

2301108 CALCULUS II

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1 Differential Equation

1.1 Seperable Equation

Definition 1.1.1 (Seperable Equation). A differential equation is called seperable if it can be written in the form

$$\frac{dy}{dx} = g(y)h(x)$$

where $g(y)$ is a function of y only and $h(x)$ is a function of x only.

$$\frac{dy}{dx} = g(y)h(x)$$

$$\frac{dy}{g(y)} = h(x)dx$$

$$\int \frac{dy}{g(y)} = \int h(x)dx$$

Definition 1.1.2 (Implicit Solution). An implicit solution to a differential equation is a solution that is not solved for y explicitly. Answer is in the form of $F(x, y) = c$.

Definition 1.1.3 (Explicit Solution). An explicit solution to a differential equation is a solution that is solved for y explicitly. Answer is in the form of $y = f(x)$.

Sometimes there are initial condition of the equation.

Theorem 1.1.1 (FTC 1). If f is continuous on $[a, b]$, then the function f defined by

$$\frac{d}{dx} \int_a^{u(x)} f(t)dt = f(u(x))u'(x)$$

is continuous on $[a, b]$ and differentiable in (a, b) , and $g'(x) = f(x)$.

Definition 1.1.4 (Integral Equation). An integral equation is an equation in which an unknown function appears under an integral sign.

We have to change the integral equation to a differential equation. Then, we can use FTC 1 to solve the differential equation.

1.2 Linear Equation

Definition 1.2.1 (Linear Equation). A differential equation is called linear if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions of x .

The term “linear” refers to the fact that the unknown function y and its derivative $\frac{dy}{dx}$ appear in the equation to the first power and are not multiplied together.

Definition 1.2.2 (Integrating Factor). The integrating factor for the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is the function $I(x)$ defined by

$$I(x) = \exp\left(\int P(x)dx\right)$$

We can solve the linear differential equation by the formula.

$$y(x) = \frac{1}{I(x)}\left[\int I(x)Q(x)dx + C\right]$$

without any additional constraint c from integrating.

1.3 Bernoulli Equation

Definition 1.3.1 (Bernoulli Equation). A differential equation is called Bernoulli if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where $P(x)$ and $Q(x)$ are continuous functions of x .

As you can see, there is y^n in the equation which is not linear.

We have to change the Bernoulli equation to a linear equation by substitution.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

divides both side by y^n

$$\frac{dy}{dx}y^{-n} + P(x)y^{1-n} = Q(x)$$

Then we substitute $u = y^{1-n}$

$$\begin{aligned}\frac{du}{dy} &= (1-n)y^{-n} \frac{dy}{dx} \\ \frac{du}{dx} \cdot \frac{1}{1-n} &= y^{-n} \frac{dy}{dx}\end{aligned}\tag{1.1}$$

Let $u = y^{1-n}$

$$\begin{aligned}\frac{du}{dx} &= (1-n)y^{-n} \frac{dy}{dx} \\ \frac{du}{dx} \cdot \frac{1}{1-n} &= y^{-n} \frac{dy}{dx}\end{aligned}\tag{1.2}$$

Then, try to substitute the equation with u .

$$\begin{aligned}\frac{du}{dx} \cdot \frac{1}{1-n} + P(x)u &= Q(x) \\ \frac{du}{dx} + (1-n)P(x)u &= (1-n)Q(x)\end{aligned}\tag{1.3}$$

Now, we can solve the equation with the linear equation method. Since the equation is linear on u .

2 Sequence

2.1 Sequence

Definition 2.1.1 (Sequence). A sequence is a function whose domain is \mathbb{N}

We are considering behavior of a sequence as n becomes infinite.

$$\lim_{n \rightarrow \infty} a_n = \begin{cases} L \in \mathbb{R} & \text{convergent} \\ \pm\infty & \text{divergent} \\ \text{morethanonevalue} & \text{divergent} \end{cases}$$

Definition 2.1.2 (Sequence Notation). The sequence $\{a_1, a_2, a_3, \dots\}$ is denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Theorem 2.1.1 (Limit of Sequence). A sequence $\{a_n\}$ converges to L .

$$\lim_{n \rightarrow \infty} a_n = L$$

or

$$a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if for every $\epsilon > 0$, there exists N such that

$$\text{if } n > N$$

then

$$|a_n - L| < \epsilon$$

Theorem 2.1.2. If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ for all $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} a_n = L$$

Since our function is discrete, we can use the limit of the function to find the limit of the sequence.

