this reason, the control limits are set up using 3σ limits.

Suppose M is a sample statistic (e.g., mean, range, proportion of defectives, etc.) that measures some quality characteristic of interest. Further suppose that μ_M and σ_M are the mean and standard error (standard deviation) of the sample statistic M, respectively. Then the centre line and control limits for controlling the quality characteristic are given by:

Centre line $(CL) = E(M) = \mu_M$

Upper control limit $(UCL) = \mu_M + 3\sigma_M$

Lower control limit $(LCL) = \mu_M - 3\sigma_M$

8. Constructing a Control Chart

The centre line and control limits are different for different charts. We shall explain the method of determining these limits while describing each type of chart.

After obtaining centre line and control limits, we construct the control chart. In a control chart, the statistic (e.g., mean, range, number of defects, etc.) is taken along the Y-axis and the sample number or time is taken along the X-axis. We represent the centre line by a solid line and the control limits by dotted lines. We plot the value of statistic for each sample against the sample number and consecutive sample points are joined by line segments.

9. Drawing Preliminary Conclusions from the Control Chart

After constructing the control chart, we draw preliminary conclusions from it. We check whether the plotted sample points lie on or in between the upper and lower control limits or some of them lie outside these control limits. If one or more points lie outside the control limits, we indicate each such point by drawing a **dotted circle** around it. The control limits on this chart are called **trial control limits**.

Recall Sec.1.5 of Unit 1. If all sample points lie on or in between the upper and lower control limits and there is no unnatural patterns of variation, the chart indicates that the process is under statistical control.

Even if a process is under statistical control, small variations may exist due to machine performances, operator performance, material characteristics, etc. Such small variations are considered to be a part of a stable process. It means that only chance causes are present in the process and no assignable causes are present.

When a control chart indicates that the process is under statistical control, periodic samples are taken from the process to determine whether it remains under statistical control continuously. When we assure that the process has been under statistical control after the analysis of the preliminary data, the trial control limits are used to control future production. Then $\overline{X}, \sigma, R$, etc. can be considered as representative of the process and their values are taken as standard values. However, when more data accumulates, the limits may be revised from time to time or whenever necessary. To ensure that the process remains under statistical control, periodic samples are taken once a week, once a month or once every 25, 50 or 100 items.

Most processes are not under statistical control at the time of the initial analysis, i.e., some points lie outside the control limits or there are long runs or unusual patterns of variation. Then the control chart indicates that the process is not under statistical control. Some assignable causes are present in the process. We investigate the reasons of assignable causes and take corrective action to eliminate them from the process. Suppose these are due to raw materials supplied by new vendor, we eliminate the cause by choosing a new vendor or maintaining the quality of raw material at vendor's point. Rectification is made by deleting the out-of-control points and calculating the revised centre line and control limits for the chart. These revised limits are known as the **revised control limits**. This procedure is continued till the process is being under statistical control.

So far you have studied the general technique for constructing a control chart. You have learnt how a control chart is constructed and how to determine whether a process is under statistical control or out of control on the basis of the control chart. We now discuss the control charts for variables.

2.3 CONTROL CHARTS FOR VARIABLES

In any production process, it is not possible to ensure that all items produced are alike in respect of certain characteristics. Some variations are always present in the items. These variations may be due to raw material quality, unskilled work force, machines, faulty equipment, etc. The existence of variation in products affects the quality of the product. These variations may be due to chance causes or assignable causes. So the aim of Statistical Process Control is to trace the sources of such variation and as far as possible try to eliminate them. If the quality characteristic in which we have to control the variation is measurable, we use control chart for variables.

In quality control, the term **variable** means the quality characteristic which can be measured, e.g., diameter of ball bearings, length of refills, weight of cricket balls, etc. The control charts based on measurements of quality characteristics are called **control charts for variables**.

When we deal with a measurable quality characteristic, it is necessary to control the central tendency (average) as well as the dispersion (variability) of the quality characteristic or the process. For central tendency, we usually apply mean and for variability we calculate range or standard deviation. Therefore, there are different control charts for variables for controlling the mean and variability of the process. The most frequently used control charts for variables are:

1. Control chart for mean

2. Control chart for variability

We generally use control chart for mean ($\overline{X}$-chart, read as X-bar chart) to control the mean quality level or process mean. We discuss this chart with examples in Secs. 2.4.

Experience reveals that process variability also affects product quality. When variation in the production process is high, that is, the produced items have a wider range of values then the quality of the product is assumed to be poor. So to improve the quality of the product we must reduce the variability.

The following charts are used to control process variability:

1. Range chart (R-chart), and

2. Standard deviation chart (S-chart).

We discuss each of these charts with examples in Secs. 2.5 and 2.6.

You may now like to check your understanding of different types of control charts by answering the following exercise.

To monitor and control variability process, we use R-chart or S-chart. But R-chart is most widely used in quality control as compared to S-chart because in quality control, we generally use samples of sizes less than 10. We know that for n ≤ 10, the range and standard deviation reflect almost the same information about the variability. However, the range is easier to calculate and interpret in comparison to the standard deviation.

E1) Choose the correct option from the following:

  i)   The control chart for variables is/are the

  a) $\overline{X}$-chart   b) R-chart   c) S-chart   d) all of these

  ii)   The control chart for process mean is the

  a) $\overline{X}$-chart   b) R-chart   c) S-chart   d) p-chart

  iii)   The control chart for process variability is the

  a) $\overline{X}$-chart   b) R-chart   c) c-chart   d) p-chart

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We now discuss the control chart for the mean.

2.4 CONTROL CHART FOR MEAN ($\overline{\overline{X}}$-CHART)

If we want to control the process mean, in any production process, we use the **control chart for mean**, i.e., the $\overline{\mathbf{X}}$-chart. With the help of the $\overline{X}$-chart, we monitor the variation in the mean of the samples that have been drawn from time to time from the process. We plot the sample means instead of individual measurements on the control chart because sample means are calculated from n individuals and give additional information. The $\overline{X}$-chart is more sensitive than the individual chart for detecting the changes in the process mean.

We now explain various steps for constructing the $\overline{X}$-chart.

Steps Involved in the Construction of the $\overline{X}$-chart

The main steps for the construction of the $\overline{X}$-chart are as follows:

Step 1: We select the measurable quality characteristic for which the $\overline{X}$-chart has to be constructed. We have discussed how to select the quality characteristic in Sec.2.2.

Step 2: Then we decide the subgroup/sample size as explained in Sec. 2.2. Generally, four or five items are selected in a sample and twenty to twenty five samples are collected for the $\overline{X}$-chart.

Step 3: After deciding the size of the sample and the number of samples, we select sample units/items randomly from the process so that each unit/item has an equal chance of being selected.

Step 4: We measure the quality characteristic decided in **Step 1** for each selected item/unit in each sample. Errors in measurement may be due to many reasons. These may be due to:

    i)   use of faulty machine,

    ii)   different methods of taking measurements,

    iii) inept use of instruments by unskilled or inexperienced work force, etc.

Therefore, care should be taken that measurements are made by experts, using standard instruments.

Step 5: Calculate the sample mean for each sample.

Suppose we measure the quality characteristic, say, X for each unit/item of the sample. Also suppose $X_1, X_2, ..., X_n$ are the measurements for the units of a sample of size n. Then sample mean is given by

$$\overline{X} = \frac{1}{n}(X_1 + X_2 + ... + X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \qquad ... (1)$$

where X_i represents the measurement of the quality characteristic of the ith unit.

Suppose there are k samples each of size n. Then we calculate the sample mean for each sample by using equation (1). Suppose the means for $1^{st}, 2^{nd}, ..., k^{th}$ samples are represented by $\overline{X}_1, \overline{X}_2, ..., \overline{X}_k$, respectively.

Step 6: Set the control limits

To find out whether the process is under statistical control or out-of-control, we set up the control limits. The 3σ control limits for the $\overline{X}$-chart are given by

Centre line = $E(\overline{X})$

Lower control limit (UCL) = $E(\overline{X}) - 3SE(\overline{X})$

Upper control limit (UCL) = $E(\overline{X}) + 3SE(\overline{X})$

where $E(\overline{X})$ and $SE(\overline{X})$ are the mean and the standard error of the sampling distribution of $\overline{X}$, respectively.

For getting the control limits of the $\overline{X}$-chart we need the sampling distribution of the sample mean. From Sec. 2.2 of Unit 2 of MST-004, we know that if $X_1, X_2, ..., X_n$ is a sample of size n drawn from a normal population (distribution) with mean μ and variance $σ^2$, the

Recall the procedure of drawing a sample from a population (process) which has been explained in Unit 1 of MST-005 (Statistical Techniques).

sample mean $\overline{X}$ is normally distributed with mean μ and variance σ^2/n. It means that, if

$$X_i \sim N(\mu, \sigma^2) \text{ then } \overline{X} \sim N(\mu, \sigma^2/n),$$

$$E(\overline{X}) = \mu \text{ and } Var(\overline{X}) = \frac{\sigma^2}{n} \qquad \dots (2)$$

We also know that

$$SE(X) = \sqrt{Var(X)}$$

$$\therefore \quad SE(\overline{X}) = \sqrt{Var(\overline{X})} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}} \qquad \dots (3)$$

Therefore, the centre line and control limits for the $\overline{X}$-chart are given as follows:

Centre line (CL) $= E(\overline{X}) = \mu$ $\qquad \dots (4a)$

Lower control limit (LCL) $= E(\overline{X}) - 3SE(\overline{X}) = \mu - 3\sigma/\sqrt{n}$

or $\qquad\qquad$ LCL $= \mu - A\sigma$ $\qquad \dots (4b)$

Upper control limit (UCL) $= E(\overline{X}) + 3SE(\overline{X}) = \mu + 3\sigma/\sqrt{n}$

or $\qquad\qquad$ UCL $= \mu + A\sigma$ $\qquad \dots (4c)$

where $A = 3/\sqrt{n}$ is a constant and depends on the size of the sample. It has been tabulated for various sample sizes in Table I given at the end of this block.

the values of μ and σ are not known. Therefore, these are estimated from the samples, which are taken when the process is thought to be under statistical control. We use their **best possible estimates**. In Unit 5 of MST-004, we have seen that the best estimators of the population mean (μ) and population standard deviation (σ) of normal population are the sample mean and sample standard deviation, respectively. So in this case, the best estimator of the process mean (μ) is the grand mean ($\overline{\overline{X}}$), that is, the mean of all sample means given by

$$\overline{\overline{X}} = \frac{1}{k}(\overline{X}_1 + \overline{X}_2 \dots, \overline{X}_k) = \frac{1}{k}\sum_{i=1}^{k}\overline{X}_i \qquad \dots (5)$$

To calculate the grand mean, we first calculate the mean of each sample from equation (1). Then we calculate the grand mean from equation (5).

Now, we need the best estimate of the unknown standard deviation σ. It can either be estimated by sample range (R) or sample standard deviation (S). So we have two cases:

Case I: When standard deviation is estimated by the sample range R.

In this case, the standard deviation σ is estimated as follows:

$$\hat{\sigma} = \overline{R}/d_2 \qquad \dots (6)$$

where $\bar{R}$ represents the mean of all sample ranges given by

$$\bar{R} = \frac{1}{k}\left(R_1 + R_2 \ldots, R_k\right) = \frac{1}{k}\sum_{i=1}^{k} R_i \qquad \ldots (7)$$

Here R_i is the sample range and is given by

$$R_i = X_{max} - X_{min} \qquad \ldots (8)$$

where X_{max} and X_{min} represent the maximum (or largest) and minimum (or smallest) measurements of the quality characteristic of a sample, respectively. Also d_2 is a constant and depends on the size of the sample. It has been tabulated for various sample sizes in Table I given at the end of this block.

