University of Lausanne
Faculty of business and economics
MSc. in Finance - Asset and Risk Management

Homework 4

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Course: *Empirical Method in Finance* – Professor: *Eric Jondeau*
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1. Introduction

The objective of this assignment is to highlight the so-called "volatility timing", i.e. the additional return an investor can expect when she is able to forecast the dynamics of expected returns and volatility correctly. The dataset includes the total return index of the S&P500 index for US stocks and the JP Morgan index for US government bonds, and the one week interest rate for cash, from January 2001 to December 2018 at weekly frequency. The one-week interest rate is annualized and expressed in %. Compute returns for stocks and bonds (and) and compute the weekly returns for the risk-free asset ( ). Be careful, do not multiply returns by 100.

2. Static Asset Allocation

2.a. Optimal Static allocation. The mean-variance criterion,

$$\max_{\alpha} \mu_p - \frac{\lambda}{2}\sigma_p^2 = \max_{\alpha} \alpha'\mu + (1 - e'\alpha)R_f - \frac{\lambda}{2}\alpha'\Sigma\alpha \tag{1}$$

Using First Order Condition (FOC) with respect to α:

$$\frac{\partial \max(.)}{\partial \alpha} = 0 \Leftrightarrow \alpha = \frac{1}{\lambda}\Sigma^{-1}(\mu' - e'R_f) \tag{2}$$

2.b. Computing the optimal allocation.

	S&P 500 Comp	JP US Bond	Risk Free
Lambda = 2	1.7946	8.4718	-9.2665
Lambda = 10	0.35893	1.6944	-1.0533

Table 1: Optimal Static Allocation for different risk aversion coefficient.

We can notice that the composition of the tangency portfolio keeps unchanged and thus the proportion of stocks and bonds among risky assets is independent of the risk aversion coefficient, we only allocate differently between the risk free and the tangency portfolio. Moreover, in both case weights allocated to risky assets are superior to one meaning that positions on risky assets are levered and this also explains the negative allocations to risk-free assets. Obviously the lever is bigger (risks are bigger) in the case of a small risk aversion coefficient.

3. Estimation of GARCH Model

3.a. Testing Non-Normality and auto-correlation of the excess returns.

Since we do not know the mean and standard error of the process, we will use a Lilliefors test and not a Kolmogorov-Smirnov one. The Lilliefors test takes the biggest difference between the theoretical and the empirical cumulative distribution function.

	Decision	P-Value	K-Stat	Critical Value
S&P 500 Comp	1	0.001	0.079279	0.029501
JP US Bond	1	0.010649	0.034181	0.029501

Table 2: Lilliefors test on S&P 500 and JP US. Bond

Thus, we can reject the null hypothesis of normality of returns at a level of significance of 5%. Therefore, the cumulative distribution function of returns for both index cannot be associated to a normal distribution functions.

We have performed a Ljungbox test allowing us to test statically if the time series are auto-correlated. The test is computed as follow :

$$\mathcal{Q} = n\,(n+2)\sum_{k=1}^{h}\frac{\hat{\rho}_k^2}{n-k} \tag{3}$$

Where : $k \in [1, ..., 4]$ is the number of lag.

This yield the following results:

	T-stat	Critical Value	P-Value
S&P 500 Comp	11.9435	9.4877	0.017777
JP US Bond	8.4638	9.4877	0.075991

Table 3: Ljungbox test on S&P 500 and JP US. Bond

A Ljungbox test aim to test the null hypothesis that serial auto correlations are equal to 0. And evidence show that we could reject H_0 for the equity index at a level of significance of 5%. However, for the bonds index we would only reject it at a significance level of 7%.

3.b. AR(1) estimation.

The auto-regressive process AR(1) assumes that the first lag operator of the process r_{t-1} is useful to forecast r_t, therefore we can write the AR(1) process as follow,

$$r_t = \Phi_0 + \Phi_1 r_{t-1} + \hat{\epsilon}_t \tag{4}$$

Where $\hat{\epsilon}_t$ are the AR(1) residuals that we will use later on.

