UNIT 3 STANDARD SAMPLING DISTRIBUTIONS-I

Structure

3.1 INTRODUCTION

In Unit 1 of this block, we have discussed the fundamentals of sampling distributions and some basic definitions related to it. In Unit 2, we have discussed the sampling distributions of various sample statistics such as sample mean, sample proportion, sample variance, etc. Sometimes statistics such as sample mean, sample proportion, sample variance, etc. may follow a particular sampling distribution. Based on this fact Prof. R.A. Fisher, Prof. G. Snedecor and some other statisticians worked in this area and found some exact sampling distributions which are often followed by some of the statistics. In present and next unit of this block, we shall discuss some important sampling distributions such as χ^2(read as chi-square), t and F. Generally, these sampling distributions are named on the name of the originator, that is, Fisher's F-distribution is named on Prof. R.A. Fisher.

This unit is divided in 9 sections. Section 3.1 is introductive in nature. In Section 3.2 the brief introduction of χ^2-distribution is given. The different properties and probability curves for various degrees of freedom n = 1, 4, 10 and 22 are described along with the mean and variance in Section 3.3. The distribution of χ^2 is used in testing the goodness of fit, independence of attributes, etc. Therefore, in Section 3.4 the important applications of χ^2-distribution are listed. In Section 3.5, the brief introduction of t-distribution is described. Properties of t-distribution, probability curve, the general formula for central moment, mean and variance are described in Section 3.6. Different applications of t-distribution are listed in Section 3.7. Unit ends by providing summary of what we have discussed in this unit in Section 3.8 and solution of exercises in Section 3.9.

Objectives

After studying this unit, you should be able to:

- explain the χ^2-distribution;

- describe the properties of χ^2-distribution;
- explain the probability curve of χ^2-distribution;
- find the mean and variance of χ^2-distribution;
- explore the applications of χ^2-distribution;
- explain the t-distribution;
- describe the properties of t-distribution;
- describe the probability curve of t-distribution;
- find the mean and variance of t-distribution; and
- explore the applications of t-distribution.

3.2 INTRODUCTION TO χ^2-DISTRIBUTION

Before describing the chi-square (χ^2) distribution, first we will discuss very useful concept "degrees of freedom". The exact sampling distributions are described with the help of degrees of freedom.

Degrees of Freedom (df)

The term degree of freedom (df) is related to the independency of sample observations. In general, the number of degree of freedom is the total number of observations minus the number of independent constraints or restrictions imposed on the observations.

For a sample of n observations, if there are k restrictions among observations (k < n), then the degrees of freedom will be (n−k).

For example, suppose there are two observations X_1, X_2 and a restriction (condition) is imposed that their total should be equal to 100. Then we can arbitrarily assign value to any one of these two as 30, 98, 52, etc. but the value of rest is automatically determined by a simple adjustment like $X_2 = (100 - X_1)$. Therefore, we can say that only one observation is independent or there is one degree of freedom.

Consider another example, suppose there are 50 observations as X_1, X_2, …, X_{50} and the restriction (condition) are imposed on these like as:

$$\text{(i)}\quad \sum_{i=1}^{50} X_i = 2000 \quad \text{(ii)}\quad \sum_{i=1}^{50} X_i^2 = 10000 \quad \text{(iii)}\quad X_5 = 2X_4$$

The restriction (i) reveals sum of observations equal to 2000 (ii) reveals sum of square of observations equal to 10000 and restriction (iii) implies fifth observation double to the fourth observation.

With these three restrictions (i), (ii) and (iii), there shall be only 47 observations independently chosen and rest three could be obtained by solving equations or by adjustments. The degrees of freedom of this set of observations of size 50 is now (50 −3) = 47.

Chi-square Distribution

The chi-square distribution is first discovered by Helmert in 1876 and later independently explained by Karl- Pearson in 1900. The chi-square distribution was discovered mainly as a measure of goodness of fit in case of frequency.

If a random sample $X_1, X_2, \ldots, X_n$ of size n is drawn from a normal population having mean μ and variance σ^2 then the sample variance can be defined as

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

or

$$\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2 = (n-1)S^2 = \nu S^2$$

where, $\nu = n - 1$ and the symbol ν read as 'nu'.

Generally, ν symbol is used to represent the general degrees of freedom. Its value may be n, n – 1, etc.

Thus, the variate $\chi^2 = \frac{\nu S^2}{\sigma^2}$, which is the ratio of sample variance multiplied by its degrees of freedom and the population variance follows the χ^2-distribution with ν degrees of freedom.

