2301108 CALCULUS II (Final Edition)

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2024

Contents

1 Partial Derivatives

1.1 Functions of Several Variables

Definition 1.1.1 (Function of Two Variables)**.** A function f of two variables is a rule that assigns to each ordered pair (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and the set of all possible values of $f(x, y)$ is the range of f.

Definition 1.1.2 (Graphs)**.** If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in space where $z = f(x, y)$.

Definition 1.1.3 (Level Curves)**.** The level curves of a function f of two variables are the curves with equations $f(x, y) = k$ where k is a constant (in range of f).

1.2 Limits and Continuity

Limits of Functions of Two Variables

Limits of functions of two variables are written as

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

Definition 1.2.1 (Limit of a Function of Two Variables)**.** Let f be a function of two variables defined on some open region that includes points arbitrarily close to (a, b) except possibly at (a, b) itself. Then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x, y) - L| < \epsilon$$

whenever (x, y) is in the domain of f and satisfies

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$$

Theorem 1.2.1 (Limit Existance)**.** *If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, then $L_1 = L_2$.*

Corollary 1.2.1 (Limit not Exists)**.** *If $f(x,y)$ approaches different values as $(x,y) \to (a,b)$ along different paths, then the limit does not exist.*

Show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2}$$

Consider path C_1 where $y = 0$

$$\begin{aligned}
\lim_{(x,y) to (0,0)} \frac{2xy}{x^2+3y^2} &= \lim_{x\to 0} \frac{2x\cdot 0}{x^2+3\cdot 0^2} \\
&= \lim_{(x,y)\to(0,0)} 0 \\
&= 0
\end{aligned}$$

Consider path C_2 where $x = y$

$$\begin{aligned}
\lim_{(x,y) to (0,0)} \frac{2xy}{x^2+3y^2} &= \lim_{x\to 0} \frac{2x^2\cdot 0}{4x^2} \\
&= \lim_{(x,y)\to(0,0)} \frac{2}{4} \\
&= \frac{1}{2}
\end{aligned}$$

Since the limit approaches different values along different paths, the limit does not exist.

Properties of Limits

Sum Law Limit of a sum is the sum of limits.
Difference Law Limit of a difference is the difference of limits.
Constant Multiple Law Limit of a constant multiple is the constant multiple of the limit.
Product Law Limit of a product is the product of limits.
Quotient Law Limit of a quotient is the quotient of limits.

Use the Squeeze Theorem to find the limit.

$$\lim_{(x,y)\to(0,0)} xy\sin\frac{1}{x^2+y^2}$$

By the Squeeze Theorem, we have

$$\begin{aligned} -1 \le \sin\frac{1}{x^2+y^2} &\le 1 \\ -xy \le xy\sin\frac{1}{x^2+y^2} &\le xy \end{aligned}$$

Since

$$\lim_{(x,y)\to(0,0)} -xy = 0$$

and

$$\lim_{(x,y)\to(0,0)} xy = 0$$

By the Squeeze Theorem, the limit is 0.

Use the Squeeze Theorem to find the limit.

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4}$$

By the Squeeze Theorem, we have

$$\begin{aligned} y^4 &\le x^4+y^4 \\ \frac{y^4}{x^4+y^4} &\le 1 \\ -1 \le \frac{y^4}{x^4+y^4} &\le 1 \\ -x \le \frac{xy^4}{x^4+y^4} &\le x \end{aligned}$$

Since

$$\lim_{(x,y)\to(0,0)} -x = 0$$

and

$$\lim_{(x,y)\to(0,0)} x = 0$$

By the Squeeze Theorem, the limit is 0.

Continuity

Definition 1.2.2 (Continuity)**.** A function f of two variables is continuous at a point (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

Determine the set of points at which the function is continuous.

$$f(x, y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Consider

$$\begin{aligned} y^2 &\leq 2x^2 + y^2 \\ \frac{y^2}{2x^2 + y^2} &\leq 1 \\ -x^2y \leq \frac{x^2y^3}{2x^2 + y^2} &\leq x^2y \end{aligned}$$

Since

$$\lim_{(x,y)\to(0,0)} -x^2y = 0$$

and

$$\lim_{(x,y)\to(0,0)} x^2y = 0$$

By the Squeeze Theorem, the limit is 0 but the function is not continuous at $(0, 0)$.

