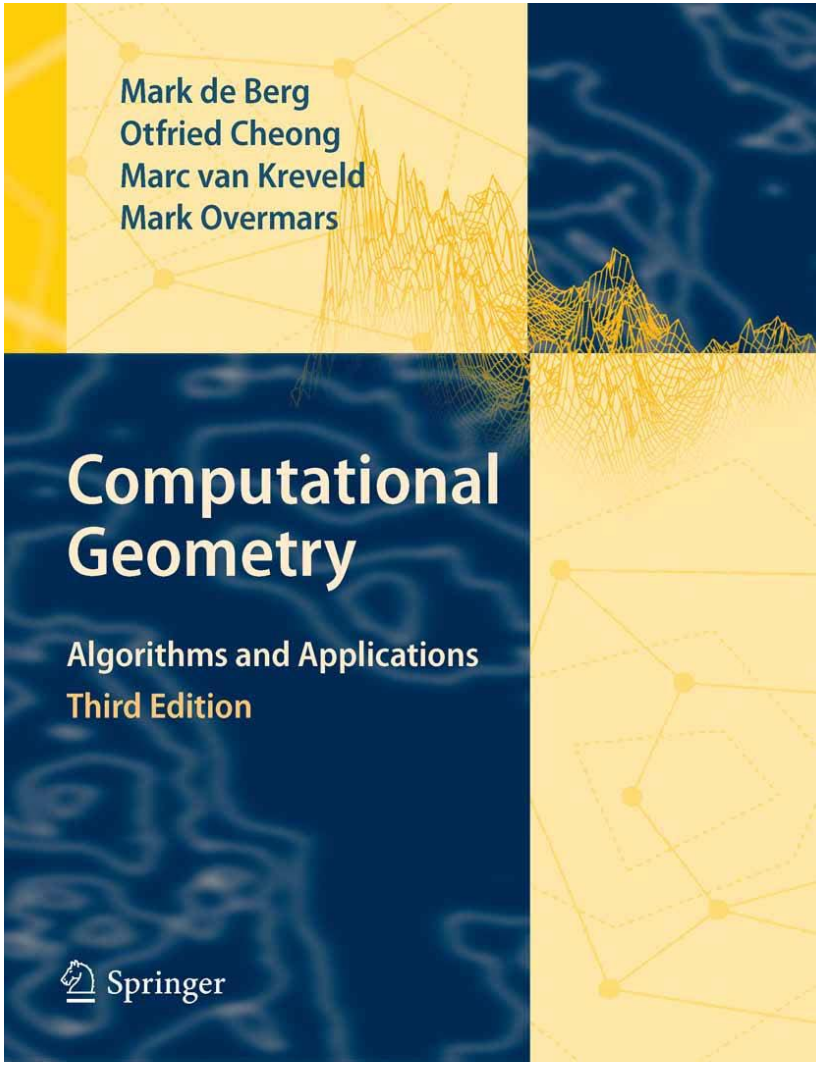
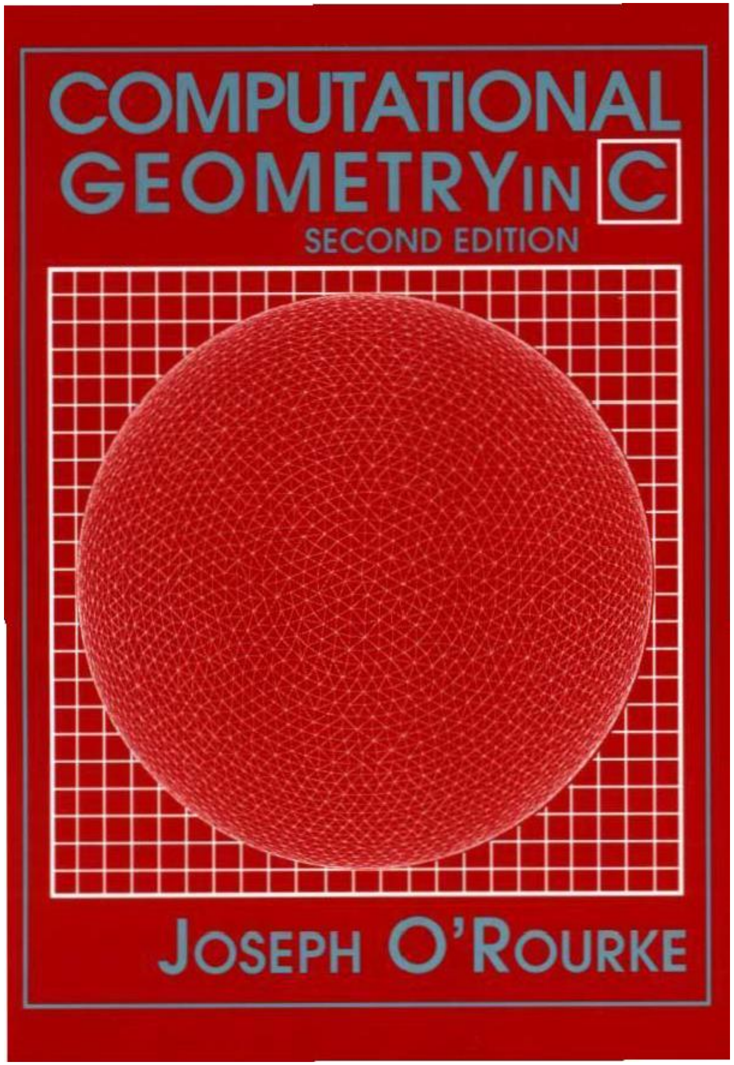
Geometry Introduction

หนังสือแนะนำ


Mark de Berg
Otfried Cheong
Marc van Kreveld
Mark Overmars
Computational Geometry
Algorithms and Applications
Third Edition
Springer


COMPUTATIONAL GEOMETRY IN C
SECOND EDITION
JOSEPH O'ROURKE

Topic

- Introduction
- Two lines Intersection Test
- Point inside polygon
- Convex hull
- Line Segments Intersection Algorithm

Geometry

<u>Components</u>

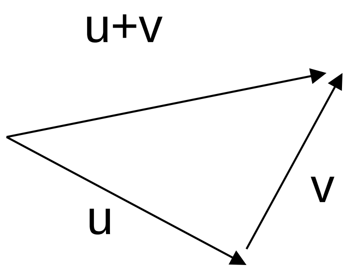
- Scalar (S)
- Point (P)
- Free vector (V)

<u>Allowed operations</u>

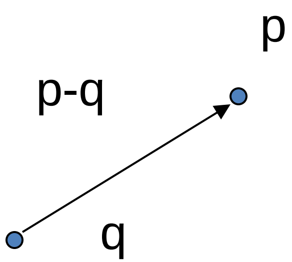
- S * V → V
- V + V → V
- P – P → V
- P + V → P

Examples

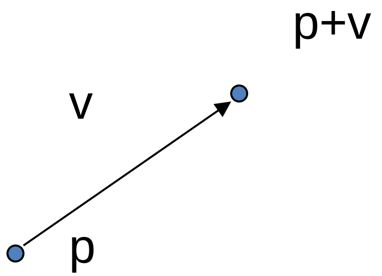
Vector addition

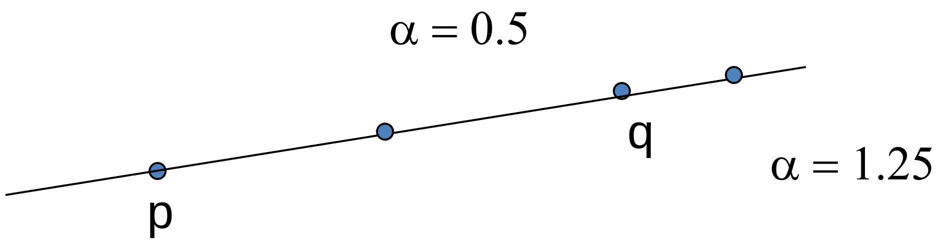


Point subtraction



Point-vector addition





จุดใดๆบนเส้นตรงที่ลากผ่านจุด p และ q สามารถสร้างได้จาก affine combination:

$$\mathbf{r} = \mathbf{p} + \alpha \cdot (\mathbf{q} - \mathbf{p})$$

Euclidean Geometry

- In affine geometry, angle and distance are not defined.
- Euclidean geometry is an extension providing an additional operation called “inner product”
- There are other types of geometry that extends affine geometry such as projective geometry, hyperbolic geometry…

Dot product is a mapping from two vectors to a real number.

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}$$

Dot product

$$\mathbf{u} \cdot \mathbf{v} = \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{d}} u_i v_i$$

Length

$$|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

Distance

$$\mathrm{dist}(\mathbf{P}, \mathbf{Q}) = |\mathbf{P} - \mathbf{Q}|$$

Angle

$$\mathrm{ang}(\mathbf{u}, \mathbf{v}) = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

Orthogonality: u and v are orthogonal when $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$

Topic

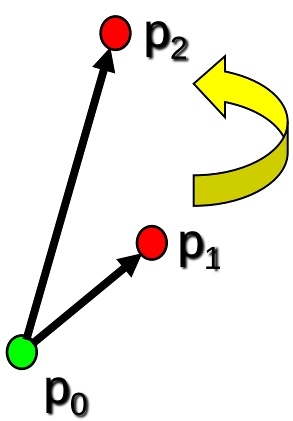
- Introduction
- Two lines Intersection Test
- Point inside polygon
- Convex hull
- Line Segments Intersection Algorithm

Cross-Product-Based Geometric Primitives

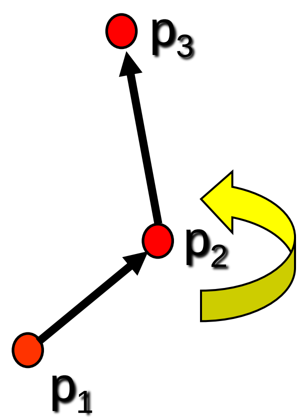
Some fundamental geometric questions:

1. Given two directed segments $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_0p_2}$, is $\overrightarrow{p_0p_1}$ clockwise from $\overrightarrow{p_0p_2}$ with respect to their common endpoint p_0?
2. Given two line segments $\overline{p_1p_2}$ and $\overline{p_2p_3}$, if we traverse $\overline{p_1p_2}$ and then $\overline{p_2p_3}$, do we make a left turn at point p_2?
3. Do line segments $\overline{p_1p_2}$ and $\overline{p_3p_4}$ intersect?

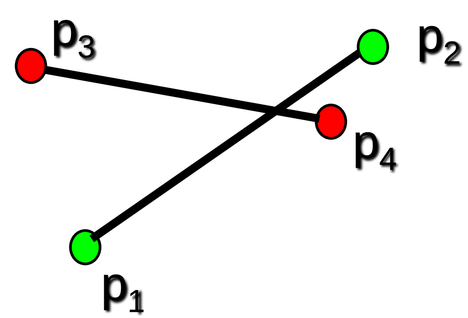
source: 91.503 textbook Cormen et al.



(1)



(2)



(3)

Cross-Product-Based Geometric Primitives: (1)

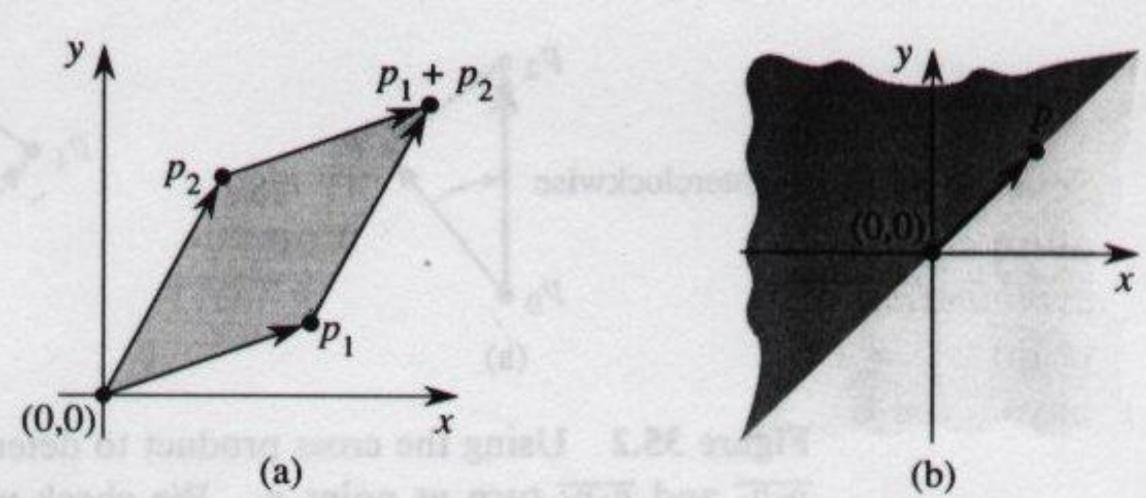
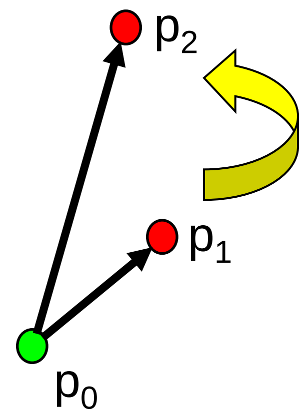


Figure 35.1 **(a)** The cross product of vectors p_1 and p_2 is the signed area of the parallelogram. **(b)** The lightly shaded region contains vectors that are clockwise from p. The darkly shaded region contains vectors that are counterclockwise from p.



(1)

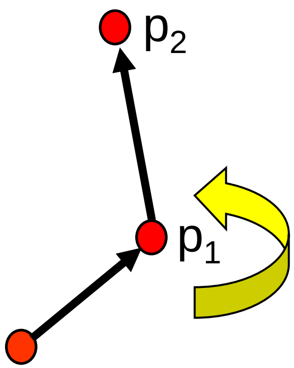
$$p_1 \times p_2 = \det\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = x_1 y_2 - x_2 y_1$$

$$(p_1 - p_0) \times (p_2 - p_0) = (x_1 - x_0)(y_2 - y_0) - (x_2 - x_0)(y_1 - y_0)$$

Advantage: less sensitive to accumulated round-off error

source: 91.503 textbook Cormen et al.

Cross-Product-Based Geometric Primitives: (2)



(2)

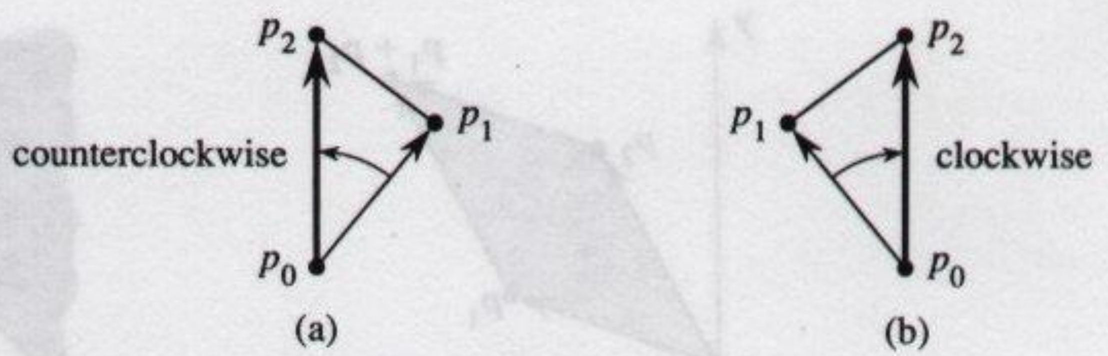


Figure 35.2 Using the cross product to determine how consecutive line segments $\overline{p_0 p_1}$ and $\overline{p_1 p_2}$ turn at point p_1. We check whether the directed segment $\overrightarrow{p_0 p_2}$ is clockwise or counterclockwise relative to the directed segment $\overrightarrow{p_0 p_1}$. (**a**) If counterclockwise, the points make a left turn. (**b**) If clockwise, they make a right turn.

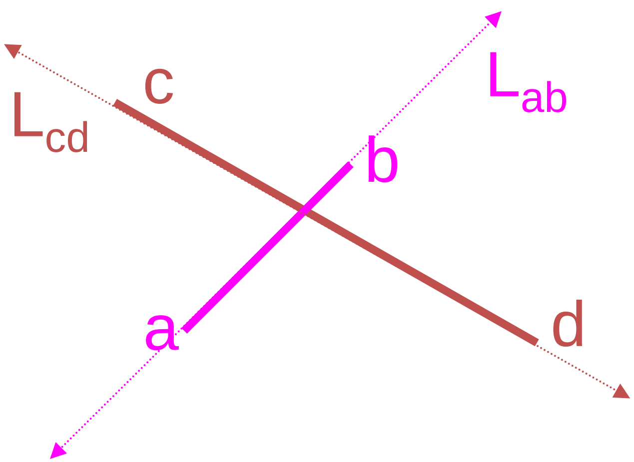
source: 91.503 textbook Cormen et al.

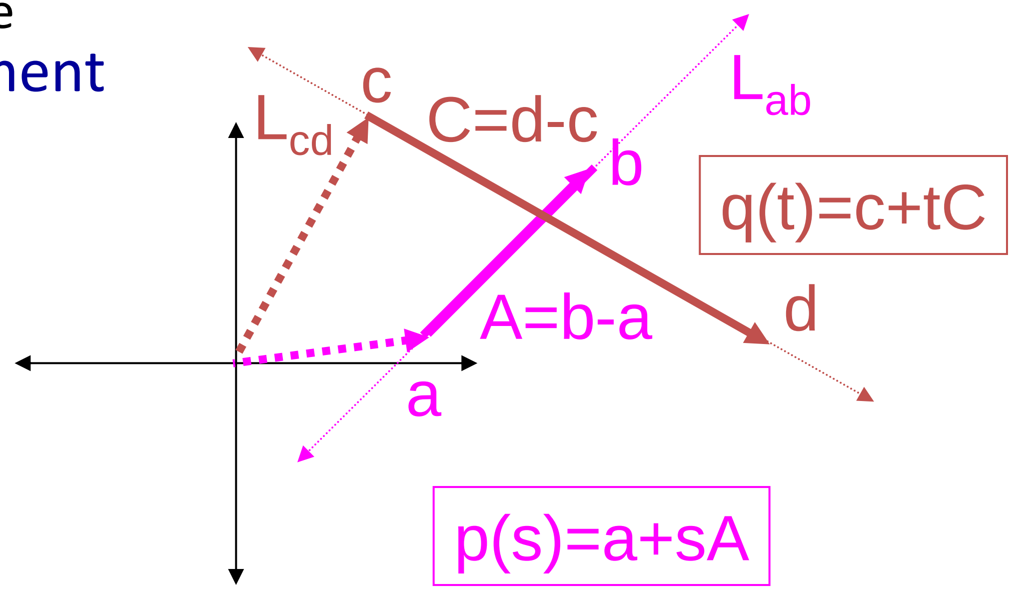
isLeft()

```
// isLeft(): tests if a point is Left|On|Right of an infinite line.
//    Input:  three points P0, P1, and P2
//    Return: >0 for P2 left of the line through P0 and P1
//            =0 for P2 on the line
//            <0 for P2 right of the line
int isLeft( Point P0, Point P1, Point P2 )
{
    return (  (P1.x - P0.x) * (P2.y - P0.y)
            - (P2.x - P0.x) * (P1.y - P0.y) );
}
```

Segment-Segment Intersection

- Finding the ***actual intersection point***
- Approach: parametric vs. slope/intercept
  - parametric generalizes to more complex intersections
    - e.g. segment/triangle
- Parameterize each segment





Intersection: values of s, t such that $p(s) = q(t)$: $a+sA=c+tC$

2 equations in unknowns s, t : 1 for x, 1 for y

source: O'Rourke, Computational Geometry in C

Assume that a = (x1,y1) b = (x2,y2) c = (x3,y3) d = (x4,y4)

$$s = \frac{(x_4 - x_3)(y_1 - y_3) - (y_4 - y_3)(x_1 - x_3)}{(y_4 - y_3)(x_2 - x_1) - (x_4 - x_3)(y_2 - y_1)}$$

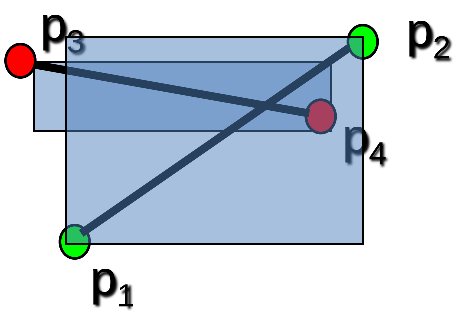
$$t = \frac{(x_2 - x_1)(y_1 - y_3) - (y_2 - y_1)(x_1 - x_3)}{(y_4 - y_3)(x_2 - x_1) - (x_4 - x_3)(y_2 - y_1)}$$

Code

```
typedef struct point { double x; double y;} point;
typedef struct line { point p1; point p2;} line;

int check_lines(line *line1, line *line2, point *hitp)
{
    double d   =   (line2->p2.y - line2->p1.y)*(line1->p2.x-line1->p1.x) -
                   (line2->p2.x - line2->p1.x)*(line1->p2.y-line1->p1.y);

    double ns =   (line2->p2.x - line2->p1.x)*(line1->p1.y-line2->p1.y) -
                  (line2->p2.y - line2->p1.y)*(line1->p1.x-line2->p1.x);

    double nt =   (line1->p2.x - line1->p1.x)*(line1->p1.y - line2->p1.y) -
                  (line1->p2.y - line1->p1.y)*(line1->p1.x - line2->p1.x);

    if(d == 0)  return 0;

    double s= ns/d;
    double t = nt/d;

    return (s >=0 && s <= 1 && t >= 0 && t <= 1));
}
```

Intersection of 2 Line Segments

Step 1:
Bounding Box
Test



(3)

Step 2: Does each segment straddle the line containing the other?

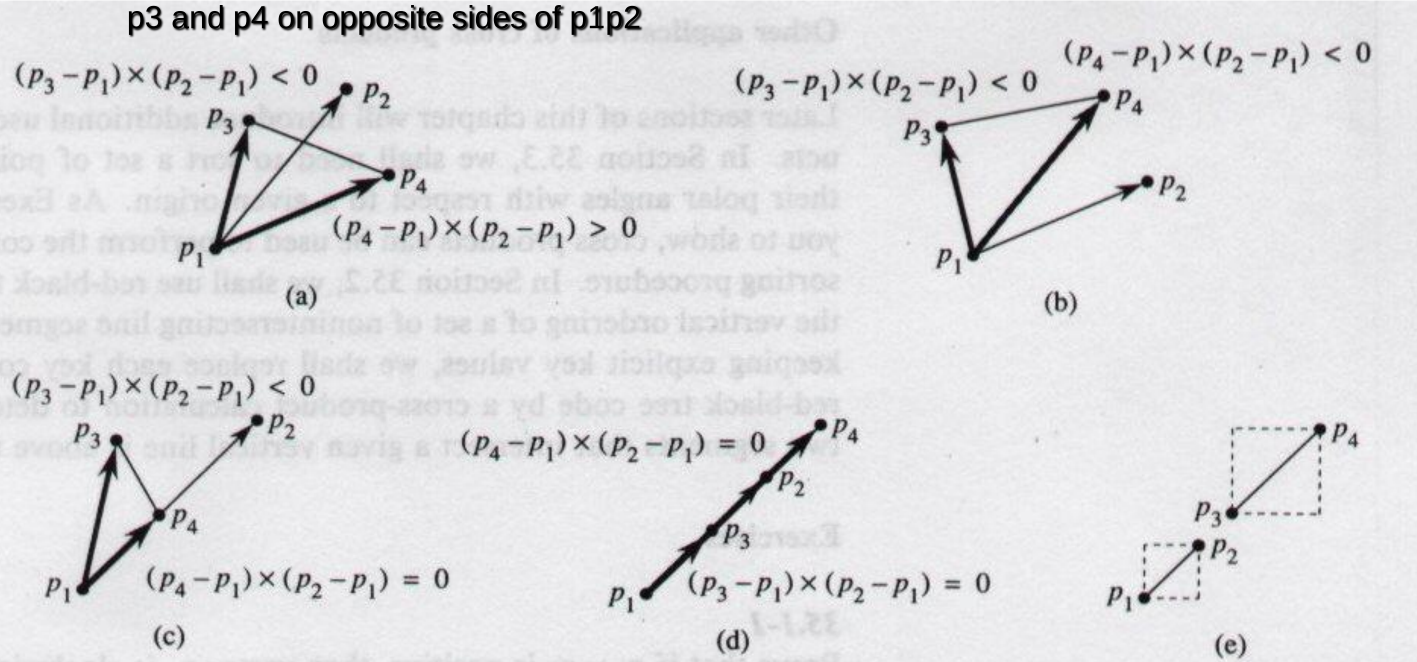


Figure 35.3 Determining whether line segment $\overline{p_3p_4}$ straddles the line containing segment $\overline{p_1p_2}$. **(a)** If it does straddle, then the signs of the cross products $(p_3 - p_1) \times (p_2 - p_1)$ and $(p_4 - p_1) \times (p_2 - p_1)$ differ. **(b)** If it does not straddle, then the signs of the cross products are the same. **(c)–(d)** Boundary cases in which at least one of the cross products is zero and the segment straddles. **(e)** A boundary case in which the segments are collinear but do not intersect. Both cross products are zero, but they would not be computed by our algorithm because the segments fail the quick rejection test—their bounding boxes do not intersect.

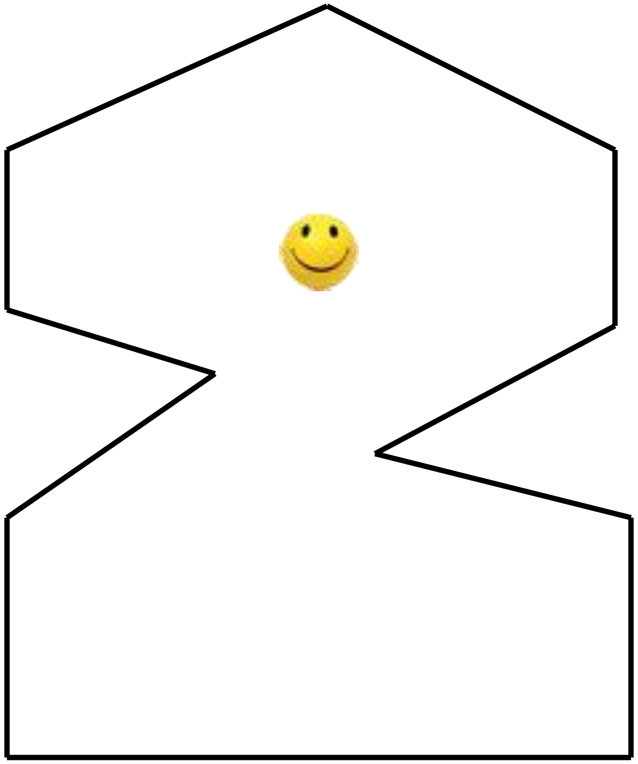
source: 91.503 textbook Cormen et al.

Topic

- Introduction
- Two lines Intersection Test
- Point inside polygon
- Convex hull
- Line Segments Intersection Algorithm

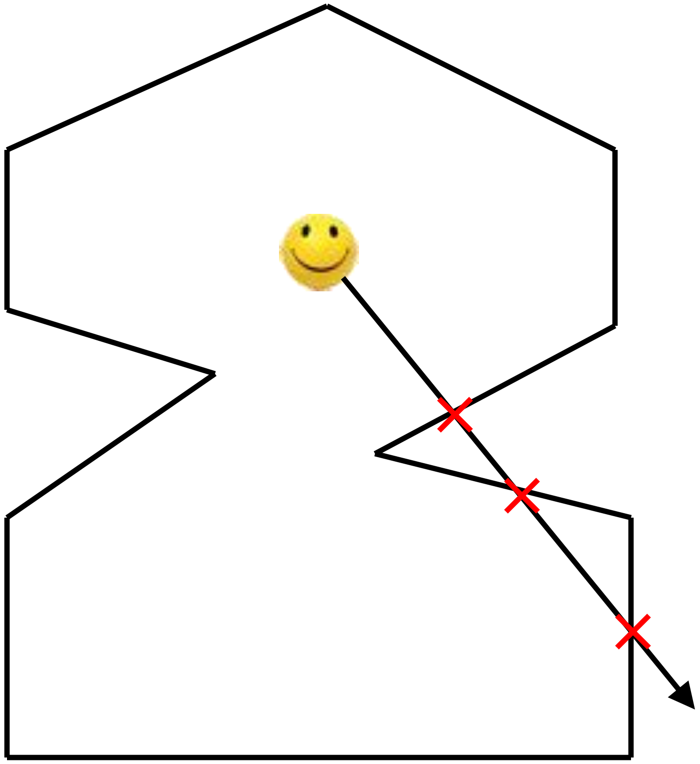
Point Inside Polygon Test

- Given a point, determine if it lies inside a polygon or not



Ray Test

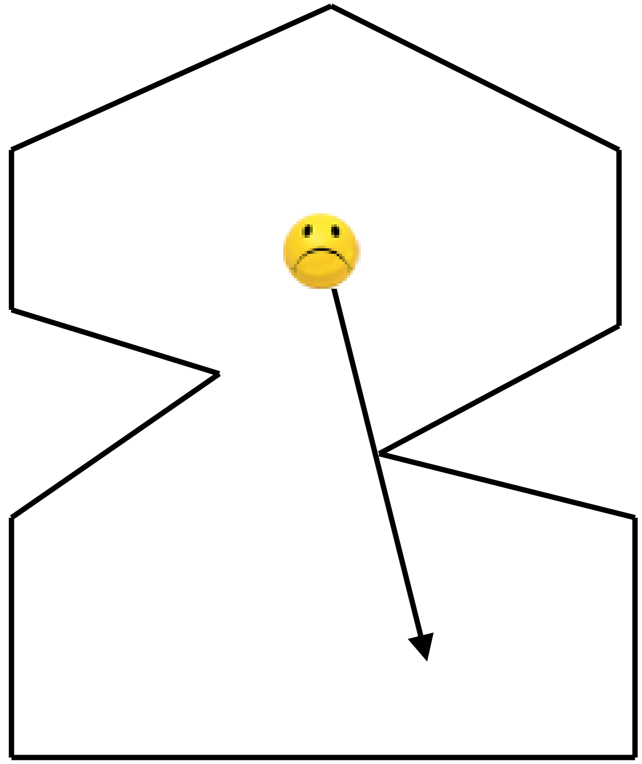
- Fire ray from point
- Count intersections
  - Odd = inside polygon
  - Even = outside polygon



Problems With Rays

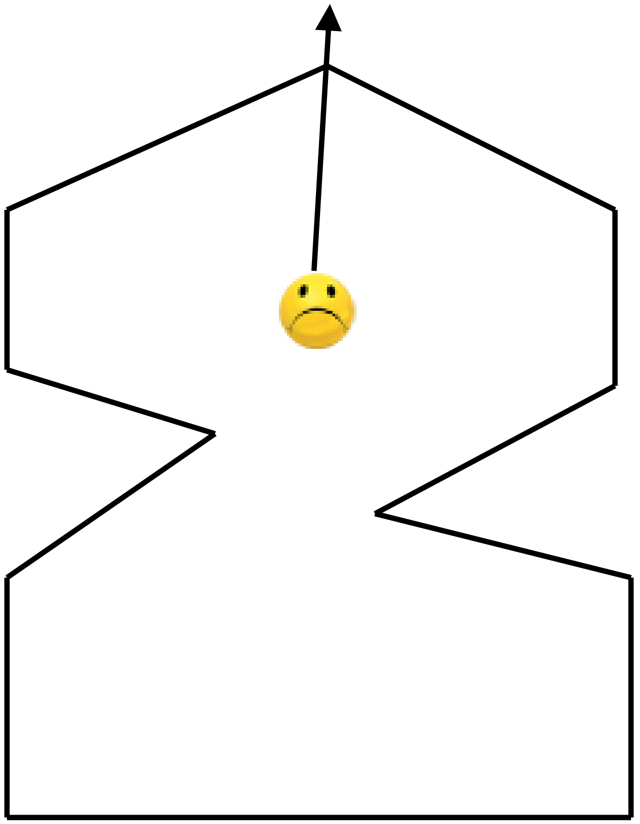
- Fire ray from point
- Count intersections
  - Odd = inside polygon
  - Even = outside polygon

- Problems
  - Ray through vertex



Problems With Rays

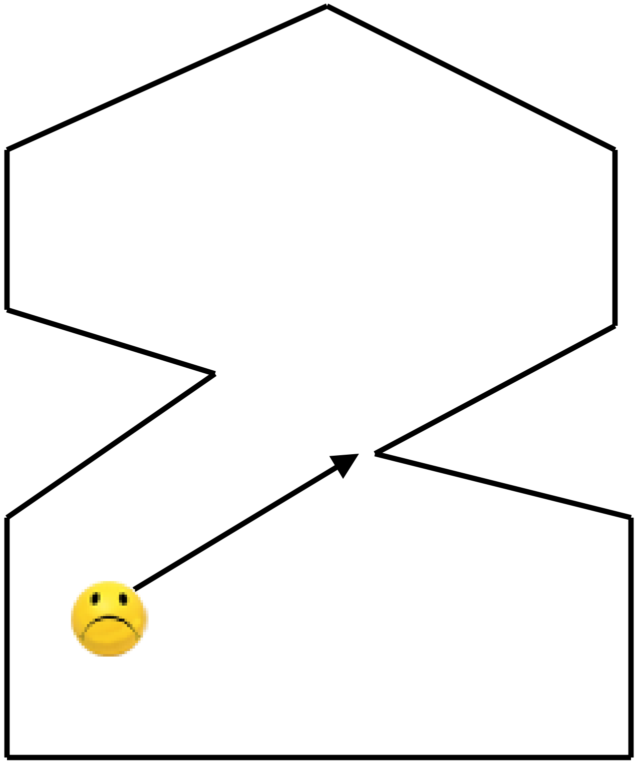
- Fire ray from point
- Count intersections
  - Odd = inside polygon
  - Even = outside polygon
- Problems
  - Ray through vertex



Problems With Rays

- Fire ray from point
- Count intersections
  - Odd = inside polygon
  - Even = outside polygon

- Problems
  - Ray through vertex
  - Ray parallel to edge



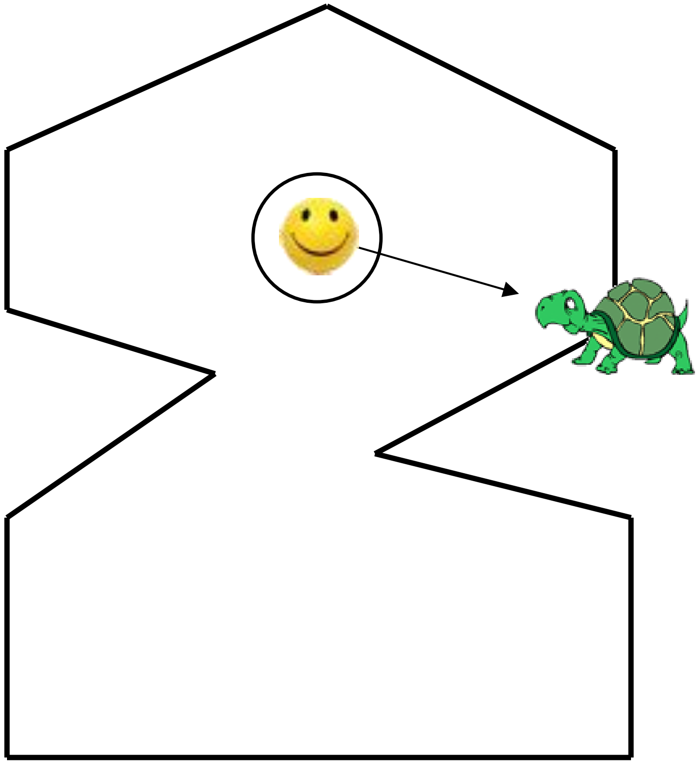
Solution

- Edge Crossing Rule
  - an upward edge includes its starting endpoint, and excludes its final endpoint;
  - a downward edge excludes its starting endpoint, and includes its final endpoint;
  - horizontal edges are excluded; and
  - the edge-ray intersection point must be strictly right of the point P.
- Use horizontal ray for simplicity in computation

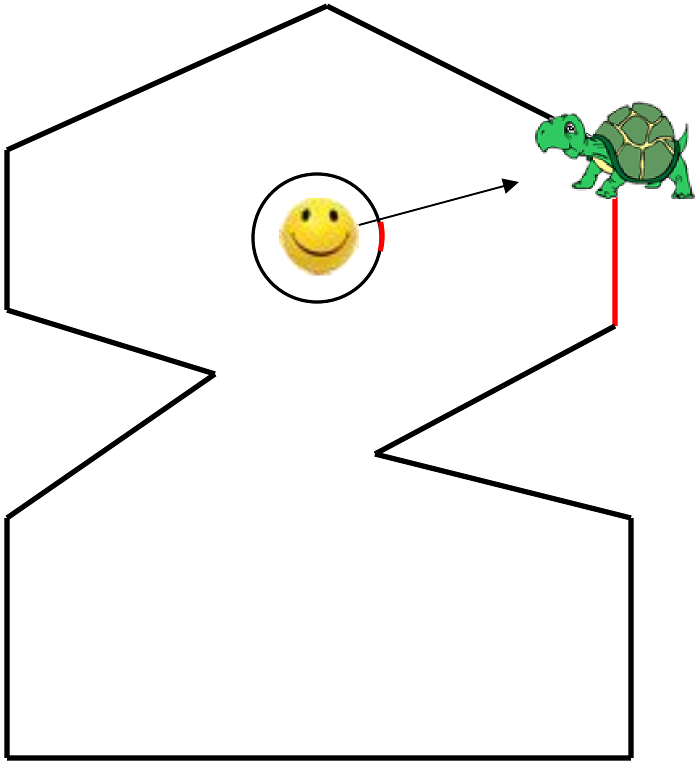
Code

```
// cn_PnPoly(): crossing number test for a point in a polygon
//      Input:   P = a point,
//               V[] = vertex points of a polygon V[n+1] with V[n]=V[0]
//      Return:  0 = outside, 1 = inside
// This code is patterned after [Franklin, 2000]
int cn_PnPoly( Point P, Point* V, int n )
{
    int    cn = 0;    // the crossing number counter

    // loop through all edges of the polygon
    for (int i=0; i<n; i++) {    // edge from V[i] to V[i+1]
       if (((V[i].y <= P.y) && (V[i+1].y  > P.y))    // an upward crossing
        || ((V[i].y  > P.y) && (V[i+1].y <= P.y))) { // a downward crossing
            // compute the actual edge-ray intersect x-coordinate
            float vt = (float)(P.y - V[i].y) / (V[i+1].y - V[i].y);
            if (P.x < V[i].x + vt * (V[i+1].x - V[i].x)) // P.x < intersect
                ++cn;   // a valid crossing of y=P.y right of P.x
        }
    }
    return (cn&1);    // 0 if even (out), and 1 if odd (in)

}
```

