2301108 CALCULUS II (Midterm Edition)

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Contents

1 Differential Equation

1.1 Seperable Equation

Definition 1.1.1 (Seperable Equation)**.** A differential equation is called seperable if it can be written in the form
$$\frac{dy}{dx} = g(y)h(x)$$
where $g(y)$ is a function of y only and $h(x)$ is a function of x only.

$$\begin{aligned} \frac{dy}{dx} &= g(y)h(x) \\ \frac{dy}{g(y)} &= h(x)dx \\ \int \frac{dy}{g(y)} &= \int h(x)dx \end{aligned}$$

Definition 1.1.2 (Implicit Solution)**.** An implicit solution to a differential equation is a solution that is not solved for y explicitly. Answer is in the form of $F(x, y) = c$.

Definition 1.1.3 (Explicit Solution)**.** An explicit solution to a differential equation is a solution that is solved for y explicitly. Answer is in the form of $y = f(x)$.

Sometimes there are initial condition of the equation.

Theorem 1.1.1 (FTC 1)**.** *If f is continuous on $[a, b]$, then the function f defined by*
$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = f(u(x))u'(x)$$
is continuous on $[a, b]$ and differentiable in (a, b), and g'(x) = f(x).

Definition 1.1.4 (Integral Equation)**.** An integral equation is an equation in which an unknown function appears under an integral sign.

We have to change the integral equation to a differential equation. Then, we can use FTC 1 to solve the differential equation.

1.2 Linear Equation

Definition 1.2.1 (Linear Equation)**.** A differential equation is called linear if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions of x.

The term "linear" refers to the fact that the unknown function y and its derivative $\frac{dy}{dx}$ appear in the equation to the first power and are not multiplied together.

Definition 1.2.2 (Integrating Factor)**.** The integrating factor for the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is the function $I(x)$ defined by

$$I(x) = \exp(\int P(x)dx)$$

We can solve the linear differential equation by the formula.

$$y(x) = \frac{1}{I(x)}[\int I(x)Q(x)dx + C]$$

without any additional constraint c from integrating.

1.3 Bernoulli Equation

Definition 1.3.1 (Bernoulli Equation)**.** A differential equation is called Bernoulli if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where $P(x)$ and $Q(x)$ are continuous functions of x.

As you can see, there is y^n in the equation which is not linear.

We have to change the Bernoulli equation to a linear equation by substitution.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

divides both side by y^n

$$\frac{dy}{dx}y^{-n} + P(x)y^{1-n} = Q(x)$$

Then we substitute $u = y^{1-n}$

$$\begin{aligned}\frac{du}{dy} &= (1-n)y^{-n}\frac{dy}{dx}\\ \frac{du}{dx}\cdot\frac{1}{1-n} &= y^{-n}\frac{dy}{dx}\end{aligned}$$

Let $u = y^{1-n}$

$$\begin{aligned}\frac{du}{dx} &= (1-n)y^{-n}\frac{dy}{dx}\\ \frac{du}{dx}\cdot\frac{1}{1-n} &= y^{-n}\frac{dy}{dx}\end{aligned}$$

Then, try to substitute the equation with u.

$$\begin{aligned}&\frac{du}{dx}\cdot\frac{1}{1-n} + P(x)u = Q(x)\\ &\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)\end{aligned}$$

Now, we can solve the equation with the linear equation method. Since the equation is linear on u.

2 Sequence

2.1 Sequence

Definition 2.1.1 (Sequence)**.** A sequence is a function whose domain is $\mathbb{N}$

We are considering behavior of a sequence as n becomes infinite.

$$\lim_{n\to\infty} a_n = \begin{cases} L \in \mathbb{R} & \text{convergent} \\ \pm\infty & \text{divergent} \\ morethanonevalue & \text{divergent} \end{cases}$$

Definition 2.1.2 (Sequence Notation)**.** The sequece $\{a_1, a_2, a_3, \dots\}$ is denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Theorem 2.1.1 (Limit of Sequence)**.** *A sequence $\{a_n\}$ converges to L.*

$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L \text{ as } n \to \infty$$

if for every $\epsilon > 0$, there exists N such that

$$\text{if } n > N$$

then

$$|a_n - L| < \epsilon$$

Theorem 2.1.2. *If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for all $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} a_n = L$$

Since our function is discrete, we can use the limit of the function to find the limit of the sequence.