1.3 Partial Derivatives

Partial Derivatives of Functions of Two Variables

Definition 1.3.1 (Partital Derivatives)**.** Partial derivative of f with respect to x at (a, b) is denoted by

$$f_x(a, b) = g'(a)$$

where

$$g(x) = f(x, b)$$

By the definition of derivative,

$$g'(a) = \lim_{h\to 0} \frac{g(a + h) - g(a)}{h}$$

and so it becomes

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Similarly, the partial derivative of f with respect to y at (a, b) is denoted by

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$$

Remark. While finding partial derivatives, treat the other variable as a constant.

Find the first partital derivatives

$$u(r, \theta) = \sin(r \cdot \cos\theta)$$

With respect to r

$$\begin{aligned} u_r &= \frac{\partial}{\partial r} \sin(r \cdot \cos\theta) \\ &= \cos(r \cdot \cos\theta) \cdot \cos\theta \end{aligned}$$

With respect to θ

$$\begin{aligned} u_\theta &= \frac{\partial}{\partial \theta} \sin(r \cdot \cos\theta) \\ &= -r \cdot \sin(r \cdot \cos\theta) \cdot \sin\theta \end{aligned}$$

1.4 Tangent Planes and Linear Approximations

Formula (Tangent Plane)**.** *Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is*

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Formula (Linear Approximation)**.** *The linear approximation to the function f at the point (a, b) is the linear function*

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Differentials

Definition 1.4.1 (Diffetentials)**.**

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Find the differential of the function.

$$z = x \ln(y^2 + 1)$$

From

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Consider z_x

$$\begin{aligned} z_x &= \frac{\partial}{\partial x} x \ln(y^2 + 1) \\ &= \ln(y^2 + 1) \end{aligned}$$

Consider z_y

$$\begin{aligned} z_y &= \frac{\partial}{\partial y} x \ln(y^2 + 1) \\ &= \frac{2xy}{y^2 + 1} \end{aligned}$$

The differential is

$$dz = \ln(y^2 + 1) dx + \frac{2xy}{y^2 + 1} dy$$

1.5 Higher Derivatives

If f is a function of two variables, then the second partial derivatives are defined as follows:

$$\begin{aligned} (f_x)_x = f_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial x^2} \\ (f_x)_y = f_{xy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial y \partial x} \\ (f_y)_x = f_{yx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial x \partial y} \\ (f_y)_y = f_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial y^2} \end{aligned}$$

Theorem 1.5.1 (Claireaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b) and that the functions f_{xy} and f_{yx} are continuous on D. Then*

$$f_{xy} = f_{yx}$$

Find all the second partial derivatives.

$$f(x,y) = \ln(ax+by)$$

Find first partial derivatives

$$\begin{aligned} f_x &= \frac{1}{ax+by} \cdot a \\ &= \frac{a}{ax+by} \\ f_y &= \frac{1}{ax+by} \cdot b \\ &= \frac{b}{ax+by} \end{aligned}$$

1.6 The Chain Rule

Formula (The Chain Rule (Case 1))**.** *Suppose that $z = f(x,y)$ is differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable function of t and*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Use the Chain Rule to find $\frac{dz}{dt}$.

$$z = \frac{x-y}{x+2y}, x = e^{\pi t}, y = e^{-\pi t}$$

Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$

$$\begin{aligned} \frac{dz}{dx} &= \frac{1 \cdot (x+2y) - (x-y) \cdot 1}{(x+2y)^2} \\ &= \frac{3y}{(x+2y)^2} \\ \frac{dz}{dy} &= \frac{-1 \cdot (x+2y) - (x-y) \cdot 2}{(x+2y)^2} \\ &= \frac{-3x}{(x+2y)^2} \end{aligned}$$

Formula (The Chain Rule (Case 2))**.** *Suppose that $z = f(x,y)$ is differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$ are both differentiable function of s and t and*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

2 Multiple Integrals

2.1 Double Integrals over Rectangles

Volume and Double Integrals

In a similar way to how we defined the definite integral of a function of one variable as the limit of the sum of the areas of rectangles, we can define the definite integral of a function of two variables over a region in the plane as the limit of the sum of the volumes of rectangular boxes.