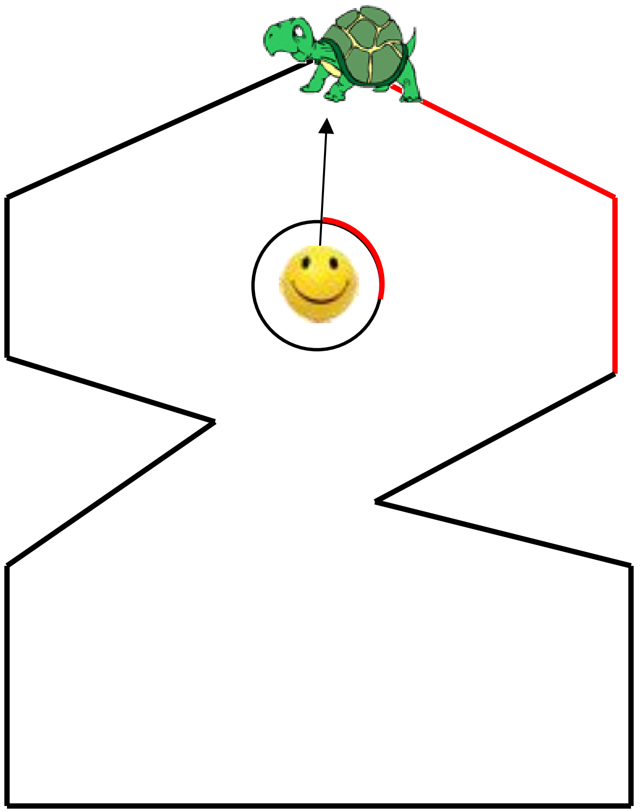
A Better Way



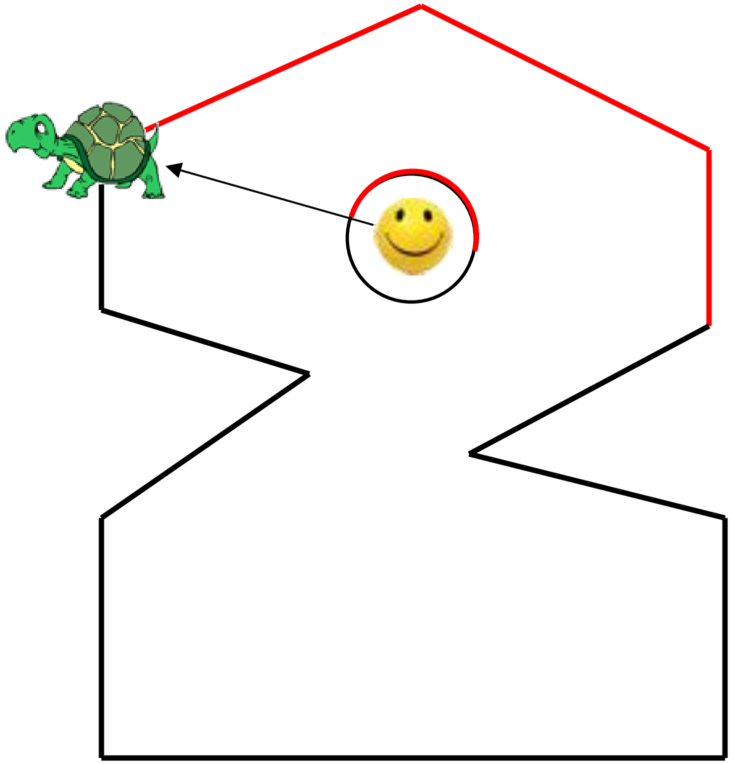
A Better Way



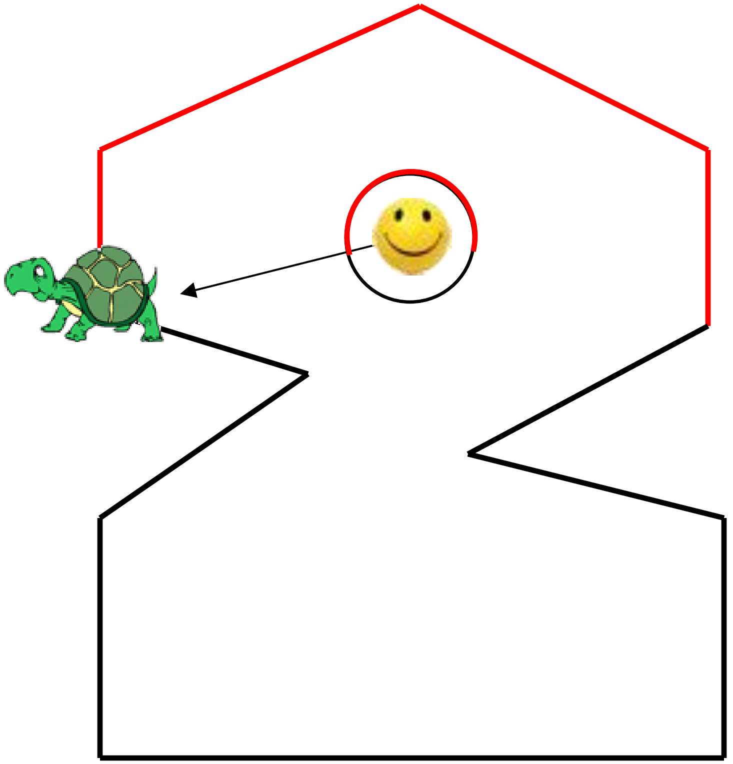
A Better Way



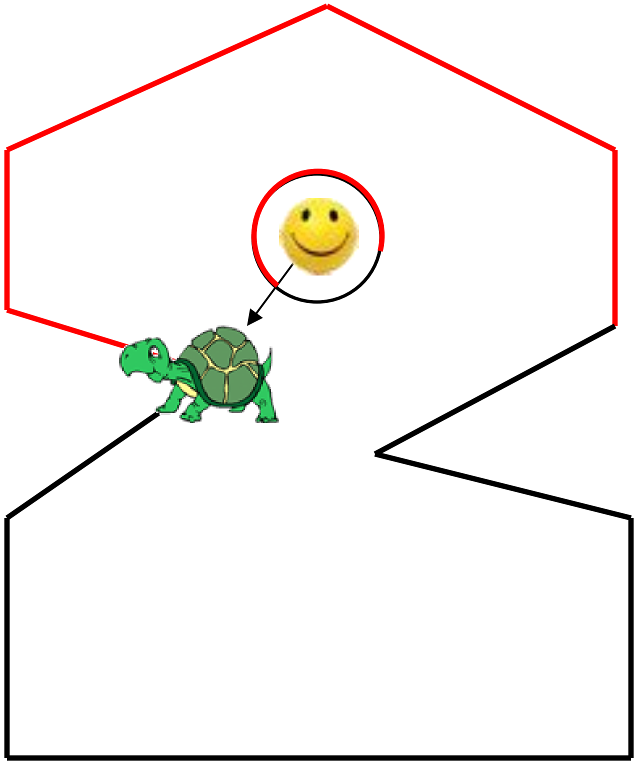
A Better Way



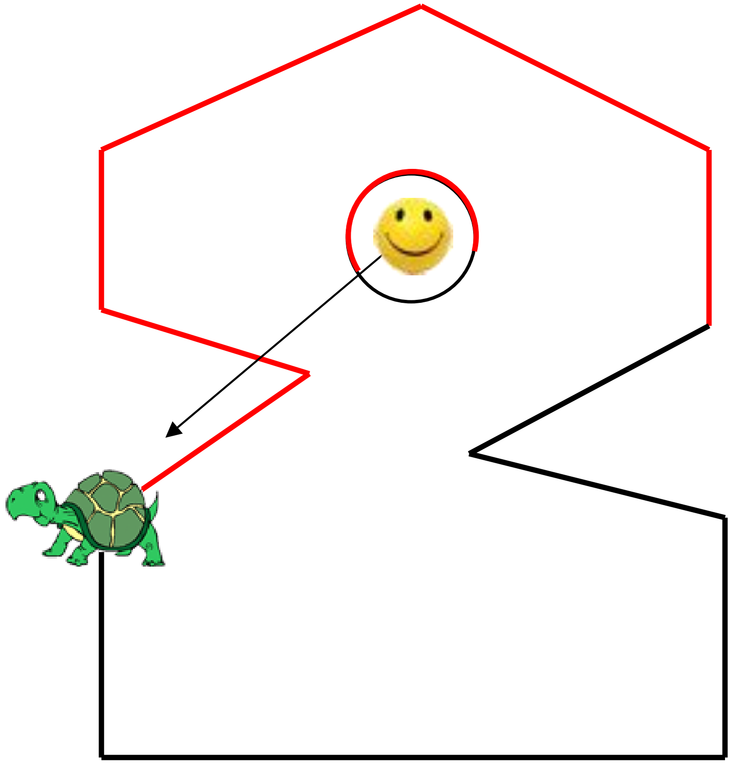
A Better Way



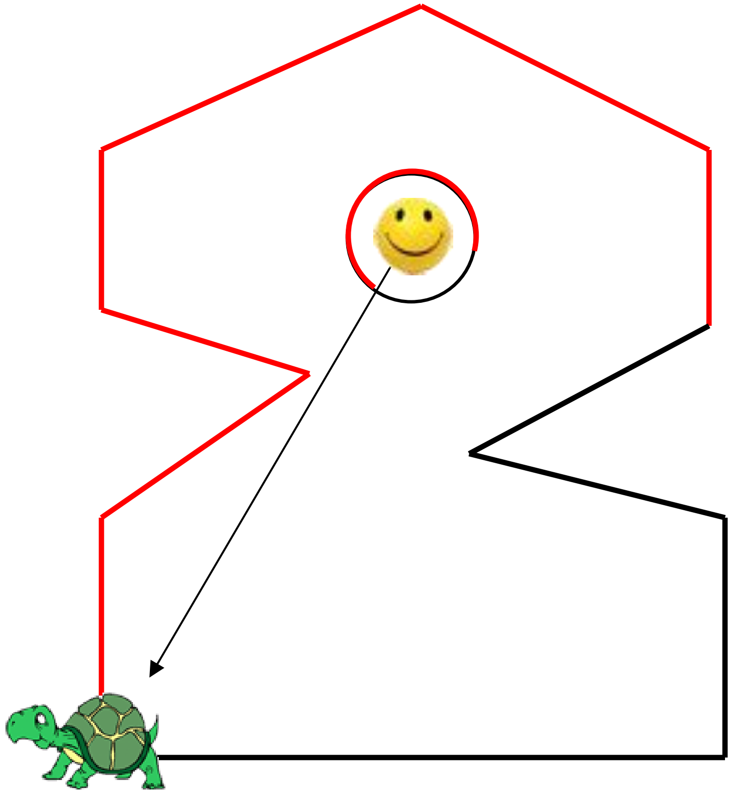
A Better Way



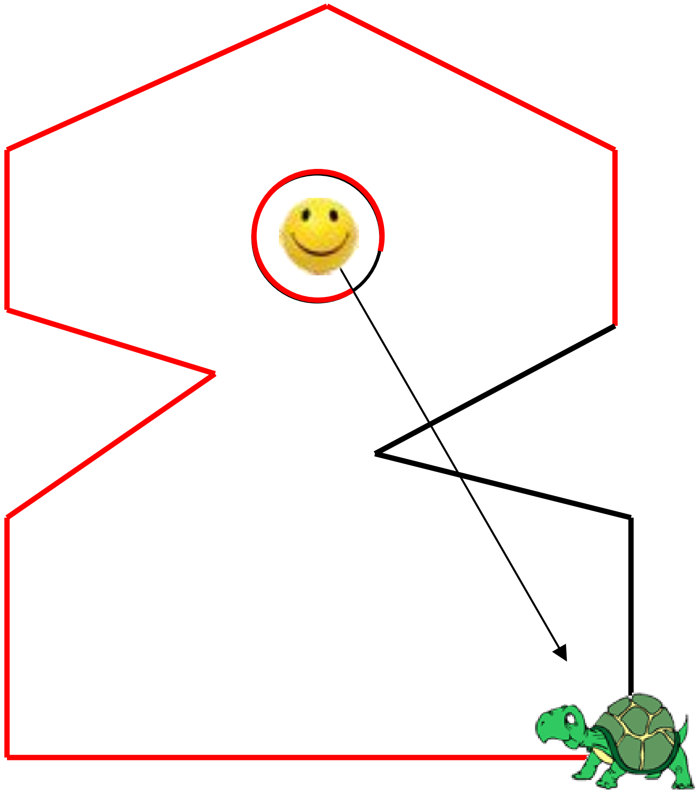
A Better Way



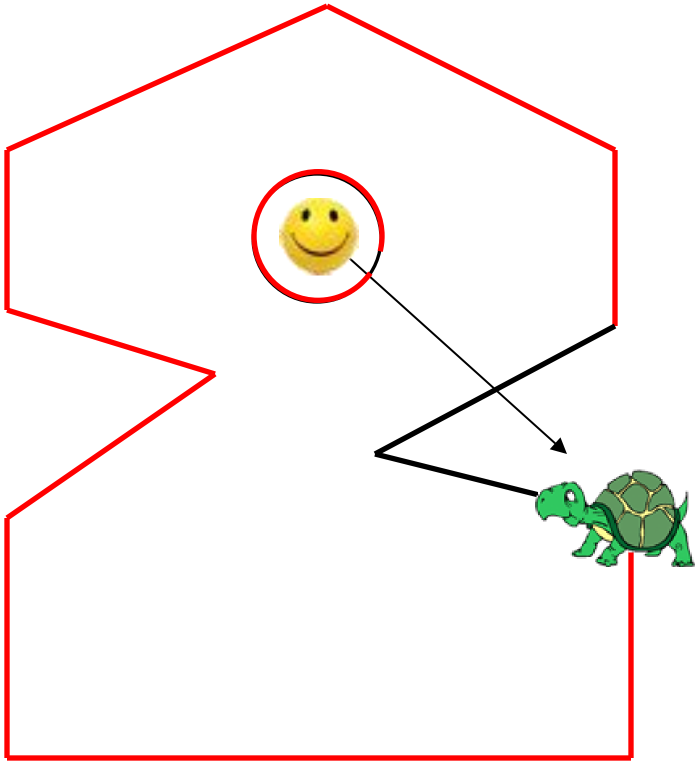
A Better Way



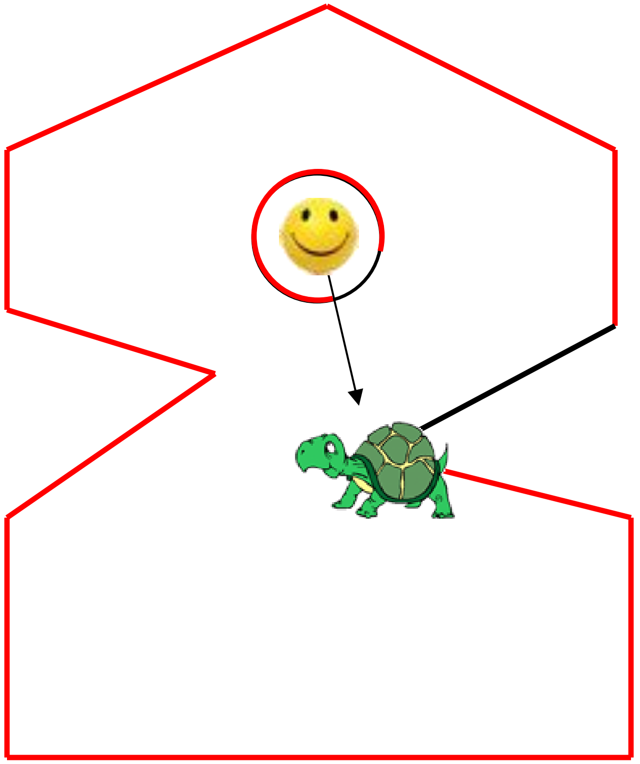
A Better Way



A Better Way

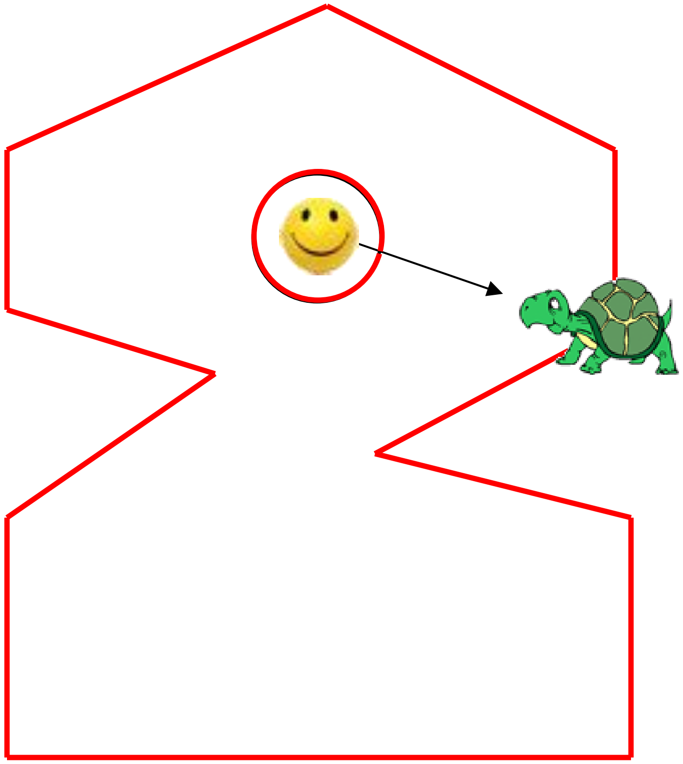


A Better Way

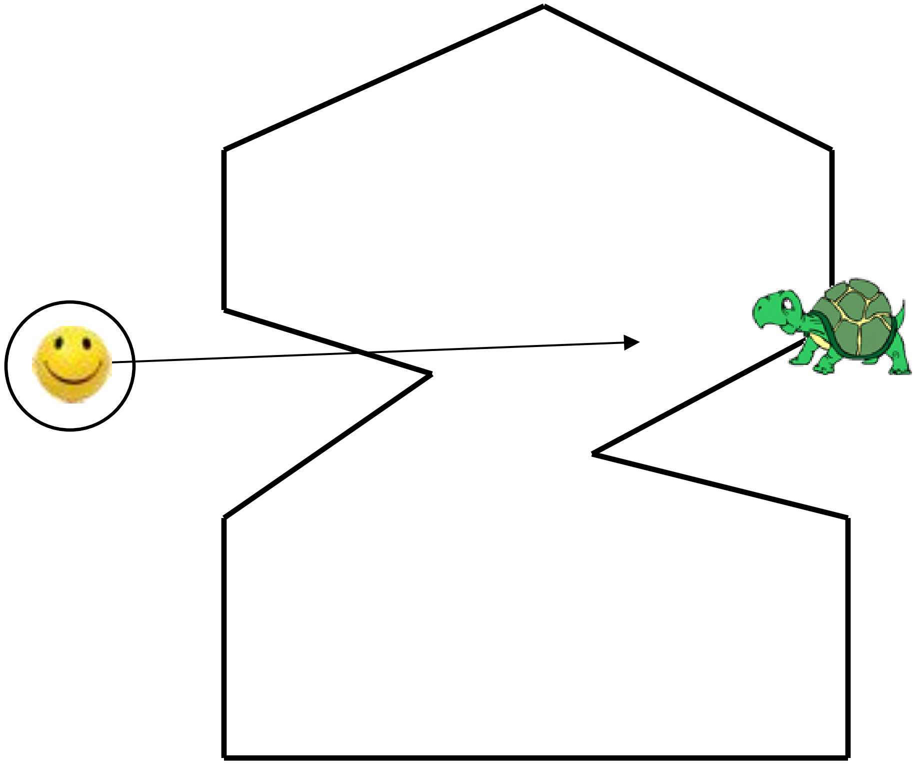


A Better Way

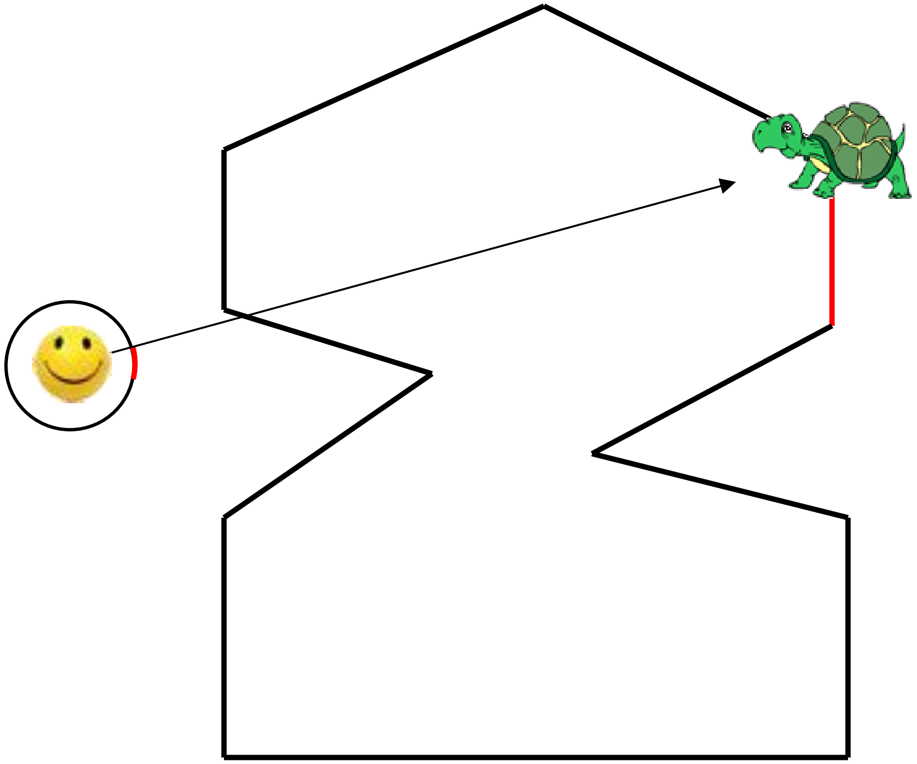
- One winding = inside



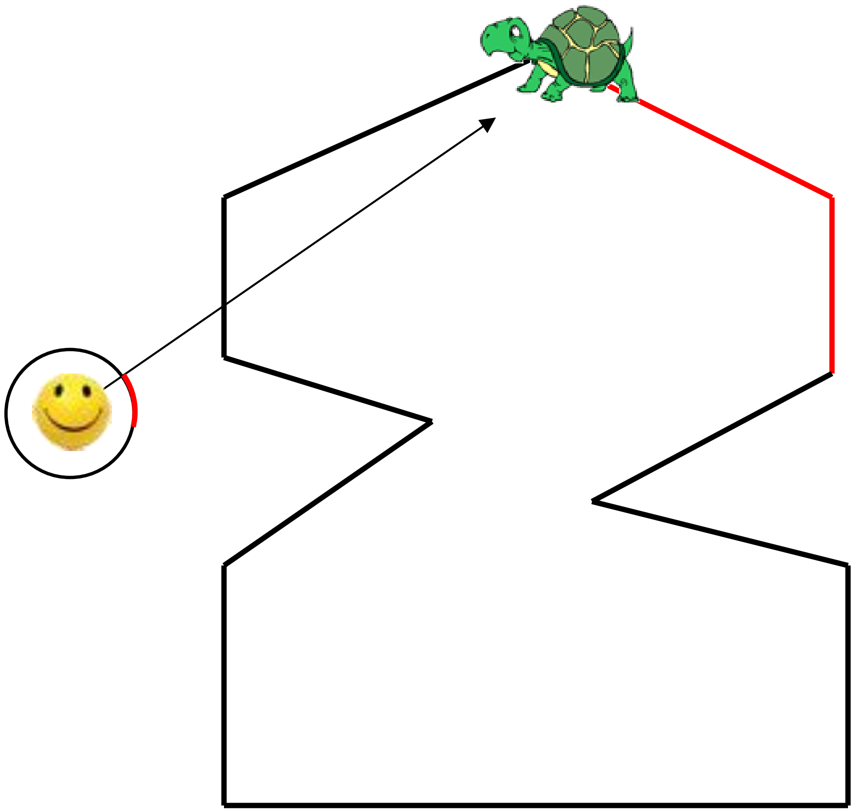
A Better Way



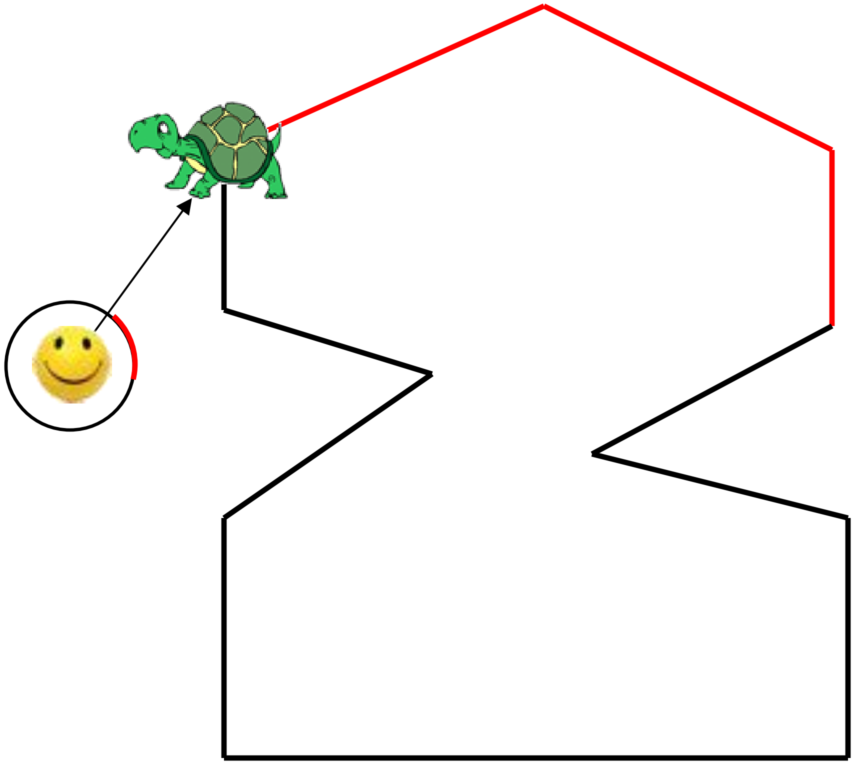
A Better Way



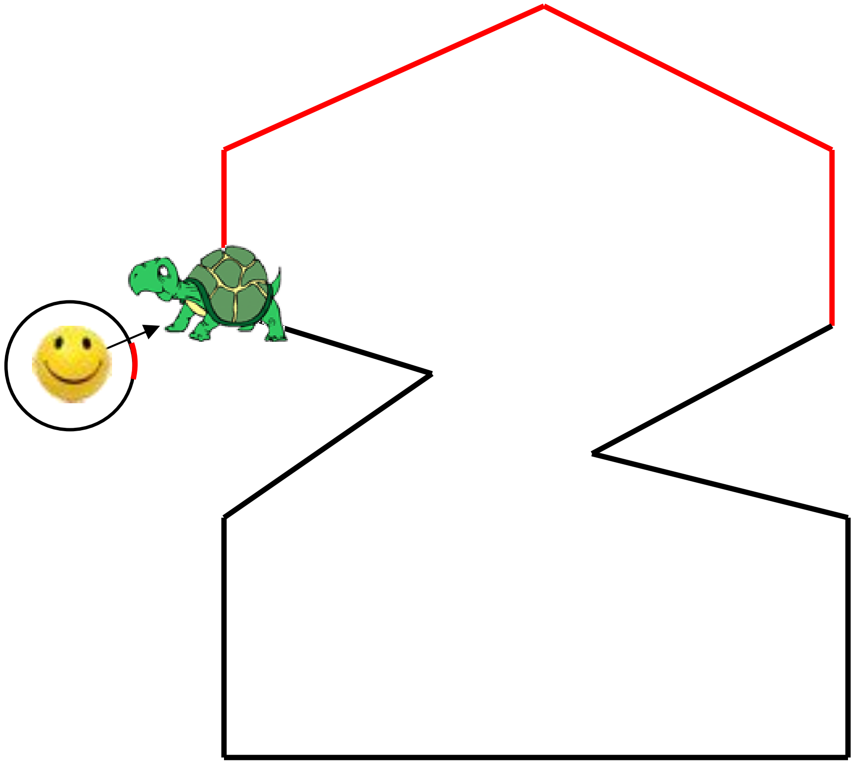
A Better Way



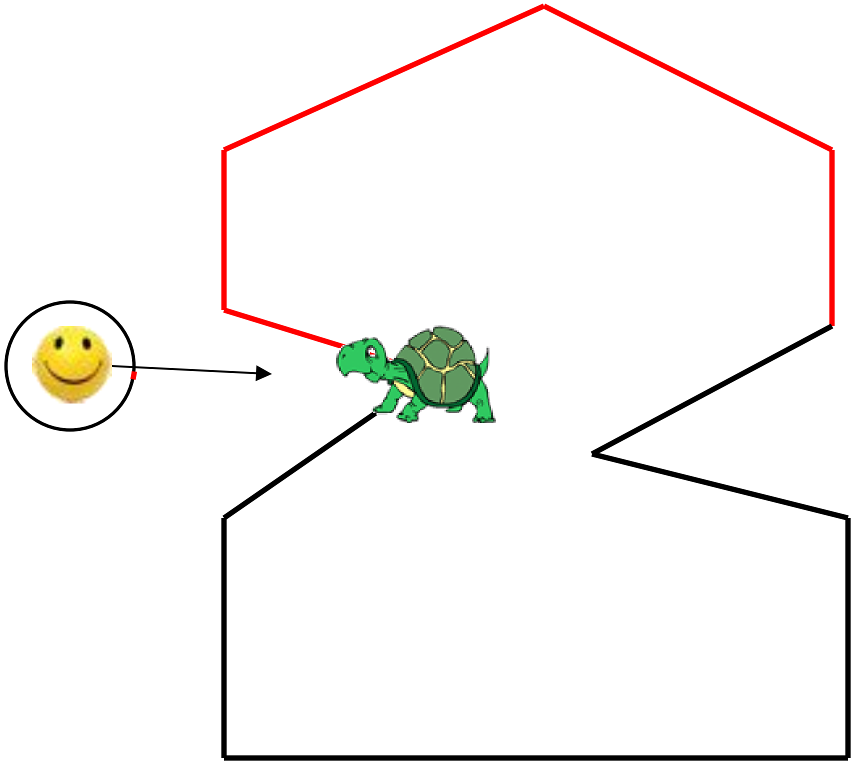
A Better Way



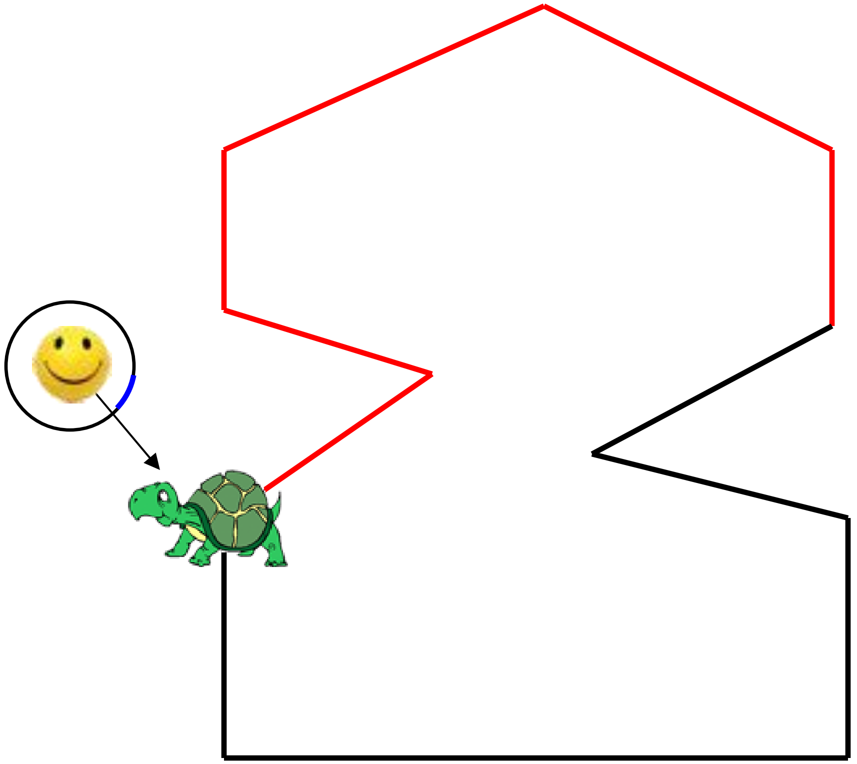
A Better Way



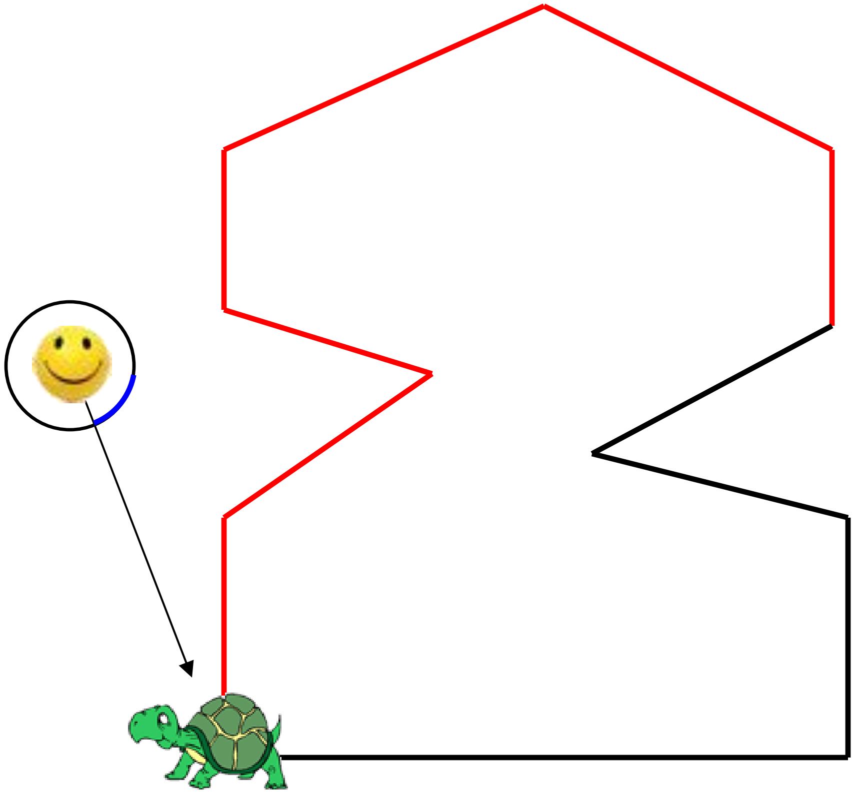
A Better Way



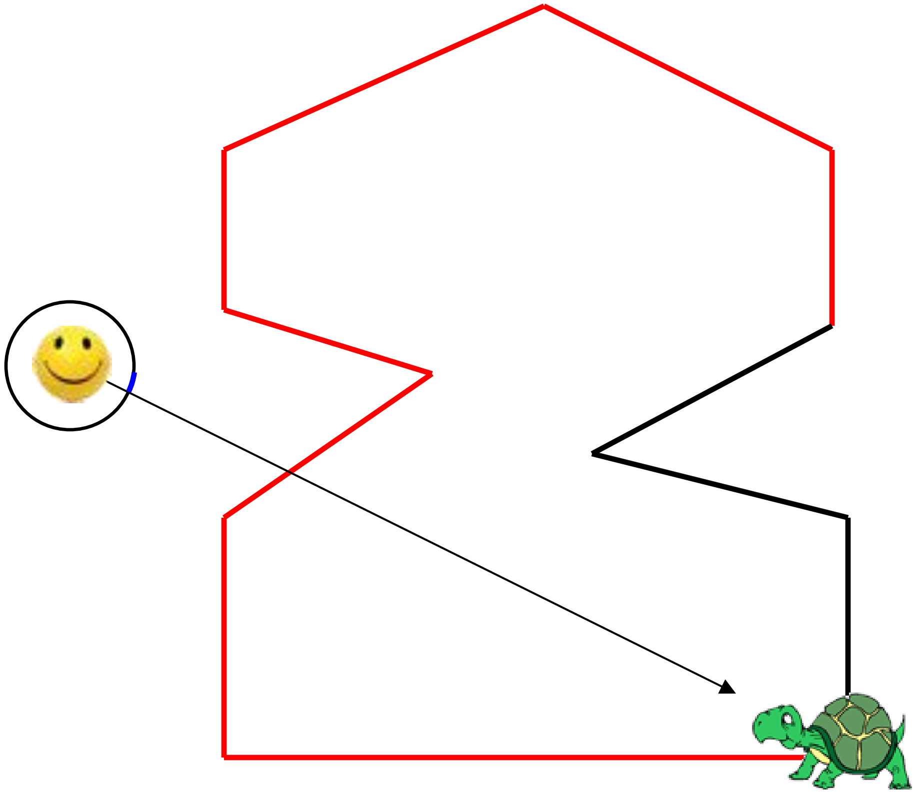
A Better Way



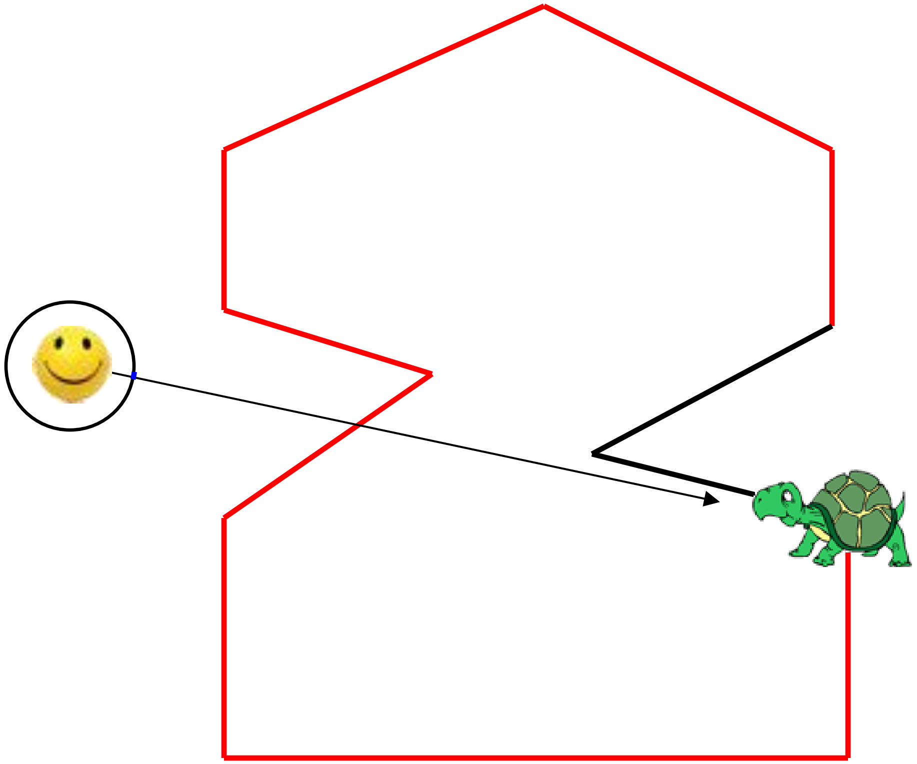
A Better Way



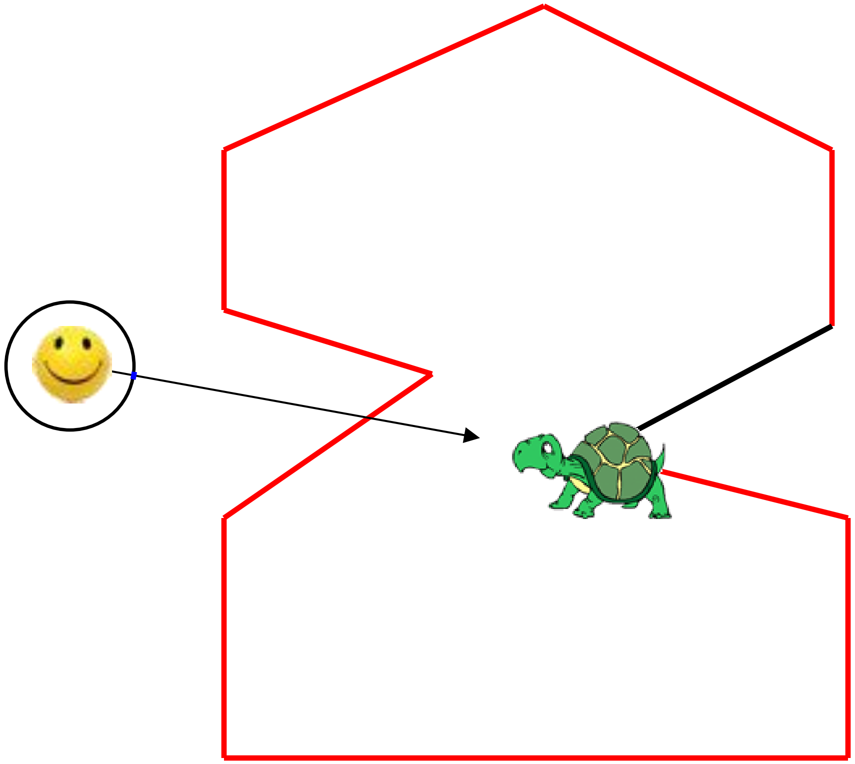
A Better Way



A Better Way

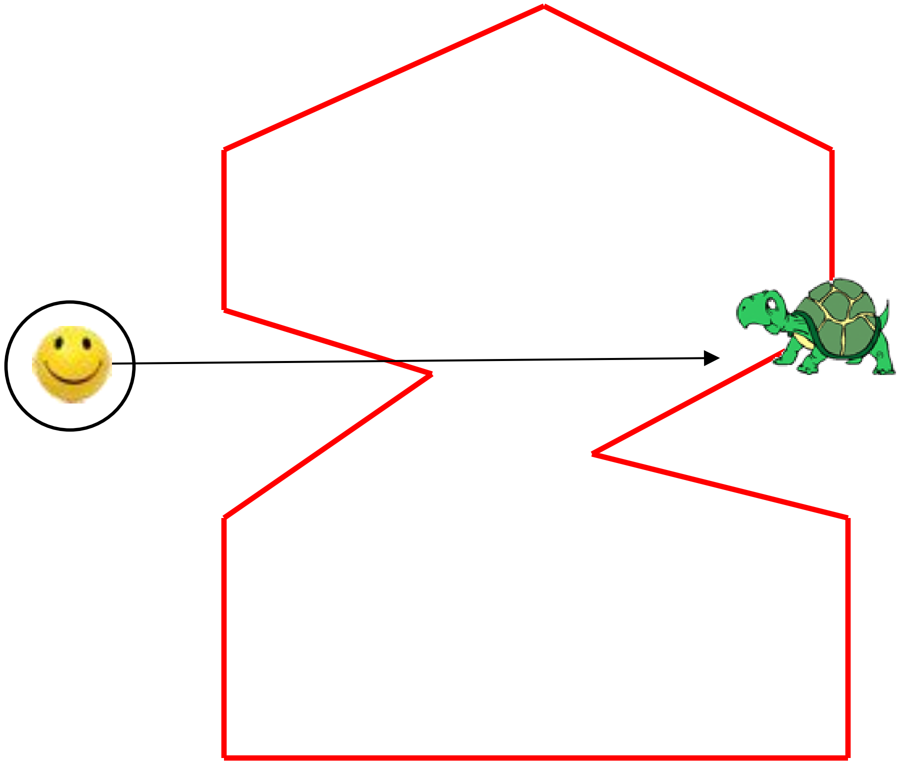


A Better Way



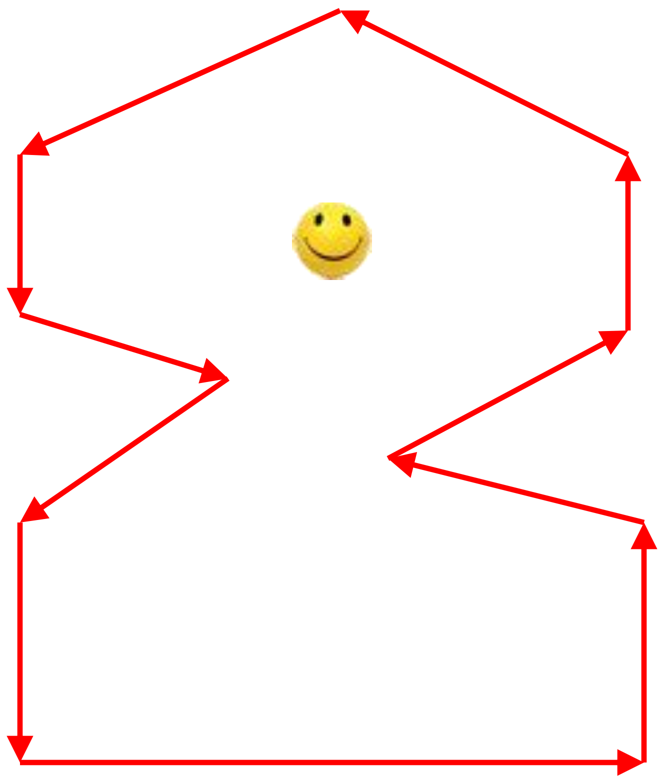
A Better Way

- zero winding = outside



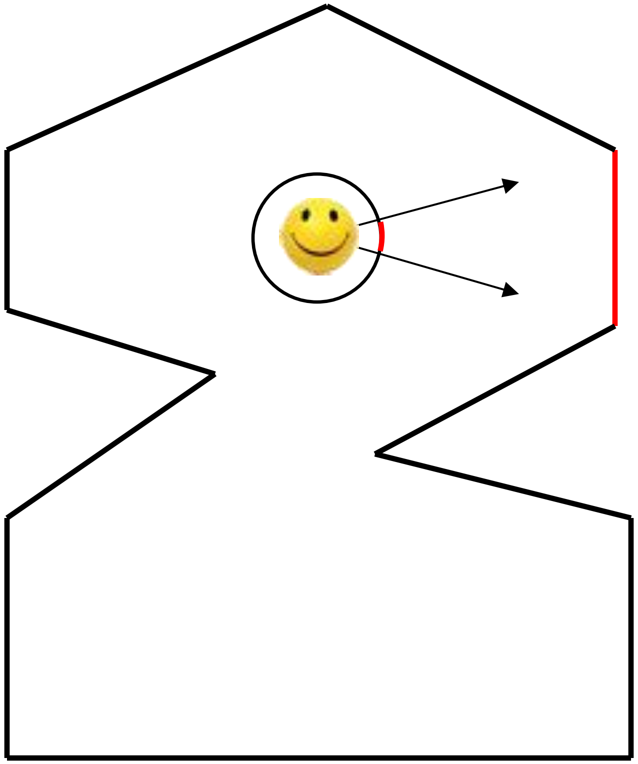
Requirements

- Oriented edges
- Edges can be processed in any order



Advantages

- Extends to 3D!
- Numerically stable
- Even works on models with holes:
  - Odd *k*: inside
  - Even *k*: outside
- No ray casting



Actual Implementation

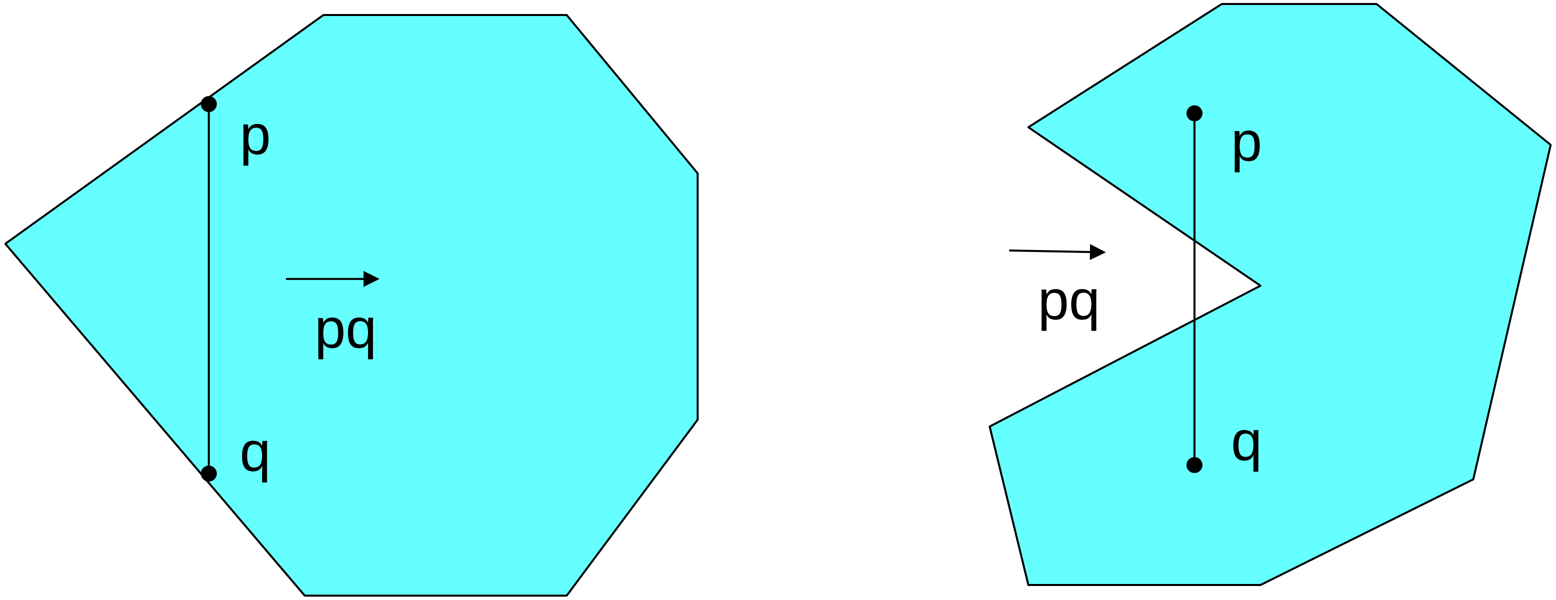
Winding Number

```
Int wn_PnPoly( Point P, Point* V, int n )
{
    int    wn = 0;    // the winding number counter

    // loop through all edges of the polygon
    for (int i=0; i<n; i++) {   // edge from V[i] to V[i+1]
        if (V[i].y <= P.y) {          // start y <= P.y
            if (V[i+1].y > P.y)      // an upward crossing
                if (isLeft( V[i], V[i+1], P) > 0)  // P left of edge
                    ++wn;            // have a valid up intersect
        }
        else {                        // start y > P.y (no test needed)
            if (V[i+1].y <= P.y)     // a downward crossing
                if (isLeft( V[i], V[i+1], P) < 0)  // P right of edge
                    --wn;            // have a valid down intersect
        }
    }
    return wn;
}
```

Topic

- Introduction
- Two lines Intersection Test
- Point inside polygon
- Convex hull
- Line Segments Intersection Algorithm

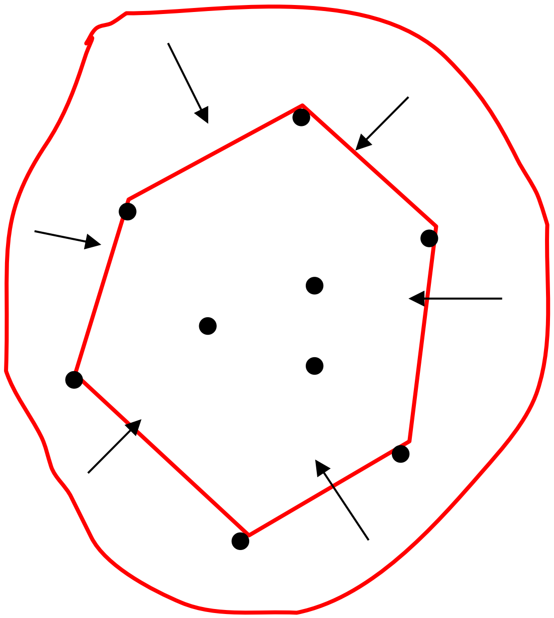
Convex Hulls



Subset of S of the plane is convex, if for all pairs p,q in S the line segment pq is completely contained in S.

