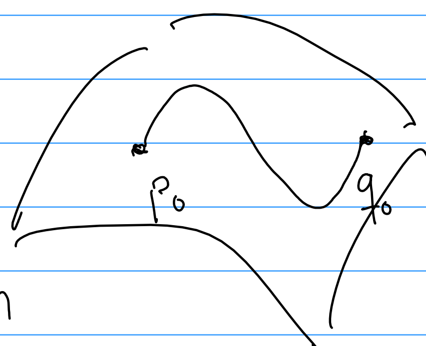
Energía:

$$\mathcal{E} = \int_a^b \|c'(t)\|^2\, dt = \int_a^b \left(E u'^2 + 2F u'v' + G v'^2 \right) dt$$

$$= \int_a^b \underbrace{(u' \;\; v')\, I \begin{pmatrix} u' \\ v' \end{pmatrix}}_{= f(u,v,u',v',t)} dt$$

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = I(u,v)$$

$$\mathcal{E} : \underbrace{\left\{ c : [a,b] \to S \subset \mathbb{R}^3 \;\middle|\; \begin{array}{l} c(t) = X(u(t),v(t)) \\ (u(t),v(t)) \text{ es curva en el esp. de parámetros} \\ \text{varían} \\ \text{tales que } c(a) = p_0 \\ \qquad\qquad\;\; c(b) = q_0 \end{array} \text{ fijos} \right\}}_{= \mathcal{X}} \to \mathbb{R}$$



Minimizar $\mathcal{E}$ es un Problema variacional

$$\mathcal{E} = \int_a^b f(u,v,u',v',t)\, dt$$

Variación de la curva: $W : [a,b] \to \mathbb{R}^2$

$$W(t) = (\alpha(t), \beta(t))$$

$$u \mapsto u + \lambda\alpha \quad , \lambda \in \mathbb{R} \qquad \alpha(a) = \beta(a) = 0$$

$$v \mapsto v + \lambda\beta \qquad \text{parámetro.} \qquad \alpha(b) = \beta(b) = 0$$

La curva $t \mapsto X(u(t) + \lambda\alpha(t), v(t) + \lambda\beta(t))$

$$\int_a^b \qquad \| \qquad = \tilde{c}(\lambda, t)$$

$$\mathcal{E}(\lambda) = \int_a^b \left\| \frac{\partial \tilde{c}}{\partial t} \right\|^2 dt$$ queremos que sea mínima en $\lambda = 0$.

$$\left. \frac{d\mathcal{E}}{d\lambda} \right|_{\lambda=0} = 0$$ cond. necesaria.

$$\mathcal{E}(\lambda) = \int_a^b f(u + \lambda a, v + \lambda b, u' + \lambda a', v' + \lambda b', t)\, dt$$

$$\left. \frac{d\mathcal{E}}{d\lambda} \right|_{\lambda=0} = \int_a^b \frac{\partial f}{\partial u} a + \frac{\partial f}{\partial v} b + \underbrace{\frac{\partial f}{\partial u'}}\, a' + \underbrace{\frac{\partial f}{\partial v'}}\, b' \; dt \overset{\forall a, b}{=} 0$$

$$\frac{\partial f}{\partial u'} a' = \frac{d}{dt}\left(\frac{\partial f}{\partial u'} a\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial u'}\right) a$$

$$\frac{\partial f}{\partial v'} b' = \frac{d}{dt}\left(\frac{\partial f}{\partial v'} b\right) - \frac{d}{dt}\left(\frac{\partial f}{\partial v'}\right) b$$

$$\int_a^b \frac{d}{dt}\left(\frac{\partial f}{\partial u'} a\right) dt = \frac{\partial f}{\partial u'} a \Big]_{t=a}^{t=b} = 0$$

a se anula en los extremos

Ojo: α, β se han transf. en a y b

No confundir con los extremos del intervalo!!