The control limits for the $\bar{X}$-chart when μ is estimated by $\left(\bar{\bar{X}}\right)$ and σ is estimated by $\bar{R}/d_2$ are given by:

Centre line (CL) $= \hat{\mu} = \bar{\bar{X}}$ $\qquad \ldots (9a)$

Lower control limit (LCL) $= \hat{\mu} - \dfrac{3}{\sqrt{n}}\hat{\sigma} = \bar{\bar{X}} - \dfrac{3}{\sqrt{n}}\dfrac{\bar{R}}{d_2}$

or $\qquad$ LCL $= \bar{\bar{X}} - A_2\bar{R}$ $\qquad \ldots (9b)$

Upper control limit (UCL) $= \hat{\mu} + \dfrac{3}{\sqrt{n}}\hat{\sigma} = \bar{\bar{X}} + \dfrac{3}{\sqrt{n}}\dfrac{\bar{R}}{d_2} = \bar{\bar{X}} + A_2\bar{R}$

or $\qquad$ UCL $= \bar{\bar{X}} + A_2\bar{R}$ $\qquad \ldots (9c)$

where $A_2 = \dfrac{3}{d_2\sqrt{n}}$ is a constant and depends on the size of the sample. It has been also tabulated for various sample sizes in Table I given at the end of this block.

Case II: When standard deviation is estimated by the sample standard deviation.

In this case, the standard deviation σ is estimated as follows:

$$\hat{\sigma} = \bar{S}/c_4 \qquad \ldots (10)$$

where $\bar{S}$ represents the mean of all sample standard deviations given by

$$\bar{S} = \frac{1}{k}\left(S_1 + S_2 \ldots, S_k\right) = \frac{1}{k}\sum_{i=1}^{k} S_i \qquad \ldots (11)$$

Here S_i is the sample standard deviation given by

$$S_i = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_i\right)^2} \qquad \ldots (12)$$

Therefore, the control limits for the $\bar{X}$-chart when μ is estimated by $\left(\bar{\bar{X}}\right)$ and σ is estimated by $\bar{S}$ are given by

Centre line (CL) = $\hat{\mu} = \bar{\bar{X}}$ … (13a)

Lower control limit (LCL) = $\hat{\mu} - \dfrac{3}{\sqrt{n}}\hat{\sigma} = \bar{\bar{X}} - \dfrac{3}{\sqrt{n}}\dfrac{\bar{S}}{c_4}$

or   LCL $= \bar{\bar{X}} - A_3\bar{S}$ … (13b)

Upper control limit (UCL) = $\hat{\mu} + \dfrac{3}{\sqrt{n}}\hat{\sigma} = \bar{\bar{X}} + \dfrac{3}{\sqrt{n}}\dfrac{\bar{S}}{c_4}$

or   UCL $= \bar{\bar{X}} + A_3\bar{S}$ … (13c)

where $A_3 = \dfrac{3}{c_4\sqrt{n}}$ is a constant and depends on the size of the sample. It has been also tabulated for various sample sizes in Table I given at the end of this block.

Note: Generally, for the $\bar{X}$-chart, we select samples of four or five units/items. So for this chart, we use the sample range to estimate the standard deviation of the process instead of the sample standard deviation because the range is easier to calculate as compared to the standard deviation. Moreover, it reflects almost the same information about variability as the standard deviation.

Step 7: Construct the $\bar{X}$-chart

After setting the centre line and control limits, we construct the $\bar{X}$-chart as explained in Step 8 of Sec.2.2. It means that we take the sample number on the X-axis (horizontal scale) and the sample mean $(\bar{X})$ on the Y-axis (vertical scale). We plot the value of sample mean for each sample against the sample number and then join the consecutive sample points by line segments.

Step 8: Interpret the result

If all sample points on the plot lie on or in between upper and lower control limits, **the process is under statistical control**. In this situation, only chance causes are present in the process. If one or more points lie outside the control limits the control chart alarms (indicates) that **the process is not under statistical control**. Some assignable causes are present in the process.

To bring the process under statistical control, we investigate the assignable causes and take corrective action to eliminate them. Once the assignable causes are eliminated from the process, we delete the out-of-control points (samples) and calculate the revised centre line and control limits for the $\bar{X}$-chart by using the remaining samples. These limits are known as **revised control limits**. To calculate the revised limits for the $\bar{X}$-chart, we first calculate new $\bar{\bar{X}}$ and new $\bar{R}$ as follows:

$$\overline{\overline{X}}_{new} = \frac{\sum\limits_{i=1}^{k} \overline{X}_i - \sum\limits_{j=1}^{d} \overline{X}_j}{k-d} \qquad \ldots (14)$$

and

$$\overline{R}_{new} = \frac{\sum\limits_{i=1}^{k} R_i - \sum\limits_{j=1}^{d} R_j}{k-d} \qquad \ldots (15)$$

where d – the number of discarded samples,

$\sum\limits_{j=1}^{d} \overline{X}_j$ – the sum of means of discarded samples, and

$\sum\limits_{j=1}^{d} R_i$ – the sum of ranges of discarded samples.

After finding the $\overline{\overline{X}}_{new}$ and $\overline{R}_{new}$, we reconstruct the centre line and control limits, replacing $\overline{\overline{X}}$ by $\overline{\overline{X}}_{new}$ and $\overline{R}$ by $\overline{R}_{new}$ in equations (9a to 9c) as follows:

Centre line (CL) = $\overline{\overline{X}}_{new}$ … (16a)

Lower control limit (UCL) = $\overline{\overline{X}}_{new} - A_2 \overline{R}_{new}$ … (16b)

Upper control limit (UCL) = $\overline{\overline{X}}_{new} + A_2 \overline{R}_{new}$ … (16c)

If we use the sample standard deviation to estimate the process standard deviation, then the revised control line and control limits are given as follows:

Centre line (CL) = $\overline{\overline{X}}_{new}$ … (17a)

Lower control limit (UCL) = $\overline{\overline{X}}_{new} - A_3 \overline{S}_{new}$ … (17b)

Upper control limit (UCL) = $\overline{\overline{X}}_{new} + A_3 \overline{S}_{new}$ … (17c)

where

$$\overline{S}_{new} = \frac{\sum\limits_{i=1}^{k} S_i - \sum\limits_{j=1}^{d} S_j}{k-d} \quad \text{and} \qquad \ldots (18)$$

$\sum\limits_{j=1}^{d} S_j$ – the sum of standard deviations of discarded samples.

Note: Generally, we control the process mean and the process variability at the same time. So we use both $\overline{X}$ and R-charts or $\overline{X}$ and S-charts together.

So far, we have discussed various steps involved in the construction of the $\overline{X}$-chart. Let us now take up some examples to illustrate this method.

Example 1: A statistical quality controller uses the $\bar{X}$-chart for monitoring a quality characteristic of a product. If the process mean (μ) and process standard deviation (σ) are 100 and 5, respectively, find the centre line and control limits for the $\bar{X}$-chart. It is given that A = 1.5 for n = 4.

Solution: Here we are given that

$$\mu = 100, \ \sigma = 5 \ \text{and} \ A = 1.5$$

Therefore, we can calculate the centre line and control limits for the $\bar{X}$-chart using equations (4a to 4c) as follows:

Centre line (CL) $= \mu = 100$

Upper control limit (UCL) $= \mu - A\sigma = 100 - 1.5 \times 5 = 100 - 7.5 = 92.5$

Lower control limit (LCL) $= \mu + A\sigma = 100 + 1.5 \times 5 = 100 + 7.5 = 107.5$

Example 2: An automatic machine (shown in Fig. 2.2) is used to fill and seal 20 ml tube of medicine. The $\bar{X}$-control chart has been used to monitor this process. The process is sampled in samples of four and the values of $\bar{X}$ and R are computed for each sample. After 25 samples, $\sum \bar{X} = 525$ and $\sum R = 90$. Estimate the process mean and standard deviation. Compute the control limits for the $\bar{X}$-chart. It is given that $d_2 = 2.088$ and $A_2 = 0.729$ for n = 4.

Solution: It is given that

$$k = 25, \ n = 4, \ \sum \bar{X} = 525, \ \sum R = 90, \ d_2 = 2.088 \ \text{and} \ A_2 = 0.729.$$

The process mean is estimated by the grand sample mean, which is calculated as follows:

$$\hat{\mu} = \bar{\bar{X}} = \frac{1}{k} \sum \bar{X} = \frac{1}{25} \times 525 = 21$$

The process standard deviation is estimated by sample range, which is calculated as follows:

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{1}{d_2} \frac{1}{k} \sum R = \frac{1}{2.088} \times \frac{1}{25} \times 90 = 1.724$$

Since the process average (μ) and process variability (σ) are unknown in this case, we use equations (9a to 9c) to calculate the centre line and the control limits for the $\bar{X}$-chart. These are given by

$$CL = \bar{\bar{X}} = 21$$

$$UCL = \bar{\bar{X}} + A_2\bar{R} = 21 + 0.729 \times 3.6 = 23.6244$$

$$LCL = \bar{\bar{X}} - A_2\bar{R} = 21 - 0.729 \times 3.6 = 18.3756$$

Example 3: A new process of producing ball bearings is started. For monitoring the outside diameter of the ball bearings, the quality controller takes the sample of five ball bearings at 10.00 AM, 12.00 PM, 2.00 PM, 4.00 PM and 6.00 PM and measures the outside diameter (in mm) of each selected ball bearing (Fig. 2.3). The results of the test over a 4-day production period are as follows:

Fig. 2.2

The CL, UCL and LCL for the $\bar{X}$-chart when μ and σ are unknown are

$$CL = \bar{\bar{X}}$$

$$UCL = \bar{\bar{X}} + A_2\bar{R}$$

$$LCL = \bar{\bar{X}} - A_2\bar{R}$$

Fig. 2.3

Day	Sample Number	Time	Observations					$\overline{X}$	R
			X_1	X_2	X_3	X_4	X_5		
Monday	1	10.00AM	52	52	50	51	51	51.2	2
	2	12.00PM	50	53	52	53	51	51.8	3
	3	2.00 PM	54	51	50	52	53	52.0	4
	4	4.00PM	56	55	53	55	53	54.4	3
	5	6.00 PM	51	52	50	53	53	51.8	3
Tuesday	6	10.00AM	50	52	51	50	51	50.8	2
	7	12.00PM	50	50	52	51	53	51.2	3
	8	2.00 PM	52	51	53	50	50	51.2	3
	9	4.00 PM	52	53	52	55	53	53.0	3
	10	6.00 PM	51	51	50	51	52	51.0	2
Wednesday	11	10.00AM	52	52	54	52	52	52.4	2
	12	12.00PM	49	48	50	50	51	49.6	3
	13	2.00 PM	52	53	54	49	52	52.0	5
	14	4.00PM	52	51	54	51	54	52.4	3
	15	6.00 PM	51	51	52	52	51	51.4	1
Thursday	16	10.00AM	50	50	51	52	51	50.8	2
	17	12.00PM	50	51	53	51	53	51.6	3
	18	2.00 PM	52	50	49	53	50	50.8	4
	19	4.00PM	52	51	54	51	51	51.8	3
	20	6.00 PM	51	51	50	52	52	51.2	2
Total								1032.4	56

Calculate and draw the centre line and control limits of the $\overline{X}$-chart. Draw the conclusion about the process, assuming assignable causes for any out-of-control point. If the process is out-of-control, calculate the revised centre line and control limits.

Solution: Since the process average (μ) and process variability (σ) are unknown, we use equations (9a to 9c) for determining the centre line and control limits for the $\overline{X}$-chart.

From Table I given at the end of this block, we have $A_2 = 0.577$ for $n = 5$.