The probability density function of χ^2-distribution with ν df is given by

$$f\left(\chi^2\right) = \frac{1}{2^{\nu/2}\left|\overline{\frac{\nu}{2}}\right.} e^{-\chi^2/2}\left(\chi^2\right)^{(\nu/2)-1}; \qquad 0 < \chi^2 < \infty \qquad \ldots (1)$$

where, $\nu = n - 1$.

The chi-square distribution can also be we described as:

If random variable X follows normal distribution with mean μ and variance σ^2, then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$, known as standard normal variate, follows normal distribution with mean 0 and variance 1. The square of a standard normal variate Z follows chi-square distribution with 1 degree of freedom (df), that is,

If $X \sim N\left(\mu, \sigma^2\right)$, then $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$ and

$$Z^2 = \left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$$

where, $\chi^2_{(1)}$ read as chi-square with one degree of freedom.

In general, if X_i's (i =1, 2,…, n) are n independent normal variates with means μ_i and variances $\sigma_i^2 (i = 1, 2, ..., n)$ then the sum of squares of n standard normal variate follows chi-square distribution with n df i.e.

If $X_i \sim N\left(\mu_i, \sigma_i^2\right)$, then $Z_i = \frac{X_i - \mu_i}{\sigma_i} \sim N(0,1)$

Therefore, $\chi^2 = \sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n}\left(\frac{X_i - \mu_i}{\sigma_i}\right)^2 \sim \chi^2_{(n)}$

The probability density function of χ^2-distribution with n df is given by

$$f\left(\chi^2\right) = \frac{1}{2^{n/2}\left|\frac{n}{2}\right.} e^{-\chi^2/2} \left(\chi^2\right)^{(n/2)-1}; \qquad 0 < \chi^2 < \infty \qquad \ldots (2)$$

Try the following exercise to make sure that you have understood the chi-square distribution.

E1) Write down the pdf of chi-square distribution in each of the following cases:

(i) 6 degrees of freedom

(ii) 10 degrees of freedom

E2) Below, in each case, the pdf of chi-square distribution is given. Obtain the degrees of freedom of each chi-square distribution:

(i) $f\left(\chi^2\right) = \frac{1}{96} e^{-\chi^2/2} \left(\chi^2\right)^3; \qquad 0 < \chi^2 < \infty$

(ii) $f\left(\chi^2\right) = \frac{1}{2} e^{-\chi^2/2}; \qquad 0 < \chi^2 < \infty$

3.3 PROPERTIES OF χ^2-DISTRIBUTION

In previous section, we have discussed the χ^2-distribution and its probability density function. Now, in this section, we shall discuss some important properties of χ^2-distribution as given below:

1. The probability curve of the chi-square distribution lies in the first quadrant because the range of χ^2-variate is from 0 to ∞.
2. Chi-square distribution has only one parameter n, that is, the degrees of freedom.
3. Chi-square probability curve is highly positive skewed.
4. Chi-square-distribution is a uni-modal distribution, that is, it has single mode.
5. The mean and variance of chi-square distribution with n df are n and 2n respectively.

After deliberating the properties of χ^2-distribution above, some of these are discussed in detailed in the subsequent sub-sections.

3.3.1 Probability Curve of χ^2-distribution

The probability distribution of all possible values of a statistic is known as the sampling distribution of that statistic. The shape of probability distribution of a statistic can be shown by the probability curve. In this sub-section, we will discuss the probability curve of chi-square distribution.

A rough sketch of the probability curve of chi-square distribution is shown in Fig. 3.1 at different values of degrees of freedom as n = 1, 4, 10 and 22.

