Necessary Conditions for Asymptotic Tracking in Nonlinear Systems

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***Abstract*—In the literature, it has been shown that if a single-input single-output analytic nonlinear plant 1) has a well-defined relative degree and 2) is minimum-phase, it is possible to achieve asymptotic tracking for an open set of output trajectories containing the origin in $C^N[0,\infty)$, the space of N-times continuously differentable functions taking values in $\mathbb{R}$. When either of these sufficient conditions is not met, various authors have investigated approximate analytic solutions, discontinuous solutions and solutions for restricted sets of trajectories. In this paper, it is shown that conditions 1) and 2) are necessary for the existence of an analytic compensator which yields asymptotic tracking for an open set of output trajectories. Analogous results are established for multi-input multi-output systems.**

I. Introduction

CONSIDER a nonlinear system of the form

$$P: \begin{array}{rcl} \dot{x} &=& f(x) + g(x)u \\ y &=& h(x) \end{array}$$

called the plant, and a class of desired output signals, denoted S. The problem of asymptotic tracking for P and S consists in finding a compensator Q such that i) the closed-loop system is internally stable and ii) for any given desired trajectory y_d belonging to S, the output of the closed-loop system asymptotically approaches y_d.

One can distinguish several qualitatively different types of problems by the amount of structure imposed on S. The most highly structured situation is when S consists of only a finite set of desired trajectories; often it may be a singleton [5], [12], [22], [32]. The resulting compensator is effectively an open-loop controller and must be changed for each trajectory. If the class of signals S is described by an exosystem, that is, an autonomous, noninitialized set of differential equations with an output, asymptotic tracking for S is usually called the regulator or servomechanism problem and has been investigated in [23], [26]. If the class of signals S is generated by a model, that is a forced, noninitialized dynamical system described by a set of differential equations, the problem is usually termed asymptotic model matching [11].

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The least structured situation which has been investigated is when S consists of an open ball around the origin of[1] $C_m^N[0,\infty)$, defined by $\sup_{t\geq 0}\{\|y_d(t)\|,\cdots,\|y_d^{(N)}(t)\|\} < \epsilon$, for $N \geq 0$ and $\epsilon > 0$ appropriately chosen, where y_d is a desired output trajectory. This is useful in applications when one cannot specify *a priori* an exhaustive finite set of trajectories to be tracked or an exosystem/model for generating the desired output trajectories. For example, this situation arises in robotics where the desired output may be described by a concatenation of signals each generated by a different exosystem. Applying the regulator theory in this case yields a controller which experiences transients at each change of the exosystem [6].

A. Motivation

Output tracking is an ubiquitous control objective. In the nonlinear arena, many sufficient conditions for solving various forms of the tracking problem have been obtained. It is current practice, when these conditions are not met, to approximate the system model by one for which the known sufficient conditions hold. This leads to an approximate solution with lower performance. In many instances, this may not be necessary because the original problem is actually solvable; it is just the known sufficient conditions which are not met.

This paper concentrates on asymptotic tracking for open sets of output trajectories. Sufficient conditions for achieving this are known. The first solutions [20], [40], [42] explicitly placed a system inverse in the closed loop to generate the tracking control. More recently, it has been proposed to first input–output linearize the system, rendering unobservable the system's zero-dynamics, and then to apply standard linear theory to the resulting system [19]. It follows that a solution to the asymptotic tracking problem may be found via these methods if

1) the regularity conditions needed to construct an input–output linearizing compensator or the system's inverse are met;
2) the system's zero-dynamics is asymptotically stable.

On the other hand, whenever the conditions 1) or 2) above are not met, only approximate solutions to tracking are known; indeed, all current approaches are based upon approximating the system by one for which a vector relative degree can be achieved and which possesses an asymptotically stable zero-dynamics [16], [18], [38].

[1] $C_m^N[0,\infty)$ denotes the space of N-times continuously differentiable functions taking values in $\mathbb{R}^m$.

The goal of this paper is to establish to what extent such conditions are necessary for the existence of an analytic compensator yielding asymptotic tracking for an open set of output trajectories. For smaller classes of desired output trajectories, e.g., S finite [5], [12], [22], [32], generated by a finite dimensional exosystem [23], [26], or a forced model [11], there do exist examples of systems which do not satisfy these properties, but for which a solution to the associated asymptotic tracking problem does exist. With the exception of asymptotic model matching, however, no general theory is available for when the methods in question can or cannot overcome singularities in the input–output map. It has been indicated in [15] that using a discontinuous control law may be advantageous when singularities are present, though it may be argued that an analytic control law is often preferable because truncated series solutions of the equations describing the compensator are often used [24].

B. Contributions of the Paper

In this paper, it is shown that, under certain controllability and invertibility assumptions, there exists an analytic compensator yielding asymptotic tracking (for an open set of output trajectories) only if 1) a vector relative degree at the origin can be achieved by dynamic compensation and 2) the system is minimum phase. More precisely, it is proven that:

i) When a compensator induces asymptotic tracking, the output of the closed-loop system exactly coincides with the desired output trajectory y_d, whenever a) y_d and its first $\bar{N}$ derivatives vanish at time $t_0 = 0$, where $\bar{N}$ is the dimension of the closed-loop system, b) y_d and its first $\bar{N}$-derivatives are uniformly sufficiently small and c) the closed-loop system is initialized at the origin (an assumed equilibrium point). The coincidence of the outputs under properties a)–c) is called exact tracking. This result is fundamental in that it allows one to analyze many aspects of asymptotic tracking on the basis of the simpler, more highly structured property of exact tracking.

ii) For a multi-input multi-output (MIMO) plant, whenever the plant has a controllable linearization and the asymptotic tracking problem is solvable, then a vector relative degree at the origin can be achieved by dynamic compensation. In the case of a single-input single output (SISO) plant, a stronger result is proven: whenever the plant is strongly accessible from the origin and the asymptotic tracking problem is solvable, then the plant possesses a well-defined relative degree at the origin.

iii) For locally controllable MIMO systems, it is established that under certain "regularity conditions" associated with system inverses and zero-dynamics, the minimum-phase property is also a necessary condition for asymptotic tracking. In the case of a SISO system, this result in combination with ii) and [19] yields a set of necessary and sufficient conditions for the solvability of asymptotic tracking.

These necessary conditions define situations where, if one seeks a solution to the asymptotic tracking problem, it is obligatory to a) leave the class of analytic compensators, b) restrict the class of trajectories to be tracked, or c) consider approximate solutions to the problem. In particular, they highlight the interest of the approximate solutions to the asymptotic tracking problem developed in [18], which approximated the plant by one for which a vector relative degree could be achieved, and those in [19], [38], which approximated the plant by one which is minimum phase.

Organization of the Paper

Section II establishes the notation and recalls some background material; some standard terminology is not explicitly defined in the paper and the reader is referred to the text books [25], [35]. In Section III, the asymptotic tracking problem is formulated, and it is shown that any solution to the asymptotic tracking problem also provides exact tracking from the origin. Section IV establishes that, for the case of SISO systems, whenever the plant is strongly accessible from the equilibrium, a well-defined relative degree is a necessary condition for achieving asymptotic tracking. For MIMO systems, under the stronger hypothesis that the linearization of the plant is controllable, it is necessary that a vector relative degree at the origin can be achieved by dynamic compensation. Both of these results are based on another result of some independent interest: a system which is strongly accessible from the origin and possesses a linear input–output behavior from the origin also has a linear input–output behavior in a neighborhood of the origin. In Section V, for locally controllable MIMO systems, it is established that under certain "regularity conditions" associated with system inverses and zero-dynamics, the minimum-phase property is also a necessary condition. Section VI contains the conclusions.

II. BACKGROUND AND NOTATION

This paper considers affine analytic systems of the form

$$P : \begin{array}{rcl} \dot{x} & = & f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i \\ y & = & h(x) \end{array} \quad (1)$$

where $x \in \mathbb{R}^n, u, y \in \mathbb{R}^m, f(0) = 0$, and the entries of f, the columns of g and the rows of h consist of analytic functions, and $h(0) = 0$. An affine analytic dynamic compensator for P is a second system

$$C : \begin{array}{rcl} \dot{\xi} & = & c(x,\xi) + d(x,\xi)v \\ u & = & \gamma(x,\xi) + \delta(x,\xi)v \end{array} \quad (2)$$

where $\xi \in \mathbb{R}^{\nu}, v \in \mathbb{R}^m, c(0,0) = 0, \gamma(0,0) = 0$, and the entries of c, d, γ and δ consist of analytic functions. The closed-loop system (1)–(2) is denoted by $P \circ C$; it has input v and output y, sometimes denoted $y^{P \circ C}$.