The Convex Hull CH(S) is the smallest convex set, which contains S.

Convex hull of a set of points in the plane



Rubber band experiment

The convex hull of a set P of points is the unique convex polygon whose vertices are points of P and which contains all points from P.

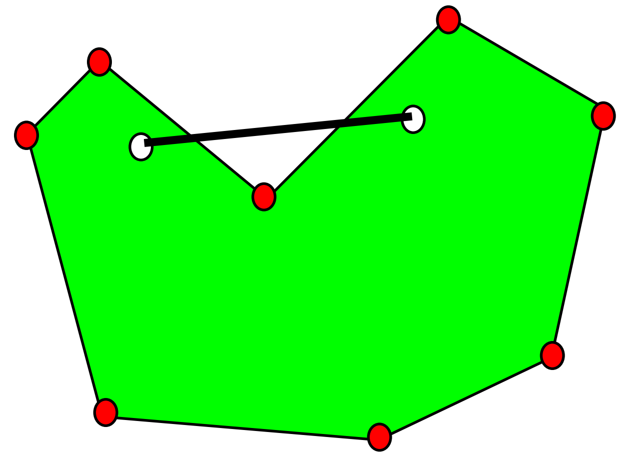
Convexity & Convex Hulls

source: O'Rourke, Computational Geometry in C

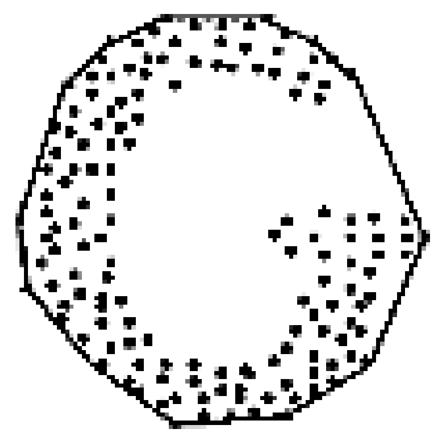
- A convex combination of points $x_1, \ldots, x_k$ is a sum of the form $\alpha_1 x_1 + \ldots + \alpha_k x_k$ where

$$\alpha_i \geq 0 \ \forall i \quad and \ \alpha_1 + \cdots + \alpha_k = 1$$

- Convex hull of a set of points is the set of all convex combinations of points in the set.



nonconvex polygon



convex hull of a point set

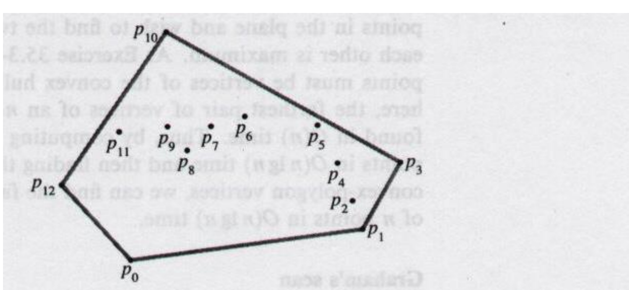
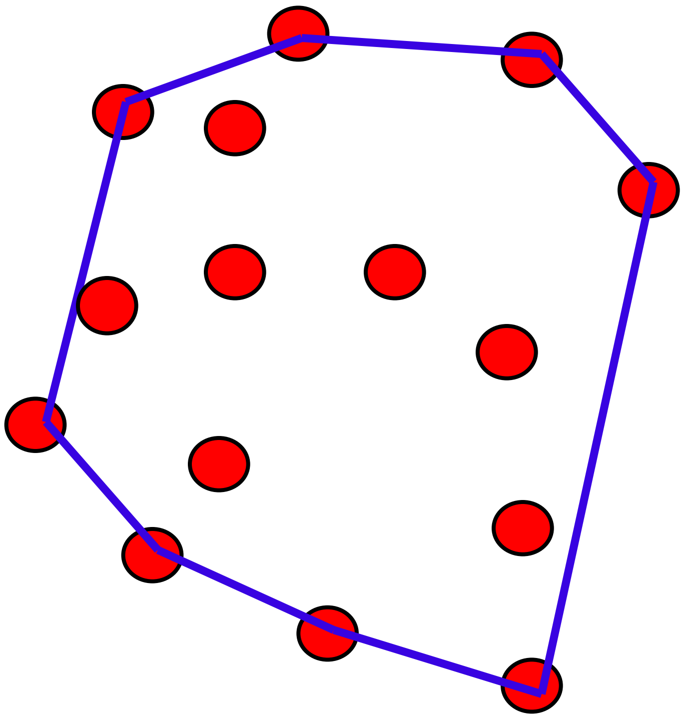


Figure 35.6 A set of points Q with its convex hull CH(Q) in gray.

source: 91.503 textbook Cormen et al.

Convex Hull

- Input:
  - Set S = $\{s_1, \ldots, s_n\}$ of n points
- Output:
  - Find its convex hull



- Many algorithms:
  - Naïve – $O(n^3)$
  - Insertion – $O(n \log n)$
  - Divide and Conquer – $O(n \log n)$
  - Gift Wrapping – $O(nh)$, h = no of points on the hull
  - Graham Scan – $O(n \log n)$

Naive Algorithms for Extreme Points

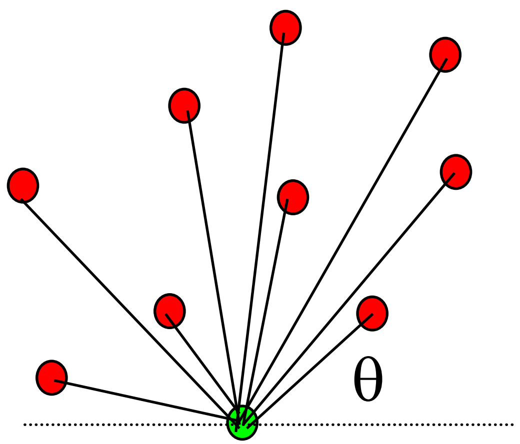
Algorithm: INTERIOR POINTS
for each i do
    for each j ≠ i do
        for each k ≠ j ≠ i do
            for each L ≠ k ≠ j ≠ i do
                if p_L in triangle(p_i, p_j, p_k)
                    then p_L is nonextreme **$O(n^4)$**

Algorithm: EXTREME EDGES
for each i do
    for each j ≠ i do
        for each k ≠ j ≠ i do
            if p_k is not left or on (p_i, p_j)
                then (p_i , p_j) is not extreme **$O(n^3)$**

source: O'Rourke, Computational Geometry in C

Algorithms: 2D Gift Wrapping

- Use one extreme edge as an anchor for finding the next



Algorithm: GIFT WRAPPING

$i_0 \leftarrow$ index of the lowest point

$i \leftarrow i_0$

repeat

  for each $j \neq i$

    Compute counterclockwise angle θ from previous hull edge

  $k \leftarrow$ index of point with smallest θ

  Output (p_i, p_k) as a hull edge

  $i \leftarrow k$

until $i = i_0$

source: O'Rourke, Computational Geometry in C

$O(n^2)$

Gift Wrapping

source: 91.503 textbook Cormen et al.

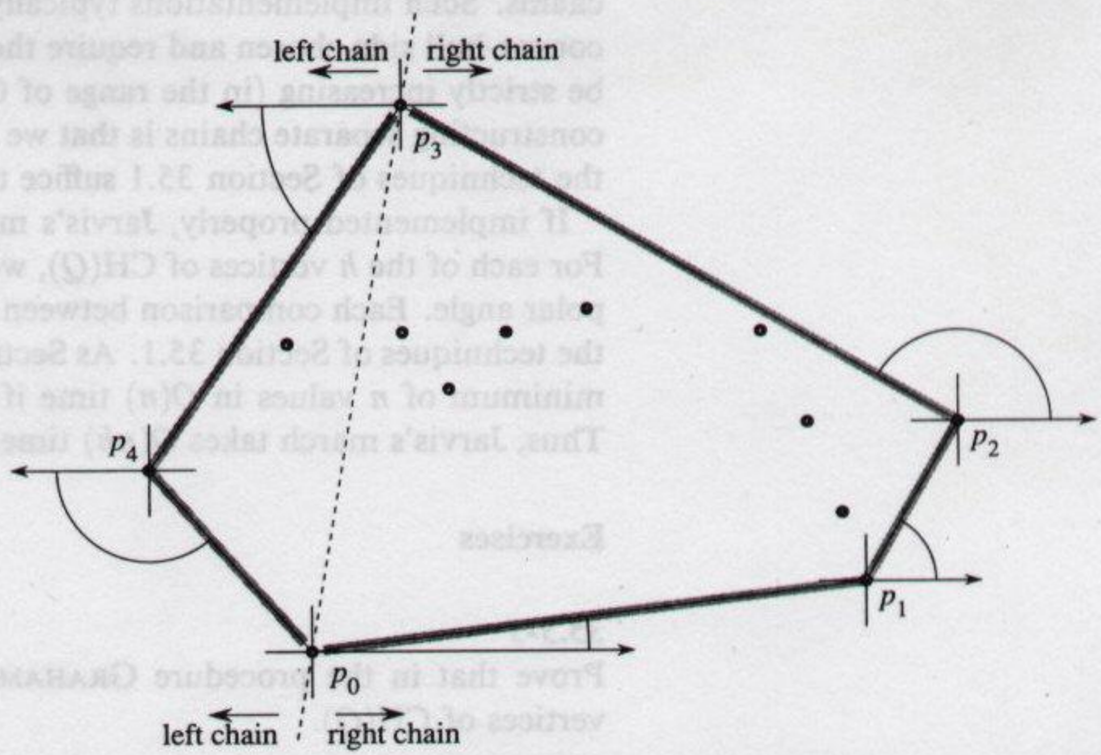
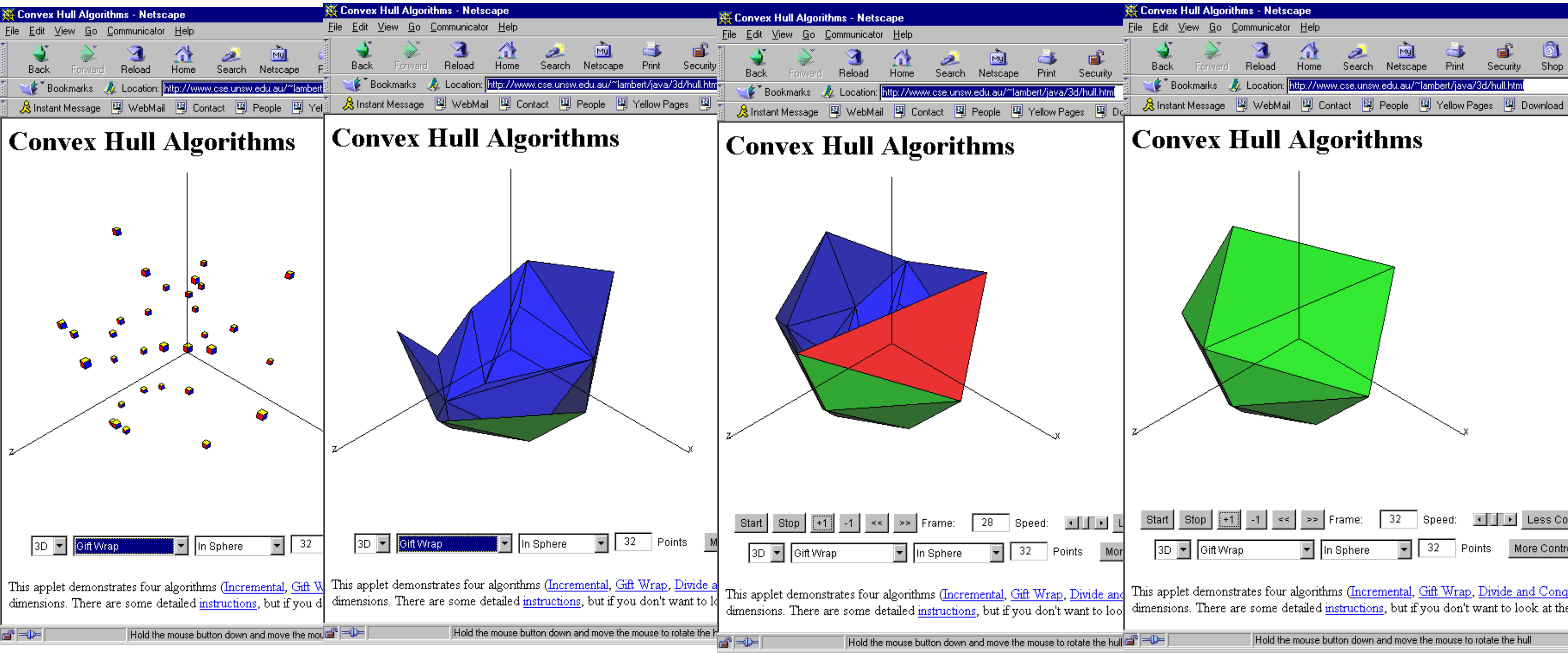


Figure 33.9 The operation of Jarvis's march. The first vertex chosen is the lowest point p_0. The next vertex, p_1, has the least polar angle of any point with respect to p_0. Then, p_2 has the least polar angle with respect to p_1. The right chain goes as high as the highest point p_3. Then, the left chain is constructed by finding least polar angles with respect to the negative x-axis.

<u>Output Sensitivity</u>: O(n^2) run-time is actually O(nh) where h is the number of vertices of the convex hull.

Algorithms: 3D Gift Wrapping



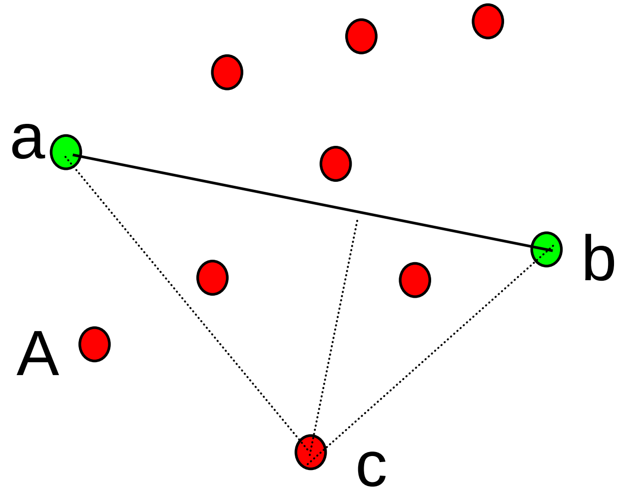
$O(n^2)$ time

[output sensitive: O(nF) for F faces on hull]

CxHull Animations: http://www.cse.unsw.edu.au/~lambert/java/3d/hull.html

Algorithms: 2D QuickHull

- Concentrate on points close to hull boundary
- Named for similarity to Quicksort



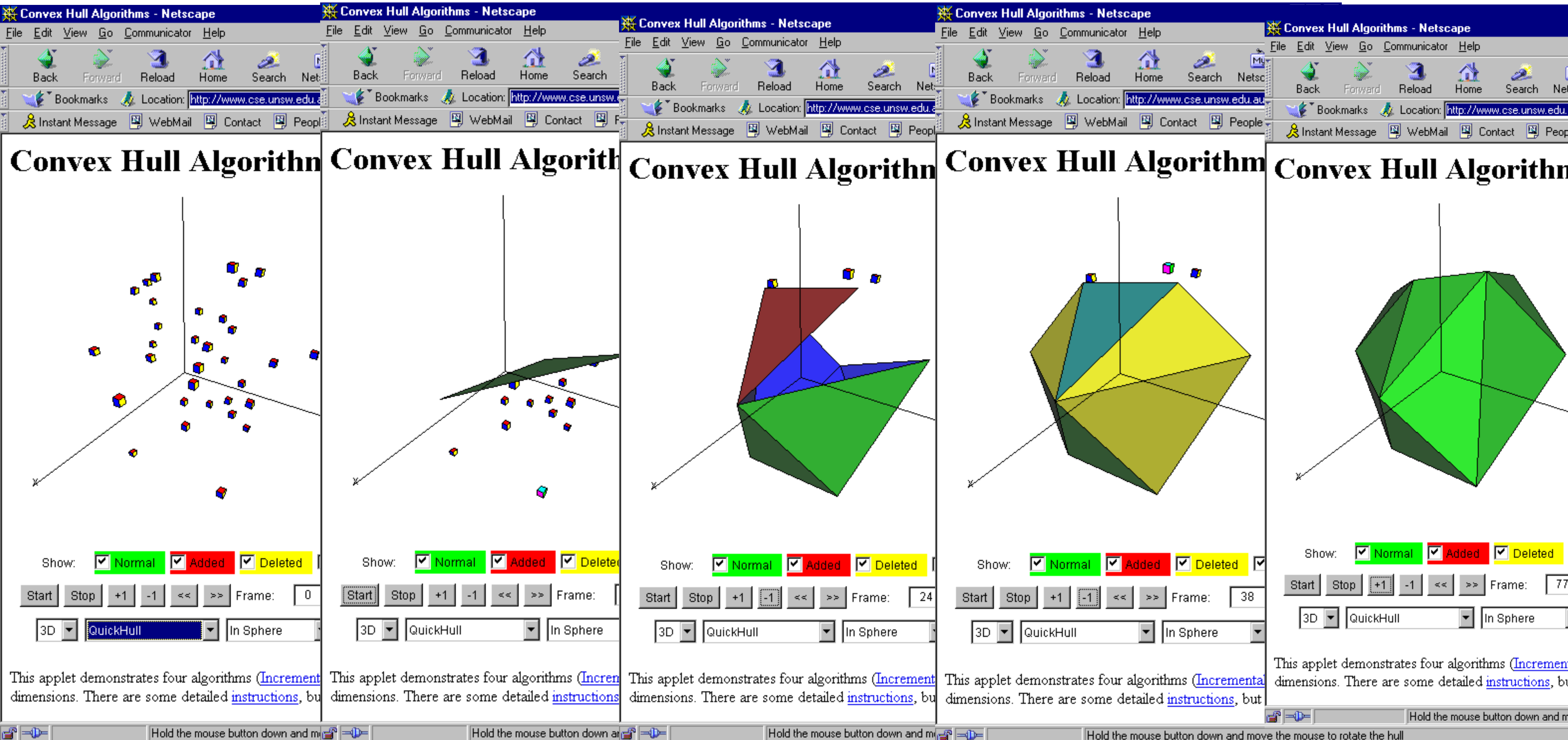
Algorithm: QUICK HULL *finds one of upper or lower hull*

```
function QuickHull(a,b,S)
    if S = ∅ return()
    else
        c ← index of point with max distance from ab
        A ← points strictly right of (a,c)
        B ← points strictly right of (c,b)
        return QuickHull(a,c,A) + (c) + QuickHull(c,b,B)
```

$O(n^2)$

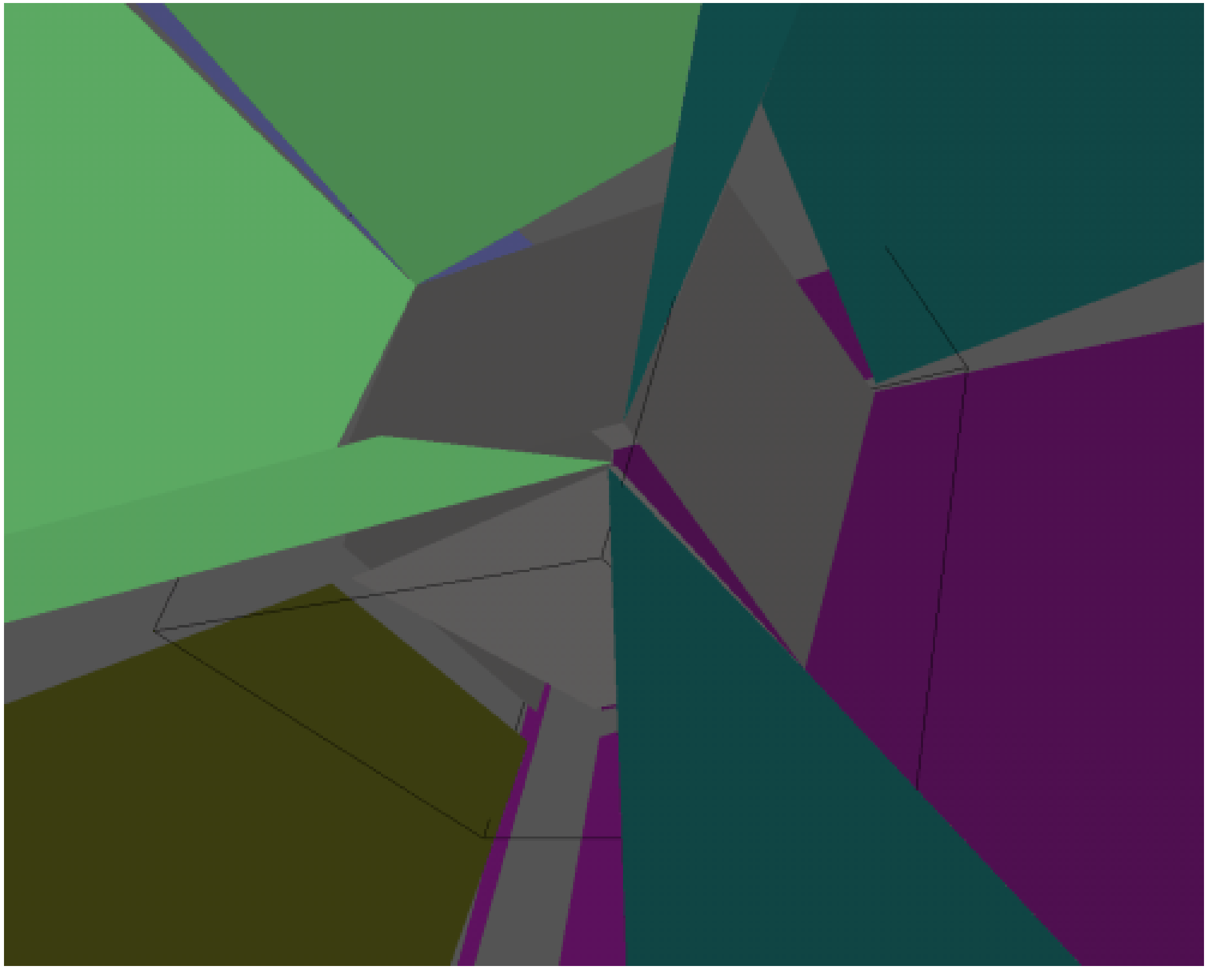
source: O'Rourke, Computational Geometry in C

Algorithms: 3D QuickHull



CxHull Animations: http://www.cse.unsw.edu.au/~lambert/java/3d/hull.html

Algorithms: >= 2D



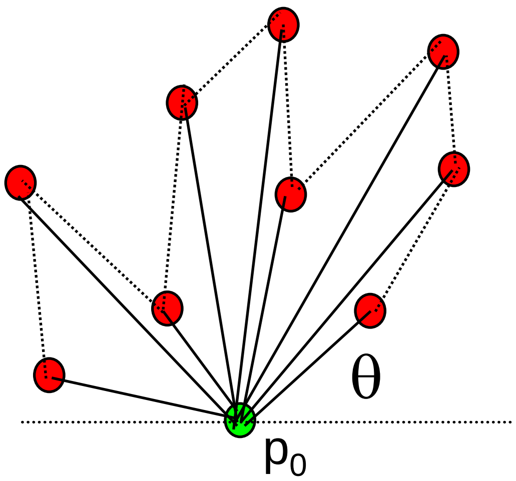
Convex Hull boundary is intersection of hyperplanes, so worst-case combinatorial size (not necessarily running time) complexity is in: $\Theta(n^{\lfloor d/2 \rfloor})$

Qhull: http://www.qhull.org/

Graham's Algorithm

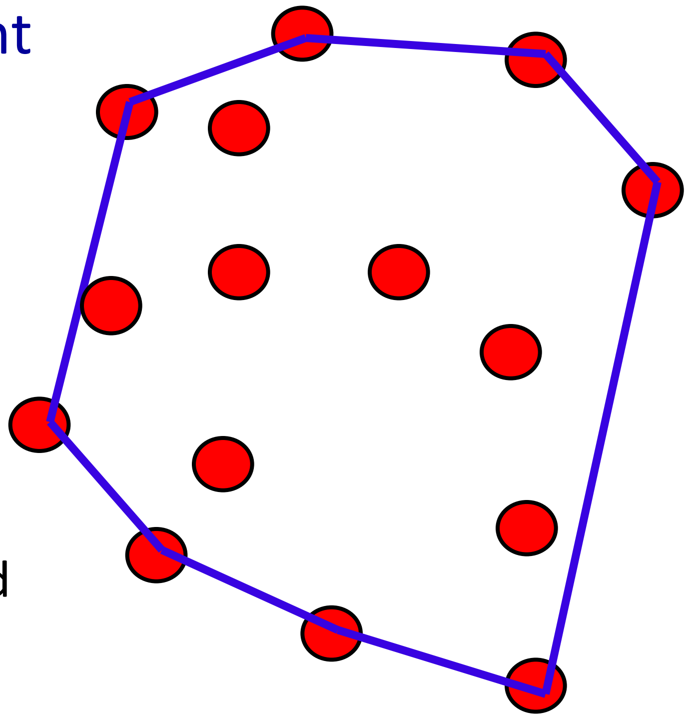
source: O'Rourke, Computational Geometry in C

- Points sorted angularly provide "star-shaped" starting point
- Prevent "dents" as you go via convexity testing



Graham Scan

- Polar sort the points around a point inside the hull
- Scan points in counter-clockwise (CCW) order
  - Discard any point that causes a clockwise (CW) turn
    - If CCW, advance
    - If !CCW, discard current point and back up



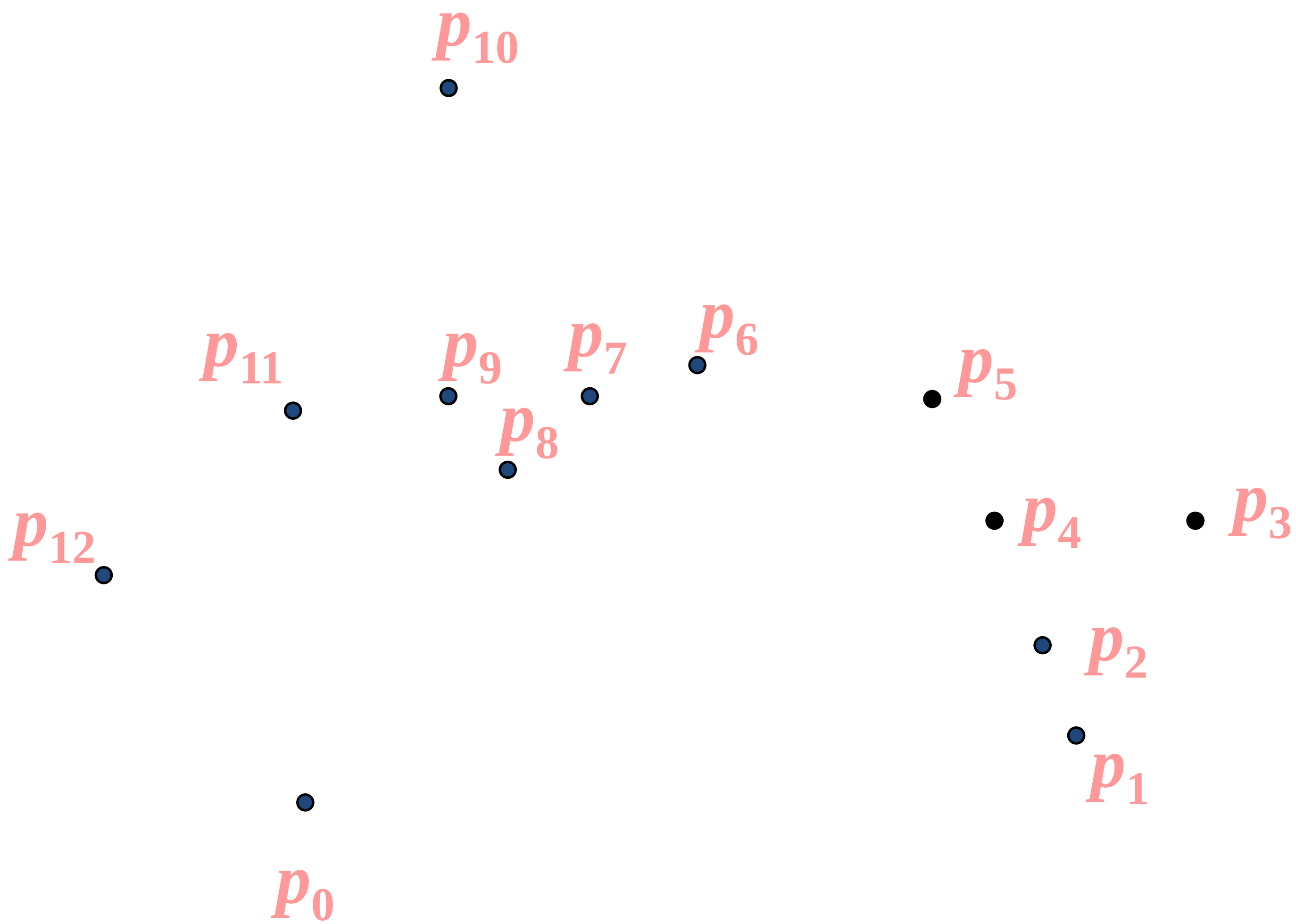
Graham Scan

GRAHAM-SCAN(Q)

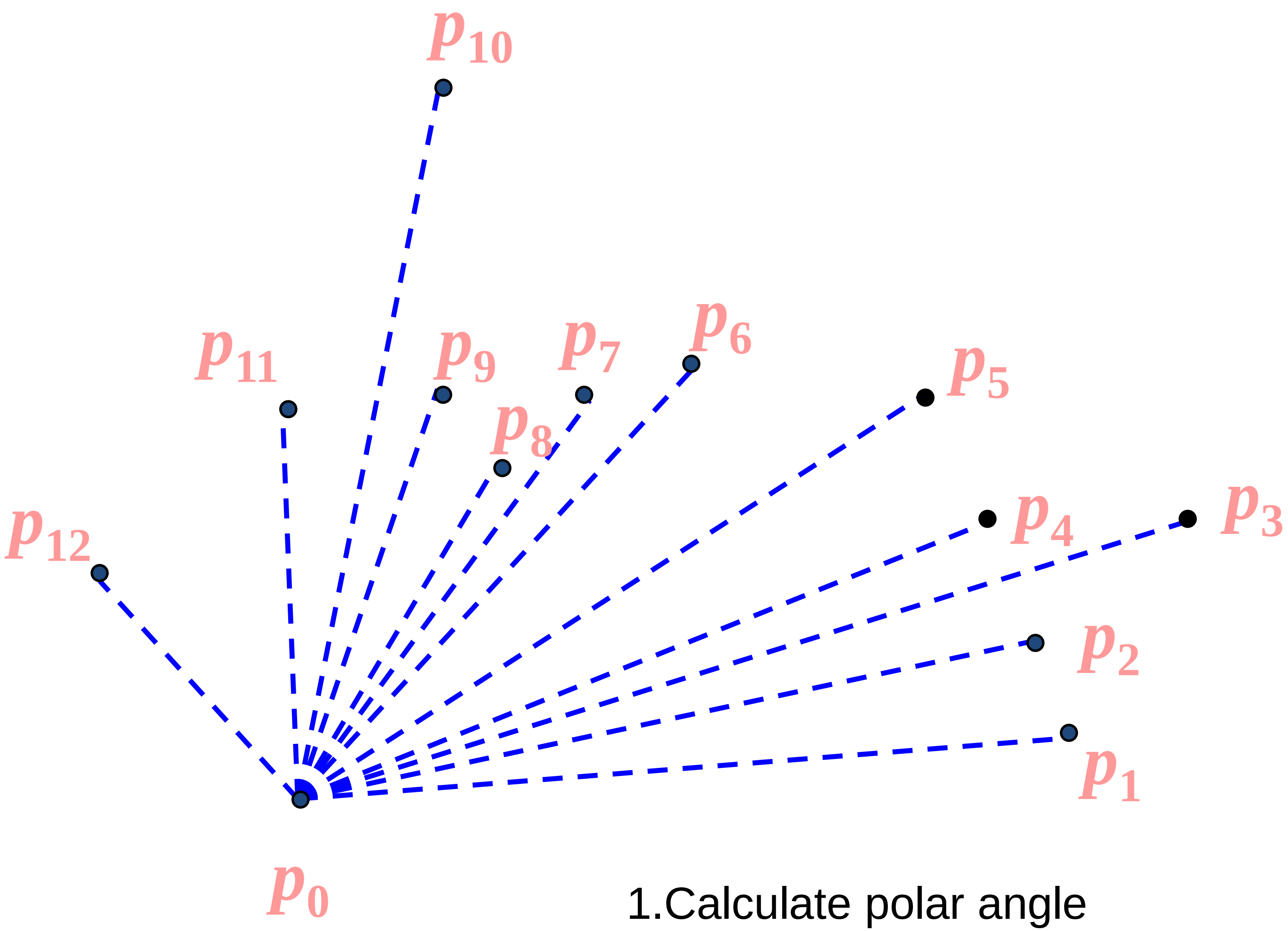
let p_0 be the point in Q with the minimum y-coordinate,
    or the leftmost such point in case of a tie
let $\langle p_1, p_2, \ldots, p_m \rangle$ be the remaining points in Q,
    sorted by polar angle in counterclockwise order around p_0
    (if more than point has the same angle, remove all but
    the one that is farthest from p_0)
top[S] $\leftarrow 0$
PUSH(p_0, S)
PUSH(p_1, S)
PUSH(p_2, S)
for $i \leftarrow 3$ **to** m
    do while the angle formed by points NEXT-TO-TOP(S),
            TOP(S), and p_i makes a nonleft turn
        do POP(S)
    PUSH(S, p_i)
return S

source: 91.503 textbook Cormen et al.

Graham-Scan : (1/11)



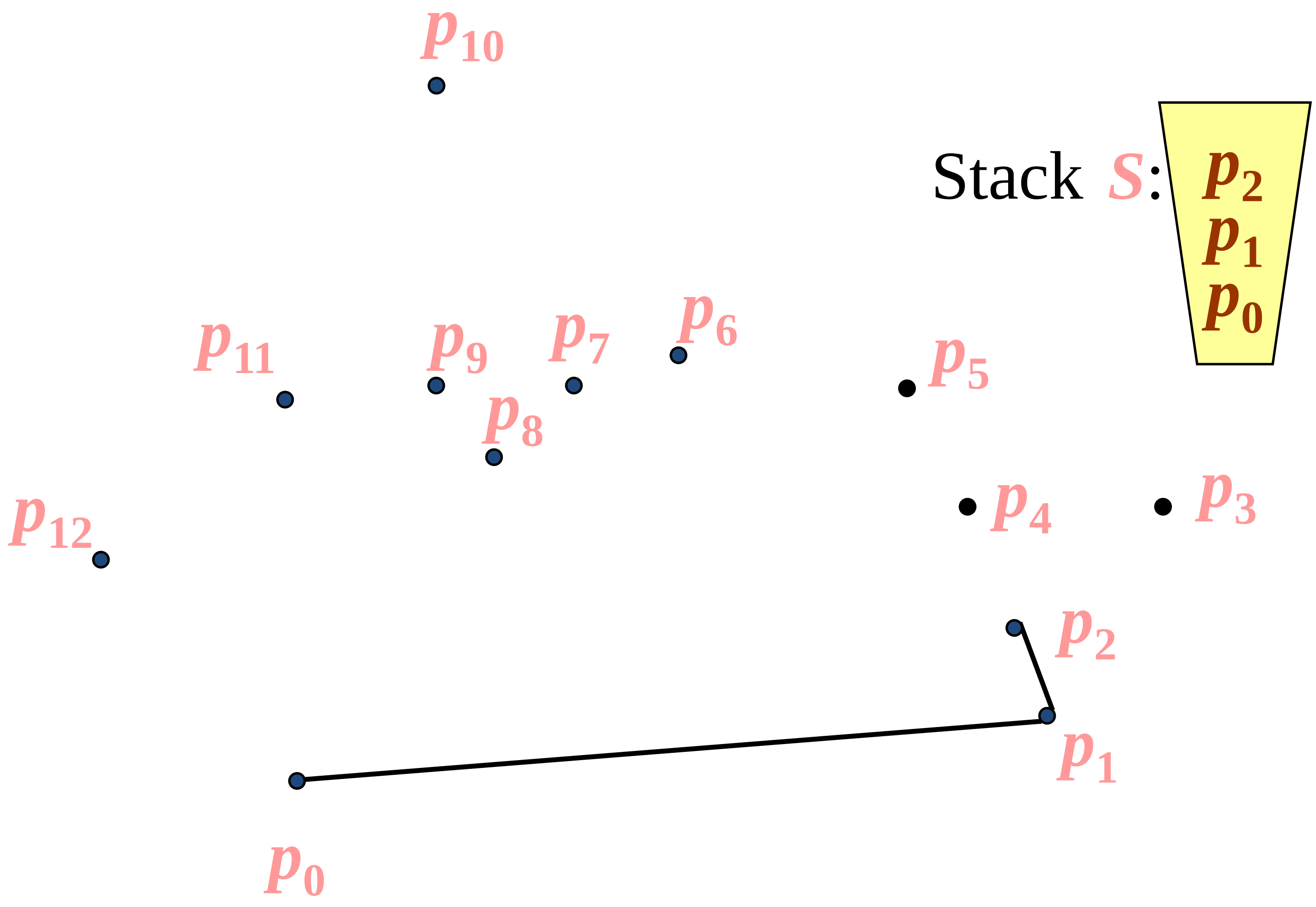
Graham-Scan :(1/11)



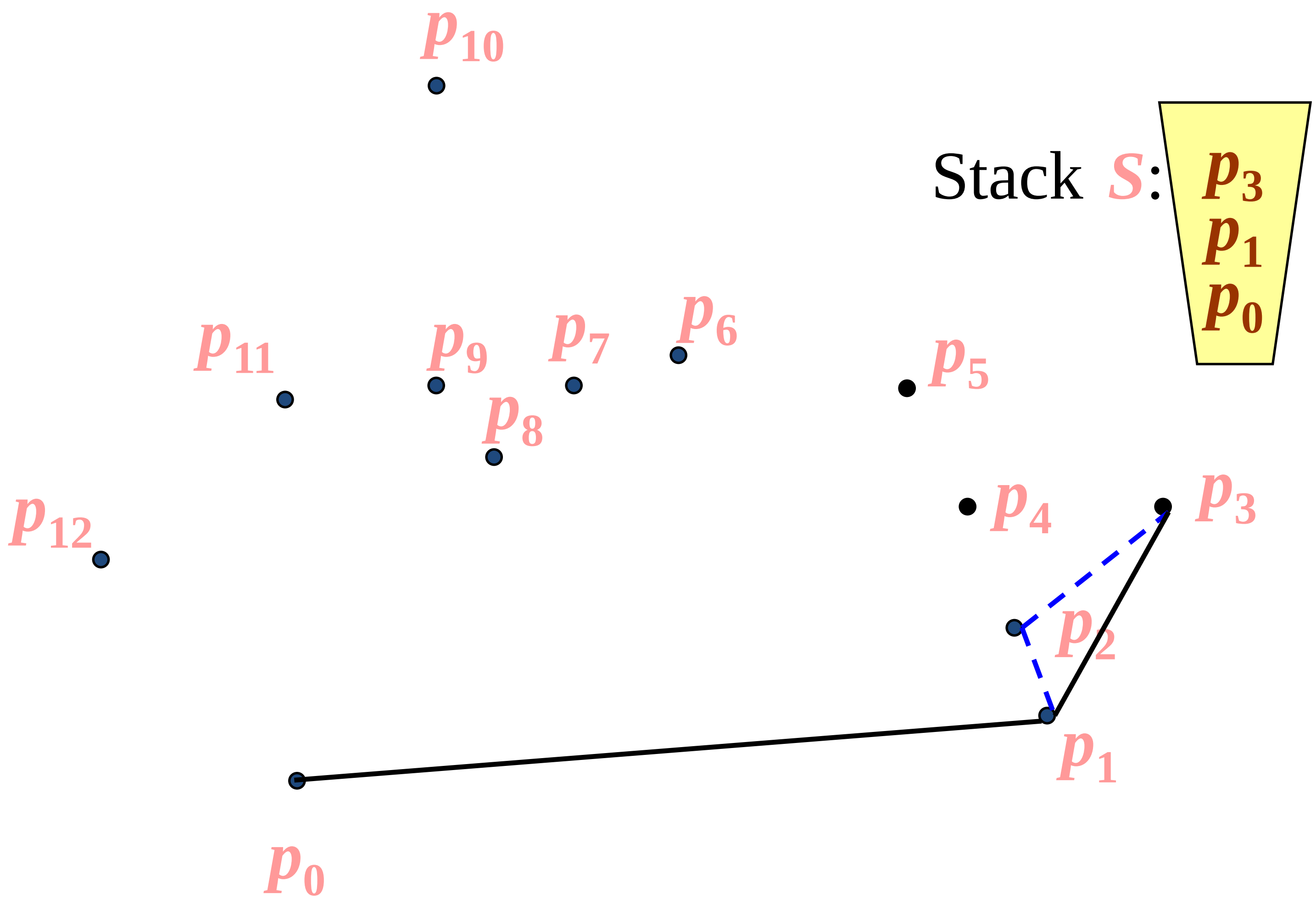
1.Calculate polar angle

2.Sorted by polar angle

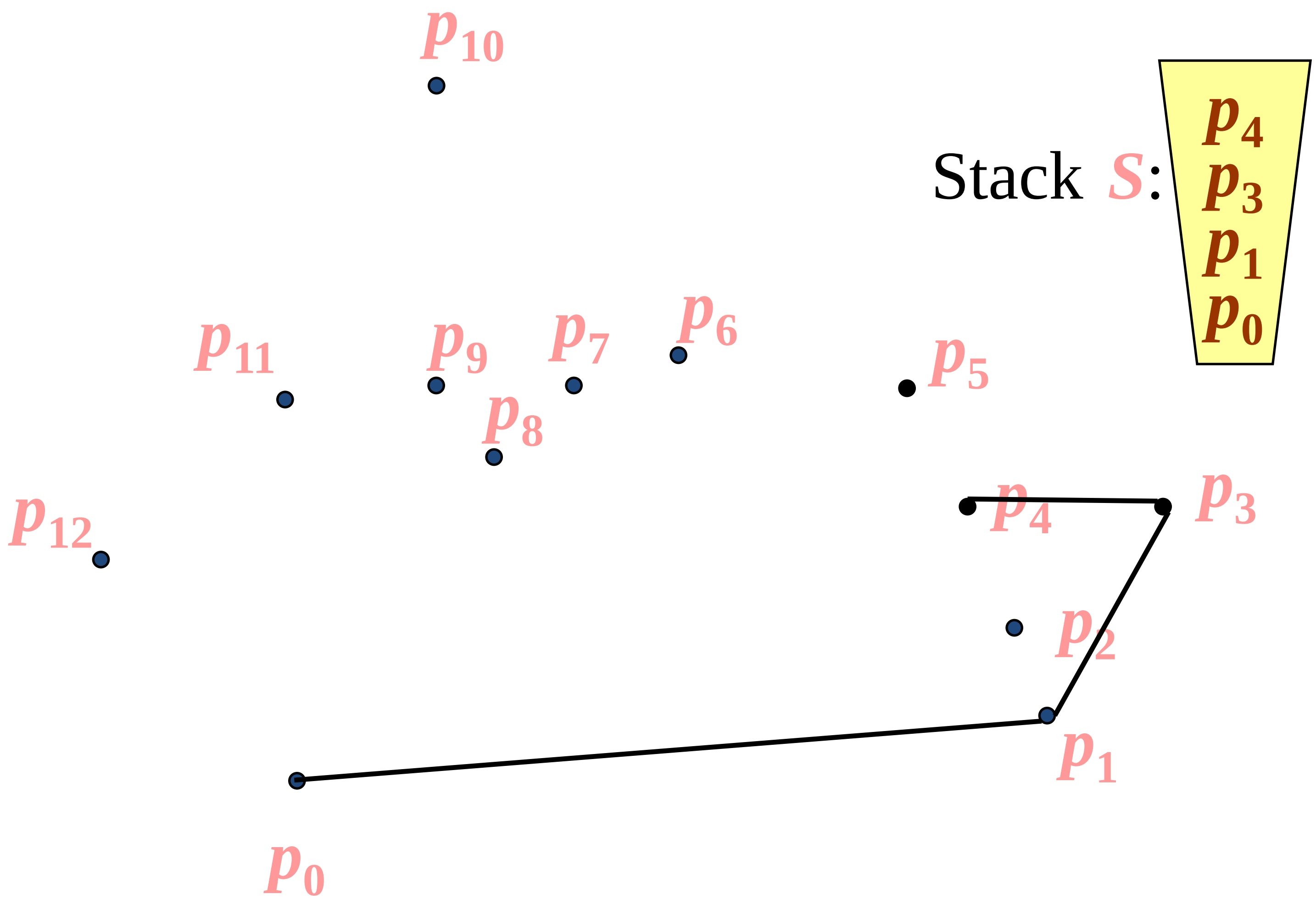
Graham-Scan : (2/11)