To calculate the centre line and control limits, we first calculate the values of $\overline{\overline{X}}$ and $\overline{R}$ using equations (5) and (7) as follows:

$$\overline{\overline{X}} = \frac{1}{k}\sum \overline{X} = \frac{1}{20}\times 1032.4 = 51.62$$

$$\overline{R} = \frac{1}{k}\sum R = \frac{1}{20}\times 56 = 2.80$$

Putting the values of $\overline{\overline{X}}$, $\overline{R}$ and A_2 in equations (9a to 9c), we have

$$CL = \overline{\overline{X}} = 51.62$$

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$$UCL = \bar{\bar{X}} + A_2\bar{R} = 51.62 + 0.577 \times 2.80 = 53.236$$
$$LCL = \bar{\bar{X}} - A_2\bar{R} = 51.62 - 0.577 \times 2.80 = 50.004$$

We now construct the $\bar{X}$-chart by taking the sample number on the X-axis and the average diameter of the ball bearing $(\bar{X})$ on the Y-axis as shown in Fig. 2.4.
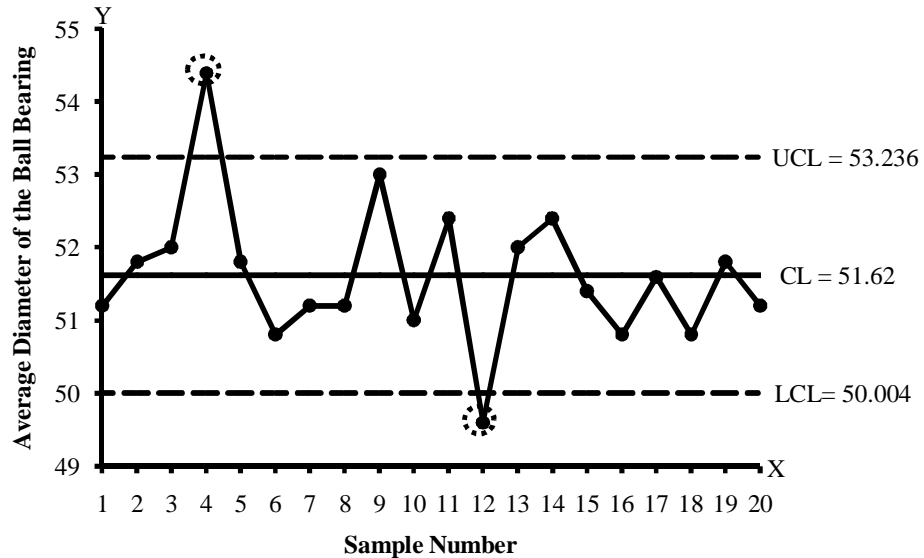

Fig. 2.4: The $\bar{X}$-chart for the average diameter of the ball bearings.

Interpretation of the result

From Fig. 2.4, we observe that the points corresponding to samples 4 and 12 lie outside the control limits. Therefore, the process is out-of-control and some assignable causes are present in the process. To bring the process under statistical control, it is necessary to investigate the assignable causes and take corrective action to eliminate them.

After eliminating the assignable causes from the process, we delete the out-of-control points (4 and 12 samples) and calculate the revised centre line and control limits for the $\bar{X}$-chart using the remaining samples. For revised limits of the $\bar{X}$-chart, we first calculate the new $\bar{X}$ and new $\bar{R}$ by using equations (14) and (15).

In our case, $k = 20, d = 2, \sum_{j=1}^{d}\bar{X}_j = 54.4 + 49.6 = 104$ and $\sum_{j=1}^{d}R_j = 3 + 3 = 6$.

$$\therefore \bar{\bar{X}}_{new} = \frac{\sum_{i=1}^{k}\bar{X}_i - \sum_{j=1}^{d}\bar{X}_j}{k-d} = \frac{1032.4 - 104}{20 - 2} = 51.578 \text{ and}$$

$$\bar{R}_{new} = \frac{\sum_{i=1}^{k}R_i - \sum_{j=1}^{d}R_j}{k-d} = \frac{56 - 6}{20 - 2} = 2.778$$

We calculate the revised centre line and control limits of the $\bar{X}$-chart using equations (16a to 16c) as follows:

$$CL = \bar{\bar{X}}_{new} = 51.578$$

$$LCL = \overline{\overline{X}}_{new} - A_2\overline{R}_{new} = 51.578 - 0.577 \times 2.778 = 49.975$$

$$UCL = \overline{\overline{X}}_{new} + A_2\overline{R}_{new} = 51.578 + 0.577 \times 2.778 = 53.181$$

You should now check your understanding about the $\overline{X}$-chart by solving the following exercises.

E2) Choose the correct option from the following:

i) The control limits for the $\overline{X}$-chart are

a) $\overline{\overline{X}} \pm A_2\overline{R}$ b) $\overline{\overline{X}} \pm 3\overline{R}$

c) $\overline{\overline{X}} \pm B_2\overline{R}$ d) $\overline{\overline{X}} \pm A\overline{R}$

ii) If one point of the $\overline{X}$-chart lies outside the control limits, the process is

a) in control   b) out-of-control   c) both (a) and (b)

iii) The $\overline{X}$-chart is used for

a) variables   b) attributes   c) defects   d) defectives

E3) Samples of size 5 are taken from a manufacturing process at regular intervals. A normally distributed quality characteristic is measured and $\overline{X}$ and R are calculated for each sample. After 20 samples, we have $\overline{\overline{X}} = 6.40$ and $\overline{R} = 0.0877$. Compute the centre line and control limits for the $\overline{X}$-chart.

E4) A factory was producing electrical bulbs. To test whether the process was under statistical control or not with respect to the average life of the bulbs, 16 samples of 3 bulbs were drawn at regular intervals and the life of each sampled bulb was measured. The measurements are given below:

Sample Number	Life of Bulb (in hour) (X)			Total	$\overline{X}$
1	1152	650	1090	2893	964.00
2	688	540	1074	2304	767.33
3	876	1342	1187	3408	1135.00
4	725	1070	847	2646	880.67
5	900	846	1254	3005	1000.00
6	1080	790	871	2747	913.67
7	952	1087	1126	3172	1055.00
8	741	786	578	2113	701.67
9	1005	1244	1140	3398	1129.67
10	1230	1005	654	2899	963.00
11	1200	870	859	2940	976.33
12	745	964	1050	2771	919.67
13	926	1172	1200	3311	1099.33
14	781	876	981	2652	879.33
15	1120	1350	750	3235	1073.33
16	880	975	768	2639	874.33

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Test whether the process is under statistical control or not by drawing the $\bar{X}$-chart when the average life and the standard deviation of the life of the bulbs are

i) known to be 1000 hour and 215 hour, respectively.

ii) unknown.

It is given that $A = 1.732$ and $A_2 = 1.023$ for $n = 3$.

In Sec.2.4, you have studied the $\bar{X}$-chart and learnt when it is used and how it is constructed. We now discuss the charts which are used for controlling the process variability. We first take up the R-chart.

2.5 RANGE CHART (R-CHART)

You have learnt in Sec. 2.3 that we use R-chart or S-chart to control the process variability and the R-chart is widely used in quality control.

The underlying logic and basic form of an R-chart are similar to the $\bar{X}$-chart. The primary difference between the $\bar{X}$-chart and the R-chart is that instead of plotting the sample means and monitoring their variation, we plot sample range and monitor the variation in sample ranges.

To get the control limits for the R-chart, we need to know the sampling distribution of the range. However, the derivation of the sampling distribution of the range is beyond the scope of this course. So we state the result without proof. If $X_1, X_2, ..., X_n$ is a random sample of size n taken from a normal population with mean μ and variance σ^2, the mean and variance of sampling distribution of range are given as follows:

$$E(R) = d_2\sigma \qquad \qquad \dots (19)$$

and

$$Var(R) = d_3^2\sigma^2 \qquad \qquad \dots (20)$$

where d_2 and d_3 are constants and depend on the size of the sample. These constants have been tabulated for various sample sizes in Table I given at the end of this block.

We also know that $SE(X) = \sqrt{Var(X)}$

Therefore, $SE(R) = \sqrt{Var(R)} = \sqrt{d_3^2\sigma^2} = d_3\sigma \qquad \qquad \dots (21)$

Although the sampling distribution of range is not a normal distribution, it is common in practice to use the $\pm 3\sigma$ limits. These limits assure that the probability of an observation outside these limits is very small. Therefore, we define CL, LCL and UCL for the R-chart as follows:

$$\text{Centre line (CL)} = E(R) = d_2\sigma \qquad\qquad\qquad\qquad \ldots (22a)$$

$$\text{Lower control limit (LCL)} = E(R) - 3\,SE(R) = d_2\sigma - 3d_3\sigma$$

$$\text{or} \qquad\qquad\qquad LCL = (d_2 - 3d_3)\sigma = D_1\sigma \qquad\qquad \ldots (22b)$$

$$\text{Upper control limit (UCL)} = E(R) + 3\,SE(R) = d_2\sigma + 3d_3\sigma$$

$$\text{or} \qquad\qquad\qquad UCL = (d_2 + 3d_3)\sigma = D_2\sigma \qquad\qquad \ldots (22c)$$

where $D_1 = d_2 - 3d_3$ and $D_2 = d_2 + 3d_3$ are constants and depend on the size of the samples. These constants have been tabulated for various sample sizes in Table I given at the end of this block.

In practice, the value of σ is not known. Therefore, it is estimated from the samples, which are taken when the process is thought to be under statistical control. Since the R-chart is used for small sample size (<10), σ is estimated by sample range (R). If $X_1, X_2, \ldots, X_n$ is a sample of size n, we estimate σ from equation (6) as:

$$\hat{\sigma} = \overline{R}/d_2$$

So the centre line and control limits for the R-chart when σ is estimated by $\overline{R}/d_2$ are given as:

$$\text{Centre line (CL)} = E(R) = d_2\hat{\sigma} = d_2\frac{\overline{R}}{d_2} = \overline{R} \qquad\qquad \ldots (23a)$$

$$\text{Lower control limit (LCL)} = E(R) - 3SE(R) = d_2\hat{\sigma} - d_3\hat{\sigma} = d_2\frac{\overline{R}}{d_2} - 3d_3\frac{\overline{R}}{d_2}$$

$$\text{or} \qquad\qquad LCL = \overline{R} - \frac{3d_3}{d_2}\overline{R} = \left(1 - \frac{3d_3}{d_2}\right)\overline{R} = D_3\overline{R} \qquad \ldots (23b)$$

$$\text{Upper control limit (UCL)} = E(R) + 3SE(R) = d_2\hat{\sigma} + d_3\hat{\sigma} = d_2\frac{\overline{R}}{d_2} + 3d_3\frac{\overline{R}}{d_2}$$

$$\text{or} \qquad\qquad UCL = \overline{R} + \frac{3d_3}{d_2}\overline{R} = \left(1 + \frac{3d_3}{d_2}\right)\overline{R} = D_4\overline{R} \qquad \ldots (23c)$$

where $D_3 = \left(1 - \frac{3d_3}{d_2}\right)$ and $D_4 = \left(1 + \frac{3d_3}{d_2}\right)$ are constants and depend on the size of the sample. These constants have been tabulated for various sample sizes in Table I given at the end of this block.

After setting the centre line and the control limits, we construct the R-chart by taking the sample number on the X-axis (horizontal scale) and the sample range (R) on the Y-axis (vertical scale). We represent the centre line by solid line and UCL and LCL by dotted lines. We plot the value of sample range for each sample against the sample number. The consecutive sample points are joined by line segments.

Interpretation of the result

If all sample points lie on or between the upper and lower control limits, the R-chart indicates that the process is under statistical control. Otherwise, it is out-of-control. To bring the process under statistical control, it is necessary to investigate the assignable causes and take corrective action to eliminate them. For this, we delete the out-of-control points and calculate the **revised centre line** and **control limits** as we did for the control chart of process mean. We

$$\text{Centre Line (CL)} = \overline{R}_{new} \qquad \dots \text{(24a)}$$

$$\text{Lower control limit (LCL)} = D_3 \overline{R}_{new} \qquad \dots \text{(24b)}$$

$$\text{Upper control limit (UCL)} = D_4 \overline{R}_{new} \qquad \dots \text{(24c)}$$

define the revised limits for the R-chart as follows:

The following points should be kept in mind while using the $\overline{X}$ and R-charts:

- In order to determine whether a process is under statistical control, we control both process mean and process variability. For that, we use the $\overline{X}$ and R-charts together. We first control the process variability. So we analyse the **R-chart before the $\overline{X}$-chart**.

