

# 2301108 CALCULUS II (Midterm Edition)

Borworntat Dendumrongkul

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# 1 Differential Equation

## 1.1 Seperable Equation

**Definition 1.1.1** (Seperable Equation). A differential equation is called seperable if it can be written in the form

$$\frac{dy}{dx} = g(y)h(x)$$

where  $g(y)$  is a function of  $y$  only and  $h(x)$  is a function of  $x$  only.

$$\frac{dy}{dx} = g(y)h(x)$$

$$\frac{dy}{g(y)} = h(x)dx$$

$$\int \frac{dy}{g(y)} = \int h(x)dx$$

**Definition 1.1.2** (Implicit Solution). An implicit solution to a differential equation is a solution that is not solved for  $y$  explicitly. Answer is in the form of  $F(x, y) = c$ .

**Definition 1.1.3** (Explicit Solution). An explicit solution to a differential equation is a solution that is solved for  $y$  explicitly. Answer is in the form of  $y = f(x)$ .

Sometimes there are initial condition of the equation.

**Theorem 1.1.1** (FTC 1). *If  $f$  is continuous on  $[a, b]$ , then the function  $f$  defined by*

$$\frac{d}{dx} \int_a^{u(x)} f(t)dt = f(u(x))u'(x)$$

*is continuous on  $[a, b]$  and differentiable in  $(a, b)$ , and  $g'(x) = f(x)$ .*

**Definition 1.1.4** (Integral Equation). An integral equation is an equation in which an unknown function appears under an integral sign.

We have to change the integral equation to a differential equation. Then, we can use FTC 1 to solve the differential equation.

## 1.2 Linear Equation

**Definition 1.2.1** (Linear Equation). A differential equation is called linear if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where  $P(x)$  and  $Q(x)$  are continuous functions of  $x$ .

The term “linear” refers to the fact that the unknown function  $y$  and its derivative  $\frac{dy}{dx}$  appear in the equation to the first power and are not multiplied together.

**Definition 1.2.2** (Integrating Factor). The integrating factor for the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is the function  $I(x)$  defined by

$$I(x) = \exp\left(\int P(x)dx\right)$$

We can solve the linear differential equation by the formula.

$$y(x) = \frac{1}{I(x)}\left[\int I(x)Q(x)dx + C\right]$$

without any additional constraint  $c$  from integrating.

## 1.3 Bernoulli Equation

**Definition 1.3.1** (Bernoulli Equation). A differential equation is called Bernoulli if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where  $P(x)$  and  $Q(x)$  are continuous functions of  $x$ .

As you can see, there is  $y^n$  in the equation which is not linear.

We have to change the Bernoulli equation to a linear equation by substitution.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

divides both side by  $y^n$

$$\frac{dy}{dx}y^{-n} + P(x)y^{1-n} = Q(x)$$

Then we substitute  $u = y^{1-n}$

$$\begin{aligned}\frac{du}{dy} &= (1-n)y^{-n}\frac{dy}{dx} \\ \frac{du}{dx} \cdot \frac{1}{1-n} &= y^{-n}\frac{dy}{dx}\end{aligned}$$

Let  $u = y^{1-n}$

$$\begin{aligned}\frac{du}{dx} &= (1-n)y^{-n}\frac{dy}{dx} \\ \frac{du}{dx} \cdot \frac{1}{1-n} &= y^{-n}\frac{dy}{dx}\end{aligned}$$

Then, try to substitute the equation with  $u$ .

$$\begin{aligned}\frac{du}{dx} \cdot \frac{1}{1-n} + P(x)u &= Q(x) \\ \frac{du}{dx} + (1-n)P(x)u &= (1-n)Q(x)\end{aligned}$$

Now, we can solve the equation with the linear equation method. Since the equation is linear on  $u$ .

## 2 Sequence

### 2.1 Sequence

**Definition 2.1.1** (Sequence). A sequence is a function whose domain is  $\mathbb{N}$

We are considering behavior of a sequence as  $n$  becomes infinite.

$$\lim_{n \rightarrow \infty} a_n = \begin{cases} L \in \mathbb{R} & \text{convergent} \\ \pm\infty & \text{divergent} \\ \text{morethanonevalue} & \text{divergent} \end{cases}$$

**Definition 2.1.2** (Sequence Notation). The sequence  $\{a_1, a_2, a_3, \dots\}$  is denoted by  $\{a_n\}$  or  $\{a_n\}_{n=1}^{\infty}$

**Theorem 2.1.1** (Limit of Sequence). A sequence  $\{a_n\}$  converges to  $L$ .

$$\lim_{n \rightarrow \infty} a_n = L$$

or

$$a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if for every  $\epsilon > 0$ , there exists  $N$  such that

$$\text{if } n > N$$

then

$$|a_n - L| < \epsilon$$

**Theorem 2.1.2.** If  $\lim_{x \rightarrow \infty} f(x) = L$  and  $f(n) = a_n$  for all  $n \in \mathbb{N}$ , then

$$\lim_{n \rightarrow \infty} a_n = L$$

Since our function is discrete, we can use the limit of the function to find the limit of the sequence.

*Remark.* We can use L'Hopital's rule to find the limit of the sequence.

**Lemma 2.1.1.** *If subsequences of  $a_n$  converge to different limits, then the sequence  $a_n$  diverges.*

$a_n = (-1)^n$ $a_n = 1, -1, 1, -1, \dots$
<p>Consider: <math>a_{2k-1} = (-1)^{2k-1}; k \in \mathbb{N}</math></p> $a_{2k-1} = (-1)^{2k-1}$ $a_{2k-1} = -1$
<p>Consider: <math>a_{2k} = (-1)^{2k}; k \in \mathbb{N}</math></p> $a_{2k} = (-1)^{2k}$ $a_{2k} = 1$
<p>Since <math>\lim_{k \rightarrow \infty} a_{2k-1} = -1</math> and <math>\lim_{k \rightarrow \infty} a_{2k} = 1</math>, the sequence <math>a_n</math> diverges.</p>

**Theorem 2.1.3** (Squeeze theorem). *Let  $a_n \leq b_n \leq c_n$  for  $n \geq n_0$ .  
If*

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$$

*then*

$$\lim_{n \rightarrow \infty} b_n = L$$

**Theorem 2.1.4.** *If  $\lim_{n \rightarrow \infty} |a_n| = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$*

*Proof.* From  $-|a_n| \leq a_n \leq |a_n|$ , we have

$$\lim_{n \rightarrow \infty} |a_n| = 0$$

by the squeeze theorem, we have  $\lim_{n \rightarrow \infty} a_n = 0$  □

**Theorem 2.1.5.** *If  $\lim_{n \rightarrow \infty} a_n = L$  and the function  $f$  is continuous at  $L$ , then*

$$\lim_{n \rightarrow \infty} f(a_n) = f(L)$$

*Remark.* If the function is continuous,

$$\lim_{n \rightarrow \infty} f(a_n) = f(\lim_{n \rightarrow \infty} a_n)$$

The sequence  $\{r^n\}$  is convergent if and only if  $-1 < r \leq 1$ .