Remark. We can use L'Hopital's rule to find the limit of the sequence.

Lemma 2.1.1. *If subsequences of a_n converge to different limits, then the sequence a_n diverges.*

$$a_n = (-1)^n$$
$$a_n = 1, -1, 1, -1, \ldots$$

Consider: $a_{2k-1} = (-1)^{2k-1}; k \in \mathbb{N}$

$$a_{2k-1} = (-1)^{2k-1}$$
$$a_{2k-1} = -1$$

Consider: $a_{2k} = (-1)^{2k}; k \in \mathbb{N}$

$$a_{2k} = (-1)^{2k}$$
$$a_{2k} = 1$$

Since $\lim_{k\to\infty} a_{2k-1} = -1$ and $\lim_{k\to\infty} a_{2k} = 1$, the sequence a_n diverges.

Theorem 2.1.3 (Squeeze theorem)**.** *Let $a_n \leq b_n \leq c_n$ for $n \geq n_0$.*
If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty c_n = L}$$

then

$$\lim_{n\to\infty} b_n = L$$

Theorem 2.1.4. *If* $\lim_{n\to\infty} |a_n| = 0$*, then* $\lim_{n\to\infty} a_n = 0$

Proof. From $-|a_n| \leq a_n \leq |a_n|$, we have

$$\lim_{n\to\infty} |a_n| = 0$$

by the squeeze theorem, we have $\lim_{n\to\infty} a_n = 0$ □

Theorem 2.1.5. *If* $\lim_{n\to\infty} a_n = L$ *and the function f is continuous at L, then*

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Remark. If the function is continuous,

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$$

The sequence $\{r^n\}$ is converget if and only if $-1 < r \leq 1$.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$$

2.2 Mathematical Induction

Definition 2.2.1 (Mathematical Induction)**.** A proof technique used to prove a statement for all positive integers.

We can use mathematical induction to find the limit of a sequence.

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{n!}$$

Consider: $a_n \geq (\frac{3}{2})^n$ and $\lim_{n\to\infty}(\frac{3}{2})^{n-1} = \infty$ so $\lim_{n\to\infty} a_n = \infty$

Proof. Proof by Mathematical Induction
Let $P(n)$: $a_n \geq (\frac{3}{2})^n$ for $n \in \mathbb{N}$
Base Case: n = 1

$$a_1 = \frac{1}{1!} = 1$$

$$(\frac{3}{2})^{1-1} = (\frac{3}{2})^0 = 1$$

Since $a_1 \geq (\frac{3}{2})^0$, $P(1)$ is true.
Inductive Step: Assume $P(k)$ is true for $k \in \mathbb{N}$

$$a_k \geq a_k \cdot \frac{2(k+1)-1}{k+1} = a_k \cdot \frac{2k+1}{k+1}$$

$$(\frac{3}{2})^k = (\frac{3}{2})^{k-1} \cdot \frac{3}{2}$$

Consider

$$a_k \cdot \frac{2k+1}{k+1} \geq (\frac{3}{2})^{k-1} \cdot \frac{3}{2}$$

So $P(k) \to P(k+1)$ is true. Such that $P(n)$ is true for all $n \in \mathbb{N}$. □

2.3 Monotonic and Bounded Sequences

Our goal is to determine if a sequence is convergent or divergent. We can tell that a sequence is "convergent" but may not know the absoulte value of the limit.

Definition 2.3.1 (Increasing Sequence)**.** A sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$.

Definition 2.3.2 (Decreasing Sequence)**.** A sequence $\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all $n \geq 1$.

Definition 2.3.3 (Monotonic Sequence)**.** A sequence $\{a_n\}$ is monotonic if it is either increasing or decreasing.

Remark. In this case, increasing and decreasing sequences are defined by "strictly" increasing and decreasing in order.