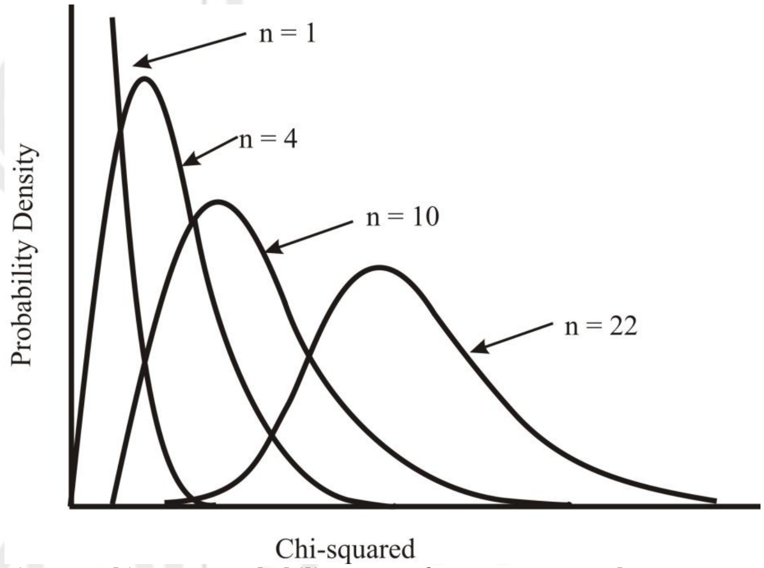

Fig. 3.1: Chi-square probability curves for n = 1, 4, 10 and 22

After looking the probability curves of χ^2-distribution for n = 1, 4, 10 and 22, one can understand that probability curve takes shape of inverse J for n = 1. The probability curve of chi-square distribution gets skewed more and more to the right as n becomes smaller and smaller. It becomes more and more symmetrical as n increases because as n tends to ∞, the χ^2-distribution becomes normal distribution.

3.3.2 Mean and Variance of χ^2-distribution

In previous sub-section, the probability curve of χ^2-distribution is discussed with some of its properties. Now in this sub-section, we will discuss the mean and variance of the χ^2-distribution. The mean and variance of χ^2-distribution is derived with the help of the moment about origin. As we have discussed in Unit 3 of MST-002 that the first order moment about origin is known as mean and central second order moment is known as variance of the distribution. So we first obtain the r^{th} order moment about origin of χ^2-distribution as

$$\mu_r' = E\left[\left(\chi^2\right)^r\right] = \int_0^\infty \left(\chi^2\right)^r f\left(\chi^2\right) d\chi^2$$

$$= \int_0^\infty \left(\chi^2\right)^r \frac{1}{2^{n/2}\left|\frac{n}{2}\right.} e^{-\chi^2/2}\left(\chi^2\right)^{(n/2)-1} d\chi^2$$

$$= \frac{1}{2^{n/2}\left|\frac{n}{2}\right.} \int_0^\infty e^{-\frac{1}{2}\chi^2}\left(\chi^2\right)^{\left(\frac{n}{2}+r\right)-1} d\chi^2$$

$$= \frac{1}{2^{n/2}\left|\frac{n}{2}\right.} \frac{\left|\frac{n}{2}+r\right.}{\left(\frac{1}{2}\right)^{\frac{n}{2}+r}} \qquad \left[\because \int_0^\infty e^{-ax}x^{n-1}dx = \frac{\left|n\right.}{a^n}\right] \qquad \ldots (3)$$

The Gamma function is discussed in Unit 15 of MST-003, according to this $\int_0^\infty e^{-ax}x^{n-1}dx = \frac{\left|n\right.}{a^n}$

Now, if we put r = 1 in the above expression given in equation (3), we get the value of first order moment about origin which is known as mean i.e.

$$\text{Mean} = \mu_1' = \frac{1}{2^{n/2}\left\lceil\frac{n}{2}\right.}\frac{\left\lceil\frac{n}{2}+1\right.}{\left(\frac{1}{2}\right)^{\frac{n}{2}+1}}$$

$$= \frac{1}{2^{n/2}\left\lceil\frac{n}{2}\right.}\frac{\frac{n}{2}\left\lceil\frac{n}{2}\right.}{\left(\frac{1}{2}\right)^{\frac{n}{2}+1}} = n \qquad \left[\because \left\lceil n+1\right. = n\left\lceil n\right.\right]$$

Similarly, if we put r = 2 in the formula of μ_r' given in equation (3) then we get the value of second order moment about origin i.e.

$$\mu_2' = \frac{1}{2^{n/2}\left\lceil\frac{n}{2}\right.}\frac{\left\lceil\frac{n}{2}+2\right.}{\left(\frac{1}{2}\right)^{\frac{n}{2}+2}}$$

$$= \frac{1}{\left\lceil\frac{n}{2}\right.}\frac{\left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right)\left\lceil\frac{n}{2}\right.}{\left(\frac{1}{2}\right)^{2}} \qquad \left[\begin{aligned}\because \left\lceil n+2\right. &= (n+1)\left\lceil n+1\right.\\ &= (n+1)n\left\lceil n\right.\end{aligned}\right]$$

$$= 2^2\left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right) = n(n+2)$$

Now, we obtain the value of variance by putting the value of first order and second order moments about origin in the formula given below

$$\text{Variance} = \mu_2' - (\mu_1')^2$$

$$= n(n+2) - (n)^2 = n^2 + 2n - n^2 = 2n$$

So the mean and variance of the chi-square distribution with n df are given by

Mean = n and Variance = 2n

Hence, we have observed that the variance is twice of the mean of chi-square distribution.

Similarly, on putting r = 3, 4,… in the formula given in equation (3) one may get higher order moments about origin such as μ_3', μ_4',...