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$$

## 2.2 Mathematical Induction

**Definition 2.2.1** (Mathematical Induction). A proof technique used to prove a statement for all positive integers.

We can use mathematical induction to find the limit of a sequence.

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{n!}$$

Consider:  $a_n \geq (\frac{3}{2})^n$  and  $\lim_{n \rightarrow \infty} (\frac{3}{2})^{n-1} = \infty$  so  $\lim_{n \rightarrow \infty} a_n = \infty$

*Proof.* Proof by Mathematical Induction  
 Let  $P(n)$ :  $a_n \geq (\frac{3}{2})^n$  for  $n \in \mathbb{N}$   
**Base Case:**  $n = 1$

$$a_1 = \frac{1}{1!} = 1$$

$$(\frac{3}{2})^{1-1} = (\frac{3}{2})^0 = 1$$

Since  $a_1 \geq (\frac{3}{2})^0$ ,  $P(1)$  is true.  
**Inductive Step:** Assume  $P(k)$  is true for  $k \in \mathbb{N}$

$$a_k \geq a_k \cdot \frac{2(k+1)-1}{k+1} = a_k \cdot \frac{2k+1}{k+1}$$

$$(\frac{3}{2})^k = (\frac{3}{2})^{k-1} \cdot \frac{3}{2}$$

Consider

$$a_k \cdot \frac{2k+1}{k+1} \geq (\frac{3}{2})^{k-1} \cdot \frac{3}{2}$$

So  $P(k) \rightarrow P(k+1)$  is true. Such that  $P(n)$  is true for all  $n \in \mathbb{N}$ . □

## 2.3 Monotonic and Bounded Sequences

Our goal is to determine if a sequence is convergent or divergent. We can tell that a sequence is “convergent” but may not know the absolute value of the limit.

**Definition 2.3.1** (Increasing Sequence). A sequence  $\{a_n\}$  is increasing if  $a_n < a_{n+1}$  for all  $n \geq 1$ .

**Definition 2.3.2** (Decreasing Sequence). A sequence  $\{a_n\}$  is decreasing if  $a_n > a_{n+1}$  for all  $n \geq 1$ .



**Definition 2.3.3** (Monotonic Sequence). A sequence  $\{a_n\}$  is monotonic if it is either increasing or decreasing.

*Remark.* In this case, increasing and decreasing sequences are defined by “strictly” increasing and decreasing in order.

**Definition 2.3.4** (Bounded Sequence). A sequence  $\{a_n\}$  is bounded above if there is a number  $M$  such that

$$a_n \leq M \text{ for } n \geq 1$$

A sequence  $\{a_n\}$  is bounded below if there is a number  $m$  such that

$$a_n \geq m \text{ for } n \geq 1$$

If a sequence is bounded above and below, then it is called a bounded sequence.

**Theorem 2.3.1** (Monotonic Sequence Theorem). *Every bounded, monotonic sequence is convergent.*

*Remark.* Techniques to check that the sequence is increasing or decreasing:

- $a_{n+1} - a_n$ 
  - If  $a_{n+1} - a_n > 0$ , then the sequence is increasing
  - If  $a_{n+1} - a_n < 0$ , then the sequence is decreasing
- $\frac{a_{n+1}}{a_n}$ 
  - If  $\frac{a_{n+1}}{a_n} > 1$ , then the sequence is increasing
  - If  $\frac{a_{n+1}}{a_n} < 1$ , then the sequence is decreasing
- Find  $f'(n)$ 
  - If  $f'(n) > 0$ , then the sequence is increasing
  - If  $f'(n) < 0$ , then the sequence is decreasing

Prove that the following sequence is convergent

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \text{ for } n \in \mathbb{N}$$

*Proof.* Since there are factorials in each term, we cannot use the derivative to determine if the sequence is increasing or decreasing.

1. Determine if the sequence is increasing or decreasing

$$\begin{aligned} a_{n+1} - a_n &= \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n+1)!}\right) - \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) \\ &= \frac{1}{(n+1)!} \end{aligned}$$

Since  $\frac{1}{(n+1)!} > 0$ , the sequence is increasing.

2. Determine if the sequence is bounded

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \cdots$$

$$\sum_{i=1}^n \frac{1}{i!} \leq \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{i-1} \leq \frac{1}{1 - \frac{1}{2}} = 2$$

So  $1 \leq a_n \leq 2$ . The sequence is bounded.

By theorem:  $\{a_n\}$  is monotonic and bounded, so it is convergent. □

**Theorem 2.3.2.** *If there are 2 or more subsequences of  $\{a_n\}$  which converges to different values then  $\{a_n\}$  is divergent.*

$\lim_{n \rightarrow \infty} |a_n| \neq 0$  and there is  $(-1)^n$  in each term.

$$\cos(n\pi) = \begin{cases} a_{2k} = 1 & \text{if } n = 2k \\ a_{2k-1} = -1 & \text{if } n = 2k - 1 \end{cases}$$

A sequence  $\{a_n\}$  is given by  $a_1 = \sqrt{2}$ ,  $a_{n+1} = \sqrt{2 + a_n}$  for  $n \geq 1$ .

