UNIT 10 POISSON DISTRIBUTION

Structure

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10.1 INTRODUCTION

In Unit 9, you have studied binomial distribution which is applied in the cases where the probability of success and that of failure do not differ much from each other and the number of trials in a random experiment is finite. However, there may be practical situations where the probability of success is very small, that is, there may be situations where the event occurs rarely and the number of trials may not be known. For instance, the number of accidents occurring at a particular spot on a road everyday is a rare event. For such rare events, we cannot apply the binomial distribution. To these situations, we apply Poisson distribution. The concept of Poisson distribution was developed by a French mathematician, Simeon Denis Poisson (1781-1840) in the year 1837.

In this unit, we define and explain Poisson distribution in Sec. 10.2. Moments of Poisson distribution are described in Sec. 10.3 and the process of fitting a Poisson distribution is explained in Sec. 10.4.

Objectives

After studing this unit, you would be able to:

- know the situations where Poisson distribution is applied;
- define and explain Poisson distribution;
- know the conditions under which binomial distribution tends to Poisson distribution;
- compute the mean, variance and other central moments of Poisson distribution;
- obtain recurrence relation for finding probabilities of this distribution; and
- know as to how a Poisson distribution is fitted to the observed data.

10.2 POISSON DISTRIBUTION

In case of binomial distributions, as discussed in the last unit, we deal with events whose occurrences and non-occurrences are almost equally important. However, there may be events which do not occur as outcomes of a definite number of trials of an experiment but occur rarely at random points of time and for such events our interest lies only in the number of occurrences and not in its non-occurrences. Examples of such events are:

i) Our interest may lie in how many printing mistakes are there on each page of a book but we are not interested in counting the number of words without any printing mistake.

ii) In production where control of quality is the major concern, it often requires counting the number of defects (and not the non-defects) per item.

iii) One may intend to know the number of accidents during a particular time interval.

Under such situations, binomial distribution cannot be applied as the value of n is not definite and the probability of occurrence is very small. Other such situations can be thought of yourself. Poisson distribution discovered by S.D. Poisson (1781-1840) in 1837 can be applied to study these situations.

Poisson distribution is a limiting case of binomial distribution under the following conditions:

i) n, the number of trials is indefinitely large, i.e. $n \to \infty$.

ii) p, the constant probability of success for each trial is very small, i.e. $p \to 0$.

iii) np is a finite quantity say 'λ'.

Definition: A random variable X is said to follow Poisson distribution if it assumes indefinite number of non-negative integer values and its probability mass function is given by:

$$p(x) = P(X = x) = \begin{cases} \dfrac{e^{-\lambda}.\lambda^{x}}{\underline{|x}}; & x = 0, 1, 2, 3, \ldots \text{ and } \lambda > 0. \\ 0; & \text{elsewhere} \end{cases}$$

where e = base of natural logarithm, whose value is approximately equal to 2.7183 corrected to four decimal places. Value of $e^{-\lambda}$ can be written from the table given in the Appendix at the end of this unit, or, can be seen from any book of log tables.

Remark 1

i) If X follows Poisson distribution with parameter λ then we shall use the notation $X \sim P(\lambda)$.

ii) If X and Y are two independent Poisson variates with parameters λ_1 and λ_2 repectively, then X + Y is also a Poisson variate with parameter $\lambda_1+\lambda_2$. This is known as **additive property of Poisson distribution**.

10.3 MOMENTS OF POISSON DISTRIBUTION

r^{th} order moment about origin of Poisson variate is

$$\mu_r' = E\left(X^r\right) = \sum_{x=0}^{\infty} x^r p(x) = \sum_{x=0}^{\infty} x^r \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$\mu_1' = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{\lfloor x} = \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x\lfloor x-1} = \sum_{x=1}^{\infty} \frac{e^{-\lambda}.\lambda^x}{\lfloor x-1}$$

$$= e^{-\lambda}\left[\frac{\lambda^1}{\lfloor 0} + \frac{\lambda^2}{\lfloor 1} + \frac{\lambda^3}{\lfloor 2} + \ldots\right]$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda^1}{\lfloor 1} + \frac{\lambda^2}{\lfloor 2} + \ldots\right]$$