Definition 2.3.4 (Bounded Sequence)**.** A sequence $\{a_n\}$ is bounded above if there is a number M such that

$$a_n \leq M \text{ for } n \geq 1$$

A sequence $\{a_n\}$ is bounded below if there is a number m such that

$$a_n \geq m \text{ for } n \geq 1$$

If a sequence is bounded above and below, then it is called a bounded sequence.

Theorem 2.3.1 (Monotonic Sequence Theorem)**.** *Every bounded, monotonic sequence is convergent.*

Remark. Techniques to check that the sequence is increasing or decreasing:

- $a_{n+1} - a_n$
  - If $a_{n+1} - a_n > 0$, then the sequence is increasing
  - If $a_{n+1} - a_n < 0$, then the sequence is decreasing
- $\frac{a_{n+1}}{a_n}$
  - If $\frac{a_{n+1}}{a_n} > 1$, then the sequence is increasing
  - If $\frac{a_{n+1}}{a_n} < 1$, then the sequence is decreasing
- Find $f'(n)$
  - If $f'(n) > 0$, then the sequence is increasing
  - If $f'(n) < 0$, then the sequence is decreasing

Prove that the following sequence is convergent

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \text{ for } n \in \mathbb{N}$$

Proof. Since there are factorials in each term, we cannot use the derivative to determine if the sequence is increasing or decreasing.
1. Determine if the sequence is increasing or decreasing

$$\begin{aligned} a_{n+1} - a_n &= (1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n+1)!}) - (1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}) \\ &= \frac{1}{(n+1)!} \end{aligned}$$

Since $\frac{1}{(n+1)!} > 0$, the sequence is increasing.
2. Determine if the sequence is bounded

$$a_n = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \ldots$$

$$\sum_{i=1}^{n} \frac{1}{i!} \leq \sum_{i=1}^{\infty} (\frac{1}{2})^{i-1} \quad \leq \frac{1}{1 - \frac{1}{2}} = 2$$

So $1 \leq a_n \leq 2$. The sequence is bounded.
By thorem: $\{a_n\}$ is monotonic and bounded, so it is convergent. □

Theorem 2.3.2. *If there are 2 or more subsequences of $\{a_n\}$ which converges to different values then $\{a_n\}$ is divergent.*

$\lim_{n\to\infty} |a_n| \neq 0$ and there is $(-1)^n$ in each term.

$$\cos{(n\pi)} = \begin{cases} a_{2k} = 1 & \text{if } n = 2k \\ a_{2k-1} = -1 & \text{if } n = 2k - 1 \end{cases}$$

A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.
(a) Find $\lim_{n\to\infty} a_n$
Let $L = \lim_{n\to\infty} a_n$

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \sqrt{2 + a_n} \\
L &= \sqrt{2 + L} \\
L^2 &= 2 + L \\
L^2 - L - 2 &= 0 \\
(L-2)(L+1) &= 0 \\
L &= 2, -1
\end{aligned}$$

Since under the square root, $a_n \geq 0$, so $L = 2$.
(b) Prove that $\{a_n\}$ is increasing and bounded above by 3. Then use the Monotonic Sequence Theorem to show that $\{a_n\}$ is convergent.

Proof. Let $P(n)$: $a_{n+1} > a_n$ and $a_n \leq 3$
Base Case: n = 1

$$a_1 = \sqrt{2} \text{ and } a_2 = \sqrt{2 + \sqrt{2}}$$

Since $\sqrt{2} < \sqrt{2 + \sqrt{2}}$ and $\sqrt{2} \leq 3$, $P(1)$ is true.
Inductive Step: Assume $P(k)$ is true for $k \in \mathbb{N}$

$$a_{k+1} > a_k \text{ and } a_k \leq 3$$

$$a_{k+2} = \sqrt{2 + a_{k+1}}$$

Since $a_{k+1} > a_k$, we have

$$\sqrt{2 + a_{k+1}} > \sqrt{2 + a_k}$$

From $a_k \leq 3$, we have

$$\begin{aligned}
a_{k+1} &= \sqrt{2 + a_k} \\
\sqrt{2 + a_k} &\leq \sqrt{2 + 3} \\
a_{k+1} &\leq \sqrt{5} \\
&< \sqrt{9} \\
&< 3
\end{aligned}$$