Example 1: What are the mean and variance of chi-square distribution with 5 degrees of freedom?

Solution: We know that the mean and variance of chi-square distribution with n degrees of freedom are

Mean = n and Variance = 2n

In our case, n = 5, therefore,

Mean = 5 and Variance = 10

To evaluate your understanding try to answer the following exercises.

E3) What are the mean and variance of chi-square distribution with 10 degrees of freedom?

E4) What are the mean and variance of chi-square distribution with pdf given below

$$f\left(\chi^2\right) = \frac{1}{96} e^{-\chi^2/2} \left(\chi^2\right)^3; \qquad 0 < \chi^2 < \infty$$

3.4 APPLICATIONS OF χ^2-DISTRIBUTION

The applications of chi-square distribution are very wide in Statistics. Some of them are listed below. The chi-square distribution is used:

1. To test the hypothetical value of population variance.
2. To test the goodness of fit, that is, to judge whether there is a discrepancy between theoretical and experimental observations.
3. To test the independence of two attributes, that is, to judge whether the two attributes are independent.

The first application listed above shall be discussed in detail in Unit 12 of this course and the remaining two applications shall be discussed in detail in Unit 16 of this course.

Now, try to write down the applications of chi-square distribution by answering the following exercise.

E5) List the applications of chi-square distribution.

3.5 INTRODUCTION TO t-DISTRIBUTION

The t-distribution was discovered by W.S. Gosset in 1908. He was better known by the pseudonym **'Student'** and hence t-distribution is called **'Student's t-distribution'**.

If a random sample $X_1, X_2, \ldots, X_n$ of size n is drawn from a normal population having mean μ and variance σ^2 then we know that the sample mean $\bar{X}$ is distributed normally with mean μ and variance σ^2/n, that is, if $X_i \sim N\left(\mu, \sigma^2\right)$ then $\bar{X} \sim N\left(\mu, \sigma^2/n\right)$, and also the variate

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

is distributed normally with mean 0 and variance 1, i.e. $Z \sim N(0, 1)$.

In general, the standard deviation σ is not known and in such a situation the only alternative left is to estimate the unknown σ^2. The value of sample variance (S^2) is used to estimate the unknown σ^2 where,

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n_1} \left(X_i - \bar{X}\right)^2$$

Thus, in this case the variate $\dfrac{\bar{X} - \mu}{S/\sqrt{n}}$ is not normally distributed whereas it follows t-distribution with (n−1) df i.e.

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{(n-1)} \qquad \ldots (4)$$

where, $S^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2$

The t-variate is a widely used variable and its distribution is called student's t-distribution on the pseudonym name 'Student' of W.S. Gosset. The probability density function of variable t with (n - 1) = ν degrees of freedom is given by

$$f(t) = \frac{1}{\sqrt{\nu}\, B\left(\frac{1}{2}, \frac{\nu}{2}\right)\left(1 + \frac{t^2}{\nu}\right)^{(\nu+1)/2}}; \qquad -\infty < t < \infty \qquad \ldots (5)$$

where, $B\left(\frac{1}{2}, \frac{\nu}{2}\right)$ is known as beta function and

$$B\left(\frac{1}{2}, \frac{\nu}{2}\right) = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{\nu}{2}}{\Gamma\frac{\nu+1}{2}} \qquad \left[\because B(a,b) = \frac{\Gamma a\,\Gamma b}{\Gamma(a+b)}\right]$$

Later on Prof. R.A. Fisher found that t-distribution can also be applied to test the regression coefficient and other practical problems. He proposed t-variate as the ratio of standard normal variate to the square root of an independent chi-square variate divided by its degrees of freedom. Therefore, if Z is a standard normal variate with mean 0 and variance 1 and χ^2 is an independent chi-square variate with n df i.e.

$$Z \sim N(0,1) \quad \text{and} \quad \chi^2 \sim \chi^2_{(n)}$$

Then Fisher's t-variate is given by

$$t = \frac{Z}{\sqrt{\chi^2/n}} \sim t_{(n)} \qquad \ldots (6)$$

That is, the t-variate follows the t-distribution with n df and has the same probability density function as student t-distribution with n df. Therefore,

$$f(t) = \frac{1}{\sqrt{n}\, B\left(\frac{1}{2}, \frac{n}{2}\right)\left(1 + \frac{t^2}{n}\right)^{(n+1)/2}}; \qquad -\infty < t < \infty \qquad \ldots (7)$$

After describing the t-distribution, we try to calculate the value of t-variate as in the given example.

Example 2: The life of light bulbs manufactured by the company A is known to be normally distributed. The CEO of the company claims that an average life time of the light bulbs is 300 days. A researcher randomly selects 25 bulbs for testing the life time and he observed the average life time of the sampled bulbs is 290 days with standard deviation of 50 days. Calculate value of t-variate.