- If the R-chart indicates that the process variability is out-of-control, we first control the process variability by investigating the assignable causes and taking corrective action to eliminate them. Only then we analyse the $\overline{X}$-chart because when the R-chart is brought under statistical control, many assignable causes for the $\overline{X}$-chart are eliminated automatically.

- If the R-chart indicates that the process variability is under statistical control, but the $\overline{X}$-chart indicates that the process mean is out-of-control, we continue to use the R-chart, but revise the $\overline{X}$-chart. The centre line and control limits of the $\overline{X}$-chart are revised by eliminating these points. As a result, the position of the control limits in the revised $\overline{X}$-chart shift. But the distance of control limits remains the same as in out-of-control case because we do not calculate $\overline{R}$ again.

- If both $\overline{X}$ and R-charts indicate that the process variability and process mean are under statistical control, $\overline{\overline{X}}$ and $\overline{R}$ can be considered as representative of the process and their values are taken as standard values.

The following examples will help you understand how to construct and interpret the $\overline{X}$ and R-charts together.

Example 4: A statistical quality controller uses the $\overline{X}$ and R-charts together for monitoring the quality characteristic of a product. Samples of size 5 are taken from the manufacturing process at regular intervals. A normally distributed quality characteristic is measured and the $\overline{X}$ and R are calculated for each sample. After 20 samples have been analysed, we have $\overline{\overline{X}} = 6.40$ and $\overline{R} = 0.0877$. Compute the centre line and control limits for the $\overline{X}$ and R-charts.

Solution: Here we are given that

$$n = 5, \ k = 20, \ \overline{\overline{X}} = 6.40, \ \overline{R} = 0.0877$$

Note that the process mean (μ) and process variability (σ) are unknown in this case. Therefore, we use equations (9a to 9c) and (23a to 23c) for calculating the centre and control limits for the $\overline{X}$-chart and the R-chart, respectively.

From Table I, we have

$$A_2 = 0.577, D_3 = 0, D_4 = 2.114 \text{ for } n = 5$$

Substituting the values of $\overline{\overline{X}}, \overline{R}$ and A_2 in equations (9a to 9c), we get the centre line and control limits of the $\overline{X}$-chart as follows:

$$CL = \overline{\overline{X}} = 6.40$$

$$UCL = \overline{\overline{X}} + A_2\overline{R} = 6.40 + 0.577 \times 0.0877 = 6.451$$

$$LCL = \overline{\overline{X}} - A_2\overline{R} = 6.40 - 0.577 \times 0.0877 = 6.349$$

Similarly, substituting the values of $\overline{R}$, D_3 and D_4 in equations (23a to 23c), we get the centre line and control limits of the R-chart as follows:

$$CL = \overline{R} = 0.0877$$

$$UCL = D_4\overline{R} = 2.114 \times 0.0877 = 0.185$$

$$LCL = D_3\overline{R} = 0 \times 0.0877 = 0$$

Example 5: A milk company uses automatic machines to fill 500 ml milk packets. A quality control inspector inspected four packets for each sample at given time intervals and measured the weight of each filled packet. The data for 20 samples are shown in the following table:

Sample Number	Weight of Filled Milk Packet (in ml)				$\overline{X}$	R
1	500	520	500	500	506.67	20
2	500	490	520	530	503.33	40
3	490	550	570	540	536.67	80
4	510	520	500	520	510.00	20
5	510	480	490	490	493.33	30
6	520	500	520	500	513.33	20
7	520	510	530	510	520.00	20
8	530	490	520	500	513.33	40
9	510	490	500	510	500.00	20
10	520	520	490	520	510.00	30
11	520	500	510	500	510.00	20
12	480	500	520	510	500.00	40
13	530	510	520	510	520.00	20
14	500	510	510	500	506.67	10
15	490	520	500	510	503.33	30
16	520	500	530	500	516.67	30
17	520	560	490	510	523.33	70
18	500	490	500	510	496.67	20
19	520	500	530	500	516.67	30
20	500	490	500	490	496.67	10
Total					10196.67	600

The CL, UCL and LCL for the $\overline{X}$-chart when μ and σ are unknown are

$$CL = \overline{\overline{X}}$$

$$UCL = \overline{\overline{X}} + A_2\overline{R}$$

$$LCL = \overline{\overline{X}} - A_2\overline{R}$$

The CL, UCL and LCL for the R-chart when σ is unknown are

$$CL = \overline{R}$$

$$UCL = D_4\overline{R}$$

$$LCL = D_3\overline{R}$$

Using the $\bar{X}$ and R-charts together, draw the conclusion about the process by assuming assignable causes for any out-of-control points. If the process is out-of-control, calculate the revised centre line and control limits to bring the process under statistical control.

Solution: Since the $\bar{X}$ and R-charts are to be used together, we draw and analyse the R-chart first.

Here, the variability of the process (σ) is unknown. Therefore, we use equations (23a to 23c) for calculating the centre and control limits for the R-chart

From Table I, we have

$$A_2 = 0.729, \ D_3 = 0 \text{ and } D_4 = 2.282 \text{ for } n = 4$$

The CL, UCL and LCL for the R-chart when σ is unknown are

$CL = \bar{R}$

$UCL = D_4\bar{R}$

$LCL = D_3\bar{R}$

We first calculate the value of $\bar{R}$ for the centre line and the control limits from the given data using equation (7) as follows:

$$\bar{R} = \frac{1}{k}\sum R = \frac{1}{20} \times 600 = 30$$

Substituting the values of $\bar{R}, D_3$ and D_4 in equations (23a to 23c) we get

$CL = \bar{R} = 30$

$UCL = D_4\bar{R} = 2.282 \times 30 = 68.46$

$LCL = D_3\bar{R} = 0 \times 30 = 0$

We now construct the R-chart by taking the sample number on the X-axis and the sample range (R) of the milk packets on the Y-axis as shown in Fig. 2.5.
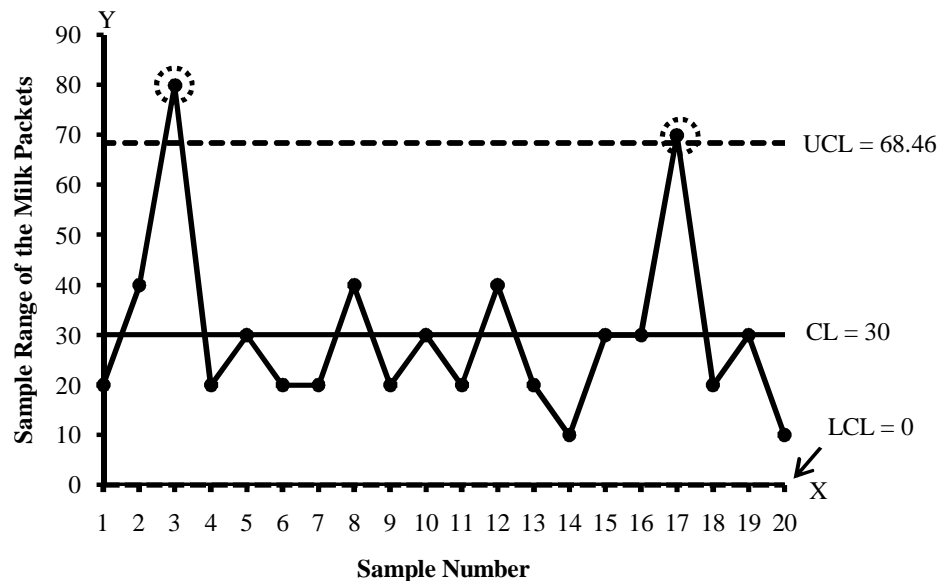

Fig. 2.5: The R-chart for milk packets.

Interpretation of the result

From Fig. 2.5, we observe that the points corresponding to samples 3 and 17 lie outside the control limits. Therefore, the process variability is out-of-control and some assignable causes are present in the process.

For calculating the revised control limits, we first delete the out-of-control points and calculate the $\overline{R}_{new}$ for the remaining samples.

In this example, $k = 20, d = 2, \sum\limits_{j=1}^{d} R_j = 80 + 70 = 150$

$$\therefore \quad \overline{R}_{new} = \frac{\sum\limits_{i=1}^{k} R - \sum\limits_{j=1}^{d} R_j}{k - d} = \frac{600 - 150}{20 - 2} = \frac{450}{18} = 25$$

Formula for calculating the new $\overline{R}$ is

$$\overline{R}_{new} = \frac{\sum\limits_{i=1}^{k} R_i - \sum\limits_{j=1}^{d} R_j}{k - d}$$

where d – the number of discarded samples, and

$\sum\limits_{j=1}^{d} R_i$ – sum of ranges of discarded samples.

We calculate the revised centre line and control limits of the R-chart using equations (24a to 24c) as follows:

$$CL = \overline{R}_{new} = 25$$

$$UCL = D_4 \overline{R}_{new} = 2.282 \times 25 = 57.05$$

$$LCL = D_3 \overline{R}_{new} = 0 \times 25 = 0$$
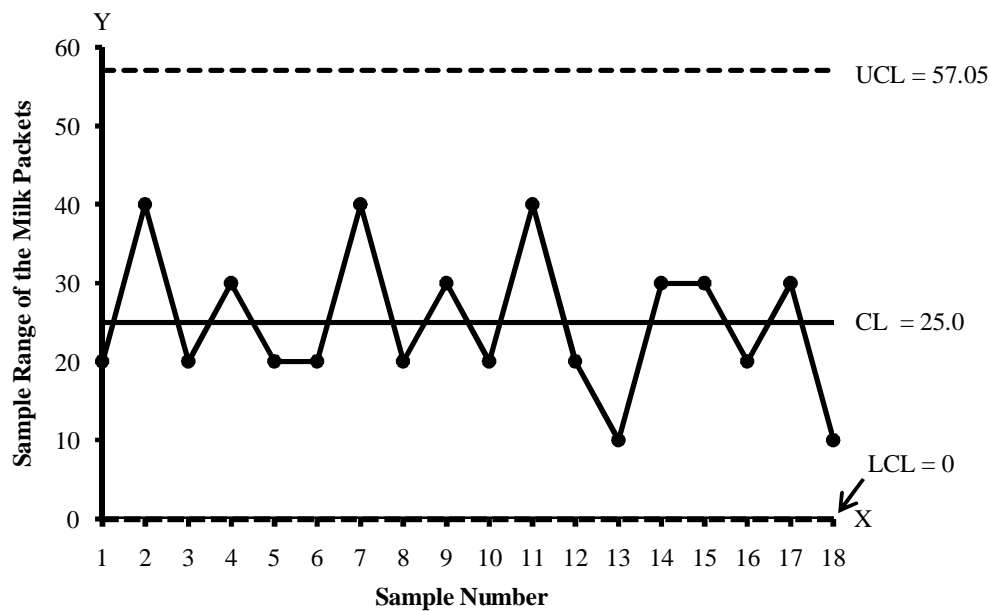
Fig. 2.6 shows the revised R-chart.

Fig. 2.6: The revised R-chart.

Interpretation of the result

The revised R-chart (shown in Fig. 2.6) indicates that all points lie within the control limits. Hence, we say that the process variability is under statistical control.

Note: If one or more points lie outside the revised control limits, we calculate the revised control limits for the R-chart again. This is continued until the process is brought under statistical control.

After controlling the process variability, we study the process mean. For this we calculate the control limits of the $\overline{X}$-chart for the remaining samples. We first calculate the new $\overline{\overline{X}}$ from equation (14).