	Intercept	tStat	Rho	tStat_{Rho}	Rsquared
S&P 500 Comp	0.0014256	1.8381	-0.062944	-1.9275	0.0039578
JP US Bond	0.00085391	4.049	-0.061629	-1.8935	0.0038198

Table 4: AR(1) estimation on S&P 500 and JP US. Bond

As we can see, expect for the Intercept of the bonds, the estimation are not significant at a level of 5%, however they are quiet close. Moreover, one must note that the predictive power of the model is very low with both Rsquared being smaller than 0.4%. This will probably be an issue when performing the dynamic Allocation.

3.c. ARCH - LM test on AR(1) residuals.

The LM test proposed by Engle (1982) will allow us to test for the (non) normality of the residuals of the AR(1) model.

The LM test statistic is equivalent to the $T \times R^2$ test statistic, where T is the sample size and R^2 is computed from the following regression:

$$\hat{\epsilon}_t^2 = a_0 + a_1\hat{\epsilon}_{t-1}^2 + ... + a_p\hat{\epsilon}_{t-p}^2 + \nu_t \tag{5}$$

Under H_0 the T^2 is asymptotically distributed as a $\chi^2(p)$.

The LM test of ARCH(4) effect, which is asked here is based on the null hypothesis $H_0 : \alpha_1 = ... = \alpha 4 = 0$ against the alternative $H_\alpha : \alpha_0 \geq 0, ..., \alpha_4 \geq 0$ with at least one strict equality.

Alternatively, we also used the Ljung-Box statistic for $\hat{\epsilon}_t^2$ with 4 lags, which is distributed as a $\chi^2(4)$.

	LM Stat	P-Value	Q Stat	PValue
S&P 500 Comp	116.2713	0	179.3059	0
JP US Bond	40.4686	3.4625e-08	47.4103	1.2525e-09

Table 5: LM (Engel) procedure on S&P 500 and JP US. Bond

The P-Value is approximately 0 for both index and using the two different methods. Therefore we can easily reject H_0 and conclude that the residuals of the AR(1) model follow an ARCH(p) model. In other words, residuals are heteroskedastic.

3.d. GARCH estimation.

Therefore, we have to run a GARCH(1,1) estimation on the stocks and the bonds index using the conditional ML technique.

The GARCH(1,1) model:

$$\epsilon_{i,t} = \sigma_{i,t} z_{i,t} \tag{6}$$

And,

$$\sigma_{i,t}^2 = \omega_i + \alpha\epsilon_{i,t-1}^2 + \beta\sigma_{i,t-1}^2 \tag{7}$$

Where $i = s, b$

Then, we have used the function provided in Kevin Sheppard's toolbox. It leads to the following estimations for parameters for ω, α and β:

	Stock	Stock tStat	Bond	Bond tStat
Omega	2.6113e-05	3.4088	5.6838e-07	1.9087
Alpha	0.20637	6.0048	0.052358	4.1351
Beta	0.75697	20.8959	0.93354	60.8117

Table 6: GARCH(1,1) estimation on S&P 500 and JP US Bond

First of all, we notice that constraints on parameters are respected as $\omega \geq 0$, $\alpha \geq 0$ and $\beta \geq 0$. One can see that ω is approximately 0 in both GARCH(1,1) estimations for stocks and bonds index. Moreover $\gamma = \alpha + \beta < 1$, also in both cases. We can interpret γ as the persistence parameter and because it is inferior to one but also because $\omega = 0$ the volatility does not explode but converge almost surely to it's unconditional level. However, we can see that the estimates of γ are very close to one and therefore, there is a strong persistence. This will be clearly shown when forecasting the volatility. On the other side, we see that the α for the SP 500 is higher than the one for the JP US Bond, this shows that the impact of innovations is higher on the stock market. Intuitively, we would also have said that innovation would have a bigger impact on stocks index, because of the larger premium and the higher volatility induced by stocks.