So $P(k) \to P(k+1)$ is true. Such that $P(n)$ is true for all $n \in \mathbb{N}$. □

3 Series

3.1 Series

Definition 3.1.1 (Series)**.** Series is the sum of the terms of a sequence.

$$a_1 + a_2 + a_3 + \cdots + a_n$$

Definition 3.1.2 (Infinite Series)**.** An infinite series is the sum of the terms of an infinite sequence.

$$a_1 + a_2 + a_3 + \cdots + a_n + \ldots$$

In this chapter we will try testing the convergence of the series.

Definition 3.1.3. Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} = a_1 + a_2 + \cdots + a_n$$

Lemma 3.1.1.

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

Sometimes s_n is expressing in telescoping sum form.

$$\begin{aligned}
\sum_{n=1}^{\infty} \ln \frac{n}{n+1} &= \sum_{n=1}^{\infty} \ln n - \ln(n+1) \\
&= \lim_{n\to\infty} (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + (\ln n - \ln(n+1)) \\
&= \lim_{n\to\infty} \ln 1 - \ln(n+1) \\
&= \lim_{n\to\infty} \ln \frac{1}{n+1} \\
&= \ln 0 \\
&= -\infty
\end{aligned}$$

Such that, $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ is divergent.

$$\begin{aligned}\sum_{n=1}^{\infty}(e^{1/n} - e^{1/(n+1)}) &= \sum_{n=1}^{\infty}(\sqrt[n]{e} - \sqrt[n+1]{e}) \\ &= \lim_{n\to\infty}(e - \sqrt{e}) + (\sqrt{e} - \sqrt[3]{e}) + (\sqrt[3]{e} - \sqrt[4]{e}) + \cdots + (\sqrt[n]{e} - \sqrt[n+1]{e}) \\ &= \lim_{n\to\infty} e - \sqrt[n+1]{e} \\ &= e - e^0 \\ &= e - 1\end{aligned}$$

Such that, $\sum_{n=1}^{\infty}(e^{1/n} - e^{1/(n+1)})$ converges to $e - 1$.

3.2 Geometric Series

Definition 3.2.1.

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \text{ where } a \neq 0$$

is defined as a geometric series which a is the first term and r is the common ratio.

$$r = \frac{a_{n+1}}{a_n},\ n \geq 1$$

Lemma 3.2.1. *If $|r| \neq 1$, we have*

$$\begin{aligned}s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ r \cdot s_n &= a_r + ar^2 + ar^3 + \cdots + ar^n \\ s_n - r \cdot s_n &= a - ar^n \\ s_n &= \frac{a(1 - r^n)}{1 - r}\end{aligned}$$

Lemma 3.2.2. *The geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1} + \ldots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}$$

If $|r| \geq 1$, the series is divergent.

Proof. If $|r| < 1$, we have

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} \\
&= \frac{a}{1-r} - \frac{a}{1-r} \cdot \lim_{n\to\infty} r^n \\
&= \frac{a}{1-r} - \frac{a}{1-r} \cdot 0 \\
&= \frac{a}{1-r}
\end{aligned}$$

If $|r| \geq 1$, we have $\lim_{n\to\infty} r^n = \infty$, such that the series is divergent. $\square$

4 Convergence Test

4.1 Test for Divergence

Theorem 4.1.1 (Test for Divergence)**.** *If the series* $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$

Corollary 4.1.1. *If* $\lim_{n\to\infty} a_n$ *does not exist or* $\lim_{n\to\infty} a_n \neq 0$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Lemma 4.1.1. *If* $\sum a_n$ *diverges and* b_n *converges, then* $\sum(a_n + b_n)$ *will be divergent.*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^n + 4^n}{e^n} &= \sum_{n=1}^{\infty} \frac{2^n}{e^n} + \sum_{n=1}^{\infty} \frac{4^n}{e^n} \\ &= \sum_{n=1}^{\infty} (\frac{2}{e})^n + (\frac{4}{e})^n\end{aligned}$$

Since

- $\frac{2}{e} < 1$ so $\sum_{n=1}^{\infty} (\frac{2}{e})^n$ is convergent
- $\frac{4}{e} > 1$ so $\sum_{n=1}^{\infty} (\frac{4}{e})^n$ is divergent

Therefore, $\sum_{n\to\infty} \frac{2^n+4^n}{e^n}$ is divergent.

