# 2301108 CALCULUS II

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## Contents

# 1 Differential Equation

## 1.1 Seperable Equation

**Definition 1.1.1** (Seperable Equation)**.** A differential equation is called seperable if it can be written in the form

$$\frac{dy}{dx} = g(y)h(x)$$

where $g(y)$ is a function of $y$ only and $h(x)$ is a function of $x$ only.

$$\frac{dy}{dx} = g(y)h(x)$$
$$\frac{dy}{g(y)} = h(x)dx$$
$$\int \frac{dy}{g(y)} = \int h(x)dx$$

**Definition 1.1.2** (Implicit Solution)**.** An implicit solution to a differential equation is a solution that is not solved for $y$ explicitly. Answer is in the form of $F(x, y) = c$.

**Definition 1.1.3** (Explicit Solution)**.** An explicit solution to a differential equation is a solution that is solved for $y$ explicitly. Answer is in the form of $y = f(x)$.

Sometimes there are initial condition of the equation.

**Theorem 1.1.1** (FTC 1)**.** *If $f$ is continuous on $[a, b]$, then the function $f$ defined by*

$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = f(u(x))u'(x)$$

*is continuous on $[a, b]$ and differentiable in (a, b), and g'(x) = f(x).*

**Definition 1.1.4** (Integral Equation)**.** An integral equation is an equation in which an unknown function appears under an integral sign.

We have to change the integral equation to a differential equation. Then, we can use FTC 1 to solve the differential equation.

## 1.2 Linear Equation

**Definition 1.2.1** (Linear Equation)**.** A differential equation is called linear if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions of $x$.

The term “linear” refers to the fact that the unknown function $y$ and its derivative $\frac{dy}{dx}$ appear in the equation to the first power and are not multiplied together.

**Definition 1.2.2** (Integrating Factor)**.** The integrating factor for the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is the function $I(x)$ defined by

$$I(x) = \exp(\int P(x)dx)$$

We can solve the linear differential equation by the formula.

$$y(x) = \frac{1}{I(x)}[\int I(x)Q(x)dx + C]$$

without any additional constraint $c$ from integrating.

## 1.3 Bernoulli Equation

**Definition 1.3.1** (Bernoulli Equation)**.** A differential equation is called Bernoulli if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where $P(x)$ and $Q(x)$ are continuous functions of $x$.

As you can see, there is $y^n$ in the equation which is not linear.

We have to change the Bernoulli equation to a linear equation by substitution.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

divides both side by $y^n$

$$\frac{dy}{dx}y^{-n} + P(x)y^{1-n} = Q(x)$$

Then we substitute $u = y^{1-n}$

$$\begin{aligned}\frac{du}{dy} &= (1-n)y^{-n}\frac{dy}{dx} \\ \frac{du}{dx}\cdot\frac{1}{1-n} &= y^{-n}\frac{dy}{dx}\end{aligned}$$

Let $u = y^{1-n}$

$$\begin{aligned}\frac{du}{dx} &= (1-n)y^{-n}\frac{dy}{dx} \\ \frac{du}{dx}\cdot\frac{1}{1-n} &= y^{-n}\frac{dy}{dx}\end{aligned}$$

Then, try to substitute the equation with $u$.

$$\begin{aligned}&\frac{du}{dx}\cdot\frac{1}{1-n} + P(x)u = Q(x) \\ &\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)\end{aligned}$$

Now, we can solve the equation with the linear equation method. Since the equation is linear on $u$.

# 2 Sequence

## 2.1 Sequence

**Definition 2.1.1** (Sequence)**.** A sequence is a function whose domain is $\mathbb{N}$

We are considering behavior of a sequence as $n$ becomes infinite.

$$\lim_{n\to\infty} a_n = \begin{cases} L \in \mathbb{R} & \text{convergent} \\ \pm\infty & \text{divergent} \\ \text{morethanonevalue} & \text{divergent} \end{cases}$$

**Definition 2.1.2** (Sequence Notation)**.** The sequece $\{a_1, a_2, a_3, \dots\}$ is denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

**Theorem 2.1.1** (Limit of Sequence)**.** *A sequence $\{a_n\}$ converges to $L$.*

$$\lim_{n\to\infty} a_n = L$$

*or*

$$a_n \to L \text{ as } n \to \infty$$

*if for every $\epsilon > 0$, there exists $N$ such that*

$$\text{if } n > N$$

*then*

$$|a_n - L| < \epsilon$$

**Theorem 2.1.2.** *If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for all $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} a_n = L$$

Since our function is discrete, we can use the limit of the function to find the limit of the sequence.