Solution: Here, we are given that

$$\mu = 300,\ n = 25,\ \bar{X} = 290 \text{ and } S = 50$$

The value of t-variate can be calculated by the formula given below

$$t = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

Therefore, we have

$$t = \frac{290 - 300}{50/\sqrt{25}} = \frac{-10}{10} = -1$$

Now, try the following exercises for your practice.

E6) Write down the pdf of t-distribution in each of the following cases:

(i) 3 degrees of freedom

(ii) 9 degrees of freedom

E7) Obtain the degrees of freedom of t-distribution whose pdf is given below:

$$f(t) = \frac{1}{\sqrt{5}\,B\left(\frac{1}{2}, \frac{5}{2}\right)\left(1 + \frac{t^2}{5}\right)^3}; \qquad -\infty < t < \infty$$

E8) If the scores on an IQ test of the students of a class are assumed to be normally distributed with a mean of 60. From the class, 15 students are randomly selected and an IQ test of the similar level is conducted. The average test score and standard deviation of test scores in the sample group are found to be 65 and 12 respectively. Calculate the value of t-variate.

3.6 PROPERTIES OF t-DISTRIBUTION

In previous section, we have discussed the t-distribution briefly. Now, we shall discuss some of the important properties of the t-distribution.

The t-distribution has the following properties:

1. The t-distribution is a uni-modal distribution, that is, t-distribution has single mode.
2. The mean and variance of the t-distribution with n df are zero and $\frac{n}{n-2}$ if $n > 2$ respectively.
3. The probability curve of t-distribution is similar in shape to the standard normal distribution and is symmetric about $t = 0$ line but flatter than normal curve.
4. The probability curve is bell shaped and asymptotic to the horizontal axis.

The properties deliberated above are some of the important properties of the t-distribution. We shall discuss some of these properties such as probability curve, mean and variance of the t-distribution in detail in the subsequent sub-sections.

3.6.1 Probability Curve of t-distribution

The probability curve of t-distribution is bell shaped and symmetric about t = 0 line. The probability curves of t-distribution is shown in Fig. 3.2 at two different values of degrees of freedom as at n = 4 and n = 12.

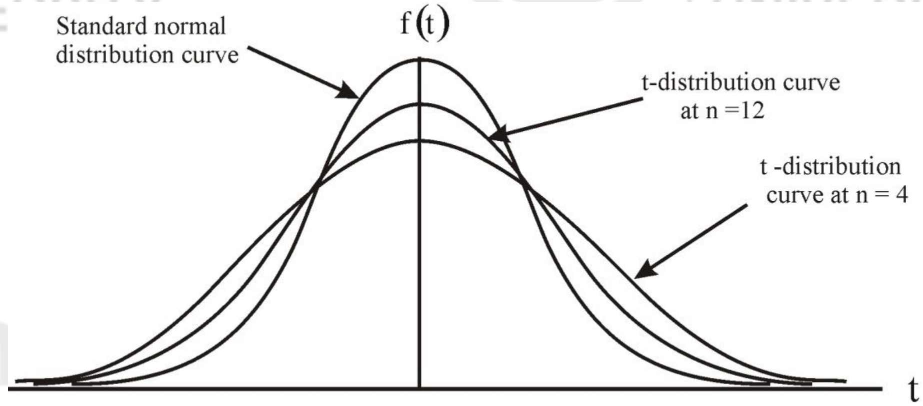

Fig. 3.2: Probability curves for t-distribution at n = 4, 12 along with standard normal curve

In the figure given above, we have drawn the probability curves of t-distribution at two different values of degrees of freedom with probability curve of standard normal distribution. By looking at the figure, one can easily understand that the probability curve of t-distribution is similar in shape to that of normal distribution and asymptotic to the horizontal-axis whereas it is flatter than standard normal curve. The probability curve of the t-distribution is tending to the normal curve as the value of n increases. Therefore, for sufficiently large value of sample size n(> 30), the t-distribution tends to the normal distribution.

3.6.2 Mean and Variance of t-distribution

In previous sub-section, we have discussed the probability curve of t-distribution. After discussing the probability curve it came to our knowledge that t-distribution is symmetrical about t = 0 line. Since f (t) is symmetrical about t = 0 line, therefore, all moments of odd order about origin vanish or become zero, i.e.

$$\mu'_{2r+1} = 0 \qquad \text{for all } r = 0, 1, 2, 3... \qquad \ldots (8)$$

In particular, for r = 0, we get

$$\mu'_1 = 0 = \text{Mean}$$

That is, the mean of the t-distribution is zero. Similarly, one can show that other odd moments about origin are zero by putting r = 1, 2,….