4.2 Integral Test

Theorem 4.2.1 (Integral Test)**.** *Let* $f(x)$ *be a continuous, positive, and decreasing function on* $[1, \infty)$*. Then the series* $\sum_{n=1}^{\infty} f(n)$ *is convergent if and only if the improper integral* $\int_1^{\infty} f(x)\,dx$ *is convergent.*

- *If* $int_1^{\infty} f(x)\,dx$ *is convergent, then* $\sum_{n=1}^{\infty} a_n$ *is convergent*
- *If* $int_1^{\infty} f(x)\,dx$ *is divergent, then* $\sum_{n=1}^{\infty} a_n$ *is divergent*

Lemma 4.2.1 (P-series)**.** *The series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if* $p > 1$ *and divergent if* $p \leq 1$*.*

$$\sum_{n=1}^{\infty} \frac{n}{n^4+1}$$

$$\begin{aligned}
f(x) &= \frac{x}{x^4+1} \\
F(x) &= \int \frac{x}{x^4+1}\,dx \\
&= \lim_{t\to\infty} \int_1^t \frac{x}{x^4+1}\,dx
\end{aligned}$$

Let $u = x^2$, then $du = 2x\,dx$

$$\begin{aligned}
\lim_{t\to\infty} \int_1^t \frac{x}{x^4+1}\,dx &= \lim_{t\to\infty} \int_{x=1}^{x=t} \frac{x}{u^2+1} \cdot \frac{du}{2x} \\
&= \lim_{t\to\infty} \frac{1}{2} \int_1^{\infty} \frac{1}{u^2+1}\,du \\
&= \lim_{t\to\infty} \frac{1}{2} \arctan u \Big|_1^t \\
&= \lim_{t\to\infty} \frac{1}{2} \arctan t - \frac{1}{2} \arctan 1 \\
&= \frac{\pi}{4} - \frac{\pi}{8} \\
&= \frac{\pi}{8}
\end{aligned}$$

$$\sum_{n=1}^{\infty} n(1+n^2)^p$$

$$\begin{aligned}
f(x) &= x(1+x^2)^p \\
F(x) &= \int_1^{\infty} x(1+x^2)^p\,dx \\
&= \lim_{t\to\infty} \int_1^t x(1+x^2)^p\,dx
\end{aligned}$$

Let $u = 1 + x^2$, then $du = 2x\,dx$

$$\begin{aligned}
\lim_{t\to\infty} \int 1^t x(1+x^2)^p\,dx &= \lim_{t\to\infty} \int_2^{\infty} x \cdot u^p \frac{du}{2x} \\
&= \lim_{t\to\infty} \frac{1}{2} \int_2^{\infty} u^p\,du \\
&= \lim_{t\to\infty} \frac{1}{2} \frac{u^{p+1}}{p+1} \Big|_2^{\infty}
\end{aligned}$$

Since $p \neq -1$, then $p + 1 > 0$. Therefore, the integral is divergent.

4.3 Comparison Test

Theorem 4.3.1 (Comparison Test)**.** *Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.*

- *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is convergent.*
- *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.*

In using the comparison test we must have some known series $\sum b_n$ for the purpose of comparison. Most of time we use one of these series:

- P-series [$\sum 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$]
- Geometric series [$\sum ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$]

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$$

$$\text{Let } a_n = \frac{1}{\sqrt{n-1}}$$

$$\text{Let } b_n = \frac{1}{\sqrt{n}}$$

$$a_n \geq b_n \text{ for all } n$$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \leq \sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$$

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent, then $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$ is divergent.