In the case that (1) is a SISO system, it is said that P has a well-defined relative degree at the origin if there exists an integer $r \geq 1$ such that $\forall x \in \mathbb{R}^n$

$$\begin{array}{l} L_g L_f^k h(x) = 0 k < r - 1 \\ L_g L_f^{r-1} h(0) \neq 0 \end{array} \quad (3)$$

r is called the relative degree of (1). Still in the case of a SISO system, the relative degree of the compensator (2) is defined on the basis of the closed-loop system $P \circ C$, viewing v as the input and u as the output. Let

$$\tilde{f} = \begin{pmatrix} f + g\gamma \\ c \end{pmatrix}, \qquad \tilde{g} = \begin{pmatrix} g\delta \\ d \end{pmatrix}.$$

If $\delta(x,\xi) \not\equiv 0$, C has a well-defined relative degree at the origin if $\delta(0,0) \neq 0$. If $\delta(x,\xi) \equiv 0$, C has a well-defined relative degree at the origin if there exists an integer $s \geq 1$ such that $\forall x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^\nu$

$$\begin{aligned} &L_{\tilde{g}} L_{\tilde{f}}^k \gamma(x,\xi) = 0, k < s-1, \\ &L_{\tilde{g}} L_{\tilde{f}}^{s-1} \gamma(0,0) \neq 0 \end{aligned} \tag{4}$$

s is called the relative degree of (2). A MIMO system of the form (1) has a vector relative degree at the origin if there exist integers $r_1, \cdots, r_m$ such that a) for all $1 \leq j \leq m$, $1 \leq i \leq m$ and $k < r_i - 1$, $L_{g_j} L_f^k h_i(x) \equiv 0$, and b) the $m \times m$ matrix, called the decoupling matrix, whose ij element is defined by $L_{g_j} L_f^{r_i - 1} h_i(x)$, has rank m when evaluated at the origin. It is said that a vector relative degree at the origin can be achieved by dynamic compensation if there exists a compensator C of the form (2) such that $P \circ C$ has a vector relative degree at the origin.

A system[2] $\sum$ of the form (1) is said to exhibit a linear input–output behavior in the sense of [28] in a neighborhood $\mathcal{O}$ of the origin if there exist $T > 0, \epsilon > 0$, a function $w_0(t, x_0)$ and matrices (F, G, H) of compatible dimensions such that

$$y(t) = w_0(t, x_0) + \int_0^t He^{F(t-\tau)} Gu(\tau)\, d\tau \tag{5}$$

for all $x_0 \in \mathcal{O}, 0 \leq t \leq T$, and all controls $u(t) \in L^\infty[0,T]$ such that $\|u\|_\infty < \epsilon$. $\sum$ is said to exhibit a linear input–output behavior from the origin if the above holds for $x_0 = 0$. Since the origin is assumed to be an equilibrium point, $w_0(t, 0) = 0$. Whenever the triple (F, G, H) corresponds to an invertible linear system, $\sum$ will be said to exhibit an invertible linear input-output behavior (from the origin). Let $F_L := \frac{\partial f(0)}{\partial x}$, $G_L := g(0)$, and $H_L := \frac{\partial h(0)}{\partial x}$ be the classical Jacobian linearization of $\sum$ about the origin, denoted $\sum_L$. Then, by standard results on Volterra series representations [31], [39], it follows that $He^{Ft}G = H_L e^{F_L t} G_L$. Therefore, the triple (F, G, H) corresponds to an invertible linear system if and only if (F_L, G_L, H_L) is invertible.

From [28], a system $\sum$ of the form (1) exhibits a linear input–output behavior in a neighborhood $\mathcal{O}$ of the origin if, and only if

$$\forall k \geq 0, 1 \leq i, j \leq m, L_{g_i} L_f^k h_j(x) \;=\; \text{constant on } \mathcal{O}. \tag{6}$$

In the case of a SISO system, this immediately implies that it has a well-defined relative degree at the origin or that the constant in (6) is zero for every k, in which case the output is not affected by the input. For a MIMO system, (6) implies that the system's nonlinear structure at infinity, as characterized in [10], coincides with the structure at infinity of $\sum_L$.

Finally, $C_m^N[0,T]$ will denote the space of N-times continuously differentiable m-vector valued functions on $[0,T]$; the norm of $c(t) \in C_m^N[0,T]$ is taken as $\sup_{0 \leq t \leq T} \{\|c(t)\|, \|\dot{c}(t)\|, \cdots, \|c^{(N)}(t)\|\}$, where $\|\cdot\|$ is any norm on $\mathbb{R}^m$. The solution of (1) corresponding to the input u and initial state $x(0) = x_0$ will be denoted by $x(t, x_0, u)$; the corresponding output will be denoted by $y(t, x_0, u)$.

[2] In the sequel, $\sum$ will often be $P \circ C$.

III. A Relation Between Asymptotic and Exact Tracking

In this section, a local version of the asymptotic tracking problem is formulated. It is proven that any compensator inducing asymptotic tracking provides exact tracking from the origin as well. This property will be crucial to the development of Sections IV and V.

Consider again the analytic plant P

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{7}$$

where the state $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, $f(0) = 0$, $h(0) = 0$ and let the class of desired output trajectories $y_d(t)$ be defined by $y_d \in C_m^N[0,\infty)$ and

$$\sup_{t \geq 0} \left\{ \|y_d(t)\|, \cdots, \|y_d^{(N)}(t)\| \right\} < \epsilon \tag{8}$$

for appropriate choices of $N \geq 0$ and $\epsilon > 0$. Let $Y_d(t) = [y_d'(t), \dot{y}_d'(t), \cdots, y_d^{(N)'}(t)]'$, where $'$ denotes transpose.

Definition 1: The asymptotic tracking problem for the plant P is to find an integer $N \geq 0$ and an $\epsilon > 0$ defining a class of output trajectories (8), an integer $q \geq 0$, an open neighborhood $\mathcal{O}_1$ of the origin in $\mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{(N+1)m}$ and two analytic functions $a : \mathcal{O}_1 \to \mathbb{R}^q$ and $\alpha : \mathcal{O}_1 \to \mathbb{R}^m$, with $a(0,0,0) = 0$, $\alpha(0,0,0) = 0$, defining the compensator Q

$$\begin{aligned} \dot{z} &= a(x, z, Y_d) \\ u &= \alpha(x, z, Y_d) \end{aligned} \tag{9}$$

such that the closed-loop $P \circ Q$ satisfies:

1) $\lim_{t \to \infty}(y^{P \circ Q}(t) - y_d(t)) = 0$, for all $(x_0, z_0) \in \mathcal{O}_2$, an open neighborhood of the origin in $\mathbb{R}^n \times \mathbb{R}^q$, and for all $y_d(t)$ satisfying (8);
2) the equilibrium (0,0) of the unforced system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha(x, z, 0) \\ \dot{z} &= a(x, z, 0) \end{aligned} \tag{10}$$

is asymptotically stable.

The main result of this section is now given.

Theorem 2: Consider the plant given by (7). Suppose that a compensator Q solves the asymptotic tracking problem with parameters N and $\tilde{\epsilon}$. Then, there exists $\epsilon < \tilde{\epsilon}$ such that whenever $P \circ Q$ is initialized at $(x_0, z_0) = (0, 0)$, then $y^{P \circ Q}(t) - y_d(t) = 0$ for $t \geq 0$, for each $y_d(t)$ in the class (8) satisfying $Y_d(0) = 0$.

Proof: See the Appendix. □

Definition 3: Based on the above result, the following properties are defined.

a) The exact tracking problem from the origin for the plant P is to find an integer $N \geq 0$, $\epsilon > 0$, and $T > 0$ defining a class of output trajectories (8), an integer $q \geq 0$, an open neighborhood $\mathcal{O}_1$, of the origin in

$\mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{(N+1)m}$, two analytic functions $a : \mathcal{O}_1 \to \mathbb{R}^q$, $\alpha : \mathcal{O}_1 \to \mathbb{R}^m$, with $a(0,0,0) = 0$, $\alpha(0,0,0) = 0$, defining the compensator (9), such that for each $y_d(t)$ in the class (8) satisfying $Y_d(0) = 0$, whenever $P \circ Q$ is initialized at $(x_0, z_0) = (0,0)$, then $y^{P\circ Q}(t) - y_d(t) = 0$ for $0 \le t \le T$.

b) If T is taken equal to ∞ in a), the previous problem is called the long-term exact tracking problem from the origin.

In terms of Definition 3, Theorem 2 states that, if a compensator Q achieves asymptotic tracking, then it also induces exact tracking from the origin. Consequently, if Q asymptotically stabilizes the closed-loop $P \circ Q$, but does not provide exact tracking, then for every $\epsilon > 0$ there is always an output trajectory satisfying (8) which cannot be asymptotically tracked. The proof of Theorem 2 given in theAppendix actually proves a stronger result which is stated in the following proposition.

Proposition 4: Consider a plant P given by (7). Suppose that a compensator Q of the form (9) asymptotically stabilizes $P \circ Q$, but that Q does not solve the long-term exact tracking problem from the origin. Then, for every $\epsilon > 0$, there exists a periodic output trajectory $y_d^\epsilon(t)$ satisfying (8) which cannot be asymptotically tracked by $P \circ Q$; that is, there exists an open neighborhood $\mathcal{O}$ of the origin of $\mathbb{R}^n \times \mathbb{R}^q$ such that, for every $(x_0, z_0) \in \mathcal{O}$, $\limsup_{t\to\infty}(y^{P\circ Q}(t) - y_d^\epsilon(t)) \neq 0$.