3.e. Volatility Forecasting.

Recursively, we found the k-step ahead forecast for σ_{t+k}^2 is:

$$\sigma_{i,t}^2(k) = \hat{\omega}_i + (\hat{\alpha}_i + \hat{\beta}_i)\sigma_{i,t}^2(k-1) \tag{8}$$

For $k = 1, ..., 52$, and for $i = s, b$.

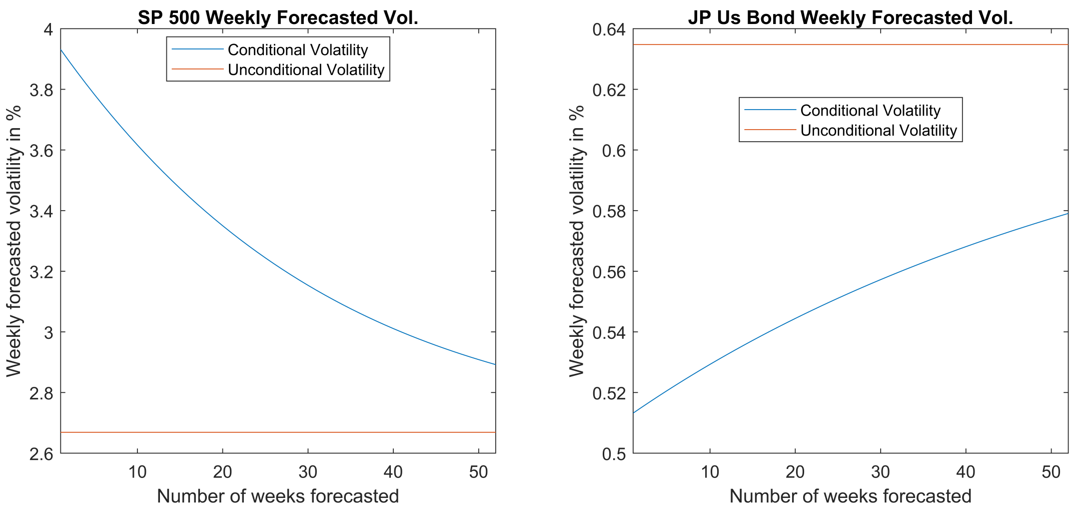

Figure 1: 52 weeks forecasted volatility. We can see that the Conditional Volatility is getting closer to the Unconditonal one. If we had forecasted for more periods, it would have converge.

We can clearly see that the Conditional Volatility is converging towards the Unconditional one for both Stocks and Bonds. Indeed, the more we forecast, the less the information already known will have an impact. However, it takes quiet some time due to the persistence implied by the model estimates as said before.

4. Dynamic Asset Allocation

4.a. Volatility Estimation.

Under stationarity, the first moment of an AR(1) process is:

$$\mu_{i,t} = \frac{\Phi_{i,0}}{1 - \Phi_{i,1}} \tag{9}$$

Using the estimated AR(1) process and the estimated parameters in 3b., we found respectively for $i = s, b$:

$$\mu_{s,t} = \frac{0.0014256}{1 + 0.062944} = 0.0013412 \tag{10}$$

And,

$$\mu_{b,t} = \frac{0.00085391}{1 + 0.061629} = 0.000804339 \tag{11}$$

The volatility is estimate using the GARCH(1,1) results as follow :

$$\Sigma_t = \begin{vmatrix} \sigma^2_{s,t} & \sigma_{sb,t} \\ \sigma_{sb,t} & \sigma^2_{b,t} \end{vmatrix} \tag{12}$$

Where $\sigma_{sb,t} = \rho_{sb} * \sigma_{b,t} * \sigma_{s,t}$

Therefore, we have a dynamic volatility with the covariance matrix being of size 2 X 2 X N.