Remark. We can use L'Hopital's rule to find the limit of the sequence.

If subsequences of a_n converge to different limits, then the sequence a_n diverges. For example

$$\begin{aligned} a_n &= (-1)^n \\ a_n &= 1, -1, 1, -1, \dots \end{aligned} \tag{2.1}$$

Consider: $a_{2k-1} = (-1)^{2k-1}; k \in \mathbb{N}$

$$\begin{aligned} a_{2k-1} &= (-1)^{2k-1} \\ a_{2k-1} &= -1 \end{aligned} \tag{2.2}$$

Consider: $a_{2k} = (-1)^{2k}; k \in \mathbb{N}$

$$\begin{aligned} a_{2k} &= (-1)^{2k} \\ a_{2k} &= 1 \end{aligned} \tag{2.3}$$

Since $\lim_{k \rightarrow \infty} a_{2k-1} = -1$ and $\lim_{k \rightarrow \infty} a_{2k} = 1$, the sequence a_n diverges.

Theorem 2.1.3 (Squeeze theorem). *Let $a_n \leq b_n \leq c_n$ for $n \geq n_0$. If*

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = L$$

then

$$\lim_{n \rightarrow \infty} b_n = L$$

Theorem 2.1.4. *If $\lim_{n \rightarrow \infty} |a_n| = 0$, then $\lim_{n \rightarrow \infty} a_n = 0$*

Proof. From $-|a_n| \leq a_n \leq |a_n|$, we have

$$\lim_{n \rightarrow \infty} |a_n| = 0$$

by the squeeze theorem, we have $\lim_{n \rightarrow \infty} a_n = 0$ □

Theorem 2.1.5. *If $\lim_{n \rightarrow \infty} a_n = L$ and the function f is continuous at L , then*

$$\lim_{n \rightarrow \infty} f(a_n) = f(L)$$

Remark. If the function is continuous,

$$\lim_{n \rightarrow \infty} f(a_n) = f(\lim_{n \rightarrow \infty} a_n)$$

The sequence $\{r^n\}$ is convergent if and only if $-1 < r \leq 1$.

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$$

2.2 Mathematical Induction

Definition 2.2.1 (Mathematical Induction). A proof technique used to prove a statement for all positive integers.

We can use mathematical induction to find the limit of a sequence. For example

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{n!} \quad (2.4)$$

Consider: $a_n \geq (\frac{3}{2})^n$ and $\lim_{n \rightarrow \infty} (\frac{3}{2})^{n-1} = \infty$ so $\lim_{n \rightarrow \infty} a_n = \infty$

Proof. Proof by Mathematical Induction

Let $P(n)$: $a_n \geq (\frac{3}{2})^n$ for $n \in \mathbb{N}$

Base Case: $n = 1$

$$a_1 = \frac{1}{1!} = 1$$
$$(\frac{3}{2})^{1-1} = (\frac{3}{2})^0 = 1$$

Since $a_1 \geq (\frac{3}{2})^0$, $P(1)$ is true.

Inductive Step: Assume $P(k)$ is true for $k \in \mathbb{N}$

$$a_k \geq a_k \cdot \frac{2(k+1)-1}{k+1} = a_k \cdot \frac{2k+1}{k+1}$$
$$(\frac{3}{2})^k = (\frac{3}{2})^{k-1} \cdot \frac{3}{2}$$

Consider

$$a_k \cdot \frac{2k+1}{k+1} \geq (\frac{3}{2})^{k-1} \cdot \frac{3}{2}$$

So $P(k) \rightarrow P(k+1)$ is true. Such that $P(n)$ is true for all $n \in \mathbb{N}$. □

2.3 Monotonic and Bounded Sequences

Our goal is to determine if a sequence is convergent or divergent. We can tell that a sequence is "convergent" but may not know the absolute value of the limit.

Definition 2.3.1 (Increasing Sequence). A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$.

Definition 2.3.2 (Decreasing Sequence). A sequence $\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all $n \geq 1$.

Definition 2.3.3 (Monotonic Sequence). A sequence $\{a_n\}$ is monotonic if it is either increasing or decreasing.