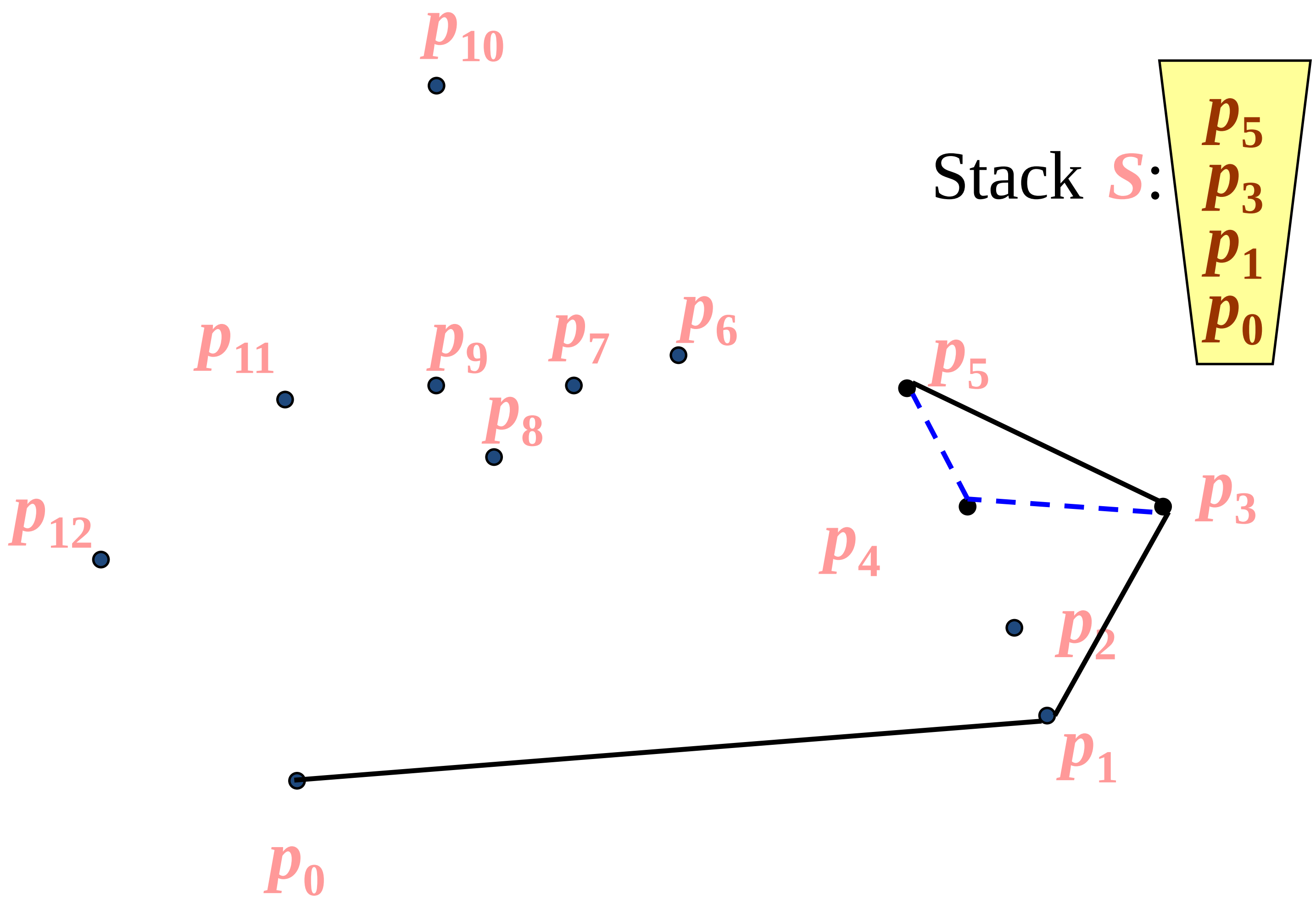
Graham-Scan : (3/11)



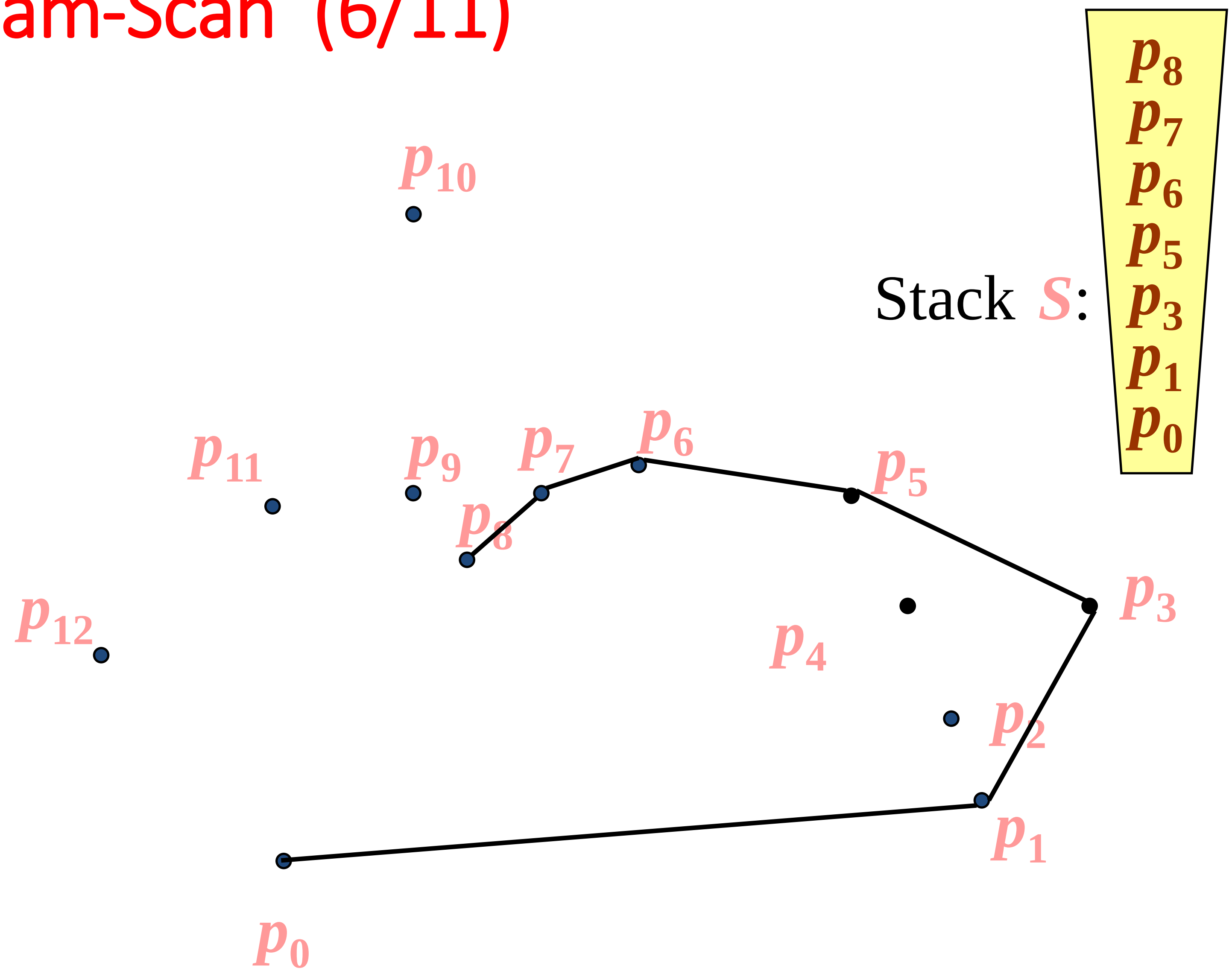
Graham-Scan : (4/11)



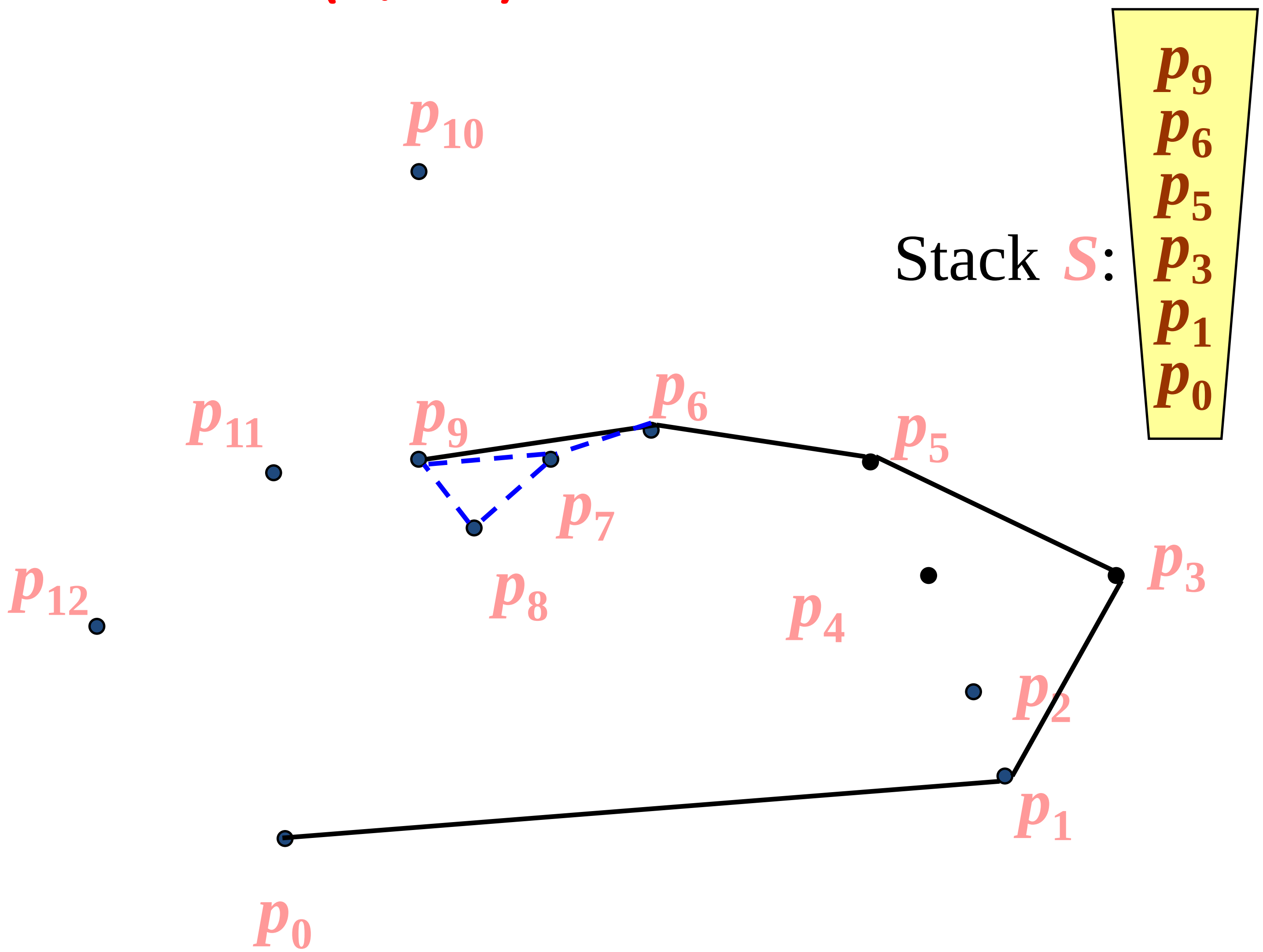
Graham-Scan (5/11)



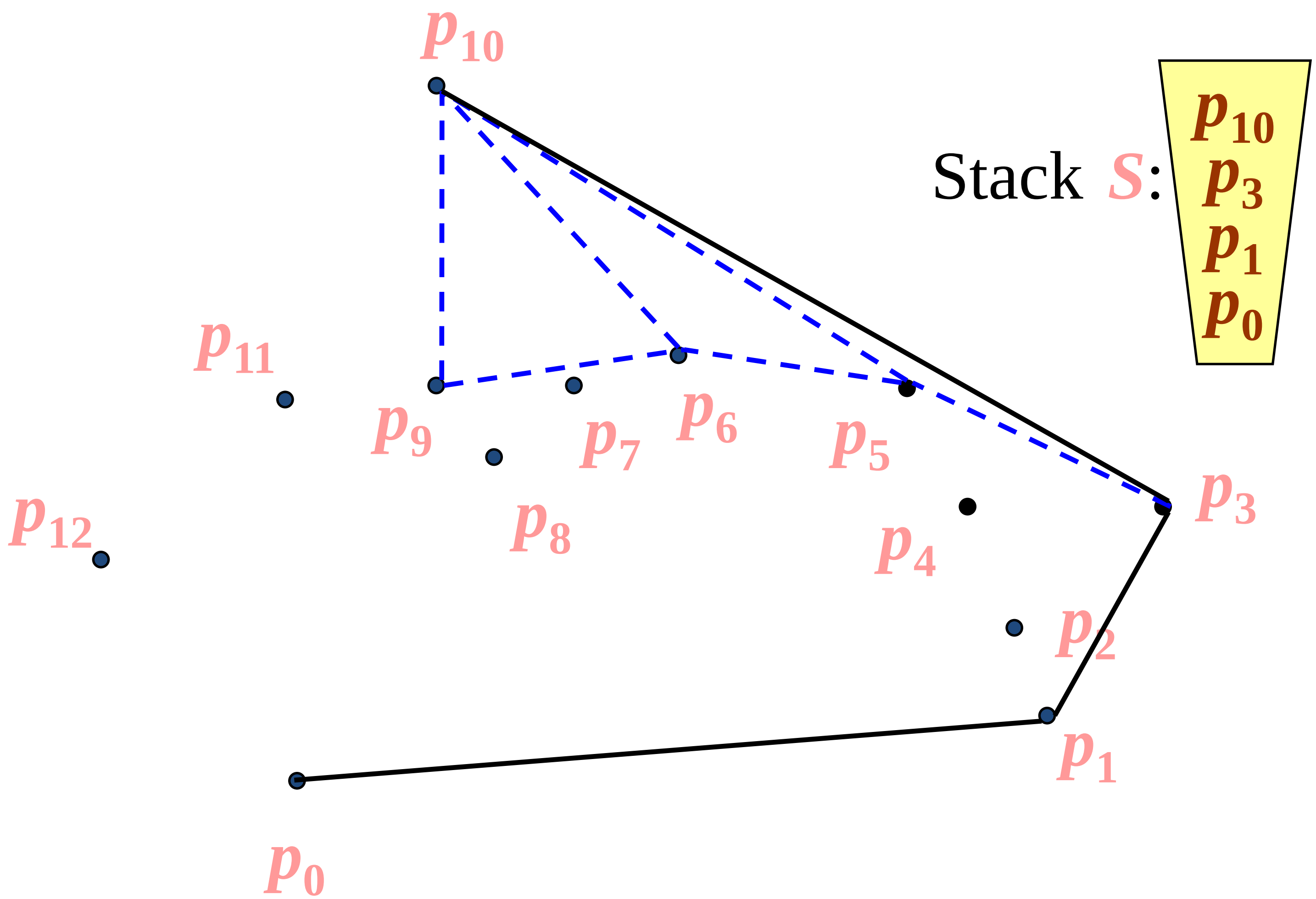
Graham-Scan (6/11)



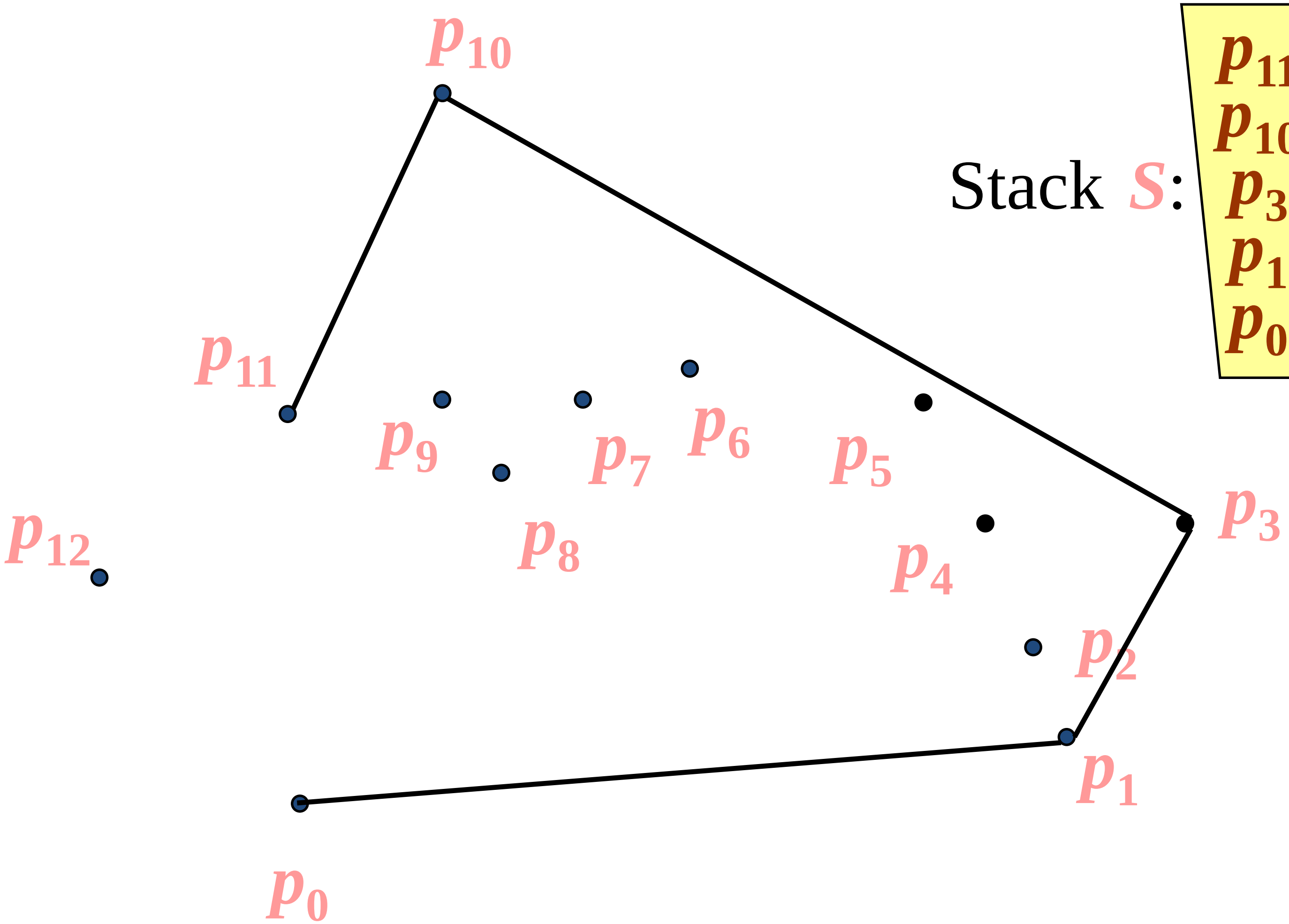
Graham-Scan (7/11)



Graham-Scan (8/11)

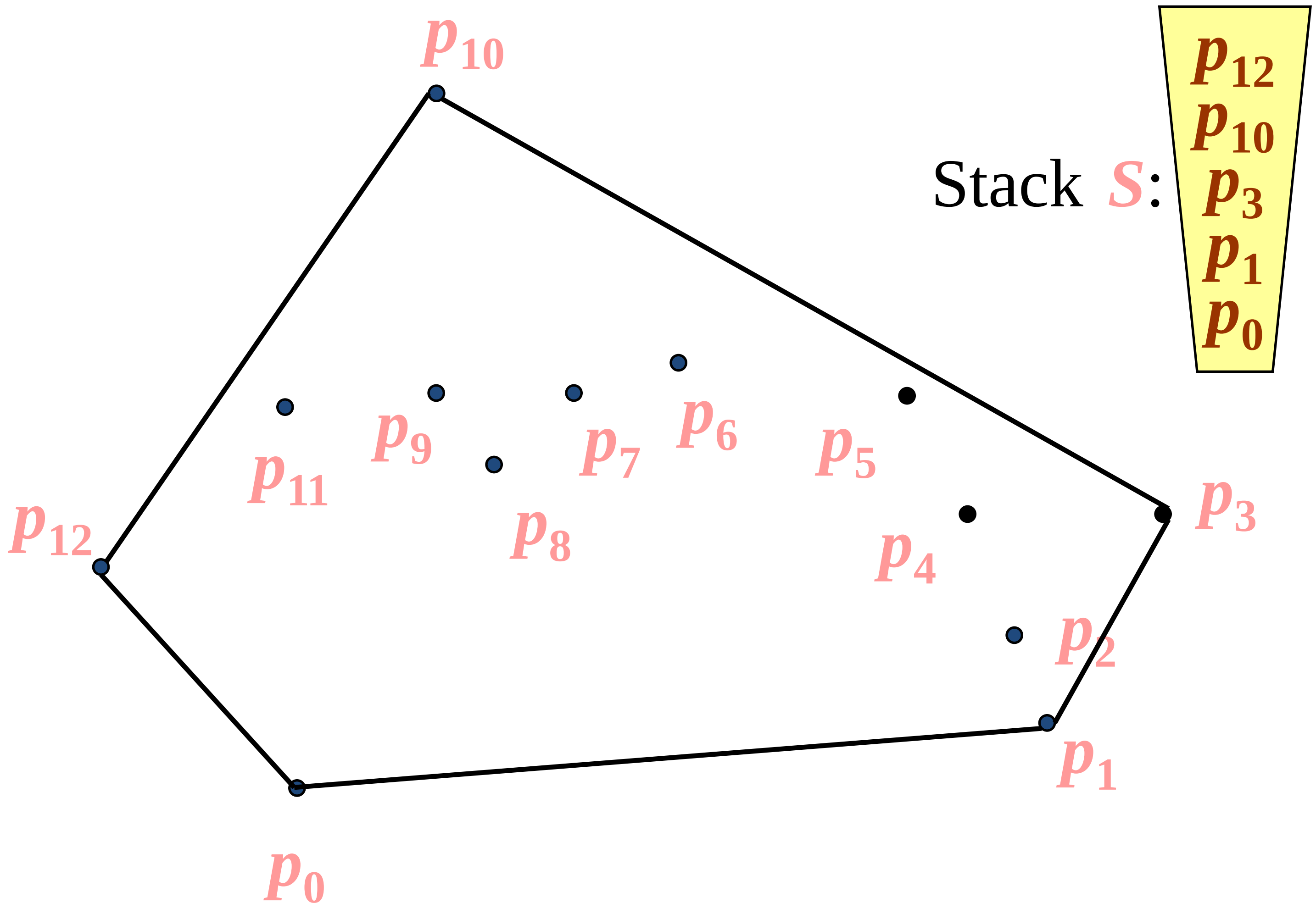


Graham-Scan (9/11)



Stack S: p_{11}, p_{10}, p_3, p_1, p_0

Graham-Scan (10/11)



Time complexity Analysis

- Graham-Scan
  - Sorting in step 2 needs $O(n \log n)$.
  - Time complexity of stack operation is $O(2n)$
  - The overall time complexity in **Graham-Scan** is $O(n \log n)$.

- Demo:
  - http://www.cse.unsw.edu.au/~lambert/java/3d/hull.html

Graham Scan

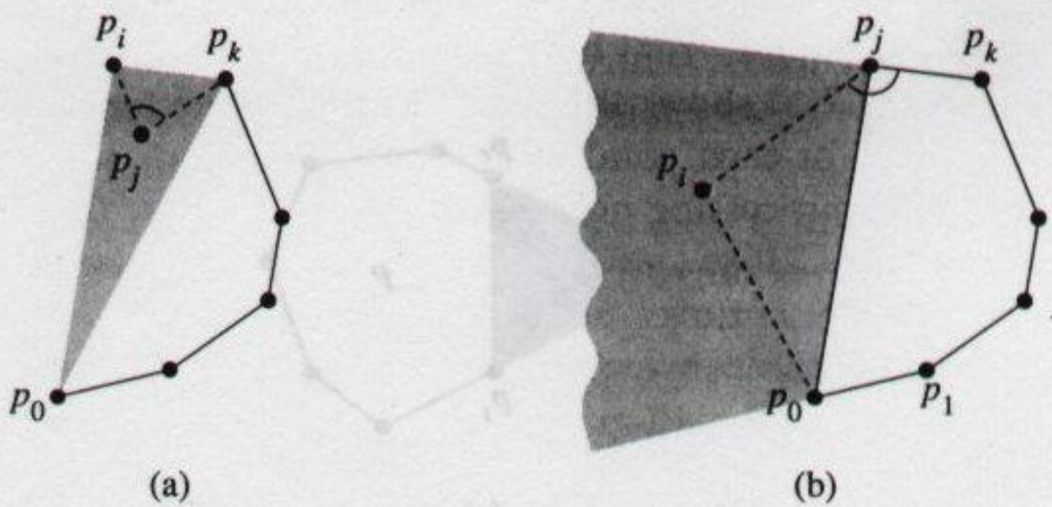


Figure 35.8 The two basic situations in the proof of correctness of GRAHAM-SCAN. **(a)** Showing that a point popped from the stack in GRAHAM-SCAN is not a vertex of CH(Q). If point p_j is popped from the stack because angle $\angle p_k p_j p_i$ makes a nonleft turn, then the shaded triangle $\triangle p_0 p_k p_i$ contains point p_j. Point p_j is therefore not a vertex of CH(Q). **(b)** If point p_i is pushed onto the stack, then there must be a left turn at angle $\angle p_k p_j p_i$. Because p_i follows p_j in the polar-angle ordering of points and because of how p_0 was chosen, p_i must be in the shaded region. If the points on the stack form a convex polygon before the push, then they must form a convex polygon afterward.

source: 91.503 textbook Cormen et al.

Graham Scan

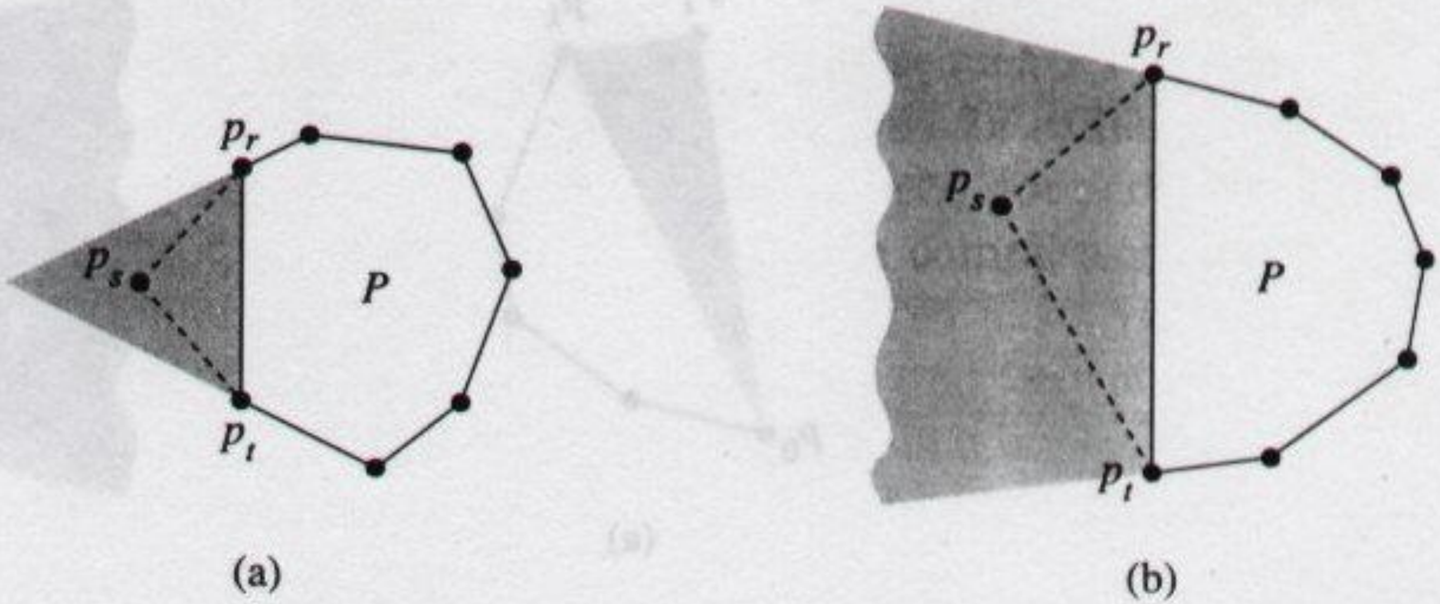


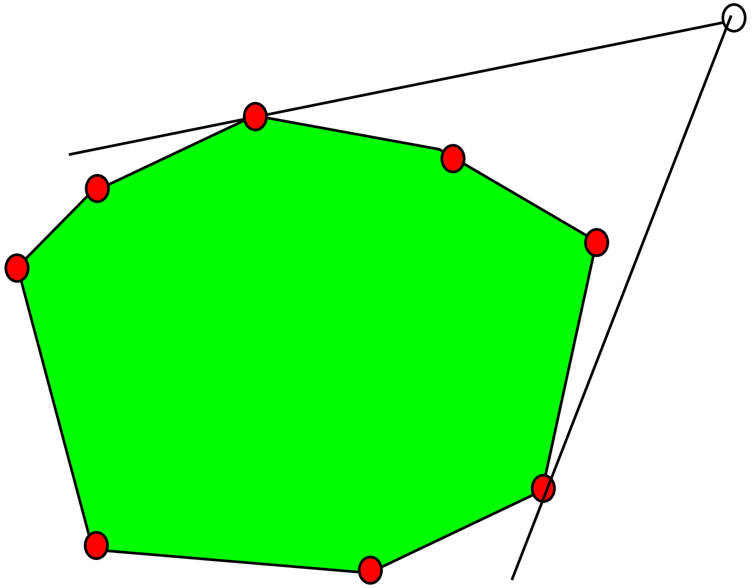
Figure 35.9 Adding a point in the shaded region to a convex polygon P yields another convex polygon. The shaded region is bounded by a side of $\overline{p_r p_t}$ and the extensions of the two adjacent sides. **(a)** The shaded region is bounded. **(b)** The shaded region is unbounded.

source: 91.503 textbook Cormen et al.

Algorithms: 2D Incremental

source: O'Rourke, Computational Geometry in C

- Add points, one at a time
  - update hull for each new point
- Key step becomes adding a single point to an existing hull.
  - Find 2 tangents
    - Results of 2 consecutive LEFT tests differ
- Idea can be extended to 3D.



Algorithm: INCREMENTAL ALGORITHM

Let $H_2 \leftarrow$ ConvexHull{p_0 , p_1 , p_2}

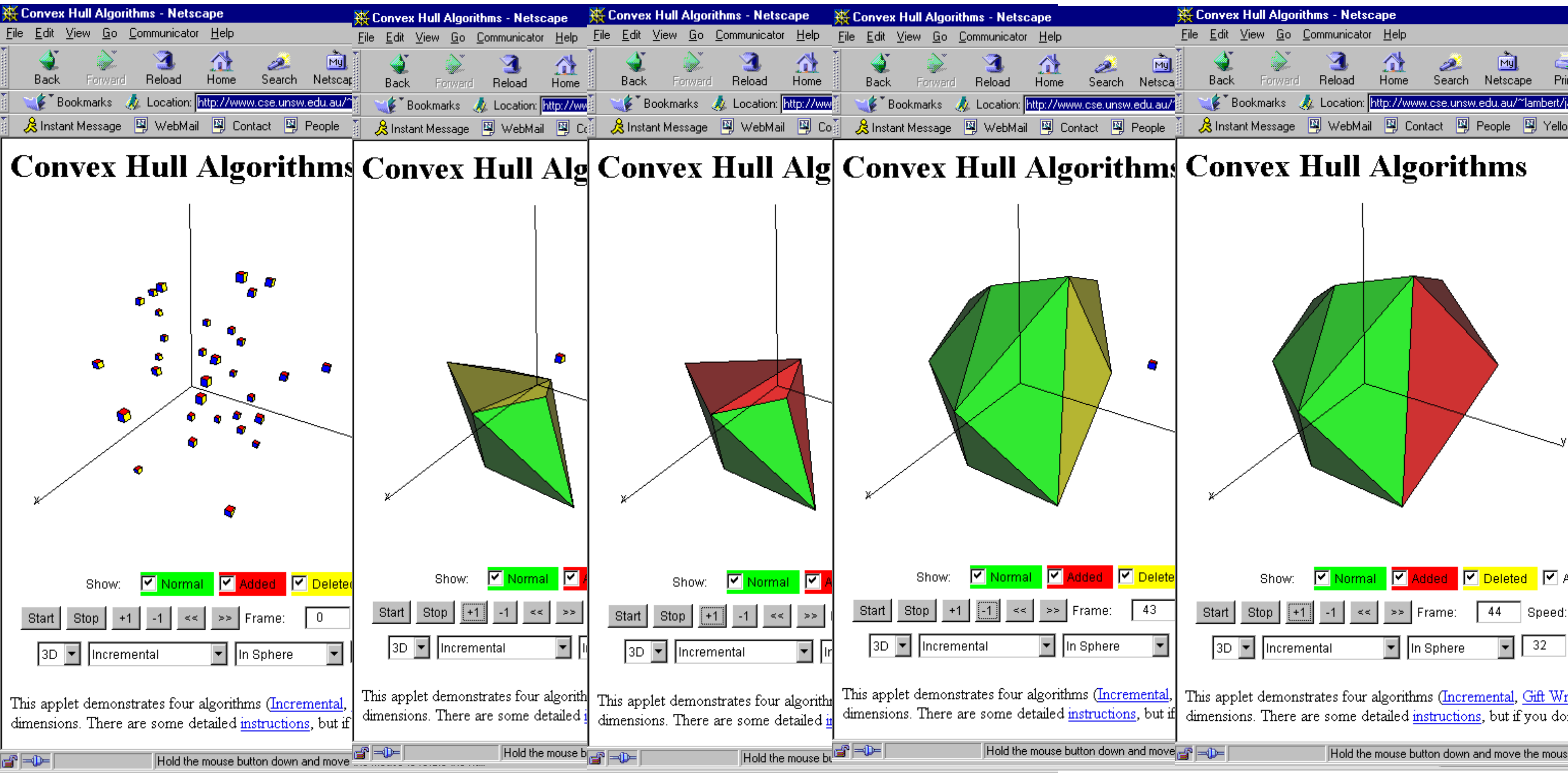
for k $\leftarrow$ 3 to n - 1 do

$H_k \leftarrow$ ConvexHull{ H_{k-1} U p_k }

$O(n^2)$

can be improved to O(nlgn)

Algorithms: 3D Incremental



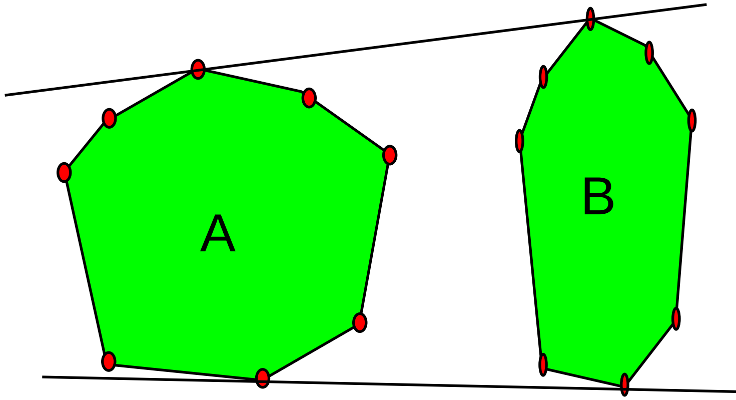
O(n^2) time

CxHull Animations: http://www.cse.unsw.edu.au/~lambert/java/3d/hull.html

Algorithms:
2D Divide-and-Conquer

source: O'Rourke, Computational Geometry in C

- Divide-and-Conquer in a geometric setting
- O(n) merge step is the challenge
  - Find upper and lower tangents
  - Lower tangent: find rightmost pt of A & leftmost pt of B; then "walk it downwards"
- Idea can be extended to 3D.

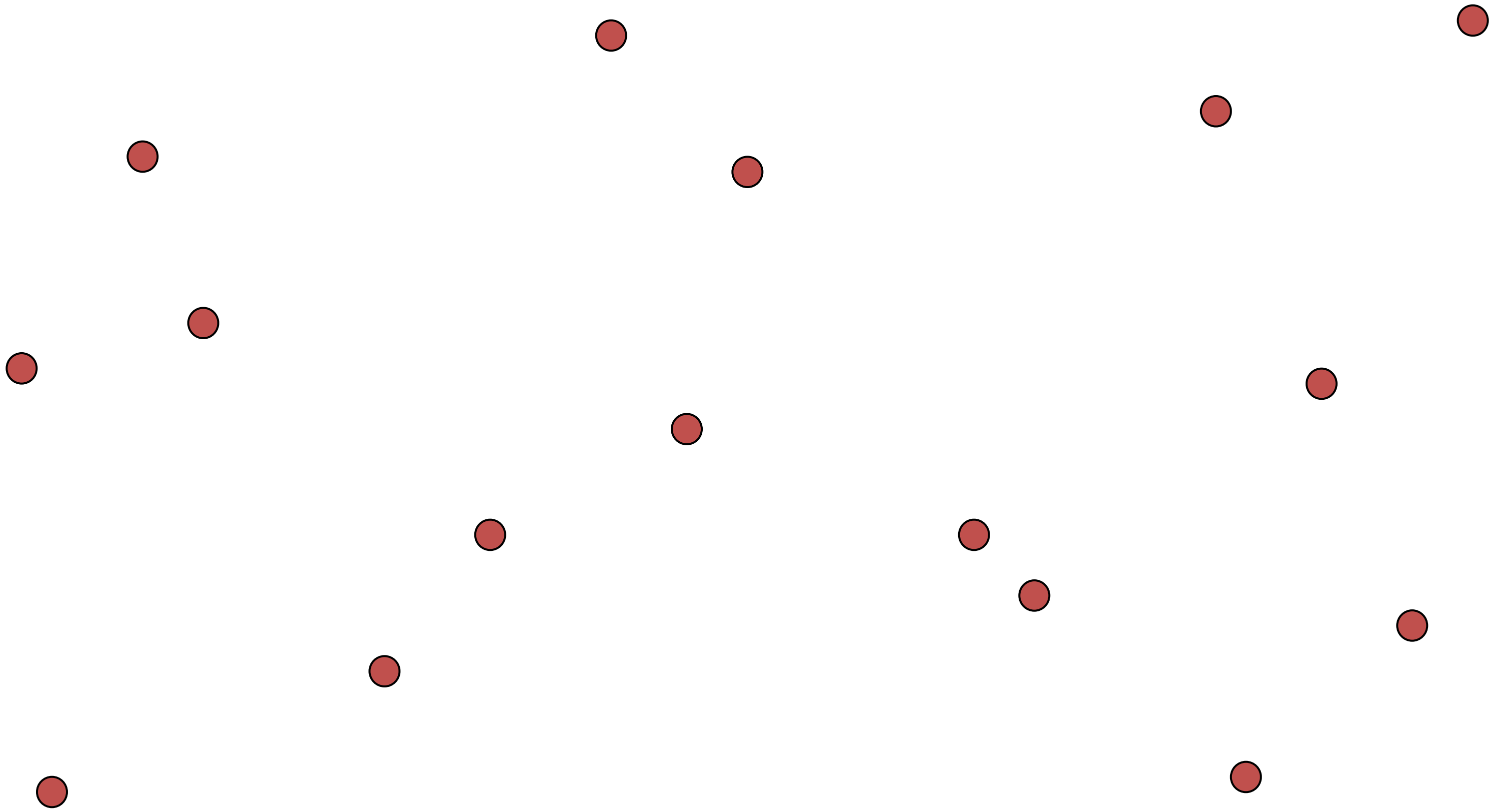


Algorithm: DIVIDE-and-CONQUER
Sort points by x coordinate
Divide points into 2 sets A and B:
A contains left $\lceil n/2 \rceil$ points
B contains right $\lfloor n/2 \rfloor$ points
Compute ConvexHull(A) and ConvexHull(B) recursively
Merge ConvexHull(A) and ConvexHull(B)

O(nlgn)

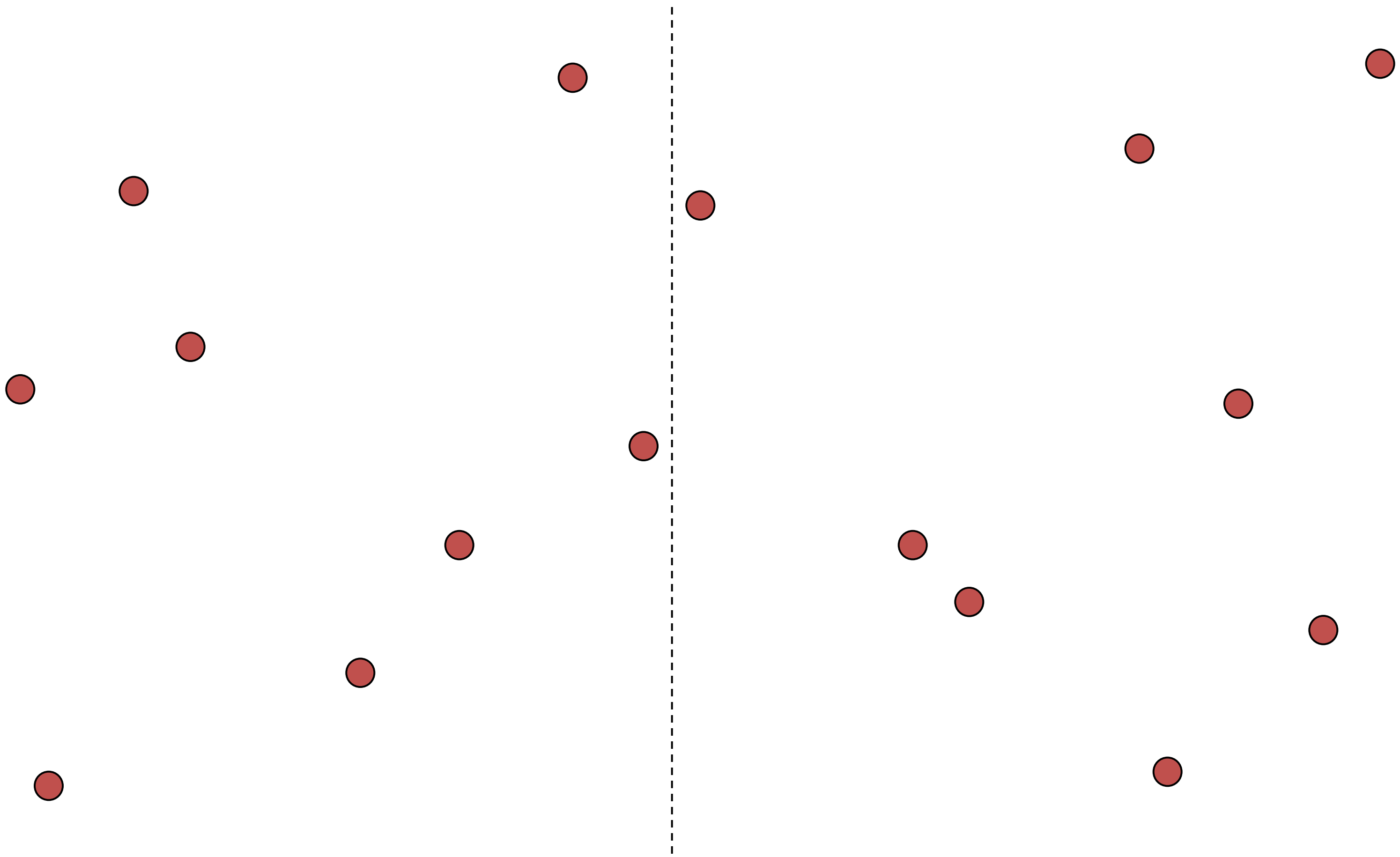
Convex Hull – Divide & Conquer

- Split set into two, compute convex hull of both, combine.