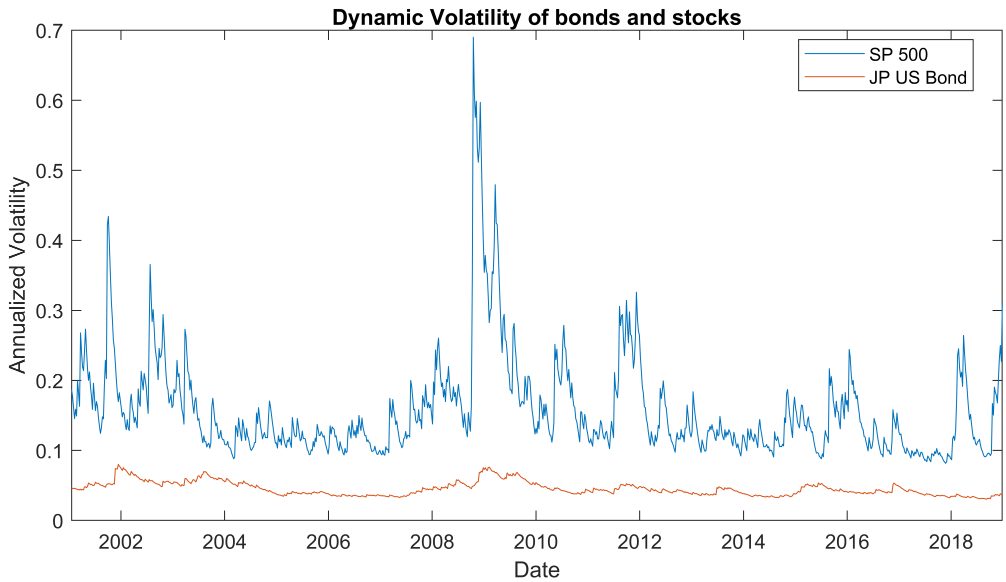

Figure 2: Volatility Dynamic using a GARCH(1,1) model.

On this figure, we can clearly see the crisis with high peak in volatility. This is also relatively strong for the bonds, however, the scale of the graph does not clearly shows it. One can see that there seem to be period where the volatility on the stocks is high and there is almost no effects on the bond market (e.g. European debt crisis).

4. b. and c. Dynamic allocation.

Recalling the expression for the optimal portfolio weight derived from 2a., we have:

$$\tilde{\alpha}_t = \frac{1}{\lambda}\Sigma_t^{-1}(\mu'_t - e'R_{f,t}) \tag{13}$$

Where,

$$\mu_{p,t} = \alpha_t\mu_t + (1 - e'\alpha_t)R_f \tag{14}$$

And,

$$\Sigma_t^{-1} = \frac{1}{\sigma^2_{s,t}\sigma^2_{b,t} - \sigma_{sb,t}\sigma_{sb,t}} \begin{vmatrix} \sigma^2_{b,t} & -\sigma_{sb,t} \\ -\sigma_{sb,t} & \sigma^2_{s,t} \end{vmatrix} \tag{15}$$