The above proposition implies that there is no advantage to formulating the asymptotic tracking problem for the smaller set S consisting of periodic trajectories satisfying (8), although this smaller set may be more appealing from a practical point-of-view.

IV. Regularity Properties of Tracking Loops

In this section, it is shown that if a plant P has a controllable Jacobian linearization and the exact tracking problem from the origin is solvable, then P can be dynamically input–output decoupled in a neighborhood of the origin. For the special case of SISO plants, the hypothesis on the controllability of the Jacobian linearization can be weakened to the strong accessibility of P from the origin and it is concluded that P necessarily has a well-defined relative degree at the origin. The development of these results is based on the observation that a compensator solving the exact tracking problem from the origin can be easily modified to yield a closed-loop system having a linear input–output behavior from the origin. Whenever the plant possesses certain controllability properties, it is always possible to construct a third compensator yielding a linear input–output behavior in an open neighborhood of the origin. This latter characteristic will make it possible to show that P necessarily has the announced properties. SISO systems are first treated in detail, and the MIMO result is obtained by simple modifications.

To establish the connection between tracking and linear input–output behavior, note that any element y_d of $C_m^N[0,T]$ can be obtained as a solution of the system on $\mathbb{R}^{Nm}$

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_N &= \xi_{N+1} \\ y_d &= \xi_1 \end{aligned} \tag{11}$$

where ξ_{N+1} is a continuous function on $[0,T]$. Therefore, if a compensator Q of the form (9) solves the exact tracking problem from the origin, the compensator

$$\overline{Q} : \begin{array}{rcl} \dot{\xi}_1 &=& \xi_2 \\ &\vdots& \\ \dot{\xi}_N &=& \xi_{N+1} \\ \dot{z} &=& a(x, z, \xi_1, \cdots, \xi_{N+1}) \\ u &=& \alpha(x, z, \xi_1, \cdots, \xi_{N+1}) \end{array} \tag{12}$$

is such that $y^{P\circ\overline{Q}}(t) = y_d(t), 0 \le t \le T$, whenever $P \circ \overline{Q}$ is initialized at 0, $\xi_{N+1}(0) = 0$ and $\sup_{0\le t\le T}\{\|\xi_1(t)\|, \cdots, \|\xi_{N+1}(t)\|\} < \epsilon$. By appending integrators to ξ_{N+1} per $\dot{\xi}_{N+1} = v$ in (12), and letting Q^a denote the resulting compensator

$$Q^a : \begin{array}{rcl} \dot{\xi}_1 &=& \xi_2 \\ &\vdots& \\ \dot{\xi}_N &=& \xi_{N+1} \\ \dot{\xi}_{N+1} &=& v \\ \dot{z} &=& a(x, z, \xi_1, \cdots, \xi_{N+1}) \\ u &=& \alpha(x, z, \xi_1, \cdots, \xi_{N+1}) \end{array} \tag{13}$$

the system $P \circ Q^a$ is affine in v and exhibits an invertible linear input–output behavior from the origin.

For a SISO system P, if $P \circ Q^a$ had an invertible linear input-output behavior in an open neighborhood of the origin, and not just from the origin, it would be easy to conclude that P has a well-defined relative degree at the origin from the following result.

Lemma 5: Consider a SISO plant P of the form (7) and an affine dynamic compensator C of the form

$$\begin{aligned} \dot{\xi} &= c(x,\xi) + d(x,\xi)v \\ u &= \gamma(x,\xi) + \delta(x,\xi)v \end{aligned} \tag{14}$$

with ξ belonging to an open neighborhood of the origin in $\mathbb{R}^\nu$, $c, d, \gamma,$ and δ being $n + \nu$ times continuously differentiable, $c(0,0) = 0$ and $\gamma(0,0) = 0$. Then, if the closed-loop system $P \circ C$ has a well-defined relative degree at the origin, both P and C have well-defined relative degrees at the origin.

Proof: See the Appendix. □

Without any further hypotheses, however, a linear input–output behavior from the origin does not imply that such a property holds on an open neighborhood of the origin. Indeed, consider the system

$$\Sigma : \begin{array}{rcl} \dot{x}_1 &=& -(x_1)^3 \\ \dot{x}_2 &=& x_3 \\ \dot{x}_3 &=& u \\ y &=& x_2 + x_1 x_3. \end{array}$$

Σ does not possess a well-defined relative degree at the origin, but clearly, when initialized at 0, the input–output behavior is linear. The problem, of course, comes from the uncontrollable part of the system.

Proposition 6: Consider a MIMO system Σ of the form (7), and suppose that it is strongly accessible from the origin. Then Σ has a linear input–output behavior on an open neighborhood of the origin if, and only if, it has a linear input–output behavior from the origin.

Proof: See the Appendix.

At this point, it is easily seen that, for a SISO plant P, if $P \circ Q^a$ were strongly accessible from the origin, then P would have a well-defined relative degree at the origin. In general, this is not the case given only the strong accessibility from the origin of P. The goal of the following is to derive from Q^a a new compensator C such that $P \circ C$ has the same input-output behavior from the origin as $P \circ Q^a$, and $P \circ C$ is strongly accessible from the origin whenever P is.

Let $\mathcal{R}^*$ be the strong accessibility distribution of $P \circ Q^a$ and let $\mathcal{M}$ be the leaf of $\mathcal{R}^*$ containing the origin. $\mathcal{M}$ is an analytic submanifold of $\mathbb{R}^n \times \mathbb{R}^q$ and is invariant under the dynamics of $P \circ Q^a$. By construction, the system $P \circ Q^a$ restricted to $\mathcal{M}$, denoted $P \circ Q^a|_{\mathcal{M}}$, is strongly accessible from the origin [43] and has a linear input–output behavior from the origin. Hence, $P \circ Q^a|_{\mathcal{M}}$ has an invertible linear input-ouput behavior in a neighborhood of the origin by Proposition 6; so, as noted in Section II, it has a well-defined relative degree at the origin (with input v and output y). Let $C = P \circ Q^a|_{\mathcal{M}}$, with input v and output u. Note that $P \circ C$ has the same input–output behavior from the origin as $P \circ Q^a|_{\mathcal{M}}$ from v to y; indeed this is just the usual way of realizing a dynamic state feedback as a precompensator. The advantage of the compensator C over Q^a is that it necessarily has a well-defined relative degree at the origin as shown in Lemma 7 below. Then, using the regularity of C, one deduces a final compensator $\tilde{C}$ of the form (14) such that $P \circ \tilde{C}$ is strongly accessible from the origin and has an invertible linear input-output behavior from the origin by Lemma 14.

Lemma 7: Consider a SISO plant P of the form (7) and an affine dynamic compensator C of the form (14). Let $\mathcal{M}$ be an analytic submanifold of $\mathbb{R}^n \times \mathbb{R}^\nu$ containing the origin and suppose that $\mathcal{M}$ is an invariant manifold for the closed-loop system $P \circ C$. Then, if $P \circ C$ restricted to $\mathcal{M}$, denoted $P \circ C|_{\mathcal{M}}$, has a well-defined relative degree at the origin with input v and output y, it also has a well-defined relative degree at the origin with input v and output u.

Proof: See the Appendix. □

Lemma 8: Consider a SISO plant P of the form (7) and suppose P is strongly accessible from the origin. Let C be a compensator of the form (14) and suppose that C has a well-defined relative degree at the origin. Then there exists a compensator $\tilde{C}$ of the form (14) such that $P \circ \tilde{C}$ is strongly accessible from the origin and has the same input–output behavior from the origin as $P \circ C$.

Proof: See the Appendix. □

Putting all this together establishes the main result of the section.

Theorem 9: Consider a SISO plant P of the form (7) and suppose P is strongly accessible from the origin. Then, if the exact tracking problem from the origin is solvable, P has a well-defined relative degree at the origin. In particular, this is the case if the asymptotic tracking problem is solvable.

Proof: Let Q be a compensator of the form (9) solving the exact tracking problem and let Q^a be as defined in (13). Then, $P \circ Q^a$ has an invertible linear input–output behavior from the origin. Let $\mathcal{M}$ be the leaf of the strong accessibility distribution of $P \circ Q^a$ passing through the origin. We define $C = P \circ Q^a|_{\mathcal{M}}$. C has a well-defined relative degree at the origin from input v to output u, by Lemma 7. Let $\tilde{C}$ be constructed as in Lemma 8. Then $P \circ \tilde{C}$ has an invertible linear input–output behavior from the origin, indeed the same as $P \circ Q^a$, and is strongly accessible from the origin. Hence, by Proposition 6, $P \circ \tilde{C}$ has a linear input–output behavior in a neighborhood of the origin, and therefore a well-defined relative degree at the origin. Finally, by Lemma 5, P has a well-defined relative degree at the origin. □

As a consequence, the following also holds.