(a) Find  $\lim_{n \rightarrow \infty} a_n$

Let  $L = \lim_{n \rightarrow \infty} a_n$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sqrt{2 + a_n}$$

$$L = \sqrt{2 + L}$$

$$L^2 = 2 + L$$

$$L^2 - L - 2 = 0$$

$$(L - 2)(L + 1) = 0$$

$$L = 2, -1$$

Since under the square root,  $a_n \geq 0$ , so  $L = 2$ .

(b) Prove that  $\{a_n\}$  is increasing and bounded above by 3. Then use the Monotonic Sequence Theorem to show that  $\{a_n\}$  is convergent.

*Proof.* Let  $P(n)$ :  $a_{n+1} > a_n$  and  $a_n \leq 3$

**Base Case:**  $n = 1$

$$a_1 = \sqrt{2} \text{ and } a_2 = \sqrt{2 + \sqrt{2}}$$

Since  $\sqrt{2} < \sqrt{2 + \sqrt{2}}$  and  $\sqrt{2} \leq 3$ ,  $P(1)$  is true.

**Inductive Step:** Assume  $P(k)$  is true for  $k \in \mathbb{N}$

$$a_{k+1} > a_k \text{ and } a_k \leq 3$$

$$a_{k+2} = \sqrt{2 + a_{k+1}}$$

Since  $a_{k+1} > a_k$ , we have

$$\sqrt{2 + a_{k+1}} > \sqrt{2 + a_k}$$

From  $a_k \leq 3$ , we have

$$a_{k+1} = \sqrt{2 + a_k}$$

$$\sqrt{2 + a_k} \leq \sqrt{2 + 3}$$

$$a_{k+1} \leq \sqrt{5}$$

$$< \sqrt{9}$$

$$< 3$$

So  $P(k) \rightarrow P(k + 1)$  is true. Such that  $P(n)$  is true for all  $n \in \mathbb{N}$ . □

## 3 Series

### 3.1 Series

**Definition 3.1.1** (Series). Series is the sum of the terms of a sequence.

$$a_1 + a_2 + a_3 + \cdots + a_n$$

**Definition 3.1.2** (Infinite Series). An infinite series is the sum of the terms of an infinite sequence.

$$a_1 + a_2 + a_3 + \cdots + a_n + \dots$$

In this chapter we will try testing the convergence of the series.

**Definition 3.1.3.** Given a series  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$ , let  $s_n$  denote its  $n$ th partial sum:

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$$

**Lemma 3.1.1.**

$$\sum_{n=1}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$$

Sometimes  $s_n$  is expressing in telescoping sum form.

$$\begin{aligned} \sum_{n=1}^{\infty} \ln \frac{n}{n+1} &= \sum_{n=1}^{\infty} \ln n - \ln(n+1) \\ &= \lim_{n \rightarrow \infty} (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + (\ln n - \ln(n+1)) \\ &= \lim_{n \rightarrow \infty} \ln 1 - \ln(n+1) \\ &= \lim_{n \rightarrow \infty} \ln \frac{1}{n+1} \\ &= \ln 0 \\ &= -\infty \end{aligned}$$

Such that,  $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$  is divergent.

$$\begin{aligned}
\sum_{n=1}^{\infty} (e^{1/n} - e^{1/(n+1)}) &= \sum_{n=1}^{\infty} (\sqrt[n]{e} - \sqrt[n+1]{e}) \\
&= \lim_{n \rightarrow \infty} (e - \sqrt{e}) + (\sqrt{e} - \sqrt[3]{e}) + (\sqrt[3]{e} - \sqrt[4]{e}) + \cdots + (\sqrt[n]{e} - \sqrt[n+1]{e}) \\
&= \lim_{n \rightarrow \infty} e - \sqrt[n+1]{e} \\
&= e - e^0 \\
&= e - 1
\end{aligned}$$

Such that,  $\sum_{n=1}^{\infty} (e^{1/n} - e^{1/(n+1)})$  converges to  $e - 1$ .

## 3.2 Geometric Series

**Definition 3.2.1.**

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \text{ where } a \neq 0$$

is defined as a geometric series which  $a$  is the first term and  $r$  is the common ratio.

$$r = \frac{a_{n+1}}{a_n}, \quad n \geq 1$$

**Lemma 3.2.1.** *If  $|r| \neq 1$ , we have*

$$\begin{aligned}
s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\
r \cdot s_n &= ar + ar^2 + ar^3 + \cdots + ar^n \\
s_n - r \cdot s_n &= a - ar^n \\
s_n &= \frac{a(1 - r^n)}{1 - r}
\end{aligned}$$

**Lemma 3.2.2.** *The geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$$

*is convergent if  $|r| < 1$  and its sum is*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}$$

*If  $|r| \geq 1$ , the series is divergent.*

*Proof.* If  $|r| < 1$ , we have

$$\begin{aligned}\lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \frac{a(1 - r^n)}{1 - r} \\&= \frac{a}{1 - r} - \frac{a}{1 - r} \cdot \lim_{n \rightarrow \infty} r^n \\&= \frac{a}{1 - r} - \frac{a}{1 - r} \cdot 0 \\&= \frac{a}{1 - r}\end{aligned}$$

If  $|r| \geq 1$ , we have  $\lim_{n \rightarrow \infty} r^n = \infty$ , such that the series is divergent. □

## 4 Convergence Test

### 4.1 Test for Divergence

**Theorem 4.1.1** (Test for Divergence). *If the series  $\sum_{n=1}^{\infty} a_n$  is convergent, then  $\lim_{n \rightarrow \infty} a_n = 0$*

**Corollary 4.1.1.** *If  $\lim_{n \rightarrow \infty} a_n$  does not exist or  $\lim_{n \rightarrow \infty} a_n \neq 0$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.*

**Lemma 4.1.1.** *If  $\sum a_n$  diverges and  $b_n$  converges, then  $\sum(a_n + b_n)$  will be divergent.*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^n + 4^n}{e^n} &= \sum_{n=1}^{\infty} \frac{2^n}{e^n} + \sum_{n=1}^{\infty} \frac{4^n}{e^n} \\ &= \sum_{n=1}^{\infty} \left(\frac{2}{e}\right)^n + \left(\frac{4}{e}\right)^n\end{aligned}$$

Since

- $\frac{2}{e} < 1$  so  $\sum_{n=1}^{\infty} \left(\frac{2}{e}\right)^n$  is convergent
- $\frac{4}{e} > 1$  so  $\sum_{n=1}^{\infty} \left(\frac{4}{e}\right)^n$  is divergent

Therefore,  $\sum_{n \rightarrow \infty} \frac{2^n + 4^n}{e^n}$  is divergent.