4.4 Limit Comparison Test

Theorem 4.4.1 (Limit Comparison Test)**.** *Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.*
If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where L is a finite positive number and $L > 0$, then either both series converge or both diverge.

$$\sum_{n=1}^{\infty} \sin^2 (\frac{1}{n})$$

Choose $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{sin^2(\frac{1}{n})}{\frac{1}{n^2}} = \lim_{n\to\infty} (\frac{\sin^2 \frac{1}{n}}{\frac{1}{n}})^2$$

Since $\lim_{n\to\infty} \frac{\sin x}{x} = 1$, then

$$\lim_{n\to\infty} (\frac{\sin^2 \frac{1}{n}}{\frac{1}{n}})^2 = 1^2 = 1 > 0$$

Since $\sum_{n=1} \infty \frac{1}{n^2}$ converges. Then, by limit comparison test $\sum_{n=1}^{\infty} \sin^2 (\frac{1}{n})$ converges.

Show that if $a_n > 0$ and $\lim_{n\to\infty} n \cdot a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof. Since $\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = L$ where $L > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. By limit comparison test, $\sum_{n=1}^{\infty} a_n$ is divergent. □

4.5 Alternating Series Test

Definition 4.5.1 (Alternating Series)**.** An alternating series is a series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

where $b_n > 0$ for all n.

Theorem 4.5.1 (Alternating Series Test)**.** *If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ satisfies*

- $b_n \geq b_{n+1}$ *for all* n
- $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$$

Let $b_n = \left|\frac{(-1)^{n+1}}{\sqrt{n+1}}\right| = \frac{1}{\sqrt{n+1}}$ 1. Determine if the series is decreasing
From:

$$\sqrt{n+1} \leq \sqrt{(n+1)+1}$$
$$\frac{1}{\sqrt{n+1}} \geq \frac{1}{\sqrt{n+2}}$$
$$b_n \geq b_{n+1}$$

Since $b_n \geq b_{n+1}$, the series is decreasing.
2. Determine if limit of b_n is 0

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$$

Since the series is decreasing and the limit of b_n is 0, the series is convergent.
Obervation:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \left|\frac{(-1)^{n+1}}{\sqrt{n+1}}\right|$$

diverges.

There are 3 ways to determine if an alternating series is convergent or divergent:

- $\sum_{n=1}^{\infty}(-1)^n \cdot b_n$ converges but $\sum_{n=1}^{\infty} |(-1)^n \; cdotb_n|$ diverges
- $\sum_{n=1}^{\infty}(-1)^n \cdot b_n$ converges and $\sum_{n=1}^{\infty} |(-1)^n \; cdotb_n|$ converges (by comparison test)
- $\sum_{n=1}^{\infty}(-1)^n \cdot b_n$ diverges and $\sum_{n=1}^{\infty} |(-1)^n \; cdotb_n|$ diverges

Estimating the Sum of an Alternating Series

A partial sum s_n of any convergent series can be used as an approximation to the total sum s. The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$.

Theorem 4.5.2 (Alternating Series Estimation Theorem). *If $s = \sum_{n=1}^{\infty}(-1)^{n-1} \cdot b_n$, where $b_n > 0$, is the sum of an alternating series that satisfies*

- $b_n \geq b_{n+1}$ *for all n*
- $\lim_{n\to\infty} b_n = 0$ *(converges)*

then $|R_n| = |s - s_n| \leq b_{n+1}$

Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.
Determine the series is convergent by the alternating series test. 1. Determine if the series is decreasing

$$\begin{aligned} b_{n+1} &= \frac{1}{(n+1)!} \\ &= \frac{1}{n!(n+1)} \qquad < b_n \\ &\qquad\qquad < \frac{1}{n!} \end{aligned}$$

2. Determine if limit of b_n is 0

$$\lim_{n\to\infty} \frac{1}{n!} = 0$$

Such that the series is convergent.
Use the alternating series estimation theorem to find the sum of the series correct to three decimal places.

$$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots \quad = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \dots$$

Note: $b_7 = \frac{1}{5040} \approx \frac{1}{5000} = 0.0002$

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$$

By alternating series estimation theorem, $|s - s_6| \le b_7 = 0.0002$
Therefore, the sum of the series is $s \approx 0.368$

Remark. For some $\sum_{n=1}^{\infty}(-1)^n \cdot b_n$ converges but sometimes

- $\sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$ diverges (most of the time)
- $\sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$ converges

$\sum_{n=1}^{\infty}(-1)^n \cdot b_n$ which is convergent relates with $\sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$ which is divergent.