Corollary 10: Consider a SISO plant P of the form (7) and let $\mathcal{L}$ be the leaf of the strong accessibility distribution of P passing through the origin. Then, if the exact tracking problem from the origin is solvable, the plant P restricted to $\mathcal{L}$ has a well-defined relative degree at the origin.

In the case of MIMO systems, the key observation is that the conclusion of Lemma 8 still holds if one assumes the stronger hypothesis that the plant has a controllable Jacobian linearization about the origin. This leads to the second important result of the Section.

Theorem 11: Consider a MIMO plant P of the form (7) and suppose that P has a controllable Jacobian linearization about the origin. Then, if the exact tracking problem from the origin is solvable, P is dynamically input-output decouplable in a neighborhood of the origin; more precisely, a vector relative degree at the origin can be achieved for P by dynamic compensation. In particular, this is the case if the asymptotic tracking problem is solvable.

Proof: Let Q be a compensator of the form (9) solving the exact tracking problem and let Q^a be as defined in (13). Then, $P \circ Q^a$ has an invertible linear input–output behavior from the origin. Therefore, the Jacobian linearization about the origin of $P \circ Q^a$, denoted $(P \circ Q^a)_L$, is invertible, implying that the Jacobian linearization of Q^a, denoted Q^a_L is also invertible. Let π be the canonical projection from the state space of $P \circ Q^a$ to $\mathbb{R}^n$, let $\mathcal{R}^*$ be the strong accessibility distribution of $P \circ Q^a$ and let $\mathcal{R}^*_L$ be the controllable subspace of $(P \circ Q^a)_L$; note that $\mathcal{R}^*_L \subset \mathcal{R}^*(0)$. By a straightforward argument, the invertibility of Q^a_L combined with the controllability of P_L, the Jacobian linearization of P, implies that $\pi(\mathcal{R}^*_L) = \mathbb{R}^n$. Thus, $\pi_*(\mathcal{R}^*(0)) = T_0\mathbb{R}^n$ showing that (23) holds in an open neighborhood of the origin. Let $\mathcal{L}$ be the leaf of $\mathcal{R}^*$ containing the origin. Then, following the proof of Lemma 8, there exists a compensator $\tilde{C}$ such that $P \circ \tilde{C} = P \circ Q^a|_{\mathcal{L}}$. Moreover, $P \circ \tilde{C}$ is strongly accessible from the origin and has the same input–output behavior from the origin as $P \circ Q^a$. Therefore, by Proposition 6, the result follows. □

Some of the results of this section are illustrated in the following (academic) examples.

Consider the plant

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y &= x_1 + (x_2)^3. \end{aligned}$$

It has a completely controllable linear dynamics but does not have a well-defined relative degree at the origin. Hence, by Theorem 9, asymptotic tracking can never be achieved for this system, no matter how many derivatives of the output trajectories are assumed to exist nor how high an order is chosen for the compensator. One is therefore obliged to: a) consider approximate solutions to the problem, b) leave the class of analytic compensators, or c) restrict the class of trajectories to be tracked.

Consider the plant

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\\ \dot{x}_3 &= -(x_2+x_3)^3 - u\\ y &= x_1 + x_2x_3 + (x_3)^2.\end{aligned}$$

It can be verified that the system does not possess a well-defined relative degree at the origin, but nevertheless, that the compensator

$$u = \ddot{y}_d + a_1(\dot{y}_d - x_2) + a_2(y_d - x_1) \qquad (15)$$

with $a_1, a_2 > 0$, yields (global) asymptotic tracking. Hence, by Corollary 10, it must be true that the system restricted to the leaf of its strong accessibility distribution through the origin has a well-defined relative degree at the origin. Indeed, this leaf is given by $\mathcal{L} = \{(x_1, x_2, -x_2) | x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$ and the system restricted to this leaf is diffeomorphic to $\dot{x}_1 = x_2, \dot{x}_2 = u, y = x_1$.

V. On a Minimum-Phase Property of Tracking Loops

In the first part of this section, it is shown that if a plant P satisfies certain "regularity conditions" associated with the existence of a left-inverse[3] at the origin, then any closed-loop system yielding asymptotic tracking must contain a copy of the full-order left-inverse at the origin of P. It is then proven, under the additional hypotheses that P has a controllable linearization and a well-defined zero-dynamics [2], [1], [25], [27], that the minimum-phase property [2], [1], [25] is a necessary condition for achieving asymptotic tracking.

From Section IV, if Q solves the exact tracking problem from the origin, then the associated closed-loop $P \circ Q^a$ has an invertible linear input–ouput behavior from the origin. Hence, the Jacobian linearization about the origin of $P \circ Q^a$ is invertible and this implies that the Jacobian linearization about the origin of P is invertible.[4] It follows from [16] that the rank of P [13], [14], ρ^*, equals m. In that case, it is known that, in a neighborhood of a generic[5] point of the state space, the system possesses a left-inverse [21], [27], [40]. To ensure that the origin is not an exceptional point, in addition to $\rho^* = m$ it is sufficient to suppose, for example, that the pair $(x_0 = 0, y \equiv 0)$ is locally strongly regular in the sense of [9]. In this case, it follows from [27], [40] that there exist $\epsilon_2 > 0$ and an open neighborhood $\mathcal{O}$ of the origin of (7) such that if $u(t)$ and $v(t)$ are two inputs so that

i) the associated state trajectories from the origin $x(t, 0, u)$ and $x(t, 0, v)$ remain in $\mathcal{O}$ for the interval of time $[0, T]$, where $T > 0$;

ii) the associated output trajectories $y(t, 0, u)$ and $y(t, 0, v)$ coincide for $t \in [0, T]$; and

iii) $\|y^{(j)}(t, 0, u)\| < \epsilon_2, 0 \le j \le n-1, t \in [0, T]$;

then $u(t) = v(t)$, for $t \in [0, T]$. Whenever i), ii), and iii) are satisfied, system (7) is said to be locally left-invertible at the origin. If, in addition, the control $u(t)$ giving rise to the trajectory $y(t, 0, u)$ can be expressed as $u(t) = u^*(x(t), y(t), \cdots, y^{(\alpha)}(t))$ for some integer $0 \le \alpha \le n-1$, the system

$$\dot{\overline{x}} = f(\overline{x}) + g(\overline{x})u^*(\overline{x}, y, \cdots, y^{(\alpha)}) \qquad (16)$$

$$u = u^*(\overline{x}, y, \cdots, y^{(\alpha)}) \qquad (17)$$

$$\overline{x}(0) = 0 \qquad (18)$$

is called the full-order left-inverse of (7) at the origin [27]. References [17], [36] discuss the fact that a left-inverse at the origin may exist under weaker conditions than strong regularity, still ensuring that the control u^* in (17) is unique. As long as this property holds, the system is said to have a well-defined left-inverse at the origin.

A zero-dynamics manifold [25] for a system P of the form (7) is a C^1-manifold Z^* of $\mathbb{R}^n$ containing the origin and satisfying i) $Z^* \subset h^{-1}(0)$, ii) $f|_{Z^*} \subset TZ^* + \operatorname{span}\{g\}$ and iii) Z^* is locally maximal with respect to i) and ii). The system P has a zero-dynamics [25] if, in addition, there exists a unique function α^* such that $f^* := f + g\alpha^*|_{Z^*}$ is tangent to Z^*, α^* is C^1 and $\alpha^*(0) = 0$. If the Jacobian linearization P_L of P is invertible, as is always the case when exact tracking is achievable, then dim span$\{g(0)\} = m$ and[6] $T_0Z^* \cap \operatorname{span}\{g(0)\} = 0$; these latter two properties ensure the existence of a unique C^1 function α^* [25]. Whenever the zero-dynamics (Z^*, f^*) is asymptotically (exponentially) stable, P is said to be (exponentially) minimum-phase [2], [1].

For a SISO system, a well-defined relative degree at the origin ensures the existence of the full-order left-inverse at the origin and of the zero-dynamics. In particular, this is true whenever asymptotic tracking can be achieved.

In the context of exact tracking, the conclusion of the following Proposition has been essentially well-known since the work of [20], [40]. The following statement formalizes the result for asymptotic tracking.

Proposition 14: Consider the system P given by (7) and suppose that P has a well-defined left-inverse at the origin. Then, if Q is any compensator of the form (9) solving the asymptotic tracking problem, the closed-loop system $P \circ Q$ contains a copy of the full-order left-inverse of P at the origin.

Proof: Since Q solves the asymptotic tracking problem, property P2) in the proof of Theorem 2 holds. Hence, by choosing $\mu > 0$ sufficiently small, there exists $\epsilon > 0$ such that the solutions of $P \circ Q$ initialized at (0,0) remain in an arbitrarily

[3] Even though we are dealing with square systems, it is still important to make the distinction between left and right inverses, unless one very carefully defines the system's range in relation to its initial condition. To see this, just consider a SISO linear system with relative degree one initialized at the origin. Then $y(0) = 0$ for all u.

[4] Related work on functional reproducibility and rank properties can be found in [37].

[5] A point belonging to the complement of a set having zero Lebesgue measure.