*Remark.* We can use L'Hopital's rule to find the limit of the sequence.

If subsequences of $a_n$ converge to different limits, then the sequence $a_n$ diverges.

$$a_n = (-1)^n$$
$$a_n = 1, -1, 1, -1, \dots$$

Consider: $a_{2k-1} = (-1)^{2k-1}; k \in \mathbb{N}$

$$a_{2k-1} = (-1)^{2k-1}$$
$$a_{2k-1} = -1$$

Consider: $a_{2k} = (-1)^{2k}; k \in \mathbb{N}$

$$a_{2k} = (-1)^{2k}$$
$$a_{2k} = 1$$

Since $\lim_{k\to\infty} a_{2k-1} = -1$ and $\lim_{k\to\infty} a_{2k} = 1$, the sequence $a_n$ diverges.

**Theorem 2.1.3** (Squeeze theorem)**.** *Let $a_n \le b_n \le c_n$ for $n \ge n_0$.*
*If*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty c_n = L}$$

*then*

$$\lim_{n\to\infty} b_n = L$$

**Theorem 2.1.4.** *If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$*

*Proof.* From $-|a_n| \le a_n \le |a_n|$, we have

$$\lim_{n\to\infty} |a_n| = 0$$

by the squeeze theorem, we have $\lim_{n\to\infty} a_n = 0$ □

**Theorem 2.1.5.** *If $\lim_{n\to\infty} a_n = L$ and the function $f$ is continuous at $L$, then*

$$\lim_{n\to\infty} f(a_n) = f(L)$$

*Remark.* If the function is continuous,

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$$

The sequence $\{r^n\}$ is converget if and only if $-1 < r \le 1$.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$$

## 2.2 Mathematical Induction

**Definition 2.2.1** (Mathematical Induction)**.** A proof technique used to prove a statement for all positive integers.

We can use mathematical induction to find the limit of a sequence.

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$$

Consider: $a_n \geq (\frac{3}{2})^n$ and $\lim_{n\to\infty}(\frac{3}{2})^{n-1} = \infty$ so $\lim_{n\to\infty} a_n = \infty$

*Proof.* Proof by Mathematical Induction
Let $P(n)$: $a_n \geq (\frac{3}{2})^n$ for $n \in \mathbb{N}$
**Base Case:** n = 1

$$a_1 = \frac{1}{1!} = 1$$

$$(\frac{3}{2})^{1-1} = (\frac{3}{2})^0 = 1$$

Since $a_1 \geq (\frac{3}{2})^0$, $P(1)$ is true.
**Inductive Step:** Assume $P(k)$ is true for $k \in \mathbb{N}$

$$a_k \geq a_k \cdot \frac{2(k+1)-1}{k+1} = a_k \cdot \frac{2k+1}{k+1}$$

$$(\frac{3}{2})^k = (\frac{3}{2})^{k-1} \cdot \frac{3}{2}$$

Consider

$$a_k \cdot \frac{2k+1}{k+1} \geq (\frac{3}{2})^{k-1} \cdot \frac{3}{2}$$

So $P(k) \to P(k+1)$ is true. Such that $P(n)$ is true for all $n \in \mathbb{N}$. □

## 2.3 Monotonic and Bounded Sequences

Our goal is to determine if a sequence is convergent or divergent. We can tell that a sequence is "convergent" but may not know the absoulte value of the limit.

**Definition 2.3.1** (Increasing Sequence)**.** A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$.

**Definition 2.3.2** (Decreasing Sequence)**.** A sequence $\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all $n \geq 1$.

**Definition 2.3.3** (Monotonic Sequence)**.** A sequence $\{a_n\}$ is monotonic if it is either increasing or decreasing.

*Remark.* In this case, increasing and decreasing sequences are defined by "strictly" increasing and decreasing in order.

**Definition 2.3.4** (Bounded Sequence)**.** A sequence $\{a_n\}$ is bounded above if there is a number $M$ such that

$$a_n \leq M \text{ for } n \geq 1$$

A sequence $\{a_n\}$ is bounded below if there is a number $m$ such that

$$a_n \geq m \text{ for } n \geq 1$$

If a sequence is bounded above and below, then it is called a bounded sequence.