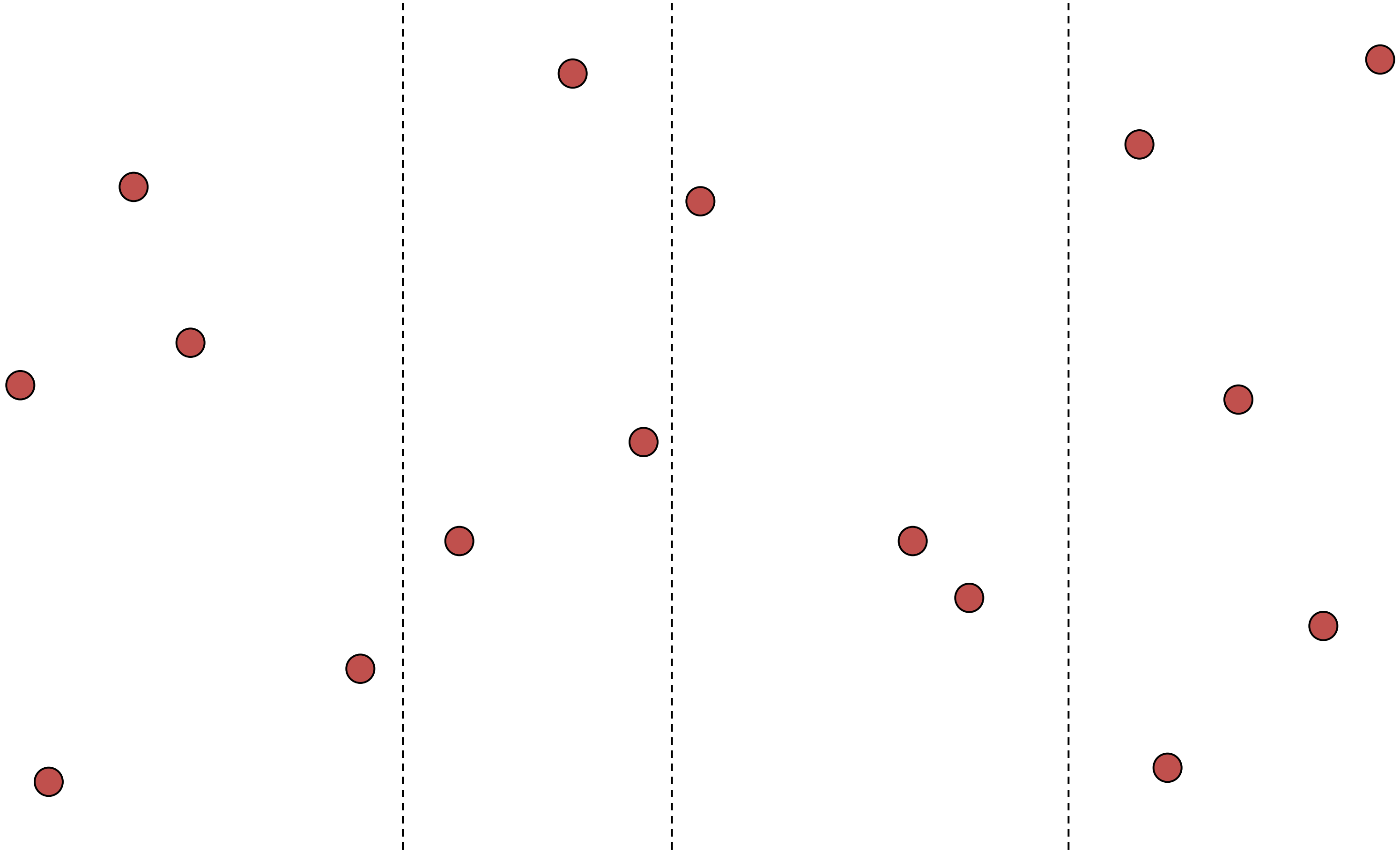
Convex Hull – Divide & Conquer

- Split set into two, compute convex hull of both, combine.



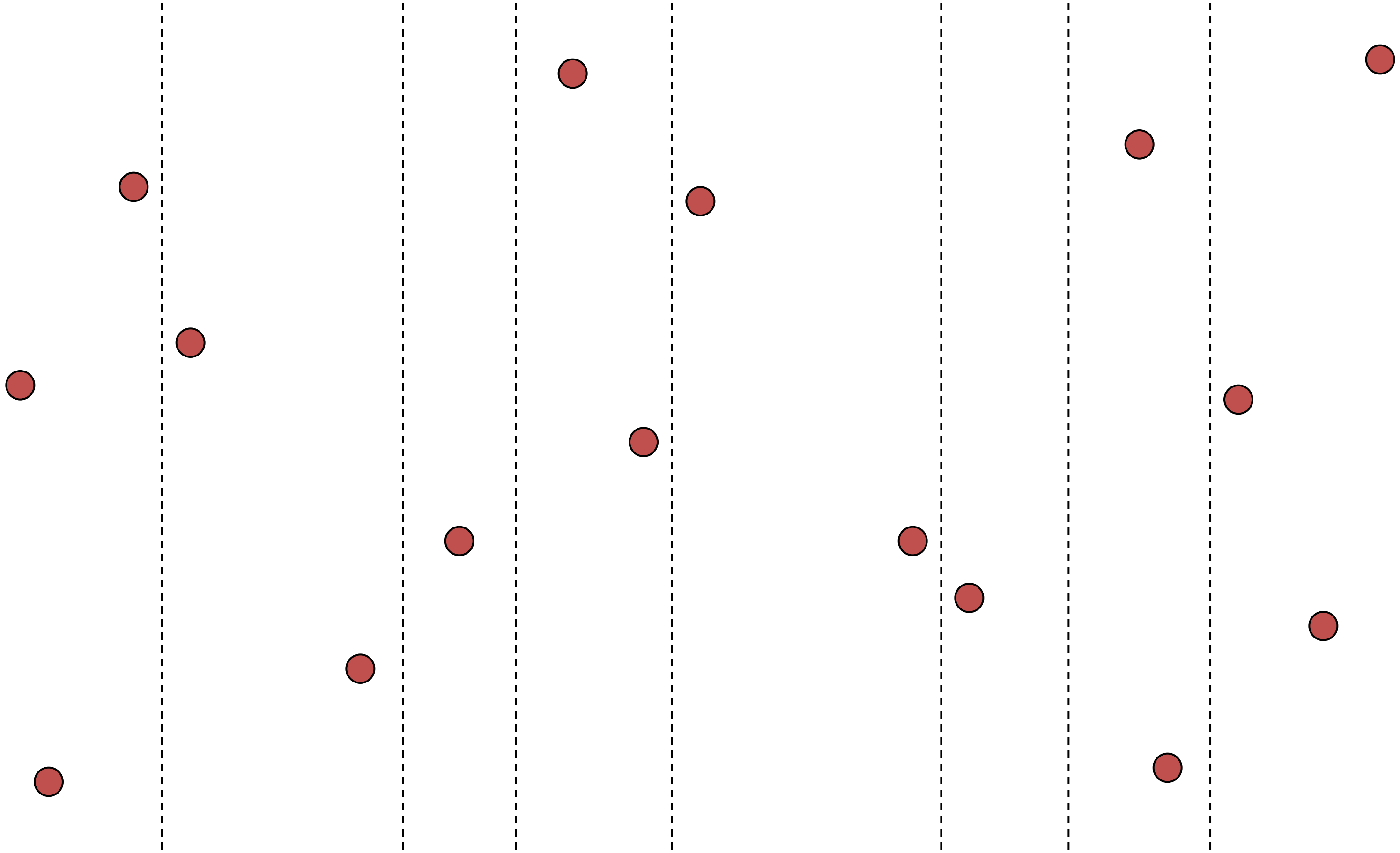
Convex Hull – Divide & Conquer

- Split set into two, compute convex hull of both, combine.



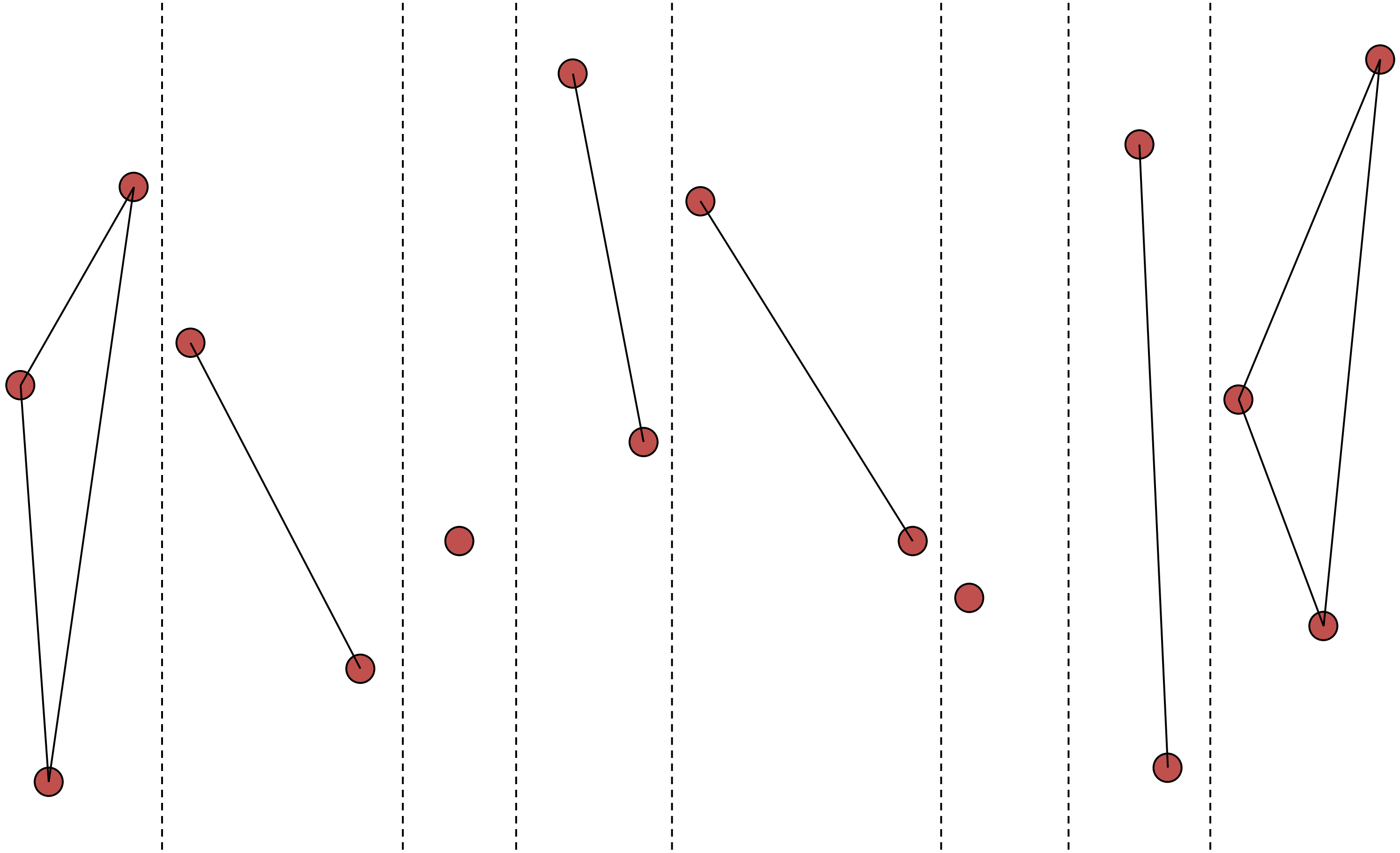
Convex Hull – Divide & Conquer

- Split set into two, compute convex hull of both, combine.



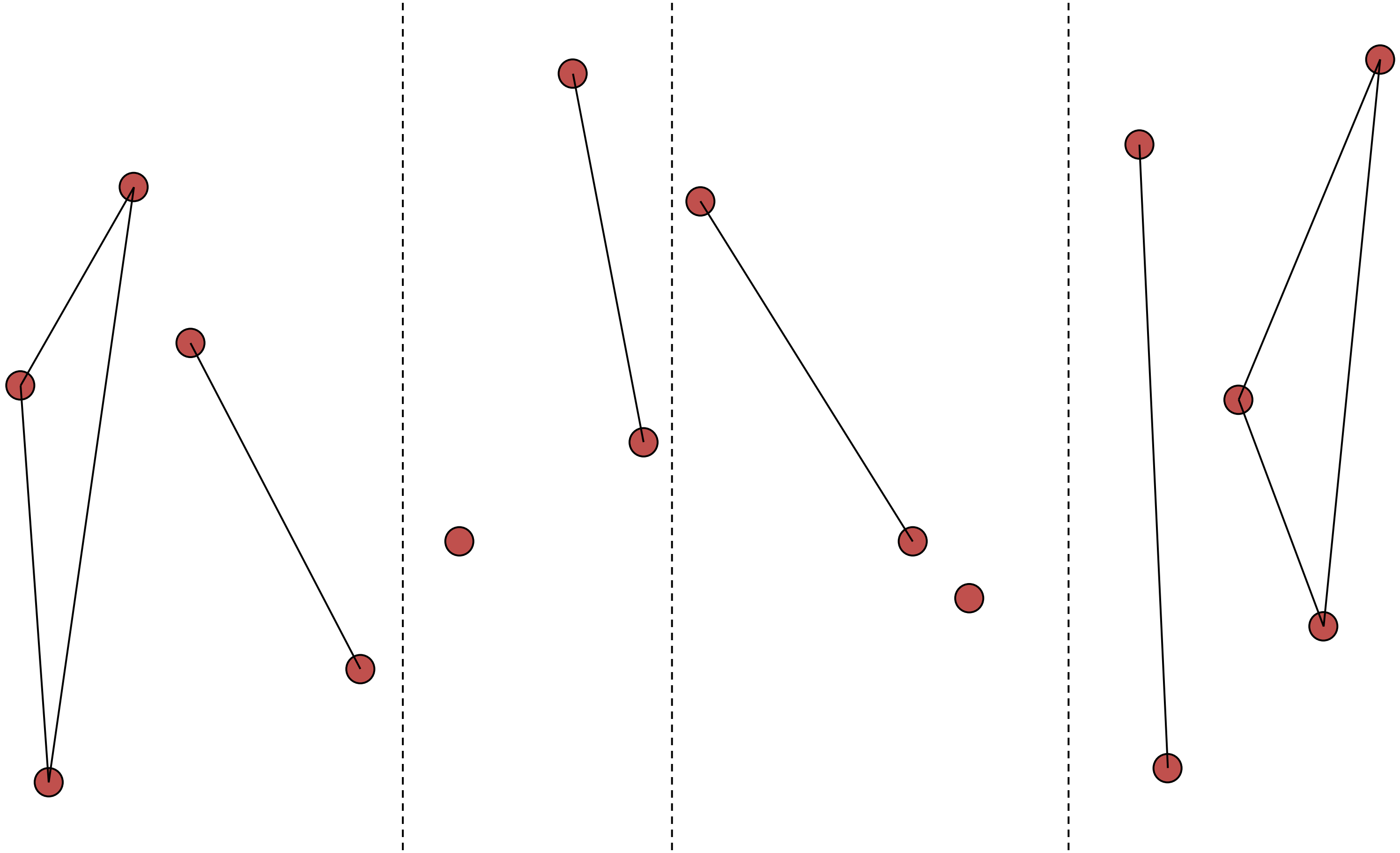
Convex Hull – Divide & Conquer

- Split set into two, compute convex hull of both, combine.



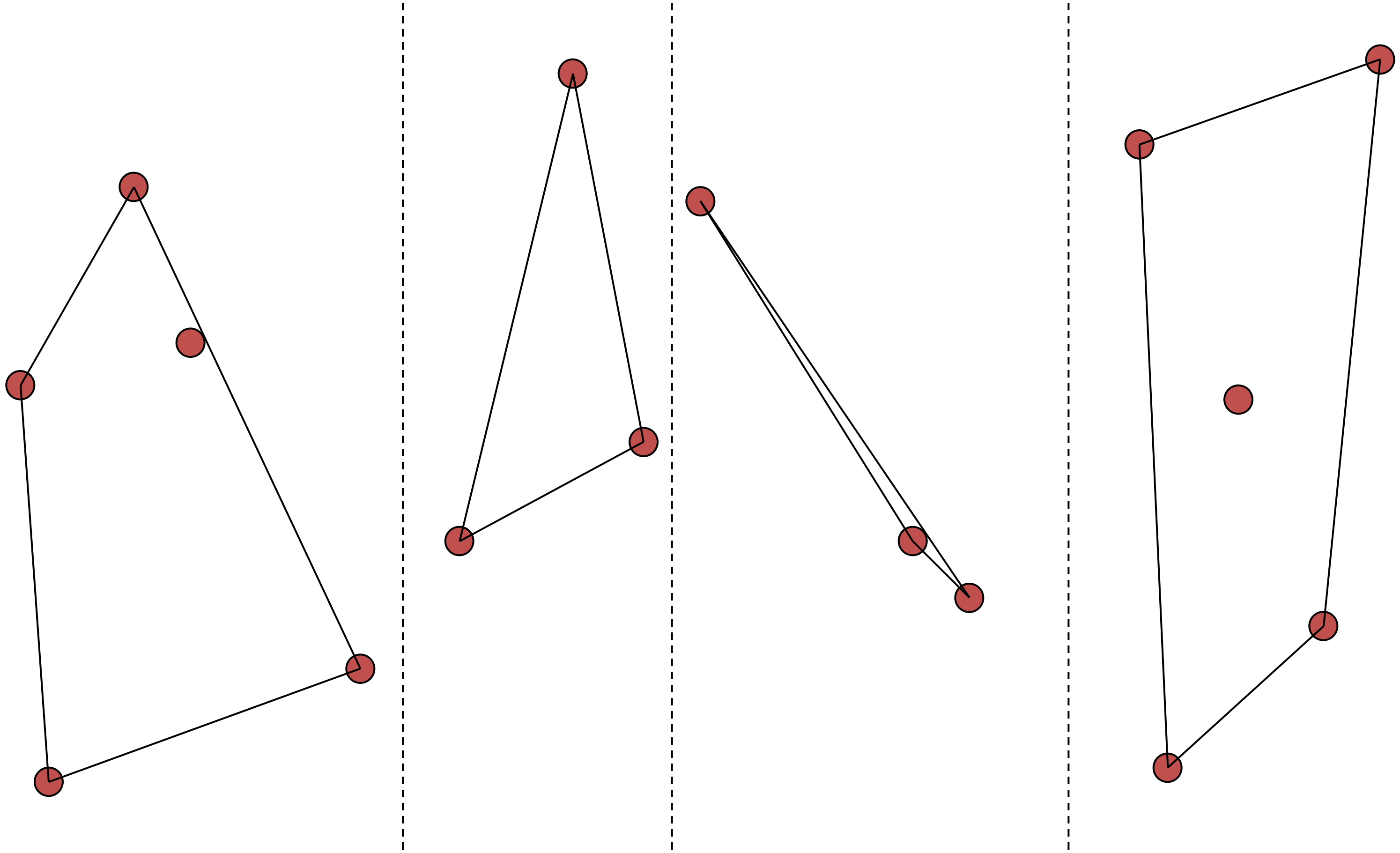
Convex Hull – Divide & Conquer

- Split set into two, compute convex hull of both, combine.



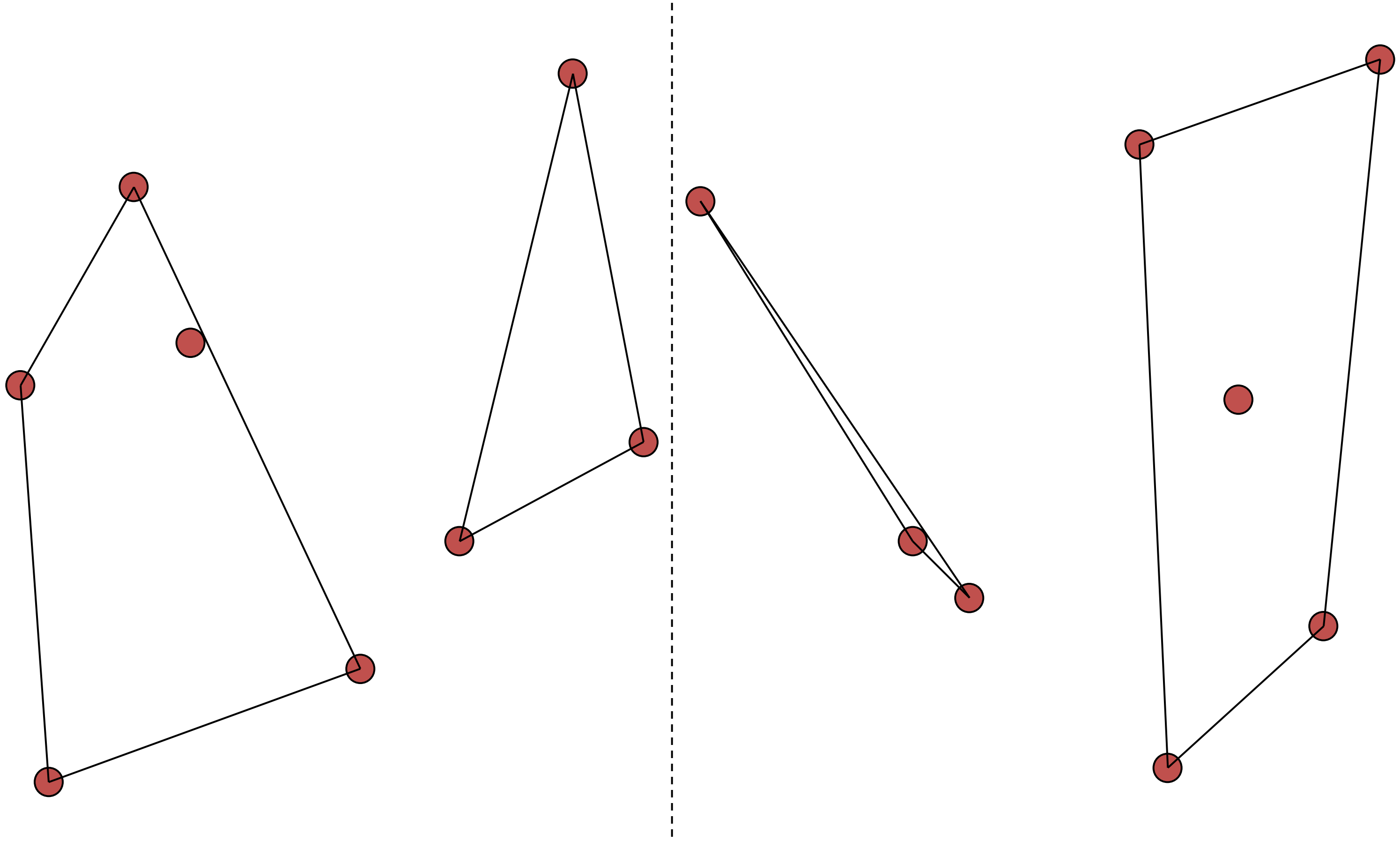
Convex Hull – Divide & Conquer

- Split set into two, compute convex hull of both, combine.



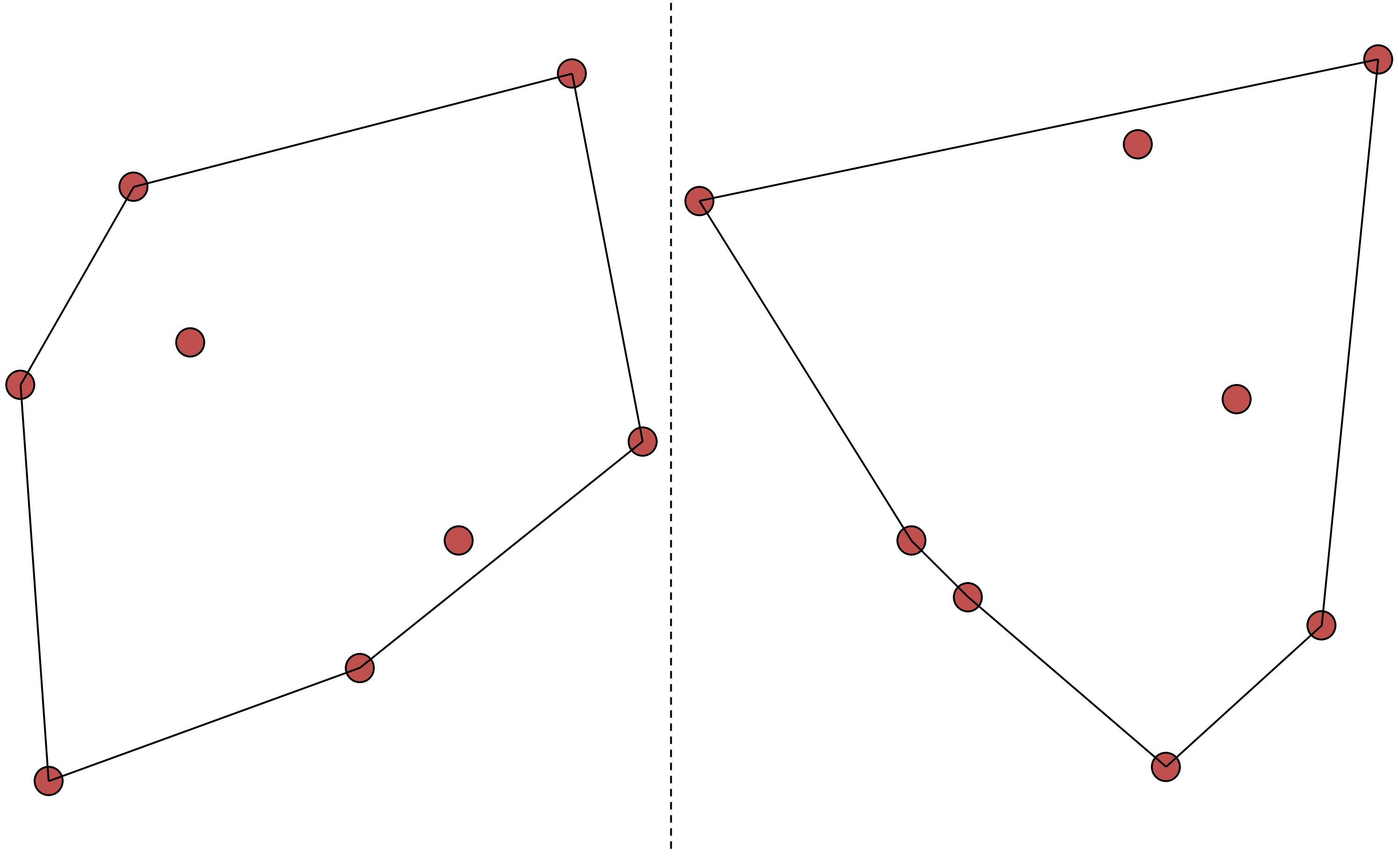
Convex Hull – Divide & Conquer

- Split set into two, compute convex hull of both, combine.



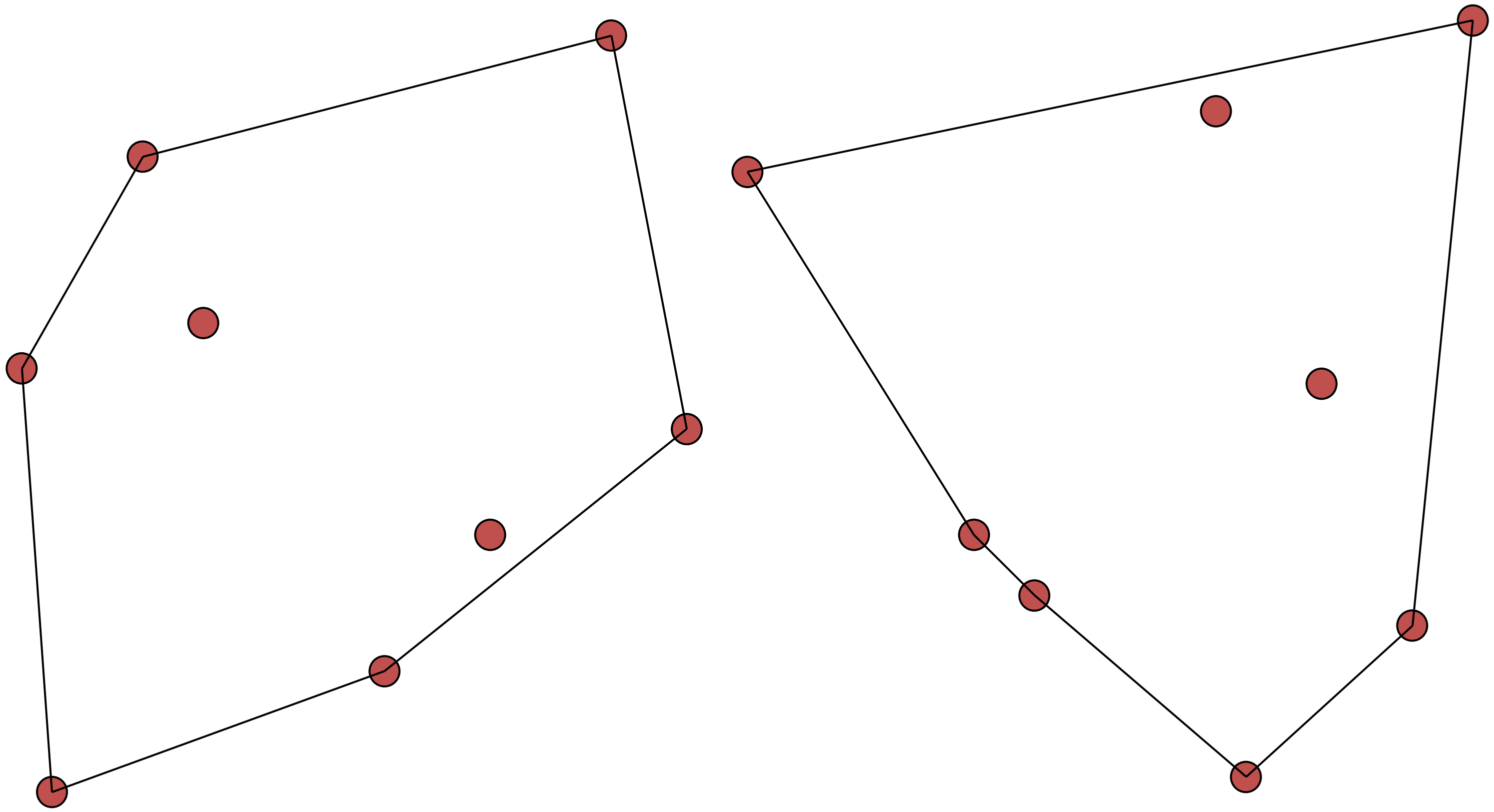
Convex Hull – Divide & Conquer

- Split set into two, compute convex hull of both, combine.



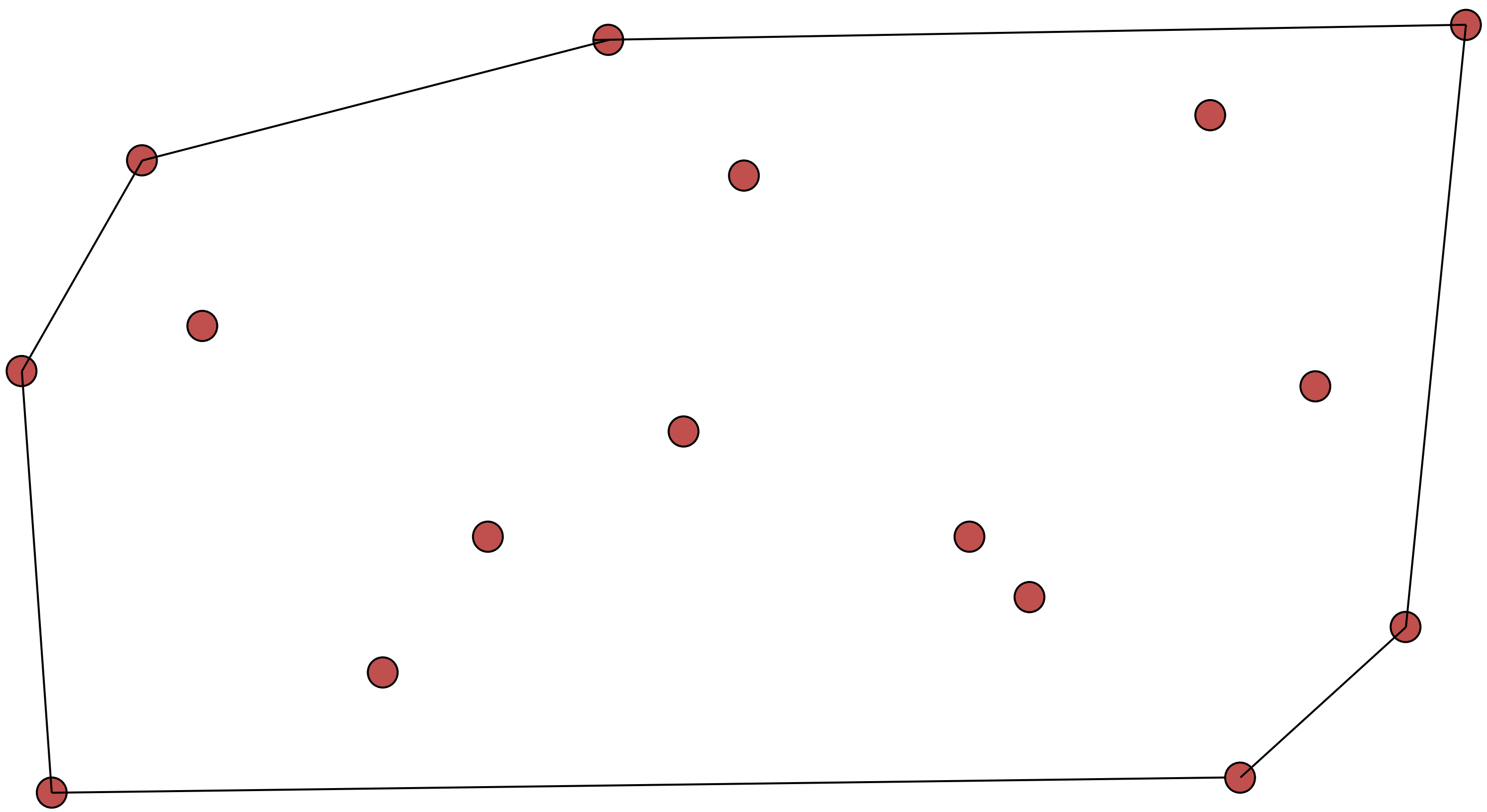
Convex Hull – Divide & Conquer

- Split set into two, compute convex hull of both, combine.



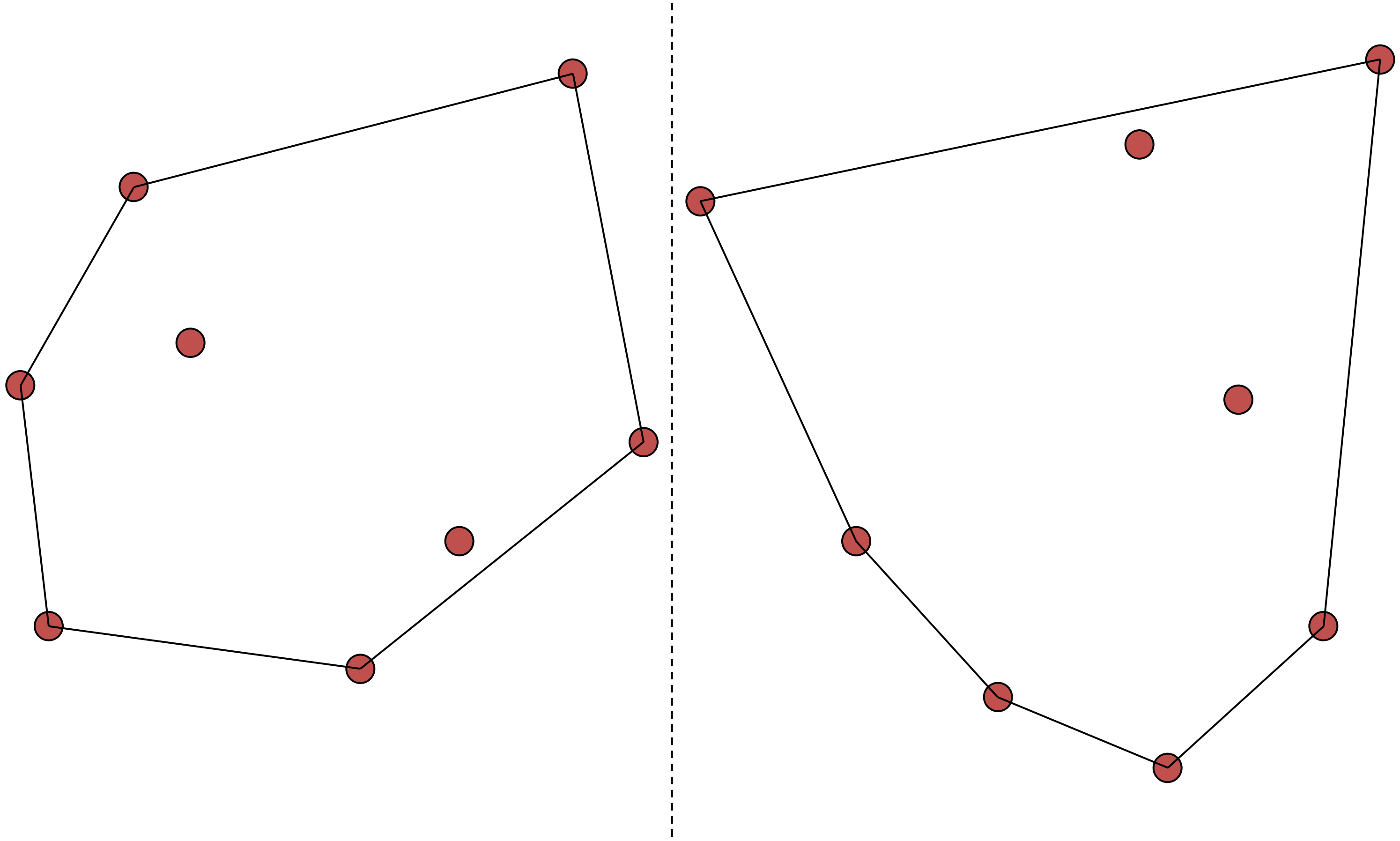
Convex Hull – Divide & Conquer

- Split set into two, compute convex hull of both, combine.



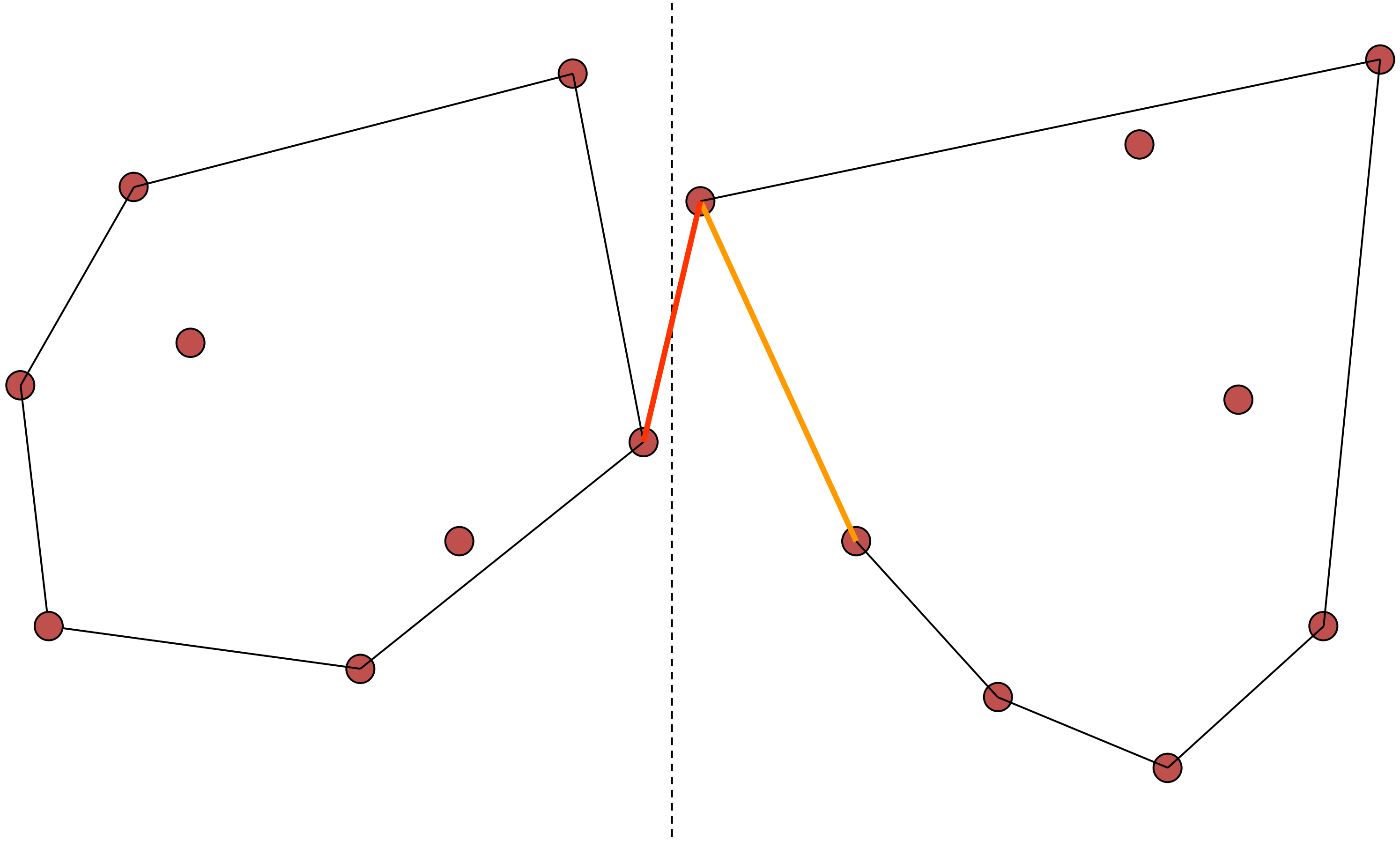
Convex Hull – Divide & Conquer

- Merging two convex hulls.



