To find the probability p_i of token i (not in top-k tokens), add bias λ to the token's logit ℓ_i and obtain the biased logprob

$$\log p'_i = \log \frac{\exp(\ell_i + \lambda)}{\exp(\ell_i + \lambda) + \sum_{j \neq i} \exp \ell_j}. \tag{1}$$

Rearranging this, we obtain

$$\begin{aligned}
p'_i &= \frac{\exp(\ell_i + \lambda)}{\exp(\ell_i + \lambda) + \sum_{j \neq i} \exp \ell_j} && (2)\\
p'_i \left(\exp(\ell_i + \lambda) + \sum_{j \neq i} \exp \ell_j \right) &= \exp(\ell_i + \lambda) && (3)\\
p'_i \exp(\ell_i + \lambda) + p'_i \sum_{j \neq i} \exp \ell_j &= \exp(\ell_i + \lambda) && (4)\\
\exp(\ell_i + \lambda) - p'_i \exp(\ell_i + \lambda) &= p'_i \sum_{j \neq i} \exp \ell_j && (5)\\
(1 - p'_i) \exp(\ell_i + \lambda) &= p'_i \sum_{j \neq i} \exp \ell_j && (6)\\
\exp \lambda \exp \ell_i \frac{1 - p'_i}{p'_i} &= \sum_{j \neq i} \exp \ell_j . && (7)
\end{aligned}$$

Note that the righthand side does not depend on the bias term, therefore we can equate the lefthand side to the the case where $\lambda = 0$ (i.e., the unbiased case) and solve for the probability p_i

$$\begin{aligned}
\exp \lambda \exp \ell_i \frac{1 - p'_i}{p'_i} &= \exp 0 \exp \ell_i \frac{1 - p_i}{p_i} && (8)\\
\exp \lambda \frac{1 - p'_i}{p'_i} &= \frac{1 - p_i}{p_i} && (9)\\
p_i &= \frac{1}{\exp \lambda \frac{1 - p'_i}{p'_i} + 1}. && (10)
\end{aligned}$$

Thus, it is possible to obtain unbiased logprobs for any token with exactly 1 API call.