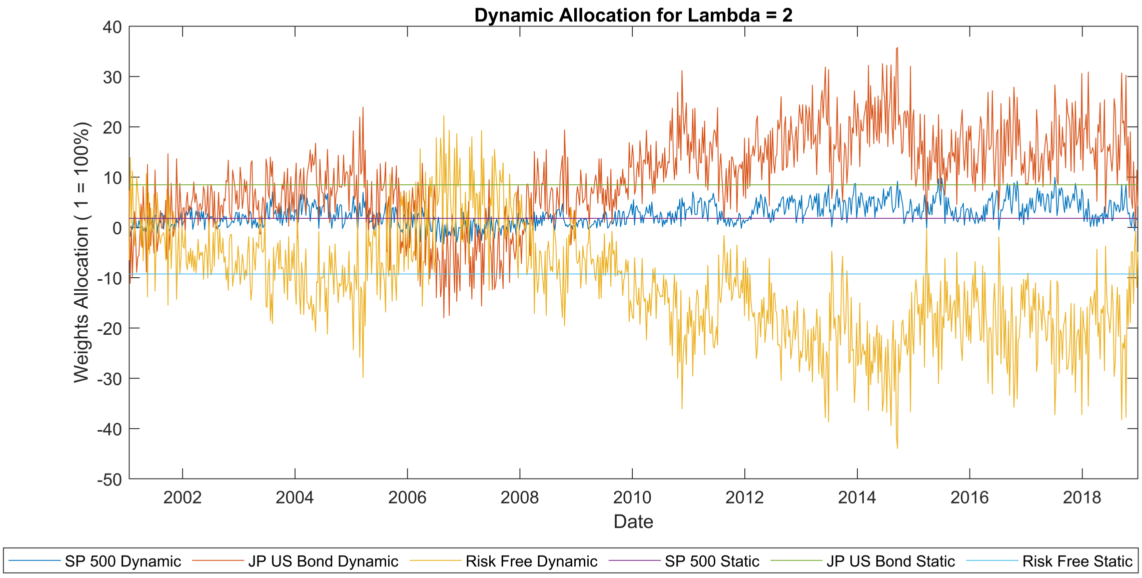

Figure 3: Dynamic Weights allocation for a risk aversion of 2.

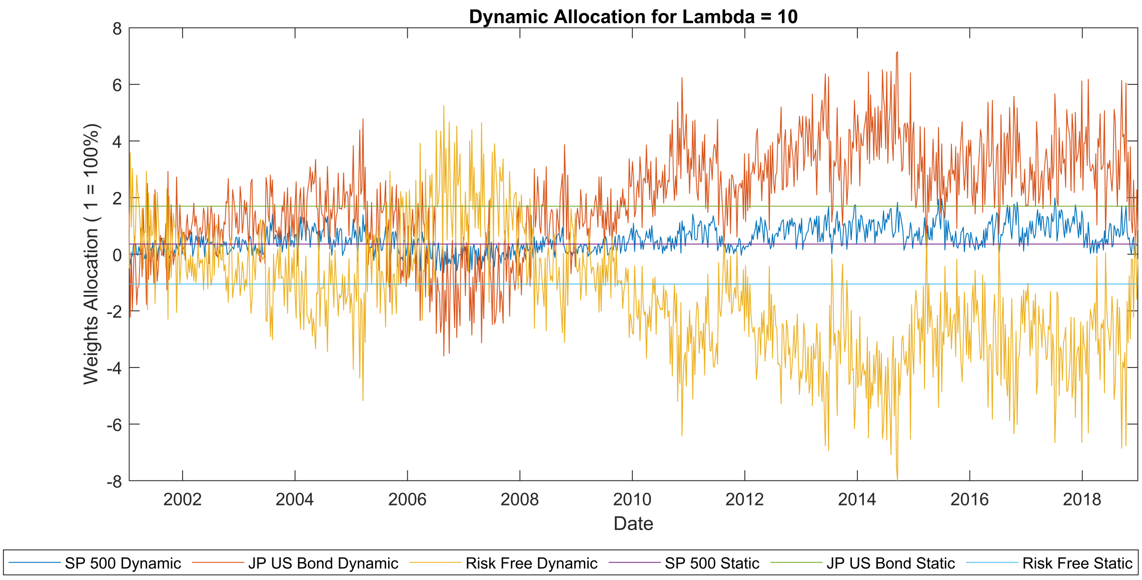

Figure 4: Dynamic Weights allocation for a risk aversion of 10.

We can clearly see that the weights are extremely noisy. Indeed, the lack of prediction of the AR(1) model render the estimation not stable.

4.d. Performance of the Portfolio.