### 4.2 Integral Test

**Theorem 4.2.1** (Integral Test). *Let  $f(x)$  be a continuous, positive, and decreasing function on  $[1, \infty)$ . Then the series  $\sum_{n=1}^{\infty} f(n)$  is convergent if and only if the improper integral  $\int_1^{\infty} f(x) dx$  is convergent.*

- If  $\int_1^{\infty} f(x) dx$  is convergent, then  $\sum_{n=1}^{\infty} a_n$  is convergent
- If  $\int_1^{\infty} f(x) dx$  is divergent, then  $\sum_{n=1}^{\infty} a_n$  is divergent

**Lemma 4.2.1** (P-series). *The series  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is convergent if  $p > 1$  and divergent if  $p \leq 1$ .*

$$\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$$

$$f(x) = \frac{x}{x^4 + 1}$$

$$\begin{aligned} F(x) &= \int \frac{x}{x^4 + 1} dx \\ &= \lim_{t \rightarrow \infty} \int_1^t \frac{x}{x^4 + 1} dx \end{aligned}$$

Let  $u = x^2$ , then  $du = 2x dx$

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_1^t \frac{x}{x^4 + 1} dx &= \lim_{t \rightarrow \infty} \int_{x=1}^{x=t} \frac{x}{u^2 + 1} \cdot \frac{du}{2x} \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \int_1^{\infty} \frac{1}{u^2 + 1} du \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \arctan u \Big|_1^t \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \arctan t - \frac{1}{2} \arctan 1 \\ &= \frac{\pi}{4} - \frac{\pi}{8} \\ &= \frac{\pi}{8} \end{aligned}$$

$$\sum_{n=1}^{\infty} n(1 + n^2)^p$$

$$f(x) = x(1 + x^2)^p$$

$$\begin{aligned} F(x) &= \int_1^{\infty} x(1 + x^2)^p dx \\ &= \lim_{t \rightarrow \infty} \int_1^t x(1 + x^2)^p dx \end{aligned}$$

Let  $u = 1 + x^2$ , then  $du = 2x dx$

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_1^t x(1 + x^2)^p dx &= \lim_{t \rightarrow \infty} \int_2^{\infty} x \cdot u^p \frac{du}{2x} \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \int_2^{\infty} u^p du \\ &= \lim_{t \rightarrow \infty} \frac{1}{2} \frac{u^{p+1}}{p+1} \Big|_2^{\infty} \end{aligned}$$

Since  $p \neq -1$ , then  $p + 1 > 0$ . Therefore, the integral is divergent.



## 4.3 Comparison Test

**Theorem 4.3.1** (Comparison Test). *Suppose that  $\sum a_n$  and  $\sum b_n$  are series with positive terms.*

- *If  $\sum b_n$  is convergent and  $a_n \leq b_n$  for all  $n$ , then  $\sum a_n$  is convergent.*
- *If  $\sum b_n$  is divergent and  $a_n \geq b_n$  for all  $n$ , then  $\sum a_n$  is divergent.*

In using the comparison test we must have some known series  $\sum b_n$  for the purpose of comparison. Most of time we use one of these series:

- P-series [ $\sum 1/n^p$  converges if  $p > 1$  and diverges if  $p \leq 1$ ]
- Geometric series [ $\sum ar^{n-1}$  converges if  $|r| < 1$  and diverges if  $|r| \geq 1$ ]

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$$

$$\text{Let } a_n = \frac{1}{\sqrt{n-1}}$$

$$\text{Let } b_n = \frac{1}{\sqrt{n}}$$

$$a_n \geq b_n \text{ for all } n$$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \leq \sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Since  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  is divergent, then  $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$  is divergent.

## 4.4 Limit Comparison Test

**Theorem 4.4.1** (Limit Comparison Test). *Suppose that  $\sum a_n$  and  $\sum b_n$  are series with positive terms.*

*If*

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$$

*where  $L$  is a finite positive number and  $L > 0$ , then either both series converge or both diverge.*

$$\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$$

Choose  $b_n = \frac{1}{n^2}$

$$\lim_{n \rightarrow \infty} \frac{\sin^2(\frac{1}{n})}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \left(\frac{\sin^2 \frac{1}{n}}{\frac{1}{n}}\right)^2$$

Since  $\lim_{n \rightarrow \infty} \frac{\sin x}{x} = 1$ , then

$$\lim_{n \rightarrow \infty} \left(\frac{\sin^2 \frac{1}{n}}{\frac{1}{n}}\right)^2 = 1^2 = 1 > 0$$

Since  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges. Then, by limit comparison test  $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$  converges.

Show that if  $a_n > 0$  and  $\lim_{n \rightarrow \infty} n \cdot a_n \neq 0$ , then  $\sum_{n=1}^{\infty} a_n$  is divergent.

*Proof.* Since  $\lim_{n \rightarrow \infty} \frac{a_n}{\frac{1}{n}} = L$  where  $L > 0$  and  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges. By limit comparison test,  $\sum_{n=1}^{\infty} a_n$  is divergent.  $\square$

## 4.5 Alternating Series Test

**Definition 4.5.1** (Alternating Series). An alternating series is a series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

where  $b_n > 0$  for all  $n$ .

**Theorem 4.5.1** (Alternating Series Test). *If the alternating series  $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$  satisfies*

- $b_n \geq b_{n+1}$  for all  $n$
- $\lim_{n \rightarrow \infty} b_n = 0$

*then the series is convergent.*

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$$

Let  $b_n = \left| \frac{(-1)^{n+1}}{\sqrt{n+1}} \right| = \frac{1}{\sqrt{n+1}}$  1. Determine if the series is decreasing

From:

$$\begin{aligned}\sqrt{n+1} &\leq \sqrt{(n+1)+1} \\ \frac{1}{\sqrt{n+1}} &\geq \frac{1}{\sqrt{n+2}} \\ b_n &\geq b_{n+1}\end{aligned}$$

Since  $b_n \geq b_{n+1}$ , the series is decreasing.