$$= e^{-\lambda}\lambda e^{\lambda} \quad \left[\because e^{\lambda} = 1 + \frac{\lambda}{\lfloor 1} + \frac{\lambda^2}{\lfloor 2} + \frac{\lambda^3}{\lfloor 3} + \ldots \text{(see Unit 2 of MST-001)}\right]$$

$$= \lambda$$

$\therefore$ Mean = λ

$$\mu_2' = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$= \sum_{x=0}^{\infty}\left[x(x-1) + x\right]\frac{e^{-\lambda}.\lambda^x}{\lfloor x} \qquad \text{[As done in Unit 9 of this Course]}$$

$$= \sum_{x=0}^{\infty}\left(x(x-1)\frac{e^{-\lambda}\lambda^x}{\lfloor x} + x\frac{e^{-\lambda}\lambda^x}{\lfloor x}\right)$$

$$= \sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x(x-1)\lfloor x-2} + \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$= e^{-\lambda}\sum_{x=2}^{\infty}\frac{\lambda^x}{\lfloor x-2} + \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{\lfloor x}$$

$$= e^{-\lambda}\left[\frac{\lambda^2}{\lfloor 0} + \frac{\lambda^3}{\lfloor 1} + \frac{\lambda^4}{\lfloor 2} + \ldots\right] + \mu_1'$$

$$= e^{-\lambda}\lambda^2\left[1 + \frac{\lambda}{\lfloor 1} + \frac{\lambda^2}{\lfloor 2} + \ldots\right] + \mu_1'$$

$$= e^{-\lambda}\lambda^2 e^{\lambda} + \mu_1'$$

$$= \lambda^2 + \lambda$$

$\therefore$ Variance of X is given as V(X) = $\mu_2 = \mu_2' - \left(\mu_1'\right)^2$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$= \lambda$$

$$\mu_3' = \sum_{x=0}^{3} x^3 p(x)$$

Writing x^3 as $x(x-1)(x-2)+3x(x-1)+x,$ we have

$\left[\begin{array}{l}\text{See Unit 9 of this course where} \\ \text{the expression of } \mu_3' \text{ is obtained}\end{array}\right]$

$$= \sum_{x=0}^{\infty}\left[x(x-1)(x-2)+3x(x-1)+x\right]\frac{e^{-\lambda}\lambda^x}{\underline{|x}}$$

$$= \sum_{x=3}^{\infty} x(x-1)(x-2)\frac{e^{-\lambda}\lambda^x}{\underline{|x}} + 3\sum_{x=2}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{\underline{|x}} + \sum_{x=1}^{\infty} x\frac{e^{-\lambda}\lambda^x}{\underline{|x}}$$

$$= e^{-\lambda}\sum_{x=3}^{\infty} x(x-1)(x-2)\frac{\lambda^x}{x(x-1)(x-2)\underline{|x-3}} + 3\left(\lambda^2\right) + (\lambda)$$

$$= e^{-\lambda}\sum_{x=3}^{\infty}\frac{\lambda^x}{\underline{|x-3}} + 3\lambda^2 + \lambda$$

$$= e^{-\lambda}\left(\frac{\lambda^3}{\underline{|0}} + \frac{\lambda^4}{\underline{|1}} + \frac{\lambda^5}{\underline{|2}} + \ldots\right) + 3\lambda^2 + \lambda$$

$$= e^{-\lambda}\lambda^3\left(1 + \frac{\lambda}{\underline{|1}} + \frac{\lambda^2}{\underline{|2}} + \ldots\right) + 3\lambda^2 + \lambda$$

$$= e^{-\lambda}\lambda^3 e^{\lambda} + 3\lambda^2 + \lambda$$

$$= \lambda^3 + 3\lambda^2 + \lambda$$

Third order central moment is

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\left(\mu_1'\right)^3$$

$$= \lambda \qquad \text{[On simplification]}$$

$$\mu_4' = \sum_{x=3}^{\infty} x^4 . \frac{e^{-\lambda}\lambda^x}{\underline{|x}}$$