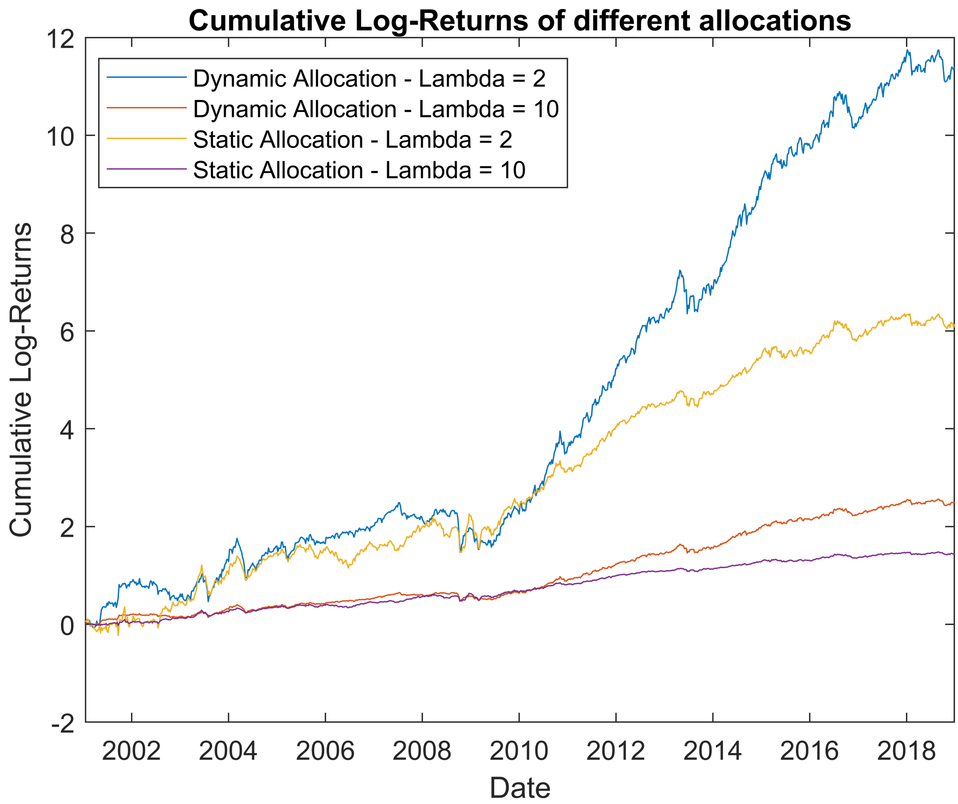

Figure 5: Cumulative returns of static and dynamic allocation for different level of risk aversion.

Overall, dynamic allocations tends to perform better than static allocations. But we can also notice negative returns for dynamic allocations during high-volatility periods and crises. After 2009, there is no doubt that dynamic allocations lead to far better performances. As this period could be characterized by a lower volatility and a increasing trending market (on both indexes, stocks and bonds). Combined, this allows the dynamic optimization to use larger leverage effects and to profit from better expected returns. Indeed, we notice that negative weight allocated to risk-free assets can double or more between ex-2009 and post-2009. Moreover, different risk aversions profile will lead to different leverage effects, for instance at the same period (in 2014), a low risk-aversion profile will reach an allocation of -4000% in the risk-free asset while a high risk-aversion will "only" reach -800%. In this case, the lower risk-aversion profile lead to superior performances by taking more risks while levering his position at dramatic levels.

	Dynamic Lambda = 2	Dynamic Lambda = 10	Static Lambda = 2	Static Lambda = 10
Annualized Mean	0.63257	0.13853	0.34087	0.08019
Annualized Volatility	0.58222	0.11633	0.39949	0.07983
Sharpe Ratio	1.0865	1.1908	0.85328	1.0045
Skewness	-0.45701	-0.47683	-0.71613	-0.71902
Kurtosis	3.3801	3.4203	5.1914	5.2142

Table 7: Statistics of static and dynamic allocation for different level of risk aversion.