The moments of even order are given by

$$\mu'_{2r} = \int_{-\infty}^{\infty} t^{2r} f(t)\,dt$$

Now, we can also write

$$\mu'_{2r} = 2\int_0^{\infty} t^{2r} f(t)\,dt$$

$$= 2\frac{1}{\sqrt{n}\,B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^{\infty} \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{(n+1)/2}}dt$$

If
ab

$$= \frac{2}{\sqrt{n}\,B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^{\infty} \frac{\left(t^2\right)^r}{\left(1+\frac{t^2}{n}\right)^{(n+1)/2}} dt \qquad \ldots (9)$$

Putting $1+\frac{t^2}{n}=\frac{1}{y} \Rightarrow t^2 = n\left(\frac{1}{y}-1\right)$

Therefore, $2t\,dt = -\frac{n}{y^2}dy \Rightarrow 2dt = -\frac{n}{ty^2}dy = \frac{n}{\sqrt{n\left(\frac{1}{y}-1\right)}y^2}dy$

Also $t = 0 \Rightarrow y = 1 \text{ and } t \to \infty \Rightarrow y \to 0$

By putting these values in equation (9), we get

$$\mu'_{2r} = -\frac{1}{\sqrt{n}\,B\left(\frac{1}{2},\frac{n}{2}\right)} \int_1^0 \frac{\left[n\left(\frac{1}{y}-1\right)\right]^r}{\left(\frac{1}{y}\right)^{(n+1)/2}} . \frac{n}{\sqrt{n\left(\frac{1}{y}-1\right)}y^2} dy$$

$$\left[\because \int_a^0 f(x)\,dx = -\int_0^a f(x)\,dx\right]$$

$$= \frac{\sqrt{n}}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 \left[n\left(\frac{1}{y}-1\right)\right]^{r-\frac{1}{2}} y^{\frac{n+1}{2}-2} dy$$

$$= \frac{n^r}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 \left(\frac{1-y}{y}\right)^{r-\frac{1}{2}} y^{\frac{n+1}{2}-2} dy$$

$$= \frac{n^r}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 (1-y)^{r-\frac{1}{2}} y^{\frac{n+1}{2}-2-r+\frac{1}{2}} dy$$

$$= \frac{n^r}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 y^{\frac{n}{2}-r-1} (1-y)^{r+\frac{1}{2}-1} dy$$

$$= \frac{n^r}{B\left(\frac{1}{2},\frac{n}{2}\right)} B\left(\frac{n}{2}-r, r+\frac{1}{2}\right) \qquad \left[\begin{array}{l}\because \text{For } a>0 \,\&\, b>0 \\ \int_0^1 x^{a-1}(1-x)^{b-1}dx \\ \qquad = B(a,b)\end{array}\right]$$

$$= n^r \frac{\left\lceil\frac{1}{2}+\frac{n}{2}\right. \left\lceil\frac{n}{2}-r\right. \left\lceil r+\frac{1}{2}\right.}{\left\lceil\frac{1}{2}\right. \left\lceil\frac{n}{2}\right. \left\lceil\frac{n}{2}-r+r+\frac{1}{2}\right.}; \quad \frac{n}{2}-r>0 \left[\because B(a,b) = \frac{\lceil a \lceil b}{\lceil a+b}\right]$$

$$\mu'_{2r} = n^r \frac{\left\lceil\frac{n}{2}-r\right. \left\lceil r+\frac{1}{2}\right.}{\left\lceil\frac{1}{2}\right. \left\lceil\frac{n}{2}\right.}; \qquad n > 2r \qquad \ldots(10)$$

Now, if we put r = 1 in the above expression given in equation (10), we get the value of second order moment about origin i.e.

$$\mu_2' = n\frac{\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(1+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)}; \qquad n > 2$$

$$= n\frac{\Gamma\left(\frac{n}{2}-1\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)} \qquad \left[\because \Gamma(n+1) = n\Gamma(n)\right]$$

$$= \frac{n}{(n-2)}; \qquad n > 2$$

Now, we obtain the value of variance by putting the value of first order and second order moments about origin in the formula given below

$$\text{Variance} = \mu_2' - \left(\mu_1'\right)^2$$

$$= \frac{n}{(n+2)} - (0)^2$$

$$= \frac{n}{(n+2)}; \qquad n > 2$$

So the mean and variance of the t-distribution with n df are given by

$$\text{Mean} = 0 \text{ and Variance } = \frac{n}{(n+2)}; \qquad n > 2$$

Now, you can try an exercise to see how much you learn about the properties of t-distribution.