Now writing $x^4 = x(x-1)(x-2)(x-3)+6x(x-1)(x-2)+7x(x-1)+x,$ and proceeding in the similar fashion as done in case of μ_3', we have

$$\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

$\therefore$ Fourth order central moment is

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4$$

$$= 3\lambda^2 + \lambda \qquad \text{[On simplification]}$$

Therefore, measures of skewness and kurtosis are given by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda}, \qquad \gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}; \text{ and}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2 + \lambda}{\lambda^2} = 3 + \frac{1}{\lambda}, \qquad \gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}.$$

Now as γ_1 is positive, therefore the Poisson distribution is always positively skewed distribution. Also as $\gamma_2 >$ ($\because$ $\lambda > 0$), the curve of the distribution is Leptokurtic.

Remark 2

i) Mean and variance of Poisson distribution are always equal. In fact this is the only discrete distribution for which Mean = Variance = the third central moment.

ii) Moments of the Poisson distribution can be deduced from those of the binomial distribution also as explained below:

For a binomial distribution,

Mean = np

Variance = npq

$\mu_3 = npq(q - p)$

$\mu_4 = npq\left[1 + 3pq(n-2)\right] = npq\left[1 + 3npq - 6pq\right]$

Now as the Poisson distribution is a limiting form of binomial distribution under the conditions:

(i) $n \to \infty$, (ii) $p \to 0$ i.e. $q \to 1$, and (iii) $np = \lambda$ (a finite quantity);

$\therefore$ Mean, Variance and other moments of the Poisson distribution are given as:

Mean = Limiting value of np = λ

Variance = Limiting value of npq

= Limiting value of (np) (q)

= (λ) (1) = λ

μ_3 = Limiting value of npq (q – p)

= Limiting value of (npq) (q – p)

= (λ) (1 – 0)

= λ

μ_4 = Limiting value of npq [1 + 3npq – 6pq]

= Limiting value of (npq) [1 + 3(npq) –6 (p)(q)]

= (λ) [1 + 3(λ) – 6 (0)(1)]

= λ[1 + 3λ] = 3λ^2 + λ

Now let's give some examples of Poisson distribution.

Example 1: It is known that the number of heavy trucks arriving at a railway station follows the Poisson distribution. If the average number of truck arrivals during a specified period of an hour is 2, find the probabilities that during a given hour

a) no heavy truck arrive,

b) at least two trucks will arrive.

Solution: Here, the average number of truck arrivals is 2

i.e. mean = 2

$\Rightarrow \lambda = 2$

Let X be the number of trucks arrive during a given hour,

$\therefore$ by Poisson distribution, we have

$$P[X = x] = \frac{e^{-\lambda}\lambda^x}{\lfloor x} = \frac{e^{-2}(2)^x}{\lfloor x}; x = 0, 1, 2, \ldots$$

Thus, the desired probabilities are:

(a) P[arrival of no heavy truck] = P[X = 0]

$$= \frac{e^{-2}2^0}{\lfloor 0}$$

$$= e^{-2}$$

$$= 0.1353 \quad \begin{bmatrix} \text{See the table given} \\ \text{in the Appendix at} \\ \text{the end of this unit} \end{bmatrix}$$

(b) P[arrival of at least two trucks] = $P[X \geq 2]$

$$= P[X = 2] + P[X = 3] + \ldots$$

$$= 1 - \left[P[X = 1] + P[X = 0]\right]$$

$$\begin{bmatrix} \because \text{ sum of all the} \\ \text{probabilities is 1} \end{bmatrix}$$

$$= 1 - \left[\frac{e^{-2}2^0}{\lfloor 0} + \frac{e^{-2}2^1}{\lfloor 1}\right]$$

$$= 1 - e^{-2}\left[\frac{2^0}{\lfloor 0} + \frac{2^1}{\lfloor 1}\right] = 1 - e^{-2}(1 + 2)$$

$$= 1 - (0.1353)(3) = 1 - 0.4059 = 0.5941$$

Note: In most of the cases for Poisson distribution, if we are to compute the probabilities of the type $P[X > a]$ or $P[X \geq a]$, we write them as

$P[X > a] = 1 - P[X \leq a]$ and

$P[X \geq a] = 1 - P[X < a]$, because n may not be definite and hence we cannot go up to the last value and hence the probability is written in terms of its complementary probability.