Remark. In this case, increasing and decreasing sequences are defined by "strictly" increasing and decreasing in order.

Definition 2.3.4 (Bounded Sequence). A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \leq M \text{ for } n \geq 1$$

A sequence $\{a_n\}$ is bounded below if there is a number m such that

$$a_n \geq m \text{ for } n \geq 1$$

If a sequence is bounded above and below, then it is called a bounded sequence.

Theorem 2.3.1 (Monotonic Sequence Theorem). *Every bounded, monotonic sequence is convergent.*

Remark. Techniques to check that the sequence is increasing or decreasing:

- $a_{n+1} - a_n$
 - If $a_{n+1} - a_n > 0$, then the sequence is increasing
 - If $a_{n+1} - a_n < 0$, then the sequence is decreasing
- $\frac{a_{n+1}}{a_n}$
 - If $\frac{a_{n+1}}{a_n} > 1$, then the sequence is increasing
 - If $\frac{a_{n+1}}{a_n} < 1$, then the sequence is decreasing
- Find $f'(n)$
 - If $f'(n) > 0$, then the sequence is increasing
 - If $f'(n) < 0$, then the sequence is decreasing

Example: Prove that the following sequence is convergent

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \text{ for } n \in \mathbb{N}$$

Proof. Since there are factorials in each term, we cannot use the derivative to determine if the sequence is increasing or decreasing.

1. Determine if the sequence is increasing or decreasing

$$\begin{aligned} a_{n+1} - a_n &= \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n+1)!}\right) - \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) \\ &= \frac{1}{(n+1)!} \end{aligned}$$

Since $\frac{1}{(n+1)!} > 0$, the sequence is increasing.

2. Determine if the sequence is bounded

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \cdots$$

$$\sum_{i=1}^n \frac{1}{i!} \leq \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{i-1} \leq \frac{1}{1 - \frac{1}{2}} = 2$$

So $1 \leq a_n \leq 2$. The sequence is bounded.

By theorem: $\{a_n\}$ is monotonic and bounded, so it is convergent. □

Theorem 2.3.2. *If there are 2 or more subsequences of $\{a_n\}$ which converges to different values then $\{a_n\}$ is divergent.*

For example: $\lim_{n \rightarrow \infty} |a_n| \neq 0$ and there is $(-1)^n$ in each term.

$$\cos(n\pi) = \begin{cases} a_{2k} = 1 & \text{if } n = 2k \\ a_{2k-1} = -1 & \text{if } n = 2k - 1 \end{cases}$$

A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.

(a) Find $\lim_{n \rightarrow \infty} a_n$

Let $L = \lim_{n \rightarrow \infty} a_n$

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \sqrt{2 + a_n} \\ L &= \sqrt{2 + L} \\ L^2 &= 2 + L \\ L^2 - L - 2 &= 0 \\ (L - 2)(L + 1) &= 0 \\ L &= 2, -1 \end{aligned}$$

Since under the square root, $a_n \geq 0$, so $L = 2$.

(b) Prove that $\{a_n\}$ is increasing and bounded above by 3. Then use the Monotonic Sequence Theorem to show that $\{a_n\}$ is convergent.

Proof. Let $P(n)$: $a_{n+1} > a_n$ and $a_n \leq 3$

Base Case: $n = 1$

$$a_1 = \sqrt{2} \text{ and } a_2 = \sqrt{2 + \sqrt{2}}$$

Since $\sqrt{2} < \sqrt{2 + \sqrt{2}}$ and $\sqrt{2} \leq 3$, $P(1)$ is true.

Inductive Step: Assume $P(k)$ is true for $k \in \mathbb{N}$

$$a_{k+1} > a_k \text{ and } a_k \leq 3$$

$$a_{k+2} = \sqrt{2 + a_{k+1}}$$

Since $a_{k+1} > a_k$, we have

$$\sqrt{2 + a_{k+1}} > \sqrt{2 + a_k}$$

From $a_k \leq 3$, we have

$$a_{k+1} = \sqrt{2 + a_k}$$

$$\sqrt{2 + a_k} \leq \sqrt{2 + 3}$$

$$a_{k+1} \leq \sqrt{5}$$

$$< \sqrt{9}$$

$$< 3$$

So $P(k) \rightarrow P(k+1)$ is true. Such that $P(n)$ is true for all $n \in \mathbb{N}$. □