**Theorem 2.3.1** (Monotonic Sequence Theorem)**.** *Every bounded, monotonic sequence is convergent.*

*Remark.* Techniques to check that the sequence is increasing or decreasing:

- $a_{n+1} - a_n$
  - If $a_{n+1} - a_n > 0$, then the sequence is increasing
  - If $a_{n+1} - a_n < 0$, then the sequence is decreasing
- $\frac{a_{n+1}}{a_n}$
  - If $\frac{a_{n+1}}{a_n} > 1$, then the sequence is increasing
  - If $\frac{a_{n+1}}{a_n} < 1$, then the sequence is decreasing
- Find $f'(n)$
  - If $f'(n) > 0$, then the sequence is increasing
  - If $f'(n) < 0$, then the sequence is decreasing

Prove that the following sequence is convergent

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \text{ for } n \in \mathbb{N}$$

*Proof.* Since there are factorials in each term, we cannot use the derivative to determine if the sequence is increasing or decreasing.
1. Determine if the sequence is increasing or decreasing

$$\begin{aligned} a_{n+1} - a_n &= (1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n+1)!}) - (1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}) \\ &= \frac{1}{(n+1)!} \end{aligned}$$

Since $\frac{1}{(n+1)!} > 0$, the sequence is increasing.
2. Determine if the sequence is bounded

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \dots$$

$$\sum_{i=1}^{n} \frac{1}{i!} \leq \sum_{i=1}^{\infty} (\frac{1}{2})^{i-1} \quad \leq \frac{1}{1 - \frac{1}{2}} = 2$$

So $1 \leq a_n \leq 2$. The sequence is bounded.
By thorem: $\{a_n\}$ is monotonic and bounded, so it is convergent. □

**Theorem 2.3.2.** *If there are 2 or more subsequences of $\{a_n\}$ which converges to different values then $\{a_n\}$ is divergent.*

$\lim_{n\to\infty} |a_n| \neq 0$ and there is $(-1)^n$ in each term.

$$\cos(n\pi) = \begin{cases} a_{2k} = 1 & \text{if } n = 2k \\ a_{2k-1} = -1 & \text{if } n = 2k - 1 \end{cases}$$

A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.
(a) Find $\lim_{n\to\infty} a_n$
Let $L = \lim_{n\to\infty} a_n$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \sqrt{2 + a_n} \\
L &= \sqrt{2 + L} \\
L^2 &= 2 + L \\
L^2 - L - 2 &= 0 \\
(L-2)(L+1) &= 0 \\
L &= 2, -1
\end{aligned}$$

Since under the square root, $a_n \geq 0$, so $L = 2$.
(b) Prove that $\{a_n\}$ is increasing and bounded above by 3. Then use the Monotonic Sequence Theorem to show that $\{a_n\}$ is convergent.

*Proof.* Let $P(n)$: $a_{n+1} > a_n$ and $a_n \leq 3$
**Base Case:** n = 1

$$a_1 = \sqrt{2} \text{ and } a_2 = \sqrt{2 + \sqrt{2}}$$

Since $\sqrt{2} < \sqrt{2 + \sqrt{2}}$ and $\sqrt{2} \leq 3$, $P(1)$ is true.
**Inductive Step:** Assume $P(k)$ is true for $k \in \mathbb{N}$

$$a_{k+1} > a_k \text{ and } a_k \leq 3$$

$$a_{k+2} = \sqrt{2 + a_{k+1}}$$

Since $a_{k+1} > a_k$, we have

$$\sqrt{2 + a_{k+1}} > \sqrt{2 + a_k}$$

From $a_k \leq 3$, we have

$$\begin{aligned}
a_{k+1} &= \sqrt{2 + a_k} \\
\sqrt{2 + a_k} &\leq \sqrt{2 + 3} \\
a_{k+1} &\leq \sqrt{5} \\
&< \sqrt{9} \\
&< 3
\end{aligned}$$

So $P(k) \to P(k+1)$ is true. Such that $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

# 3 Series

## 3.1 Series

**Definition 3.1.1** (Series)**.** Series is the sum of the terms of a sequence.

$$a_1 + a_2 + a_3 + \cdots + a_n$$

**Definition 3.1.2** (Infinite Series)**.** An infinite series is the sum of the terms of an infinite sequence.

$$a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

In this chapter we will try testing the convergence of the series.