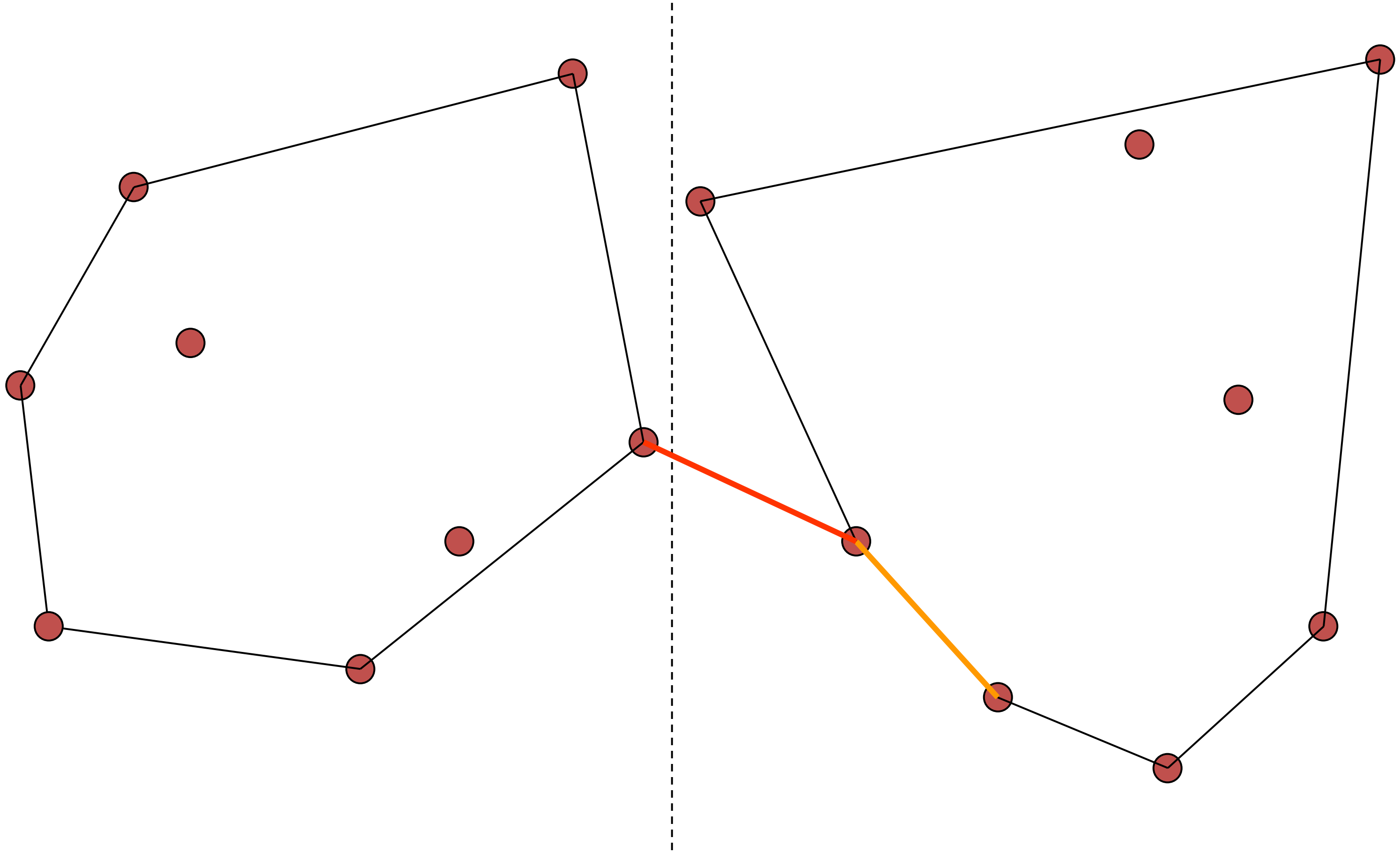
Convex Hull – Divide & Conquer

- Merging two convex hulls: (i) Find the lower tangent.



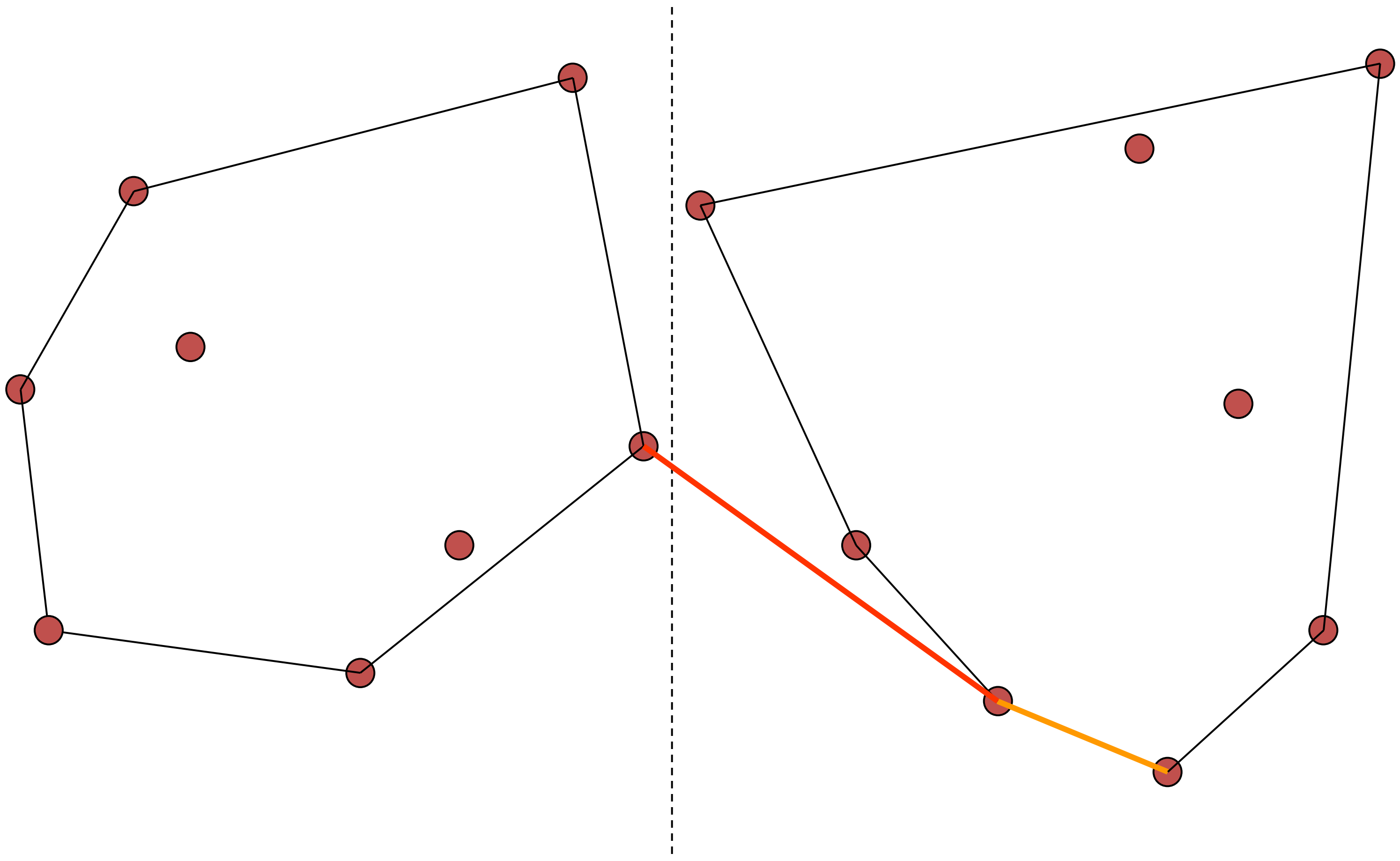
Convex Hull – Divide & Conquer

- Merging two convex hulls: (i) Find the lower tangent.



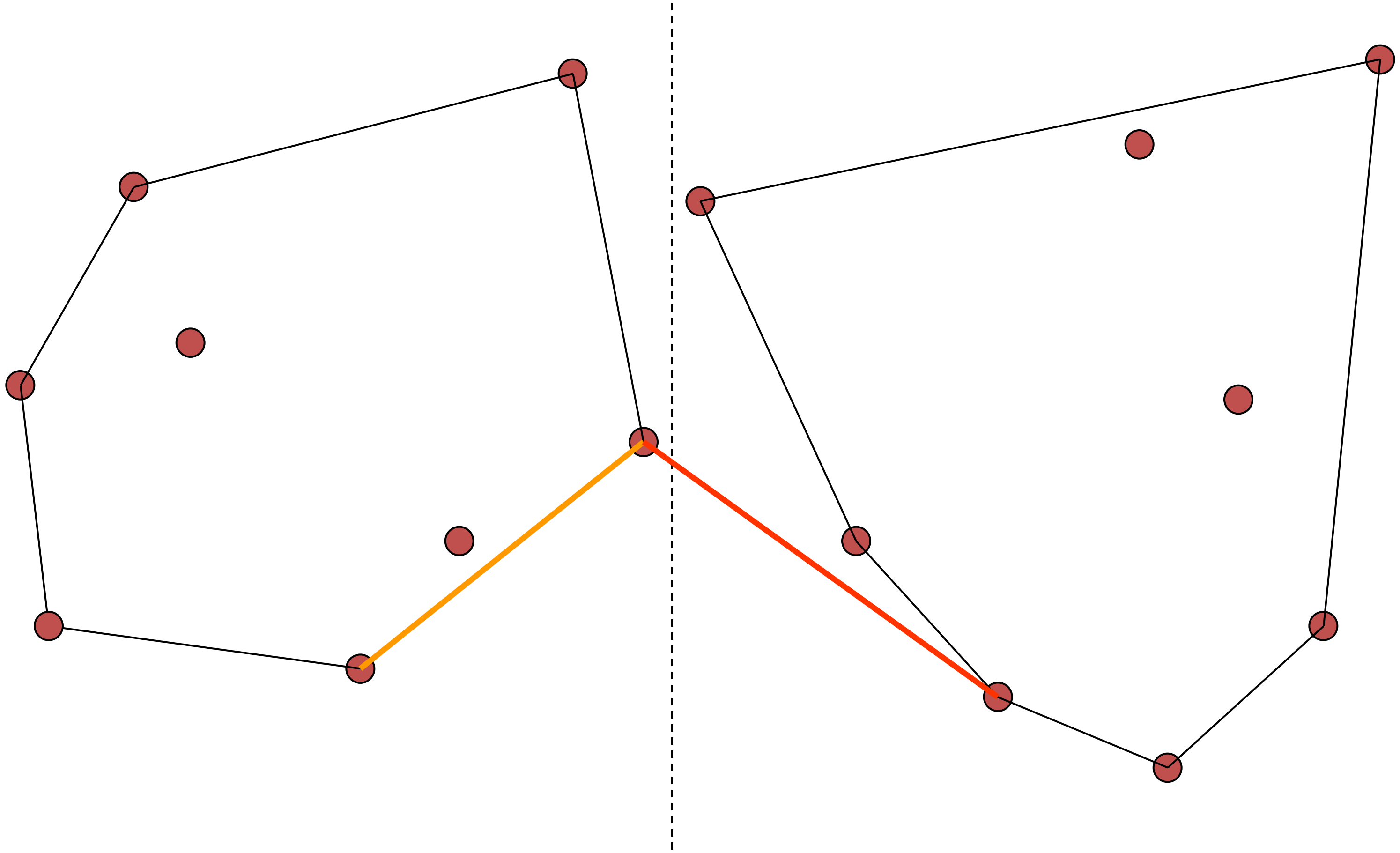
Convex Hull – Divide & Conquer

- Merging two convex hulls: (i) Find the lower tangent.



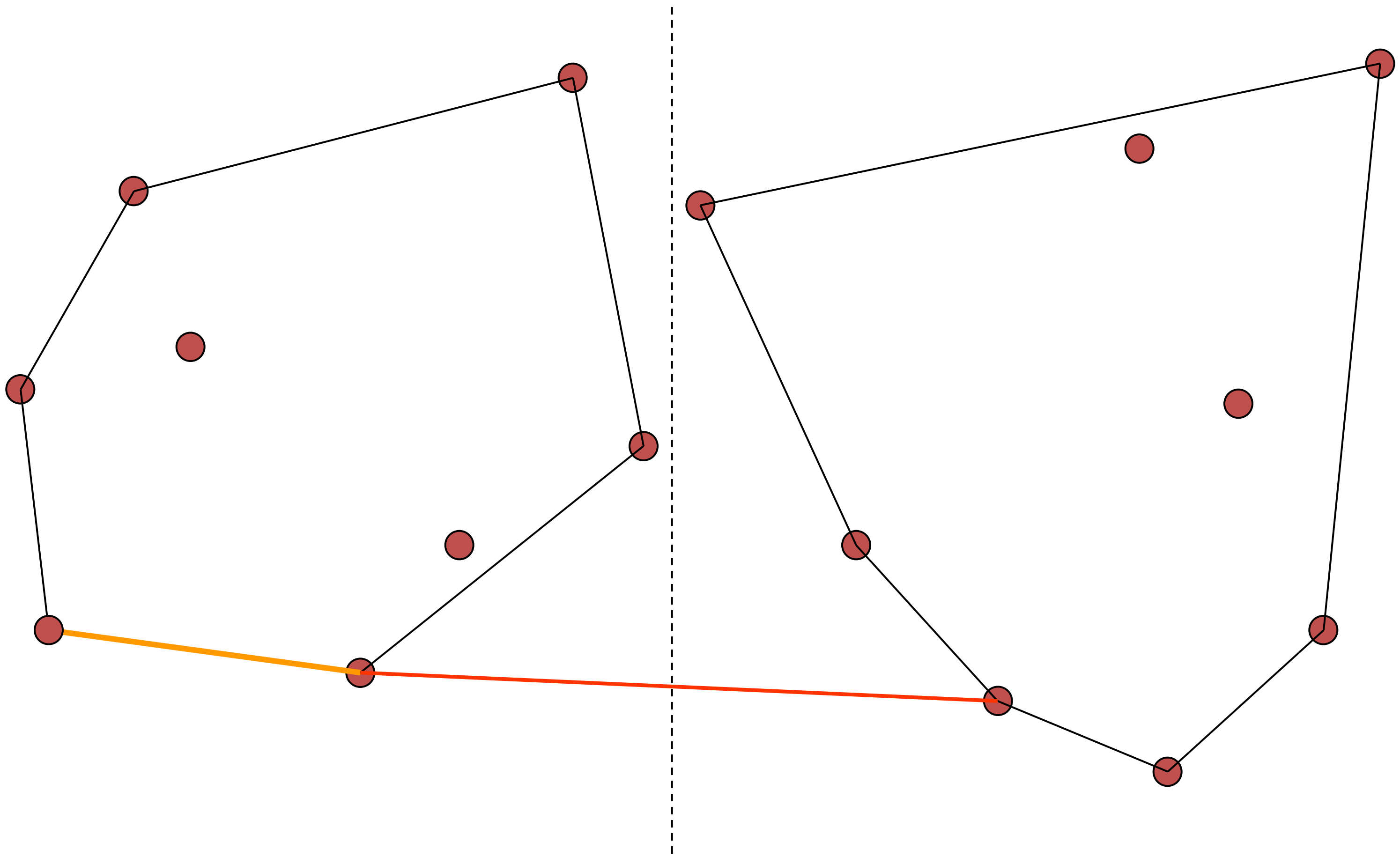
Convex Hull – Divide & Conquer

- Merging two convex hulls: (i) Find the lower tangent.



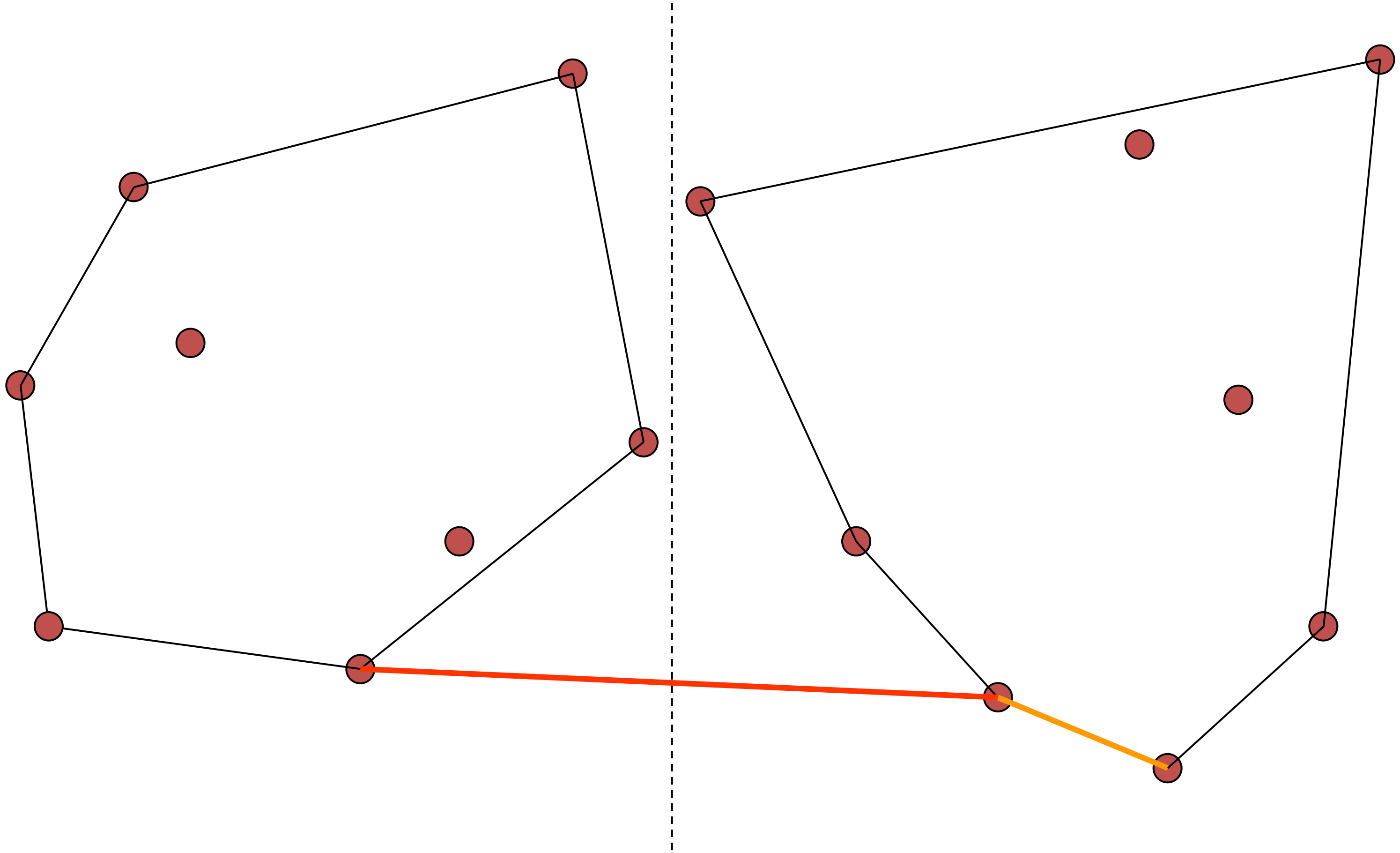
Convex Hull – Divide & Conquer

- Merging two convex hulls: (i) Find the lower tangent.



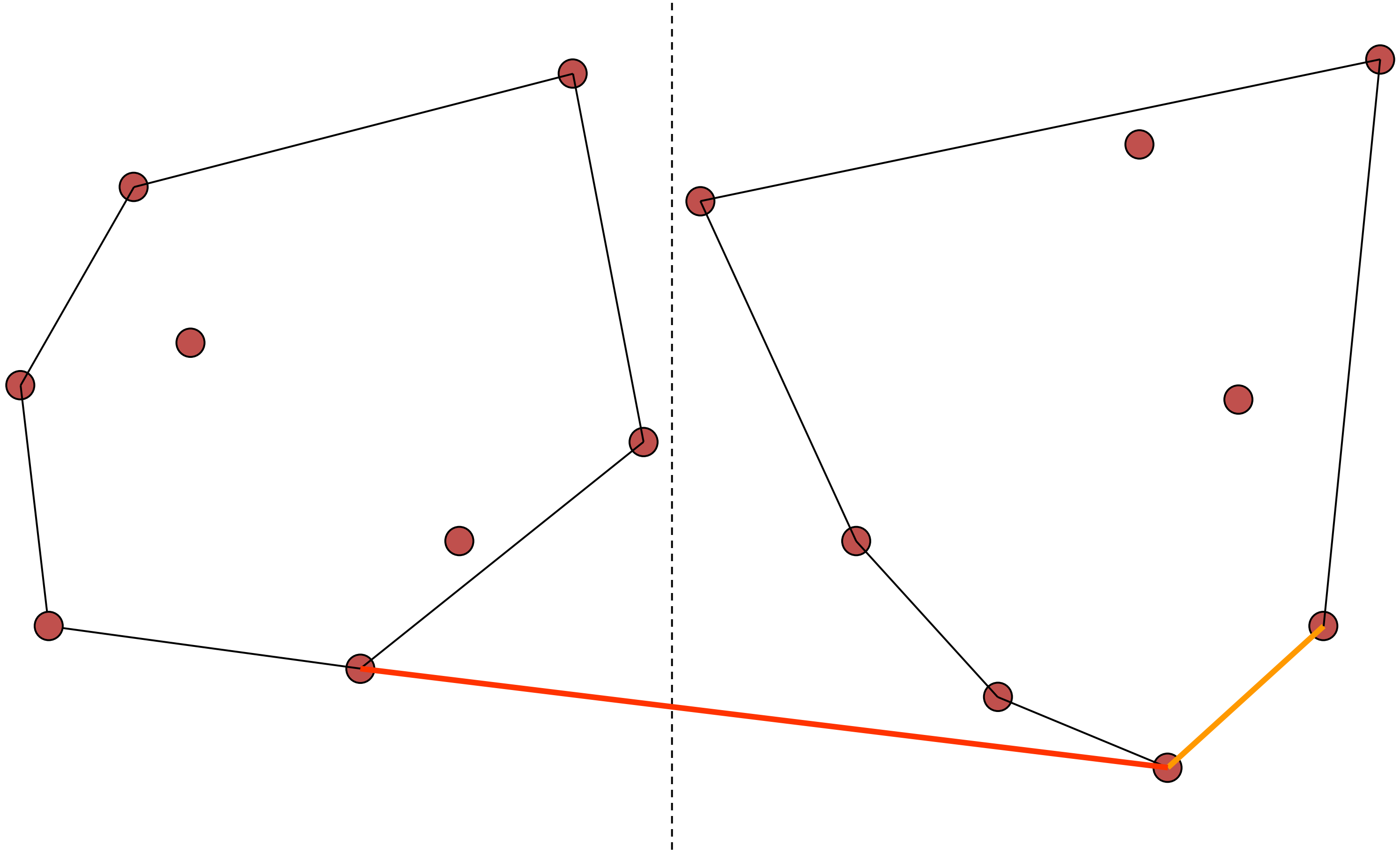
Convex Hull – Divide & Conquer

- Merging two convex hulls: (i) Find the lower tangent.



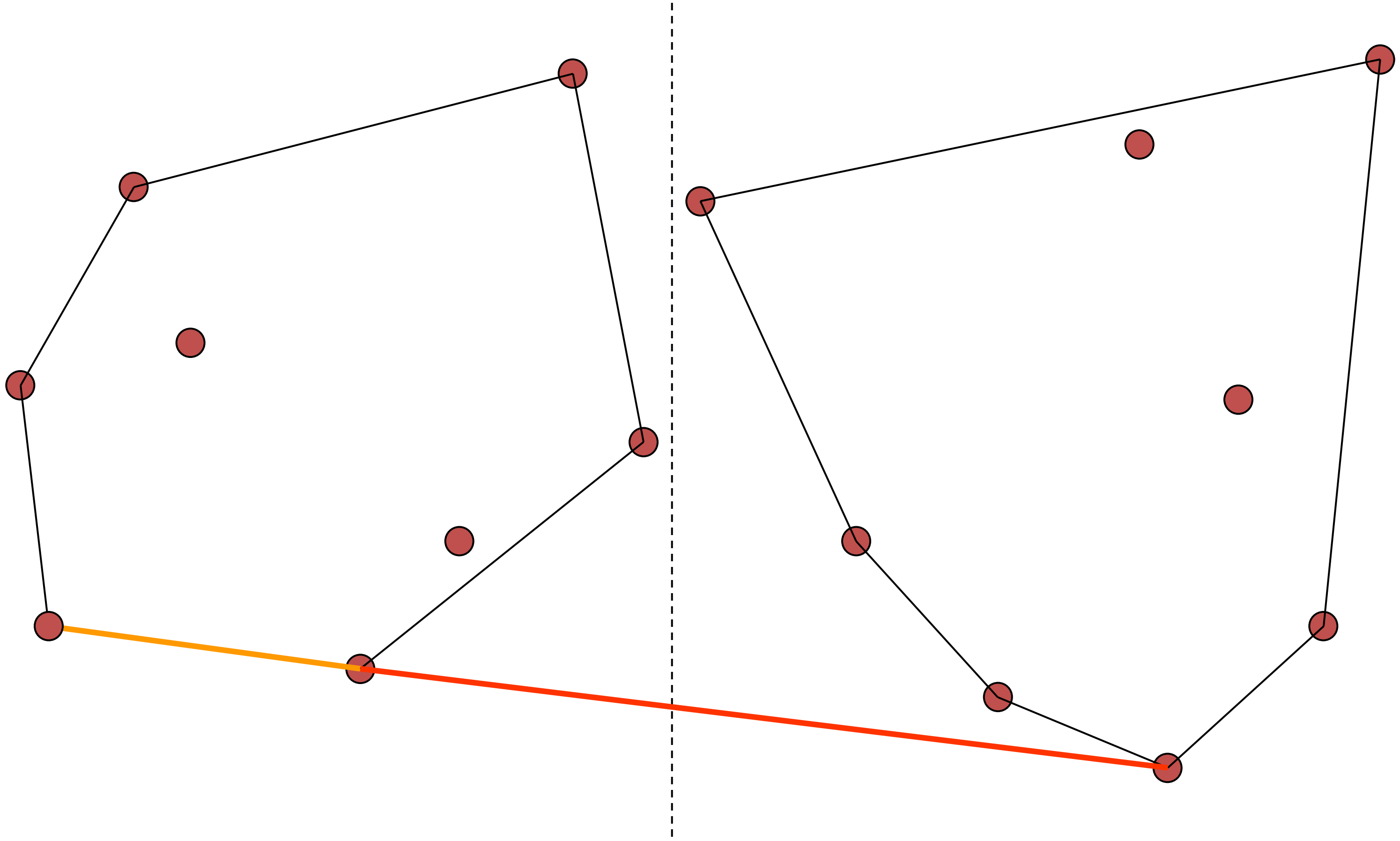
Convex Hull – Divide & Conquer

- Merging two convex hulls: (i) Find the lower tangent.



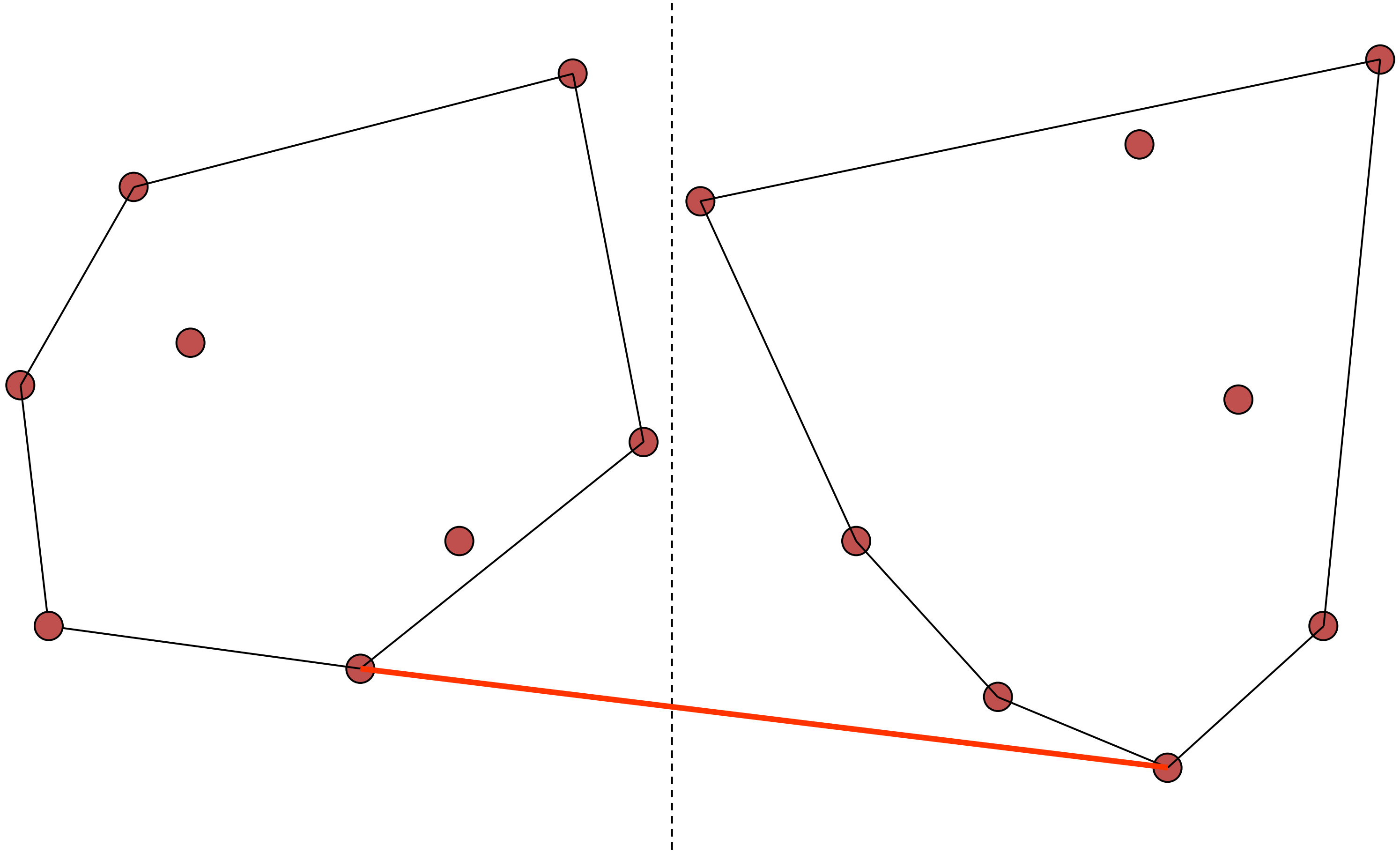
Convex Hull – Divide & Conquer

- Merging two convex hulls: (i) Find the lower tangent.



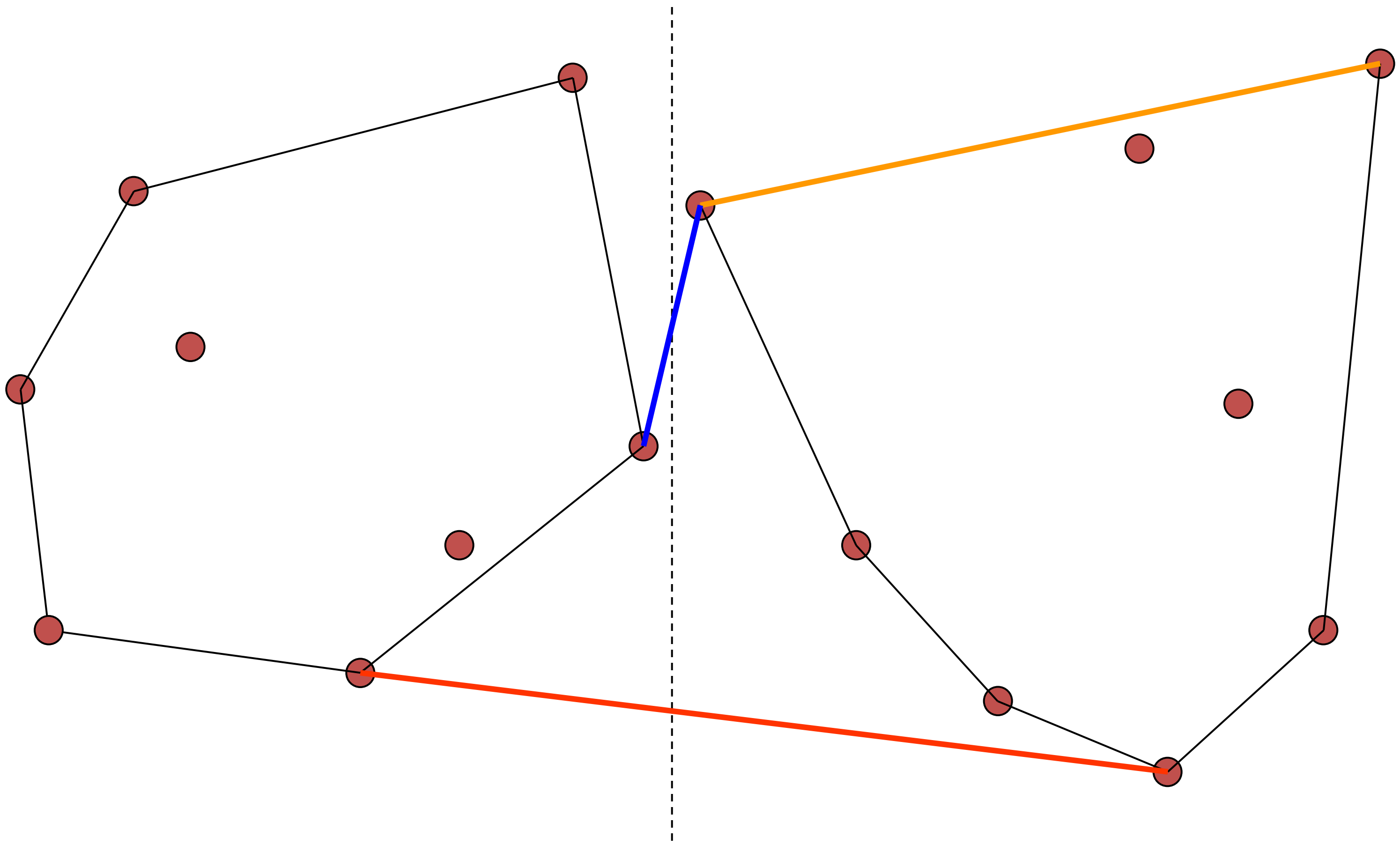
Convex Hull – Divide & Conquer

- Merging two convex hulls: (i) Find the lower tangent.



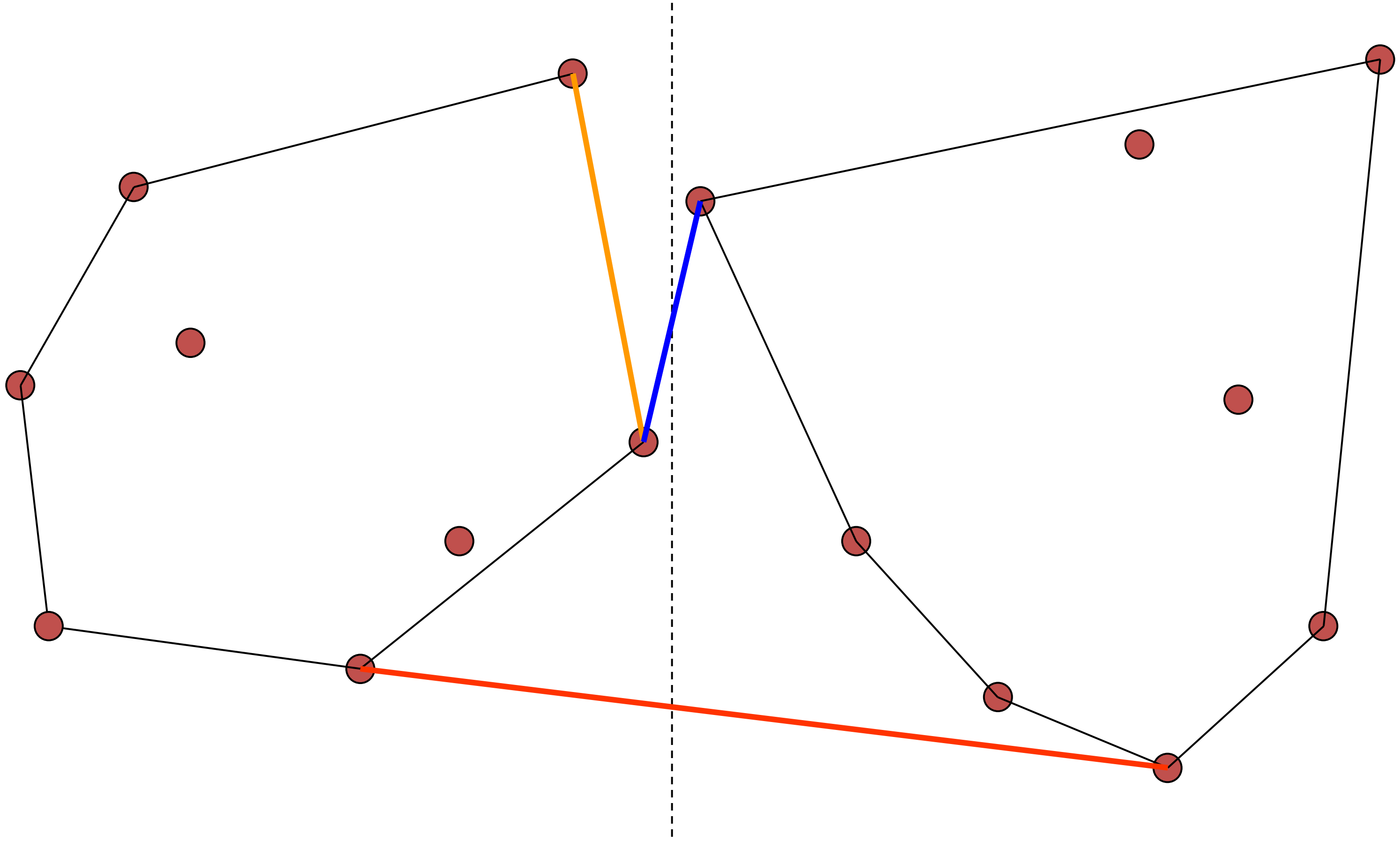
Convex Hull – Divide & Conquer

- Merging two convex hulls: (ii) Find the upper tangent.



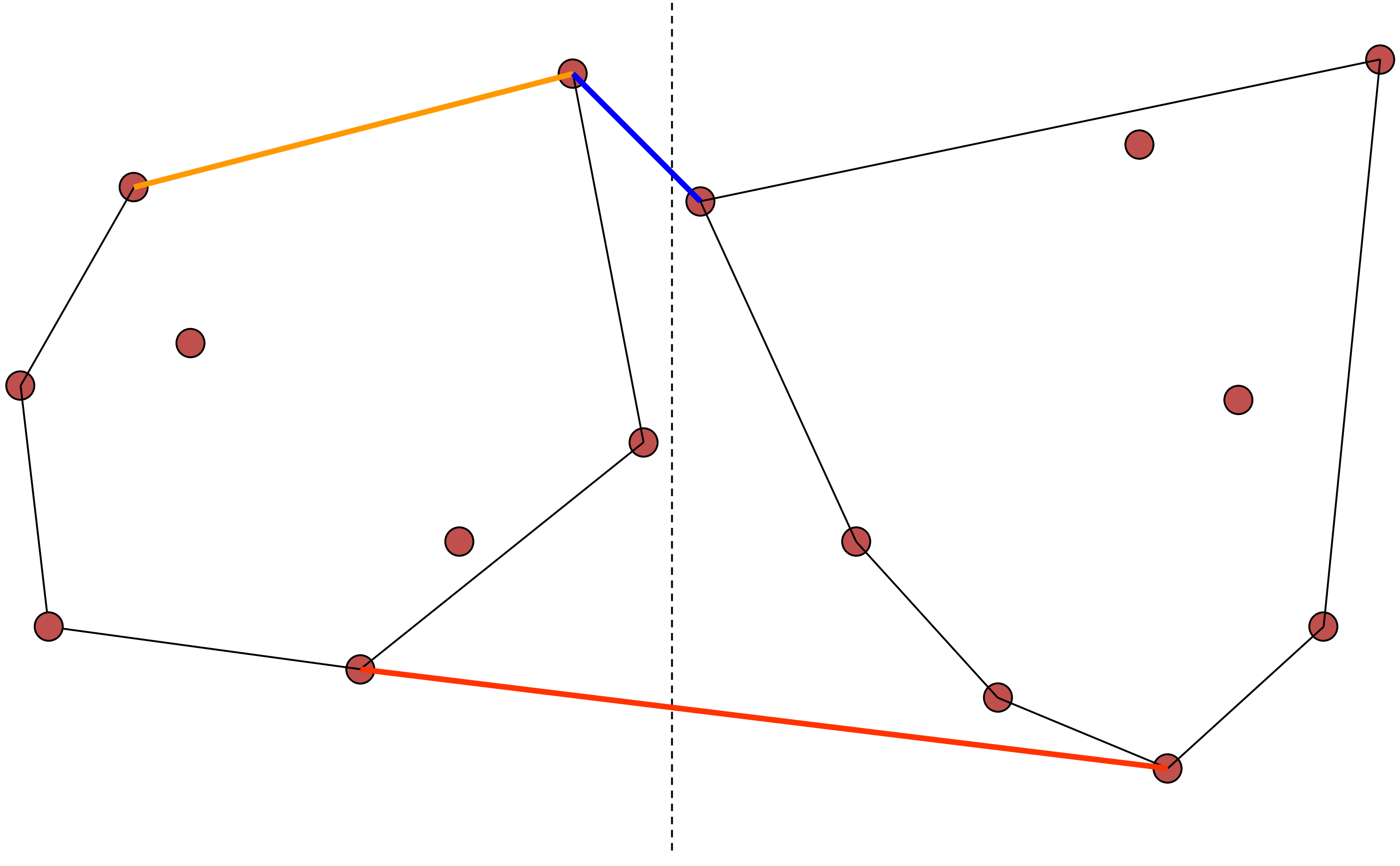
Convex Hull – Divide & Conquer

- Merging two convex hulls: (ii) Find the upper tangent.