In this example, $k = 20, d = 2, \sum\limits_{j=1}^{d} \overline{X}_j = 536.67 + 523.33 = 1060$

$$\therefore \quad \bar{\bar{X}}_{new} = \frac{\sum\limits_{i=1}^{k}\bar{X} - \sum\limits_{j=1}^{d}\bar{X}_j}{k-d} = \frac{10196.67 - 1060}{20 - 2} = \frac{9136.67}{18} = 507.593$$

We now calculate the revised centre line and control limits of the $\bar{X}$-chart using equations (16a to 16c) as follows:

$$CL = \bar{\bar{X}}_{new} = 507.593$$

$$UCL = \bar{\bar{X}}_{new} + A_2\bar{R}_{new} = 507.593 + 0.729 \times 25 = 525.818$$

$$LCL = \bar{\bar{X}}_{new} - A_2\bar{R}_{new} = 507.593 - 0.729 \times 25 = 489.368$$

We can construct the $\bar{X}$-chart by taking sample the number on the X-axis and the average weight of the milk packets on the Y-axis as shown in Fig. 2.7.
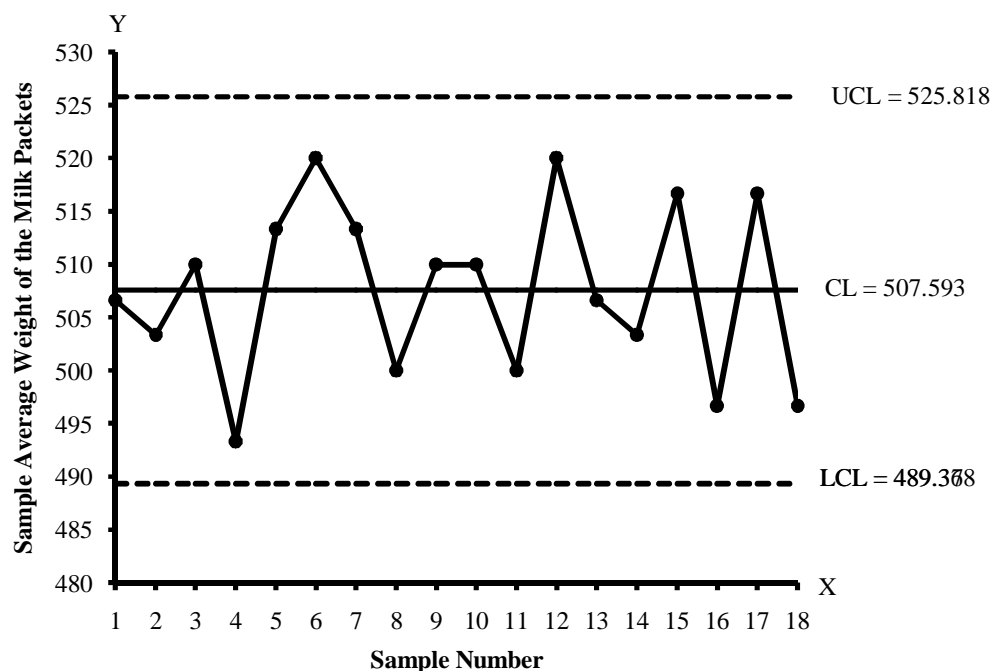

Fig. 2.7: The $\bar{X}$-chart.

Interpretation of the result

Since no point lies outside the control limits of the $\bar{X}$-chart (shown in Fig. 2.7), it indicates that the process mean is under statistical control.

You may now like to construct the $\bar{X}$-chart and the R-chart for practice.

E5) Choose the correct option from the following:

i) The upper control limit for the R-chart is

a) $D_4\bar{R}$ b) $D_3\bar{R}$ c) $\bar{R}$ d) $\bar{\bar{X}} + D_4\bar{R}$

ii) Which control chart is used for controlling the process variability?

a) $\bar{\bar{X}}$-chart   b) R-chart   c) p-chart   d) c-chart

E6) A high-voltage power station supplies a nominal output voltage of 250 V. A random sample of five outputs is taken each day and the mean and range calculated for all samples are tabulated below:

Day	1	2	3	4	5	6	7	8
$\overline{X}$	260	240	250	260	290	240	260	250
R	20	20	60	30	20	60	50	30
Day	9	10	11	12	13	14	15	16
$\overline{X}$	240	300	240	260	250	270	250	240
R	50	40	50	20	20	60	22	40

Test whether the process is under statistical control or not by drawing the $\overline{X}$ and R-charts.

In this section, you have learnt about the R-chart and how it is constructed. We now discuss the S-chart which is also used for controlling process variability.

2.6 STANDARD DEVIATION CHART (S-CHART)

In Secs. 2.4 and 2.5, you have studied the control charts for process mean ($\overline{X}$-chart) and process variability (R-chart). Although the R-chart is easy to calculate and explain, for large sample size (usually greater than 10) it does not give a correct picture of the process variability. In Unit 2 of MST-002 entitled Descriptive Statistics, you have studied that the range depends only on maximum and minimum values of the data whereas the standard deviation depends on all observations. As the number of observations increases, the range of the data fluctuates more than the standard deviation. So, when the number of observations increases (≥ 10), we use standard deviation chart (S-chart) instead of the range chart (R-chart).

To get the control limits for the S-chart, we need to know the sampling distribution of the standard deviation (S). Since the derivation of the sampling distribution of S is beyond the scope of this course, we state the result without proof. If $X_1, X_2, ..., X_n$ is a random sample of size n drawn from a normal distribution with mean μ and variance σ^2,

$$E(S) = c_4 \sigma \qquad \qquad \dots (25)$$

and

$$Var(S) = \left(1 - c_4^2\right)\sigma^2 \qquad \qquad \dots (26)$$

where c_4 is a constant and depends on the size of the sample. It has been tabulated for various sample sizes in Table I given at the end of this block.

We also know that $SE(X) = \sqrt{Var(X)}$

$$\therefore \quad SE(S) = \sqrt{Var(S)} = \sqrt{\left(1 - c_4^2\right)\sigma^2} = \sigma\sqrt{1 - c_4^2} \qquad \dots (27)$$

Although the sampling distribution of the standard deviation is not normal, it is common to use the $\pm 3\sigma$ limits. These limits assure that the probability of an observation outside these limits is very small. Therefore, we can calculate the CL, LCL and UCL for the S-chart as follows:

$$\text{Centre line (CL)} = E(S) = c_4\sigma \qquad\qquad \dots (28a)$$

$$\text{Lower control limit (LCL)} = E(S) - 3SE(S) = c_4\sigma - 3\sigma\sqrt{1-c_4^2}$$

$$\text{or} \qquad\qquad \text{LCL} = \left(c_4 - 3\sqrt{1-c_4^2}\right)\sigma = B_5\sigma \qquad\qquad \dots (28b)$$

$$\text{Upper control limit (UCL)} = E(S) + 3SE(S) = c_4\sigma + 3\sigma\sqrt{1-c_4^2}$$

$$\text{or} \qquad\qquad \text{UCL} = \left(c_4 + 3\sqrt{1-c_4^2}\right)\sigma = B_6\sigma \qquad\qquad \dots (28c)$$

where $B_5 = c_4 - 3\sqrt{1-c_4^2}$ and $B_6 = c_4 + 3\sqrt{1-c_4^2}$ are constants and depend on the size of the sample. These constants have been tabulated for various sample sizes in Table I given at the end of this block.

In practice, the value of σ is not known. Therefore, it is estimated from the samples which are taken when the process is thought to be under statistical control. Since the S-chart is used for large sample size (≥ 10), σ is estimated by the sample standard deviation (S) as we have described in Case II of Sec. 2.4.

Hence, the control limits for the S-chart when σ is estimated by the average sample standard deviation are given by

$$\text{CL} = E(S) = c_4\hat{\sigma} = c_4\frac{\overline{S}}{c_4} = \overline{S} \qquad\qquad \dots (29a)$$

$$\text{LCL} = E(S) - 3SE(S) = c_4\hat{\sigma} - 3\hat{\sigma}\sqrt{1-c_4^2}$$

$$\text{or} \qquad \text{LCL} = c_4\frac{\overline{S}}{c_4} - 3\frac{\overline{S}}{c_4}\sqrt{1-c_4^2} = \left(1 - \frac{3}{c_4}\sqrt{1-c_4^2}\right)\overline{S} = B_3\overline{S} \dots (29b)$$

$$\text{UCL} = E(S) + 3SE(S) = c_4\hat{\sigma} + 3\hat{\sigma}\sqrt{1-c_4^2}$$

$$\text{or} \qquad \text{UCL} = c_4\frac{\overline{S}}{c_4} + 3\frac{\overline{S}}{c_4}\sqrt{1-c_4^2} = \left(1 + \frac{3}{c_4}\sqrt{1-c_4^2}\right)\overline{S} = B_4\overline{S} \dots (29c)$$

where $B_3 = 1 - \frac{3}{c_4}\sqrt{1-c_4^2}$ and $B_4 = 1 + \frac{3}{c_4}\sqrt{1-c_4^2}$ are constants and depend on the size of the sample. These constants have been tabulated for various sample sizes in Table I given at the end of this block.

The construction of the S-chart is similar to the R-chart. We take the sample number on the X-axis (horizontal scale) and the sample standard deviation on the Y-axis (vertical scale). We represent the centre line by solid line and UCL and LCL by dotted lines. We also plot the value of sample standard deviation

for each sample against the sample number. The consecutive sample points are joined by line segments.

Interpretation of the result

If all sample points lie on or in between the upper and lower control limits, the S- chart indicates that the process is under statistical control. Otherwise the process is out-of-control. To bring the process under statistical control it is necessary to investigate the assignable causes and take corrective action to eliminate them. For this, we delete the out-of-control points (samples) and calculate the revised centre line and control limits as was done in the $\overline{X}$-chart. These limits are known as the **revised control limits**. For revised limits for the S-chart, we first calculate $\overline{S}_{new}$ by using the following formula:

$$\overline{S}_{new} = \frac{\sum_{i=1}^{k} S - \sum_{j=1}^{d} S_j}{k-d} \qquad \ldots (30)$$

where

d – the number of discarded samples, and

$\sum_{j=1}^{d} S_j$ – the sum of standard deviations of discarded samples.

We reconstruct the centre line and control limits of the S-chart by replacing $\overline{S}$ by $\overline{S}_{new}$ in equations (29a to 29c) as follows:

Centre Line (CL) = $\overline{S}_{new}$ ... (30a)

Lower control limit (LCL) = $B_3\overline{S}_{new}$ ... (30b)

Upper control limit (UCL) = $B_4\overline{S}_{new}$ ... (30c)

Let us illustrate the construction and interpretation of the S-chart with the help of an example.

Example 6: The mean and standard deviation of 15 samples of size 12 for a production process are given in the following table:

Sample No	Mean	Standard Deviation	Sample No	Mean	Standard Deviation
1	9.5	1.2	9	8.75	2.1
2	7.5	1.5	10	9.75	3.5
3	10.0	2.6	11	7.25	2.6
4	8.25	3.2	12	10.25	1.1
5	7.25	1.9	13	9.25	2.2
6	9.0	2.6	14	10.0	1.6
7	10.5	5.4	15	10.5	3.2
8	9.25	2.4			

Determine the control limits for the $\overline{X}$ and S-charts. Comment on whether the process variability and process mean are under statistical control or not. Use the quality control factors $A_3 = 0.886$, $B_3 = 0.354$, $B_4 = 1.646$ for n = 12.

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Process Control

The CL, UCL and LCL for the $\bar{X}$-chart when σ is estimated by S are

$CL = \bar{\bar{X}}$

$UCL = \bar{\bar{X}} + A_3\bar{S}$

$LCL = \bar{\bar{X}} - A_3\bar{S}$

The CL, UCL and LCL for the S-chart when σ is estimated by S are

$CL = \bar{S}$

$UCL = B_4\bar{S}$

$LCL = B_3\bar{S}$

Solution: Here, we want to control the process variability and process mean. Since the sample size is greater than 10, we use S and $\bar{X}$-charts.

Since sample size n > 10, we use equations (13a to13c) and (29a to 29c) for calculating the centre and control limits for the $\bar{X}$-chart and S-chart, respectively.