**Definition 3.1.3.** Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$, let $s_n$ denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} = a_1 + a_2 + \cdots + a_n$$

**Lemma 3.1.1.**

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

Sometimes $s_n$ is expressing in telescoping sum form. For example

$$\begin{aligned}
\sum_{n=1}^{\infty} \ln \frac{n}{n+1} &= \sum_{n=1}^{\infty} \ln n - \ln (n+1) \\
&= \lim_{n\to\infty} (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + (\ln n - \ln (n+1)) \\
&= \lim_{n\to\infty} \ln 1 - \ln (n+1) \\
&= \lim_{n\to\infty} \ln \frac{1}{n+1} \\
&= \ln 0 \\
&= -\infty
\end{aligned}$$

Such that, $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ is divergent.

$$\begin{aligned}
\sum_{n=1}^{\infty}(e^{1/n} - e^{1/(n+1)}) &= \sum_{n=1}^{\infty}(\sqrt[n]{e} - \sqrt[n+1]{e}) \\
&= \lim_{n\to\infty} (e - \sqrt{e}) + (\sqrt{e} - \sqrt[3]{e}) + (\sqrt[3]{e} - \sqrt[4]{e}) + \cdots + (\sqrt[n]{e} - \sqrt[n+1]{e}) \\
&= \lim_{n\to\infty} e - \sqrt[n+1]{e} \\
&= e - e^0 \\
&= e - 1
\end{aligned}$$

Such that, $\sum_{n=1}^{\infty}(e^{1/n} - e^{1/(n+1)})$ converges to $e - 1$.

## 3.2 Geometric Series

**Definition 3.2.1.**

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \text{ where } a \neq 0$$

is defined as a geometric series which $a$ is the first term and $r$ is the common ratio.

$$r = \frac{a_{n+1}}{a_n},\ n \geq 1$$

**Lemma 3.2.1.** *If $|r| \neq 1$, we have*

$$\begin{aligned}
s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\
r \cdot s_n &= a_r + ar^2 + ar^3 + \cdots + ar^n \\
s_n - r \cdot s_n &= a - ar^n \\
s_n &= \frac{a(1 - r^n)}{1 - r}
\end{aligned}$$

**Lemma 3.2.2.** *The geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \ldots$$

*is convergent if $|r| < 1$ and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}$$

*If $|r| \geq 1$, the series is divergent.*

*Proof.* If $|r| < 1$, we have

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} \\
&= \frac{a}{1-r} - \frac{a}{1-r} \cdot \lim_{n\to\infty} r^n \\
&= \frac{a}{1-r} - \frac{a}{1-r} \cdot 0 \\
&= \frac{a}{1-r}
\end{aligned}$$

If $|r| \geq 1$, we have $\lim_{n\to\infty} r^n = \infty$, such that the series is divergent. $\square$

# 4 Convergence Test

## 4.1 Test for Divergence

**Theorem 4.1.1** (Test for Divergence)**.** *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$

**Corollary 4.1.1.** *If* $\lim_{n\to\infty} a_n$ *does not exist or* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

**Lemma 4.1.1.** *If* $\sum a_n$ *diverges and* $b_n$ *converges, then* $\sum(a_n + b_n)$ *will be divergent.*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2^n + 4^n}{e^n} &= \sum_{n=1}^{\infty} \frac{2^n}{e^n} + \sum_{n=1}^{\infty} \frac{4^n}{e^n} \\ &= \sum_{n=1}^{\infty} (\frac{2}{e})^n + (\frac{4}{e})^n \end{aligned}$$

Since

- $\frac{2}{e} < 1$ so $\sum_{n=1}^{\infty} (\frac{2}{e})^n$ is convergent
- $\frac{4}{e} > 1$ so $\sum_{n=1}^{\infty} (\frac{4}{e})^n$ is divergent

Therefore, $\sum_{n\to\infty} \frac{2^n+4^n}{e^n}$ is divergent.

## 4.2 Integral Test

**Theorem 4.2.1** (Integral Test)**.** *Let* $f(x)$ *be a continuous, positive, and decreasing function on* $[1, \infty)$*. Then the series* $\sum_{n=1}^{\infty} f(n)$ *is convergent if and only if the improper integral* $\int_1^{\infty} f(x)\,dx$ *is convergent.*

- *If* $int_1^{\infty} f(x)\,dx$ *is convergent, then* $\sum_{n=1}^{\infty} a_n$ *is convergent*
- *If* $int_1^{\infty} f(x)\,dx$ *is divergent, then* $\sum_{n=1}^{\infty} a_n$ *is divergent*

**Lemma 4.2.1** (P-series)**.** *The series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if* $p > 1$ *and divergent if* $p \leq 1$*.*

$$\sum_{n=1}^{\infty} \frac{n}{n^4+1}$$

$$\begin{aligned} f(x) &= \frac{x}{x^4+1} \\ F(x) &= \int \frac{x}{x^4+1}\,dx \\ &= \lim_{t\to\infty} \int_1^t \frac{x}{x^4+1}\,dx \end{aligned}$$