E9) What are the mean and variance of t-distribution with 20 degrees of freedom?

3.7 APPLICATIONS OF t-DISTRIBUTION

In previous section of this unit, we have discussed some important properties of t-distribution and derived its mean and variance. Now, you may be interested to know the applications of the t-distribution. In this section, we shall discuss the important applications of the t-distribution. The t-distribution has wide number of applications in Statistics. The t-distribution is used:

1. To test the hypothesis about the population mean.
2. To test the hypothesis about the difference of two population means of two normal populations.
3. To test the hypothesis that population correlation coefficient is zero.

The applications of the t-distribution listed above are discussed further in Unit 12 of this course.

Now, it is time to write down the main applications of t-distribution by answering the following exercise.

E10) Write any three applications of t-distribution.

We now end this unit by giving a summary of what we have covered in it.

3.8 SUMMARY

In this unit, we have covered the following points:

1. The chi-square distribution.
2. The properties of χ^2-distribution.
3. The probability curve of χ^2-distribution.
4. Mean and variance of χ^2-distribution.
5. Applications of χ^2-distribution.
6. Student's-t and Fisher's t-distributions.
7. Properties of t-distribution.
8. The probability curve of t-distribution.
9. Mean and variance of t-distribution.
10. Applications of t-distribution.

3.9 SOLUTIONS / ANSWERS

E1) We know that the probability density function of χ^2-distribution with n df is given by

$$f\left(\chi^2\right)=\frac{1}{2^{n/2}\left\lceil\frac{n}{2}\right.}e^{-\chi^2/2}\left(\chi^2\right)^{(n/2)-1}\;; \qquad 0<\chi^2<\infty$$

(i) In our case, n = 6, therefore, for n = 6 df the pdf of χ^2-distribution is given by

$$f\left(\chi^2\right)=\frac{1}{2^{6/2}\left\lceil\frac{6}{2}\right.}e^{-\chi^2/2}\left(\chi^2\right)^{(6/2)-1}\;; \qquad 0<\chi^2<\infty$$

$$=\frac{1}{8\times\lceil 3}e^{-\chi^2/2}\left(\chi^2\right)^2$$

$$=\frac{1}{16}e^{-\chi^2/2}\left(\chi^2\right)^2; \qquad 0<\chi^2<\infty \qquad \left[\because \lceil 3=\lfloor 2=2\right]$$

(ii) Similarly, for n = 10 df the pdf of χ^2-distribution is given by

$$f\left(\chi^2\right)=\frac{1}{2^{10/2}\left\lceil\frac{10}{2}\right.}e^{-\chi^2/2}\left(\chi^2\right)^{(10/2)-1}\;; \qquad 0<\chi^2<\infty$$

$$=\frac{1}{32\times\lceil 5}e^{-\chi^2/2}\left(\chi^2\right)^4$$

$$= \frac{1}{768} e^{-\chi^2/2} \left(\chi^2\right)^4; \qquad 0 < \chi^2 < \infty \quad \left[\because \lceil 5 = \lfloor 4 = 24\right]$$

E2) We know that the probability density function of χ^2-distribution with n df is given by

$$f\left(\chi^2\right) = \frac{1}{2^{n/2}\left\lceil\frac{n}{2}\right.} e^{-\chi^2/2} \left(\chi^2\right)^{(n/2)-1}; \qquad 0 < \chi^2 < \infty \quad \ldots (11)$$

(i) Here, we are given that

$$f\left(\chi^2\right) = \frac{1}{96} e^{-\chi^2/2} \left(\chi^2\right)^3; \qquad 0 < \chi^2 < \infty$$

The given pdf can be arranged as

$$f\left(\chi^2\right) = \frac{1}{2^{8/2}\left\lceil\frac{8}{2}\right.} e^{-\chi^2/2} \left(\chi^2\right)^{\frac{8}{2}-1} \quad \left[\because 96 = 16 \times 6 = 2^{8/2}\left\lceil\frac{8}{2}\right.\right]$$

Comparing with equation (11), we get n = 8. So degree of freedom of chi-square distribution is 8.

(ii) Here, we are given that

$$f\left(\chi^2\right) = \frac{1}{2} e^{-\chi^2/2}; \qquad 0 < \chi^2 < \infty$$

The given pdf can be arranged as

$$f\left(\chi^2\right) = \frac{1}{2^{2/2}\left\lceil\frac{2}{2}\right.} e^{-\chi^2/2} \left(\chi^2\right)^{\frac{2}{2}-1} \quad \left[\because 2 = 2 \times 1 = 2^{2/2}\left\lceil\frac{2}{2}\right.\right]$$

Comparing with equation (11), we get n = 1. So degree of freedom of chi-square distribution is 1.