Example 2: If the probability that an individual suffers a bad reaction from an injection of a given serum is 0.001, determine the probability that out of 500 individuals

i) exactly 3,

ii) more than 2

individuals suffer from bad reaction

Solution: Let X be the Poisson variate, "Number of individuals suffering from bad reaction". Then,

n = 1500, p = 0.001,

$\therefore\ \lambda = np = (1500)\ (0.001) = 1.5$

$\therefore$ By Poisson distribution,

$$P[X = x] = \frac{e^{-\lambda}\lambda^{x}}{\underline{|x}}, x = 0, 1, 2, \ldots$$

$$= \frac{e^{-1.5}.(1.5)^{x}}{\underline{|x}}; x = 0, 1, 2, \ldots$$

Thus,

i) The desired probability = P[X = 3]

$$= \frac{e^{-1.5}.(1.5)^{3}}{\underline{|3}}$$

$$= \frac{(0.2231)(3.375)}{6} \quad = 0.1255$$

$$\left[\begin{array}{l} \because\ e^{-0.5} = 0.6065, e^{-1} = 0.3679, \text{so} \\ e^{-1.5} = e^{-1} \times e^{-0.5} = (0.3679)(0.6065) = 0.2231) \\ \text{See the table given in the Appendix} \\ \ \text{at the end of this unit} \end{array}\right]$$

ii) The desired probability $= P[X > 2]$

$$= 1 - P[X \leq 2]$$

$$= 1 - [P[X = 2] + P[X = 1] + P[X = 0]]$$

$$= 1 - \left[\frac{e^{-1.5}.(1.5)^{2}}{\underline{|2}} + \frac{e^{-1.5}.(1.5)^{1}}{\underline{|1}} + \frac{e^{-1.5}.(1.5)^{0}}{\underline{|0}}\right]$$

$$= 1 - e^{-1.5}\left[\frac{2.25}{2} + 1.5 + 1\right] = 1 - (3.625)e^{-1.5}$$

$$= 1 - (3.625)(0.2231) = 1 – 0.8087 = 0.1913$$

Example 3: If the mean of a Poisson distribution is 1.44, find the values of variance and the central moments of order 3 and 4.

Solution: Here, mean = 1.44

$\Rightarrow \lambda = 1.44$

Hence, Variance = λ = 1.44

$\mu_3 = \lambda = 1.44$

$\mu_4 = 3\lambda^2 + \lambda = 3\,(1.44)^2 + 1.44 = 7.66.$

Example 4: If a Poisson variate X is such that P[X = 1] = 2P[X = 2], find the mean and variance of the distribution.

Solution: Let λ be the mean of the distribution, hence by Poisson distribution,

$$P[X = x] = \frac{e^{-\lambda}\lambda^x}{\lfloor x}; x = 0, 1, 2, ...$$

Now, $P[X = 1] = 2P[X = 2]$

$$\Rightarrow \frac{e^{-\lambda}.\lambda^1}{\lfloor 1} = 2\frac{e^{-\lambda}.\lambda^2}{\lfloor 2}$$

$\Rightarrow \lambda = \lambda^2 \Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda(\lambda - 1) = 0 \Rightarrow \lambda = 0, 1$

But λ = 0 is rejected

[$\because$ if λ = 0 then either n = 0 or p = 0 which implies that Poisson distribution does not exist in this case.]

$\therefore \lambda = 1$

Hence mean = λ = 1, and

Variance = λ = 1.

Example 5: If X and Y be two independent Poisson variates having means 1 and 2 respectively, find P[X + Y < 2].

Solution: As X ~ P(1), Y ~ P(2), therefore,

X + Y follows Poisson distribution with mean = 1 + 2 = 3.