[6] $T_0Z^* \subset V^*$, where V^* is the maximal controlled invariant subspace contained in the kernel of the output map of P_L.

small neighborhood of the origin whenever $\|y_d^{(j)}(t)\| < \epsilon$, for $0 \leq j \leq \max(N, n-1)$, for all $t \geq 0$. By Theorem 2, Q also solves the long-term exact tracking problem from the origin. Hence, whenever $Y_d(0) = 0$ and $x_0 = 0, z_0 = 0$, one has $y_d(t) = y(t)$, for all $t \geq 0$. Under the hypothesis that P has a well-defined left-inverse at the origin, the control $u(t)$ giving rise to $y(t) \equiv y_d(t)$ is unique and can be expressed as $u(t) = u^*(x(t), y_d(t), \cdots, y_d^{(\alpha)}(t))$. Hence, a subset of the state trajectories of $P \circ Q$ initialized at (0,0) is evolving according to (16)–(18). □

Consider the system P described by

$$P: \begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& u \\ \dot{\eta} &=& \lambda\eta^3 + (x_1)^2 \\ y &=& x_1 \end{array} \tag{19}$$

where $\lambda \in \mathbb{R}$ is to be chosen. The system has a well-defined relative degree at the origin and is strongly accessible from the origin, though not locally controllable. One calculates the full-order left-inverse at the origin to be

$$\begin{array}{rcl} \dot{\overline{x}}_1 &=& \overline{x}_2 \\ \dot{\overline{x}}_2 &=& \ddot{y} \\ \dot{\eta} &=& \lambda\eta^3 + (y)^2 \\ u &=& \ddot{y}_d \end{array}$$

with $\overline{x}_1(0) = \overline{x}_2(0) = \eta(0) = 0$. Hence, if $\lambda > 0$, the $C^\infty[0, \infty)$ trajectory $y_d^\epsilon(t) = \epsilon e^{-1/t^2}$, for any $\epsilon > 0$, induces an unbounded response. Since $y_d^{\epsilon(k)}(0) = 0$ for all $k \geq 0$, it follows from Theorem 2 and Proposition 14 that, if Q were any solution to the asymptotic tracking problem, $P \circ Q$ would have also an unbounded response from the origin when excited by $y_d^\epsilon(t)$, leading to a contradiction. One concludes therefore that, for $\lambda > 0$, there does not exist a solution to the asymptotic tracking problem. Of course, this does not preclude the possibility of tracking a particular trajectory with bounded internal behavior [30], [32]; it just means that no matter how small ϵ is chosen nor how large N is, there always exists a trajectory in the class (8) leading to unbounded internal behavior.

If $\lambda < 0$, then it is easily checked that the compensator (15) of Example 3 yields (global) asymptotic tracking with bounded-input, bounded-state stability. Were one to analyze only the system's Jacobian linearization about the origin

$$\begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& u \\ \dot{\eta} &=& 0 \\ y &=& x_1 \end{array}$$

however, then clearly no conclusion could be drawn on the possibility or impossibility of achieving asymptotic tracking. Whence the interest of the result of Proposition 14, as opposed to just knowing that all of the zeros of the system's linearization must be cancelled in any tracking closed loop.

One would like to conclude from the above that if the asymptotic tracking problem is solvable, then P must be minimum-phase. Without further hypotheses, this seems to be problematic. For instance, consider the nonminimum phase, strongly accessible, asymptotically steerable to the origin, plant

$$\begin{array}{rcl} \dot{x} &=& -\eta + u \\ \dot{\eta} &=& \eta^2 - x^2 \\ y &=& x. \end{array} \tag{20}$$

Its full-order left-inverse at the origin is

$$\begin{array}{rcl} \dot{\overline{x}} &=& \dot{y} \\ \dot{\eta} &=& \eta^2 - y^2 \\ u &=& \eta + \dot{y}. \end{array} \tag{21}$$

All the solutions of (21) with initial conditions $(\overline{x}(0), \eta(0)) = (0, 0)$, and $y = y_d \in C^1[0, \infty)$ and $y_d(0) = 0$, are bounded. Moreover, if y_d has compact support, $\lim_{t\to\infty}(x(t), \eta(t)) = 0$. Neither of these properties is inconsistent with asymptotic stability of (10), for any compensator Q. The point is that starting from the origin, one cannot drive the state of the left-inverse system to the unstable manifold of the zero-dynamics while maintaining exact tracking. This leads to the next result, where the essential change consists in strengthening the hypothesis of strong accessibility from the origin to controllability of the plant's Jacobian linearization; this hypothesis could be replaced by small-time local controllability of the plant if the latter were known to be invariant under the addition of integrators on the input channels.[7]

Theorem 16: Consider a plant P of the form (7) and suppose

a) the Jacobian linearization of P about the origin is controllable;
b) P possesses a zero-dynamics manifold Z^*;
c) P is locally left-invertible at the origin.

Then, if the asymptotic tracking problem is solvable, P is minimum-phase. Moreover, if it is solvable with exponential stability (i.e., (10) is exponentially stable), then P is exponentially minimum-phase.

Proof: See the Appendix. □

A solution to the asymptotic tracking problem is provided in [19] using a class of static state feedbacks. In the special case of SISO plants, their results can be stated as follows. Suppose

H1) P has a well-defined relative degree at the origin; and
H2) P is minimum-phase.

Then, the class of feedback laws

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \left[y_d^{(r)} - L_f^r h(x) + \sum_{i=0}^{r-1} \alpha_i \left(y_d^{(i)} - L_f^i h(x) \right) \right]$$

where r is the relative degree and α_i are real coefficients such that the polynomial $s^r + \alpha_{r-1}s^{r-1} + \cdots + \alpha_0$ is Hurwitz, solves the asymptotic tracking problem. On the other hand, if one supposes that the Jacobian linearization of the plant P is controllable, then the results of Section IV and this section establish the necessity of conditions H1) and H2) for the existence of a dynamic compensator of the form

[7] This problem has been recently studied in [4], [44]; the results are still incomplete.

(9) solving the asymptotic tracking problem. Indeed, from Corollary 14, it follows that H1) must hold. By [25] and [27], one deduces that P therefore possesses a zero-dynamics as well as a well-defined full-order left-inverse at the origin. Hence, by Theorem 21, H2) also holds. This is formalized in the following corollary.

Corollary 17: Consider a SISO plant P of the form (7) and suppose that its Jacobian linearization about the origin is controllable. Then, the asymptotic tracking problem is solvable if and only if H1) and H2) hold. □

The main stumbling block for obtaining an extension of Corollary 17 to MIMO nonlinear systems involves a subtlety of dynamic state variable feedback: if the feedback is constructed by dynamic extension (cf. [7]), it is known that the zero dynamics is preserved [25]; if a more general dynamic feedback is used, then the "fate" of the system's zero dynamics is unknown. Recent results in [36] show that feedbacks more general than those provided by dynamic extension are often needed for dynamic input–output linearization.

If P is a linear, controllable, nonminimum phase plant, then any linear compensator Q of the form (9) which asymptotically stabilizes the closed-loop system can only achieve asymptotic tracking for isolated periodic trajectories. More precisely, $\forall\epsilon > 0$, and $\forall\Omega_L > 0$, $\forall\Omega_U > 0$ such that $\Omega_L < \Omega_U$, there exists a trajectory of the form $y_d^\epsilon(t) = \epsilon\sin(\omega t)$, with $\Omega_L \le \omega \le \Omega_U$, which cannot be asymptotically tracked by $P\circ Q$. The following corollary is a partial extension of the above linear result.

Corollary 18: Consider a SISO plant P of the form (7) and suppose that

1) the Jacobian linearization of P about the origin is controllable;
2) P has well-defined relative degree;
3) P is nonminimum phase.

Suppose furthermore that Q is any compensator of the form (9) such that $P\circ Q$ is asymptotically stable (more precisely, (10) is asymptotically stable). Then $\forall\epsilon > 0$ there exists a periodic trajectory $y_d^\epsilon(t)$ belonging to the class (8) which cannot be asymptotically tracked by $P\circ Q$.

Proof: By Corollary 17, the asymptotic tracking problem is not solvable. Hence, by Theorem 10, the long-term exact tracking problem from the origin is not solvable. Therefore, by Proposition 4, the result follows. □

VI. Conclusions

This paper has established a set of necessary conditions for the solvability of the asymptotic tracking problems. The conditions are easiest to state in the case of SISO systems: Suppose that an analytic plant has a controllable linearization at the origin; if the asymptotic tracking problem is solvable with an analytic compensator, then the plant has a well-defined relative degree at the origin and is minimum phase. Under some additional hypotheses involving invertibility, analogous results were obtained for square MIMO systems.

These necessary conditions are important from a practical viewpoint because they specify precise situations where, to achieve asymptotic tracking, it is necessary to either a) leave the class of analytic compensators, b) restrict the class of trajectories to be tracked, or c) seek an approximate solution.