E3) We know that the mean and variance of chi-square distribution with n degrees of freedom are

Mean = n and Variance = 2n

In our case, n = 10, therefore,

Mean = 10 and Variance = 20

E4) Here, we are given that

$$f\left(\chi^2\right) = \frac{1}{96} e^{-\chi^2/2} \left(\chi^2\right)^3; \qquad 0 < \chi^2 < \infty$$

As similar as (i) of **E2)**, we get n = 8, therefore,

Mean = n = 8 and Variance = 2n = 16.

E5) Refer Section 3.4.

E6) We know that the probability density function of t-distribution with n df is given by

$$f(t)=\frac{1}{\sqrt{n}\,B\left(\frac{1}{2},\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{(n+1)/2}};\qquad -\infty<t<\infty$$

(i) In our case, n = 3, therefore, for n = 3 df the pdf of t-distribution is given by

$$f(t)=\frac{1}{\sqrt{3}\,B\left(\frac{1}{2},\frac{3}{2}\right)\left(1+\frac{t^2}{3}\right)^{(3+1)/2}};\qquad -\infty<t<\infty$$

$$=\frac{\Gamma\left(\frac{1}{2}+\frac{3}{2}\right)}{\sqrt{3}\left(\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)\right)\left(1+\frac{t^2}{3}\right)^{2}}$$

$$=\frac{\Gamma 2}{\sqrt{3}\left(\Gamma\left(\frac{1}{2}\right)\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)\left(1+\frac{t^2}{3}\right)^{2}}\qquad\left[\because \Gamma\left(\frac{n}{2}\right)=\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)\right]$$

$$=\frac{2}{\sqrt{3}\pi\left(1+\frac{t^2}{3}\right)^{2}};\qquad -\infty<t<\infty\quad\left[\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\text{ and }\Gamma 2=1\right]$$

(ii) Similarly, for n = 9 df the pdf of t-distribution is given by

$$f(t)=\frac{1}{\sqrt{9}\,B\left(\frac{1}{2},\frac{9}{2}\right)\left(1+\frac{t^2}{9}\right)^{(9+1)/2}};\qquad -\infty<t<\infty$$

$$=\frac{\Gamma\left(\frac{1}{2}+\frac{9}{2}\right)}{3\left(\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{9}{2}\right)\right)\left(1+\frac{t^2}{9}\right)^{5}}$$

$$=\frac{\Gamma 5}{3\left(\Gamma\left(\frac{1}{2}\right)\times\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right)\left(1+\frac{t^2}{9}\right)^{5}}\qquad\left[\because \Gamma\left(\frac{n}{2}\right)=\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)\right]$$

$$=\frac{24}{3\left(\frac{105}{16}\right)\pi\left(1+\frac{t^2}{9}\right)^{5}}\qquad\left[\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\text{ and }\Gamma 5=\lfloor 4=24\right]$$

$$=\frac{128}{105\pi\left(1+\frac{t^2}{9}\right)^{5}};\qquad -\infty<t<\infty$$

E7) We know that the probability density function of t-distribution with n df is given by

$$f(t)=\frac{1}{\sqrt{n}B\left(\frac{1}{2},\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{(n+1)/2}};\quad -\infty<t<\infty \quad \dots (12)$$

Here, we are given that

$$f(t)=\frac{1}{\sqrt{5}\,B\left(\frac{1}{2},\frac{5}{2}\right)\left(1+\frac{t^2}{5}\right)^{3}};\quad -\infty<t<\infty$$

$$=\frac{1}{\sqrt{5}\,B\left(\frac{1}{2},\frac{5}{2}\right)\left(1+\frac{t^2}{5}\right)^{(5+1)/2}};\quad -\infty<t<\infty$$

Comparing with equation (12), we get n = 5. So degrees of freedom of t-distribution are 5.

E8) Here, we are given that

$$\mu=60,\ n=15,\ \bar{X}=65 \text{ and } S=12$$

We know that the t-variate is

$$t=\frac{\bar{X}-\mu}{S/\sqrt{n}}$$

Therefore, we have

$$t=\frac{65-60}{12/\sqrt{15}}=\frac{5}{4.39}=1.14$$

E9) We know that the mean and variance of t-distribution with n degrees of freedom are

$$\text{Mean} = 0 \text{ and Variance } =\frac{n}{(n+2)};\qquad n>2$$

In our case, n = 20, therefore,

$$\text{Mean} = 0 \text{ and Variance } =\frac{20}{(20+2)}=\frac{10}{11}$$

E10) Refer Section 3.7.