It is given that

$$n = 12, k = 15, A_3 = 0.886, B_3 = 0.354 \text{ and } B_4 = 1.646$$

We first calculate the values of $\bar{\bar{X}}$ and $\bar{S}$ for centre line and control limits using equations (5) and (11) as follows:

$$\bar{\bar{X}} = \frac{1}{k}\sum \bar{X} = \frac{1}{15}(9.5 + 7.5 + ... + 10.5) = \frac{1}{15} \times 137 = 9.133$$

$$\bar{S} = \frac{1}{k}\sum S = \frac{1}{15}(1.2 + 1.5 + ... + 3.2) = \frac{1}{15} \times 37.1 = 2.473$$

To check whether the process is under statistical control or not, we first draw and analyse the S-chart.

Substituting the values of $\bar{S}, B_3$ and B_4 in equations (29a to 29c), we get the centre line and control limits of the S-chart as follows:

$CL = \bar{S} = 2.473$

$UCL = B_4\bar{S} = 1.646 \times 2.473 = 4.071$

$LCL = B_3\bar{S} = 0.354 \times 2.473 = 0.875$

We construct the S-chart by taking the sample number on the X-axis and the sample standard deviation (S) on the Y-axis as shown in Fig. 2.8.
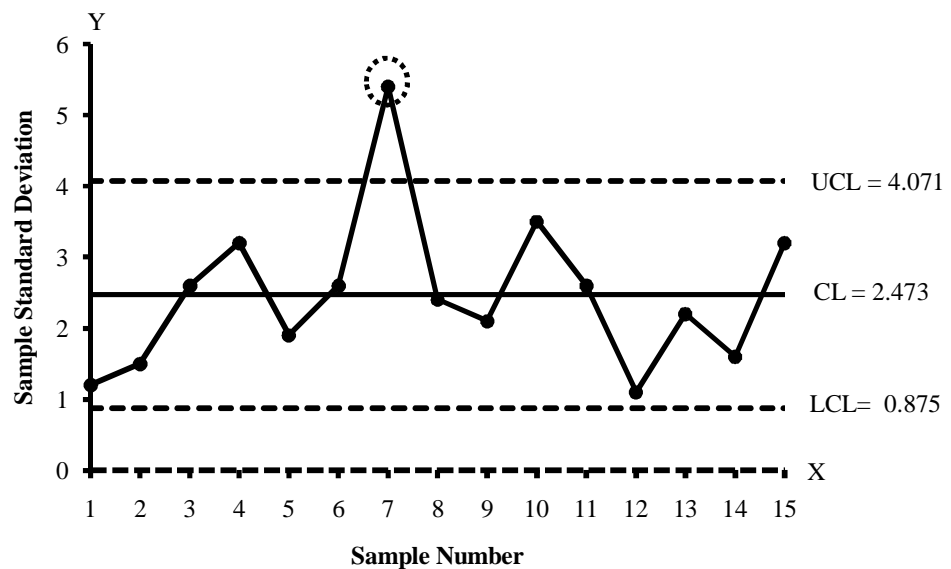

Fig. 2.8: The S-chart for Example 6.

Interpretation of the result

The S-chart shown in Fig. 2.8 indicates that the point corresponding to sample 7 lies outside the upper control limits. Therefore, the process variability is

out-of-control and some assignable causes are present in the process. To bring the process under statistical control it is necessary to investigate the assignable causes and take corrective action to eliminate them.

We calculate the centre line and control limits of the $\overline{X}$-chart as follows:

$$CL = \overline{\overline{X}} = 9.133$$

$$UCL = \overline{\overline{X}} + A_3\overline{S} = 9.133 + 0.886 \times 2.473 = 11.324$$

$$LCL = \overline{\overline{X}} - A_3\overline{S} = 9.133 - 0.886 \times 2.473 = 6.942$$

We construct the $\overline{X}$-chart by taking the sample number on the X-axis and the sample average ($\overline{X}$) on the Y-axis as shown in Fig. 2.9.
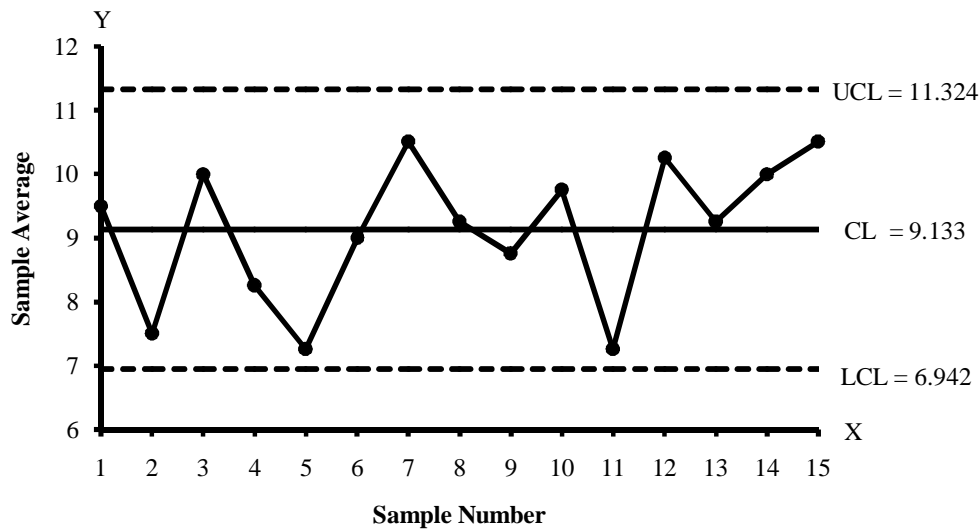

Fig. 2.9: The $\overline{X}$-chart for Example 6.

Interpretation of the result

The $\overline{X}$-chart shown in Fig. 2.9 indicates that all the sample points lie within the control limits. So the process mean is under statistical control.

Hence, with the help of S and $\overline{X}$-chart we conclude that the process variability is not under statistical control, whereas the process average is under statistical control.

You may like to check your understanding about the S-chart by answering the following exercise.

E7) In a factory, quality is controlled using mean ($\overline{X}$) and standard deviation (S) charts. Twenty samples are chosen, each having 10 units for which $\sum \overline{X} = 645$ and $\sum S = 6$. Determine the 3σ limits for the $\overline{X}$ and S-charts. It is given that $A_3 = 0.97$, $B_3 = 0.284$ and $B_4 = 1.716$ for n = 10.

So far we have discussed one aspect of statistical process control, that is, to ensure that the process is under statistical control with the help of control chart. If a process is under statistical control, it merely indicates that there are no assignable causes of variations and that it is stable. Once a process is under statistical control, the next step is to check whether the process is able to meet

the **specifications** or not. For this, we use process capability analysis, which we discuss in the next section.

2.7 PROCESS CAPABILITY ANALYSIS

Variation in manufactured products is inevitable and it is a fact of nature and industrial life. Even if the production process is well designed or carefully maintained, no two objects are identical. The difference in two items/units may range from very large to very small or it may even be undetectable depending on the sources of variation.

Sometimes, it is possible that the process is under statistical control, but the customer is not satisfied with the product. Such a situation may arise when the product does not conform to its specifications. Then we need to carry out the process capability analysis to determine whether a product conforms to its specifications.

To conduct the process capability study, it is important to distinguish between the specification limits of a product, the control limits of the process and the natural tolerance limits of the product. We now explain these limits.

Specification Limits

The limits specified by the consumer for the value of a variable characteristic of the product or set by the management or the manufacturing engineers at the design and development stages of the product are called **specification limits**. Thus, the specification limits are determined externally. The upper value of the specification limits is called upper specification limit (USL) and the lower value of the specification limits is called lower specification limit (LSL). If a product fails to meet the specifications, it is called a **defective product**. This means that any product manufactured outside these limits would have to be scrapped and reworked. Therefore, the process should be set in a way that these limits are met (if possible) by all products.

For example, the weight of a cricket ball may be set by the customer or by the manufacturing engineers as 160.0 ± 3.0 gram. Then the weight of a cricket ball produced by the manufacturer can vary from $157.0 (= 160.0 - 3.0)$ gram to $163.0 (= 160.0 + 3.0)$ gram. Thus, the USL is 163.0 gram and LSL is 157.0 gram. If any ball produced by the manufacturer has weight greater than 163.0 gram or less than 157.0 gram, it may have to be scrapped.

Control Limits

In Sec. 2.2, you have learnt about the control limits for a process. You know that the control limits are usually determined by samples collected from a process. If a sample point lies outside the control limits, the process is said to be **out-of-control**. However, even in this case the individual product may not necessarily be defective because in the control charts, we plot a statistic such as the sample mean, range instead of individual measurement of the quality characteristic. It means that a single product that falls outside the control limits will neither cause the process to be out-of-control nor defective.

Natural Tolerance Limits

The natural tolerance limits (NTLs) are the limits usually determined by individual measurements of the characteristic of the product manufactured. These limits represent the natural variability of products in the process and are set at the 3σ limits from the mean, i.e., $\mu \pm 3\sigma$ where μ and σ are the process

mean and standard deviation, respectively, for individual measurements of the characteristic of the products produced. Thus, $\mu + 3\sigma$ is known as the **upper natural tolerance limit** (UNTL) and $\mu - 3\sigma$ is known as the **lower natural tolerance limit** (LNTL). If μ and σ are unknown, their values are estimated by the grand sample mean and range or standard deviation, respectively, as we have described in Sec.2.4 for the $\overline{X}$-chart.

It is not that there is no relationship between the specification limits, control limits and natural tolerance limits just because the specification limits are specified by the customer and the control limits and natural tolerance limits are driven by the variability of the process.

We now distinguish between the specification limits of a product, the control limits of the manufacturing process and the natural tolerance limits of the product with the help of an example.

Example 7: A cricket ball manufacturing company wants to control the weight of the ball. Twenty-five samples, each of size 4 were collected. The sum of averages and sum of sample ranges were found to be 4010 gram and 72 gram, respectively. If the customer specified the weight of the ball as 160 ± 3.0 gram, find the estimates of process mean and standard deviation. Also find the specification limits, control limits and natural tolerance limits. It is given that, $d_2 = 2.088$ and $A_2 = 0.729$ for $n = 4$.

Solution: It is given that

$$k = 25, \ n = 4, \ \sum \overline{X} = 4010, \ \sum R = 72, d_2 = 2.059 \text{ and } A_2 = 0.729$$

The process mean is estimated by the grand sample mean which is calculated as follows:

$$\hat{\mu} = \overline{\overline{X}} = \frac{1}{k} \sum \overline{X} = \frac{1}{25} \times 4010 = 160.4$$

The process standard deviation is estimated by the sample range which is calculated as follows:

$$\hat{\sigma} = \frac{\overline{R}}{d_2} = \frac{1}{d_2} \frac{1}{k} \sum R = \frac{1}{2.059} \times \frac{1}{25} \times 72 = 1.399$$

It is given that the customer specified the weight of the ball as:

160.0 ± 3.0 gram. Therefore,

The targeted or aimed value $(\mu_0) = 160.0$

Upper specification limit (USL) $= 160.0 + 3.0 = 163.0$

Lower specification limit (USL) $= 160.0 - 3.0 = 157.0$

Since the manufacturer has to control the mean weight of the balls, we use $\overline{X}$-chart. Here μ and σ are unknown, therefore, we can calculate the centre line and control limits for the $\overline{X}$-chart using equations (9a to 9c) as follows:

Centre line (CL) $= \overline{\overline{X}} = 160.4$

Upper control limit (UCL) $= \overline{\overline{X}} + A_2\overline{R} = 160.4 + 0.729 \times 2.88 = 162.50$

Lower control limit (LCL) $= \overline{\overline{X}} - A_2\overline{R} = 160.4 - 0.729 \times 2.88 = 158.30$

The natural tolerance limits can be obtained as follows:

Process mean $(\mu) = \hat{\mu} = \overline{\overline{X}} = 160.4$

Upper natural tolerance limit (UNTL) $= \hat{\mu} + 3\hat{\sigma} = 160.4 + 3 \times 1.399 = 164.597$

Lower natural tolerance limit (LNTL) $= \hat{\mu} - 3\hat{\sigma} = 160.4 - 3 \times 1.399 = 156.203$

So far we have explained the specification limits, control limits and natural tolerance limits and basic difference among them. We now discuss process capability.