The open-loop counterpart to the problem treated in this paper is functional reproducibility [36], [37], [20], [41], [42]. A remaining open question is whether a well-defined relative degree is a necessary condition for a system initialized at an equilibrium to be able to reproduce a set of output trajectories defined by $\sup_{t\ge0}\{\|y_d(t)\|,\cdots,\|y_d^{(N)}(t)\|\} < \epsilon$, $y_d(0) = \cdots = y_d^{(N)}(0) = 0$.

Appendix

A. Lemma 19:

Let $c(t) \in C_m^N[0,T]$ and denote the norm on $C_m^N[0,T]$ by $|||c(t)||| := \sup_{0\le t\le T}\{\|c(t)\|,\cdots,\|c^{(N)}(t)\|\}$, where $\|\cdot\|$ is some norm on $\mathbb{R}^m$.

Lemma 19: There exists a constant $K < \infty$ such that for every $0 < T_1 < \infty$ and for each $c(t) \in C_m^N[0,T_1]$ there exists an extension denoted $\bar{c}(t)$ belonging to $C_m^N[0,T_2]$, where $T_2 = T_1 + 1$, satisfying

1) $|||\bar{c}(t)||| \le K\cdot|||c(t)|||$ and
2) $\bar{c}(T_2) = 0,\cdots,\bar{c}^{(N)}(T_2) = 0$.

Proof of Lemma 19: Consider the controllable linear system

$$\Sigma:\quad\begin{aligned}\dot{\xi}_1 &= \xi_2\\ &\vdots\\ \dot{\xi}_N &= \xi_{N+1}\\ \dot{\xi}_{N+1} &= u\\ d &= \xi_1\end{aligned}$$

where each $\xi_i \in \mathbb{R}^m$ and $u(t)$ is continuous. Choosing the initial conditions as $\xi_i(T_1) = c^{(i-1)}(T_1)$, $1\le i\le N+1$ and applying a continuous control that drives the system to the origin at time T_2 guarantees that

$$\bar{c}(t) := \begin{cases} c(t) & 0\le t\le T_1\\ d(t) & T_1 < t \le T_2\end{cases}$$

is an extension of $c(t)$ to $C_m^N[0,T_2]$. It remains to choose $u(t)$ so that an estimate of the norm of $\bar{c}(t)$ can be made. One possible choice is the so-called "minimum energy" control for Σ [3, pp. 401–402]. To define this control, let

$$A = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0\\ \vdots & & & \vdots & \\ 0 & & & & I_m\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix},\qquad B = \begin{bmatrix}0\\ \vdots\\ 0\\ I_m\end{bmatrix}$$

where I_m is the $m\times m$ identity matrix, and let

$$W(t_0,t_1) = \int_{t_0}^{t_1} e^{A(t_0-\tau)}BB'e^{A'(t_0-\tau)}d\tau$$

which is invertible for each $t_1 > t_0$ since Σ is controllable. Let $x = [\xi_1',\cdots,\xi_{N+1}']'$. Then the minimum energy control on the interval $[T_1,T_2]$ is

$$u(t) = -B'e^{A'(T_1-t)}W^{-1}(T_1,T_2)x(T_1).$$

It follows that with this control, $|||\bar{c}(t)||| \leq K|||c(t)|||$ for

$$K := \sup_{\substack{T_1 \leq t \leq T_2 \\ \|x_0\|_{\max} \leq 1}} \|e^{A(t-T_1)}x_0 - \int_{T_1}^{t} e^{A(t-\tau)}BB'e^{A'(T_1-\tau)}d\tau W^{-1}(T_1,T_2)x_0\|_{\max}$$

where $\|x\|_{\max} = \|(\xi_1', \cdots, \xi_{N+1}')'\|_{\max} = \max\{\|\xi_1\|, \cdots, \|\xi_{N+1}\|\}$. The boundedness of K is an easy consequence of the Weierstrass Theorem. □

B. Proof of Theorem 2

The idea of the proof is to show that if a given compensator does not solve the long-term exact tracking problem from the origin, it cannot solve the asymptotic tracking problem. Let Q be any compensator of the form (9) such that (10) is asymptotically stable, and let $N \geq 0$ be the number of derivatives of y_d appearing in (9). If $y_d(t)$ is a given desired output trajectory, let $y(t, x_0, z_0, Y_d(t))$ denote the resulting output of the closed-loop system $P \circ Q$ initialized at $x(0) = x_0, z(0) = z_0$. Also let $\|Y_d(t)\| = \max\left\{\|y_d(t)\|, \cdots, \|y_d^{(N)}(t)\|\right\}$.

Since (10), that is, the closed-loop system $P \circ Q$, is asymptotically stable, by [29, p. 168] and [25, p. 444] the following hold:

P1) There exists $\delta_1 > 0$, such that, $\forall \eta > 0$, there exists $0 < \overline{T} < \infty$ such that if $\|(x_0, z_0)\| < \delta_1$ and $y_d(t) \equiv 0$, then the solution of $P \circ Q$ (i.e., of (10)), initialized at $t_0 = 0$ at the point (x_0, z_0), satisfies $\|(x(t), z(t))\| < \eta$, for all $t \geq \overline{T}$.

P2) $\forall \mu > 0$, there exists $\epsilon_1 > 0$ such that if $\sup_{t \geq 0} \|Y_d(t)\| < \epsilon_1$, then the solution of $P \circ Q$ initialized at (0,0) satisfies $\|(x(t), z(t))\| < \mu$, for all $t \geq 0$.

Suppose now that Q does not solve the long-term exact tracking problem from the origin. The above properties will be used to construct an output trajectory for which the compensator Q does not provide asymptotic tracking.

Let δ_1 be as in P1) and set $\mu = \delta_1$; choose ϵ_1 according to P2). Let $K < \infty$ be as in Lemma 19, and for an arbitrary $\bar{\epsilon} > 0$, define $\epsilon = \min\{\frac{\bar{\epsilon}}{K+1}, \frac{\epsilon_1}{K+1}\}$. Since Q does not provide long-term exact tracking from the origin, for each $\epsilon > 0$, there exists an output trajectory, denoted $y_d^\epsilon(t)$, with $Y_d^\epsilon(0) = 0$ and $\sup_{t \geq 0} \|Y_d^\epsilon(t)\| < \epsilon$, such that $\sup_{t \geq 0} \|y_d^\epsilon(t) - y(t, 0, 0, Y_d^\epsilon(t))\| > 0$. Let $0 < T^\epsilon < \infty$ be such that

$$\gamma^\epsilon := \sup_{0 \leq t \leq T^\epsilon} \|y_d^\epsilon(t) - y(t, 0, 0, Y_d^\epsilon(t))\| > 0.$$

Note that γ^ϵ is necessarily finite.

Lemma 20: Let $y_d^\epsilon(t)$ and γ^ϵ be as above. Then, there exists $\eta^\epsilon > 0$ such that for each (x_0, z_0) satisfying $\|(x_0, z_0)\| < \eta^\epsilon$

$$\sup_{0 \leq t \leq T^\epsilon} \|y_d^\epsilon(t) - y(t, x_0, z_0, Y_d^\epsilon(t))\| \geq \frac{\gamma^\epsilon}{2}.$$

Proof: See next subsection of this Appendix. □

Continuing with the proof of the theorem, the idea is to construct from $y_d^\epsilon(t)$, $0 \leq t \leq T^\epsilon$, an admissible output trajectory that cannot be asymptotically tracked by $P \circ Q$; this is done in two steps. By Lemma 19, there exists an extension to $C_m^N[0, T^\epsilon + 1]$ of $y_d^\epsilon(t)$ restricted to $[0, T^\epsilon]$, denoted $\bar{y}_d^\epsilon(t)$, such that $\bar{y}_d^{\epsilon(i)}(T^\epsilon + 1) = 0$, $0 \leq i \leq N$, and $\sup_{0 \leq t \leq T^\epsilon + 1} \|\overline{Y}_d^\epsilon(t)\| \leq K \sup_{0 \leq t \leq T^\epsilon} \|Y_d^\epsilon(t)\|$. The next step consists in extending $\bar{y}_d^\epsilon$ to $[0, \infty)$ in such a way that the asymptotic stability of the unforced system can be exploited.