Let X + Y = W. Hence, probability function of W is

$$P[W = w] = \frac{e^{-3}.3^w}{\lfloor w}; w = 0, 1, 2, \ldots.$$

Thus, the required probability= P[X + Y < 2]

$$= P[W < 2]$$

$$= P[W = 0] + P[W = 1]$$

$$= \frac{e^{-3}.3^0}{\lfloor 0} + \frac{e^{-3}.3^1}{\lfloor 1}$$

$= (0.0498)(1 + 3)$ [From Table, $e^{-3} = 0.0498$]

$= 0.1992.$

You may now try these exercises.

E1) Assume that the chance of an individual coal miner being killed in a mine accident during a year is $\frac{1}{1400}$. Use the Poisson distribution to calculate the probability that in a mine employing 350 miners, there will be at least one fatal accident in a year. (use $e^{-0.25} = 0.78$)

E2) The mean and standard deviation of a Poisson distribution are 6 and 2 respectively. Test the validity of this statement.

E3) For a Poisson distribution, it is given that P[X = 1] = P[X = 2], find the value of mean of distribution. Hence find P[X = 0] and P[X = 4].

We now explain as to how the Poisson distribution is fitted to the observed data.

10.4 FITTING OF POISSON DISTRIBUTION

To fit a Poisson distribution to the observed data, we find the theoretical (or expected) frequencies corresponding to each value of the Poisson variate. Process of finding the probabilities corresponding to each value of the Poisson variate becomes easy if we use the recurrence relation for the probabilities of Poisson distribution. So, in this section, we will first establish the recurrence relation for probabilities and then define the Poisson frequency distribution followed by the process of fitting a Poisson distribution.

Recurrence Formula for the Probabilities of Poisson Distribution

For a Poisson distribution with parameter λ, we have

$$p(x) = \frac{e^{-\lambda}\lambda^x}{\lfloor x} \qquad \ldots (1)$$

Changing x to x + 1, we have

$$p(x+1) = \frac{e^{-\lambda}\lambda^{x+1}}{\lfloor x+1} \qquad \ldots (2)$$

Dividing (2) by (1), we have

$$\frac{p(x+1)}{p(x)} = \frac{\left(\frac{e^{-\lambda}\lambda^{x+1}}{\lfloor x+1}\right)}{\left(\frac{e^{-\lambda}\lambda^{x}}{\lfloor x}\right)} = \frac{\lambda}{x+1}$$

$$\Rightarrow p(x+1) = \frac{\lambda}{x+1} p(x) \qquad \ldots (3)$$

This is the recurrence relation for probabilities of Poisson distribution. After obtaining the value of p(0) using Poisson probability function i.e.

$p(0) = \frac{e^{-\lambda}\lambda^0}{\lfloor 0}= e^{-\lambda}$, we can obtain p(1), p(2), p(3),…, on putting x = 0, 1, 2, …. successively in (3).

Poisson Frequency Distribution

If an experiment, satisfying the requirements of Poisson distribution, is repeated N times, then the expected frequency of getting x successes is given by

$$f(x) = N.P[X = x] = N.\frac{e^{-\lambda}\lambda^x}{\lfloor x}; x = 0, 1, 2, \ldots$$

Example 5: A manufacturer, who produces medicine bottles, finds that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes from the producer of bottles. Using Poisson distribution, find how many boxes will contain at least two defective bottles.

Solution: Let X be the Poisson variate, "the number of defective bottles in a box". Here, number of bottles in a box (n) = 500, therefore, the probability (p) of a bottle being defective is

$$p = 0.1\% = \frac{0.1}{100} = 0.001$$

Number of boxes (N) = 100

$$\lambda = np = 500 \times .001 = 0.5$$

Using Poisson distribution, we have

$$P[X = x] = \frac{e^{-\lambda}\lambda^x}{\lfloor x}; x = 0, 1, 2, \ldots$$

$$= \frac{e^{-0.5}(0.5)^x}{\lfloor x}; x = 0, 1, 2, \ldots$$

$\therefore$ Probability that a box contain at least two defective bottles

$$= P[X \geq 2]$$

$$= 1 - P[X < 2]$$

$$= 1 - \left[P[X = 0] + P[X = 1]\right]$$

$$= 1 - \left[\frac{e^{-0.5}(0.5)^0}{\lfloor 0} + \frac{e^{-0.5}(0.5)^1}{\lfloor 1}\right] = 1 - e^{-0.5}[1 + 0.5]$$

= 1 – (0.6065) (1.5) = 1 – 0.90975 = 0.09025.