Process Capability

Process capability is a measure of the ability of a process to meet specifications. It tells us how good an individual product is. Even though the process may seem to be under statistical control, it is necessary to see whether the process is capable of manufacturing products within the specific limits or not. This can be done by process capability analysis.

Process capability may be defined as the spread of a quality characteristic measurement (i.e., natural variability) when the process is under statistical control (stable).

If the standard deviation of the stable process is σ, the process capability is defined as follows:

Process capability $= 6\sigma$

This indicates that products can be manufactured within $\pm 3\sigma$ limits. Generally, standard deviation of a process is not known and we estimate σ by range or standard deviation as follows:

$$\hat{\sigma} = \frac{\overline{R}}{d_2} \text{ or } \hat{\sigma} = \frac{\overline{S}}{c_4}$$

To test whether the process is capable of manufacturing the product within the specific limits, we compare process capability with tolerance. Tolerance is defined as the difference between specification limits, i.e.,

Tolerance = USL – LSL

When we compare process capability with tolerance, the following situations may arise:

Case I: $6\sigma < \text{USL} - \text{LSL}$

In this case, the process is capable of producing products of desired specifications, even though the process is under statistical control or out-of-control.

Case II: $6\sigma = \text{UCL} - \text{LSL}$

In this case, the process is capable of meeting specifications only if no shift in the process mean or a change in process variability takes place.

Case III: $6\sigma > \text{USL} - \text{LSL}$

In this situation, the products do not meet the desired specifications, even though the process is under statistical control.

The relation of process capability with the specification limits can be understood from the following example of your daily life.

Consider a car and a garage (Fig. 2.10). The garage defines the specification limits and the car defines the output of the process. If the car is only a little bit smaller than the garage, we can park it right in the middle of the garage (centre of the specification). If the car is wider than the garage, it does not matter if we have it centred; it will not fit. If the car is a lot smaller than the garage, it does not matter if we park it exactly in the middle; it will fit and we have plenty of room on either side.

If we have a process in control with little variation, we should be able to (park the car easily within the garage) manufacture a product that meets its specifications as well as customer requirements.

Now, we take up an example based on process capability.

Example 8: A manufacturing process produces a certain type of bolt of mean diameter 2 inche with a standard deviation of 0.05 inche. The lower and upper specification limits of the process are 1.90 and 2.05 inche. Calculate the process capability. Does it appear that the manufacturing process is capable of meeting the specification requirements?

Solution: We know the process capability (PC) of any process is defined as

$PC = 6\sigma$ where σ is the standard deviation of the process.

It is given that

$$\mu = 2, \sigma = 0.05, \ USL = 2.05 \text{ and } LSL = 1.90$$

$$\therefore \ PC = 6\sigma = 6 \times 0.05 = 0.30$$

Now we have to test whether the process is capable of meeting the required specifications. We first calculate tolerance and then compare PC with the tolerance:

$$\text{Tolerance} = USL - LSL$$

$$= 2.05 - 1.90 = 0.15$$

Since process capability > tolerance, the products do not meet the desired specifications.

Now, you can check your understanding of process capability by answering the following exercise.

E8)   An automatic machine is used to fill and seal 20 ml tube of medicine. The process is sampled in samples of four and the values of $\overline{X}$ and R are computed for each sample. After 25 samples, $\sum \overline{X} = 525$ and $\sum R = 30.$ If the specification limits are 20 ± 1.5, what are the conclusions regarding the ability of the process to produce medicine tubes conforming to specifications? It is given that $d_2 = 2.059$ for $n = 4$.

We end this unit by giving a summary of what we have covered in it.

2.8   SUMMARY

1.   **Pareto diagram** is a bar diagram, used for selection of quality characteristic for control in statistical quality control.

2.   If the characteristic to be controlled is **measureable** such as weight, length, diameter, etc., **control chart for variables** can be used.

Fig. 2.10

3. If a characteristic **cannot be measured**, e.g., colour, surface roughness, etc., **control chart for attributes** can be used.

4. A **rational subgroup** is a small set of items that are produced under similar conditions within a relatively short time, such that the variation within the group is only due to chance causes.

5. If all sample points lie on or in between the upper and lower control limits and there is no unnatural pattern of variation, the control chart indicates that the process is under statistical control.

6. If some points lie outside the control limits or there is a long run or unnatural patterns of variation, the control chart indicates that the process is not under statistical control.

7. If we wish to control the **process mean**, we use **control chart for mean** ($\bar{X}$-chart). If we wish to control **process variability**, we use the **R-chart and S-chart**.

8. The centre line and control limits of the $\bar{X}$-chart are given as

   $$CL = \bar{\bar{X}}$$

   $$LCL = \bar{\bar{X}} - A_2 \bar{R}$$

   $$UCL = \bar{\bar{X}} + A_2 \bar{R}$$

   where $\bar{\bar{X}} = \dfrac{1}{k}\sum_{i=1}^{k} \bar{X}_i$, $\bar{R} = \dfrac{1}{k}\sum_{i=1}^{k} R_i$ and A_2 is a constant.

9. The centre line and control limits of the R-chart are given as

   $$CL = \bar{R}$$

   $$LCL = D_3 \bar{R}$$

   $$UCL = D_4 \bar{R}$$

   where D_3 and D_4 are constants.

10. The centre line and control limits of the S-chart are given as

    $$CL = \bar{S}$$

    $$LCL = B_3 \bar{S}$$

    $$UCL = B_4 \bar{S}$$

    where $\bar{S} = \dfrac{1}{k}\sum_{i=1}^{k} S_i$, $S_i = \sqrt{\dfrac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}_i\right)^2}$ and D_3 and D_4 are constants.

11. The **specification limits** are the limits specified by the consumer or set by the management or the manufacturing engineers.

12. The **natural tolerance limits** (NTLs) are the limits usually determined by individual measurements of the characteristic of the product manufactured. These limits represent the natural variability of products in the process and are set at the 3σ limits.

13. **Process capability** is a measure of the ability of a process to meet specifications.

2.9   SOLUTION /ANSWERS

E1)   i)   Option (d) is the correct option because we know that control
charts for variables are the $\overline{X}$-chart, R-chart and S-chart.

ii)   Option (a) is the correct option because we use the $\overline{X}$-chart for
controlling the process mean, whereas the R-chart and the S-chart
are used for controlling the process variability. We use the p-chart
for controlling fraction defective of the process. You will study
the p-chart in Unit 3.

iii)   Option (b) is the correct option because the R-chart and the
S-chart are used for controlling process variability. Whereas the
$\overline{X}$-chart is used for process mean and the p-chart for fraction
defective.

E2)   i)   Option (a) is the correct option because we know that the control
limits for the $\overline{X}$-chart are $\overline{\overline{X}} \pm A_2 \overline{R}$.

ii)   Option (b) is the correct option because we know that if one or
more points lie outside the control limits, process is said to be
out-of-control.

iii)   Option (a) is the correct option because we know that the $\overline{X}$, R
and S-charts are control charts for variables, whereas the p, np, c
and u-charts are control charts for attributes.

E3)   It is given that

$$n = 5, k = 20, \overline{\overline{X}} = 6.40, \overline{R} = 0.0877$$

Since the process average (μ) and the process variability (σ) are
unknown, in this case, we use equations (9a to 9c) to calculate the centre
line and control limits of the $\overline{X}$-chart.

From Table I given at the end of this block, we have $A_2 = 0.577$ for n = 5.

Therefore, we can calculate the centre line and control limits for the
$\overline{X}$-chart as follows:

$$CL = \overline{\overline{X}} = 6.40$$

$$UCL = \overline{\overline{X}} + A_2 \overline{R} = 6.40 + 0.577 \times 0.0877 = 6.451$$

$$LCL = \overline{\overline{X}} - A_2 \overline{R} = 6.40 - 0.577 \times 0.0877 = 6.349$$

The CL, UCL and
LCL for the $\overline{X}$-chart
when μ and σ are
unknown are

$CL = \overline{\overline{X}}$

$UCL = \overline{\overline{X}} + A_2 \overline{R}$

$LCL = \overline{\overline{X}} - A_2 \overline{R}$

E4)   i)   It is given that

$$\mu = 1000, \ \sigma = 215, \ k = 16 \ \text{and} \ n = 3$$

To draw the $\overline{X}$-chart, we first calculate the centre line and control
limits when the average life and the standard deviation of the life of
bulbs are known as follows:

Centre line $(CL) = \mu = 1000$

Upper control limit $(UCL) = \mu + A\sigma = 1000 + 1.732 \times 215 = 1372.380$

Lower control limit (LCL) $= \mu - A\sigma = 1000 - 1.732 \times 215 = 627.620$

To test whether the process under statistical control or not, we plot the sample mean $(\overline{X})$ against the sample number by taking the sample number on the X-axis and the sample average life of the bulb $(\overline{X})$ on the Y-axis as shown in Fig. 2.11.
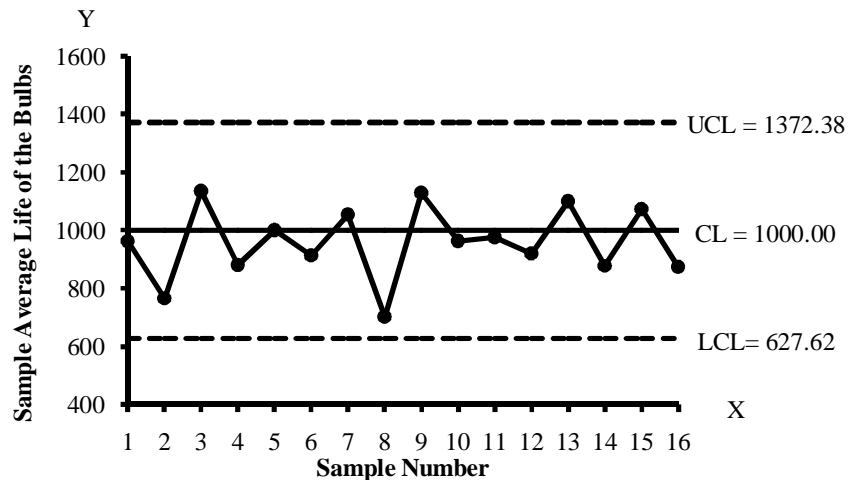

Fig. 2.11: The $\overline{X}$-chart for average life of the bulbs.

Interpretation of the result

From Fig. 2.11, we observe that no point lies outside the control limits of the $\overline{X}$-chart and also there is no specific pattern of the sample points on the chart. So this indicates that the process is under statistical control with respect to the average life of the bulbs.

ii) In this case, the average life and the standard deviation of the life of the bulbs are not known. So we use equations (9a to 9c) to calculate the centre line and control limits of the $\overline{X}$-chart.