$$= \int_a^b \frac{\partial f}{\partial u} a + \frac{\partial f}{\partial v} b - \frac{d}{dt}\left(\frac{\partial f}{\partial u'}\right) a - \frac{d}{dt}\left(\frac{\partial f}{\partial v'}\right) b \, dt = 0$$

$$\forall a, b : I \to \mathbb{R} \Rightarrow \boxed{\begin{aligned} \frac{\partial f}{\partial u} &= \frac{d}{dt}\left(\frac{\partial f}{\partial u'}\right) \\ \frac{\partial f}{\partial v} &= \frac{d}{dt}\left(\frac{\partial f}{\partial v'}\right) \end{aligned}}$$

(que se anulen en los extremos)

Ecuaciones de Euler-Lagrange

$$f = (\underline{\underline{u'}}\; v')\, I \begin{pmatrix} \underline{\underline{u'}} \\ v' \end{pmatrix}$$

$$\frac{\partial f}{\partial u'} = 2(1\ \ 0)\, \underset{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}{\underset{\parallel}{I}} \begin{pmatrix} u' \\ v' \end{pmatrix} = 2(E, F)\begin{pmatrix} u' \\ v' \end{pmatrix} = 2(Eu' + Fv')$$

$$\frac{\partial f}{\partial v'} = 2(Fu' + Gv')$$

$$\frac{\partial f}{\partial u} = (u'\ v')\, I_u \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$\frac{\partial f}{\partial v} = (u'\ v')\, I_v \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial u} &= \frac{d}{dt}\left(\frac{\partial f}{\partial u'}\right) \\ \frac{\partial f}{\partial v} &= \frac{d}{dt}\left(\frac{\partial f}{\partial v'}\right) \end{aligned}}$$

$$\boxed{\begin{aligned} 2\frac{d}{dt}(E\underline{\underline{u'}} + F\underline{\underline{v'}}) &= (u'\ v')\, I_u \begin{pmatrix} u' \\ v' \end{pmatrix} \\ 2\frac{d}{dt}(Fu' + Gv') &= (u'\ v')\, I_v \begin{pmatrix} u' \\ v' \end{pmatrix} \end{aligned}}$$

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = U' \; ; \quad Eu' + Fv' = U'^T \begin{pmatrix} E \\ F \end{pmatrix}$$

$$Fu' + Gv' = U'^T \begin{pmatrix} F \\ G \end{pmatrix}$$

$$2\frac{d}{dt}\left(U'^T \underbrace{\begin{pmatrix} E \\ F \end{pmatrix}}\right) = U'^T I_u U'$$

$$2\frac{d}{dt}\left(U'^T \underbrace{\begin{pmatrix} F \\ G \end{pmatrix}}\right) = U'^T I_v U'$$

$$2\frac{d}{dt}(E u' + F v') = (u' \; v') I_u \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$2\frac{d}{dt}(F u' + G v') = (u' \; v') I_v \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$\frac{d}{dt}\left(U'^T \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{= I}\right) = \frac{1}{2}\begin{pmatrix} U'^T I_u U' \\ U'^T I_v U' \end{pmatrix}^T$$

$$\shortparallel$$

$$\left(\underline{U''^T} \underbrace{I} + \underbrace{U'^T \cdot (I_u \cdot u' + I_v v')}\right)$$

$$U''^T I = \frac{1}{2}\begin{pmatrix} U'^T I_u U' \\ U'^T I_v U' \end{pmatrix}^T - U'^T (I_u u' + I_v v')$$

$$U''^T = \underbrace{\left[\frac{1}{2} \begin{pmatrix} U'^T I_u U' \\ U'^T I_v U' \end{pmatrix}^T - U'^T \left(I_u u' + I_v v' \right) \right] I^{-1}}$$

$$\left.\begin{array}{l} U(t_0) = U_0 \\ U'(t_0) = U'_0 \end{array}\right\} \text{cond. inic.}$$

Dada una superficie $X = X(u, v)$ define la geodésica con cond. inic. dadas.