Hence, the expected number of boxes containing at least two defective bottles

$= N.P[X \geq 2]$

$= (100)\ (0.09025)$

$= 9.025$

Process of Fitting a Poison Distribution

For fitting a Poisson distribution to the observed data, you are to proceed as described in the following steps.

- First we obtain mean of the given distribution i.e. $\frac{\sum fx}{\sum f}$, being mean, take this as the value of λ.
- Next we obtain $p(0) = e^{-\lambda}$ [Use table given in Appendix at the end of this unit.]
- The recurrence relation $p(x+1) = \frac{\lambda}{x+1} p(x)$ is then used to compute the values of p(1), p(2), p(3), …
- The probabilities obtained in the preceding two steps are then multiplied with N to get expected/theoretical frequencies i.e. $f(x) = N.P[X = x]; x = 0, 1, 2, ...$

Example 6: The following data give frequencies of aircraft accidents experienced by 2480 pilots during a certain period:

Number of Accidents	0	1	2	3	4	5
Frequencies	1970	422	71	13	3	1

Fit a Poisson distribution and calculate the theoretical frequencies.

Solution: Let X be the number of accidents of the pilots. Let us first obtain the mean number of accidents as follows:

Number of Accidents (X)	Frequency (f)	f X
0	1970	0
1	422	422
2	71	142
3	13	39
4	3	12
5	1	5
Total	2480	620

$\therefore$ Mean $= \lambda = \dfrac{\sum fx}{\sum f} = \dfrac{620}{2480}$

$\Rightarrow \lambda = 0.25$

$\therefore$ by Poisson distribution,

$p(0) = e^{-\lambda} = e^{-0.25}$

$= 0.7788$ $\left[\begin{array}{l}\text{See table given in the Appendix}\\ \text{at the end of this unit}\end{array}\right]$

Now, using the recurrence relation for probabilities of Poisson distribution i.e. $p(x+1) = \dfrac{\lambda}{x+1} p(x)$ and then multiplying each probability with N, we get the expected frequencies as shown in the following table

Number of Accidents (X) (1)	$\dfrac{\lambda}{x+1} = \dfrac{0.25}{x+1}$ (2)	$p(x) = P[X = x]$ (3)	Expected/ Theoretical frequency f(x) = 2480p(x) (4)
0	$\dfrac{0.25}{0+1} = 0.25$	p(0) = 0.7788	1931. 4 □ 1931
1	$\dfrac{0.25}{1+1} = 0.125$	p(1) = 0.25 × 0.7788 = 0.1947	482. 9 □ 483
2	$\dfrac{0.25}{2+1} = 0.0833$	p(2) = 0.125 × 0.1947 = 0.0243	60.3 □ 60
3	$\dfrac{0.25}{3+1} = 0.0625$	p(3)= 0.0833 × 0.0243 = 0.0020	4.96 □ 5
4	$\dfrac{0.25}{4+1} = 0.05$	p(4)= 0.0625 × 0.0020 = 0.0001	0.248 □ 0
5	$\dfrac{0.25}{5+1} = 0.0417$	p(5)= 0.05 × 0.0001 = 0.000005	0

You can now try the following exercises

E4) In a certain factory turning out fountain pens, there is a small chance, $\dfrac{1}{500}$, for any pen to be defective. The pens are supplied in packets of 10. Calculate the approximate number of packets containing (i) one defective (ii) two defective pens in a consignment of 20000 packets.

E5) A typist commits the following mistakes per page in typing 100 pages. Fit a Poisson distribution and calculate the theoretical frequencies.

Mistakes per page(X)	0	1	2	3	4	5
Frequency (f)	42	33	14	6	4	1

We now conclude this unit by giving a summary of what we have covered in it.