2. Determine if limit of  $b_n$  is 0

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+1}} = 0$$

Since the series is decreasing and the limit of  $b_n$  is 0, the series is convergent.

Obervation:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{\sqrt{n+1}} \right|$$

diverges.

There are 3 ways to determine if an alternating series is convergent or divergent:

- $\sum_{n=1}^{\infty} (-1)^n \cdot b_n$  converges but  $\sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$  diverges
- $\sum_{n=1}^{\infty} (-1)^n \cdot b_n$  converges and  $\sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$  converges (by comparison test)
- $\sum_{n=1}^{\infty} (-1)^n \cdot b_n$  diverges and  $\sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$  diverges

## Estimating the Sum of an Alternating Series

A partial sum  $s_n$  of any convergent series can be used as an approximation to the total sum  $s$ . The error involved in using  $s \approx s_n$  is the remainder  $R_n = s - s_n$ .

**Theorem 4.5.2** (Alternating Series Estimation Theorem). *If  $s = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot b_n$ , where  $b_n > 0$ , is the sum of an alternating series that satisfies*

- $b_n \geq b_{n+1}$  for all  $n$
- $\lim_{n \rightarrow \infty} b_n = 0$  (converges)

then  $|R_n| = |s - s_n| \leq b_{n+1}$

Find the sum of the series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$  correct to three decimal places.

Determine the series is convergent by the alternating series test. 1. Determine if the series is decreasing

$$\begin{aligned} b_{n+1} &= \frac{1}{(n+1)!} \\ &= \frac{1}{n!(n+1)} < b_n \\ &< \frac{1}{n!} \end{aligned}$$

2. Determine if limit of  $b_n$  is 0

$$\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$$

Such that the series is convergent.

Use the alternating series estimation theorem to find the sum of the series correct to three decimal places.

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \dots$$

Note:  $b_7 = \frac{1}{5040} \approx \frac{1}{5000} = 0.0002$

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$$

By alternating series estimation theorem,  $|s - s_6| \leq b_7 = 0.0002$

Therefore, the sum of the series is  $s \approx 0.368$

*Remark.* For some  $\sum_{n=1}^{\infty} (-1)^n \cdot b_n$  converges but sometimes

- $\sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$  diverges (most of the time)
- $\sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$  converges

$\sum_{n=1}^{\infty} (-1)^n \cdot b_n$  which is convergent relates with  $\sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$  which is divergent.

**Lemma 4.5.1.** *If we know that  $\pm \sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$  converges, then  $\sum_{n=1}^{\infty} (-1)^n \cdot b_n$  also converges.*

*Proof.*

$$-\sum_{n=1}^{\infty} |(-1)^n \cdot b_n| \leq \sum_{n=1}^{\infty} (-1)^n \cdot b_n \leq \sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$$

Since we know that  $\pm \sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$  converges. Then by squeeze theorem,  $\sum_{n=1}^{\infty} (-1)^n \cdot b_n$  also converges.  $\square$

## 4.6 Absolute Convergence and Conditional Convergence

**Definition 4.6.1** (Absolute Convergence). A series  $\sum_{n=1}^{\infty} a_n$  is called absolutely convergent if the series of the absolute values  $\sum_{n=1}^{\infty} |a_n|$  is convergent.

**Definition 4.6.2** (Conditional Convergence). A series  $\sum_{n=1}^{\infty} a_n$  is called conditionally convergent if the series is convergent but not absolutely convergent; that is, if  $\sum_{n=1}^{\infty} a_n$  is convergent but  $\sum_{n=1}^{\infty} |a_n|$  is divergent.

**Theorem 4.6.1.** *If a series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, then it is convergent.*

$$\sum_{n=1}^{\infty} \sin n$$

Consider:

$$\sum_{n=1}^{\infty} |\sin n| = |\sin 1| + |\sin 2| + |\sin 3| + \dots$$

There are two cases

- If  $\sum_{n=1}^{\infty} |\sin n|$  converges, then  $\sum_{n=1}^{\infty} \sin n$  is convergent
- If  $\sum_{n=1}^{\infty} |\sin n|$  diverges, then we cannot conclude that  $\sum_{n=1}^{\infty} \sin n$  is divergent

In cases where  $\sum_{n=1}^{\infty} |a_n|$  diverges but  $\sum_{n=1}^{\infty} a_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  is conditionally convergent.

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{3n+2}$$

Since  $\cos n\pi = (-1)^n$ , then

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{3n+2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$$

Consider:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{3n+2} \right| = \sum_{n=1}^{\infty} \frac{1}{3n+2}$$

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{3n+2}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{3n+2} = \frac{1}{3} > 0$$

and  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges from limit comparison test. Such that  $\sum_{n=1}^{\infty} \frac{1}{3n+2}$  diverges.

Consider:  $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$  which is an alternating series where  $b_n = \frac{1}{3n+2}$

Let  $f(x) = \frac{1}{3x+2}$

$$f'(x) = \frac{-3}{(3x+2)^2} < 0$$

So  $b_n$  is decreasing.

$$\lim_{n \rightarrow \infty} \frac{1}{3n+2} = 0$$

Therefore,  $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$  is convergent.

Since  $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$  is convergent and  $\sum_{n=1}^{\infty} \frac{1}{3n+2}$  diverges, then  $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$  is conditionally convergent.

## 4.7 Ratio Test

*Remark.* This method might be good for series with factorials.

This test is very useful in determining whether a given series is absolutely convergent.

**Theorem 4.7.1** (Ratio Test). *There are three possible outcomes for the series  $\sum_{n=1}^{\infty} a_n$ :*

- If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.
- If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$  or  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.
- If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ , then the test is inconclusive.

$$\sum_{n=1}^{\infty} \frac{n!}{100^n}$$

Consider:

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)! \cdot 100^n}{100^{n+1} \cdot n!} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{100} \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \frac{n+1}{100} = \infty$ , then the series  $\sum_{n=1}^{\infty} \frac{n!}{100^n}$  is divergent.

*Remark.* Above example shows that factorial is bigger than exponential.