In P1), set $\eta = \eta^\epsilon$ of Lemma 20 and choose $\bar{T}$ accordingly. Define $a_0 = 0$, $a_{n+1} = a_n + T^\epsilon + \bar{T} + 1$ and $b_n = a_n + T^\epsilon + 1$ for $n \geq 0$. Set

$$\tilde{y}_d^\epsilon(t) = \begin{cases} \bar{y}_d^\epsilon(t - a_i) & a_i \leq t \leq b_i \\ 0 & b_i \leq t < a_{i+1}. \end{cases}$$

Note that $\tilde{y}_d^\epsilon$ consists of copies of $\bar{y}_d^\epsilon$ shifted in time and separated by intervals of length $\bar{T}$ over which it is zero; note also that $\tilde{y}_d^\epsilon$ is $(T^\epsilon + 1 + \bar{T})$-periodic. By construction, $\tilde{y}_d^\epsilon(t) \in C_m^N[0, \infty)$ and $\sup_{t \geq 0} \|\tilde{Y}_d^\epsilon(t)\| \leq K\epsilon < \min(\bar{\epsilon}, \epsilon_1)$. Thus, by P2), the solution $(x(t), z(t))$ of $P \circ Q$ corresponding to $\tilde{y}_d^\epsilon$ and $(x_0, z_0) = (0, 0)$ satisfies $\|(x(t), z(t))\| < \delta_1$, for all $t \geq 0$. Therefore, for any n, $\|(x(b_n), z(b_n))\| < \delta_1$. Since $\tilde{y}_d^\epsilon(t) = 0$ for $b_n \leq t \leq a_{n+1}$ and $a_{n+1} - b_n \geq \overline{T}$, then $\|(x(a_{n+1}), z(a_{n+1}))\| < \eta^\epsilon$. Hence, by Lemma 20, since $\tilde{y}_d^\epsilon(t) = y_d^\epsilon(t - a_{n+1})$ for $a_{n+1} \leq t \leq a_{n+1} + T^\epsilon$, it follows that

$$\sup_{a_{n+1} \leq t \leq a_{n+1} + T^\epsilon} \|\tilde{y}_d^\epsilon(t) - y(t, 0, 0, \tilde{Y}_d^\epsilon(t))\| \geq \frac{\gamma^\epsilon}{2}.$$

That is, the asymptotic tracking problem is not solved by Q. □

C. Proof of Lemma 20

For $0 \leq t \leq T^\epsilon$, $y(t, x_0, z_0, Y_d^\epsilon(t))$ is a continuous function of (x_0, z_0). Hence, there exists $\eta^\epsilon > 0$ such that whenever $\|(x_0, z_0)\| < \eta^\epsilon$,

$$\sup_{0 \leq t \leq T^\epsilon} \|y(t, 0, 0, Y_d^\epsilon(t)) - y(t, x_0, z_0, Y_d^\epsilon(t))\| \leq \frac{\gamma^\epsilon}{4}.$$

By the triangle inequality

$$\sup_{0 \leq t \leq T^\epsilon} \|y_d^\epsilon(t) - y(t, x_0, z_0, Y_d^\epsilon(t))\| \geq \frac{\gamma^\epsilon}{2}.$$

□

D. Proof of Lemma 5

For a SISO system Σ of the form (7), let n_Σ denote its relative degree and n_Σ^L that of its Jacobian linearization at the origin (if either exists). Σ has a well-defined relative degree if, and only if, $n_\Sigma = n_\Sigma^L$; in general, $n_\Sigma \leq n_\Sigma^L$. Let $\bar{y}$ denote the output of the closed-loop system $P \circ C$. By the chain rule, if either P or C does not have a relative degree, then $P \circ C$ does not have a relative degree; the same is true of their linearizations. Hence, the existence of $n_{P \circ C}$ (respectively, $n_{P \circ C}^L$) implies the existence of n_P and n_C (respectively, n_P^L and n_C^L).

A simple computation gives

$$\bar{y}^{(n_P)} = L_f^{n_P} h + L_g L_f^{n_P - 1} h u$$

where $u = \gamma + \delta v$ and $L_g L_f^{n_P-1} h \not\equiv 0$. If $n_C = 0$, then $\delta \not\equiv 0$ and, clearly, $n_{P \circ C} = n_P + n_C$. If $n_C > 0$, then $\delta \equiv 0$ and $u^{(n_C)} = \tilde{\gamma} + \tilde{\delta} v$ for some $\tilde{\delta} \not\equiv 0$. Therefore, by the chain rule

$$\bar{y}^{(n_P + n_C)} = * + L_g L_f^{n_P - 1} h \tilde{\delta} v.$$

Hence, $n_{P \circ C} = n_P + n_C$. The same is obviously true for the Jacobian linearizations, that is $n_{P \circ C}^L = n_P^L + n_C^L$. Since, by hypothesis, $P \circ C$ has a well-defined relative degree, $n_{P \circ C} = n_{P \circ C}^L$. Therefore, $n_P + n_C = n_P^L + n_C^L$; because $n_P \leq n_P^L$ and $n_C \leq n_C^L$, one concludes $n_P = n_P^L$ and $n_C = n_C^L$. □

E. Proof of Proposition 6

The necessity being obvious, only sufficiency is proved. Let $T > 0$, $\epsilon > 0$ and (F, G, H) be such that $y(t, 0, u) = \int_0^t H e^{F(t-\tau)} G u(\tau) d\tau$ for all $0 \leq t \leq T$ and for all $u \in L^\infty[0, T]$ such that $\|u\|_\infty < \epsilon$. T is assumed to be small enough so that solutions of Σ are unique on $[0, T]$. Let R denote the accessible set from the origin at time $\overline{T} := T/2$; that is $R = \{p \in \mathbb{R}^n | \exists\ u \in L^\infty[0, \overline{T}],\ \|u\|_\infty < \epsilon$ such that $p = x(\overline{T}, 0, u)\}$. By [45], R has a nonempty interior, so let $\mathcal{O}$ be a simply connected nonempty open subset of R. We now claim that there exists a real-valued function $w_0(t, p)$ defined on $[0, \overline{T}] \times \mathcal{O}$ such that, for each $p \in \mathcal{O}$, for each $0 \leq t \leq \overline{T}$ and for $u \in L^\infty[0, \overline{T}]$ such that $\|u\|_\infty < \epsilon$

$$y(t, p, u) = w_0(t, p) + \int_0^t H e^{F(t-\tau)} G u(\tau) d\tau.$$

To show this, let $p \in \mathcal{O}$. Choose a control $u^p(t) \in L^\infty[0, \overline{T}]$ such that $\|u^p\|_\infty < \epsilon$ and $x(\overline{T}, 0, u^p) = p$. Define $\tilde{u}(t)$ on $[-\overline{T}, \overline{T}]$ by

$$\tilde{u}(t) = \begin{cases} u^p(t + \overline{T}) & -\overline{T} \leq t \leq 0 \\ u(t) & 0 \leq t \leq \overline{T} \end{cases}$$

where $u(t) \in L^\infty[0, \overline{T}]$ and $\|u\|_\infty < \epsilon$ Since Σ is time-invariant, for each $-\overline{T} \leq t \leq \overline{T}$, $y(t, 0, \tilde{u}) = \int_{-\overline{T}}^t H e^{F(t-\tau)} G \tilde{u}(\tau) d\tau$. Since the solutions of Σ are unique on $[0, T]$, $y(t, p, u) = y(t, 0, \tilde{u})$ for $0 \leq t \leq \overline{T}$. Thus, for $0 \leq t \leq \overline{T}$

$$\begin{aligned} y(t, p, u) = y(t, 0, \tilde{u}) = & \\ & \int_{-\overline{T}}^0 H e^{F(t-\tau)} G u^p(\tau) d\tau + \\ & \int_0^t H e^{F(t-\tau)} G u(\tau) d\tau. \end{aligned}$$

The claim follows upon defining, for $0 \leq t \leq \overline{T}$

$$w_0(t, p) = \int_{-\overline{T}}^0 H e^{F(t-\tau)} G u^p(\tau)\, d\tau.$$

Now, continuing with the proof of Proposition 6, by [28], for each $k \geq 0$, $L_g L_f^k h(x)$ is a constant function on $\mathcal{O}$. Since Σ is analytic, it follows that $L_g L_f^k h(x)$ is constant everywhere, in particular on an open neighborhood of the origin. □

It is interesting to note that $w_0(t, p)$ in the proof of the previous proposition is independent of the particular choice of control driving the system from the origin at time zero to the point p at time $\overline{T} = T/2$. This is because

$$w_0(t, p) = y(t, p, u) - \int_0^t H e^{F(t-\tau)} G u(\tau) d\tau$$

for all $0 \leq t \leq \overline{T}$ and all $u \in L^\infty[0, \overline{T}]$, $\|u\|_\infty < \epsilon$.

F. Proof of Lemma 7

Let n_1 be the relative degree from v to y of $P \circ C|_{\mathcal{M}}$, n_2 the relative degree from v to u of $P \circ C|_{\mathcal{M}}$. Let n_3 be the smallest integer $k \geq 1$ such that

$$L_g L_f^{k-1} h(x_0) \neq 0 \text{ for some } (x_0, z_0) \in \mathcal{M}.$$

Following the proof of Lemma 5, the chain rule yields $n_1 = n_2 + n_3$, replacing $n_{P \circ C} = n_P + n_C$. The rest of the proof is the same. □

G. Proof of Lemma 8

Let $\mathcal{R}^*$ be the strong accessibility distribution of the closed-loop system $P \circ C$

$$P \circ C : \begin{array}{rcl} \dot{x} & = & f(x) + g(x) u \\ \dot{\xi} & = & c(x, \xi) + d(x, \xi) v \\ u & = & \gamma(x, \xi) + \delta(x, \xi) v \\ y & = & h(x). \end{array} \tag{22}$$

The goal of the first part of the proof is to show that, in an open neighborhood of the origin of (22)

$$\pi_*(\mathcal{R}^*) = T\mathbb{R}^n \tag{23}$$

where $\pi : \mathbb{R}^n \times \mathbb{R}^\nu \to \mathbb{R}^n$ is the canonical projection.