10.5 SUMMARY

The following main points have been covered in this unit:

1. A random variable X is said to follow **Poisson distribution** if it assumes indefinite number of non-negative integer values and its probability mass function is given by:

$$p(x) = P(X = x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{\lfloor x}; & x = 0, 1, 2, 3, \ldots \text{ and } \lambda > 0. \\ 0; & \text{elsewhere} \end{cases}$$

2. For Poisson distribution, **Mean = Variance =** $\boldsymbol{\mu_3} = \lambda$, $\boldsymbol{\mu_4} = 3\lambda^2 + \lambda$

3. $\beta_1 = \dfrac{1}{\lambda}$, $\gamma_1 = \dfrac{1}{\sqrt{\lambda}}$, $\beta_2 = 3 + \dfrac{1}{\lambda}$, $\gamma_2 = \dfrac{1}{\lambda}$ for this distribution .

4. **Recurrence relation for probabilities of Poisson distribution** is

$$p(x+1) = \frac{\lambda}{x+1}.p(x),\ x = 0, 1, 2, 3, \ldots$$

5. **Expected frequencies for a Poisson distribution** are given by

$$f(x) = N.P[X = x] = N.\frac{e^{-\lambda}\lambda^x}{\lfloor x}; x = 0, 1, 2, \ldots$$

If you want to see what our solutions/answers to the exercises in the unit are, we have given them in the following section.

10.6 SOLUTIONS/ANSWERS

E1) Let X be the Poisson variable "Number of fatal accidents in a year".

Here n = 350, $p = \dfrac{1}{1400}$

$$\Rightarrow \lambda = np = (350)\left(\frac{1}{1400}\right) = 0.25.$$

By Poisson distribution,

$$P[X = x] = \frac{e^{-\lambda}.\lambda^x}{\lfloor x}, x = 0, 1, 2, \ldots$$

$$= \frac{e^{-0.25}(0.25)^x}{\lfloor x}, x = 0, 1, 2, \ldots$$

Therefore, P [at least one fatal accident]

$$= P[X \geq 1] = 1 - P[X < 1] = 1 - P[X = 0]$$

$$= 1 - \frac{e^{-0.25}(0.25)^0}{\lfloor 0} = 1 - e^{-0.25} = 1 - 0.78 = 0.22$$

E2) As mean = 6, therefore, $\lambda = 6$.

As standard deviation is 2, therefore, variance = 4 $\Rightarrow$ $\lambda = 4$.

We get two different values of λ, which is impossible. Hence, the statement is invalid.

E3) Let λ be the mean of the distribution,

$\therefore$ by Poisson distribution, we have

$$P[X = x] = \frac{e^{-\lambda}\lambda^x}{\lfloor x}; x = 0, 1, 2, 3, \ldots$$

Given that P[X = 1] = P[X = 2],

$$\therefore \frac{e^{-\lambda}\lambda^1}{\lfloor 1} = \frac{e^{-\lambda}\lambda^2}{\lfloor 2}$$

$$\Rightarrow \lambda = \frac{\lambda^2}{2} \Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda(\lambda - 2) = 0$$

$\Rightarrow \lambda = 0, 2.$

$\lambda = 0$ is rejected,

$\therefore \lambda = 2$

Hence, Mean = 2.

Now, $P[X = 0] = \frac{e^{-\lambda}\lambda^0}{\lfloor 0} = e^{-\lambda} = e^{-2} = 0.1353,$

[See table given in the Appendix at the end of this unit.]

and $P[X = 4] = \frac{e^{-\lambda}\lambda^4}{\lfloor 4} = \frac{e^{-2}(2)^4}{\lfloor 4} = \frac{e^{-2}(16)}{24} = \frac{2}{3}(0.1353)$

$= 2(0.0451)$

$= 0.0902.$

E4) Here $p = \frac{1}{500}, \; n = 10, \; N = 20000,$

$\therefore \lambda = np = 10 \times \frac{1}{500} = 0.02$

By Poisson frequency distribution

$f(x) = N.P[X = x]$

$= (20000) \frac{e^{-\lambda} \lambda^x}{\lfloor x}; x = 0, 1, 2, ...$

Now,

i) The number of packets containing one defective

$= f(1)$

$= (20000) \frac{e^{-0.02}.(0.02)^1}{\lfloor 1}$

$= (20000)\ (0.9802)\ (0.02)$ [See the table given in the Appendix]