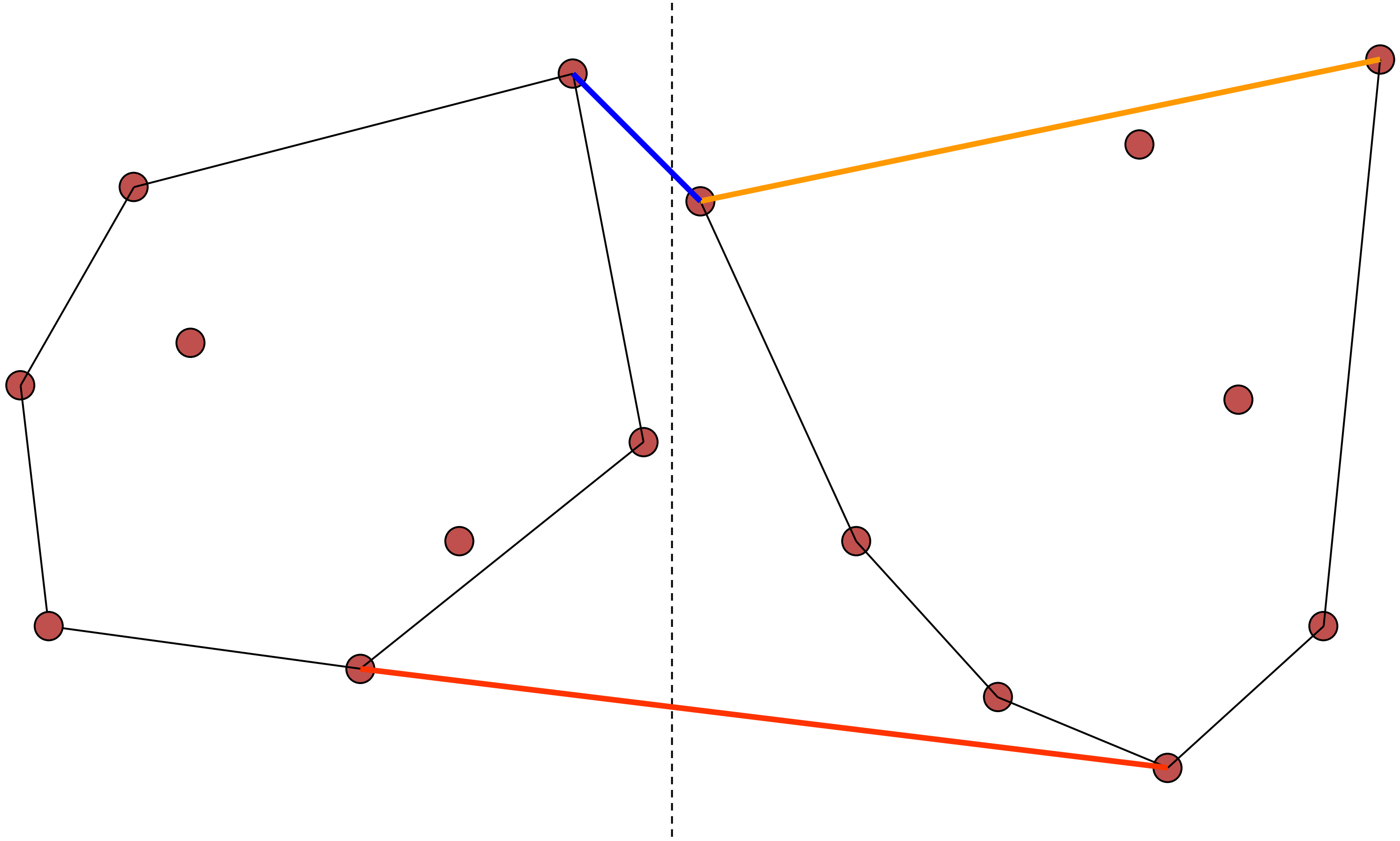
Convex Hull – Divide & Conquer

- Merging two convex hulls: (ii) Find the upper tangent.



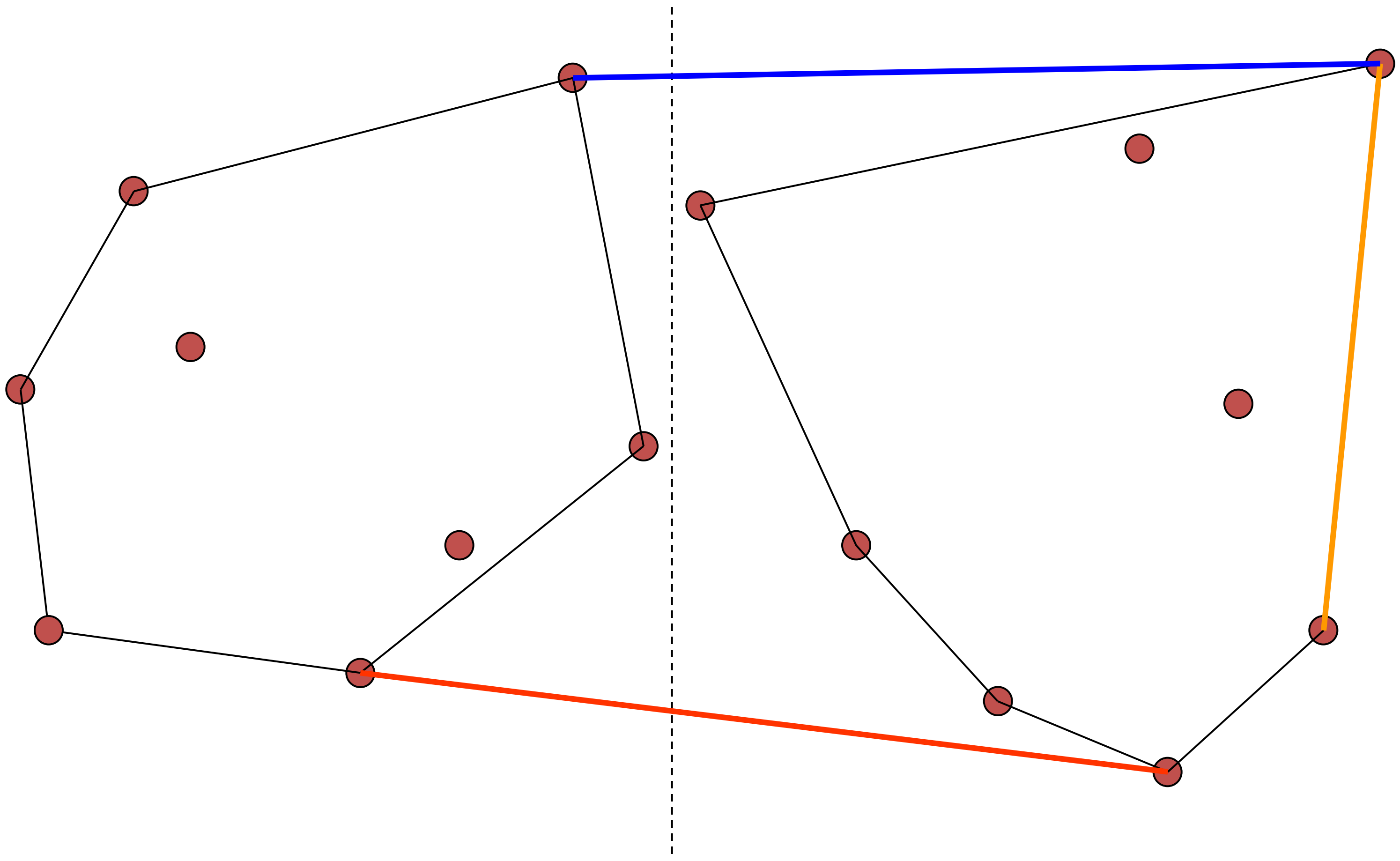
Convex Hull – Divide & Conquer

- Merging two convex hulls: (ii) Find the upper tangent.



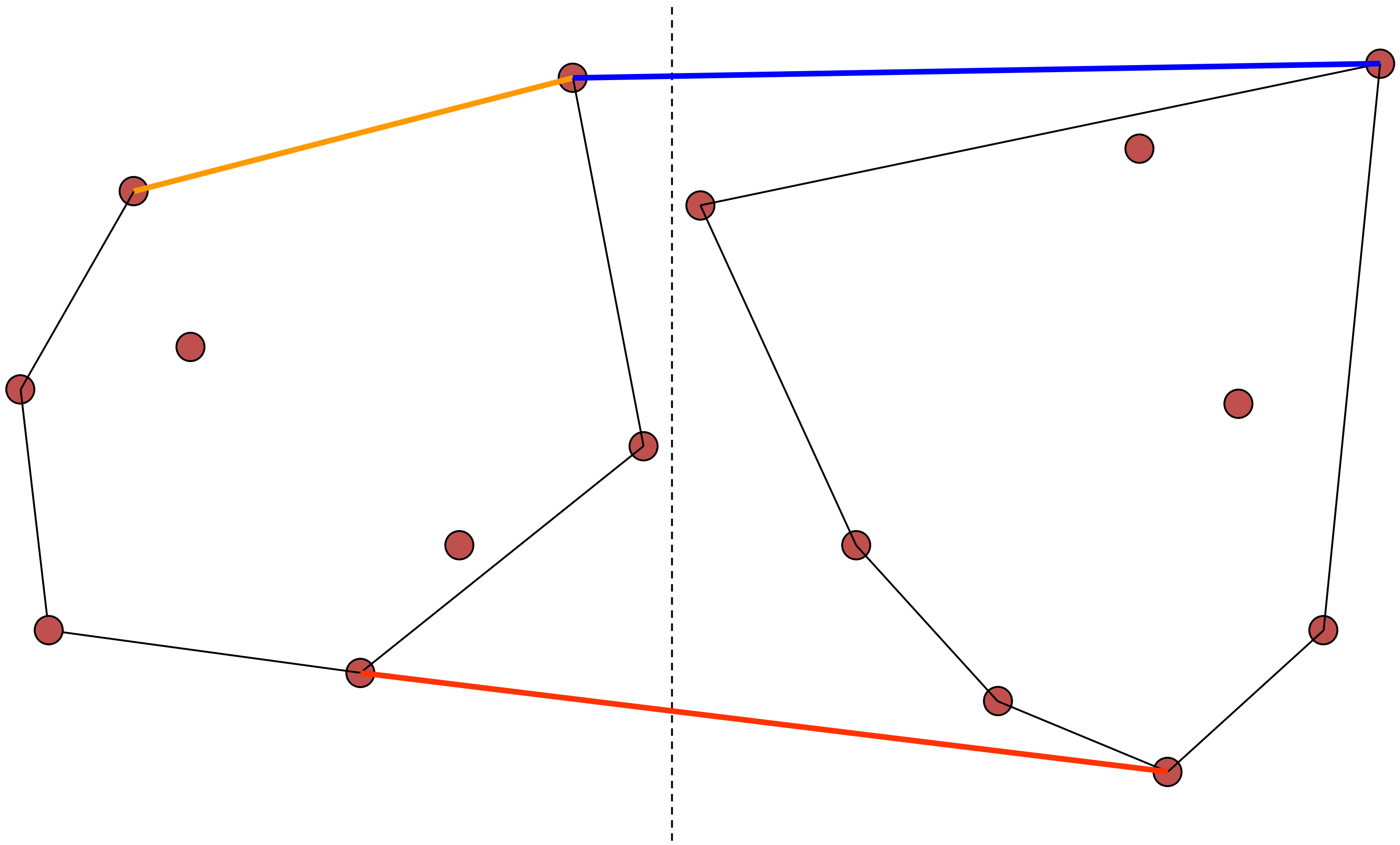
Convex Hull – Divide & Conquer

- Merging two convex hulls: (ii) Find the upper tangent.



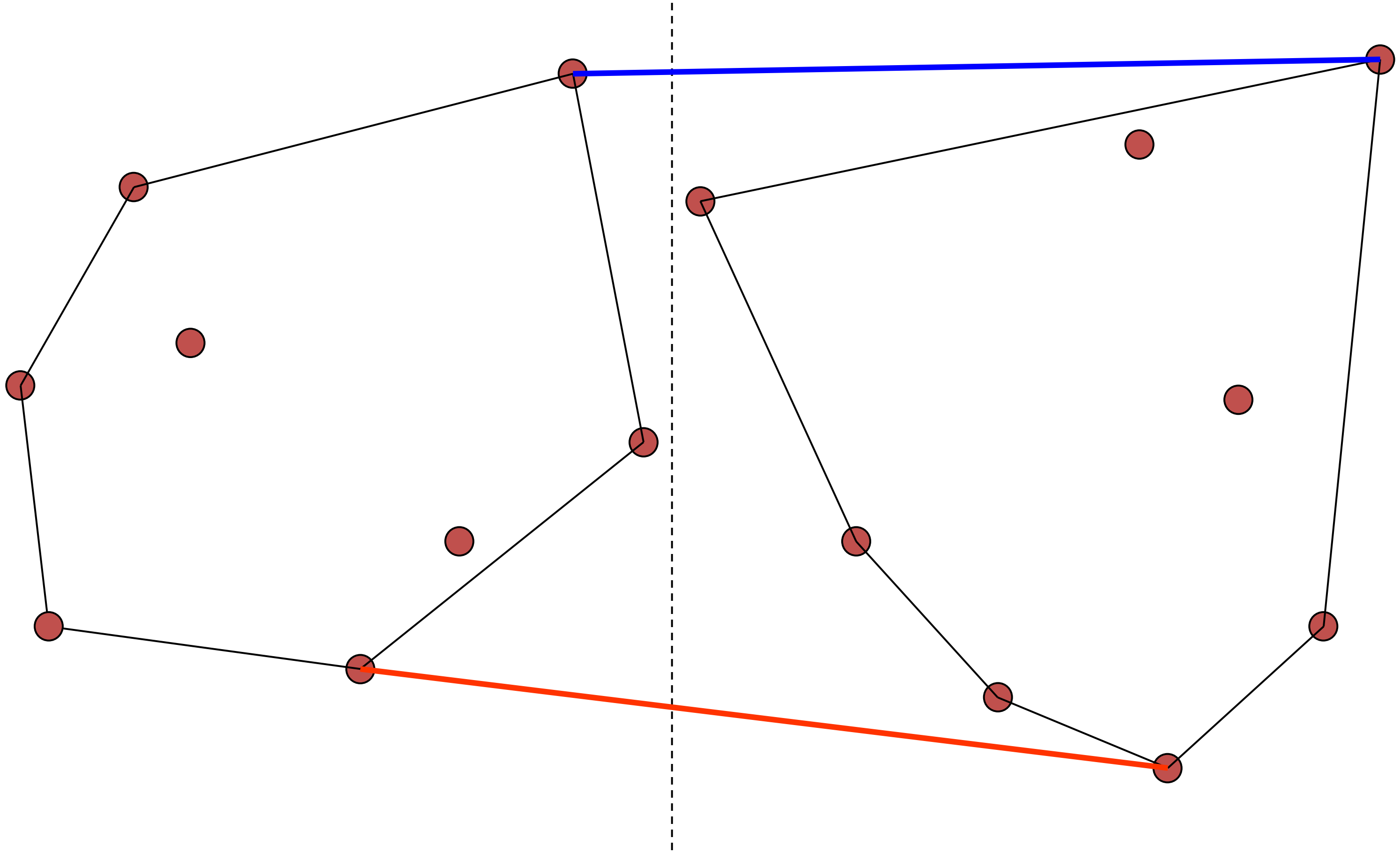
Convex Hull – Divide & Conquer

- Merging two convex hulls: (ii) Find the upper tangent.



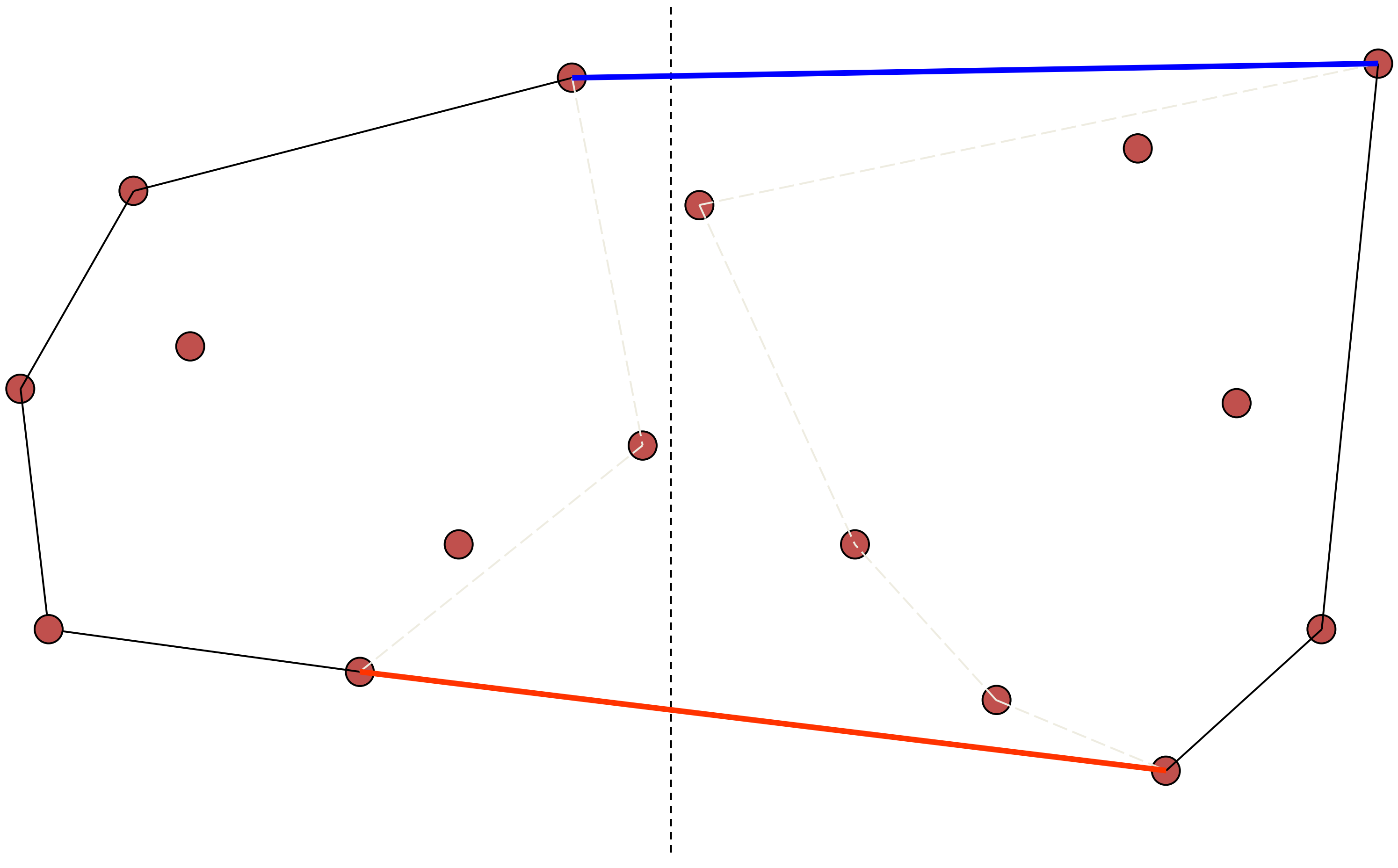
Convex Hull – Divide & Conquer

- Merging two convex hulls: (ii) Find the upper tangent.

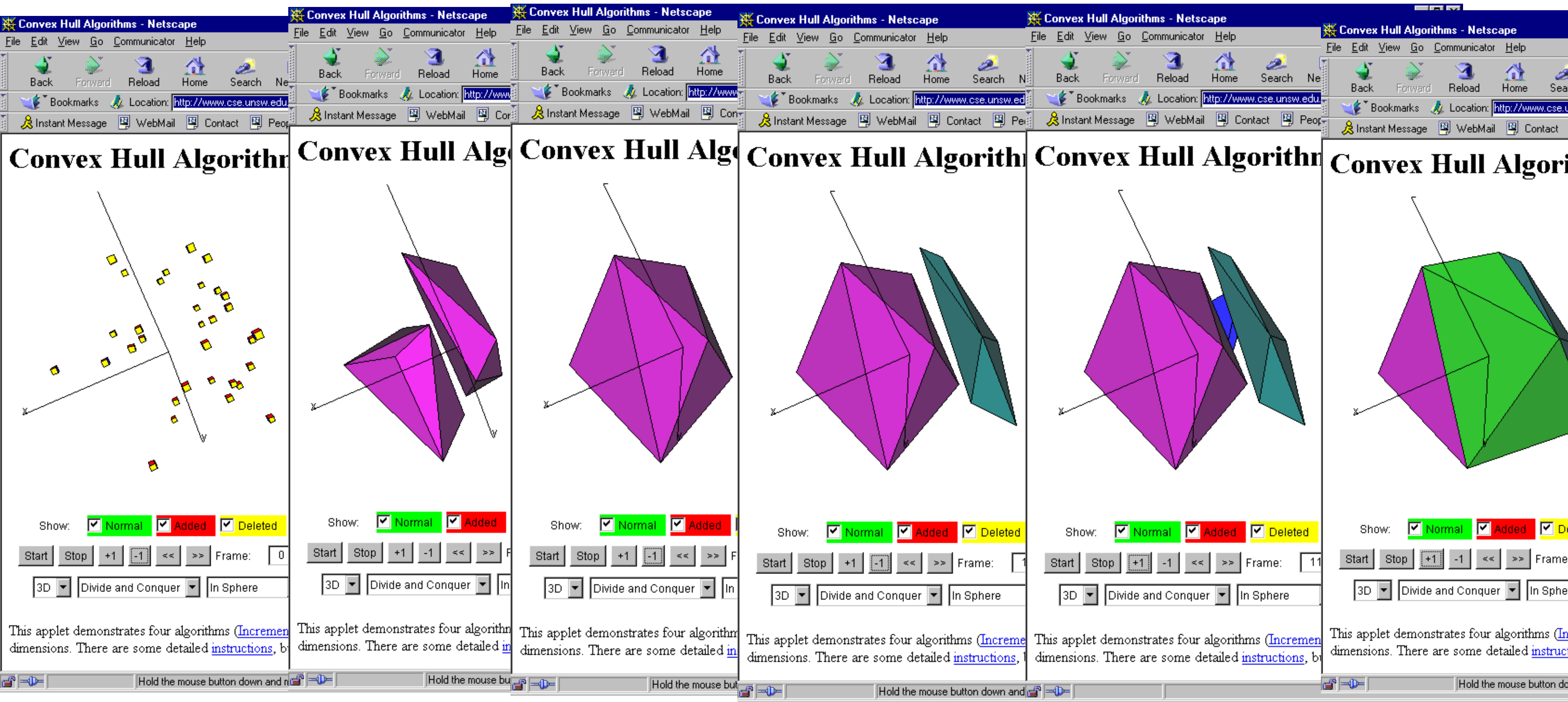


Convex Hull – Divide & Conquer

- Merging two convex hulls: (iii) Eliminate non-hull edges.



Algorithms:
3D Divide and Conquer



O(n log n) time !

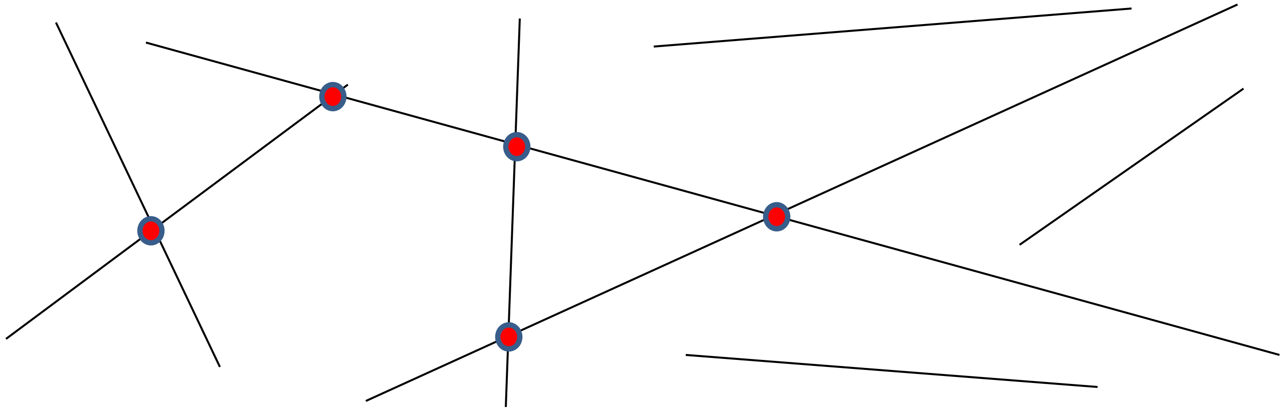
CxHull Animations: http://www.cse.unsw.edu.au/~lambert/java/3d/hull.html

Topic

- Introduction
- Two lines Intersection Test
- Point inside polygon
- Convex hull
- Line Segments Intersection Algorithm

Line segment intersection

- Input:
  - Set $S = \{s_1, \ldots, s_n\}$ of n line segments, $s_i = (x_i, y_i)$



- Output:
  - k = All intersection points among the segments in S

Line segment intersection

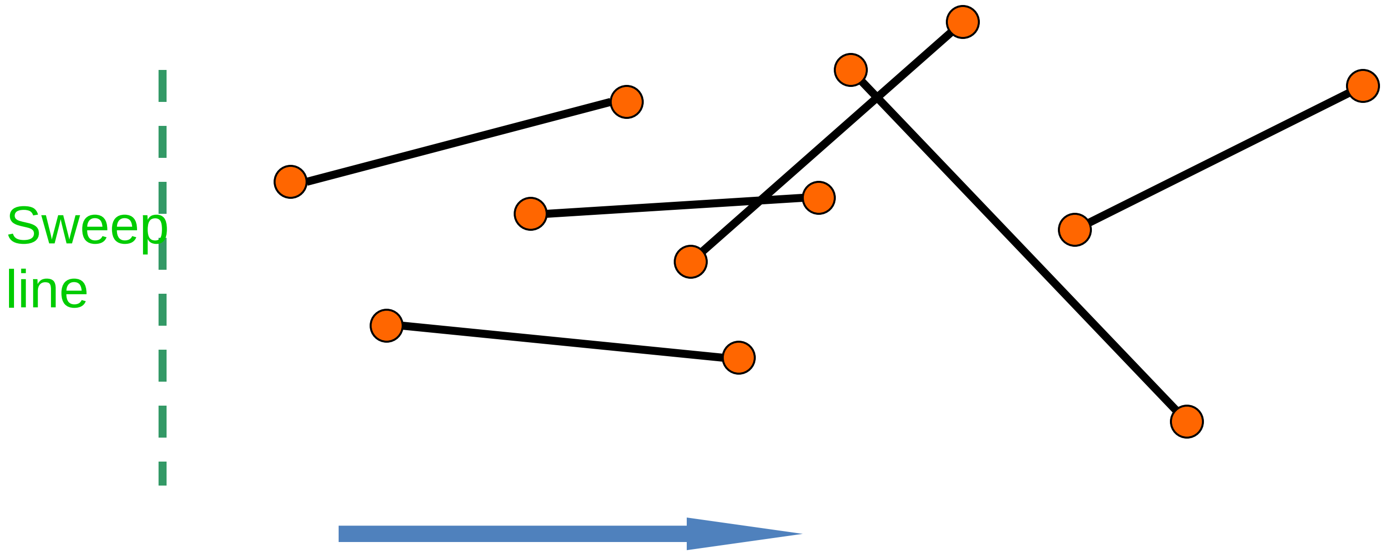
- Worst case:
  - $k = n(n-1)/2 = O(n^2)$ intersections

- Sweep line algorithm (near optimal algorithm):
  - $O(n \log n + k)$ time and $O(n)$ space
  - O(n) space

Sweep Line Algorithm

Avoid testing pairs of segments that are far apart.

Idea: *imagine* a vertical sweep line passes through the given set of line segments, from left to right.



Assumption on Non-degeneracy

No segment is vertical. // the sweep line always hits a segment at
// a point.

If a segment is vertical, imagine we rotate it clockwise by a tiny angle.
This means:

For each vertical segment, we will consider its lower
endpoint before upper point.

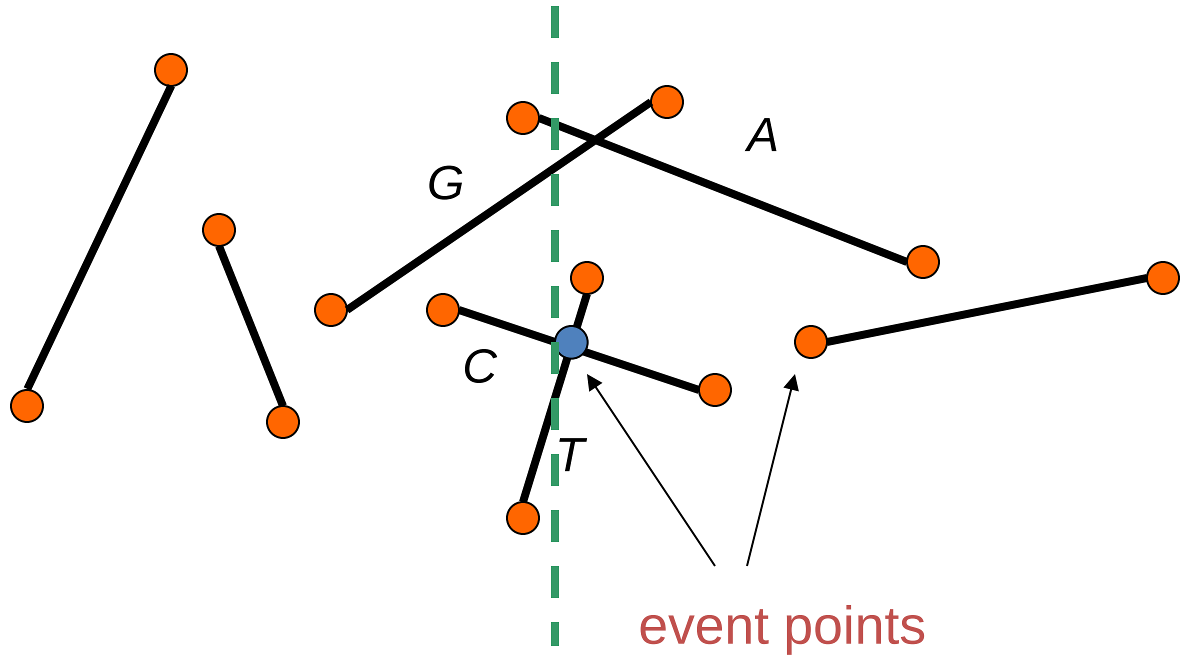
Sweep Line Status

The set of segments intersecting the sweep line.

It changes as the sweep line moves, but *not continuously*.

Updates of status happen only at *event points*.

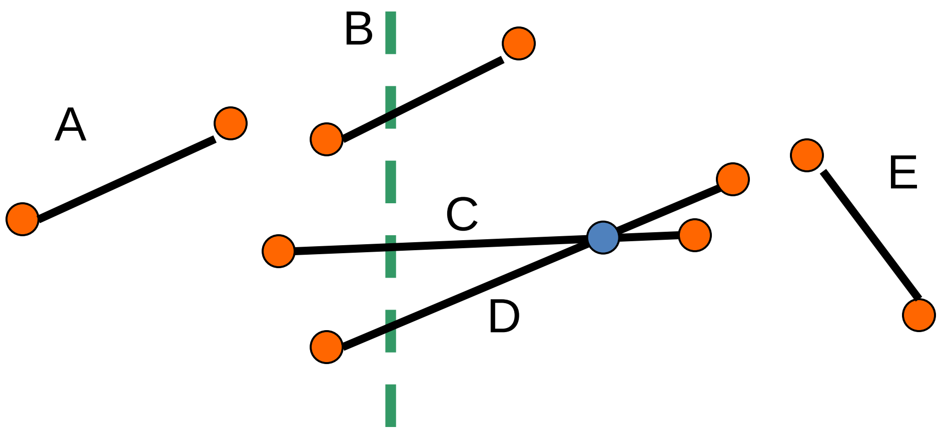
- left endpoints
- right endpoints
- intersections



Ordering Segments

A *total order* over the segments that intersect the current position of the sweep line:

- Based on which parts of the segments we are currently interested in



B > C > D
(A and E not in the ordering)

C > D
(B drops out of the ordering)

D > C
(C and D swap their positions)

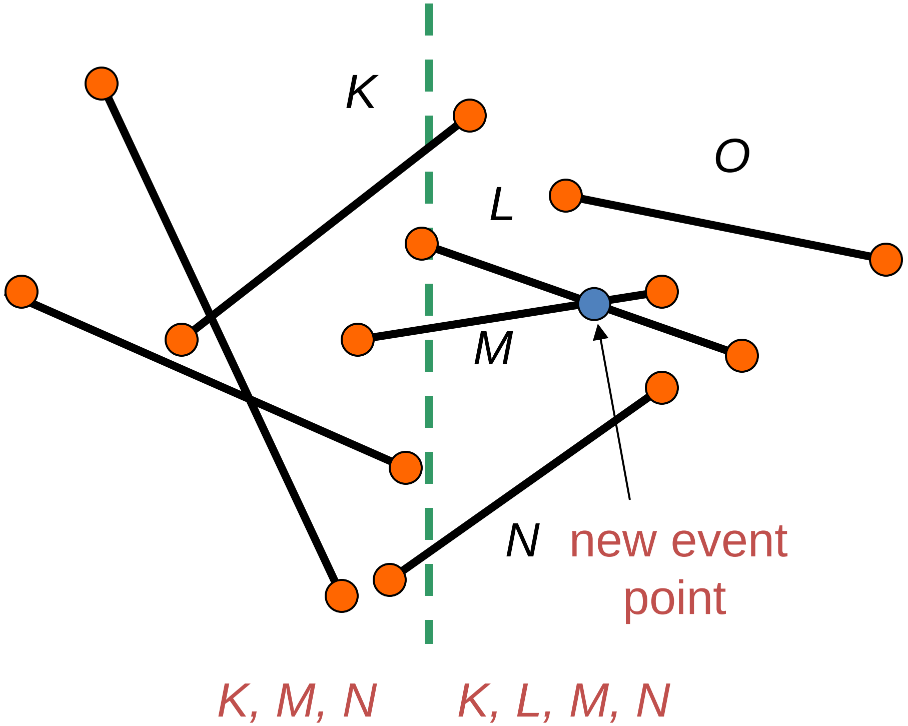
At an event point, the sequence of segments changes:

♦ Update the status.

♦ Detect the intersections.

Status Update (1)

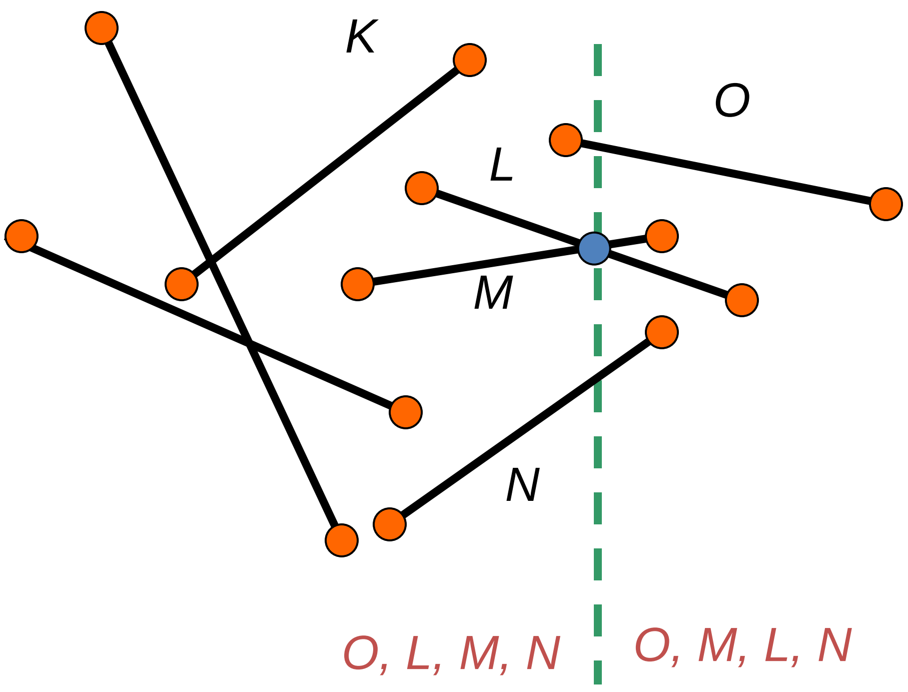
Event point is the left endpoint of a segment.



- A new segment *L* intersecting the sweep line
- Check if *L* intersects with the segment above (*K*) and the segment below (*M*).
- Intersection(s) are new event points.

Status Update (2)

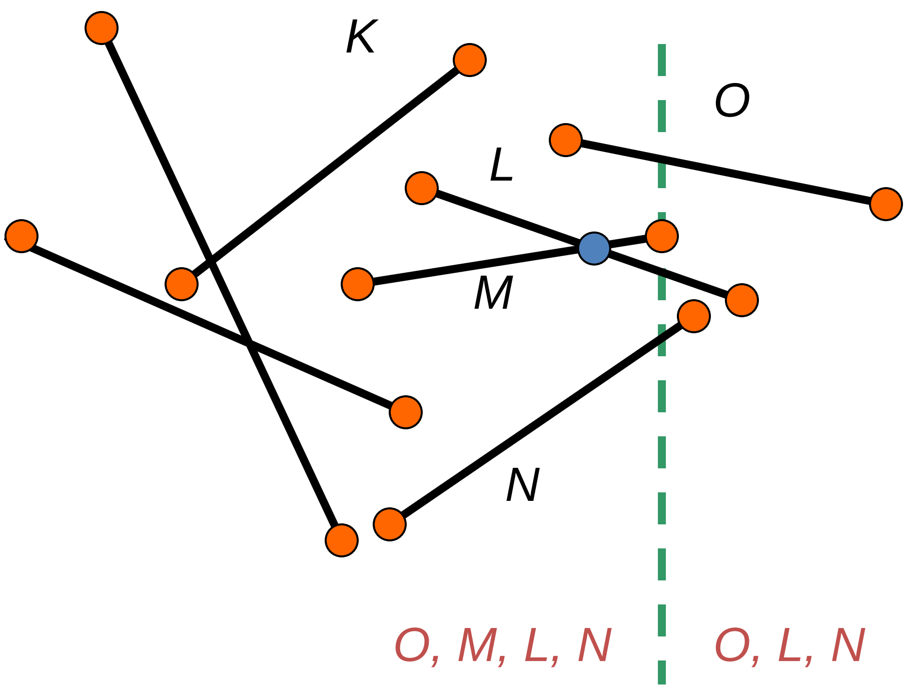
Event point is an intersection.



- The two intersecting segments (*L* and *M*) change order.
- Check intersection with new neighbors (*M* with *O* and *L* with *N*).
- Intersection(s) are new event points.

Status Update (3)

Event point is a lower endpoint of a segment.



- The two neighbors (*O* and *L*) become adjacent.
- Check if they (*O* and *L*) intersect.
- Intersection is new event point.

Data Structure for Event Queue

Ordering of event points:

- by x-coordinates
- by y-coordinates in case of a tie in x-coordinates.

Supports the following operations on a segment s.

- fetching the next event // $O(\log m)$
- inserting an event // $O(\log m)$

Every event point p is stored with all segments starting at p.

Data structure: balanced binary search tree (e.g., red-black tree).

m = #event points in the queue

Data Structure for Sweep-line Status

- Describes the relationships among the segments intersected by the sweep line.

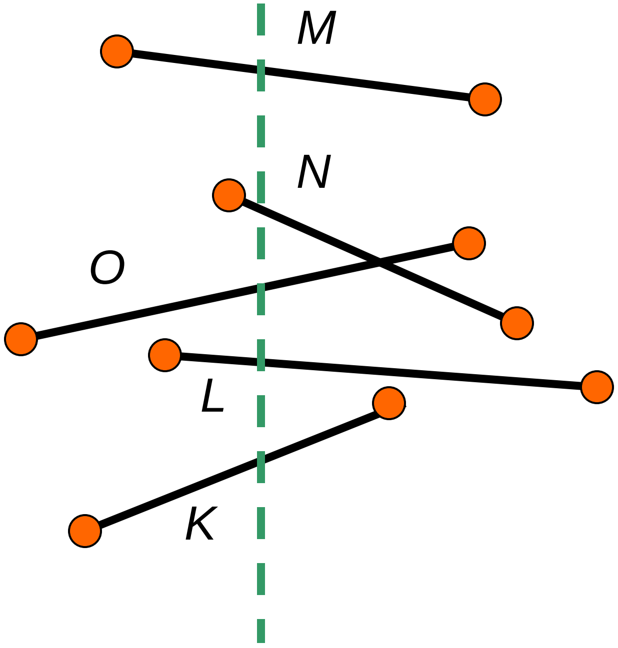
- Use a balanced binary search tree *T* to support the following operations on a segment *s*.

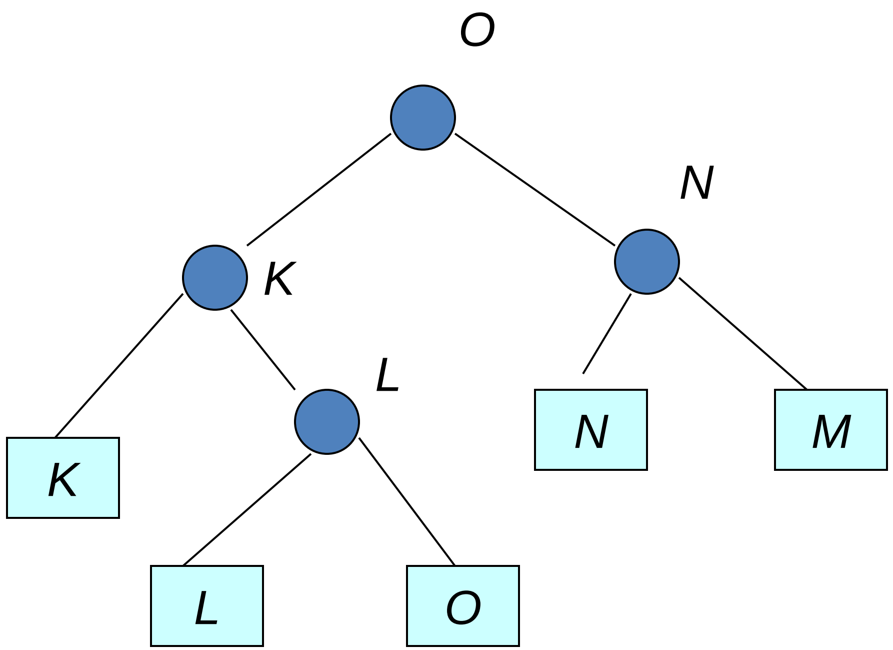
  Insert(*T*, *s*)
  Delete(*T*, *s*)
  Above(*T*, *s*)   // segment immediately above *s*
  Below(*T*, *s*)   // segment immediately below *s*

- *e.g*, Red-black trees, splay trees (key comparisons replaced by cross-product comparisons).

- $O(\log n)$ for each operation.

An Example

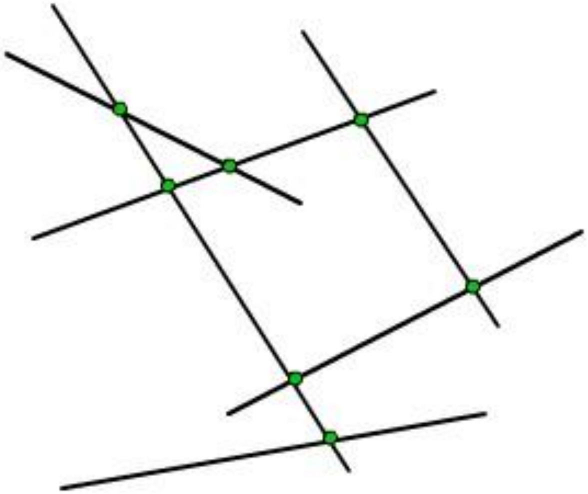




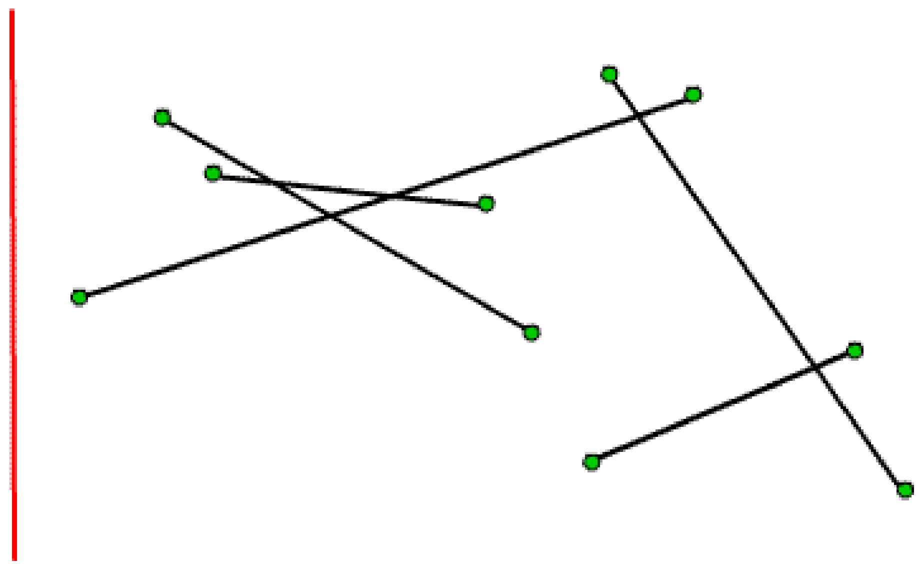
- The bottom-up order of the segments correspond to the *left-to-right* order of the leaves in the tree *T*.
- Each internal node stores the segment from the rightmost leaf in its left subtree.