## 4.8 Root Test

**Theorem 4.8.1** (Root Test). *There are three possible outcomes for the series  $\sum_{n=1}^{\infty} a_n$ :*

- If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L < 1$ , then the series  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent.
- If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L > 1$  or  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \infty$ , then the series  $\sum_{n=1}^{\infty} a_n$  is divergent.
- If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = 1$ , then the test is inconclusive.

$$\sum_{n=1}^{\infty} \left( \frac{1-n}{2+3n} \right)^n$$

Consider:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{1-n}{2+3n} \right)^n \right|} &= \lim_{n \rightarrow \infty} \left| \frac{1-n}{2+3n} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n-1}{2+3n} = \frac{1}{3} < 1 \end{aligned}$$

By root test:  $\sum_{n=1}^{\infty} \left( \frac{1-n}{2+3n} \right)^n$  is absolutely convergent.

$$\sum_{n=1}^{\infty} \left( \frac{-2n}{n+1} \right)^{5n}$$

Consider:

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{-2n}{n+1} \right)^{5n} \right|} &= \lim_{n \rightarrow \infty} \left( \frac{2n}{n+1} \right)^5 \\ &= 2^5 > 1 \end{aligned}$$

By root test:  $\sum_{n=1}^{\infty} \left( \frac{-2n}{n+1} \right)^{5n}$  is divergent.

## 4.9 Strategy for Testing Series

1. **Test for divergence**  $\lim_{n \rightarrow \infty} a_n \neq 0 \rightarrow$  divergent.

2. **P-series**  $\frac{1}{n^p}$  converges if and only if  $p > 1$ .

3. **Geometric series**

- find  $r$
- converges if  $|r| < 1$  else diverges

4. **Comparison tests**

- $a_n \leq b_n$  and  $b_n$  converges such that  $a_n$  converges  $\equiv a_n \leq b_n$  and  $a_n$  diverges such that  $b_n$  diverges

- Limit comparison

If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = C > 0$$

then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Leftrightarrow \sum_{n=1}^{\infty} b_n \text{ converges} \equiv \sum_{n=1}^{\infty} a_n \text{ diverges} \Leftrightarrow \sum_{n=1}^{\infty} b_n \text{ diverges}$$

5. **Alternating series test**

- Find real  $b_n$  and  $\text{sgn}(a_n)$
- Determine if  $b_n$  is decreasing  $b_{n+1} \leq b_n$
- Determine if  $\lim_{n \rightarrow \infty} b_n = 0$

If satisfies all conditions, then the series is convergent.

6. **Ratio test**

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L \begin{cases} < 1 & \text{absolutely convergent} \\ > 1 & \text{divergent} \\ = 1 & \text{inconclusive} \end{cases}$$

7. **Root test**

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L \begin{cases} < 1 & \text{absolutely convergent} \\ > 1 & \text{divergent} \\ = 1 & \text{inconclusive} \end{cases}$$



### 8. Integral test

If  $f(x)$  is continuous, positive, and decreasing on  $[1, \infty)$ , then

$$\int_1^{\infty} f(x) \, dx \text{ converges} \Leftrightarrow \sum_{n=1}^{\infty} f(n) \text{ converges}$$

## 5 Power Series

We can write polynomial in form  $P(x) = \sum_{i=0}^n c_i \cdot x^i$  where  $c_i$  is coefficient and  $x$  is variable.

As you can see, there is a stop position since there is  $n$  in summation.

So, we can write power series in form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

where  $x$  is variable,  $c_n$  is coefficient.

**Theorem 5.0.1** (Power Series). *If  $\sum_{n=0}^{\infty} c_n x^n$  converges, then we can write it as*

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_n x^n + \dots$$

How to check that  $\sum_{n=0}^{\infty} c_n x^n$  converges?

1. Ratio test

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{c_{n+1} x^{n+1}}{c_n \cdot x^n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \cdot x \right| \\ &= |x| \cdot \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| < 1 \end{aligned}$$

Let  $\alpha = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$

$$|x| \cdot \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| < 1$$

$$|x| \cdot \alpha < 1$$

$$|x| < \frac{1}{\alpha}$$

2. Root test

$$\lim_{n \rightarrow \infty} \sqrt[n]{|c_n x^n|} = |x| \cdot \lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} < 1$$

Let  $\alpha = \lim_{n \rightarrow \infty} \sqrt[n]{|c_n x^n|}$

$$|x| \cdot \lim_{n \rightarrow \infty} \sqrt[n]{|c_n|} < 1$$

$$|x| \cdot \alpha < 1$$

$$|x| < \frac{1}{\alpha}$$

**Definition 5.0.1** (General form of power series).

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

where  $a$  is center of power series.

**Theorem 5.0.2** (Radius of Convergence).

$$R = \frac{1}{\alpha} = \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$$

where  $\alpha$  is the limit of ratio of coefficients.

If  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = \alpha$  where  $\alpha \in \mathbb{R} - \{0\}$   
then

$$\sum_{n=0}^{\infty} c_n(x-a)^n \text{ converges} \Leftrightarrow |x-a| < \frac{1}{\alpha}$$

Since  $\frac{1}{\alpha}$  is radius of convergence.

So that  $x \in (a - \frac{1}{\alpha}, a + \frac{1}{\alpha})$  makes  $\sum_{n=0}^{\infty} c_n(x-a)^n$  converges. (By Ratio Test)

We have to check that endpoints of interval of convergence.