$= 392.08 \approx 392$; and

ii) The number of packets containing two defectives

$= f(2) = 20000 \frac{e^{-0.02}(0.02)^2}{\lfloor 2}$

$= (20000) \frac{(0.9802)(0.0004)}{2} = 3.9208 \approx 4$

E5) The mean of the given distribution is computed as follows

X	f	fX
0	42	0
1	33	33
2	14	28
3	6	18
4	4	16
5	1	5
Total	100	100

$\therefore$ Mean $\lambda = \frac{\sum fx}{\sum f} = \frac{100}{100} = 1$

$\Rightarrow p(0) = e^{-\lambda} = e^{-1} = 0.3679.$

Now, we obtain p(1), p(2), p(3), p(4), p(5) using the recurrence relation for probabilities of Poisson distribution i.e.

$p(x+1)=\frac{\lambda}{x+1}p(x); x=0, 1, 2, 3, 4$ and then obtain the expected frequencies as shown in the following table:

X	$\frac{\lambda}{x+1}=\frac{1}{x+1}$	$p(x)$	Expected/Theoretical frequency $f(x)=N.P(X=x)$ $=100.P(X=x)$
0	$\frac{1}{0+1}=1$	$p(0)=0.3679$	36.79 □ 37
1	$\frac{1}{1+1}=0.5$	$p(1)=1\times 0.3679=0.3679$	36.79 □ 37
2	$\frac{1}{2+1}=0.3333$	$p(2)=0.5\times 0.3679=0.184$	18.4 □ 18
3	$\frac{1}{3+1}=0.25$	$p(3) = 0.3333\times 0.184 = 0.0613$	6.13 □ 6
4	$\frac{1}{4+1}=0.2$	$p(4)=0.25 \times 0.0613 = 0.0153$	1.53 □ 2
5	$\frac{1}{5+1}=0.1667$	$p(5)=0.2 \times 0.0153 = 0.0031$	0.3 □ 0

Appendix

Value of $e^{-\lambda}$ (For Computing Poisson Probabilities)

($0 < \lambda < I$)

λ	0	1	2	3	4	5	6	7	8	9
0.0	1.0000	0.9900	0.9802	0.9704	0.9608	0.9512	0.9418	0.9324	0.9231	0.9139
0.1	0.9048	0.8958	0.8860	0.8781	0.8694	0.8607	0.8521	0.8437	0.8353	0.8270
0.2	0.7187	0.8106	0.8025	0.7945	0.7866	0.7788	0.7711	0.7634	0.7558	0.7483
0.3	0.7408	0.7334	0.7261	0.7189	0.7118	0.7047	0.6970	0.6907	0.6839	0.6771
0.4	06703	0.6636	0.6570	0.6505	0.6440	0.6376	0.6313	0.6250	0.6188	0.6125
0.5	0.6065	0.6005	0.5945	0.5886	0.5827	0.5770	0.5712	0.5655	0.5599	0.5543
0.6	0.5448	0.5434	0.5379	0.5326	0.5278	0.5220	0.5160	0.5113	0.5066	0.5016
0.7	0.4966	0.4916	0.4868	0.4810	0.4771	0.4724	0.4670	0.4630	0.4584	0.4538
0.8	0.4493	0.4449	0.4404	0.4360	0.4317	0.4274	0.4232	0.4190	0.4148	0.4107
0.9	0.4066	0.4026	0.3985	0.3946	0.3906	0.3867	0.3829	0.3791	0.3753	0.3716
(λ=1, 2, 3, ...,10)										
λ	1	2	3	4	5	6	7	8	9	10
$e^{-\lambda}$	0.3679	0.1353	0.0498	0.0183	0.0070	0.0028	0.0009	0.0004	0.0001	0.00004

Note: To obtain values of $e^{-\lambda}$ for other values of λ, use the laws of exponents i.e.

$e^{-(a+b)} = e^{-a}.e^{-b}$ e. g. $e^{-2.25} = e^{-2}.e^{-0.25} = (0.1353)(0.7788) = 0.1054.$