CS217 – Algorithm Design and Analysis
Homework 3

Not Strong Enough

March 23, 2020

1

Let $B_{i,j,k}$ be an indicator variable which is 1 if i is a common ancestor of j and k in the quicksort tree. That is, if both j and k appear in the subtree $T(\pi)$ rooted at i.
What is $\mathbb{E}[B_{i,j,k}]$? Give a succinct formula for this.

Solution. Let's go down through the path that the algorithm visits.

Suppose we are at the node whose pivot is p. If $p > \max\{i,j,k\}$ or $p < \min\{i,j,k\}$, then i,j,k are all in the left subtree or all in the right subtree, which means we have to check one of the subtrees to determine $B_{i,j,k}$.

If $\min\{i,j,k\} \leqslant p \leqslant \max\{i,j,k\}$, we can determine $B_{i,j,k}$ without going downwards, but we have to discuss some cases:

1. If $p = i$, then $B_{i,j,k} = 1$ because i is the ancestor of j and k.
2. If the condition of case 1 doesn't hold and $p \neq j$ and $p \neq k$, then i,j,k are not in the same subtree. So $B_{i,j,k} = 0$.
3. If the condition of both case 1 and 2 don't hold, then either j or k is the ancestor of i. So $B_{i,j,k} = 0$.

In conclusion, $B_{i,j,k} = 1$ if and only if i appears first among $[\min\{i,j,k\}, \max\{i,j,k\}]$ in the input array. So $\mathbb{E}[B_{i,j,k}] = \frac{1}{|[\min\{i,j,k\},\max\{i,j,k\}]|} = \frac{1}{\max\{i,j,k\}-\min\{i,j,k\}+1}$. $\square$

2

Let $C(\pi, k)$ be the number of comparisons made by QuickSelect when given π as input. Design a formula for $C(\pi, k)$ with the help of the indicator variables $A_{i,j}$ and $B_{i,j,k}$ (analogous to the formula $\sum_{i \neq j} A_{i,j}$) for the number of comparisons made by quicksort).

Solution. Observe that two numbers i, j will be compared if and only if $j \neq i$ and j and k are in the subtree of i. An interpretation is that i will be a pivot if and only if i is on the path from root to node k, which is equal to k is in the subtree rooted at i. As a pivot, i will be compared to every node in the subtree rooted at i.

So $C(\pi, k) = \sum_{i \neq j} B_{i,j,k}$. □