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

We have function $f(x, y) \geq 0$. The graph of f is a surface with equation $z = f(x, y)$. Let S be the solid that lies above R and under the graph of f, that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 | 0 \leq z \leq f(x, y), (x, y) \in R\}$$

Definition 2.1.1 (Double Integral)**.** The double integral of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

if the limit exists.

Iterated Integrals

Suppose that f is a function of two variables that is integrable over the rectangle $R = [a, b] \times [c, d]$. We can evaluate the double integral of f over R by evaluating two single integrals.

$$A(x) = \int_c^d f(x, y)\, dy$$

If we now integrate $A(x)$ from a to b, we get the double integral of f over R.

$$\int_a^b A(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx = \iint_R f(x,y)\,dA$$

Theorem 2.1.1 (Fubini's Theorem)**.** *If f is continuous on the rectangle*

$$R = \{(x,y)|a \le x \le b, c \le y \le d\}$$

then,

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx = \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy$$

If $f(x,y) \ge 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$

2.2 Double Integrals over General Regions

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$$

If F is integrable over R, then we define the double integral of f over D by

Formula.

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

where F is given by the formula above.

A plane region D is called **simple** if it is bounded and can be expressed as

$$D = \{(x,y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

Definition 2.2.1. If f is continuous on a simple region D described by

$$D = \{(x,y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

We also consider plane regions of the form $D = \{(x,y)|c \le y \le d, h_1(y) \le x \le h_2(y)\}$.

Definition 2.2.2. If f is continuous on a simple region D described by

$$D = \{(x, y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

$$\begin{aligned}
\int_0^2 \int_0^{y^2} x^2 y\, dx\, dy &= \int_0^2 \left[\int_0^{y^2} x^2 y\, dx\right] dy \\
&= \int_0^2 y \int_0^{y^2} x^2\, dx\, dy \\
&= \int_0^2 y \left[\frac{x^3}{3}\Big|_0^{y^2}\right] \\
&= \frac{1}{3}\int_0^2 y \cdot y^6\, dy \\
&= \frac{1}{3}\int_0^2 y^7\, dy \\
&= \frac{1}{3}\left[\frac{y^8}{8}\Big|_0^2\right] \\
&= \frac{1}{3} \cdot \frac{2^8}{8} \\
&= \frac{2^5}{3} = \frac{32}{3}
\end{aligned}$$

Changing the Order of Integration

Fubini's Theorem tell us that we can express a double integral as an iterated integral in two different orders. Sometimes one order is much easir to evaluate than the other or it may be the only way to evaluate the integral.

Remark. This method require us to draw the region of integration.

Evaluate the integral by reversing the order of integration.

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$$

$$\begin{aligned}
\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy &= \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx \\
&= \int_0^3 e^{x^2} \cdot y\Big|_{y=0}^{y=x/3}\,dx \\
&= \int_0^3 e^{x^2} \cdot \frac{x}{3}\,dx \\
&= \frac{1}{3}\int_0^3 x \cdot e^{x^2}\,dx
\end{aligned}$$

Let $u = x^2$, then $du = 2x\,dx$

$$\begin{aligned}
\frac{1}{3}\int_0^3 x \cdot e^{x^2}\,dx &= \frac{1}{3}\int_0^9 x \cdot e^u\,\frac{du}{2x} \\
&= \frac{1}{6}\int_0^9 e^u\,du \\
&= \frac{1}{6}e^u\Big|_0^9 \\
&= \frac{1}{6}(e^9 - 1)
\end{aligned}$$

2.3 Applications of Double Integrals

Density and Mass

Formula (Mass). *Mass = Density × Volume*

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x,y)\,dA$$

Moments and Centers of Mass

Formula (Moments). *The moment about the y-axis is*

$$M_y = \iint_D x\rho(x,y)\,dA$$

The moment about the x-axis is

$$M_x = \iint_D y\rho(x,y)\,dA$$

Formula (Center of Mass). *The x-coordinate of the center of mass is*

$$\bar{x} = \frac{M_y}{m}$$

The y-coordinate of the center of mass is

$$\bar{y} = \frac{M_x}{m}$$