If the relative degree $\overline{n}$ of the compensator is equal to zero, then $\delta(0, 0) \neq 0$ because the relative degree at the origin is well defined. Then, applying the invertible static state feedback $v = [-\gamma + w]/\delta$ to $P \circ C$ results in

$$\begin{aligned} \dot{x} &= f(x) + g(x) w \\ \dot{\xi} &= * \end{aligned}$$

from which it is clear that (23) holds.

On the other hand, if the relative degree of C is greater than zero, let

$$\tilde{f} = \begin{pmatrix} f + g\gamma \\ c \end{pmatrix}, \tilde{g} = \begin{pmatrix} 0 \\ d \end{pmatrix}$$

and compute

$$\begin{aligned} u^{(\overline{n})} &= L_{\tilde{f}}^{\overline{n}} \gamma + \left(L_{\tilde{g}} L_{\tilde{f}}^{\overline{n}-1} \gamma \right) v \\ &= \gamma_1(x, \xi) + \delta_1(x, \xi) v. \end{aligned}$$

Since C has a well-defined relative degree, $\delta_1(0, 0) \neq 0$. By standard arguments [25], the differentials $\left\{ dx, d\gamma_1|_{(0,0)}, \cdots, dL_{\tilde{f}}^{\overline{n}-1} \gamma_1|_{(0,0)} \right\}$ are linearly independent, for, otherwise, the relative degree of the linearization of C would not be defined. Hence, without loss of generality, it can be supposed that the coordinates for the compensator have been chosen such that $(x, \xi) = (x, u, \dot{u}, \cdots, u^{(\overline{n}-1)}, \overline{\xi})$, for some $\overline{\xi}$. Applying the invertible static state feedback $v = [-\gamma_1 + w]/\delta_1$ yields

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ u^{(\overline{n})} &= w \\ \dot{\overline{\xi}} &= * \end{aligned}$$

from which it follows that (23) holds, because the strong accessibility from the origin of P implies the strong accessibility of the subsystem

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ u^{(\overline{n})} &= w. \end{aligned}$$

The result of the lemma is now established. Since $\mathcal{R}^*$ is an analytic and involutive distribution, by Nagano's theorem [33] it is integrable. Let $\mathcal{L}$ be the leaf of $\mathcal{R}^*$ containing the origin and let $\mu \geq n$ be its dimension. By (23), there exists a cubic coordinate chart (ψ, U) such that

$$\mathcal{L} \cap U = \left\{ (\psi_1, \psi_2, \cdots, \psi_{n+\nu}) | \|\psi_i\| < \epsilon, \right. \\ \left. i = 1, \cdots, \mu; \psi_i = 0, i = \mu + 1, \cdots, n + \nu \right\}$$

and $\psi_i = x_i, i = 1, \cdots, n$. Consider the restriction of the system $P \circ C$ to $\mathcal{L}$, denoted $P \circ C|_{\mathcal{L}}$. By [43], $P \circ C|_{\mathcal{L}}$ is strongly accessible from the origin. Moreover, $P \circ C|_{\mathcal{L}}$ in the coordinates $\psi_1, \cdots, \psi_\mu$ takes the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ \dot{\tilde{\xi}} &= \tilde{c}(x, \tilde{\xi}) + \tilde{d}(x, \tilde{\xi})v \\ u &= \tilde{\gamma}(x, \tilde{\xi}) + \tilde{\delta}(x, \tilde{\xi})v \\ y &= h(x). \end{aligned}$$

Let $\tilde{C}$ be the compensator

$$\tilde{C} : \begin{array}{rcl} \dot{\tilde{\xi}} &=& \tilde{c}(x, \tilde{\xi}) + \tilde{d}(x, \tilde{\xi})v \\ u &=& \tilde{\gamma}(x, \tilde{\xi}) + \tilde{\delta}(x, \tilde{\xi})v. \end{array}$$

Then $P \circ \tilde{C} = P \circ C|_{\mathcal{L}}$; hence $P \circ \tilde{C}$ has the same input–output behavior from the origin as $P \circ C$, and $P \circ \tilde{C}$ is strongly accessible from the origin. □

H. Proof of Theorem 16

Let Q be a solution to the asymptotic tracking problem corresponding to the class of trajectories (8) for some N and ϵ; it can be supposed, without loss of generality, that $N \geq n - 1$ and that ϵ is sufficiently small so that local left-invertibility holds for the class (8). Define $y^{(0)}(x) = h(x)$ and, recursively

$$y^{(k+1)}(x, u, \cdots, u^{(k)}) = \frac{\partial y^{(k)}}{\partial x} \cdot [f + gu] + \sum_{i=0}^{k-1} \frac{\partial y^{(k)}}{\partial u^{(i)}} u^{(i+1)}.$$

By assumption b) and the fact that the asymptotic tracking problem is solvable, P possesses a zero-dynamics (Z^*, f^*). For each $k \geq 0$, define $\bar{u}^{(k)}(\zeta) = L_{f^*}^k \alpha^*(\zeta)$, for all $\zeta \in Z^*$ and set $\phi_0(\zeta) = \zeta$ and $\phi_k(\zeta) = (\zeta, \overline{u}^{(0)}(\zeta), \cdots, \overline{u}^{(k-1)}(\zeta))$. Since Z^* is invariant under f^* and is contained in $h^{-1}(0), y^{(k)}(\phi_k(\zeta)) = 0$ for each $\zeta \in Z^*$ and $k \geq 0$. Since $\zeta = 0$ is an equilibrium of $f^*, \phi_k(0) = 0$.

The idea of the proof is to construct a trajectory y_d in the class (8) which is exactly trackable from the origin and drives the state of the closed-loop system to a desired point ζ_0 of the zero-dynamics manifold of P in some finite time T, after which the trajectory is identically zero. This will impose that a subset of the closed-loop system's state components evolves according to the zero-dynamics initialized at ζ_0. From this, it will follow that the zero-dynamics of P must have the same stability properties as $P \circ Q$ with $y_d \equiv 0$.

Choose $T_1 > 0$ such that all solutions of $P \circ Q$ initialized at the origin remain within some estimate of the region of attraction of (10) for $0 \leq t \leq T_1$. Consider the extended system on $\mathbb{R}^{n+(N+1)m}$

$$\Sigma^e \quad \begin{array}{rcl} \dot{x} &=& f(x) + g(x)u \\ \dot{u} &=& u^1 \\ \vdots && \\ \dot{u}^N &=& v \\ y &=& h(x) \end{array} \tag{24}$$

having state $\xi = (x, u, u^1, \cdots, u^N)$ with $\|v(t)\| < \epsilon$ for each $t \geq 0$. Choose $T_2 > 0$ such that all solutions of Σ^e initialized at the origin give rise to output trajectories satisfying $\|y^{(k)}(t)\| < \epsilon, 0 \leq t \leq T_2, 0 \leq k \leq N$. Set $T = \min\{T_1, T_2\}$.

The system Σ^e has a controllable Jacobian linearization about the origin, since P does. Consequently, by Proposition 3.3 in [35, p. 74], there exists an open neighborhood $\mathcal{O}$ of the origin of the state-space of Σ^e such that $\forall p \in \mathcal{O}, \exists v(t)$, with $\|v(t)\| < \epsilon$, yielding $\xi(T, 0, v) = p$. Choose an open neighborhood U of the origin in Z^* such that $\forall \zeta \in U$, $\phi_{N+1}(\zeta) \in \mathcal{O}$. Let $\zeta_0 \in U$, define $p = \phi_{N+1}(\zeta_0)$ and let $v^p(t)$ be the corresponding control driving ξ from the origin to p at time T. By construction, the resulting output trajectory $y^p(t) \in C_m^N[0, T]$ and satisfies $Y^p(0) = Y^p(T) = 0$. Extend $y^p(t)$ to $[0, \infty)$ by defining

$$y_d^p(t) = \begin{cases} y^p(t) & 0 \leq t \leq T \\ 0 & t > T. \end{cases}$$

The trajectory $y_d^p(t)$ belongs to the class (8), is exactly trackable from the origin, and is identically zero for $t \geq T$. Let $y_d^p(t)$ be the input to $P \circ Q$ initialized at the origin. Since P is locally left-invertible at the origin, the control u generated by the compensator Q has to be $v^p(t)$ for $0 \leq t \leq T$. Therefore, in the closed-loop $P \circ Q$, $x(T) = \zeta_0$. Since P has a zero-dynamics and $y_d^p(t) = 0$ for $t \geq T$, the control generated by the compensator Q has to be $\alpha^*(\zeta(t))$ for $t \geq T$, where $\zeta(t)$

is the solution of the zero-dynamics initialized at ζ_0. Thus, for $t \geq T$, $\zeta(t) = x(t)$. Since $P \circ Q$ is evolving according to (10) for $t \geq T$, the result follows. □

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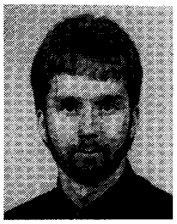

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