Line segment intersection

Input: n line segments
Output: all intersection points



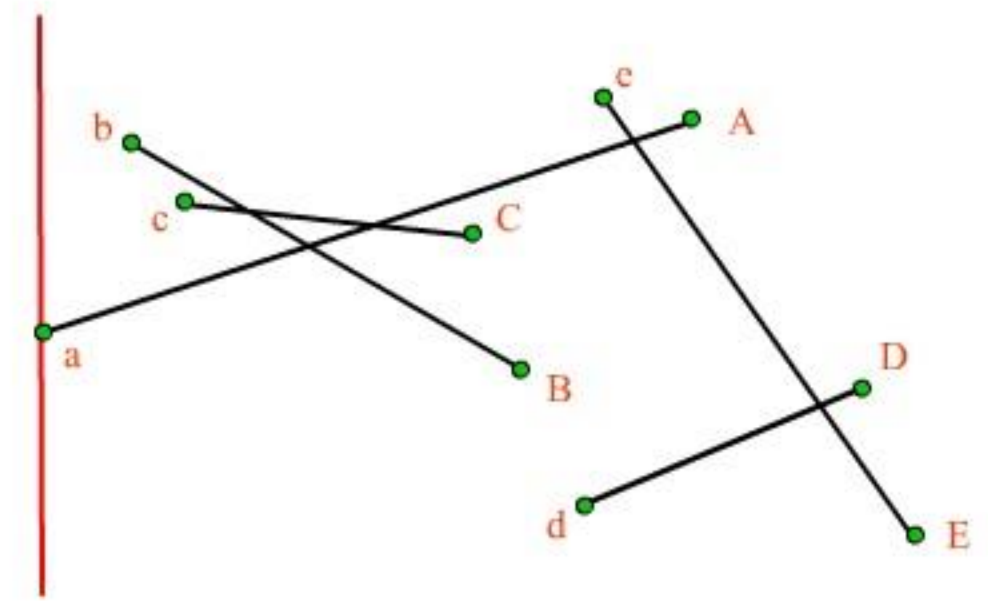
Sweeping…



Let's trace...

Intersect:

aA



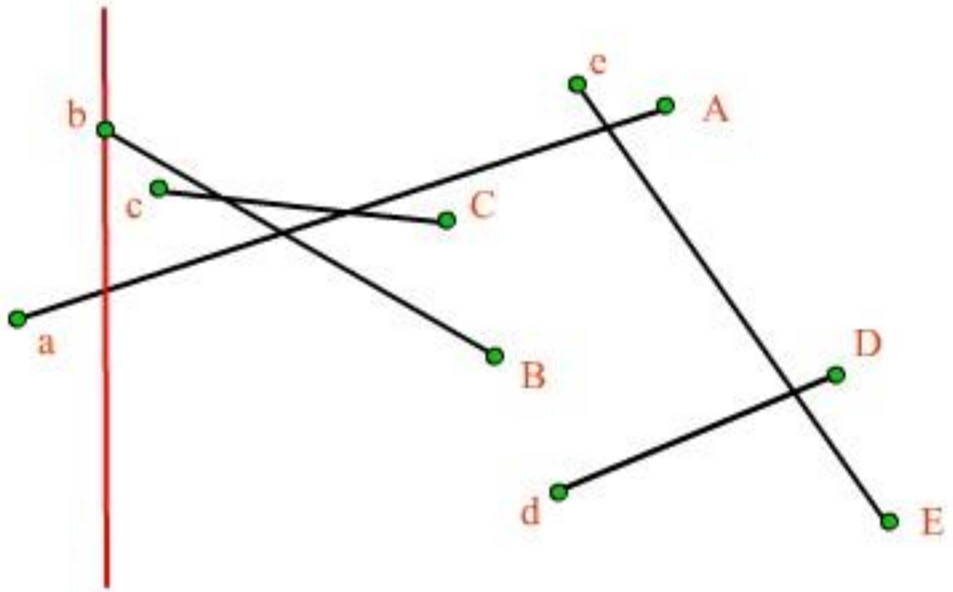
Event: a b c C B d e A D E

Let's trace…

Intersect:

aA

Insert ab
Add bB



Event: b c C B d e A D E

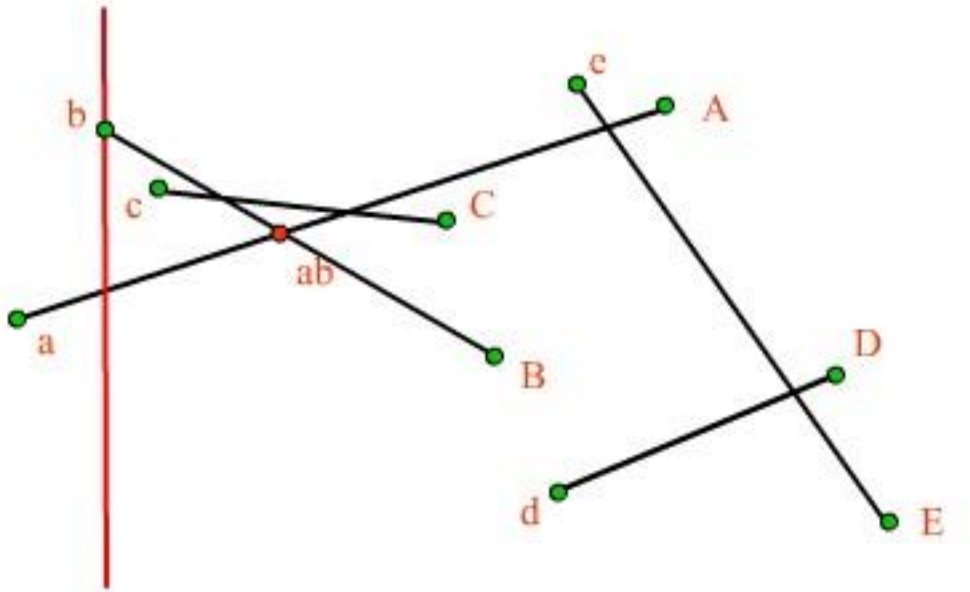
Key: two segments intersect, they must be adjacent in the intersection list at certain moment.

Let's trace…

Intersect:

bB

aA



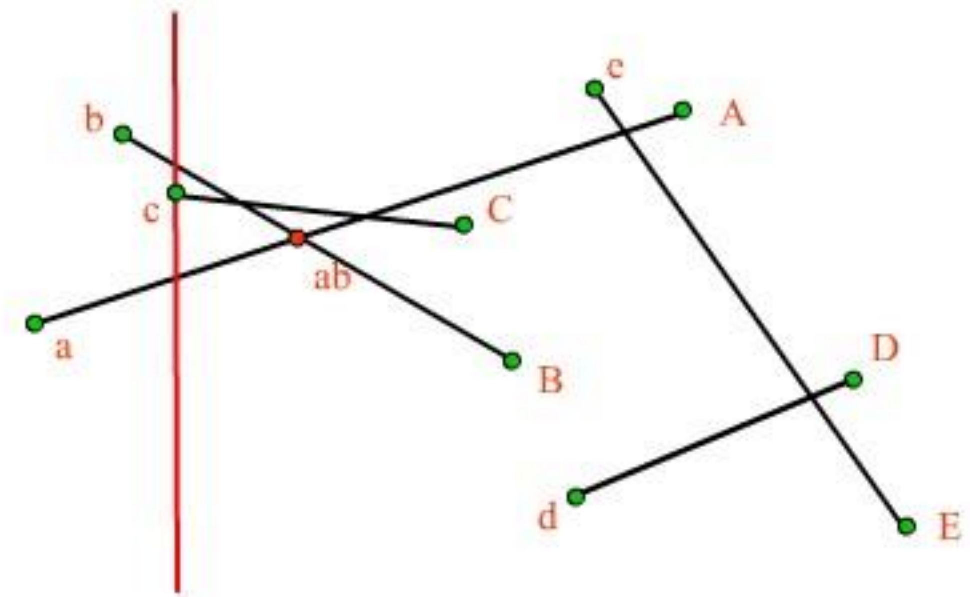
Event: b c ab C B d e A D E

Let’s trace …

Intersect:

bB

aA



Insert bc
Insert ac
Add cC

Event: c ab C B d e A D E

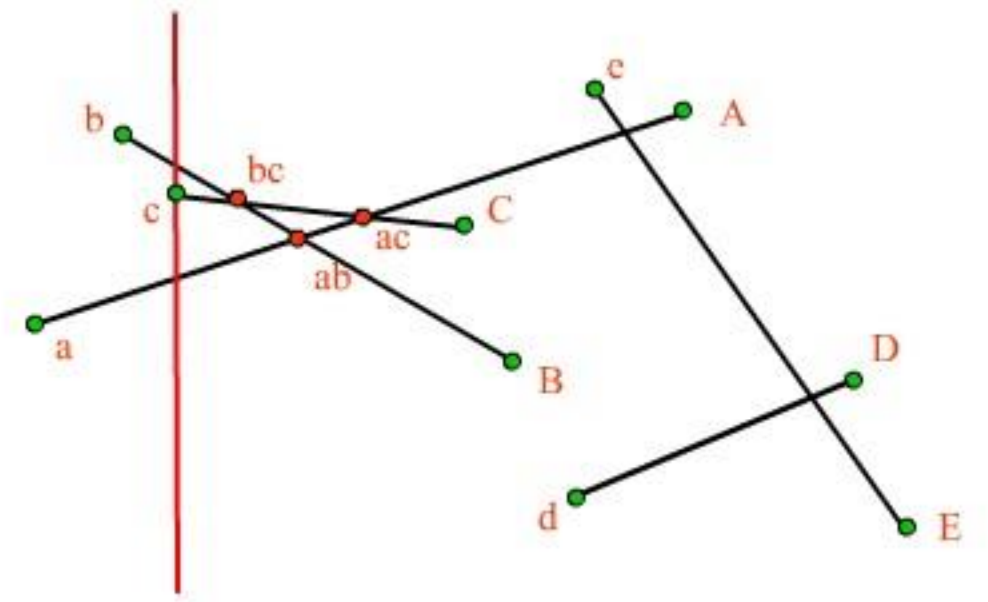
Let's trace …

Intersect:

bB

cC

aA



Event: c bc ab ac C B d e A D E

Let’s trace …

Intersect:

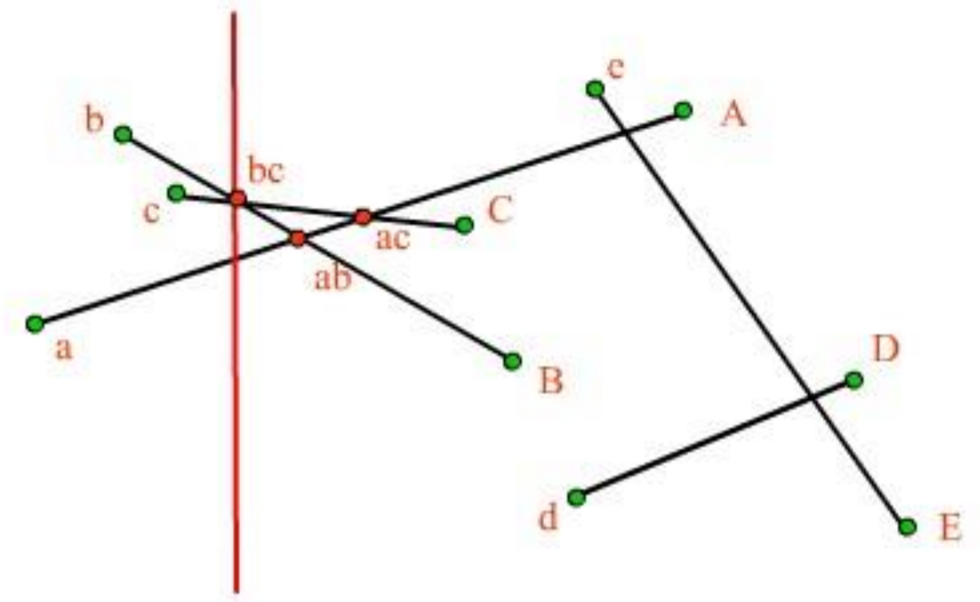
bB

cC

aA

Count bc

Swap bB-cC



Event: bc ab ac C B d e A D E

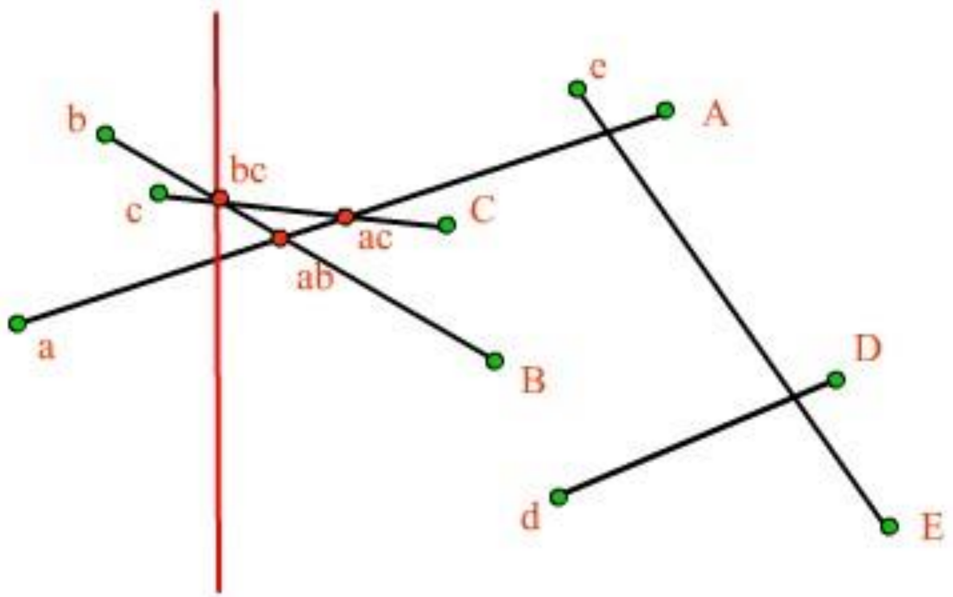
Let's trace …

Intersect:

cC

bB

aA



Event: bc ab ac C B d e A D E

Let’s trace …

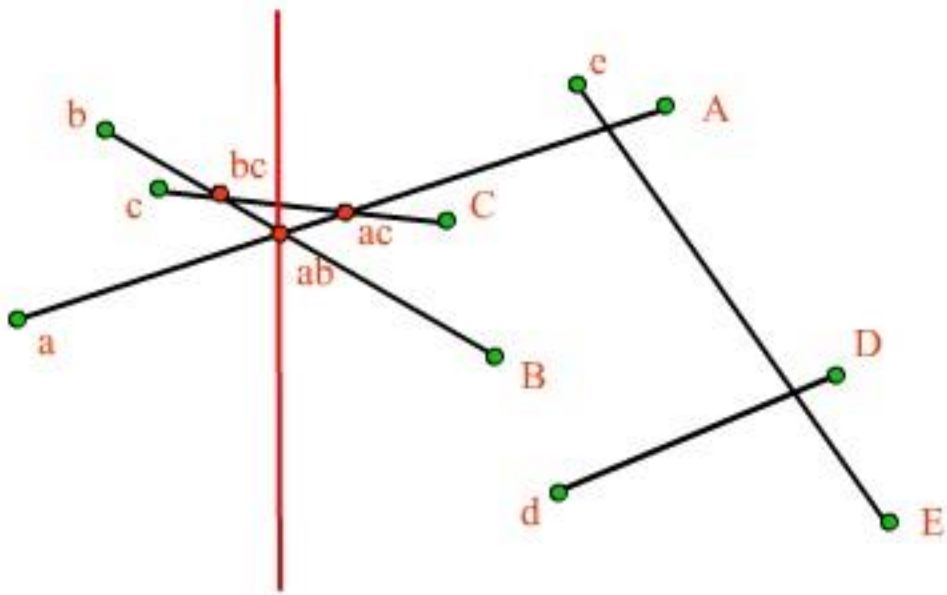
Intersect:

cC

bB

aA

Count ab
Swap aA-bB



Event: ab ac C B d e A D E

Let's trace …

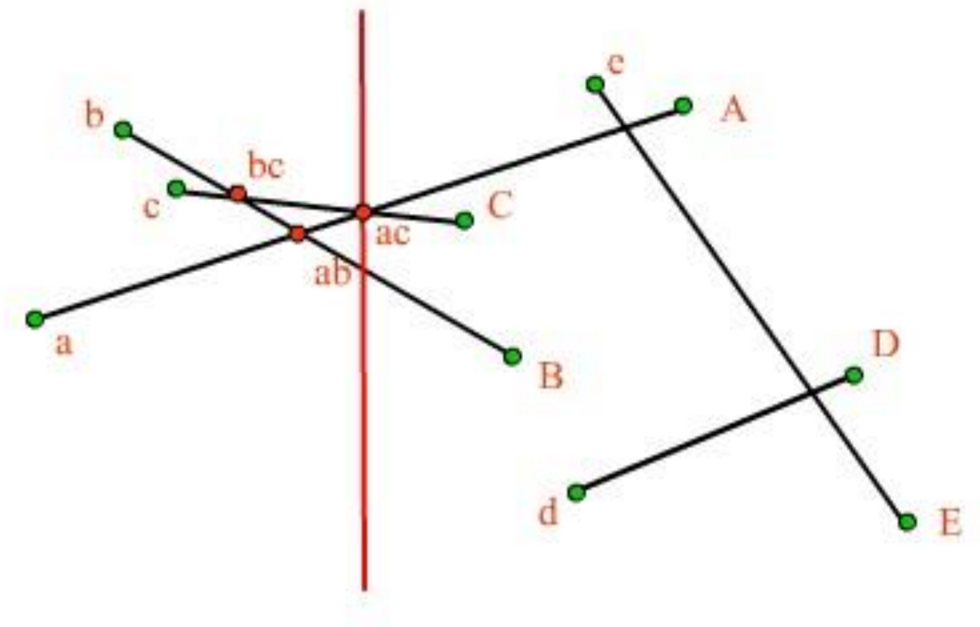
Intersect:

cC

aA

bB

Count ac
Swap aA-cC



Event: ac C B d e A D E

Let's trace ...

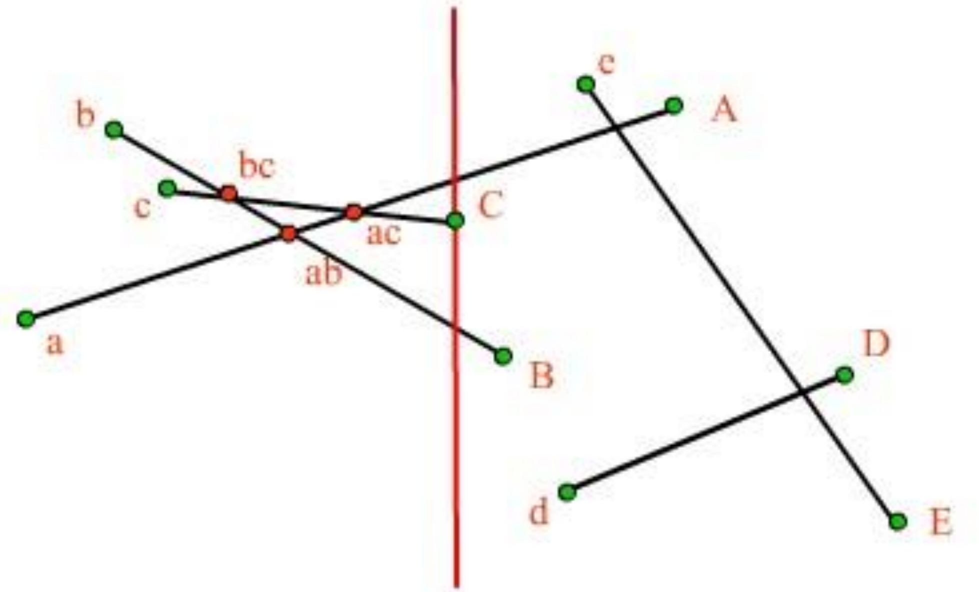
Intersect:

Remove cC

aA

cC

bB



Event: C B d e A D E

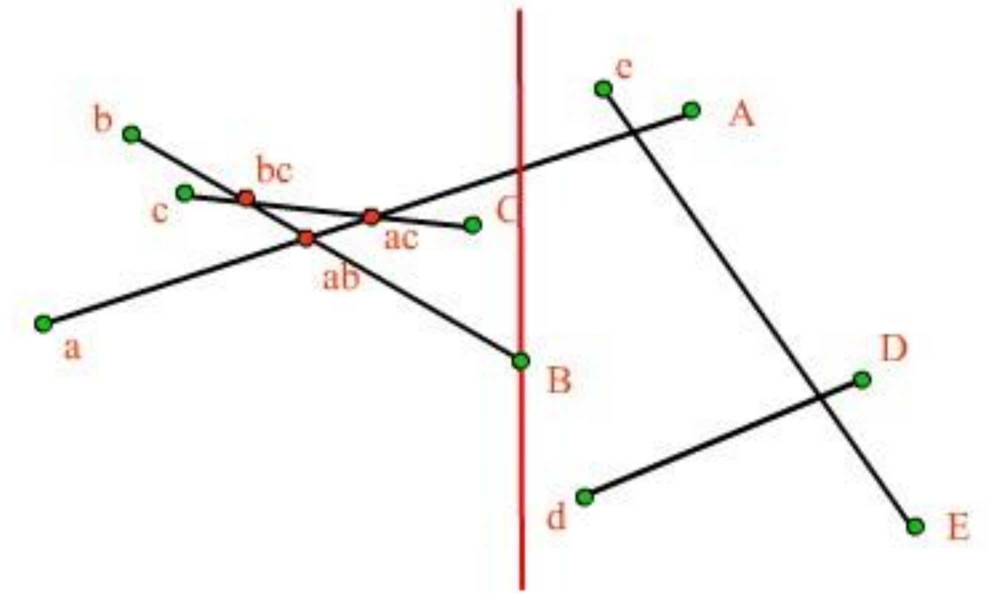
Let's trace …

Intersect:

Remove bB

aA

bB



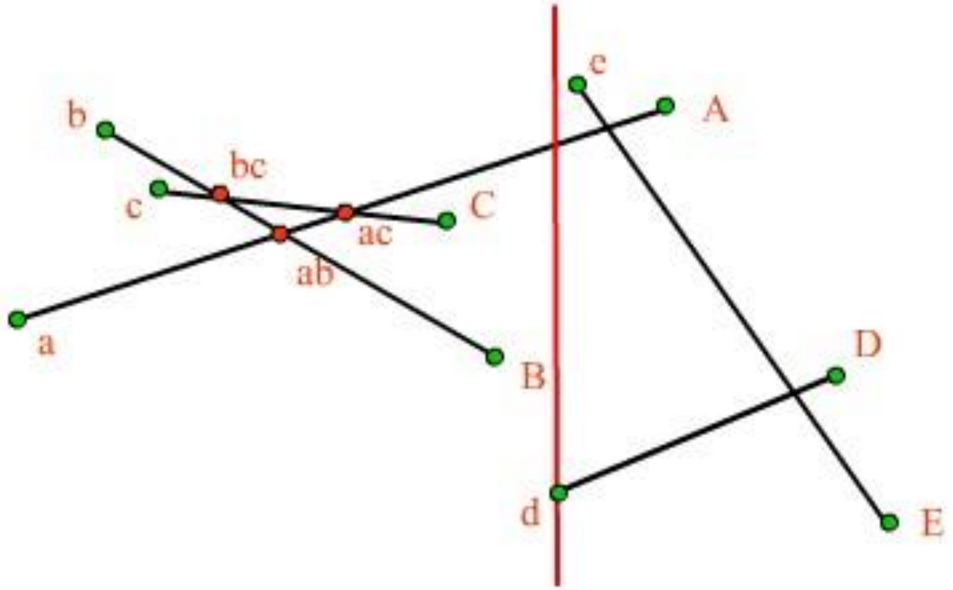
Event: B d e A D E

Let's trace …

Intersect:

Add dD

aA



Event: d e A D E

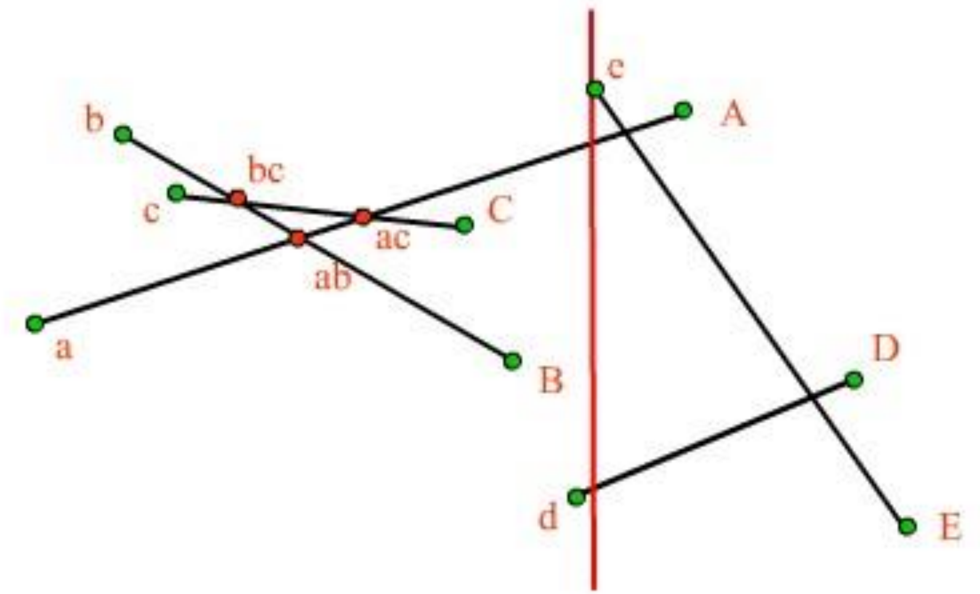
Let's trace …

Intersect:

aA

dD

Add eE
Insert ae



Event: e A D E

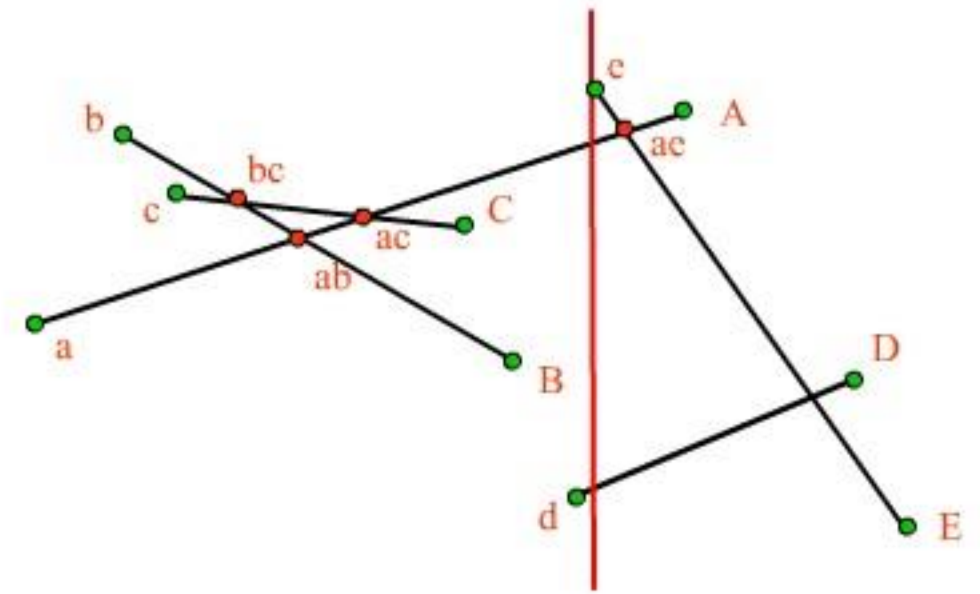
Let’s trace …

Intersect:

eE

aA

dD



Event: e ae A D E

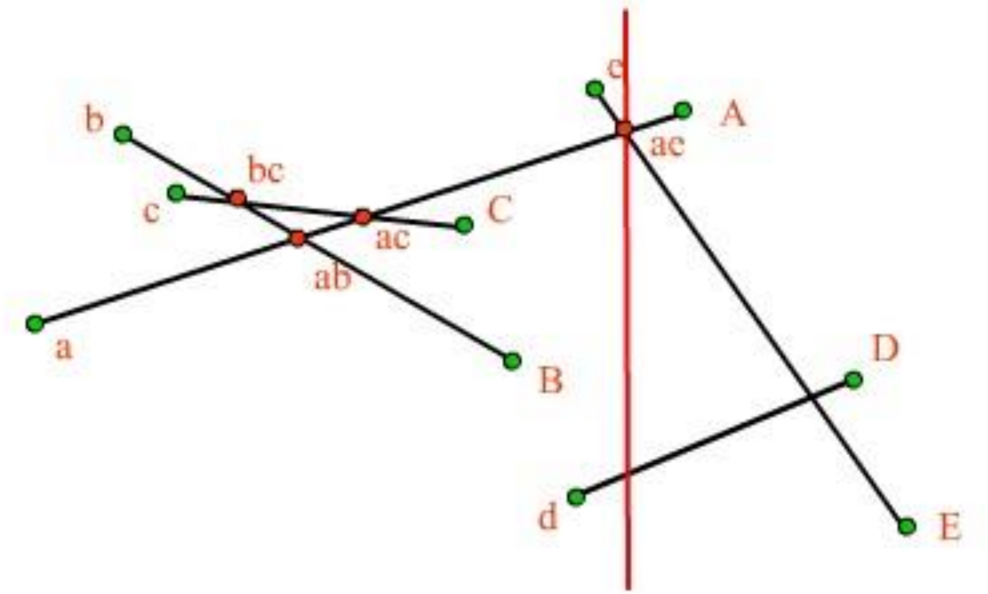
Let’s trace …

Intersect:

eE

aA

dD



Count ae
Swap eE-aA
Insert de

Event: ae A D E

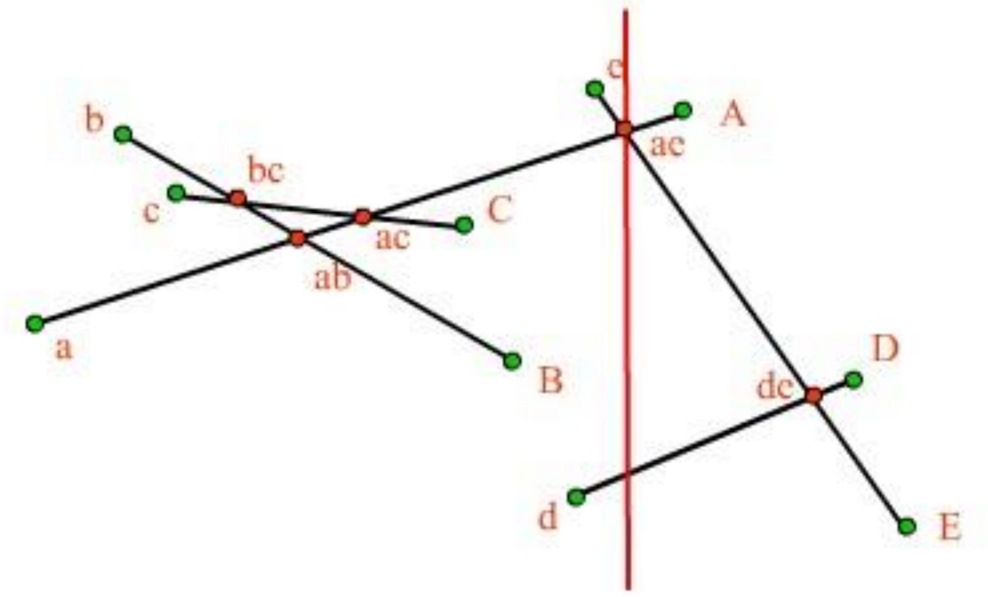
Let's trace …

Intersect:

aA

eE

dD



Event: ae A de D E

Let’s trace …

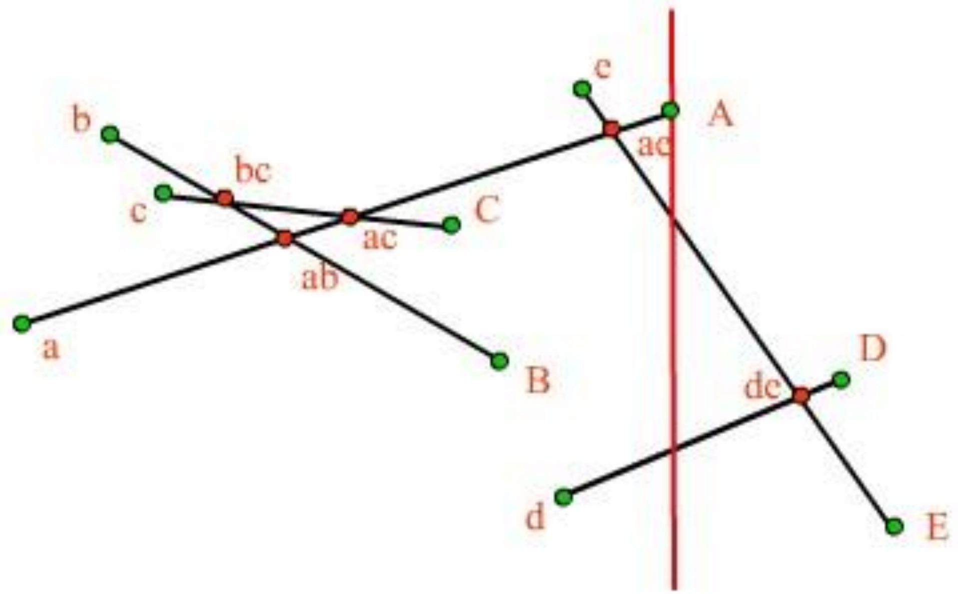
Intersect:

Remove aA

aA

eE

dD



Event: A de D E

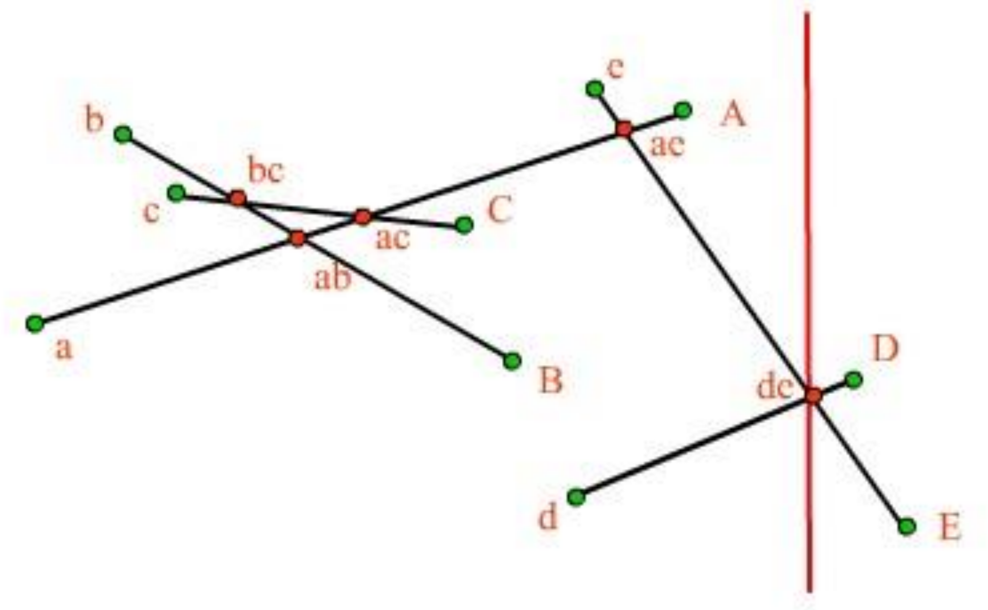
Let's trace …

Intersect:

eE

dD

Count de
Swap dD-eE



Event: de D E

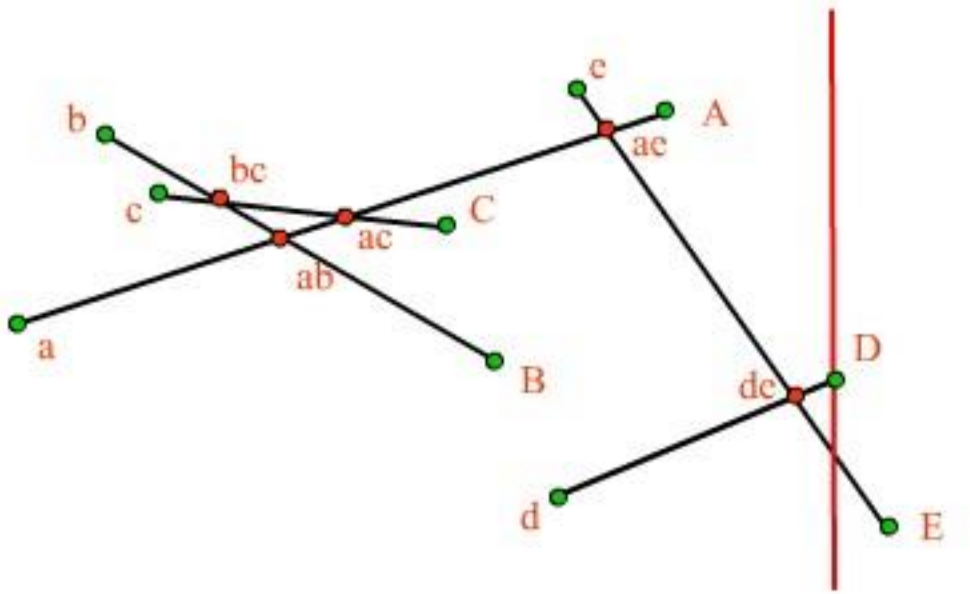
Let’s trace …

Intersect:

Remove dD

dD

eE



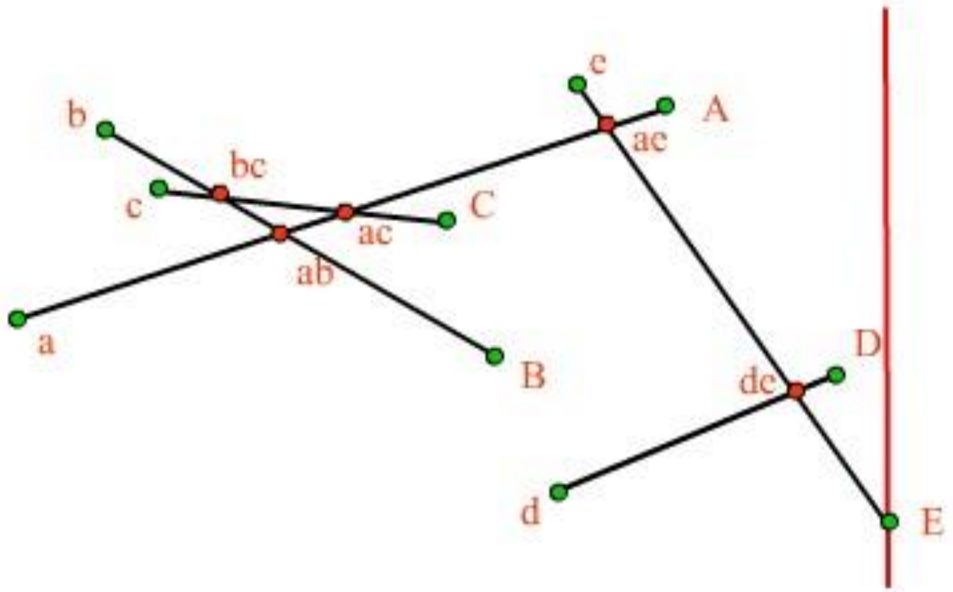
Event: D E

Let's trace ...

Intersect:

Remove eE

eE



Event: E

The Algorithm

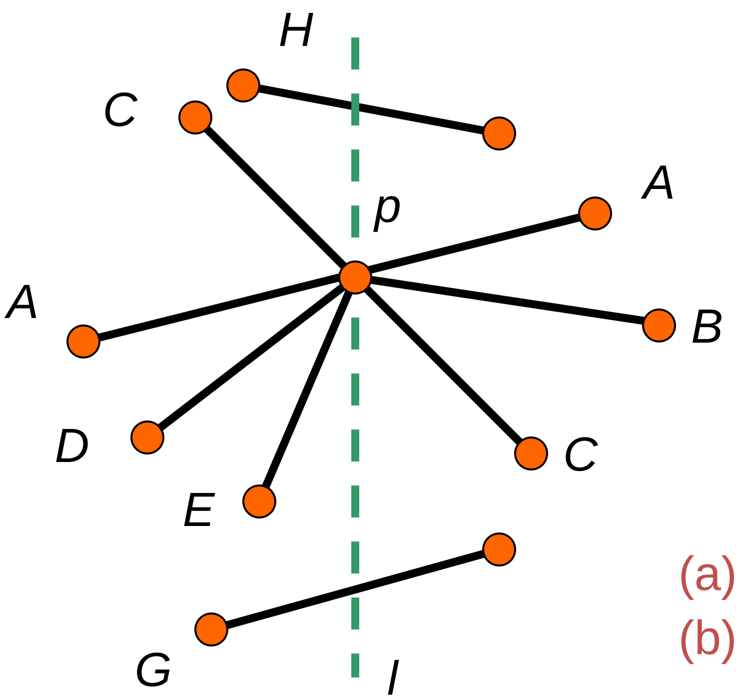
FindIntersections(S)
Input: a set S of line segments
Ouput: all intersection points and for each intersection the
segment containing it.

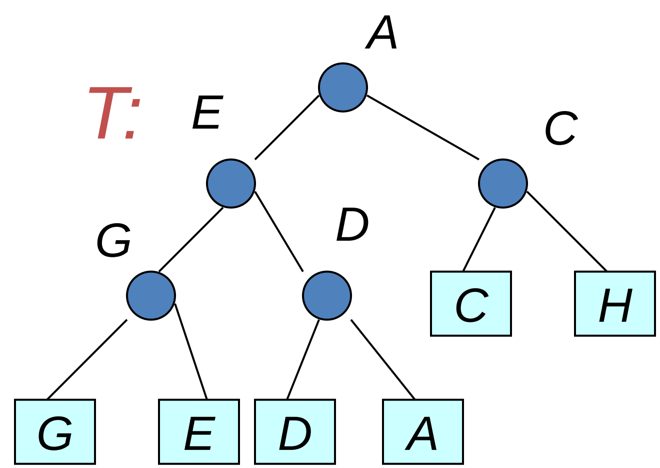
1. $Q \leftarrow \emptyset$ // initialize an empty event queue
2. Insert the segment endpoints into Q // store with every left endpoint
   // the corresponding segments
3. $T \leftarrow \emptyset$ // initialize an empty status structure
4. while $Q \neq \emptyset$
5. do extract the next event point p
6. $Q \leftarrow Q - \{p\}$
7. HandleEventPoint(p)

Handling Event Points

Status updates (1) – (3) presented earlier.

Degeneracy: several segments are involved in one event point (tricky).





(a) Delete *D*, *E*, *A*, *C*

(b) Insert *B*, *A*, *C*