**Corollary 5.0.1.** *There are four cases for endpoints of interval of convergence.*

- $x = a - \frac{1}{\alpha}, a + \frac{1}{\alpha}$  diverges
- $x = a - \frac{1}{\alpha}$  diverges,  $x = a + \frac{1}{\alpha}$  converges
- $x = a - \frac{1}{\alpha}$  converges,  $x = a + \frac{1}{\alpha}$  diverges
- $x = a - \frac{1}{\alpha}, a + \frac{1}{\alpha}$  converges

If  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = 0$

Such that  $|x-a| \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| < 1$  converges (by Ratio Test)

So that,  $\sum_{n=0}^{\infty} c_n(x-a)^n$  converges to  $\forall x \in \mathbb{R}$

- Radius of Convergence:  $\infty$
- Interval of Convergence:  $(-\infty, \infty)$

If  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = +\infty$

Such that  $|x-a| \lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right| = \infty > 1$  diverges So that,  $\forall x \in \mathbb{R} - \{a\}$  diverges

- Radius of Convergence: 0
- Interval of Convergence:  $\{a\}$

$$\sum_{n=1}^{\infty} (-1)^n n x^n$$

(Ratio Test) Consider:  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} (n+1) x^{n+1}}{(-1)^n n x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-1)(n+1)(x)}{n} \right| \\ &= |x| \lim_{n \rightarrow \infty} \frac{n+1}{n} \\ &= |x| \cdot 1 \end{aligned}$$

Expected:  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &< 1 \\ |x| &< 1 \end{aligned}$$

Consider: When  $x = -1$

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^n n x^n &= \sum_{n=1}^{\infty} (-1)^n n \cdot (-1)^n \\ &= \sum_{n=1}^{\infty} (-1)^{2n} \cdot n \\ &= \sum_{n=1}^{\infty} n \quad (\text{diverges}) \end{aligned}$$

Consider: When  $x = 1$

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^n n x^n &= \sum_{n=1}^{\infty} (-1)^n n \cdot (1)^n \\ &= \sum_{n=1}^{\infty} (-1)^n \cdot n \quad (\text{diverges by Divergence Test}) \end{aligned}$$

- Radius of Convergence: 1
- Interval of Convergence:  $(-1, 1)$

## 5.1 Representation of Functions as Power Series

**Theorem 5.1.1** (Sum of Geometric Series).

$$\frac{1}{1-x} = 1 + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \text{ where } |x| < 1$$

Such that

$$\frac{1}{1-\square} = \sum_{n=0}^{\infty} \square^n \text{ where } |\square| < 1$$

Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{5}{1-4x^2}$$

1. Write in form of power series

$$\begin{aligned} \frac{5}{1-4x^2} &= 5 \left[ \frac{1}{1-4x^2} \right] \\ &= 5 \sum_{n=0}^{\infty} (4x^2)^n \end{aligned}$$

2. Find the radius of convergence

From 1.  $|4x^2| < 1$

$$|4x^2| < 1$$

$$|x^2| < \frac{1}{4}$$

$$|x| < \frac{1}{2}$$

Interval of Convergence:  $(-\frac{1}{2}, \frac{1}{2})$

Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{4}{2x+3}$$

1. Write in form of power series

$$\begin{aligned} \frac{4}{2x+3} &= 4 \left[ \frac{1}{2x+3} \right] \\ &= \frac{4}{3} \left[ \frac{1}{1 + \frac{2x}{3}} \right] \\ &= \frac{4}{3} \left[ \frac{1}{1 - \left(-\frac{2x}{3}\right)} \right] \\ &= \frac{4}{3} \sum_{n=0}^{\infty} \left( \frac{-2x}{3} \right)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{2^{n+2} \cdot x^n}{3^{n+1}} \end{aligned}$$

2. Find the radius of convergence

From 1.  $\left| \frac{-2x}{3} \right| < 1$

$$\begin{aligned} \left| \frac{-2x}{3} \right| &< 1 \\ |x| &< \frac{3}{2} \end{aligned}$$

Interval of Convergence:  $\left(-\frac{3}{2}, \frac{3}{2}\right)$

## 5.2 Differentiation and Integration of Power Series

**Theorem 5.2.1** (Differentiation of Power Series). If  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$  and  $f(x)$  is differentiable at  $x$ , then

$$f'(x) = \sum_{n=1}^{\infty} n \cdot c_n(x-a)^{n-1}$$

where  $a$  is center of power series.

**Theorem 5.2.2** (Integration of Power Series). If  $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$  and  $f(x)$  is integrable at  $x$ , then

$$\int f(x) dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$$

where  $C$  is constant and  $a$  is center of power series.

**Corollary 5.2.1.** *We can rewrite differentiation and integration of power series as*

$$\frac{d}{dx} \left[ \sum_{n=0}^{\infty} c_n(x-a)^n \right] = \sum_{n=0}^{\infty} \frac{d}{dx} [c_n(x-a)^n]$$

$$\int \left[ \sum_{n=0}^{\infty} c_n(x-a)^n \right] dx = \sum_{n=0}^{\infty} \int [c_n(x-a)^n] dx$$

Bessel function of order 0

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

(a) Find the domain of  $J_0(x)$

By Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} \cdot x^{2n+2}}{2^{n+1} \cdot ((n+1)!)^2} \cdot \frac{2^n \cdot (n!)^2}{(-1)^n x^{2n}} \right| < 1$$

$$\lim_{n \rightarrow \infty} \left| \frac{(-1) \cdot x^2}{2 \cdot (n+1)^2} \right| < 1$$

$$x^2 \cdot \lim_{n \rightarrow \infty} \left| \frac{1}{2(n+1)^2} \right| < 1$$

$$0 < 1$$

Since  $0 < 1$ ,  $J_0(x)$  converges to  $\forall x \in \mathbb{R}$

(b) Find the derivative of  $J_0(x)$

By Differentiation of Power Series

$$J'_0(x) = \sum_{n=1}^{\infty} n \cdot \frac{(-1)^n x^{2n-1}}{2^{2n-1} (n-1)!^2}$$

$$= \sum_{n=0}^{\infty} (n+1) \cdot \frac{(-1)^{n+1} x^{2n+1}}{2^{2n+1} n!^2}$$

**Lemma 5.2.1.**

$$\frac{1}{(1-\square)^2} = \sum_{n=1}^{\infty} n \square^{n-1} \text{ where } |\square| < 1$$

*Proof.* From Theorem 5.1.1

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$= (1-x)^{-1}$$

Differentiate both sides

$$\begin{aligned}\sum_{n=1}^{\infty} n \cdot x^{n-1} &= (1-x)^{-2} \\ &= \frac{1}{(1-x)^2}\end{aligned}$$

Such that,  $\frac{1}{(1-\square)^2} = \sum_{n=1}^{\infty} n \square^{n-1}$  where  $|\square| < 1$

□

**Lemma 5.2.2.**

$$-\ln(1-\square) = \sum_{n=0}^{\infty} \frac{\square^{n+1}}{n+1} \text{ where } |\square| < 1$$