Lemma 4.5.1. *If we know that* $\pm\sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$ *converges, then* $\sum_{n=1}^{\infty}(-1)^n \cdot b_n$ *also converges.*

Proof.

$$-\sum_{n=1}^{\infty} |(-1)^n \cdot b_n| \le \sum_{n=1}^{\infty}(-1)^n \cdot b_n \le \sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$$

Since we know that $\pm\sum_{n=1}^{\infty} |(-1)^n \cdot b_n|$ converges. Then by squeeze theorem, $\sum_{n=1}^{\infty}(-1)^n \cdot b_n$ also converges. □

4.6 Absolute Convergence and Conditional Convergence

Definition 4.6.1 (Absolute Convergence)**.** A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of the absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Definition 4.6.2 (Conditional Convergence)**.** A series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if the series is convergent but not absolutely convergent; that is, if $\sum_{n=1}^{\infty} a_n$ is convergent but $\sum_{n=1}^{\infty} |a_n|$ is divergent.

Theorem 4.6.1. *If a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.*

$$\sum_{n=1}^{\infty} \sin n$$

Consider:

$$\sum_{n=1}^{\infty} |\sin n| = |\sin 1| + |\sin 2| + |\sin 3| + \dots$$

There are two cases

- If $\sum_{n=1}^{\infty} |\sin n|$ converges, then $\sum_{n=1}^{\infty} \sin n$ is convergent
- If $\sum_{n=1}^{\infty} |\sin n|$ diverges, then we cannot conclude that $\sum_{n=1}^{\infty} \sin n$ is divergent

In cases where $\sum_{n=1}^{\infty} |a_n|$ diverges but $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{3n+2}$$

Since $\cos n\pi = (-1)^n$, then

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{3n+2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$$

Consider:

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{3n+2} \right| = \sum_{n=1}^{\infty} \frac{1}{3n+2}$$

$$\lim_{n\to\infty} \frac{\frac{1}{3n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3} > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges from limit comparison test. Such that $\sum_{n=1}^{\infty} \frac{1}{3n+2}$ diverges.
Consider: $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$ which is an alternating series where $b_n = \frac{1}{3n+2}$
Let $f(x) = \frac{1}{3x+2}$

$$f'(x) = \frac{-3}{(3x+2)^2} < 0$$

So b_n is decreasing.

$$\lim_{n\to\infty} \frac{1}{3n+2} = 0$$

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$ is convergent.
Since $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$ is convergent and $\sum_{n=1}^{\infty} \frac{1}{3n+2}$ diverges, then $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$ is conditionally convergent.

4.7 Ratio Test

Remark. This method might be good for series with factorials.

This test is very useful in determining whether a given series is absolutely convergent.

Theorem 4.7.1 (Ratio Test)**.** *There are three possible outcomes for the series $\sum_{n=1}^{\infty} a_n$:*

- *If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.*
- *If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.*
- *If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the test is inconclusive.*

$$\sum_{n=1}^{\infty} \frac{n!}{100!}$$

Consider:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{100^{n+1}}}{\frac{n!}{100^n}} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)! \cdot 100^n}{100^{n+1} \cdot n!} \right| \\
&= \lim_{n\to\infty} \frac{n+1}{100}
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{n+1}{100} = \infty$, then the series $\sum_{n=1}^{\infty} \frac{n!}{100!}$ is divergent.

Remark. Above example shows that factorial is bigger than exponential.