Let $u = x^2$, then $du = 2x\,dx$

$$\begin{aligned} \lim_{t\to\infty} \int_1^t \frac{x}{x^4+1}\,dx &= \lim_{t\to\infty} \int_{x=1}^{x=t} \frac{x}{u^2+1} \cdot \frac{du}{2x} \\ &= \lim_{t\to\infty} \frac{1}{2} \int_1^{\infty} \frac{1}{u^2+1}\,du \\ &= \lim_{t\to\infty} \frac{1}{2} \arctan u \Big|_1^t \\ &= \lim_{t\to\infty} \frac{1}{2} \arctan t - \frac{1}{2} \arctan 1 \\ &= \frac{\pi}{4} - \frac{\pi}{8} \\ &= \frac{\pi}{8} \end{aligned}$$

$$\sum_{n=1}^{\infty} n(1+n^2)^p$$

$$\begin{aligned} f(x) &= x(1+x^2)^p \\ F(x) &= \int_1^{\infty} x(1+x^2)^p\,dx \\ &= \lim_{t\to\infty} \int_1^t x(1+x^2)^p\,dx \end{aligned}$$

Let $u = 1 + x^2$, then $du = 2x\,dx$

$$\begin{aligned} \lim_{t\to\infty} \int 1^t x(1+x^2)^p\,dx &= \lim_{t\to\infty} \int_2^{\infty} x \cdot u^p \frac{du}{2x} \\ &= \lim_{t\to\infty} \frac{1}{2} \int_2^{\infty} u^p\,du \\ &= \lim_{t\to\infty} \frac{1}{2} \frac{u^{p+1}}{p+1} \Big|_2^{\infty} \end{aligned}$$

Since $p \neq -1$, then $p + 1 > 0$. Therefore, the integral is divergent.

## 4.3 Comparison Test

**Theorem 4.3.1** (Comparison Test)**.** *Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.*

- *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n$, then $\sum a_n$ is convergent.*
- *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n$, then $\sum a_n$ is divergent.*

In using the comparison test we must have some known series $\sum b_n$ for the purpose of comparison. Most of time we use one of these series:

- P-series [$\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$]
- Geometric series [$\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$]

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$$

$$\text{Let } a_n = \frac{1}{\sqrt{n-1}}$$

$$\text{Let } b_n = \frac{1}{\sqrt{n}}$$

$$a_n \geq b_n \text{ for all } n$$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \leq \sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent, then $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$ is divergent.

## 4.4 Limit Comparison Test

**Theorem 4.4.1** (Limit Comparison Test)**.** *Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.*
*If*

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L$$

*where $L$ is a finite positive number and $L > 0$, then either both series converge or both diverge.*

$$\sum_{n=1}^{\infty} \sin^2(\frac{1}{n})$$

Choose $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{sin^2(\frac{1}{n})}{\frac{1}{n^2}} = \lim_{n\to\infty} (\frac{\sin^2 \frac{1}{n}}{\frac{1}{n}})^2$$

Since $\lim_{n\to\infty} \frac{\sin x}{x} = 1$, then

$$\lim_{n\to\infty} (\frac{\sin^2 \frac{1}{n}}{\frac{1}{n}})^2 = 1^2 = 1 > 0$$

Since $\sum_{n=1} \infty \frac{1}{n^2}$ converges. Then, by limit comparison test $\sum_{n=1}^{\infty} \sin^2(\frac{1}{n})$ converges.

Show that if $a_n > 0$ and $\lim_{n\to\infty} n \cdot a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

*Proof.* Since $\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = L$ where $L > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. By limit comparison test, $\sum_{n=1}^{\infty} a_n$ is divergent. □

## 4.5 Alternating Series Test

**Definition 4.5.1** (Alternating Series)**.** An alternating series is a series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

where $b_n > 0$ for all $n$.

**Theorem 4.5.1** (Alternating Series Test)**.** *If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies*

- $b_n \geq b_{n+1}$ *for all* $n$
- $\lim_{n\to\infty} b_n = 0$

*then the series is convergent.*

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$$

Let $b_n = \left|\frac{(-1)^{n+1}}{\sqrt{n+1}}\right| = \frac{1}{\sqrt{n+1}}$ 1. Determine if the series is decreasing
From:

$$\sqrt{n+1} \leq \sqrt{(n+1)+1}$$
$$\frac{1}{\sqrt{n+1}} \geq \frac{1}{\sqrt{n+2}}$$
$$b_n \geq b_{n+1}$$

Since $b_n \geq b_{n+1}$, the series is decreasing.
2. Determine if limit of $b_n$ is 0

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$$

Since the series is decreasing and the limit of $b_n$ is 0, the series is convergent.
Obervation:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{\sqrt{n+1}}\right|$$

diverges.