For calculating the centre line and control limits of the $\overline{X}$-chart, we first calculate the grand mean $(\overline{\overline{X}})$, range (R) and mean of range $(\overline{R})$ by the formulas:

$$\overline{\overline{X}} = \frac{1}{k}\sum \overline{X}, \ R = Max(X) - Min(X) \text{ and } \overline{R} = \frac{1}{k}\sum R$$

Sample Number	Life of Bulb (in hour) (X)			Total	$\overline{\mathbf{X}}$	R
1	1152	650	1090	2893	964.00	502
2	688	540	1074	2304	767.33	534
3	876	1342	1187	3408	1135.00	466
4	725	1070	847	2646	880.67	345
5	900	846	1254	3005	1000.00	408
6	1080	790	871	2747	913.67	290
7	952	1087	1126	3172	1055.00	174
8	741	786	578	2113	701.67	208
9	1005	1244	1140	3398	1129.67	239

10	1230	1005	654	2899	963.00	576
11	1200	870	859	2940	976.33	341
12	745	964	1050	2771	919.67	305
13	926	1172	1200	3311	1099.33	274
14	781	876	981	2652	879.33	200
15	1120	1350	750	3235	1073.33	600
16	880	975	768	2639	874.33	207
Total					15332.33	5669

From the above table, we have

$$\bar{\bar{X}} = \frac{1}{k}\sum \bar{X} = \frac{1}{16}\times 15332.33 = 958.271$$

$$\bar{R} = \frac{1}{k}\sum R = \frac{1}{16}\times 5669 = 354.313$$

We can calculate the centre line and control limits of the $\bar{X}$-chart as follows:

$$CL = \bar{\bar{X}} = 958.271$$

$$UCL = \bar{\bar{X}} + A_2\bar{R} = 958.271 + 1.023 \times 354.313 = 1320.733$$

$$LCL = \bar{\bar{X}} - A_2\bar{R} = 958.27 - 1.023 \times 354.313 = 595.809$$

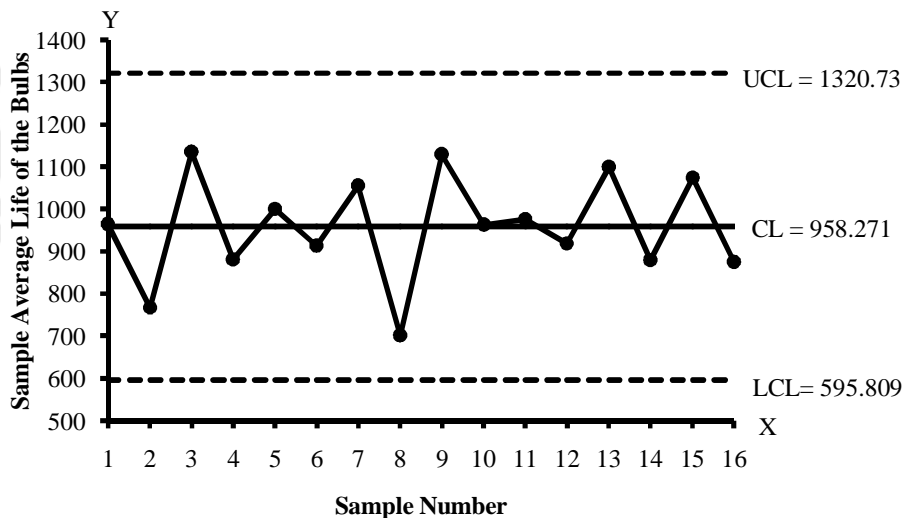We draw the $\bar{X}$-chart as in case (i) (see Fig. 2.12).

Fig. 2.12: The $\bar{X}$-chart for average life of the bulbs.

Interpretation of the result

From Fig. 2.12, we observe that no point lies outside the control limits of the $\bar{X}$-chart and there is no specific pattern of the sample points on the chart. So this indicates that the process mean is under statistical control with respect to the average life of the bulbs.

E5) i) Option (a) is the correct option because the UCL and LCL of the R-chart are $D_4\bar{R}$ and $D_3\bar{R}$.

ii) Option (b) is the correct option because we know that for controlling the process variability, we use either the R-chart or the S-chart. The $\overline{X}$-chart is used for the process mean, the p-chart is used for the fraction defective and the c-chart is used for the number of defects.

E6) Here we are given that

$$n = 5 \text{ and } k = 20$$

Note that the process mean (μ) and the process variability (σ) are unknown in this case. Therefore, we use equations (9a to 9c) and (23a to 23c) for calculating the centre and control limits for the $\overline{X}$-chart and R-chart, respectively.

From Table I, we have

$$A_2 = 0.577, D_3 = 0, D_4 = 2.114 \text{ for } n = 5$$

We first calculate $\overline{\overline{X}}$ and $\overline{R}$ using equations (5) and (7) as follows:

$$\overline{\overline{X}} = \frac{1}{k}\sum \overline{X} = \frac{1}{16}(260 + 240 + ... + 240) = \frac{1}{16} \times 4100 = 256.25$$

$$\overline{R} = \frac{1}{k}\sum R = \frac{1}{16}(20 + 20 + ... + 40) = \frac{1}{16} \times 592 = 37$$

To check whether the process is under statistical control or not, we first draw and analyse the R-chart.

Substituting the values of $\overline{R}$, D_3 and D_4 in equations (23a to 23c), we get the centre line and control limits of the R-chart as follows:

$$CL = \overline{R} = 37$$

$$UCL = D_4\overline{R} = 2.117 \times 37 = 78.218$$

$$LCL = D_3\overline{R} = 0 \times 37 = 0$$

We now construct the R-chart by taking the sample number on the X-axis and the sample voltage range (R) on the Y-axis as shown in Fig. 2.13.
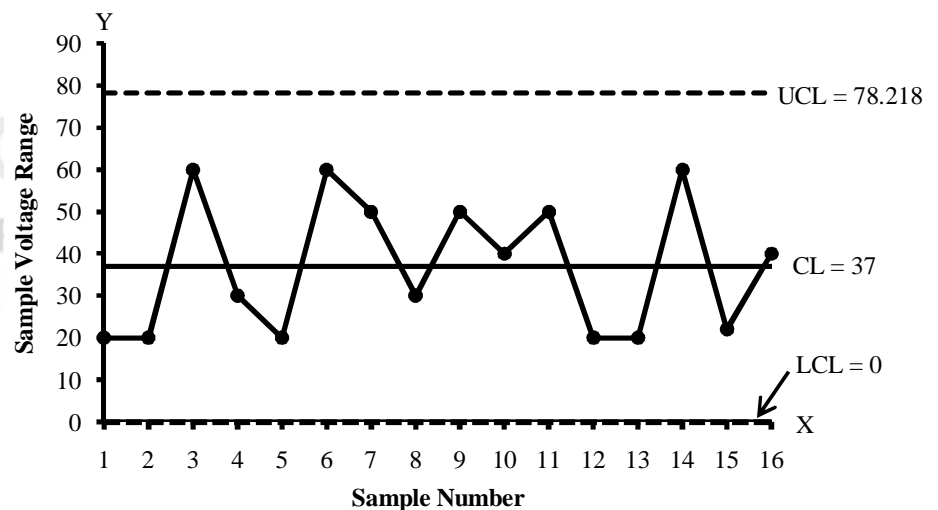

Sample Number

Fig. 2.13: The R-chart for process variability.

Interpretation of the result

The R-chart (shown in Fig. 2.13) indicates that all the points lie within
the control limits and there is no specific pattern of the sample points on
the chart. So the process variability is under statistical control.

We now check whether the process mean is under statistical control or
not using the $\overline{X}$-chart.

Substituting the values of $\overline{\overline{X}}, \overline{R}$ and A_2 in equations (9a to 9c), we get the
centre line and control limits of the $\overline{X}$-chart as follows:

$$CL = \overline{\overline{X}} = 256.25$$

$$UCL = \overline{\overline{X}} + A_2\overline{R} = 256.25 + 0.577 \times 37 = 277.599$$

$$LCL = \overline{\overline{X}} - A_2\overline{R} = 256.25 - 0.577 \times 37 = 234.901$$

We construct the $\overline{X}$-chart by taking the sample number on the X-axis and
the sample average voltage ($\overline{X}$) on the Y-axis as shown in Fig. 2.14.
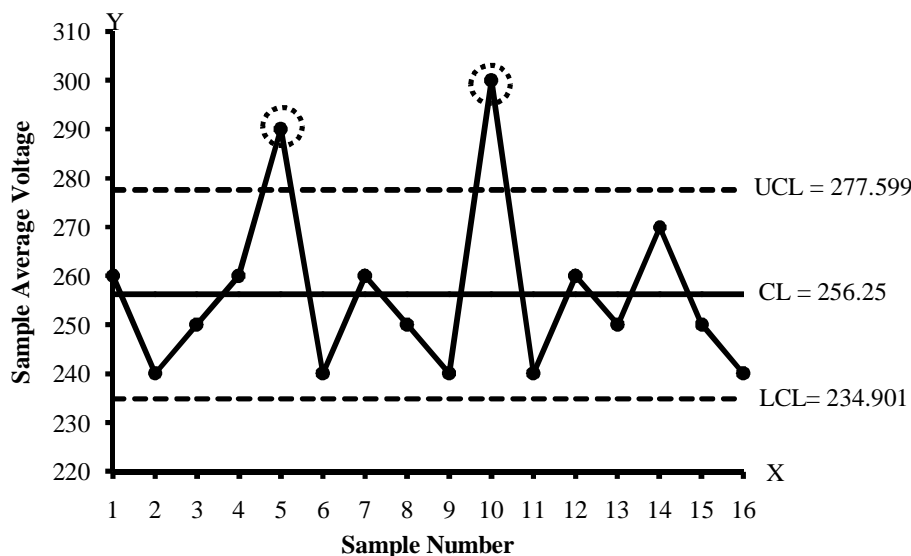
Fig. 2.14:The $\overline{X}$-chart for process mean.

Interpretation of the result

The $\overline{X}$-chart (shown in Fig. 2.14) indicates that the points
corresponding to samples 5 and 10 lie outside the control limits.
Therefore, the process average is out-of-control and some assignable
causes are present in the process.

Hence, we conclude that the process variability is under statistical
control, whereas the process average is out-of-control.

E7) It is given that

$$k = 20, \ n = 10, \ \sum \overline{X} = 645, \ \sum S = 6$$

Since the sum of sample standard deviations is given and the sample size
is 10 so in this case we use equations (13a to 13c) and (29a to 29c) to

The CL, UCL and LCL for the $\overline{X}$-chart when σ is estimated by S are

$CL = \overline{\overline{X}}$

$UCL = \overline{\overline{X}} + A_3\overline{S}$

$LCL = \overline{\overline{X}} - A_3\overline{S}$

The CL, UCL and LCL for the S-chart when σ is estimated by S are

$CL = \overline{S}$

$UCL = B_4\overline{S}$

$LCL = B_3\overline{S}$

calculate the centre line and control limits of the $\overline{X}$-chart and S-chart, respectively.

We first calculate $\overline{\overline{X}}$ and $\overline{S}$ using equations (5) and (11) as follows:

$$\overline{\overline{X}} = \frac{1}{k}\sum\overline{X} = \frac{1}{20}\times 645 = 32.25 \text{ and } \overline{S} = \frac{1}{k}\sum S = \frac{1}{20}\times 6 = 0.30$$

Substituting the values of $\overline{\overline{X}}$, A_3 and $\overline{S}$ in equations (13a to 13c) we get the centre line and control limits of the $\overline{X}$-chart as follows:

$$CL = \overline{\overline{X}} = 32.25$$

$$UCL = \overline{\overline{X}} + A_3\overline{S} = 32.25 + 0.975\times 0.30 = 32.543$$

$$LCL = \overline{\overline{X}} - A_3\overline{S} = 32.25 - 0.975\times 0.30 = 31.958$$

Similarly, substituting the values of $\overline{S}$, B_3 and B_4 in equations (29a to 29c) we get the centre line and control limits of the S-chart as follows:

$$CL = \overline{S} = 0.30$$

$$UCL = B_4\overline{S} = 1.716\times 0.30 = 0.515$$

$$LCL = B_3\overline{S} = 0.284\times 0.30 = 0.085$$

E8) Here we want to check the ability of the process. For this, we have to compare the process capability (PC) with tolerance.

We know the process capability (PC) and tolerance (T) of any process are defined as

$$PC = 6\hat{\sigma} = 6\frac{\overline{R}}{d_2}$$

and

$$T = USL - LSL T$$

It is given that

$$n = 4, k = 25, \sum R = 30, d_2 = 2.059$$

Therefore,

$$USL = 20 + 1.5 = 21.5 \text{ and } LSL = 20 - 1.5 = 18.5$$

$$\overline{R} = \frac{1}{k}\sum R = \frac{1}{25}\times 30 = 1.2$$

Therefore, we can calculate the process capability and tolerance as follows:

$$PC = 6\frac{\overline{R}}{d_2} = 6\times\frac{1.2}{2.059} = 3.50$$

$$T = USL - LSL$$
$$= 21.5 - 18.5 = 3.0$$

Since process capability is greater than tolerance, the tubes of medicine do not meet the desired specifications.