4.8 Root Test

Theorem 4.8.1 (Root Test)**.** *There are three possible outcomes for the series* $\sum_{n=1}^{\infty} a_n$*:*

- *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*
- *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ *or* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*
- *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$*, then the test is inconclusive.*

$$\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n} \right)^n$$

Consider:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{1-n}{2+3n} \right)^n \right|} &= \lim_{n\to\infty} \left| \frac{1-n}{2+3n} \right| \\
&= \lim_{n\to\infty} \frac{n-1}{2+3n} \qquad = \frac{1}{3} \quad < 1
\end{aligned}$$

By root test: $\sum_{n=1}^{\infty} \left(\frac{1-n}{2+3n} \right)^n$ is absolutely convergent.

$$\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1} \right)^{5n}$$

Consider:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{-2n}{n+1} \right)^{5n} \right|} &= \lim_{n\to\infty} \left(\frac{2n}{n+1} \right)^5 \\
&= 2^5 \qquad\qquad > 1
\end{aligned}$$

By root test: $\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1} \right)^{5n}$ is divergent.

4.9 Strategy for Testing Series

1. **Test for divergence** $\lim_{n\to\infty} a_n \neq 0 \to$ divergent.

2. **P-series** $\frac{1}{n^p}$ converges if and only if $p > 1$.

3. **Geometric series**

   - find r

   - converges if $|r| < 1$ else diverges

4. **Comparison tests**

   - $a_n \leq b_n$ and b_n converges such that a_n converges $\equiv a_n \leq b_n$ and a_n diverges such that b_n diverges

   - Limit comparison
     If
     $$\lim_{n\to\infty} \frac{a_n}{b_n} = C > 0$$
     then
     $$\sum_{n=1}^{\infty} a_n \text{ converges} \Leftrightarrow \sum_{n=1}^{\infty} b_n \text{ converges} \equiv \sum_{n=1}^{\infty} a_n \text{ diverges} \Leftrightarrow \sum_{n=1}^{\infty} b_n \text{ diverges}$$

5. **Alternating series test**

   - Find real b_n and $\operatorname{sgn}(a_n)$

   - Determine if b_n is decreasing $b_{n+1} \leq b_n$

   - Determine if $\lim_{n\to\infty} b_n = 0$

   If satisfies all conditions, then the series is convergent.

6. **Ratio test**
   $$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \begin{cases} < 1 & \text{absolutely convergent} \\ > 1 & \text{divergent} \\ = 1 & \text{inconclusive} \end{cases}$$

7. **Root test**
   $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \begin{cases} < 1 & \text{absolutely convergent} \\ > 1 & \text{divergent} \\ = 1 & \text{inconclusive} \end{cases}$$

8. **Integral test**
If $f(x)$ is continuous, positive, and decreasing on $[1, \infty)$, then

$$\int_1^\infty f(x)\,dx \text{ converges} \Leftrightarrow \sum_{n=1}^\infty f(n) \text{ converges}$$

5 Power Series

We can write polynomial in form $P(x) = \sum_{i=0}^{n} c_i \cdot x^i$ where c_i is coefficient and x is variable.

As you can see, there is a stop position since there is n in summation.

So, we can write power series in form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

where x is variable, c_n is coefficient.

Theorem 5.0.1 (Power Series)**.** *If $\sum_{n=0}^{\infty} c_n x^n$ converges, then we can write it as*

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_n x^n + \dots$$

How to check that $\sum_{n=0}^{\infty} c_n x^n$ converges? 1. Ratio test

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{c_{n+1} x^{n+1}}{c_n \cdot x^n} \right| &= \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \cdot x \right| \\ &= |x| \cdot \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| \quad < 1\end{aligned}$$

Let $\alpha = |x| \cdot \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| \neq 0$ such that α converges when $|x| < \frac{1}{\alpha}$ 2. Root test

$$\lim_{n\to\infty} \sqrt[n]{|c_n x^n|} = |x| \cdot \lim_{n\to\infty} \sqrt[n]{|c_n|} < 1$$

Let $\alpha = |x| \cdot \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| \neq 0$ such that α converges when $|x| < \frac{1}{\alpha}$