As we can see, looking at the risk-reward ratio, i.e the Sharpe ratio, the dynamic allocation is performing better than the static ones. Surprisingly, in both scenario, the more averse allocation is performing better on a risk-reward basis. Even-though, the absolute performance is way higher for the more risky allocation.

As we saw earlier, the very noisy allocation of the weights might imply relatively high fees in financial markets, therefore this might change our conclusion. We will look at this point now.

4.e. Introduction of Fees.

The fees were compute as follow for the dynamic allocation (the static allocation does not have fees in our model, however, this is a small hypothesis) :

$$TC_t = [\mid \alpha_{s,t} - \alpha_{s,t-1} \mid + \mid \alpha_{b,t} - \alpha_{b,t-1} \mid] * \tau \tag{16}$$

The process we have used to estimate whether the dynamic allocation would still perform well with fees is to find τ such that the cumulative returns of the strategies is equal (as in Figure 6.). This "optimal" fee is 0.0729%. Therefore, we can see that this a very small fee and therefore, with fees that are bigger than this, the dynamic allocation would perform less (in absolute return). Thus, due to the very noisy estimation of the process, an dynamic allocation will not necessarily be interesting to use.

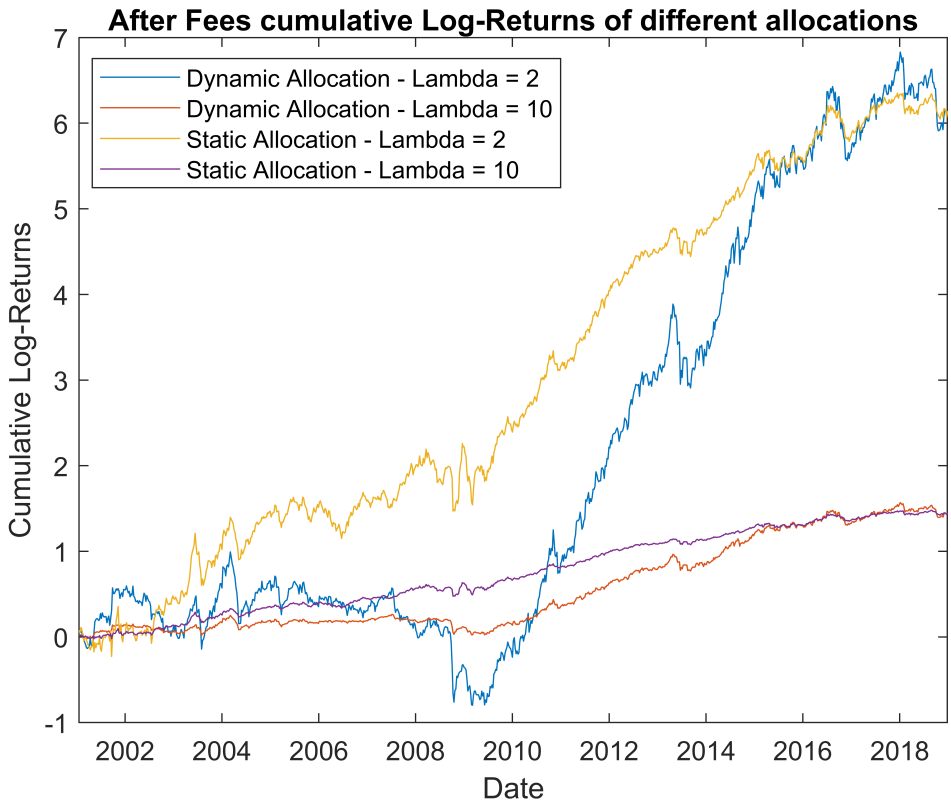

Figure 6: After Fees cumulative returns of static and dynamic allocation for different level of risk aversion