*Proof.* From Theorem 5.1.1

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Integrate both sides

$$\begin{aligned}\sum_{n=0}^{\infty} \int x^n dx &= \int \frac{1}{1-x} dx \\ \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} &= -\ln(1-x)\end{aligned}$$

Such that,  $-\ln(1-\square) = \sum_{n=0}^{\infty} \frac{\square^{n+1}}{n+1}$  where  $|\square| < 1$

□

**Lemma 5.2.3.**

$$\ln(1+\square) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{\square^n}{n} \text{ where } |\square| < 1$$

*Proof.* From Lemma 5.2.2

$$\begin{aligned}-\ln(1-\square) &= \sum_{n=0}^{\infty} \frac{\square^{n+1}}{n+1} \\ &= \sum_{n=1}^{\infty} \frac{\square^n}{n}\end{aligned}$$

Such that,  $\ln(1+\square) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{\square^n}{n}$  where  $|\square| < 1$

□

**Lemma 5.2.4.**

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} \text{ where } |x| < 1$$



*Proof.* From  $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$  and Theorem 5.1.1

$$\begin{aligned} \frac{1}{1-(-x^2)} &= \sum_{n=0}^{\infty} (-x^2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n} \\ \int \frac{1}{1+x^2} dx &= \sum_{n=0}^{\infty} \int (-1)^n \cdot x^{2n} dx \\ \arctan x &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} \end{aligned}$$

Such that,  $\arctan x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$  where  $|x| < 1$  □

$$\pi = 2\sqrt{3} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$$

*Proof.* From lemma 5.2.4

$$\begin{aligned} \arctan \frac{1}{\sqrt{3}} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \left( \frac{1}{\sqrt{3}} \right)^{2n+1} \\ \frac{\pi}{6} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \frac{1}{3^n \sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\ \pi &= 2\sqrt{3} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)} \end{aligned}$$

Such that,  $\pi = 2\sqrt{3} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$  □

## 5.3 Taylor and Maclaurin Series

**Definition 5.3.1** (Taylor Series). Power series which begins from

$$f(x) = \begin{cases} \sum_{n=0}^{\infty} c_n(x-a)^n & a \neq 0 \text{ (TaylorSeries)} \\ \sum_{n=0}^{\infty} c_n x^n & a = 0 \text{ (MaclaurinSeries)} \end{cases}$$

**Theorem 5.3.1** (coefficient of Taylor Series). *If  $f$  has a power series representation (expansion) at  $a$ , that is, if*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

then its coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

Given  $x = a$  then

$$f(a) = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$$

$$f'(a) = c_1$$

$$f''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3c_4(x-a)^2 + \dots$$

$$f''(a) = 2c_2$$

$$f'''(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + 5 \cdot 4 \cdot 3c_5(x-a)^2 + \dots$$

$$f'''(a) = 3 \cdot 2c_3$$

...

So that,

$$c_n = \frac{f^{(n)}(a)}{n!}$$

**Corollary 5.3.1** (Taylor Series).

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\ &= f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots \end{aligned}$$

**Corollary 5.3.2** (Maclaurin Series).

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots \end{aligned}$$

**Lemma 5.3.1.**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ where } x \in \mathbb{R}$$

*Proof.* From Theorem 5.3.1

$$\begin{aligned}
 f(x) &= e^x \\
 f^{(n)}(x) &= e^x \\
 &= e^0 \\
 &= 1 \\
 c_n &= \frac{f^{(n)}(0)}{n!} \\
 &= \frac{1}{n!} \\
 f(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!}
 \end{aligned}$$

Such that,  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  where  $x \in \mathbb{R}$  □

**Theorem 5.3.2.** If  $f(x) = T_n(x) + R_n(x)$ , where  $T_n(x)$  is the  $n$ th Taylor polynomial and  $R_n(x)$  is the remainder, then

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

**Theorem 5.3.3** (Taylor's Inequality). If  $|f^{(n+1)}(x)| \leq M$  for  $|x - a| \leq d$ , then the remainder  $R_n(x)$  in the Taylor series satisfies

$$|R_n(x)| \leq \frac{M|x - a|^{n+1}}{(n+1)!}$$

$$R_n(x) = |f(x) - T_n(x)| \leq \frac{M}{(n+1)!} \cdot |x - a|^{n+1}$$

**Lemma 5.3.2.**

$$\begin{aligned}
 \sin x &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} \text{ where } x \in \mathbb{R} \\
 &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots
 \end{aligned}$$

**Lemma 5.3.3.**

$$\begin{aligned}
 \cos x &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} \text{ where } x \in \mathbb{R} \\
 &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots
 \end{aligned}$$

**Theorem 5.3.4** (Euler's Formula).

$$e^{ix} = \cos x + i \sin x$$

**Theorem 5.3.5** (Binomial Theorem).

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots$$

where  $k \in \mathbb{Q}^+$ ,  $n \in \mathbb{N}_0$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$= 1 + x + x^2 + x^3 + \dots$	$ x  < 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$x \in \mathbb{R}$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$	$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$x \in \mathbb{R}$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$	$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$x \in \mathbb{R}$
$\arctan x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$	$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$	$ x  < 1$
$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n}$	$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$	$ x  < 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$	$= 1 + kx + \frac{k(k-1)}{2!} x^2 + \dots$	$ x  < 1$

Find the Taylor series for  $f(x) = \cos x$ ,  $a = \pi/2$

Taylor Series form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Solution:

$$\begin{aligned}
 \cos x &= \cos \left[ \frac{\pi}{2} + \left( x - \frac{\pi}{2} \right) \right] \\
 &= \cos \frac{\pi}{2} \cdot \cos \left( x - \frac{\pi}{2} \right) - \sin \frac{\pi}{2} \cdot \sin \left( x - \frac{\pi}{2} \right) \\
 &= -\sin \left( x - \frac{\pi}{2} \right) \\
 &= -\sum_{n=0}^{\infty} \frac{(-1)^n \cdot \left[ x - \frac{\pi}{2} \right]^{2n+1}}{(2n+1)!} \quad ; x \in \mathbb{R}
 \end{aligned}$$