CS217 – Algorithm Design and Analysis
Homework 7

Not Strong Enough

June 15, 2020

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Show that the three versions of Farkas Lemma presented in class are all equivalent:

$$(\neg\exists \mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T\mathbf{A} = \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0) \tag{1}$$

$$(\neg\exists \mathbf{x} \geq \mathbf{0} : \mathbf{A}\mathbf{x} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T\mathbf{A} \geq \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0) \tag{2}$$

$$(\neg\exists \mathbf{x} \geq \mathbf{0} : \mathbf{A}\mathbf{x} = \mathbf{b}) \iff (\exists \mathbf{y} : \mathbf{y}^T\mathbf{A} \geq \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0) \tag{3}$$

Proof. (1) $\Rightarrow$ (3):

Let $\mathbf{A}' = \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \\ -\mathbf{I} \end{bmatrix}, \mathbf{b}' = \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \\ \mathbf{0} \end{bmatrix}$. Then

$$\begin{aligned}
(\neg\exists \mathbf{x} \geq \mathbf{0} : \mathbf{A}\mathbf{x} = \mathbf{b}) &\iff (\neg\exists \mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}, -\mathbf{A}\mathbf{x} \leq -\mathbf{b}, -\mathbf{x} \leq \mathbf{0}) \\
&\iff (\neg\exists \mathbf{x} : \mathbf{A}'\mathbf{x} \leq \mathbf{b}') \\
&\iff (\exists \mathbf{y}' \geq \mathbf{0} : \mathbf{y}'^T\mathbf{A}' = \mathbf{0}, \mathbf{y}'^T\mathbf{b}' < 0) && \text{(derived from(1))} \\
&\iff (\exists \mathbf{y}_1, \mathbf{y}_2, \mathbf{z} \geq \mathbf{0} : (\mathbf{y}_1^T - \mathbf{y}_2^T)\mathbf{A} - \mathbf{z} = \mathbf{0}, (\mathbf{y}_1^T - \mathbf{y}_2^T)\mathbf{b} < 0) && (\text{Let } \mathbf{y}' = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{z} \end{bmatrix}) \\
&\iff (\exists \mathbf{z} \geq \mathbf{0}, \mathbf{y} : \mathbf{y}^T\mathbf{A} = \mathbf{z}, \mathbf{y}^T\mathbf{b} < 0) && (\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2) \\
&\iff (\exists \mathbf{y} : \mathbf{y}^T\mathbf{A} \geq \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0)
\end{aligned}$$

(3) $\Rightarrow$ (2):

let $\mathbf{A}' = \begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix}$.

$$\begin{aligned}
(\neg\exists \mathbf{x} \geq \mathbf{0} : \mathbf{A}\mathbf{x} \leq \mathbf{b}) &\iff (\neg\exists \mathbf{x}, \mathbf{z} \geq \mathbf{0} : \mathbf{A}\mathbf{x} + \mathbf{z} = \mathbf{b}) && (\mathbf{A}\mathbf{x} \leq \mathbf{b} \iff \exists \mathbf{z} \geq \mathbf{0}, \mathbf{A}\mathbf{x} + \mathbf{z} = \mathbf{b}) \\
&\iff (\neg\exists \mathbf{x}' \geq \mathbf{0} : \mathbf{A}'\mathbf{x}' = \mathbf{b}) && (\text{let } \mathbf{x}' = \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix}) \\
&\iff (\exists \mathbf{y} : \mathbf{y}^T\mathbf{A}' \geq \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0) && \text{(derived from(3))} \\
&\iff (\exists \mathbf{y} : \mathbf{y}^T\mathbf{A} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0) \\
&\iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T\mathbf{A} \geq \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0)
\end{aligned}$$

(2) $\Rightarrow$ (1):
Let $\mathbf{A}' = \begin{bmatrix}\mathbf{A} & -\mathbf{A}\end{bmatrix}$.

$$
\begin{aligned}
(\neg\exists \mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}) &\iff (\neg\exists \mathbf{x}_1, \mathbf{x}_2 \geq \mathbf{0} : \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) \leq \mathbf{b}) \\
&\iff (\neg\exists \mathbf{x}' \geq \mathbf{0} : \mathbf{A}'\mathbf{x}' \leq \mathbf{b}) && (\text{let } \mathbf{x}' = \begin{bmatrix}\mathbf{x}_1 \\ \mathbf{x}_2\end{bmatrix}) \\
&\iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T\mathbf{A}' \geq \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0) && (\text{derived from}(2)) \\
&\iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T\mathbf{A} \geq \mathbf{0}, -\mathbf{y}^T\mathbf{A} \geq \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0) \\
&\iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T\mathbf{A} = \mathbf{0}, \mathbf{y}^T\mathbf{b} < 0)
\end{aligned}
$$

□

2

Let d be the shortest path distance from s to t in the directed graph G, where distance means sum of the $c(e)$ along the path. Show that $\mathrm{opt}(MCF) = d$.

Proof. First we show that $\mathrm{opt}(MCF) \leqslant d$. This is quite easy. Suppose $(v_1, \ldots v_k)$ where $k \geqslant 1$ and $(v_i, v_{i+1}) \in E, v_1 = s, v_k = t$ is a shortest path in graph G. Then we can write a feasible solution to MCF by setting all $f(v_i, v_{i+1}) = 1$ and setting the flow of edges which are not in the shortest path to 0. (Obviously the two constraints of MCF are both satisfied.) Note that the value of this solution is exactly d, since $\sum_{e \in E} c(e) f(e) = \sum_{i=1}^{k-1} (c(v_i, v_{i+1}) \cdot 1) = d$. Hence $\mathrm{opt}(MCF) \leqslant d$.

Next we show that $\mathrm{opt}(MCF) \geqslant d$. We prove by contradiction. Assume that $\mathrm{opt}(MCF) < d$. Suppose we have a feasible solution f^* with $\mathrm{val}(f^*) < d$. The idea is first to split the flow f^* into some *single-path-flows*. And then we will show that among such $s - t$-paths, at least one path P^* satisfies that $\sum_{e \in P^*} c(e) < d$. So d is not the shortest path distance from s to t, and it leads to a contradiction.

Now we focus on how to split f^* into multiple single-path-flows. We first choose an edge (u, v) such that $f(u, v)$ is positive and minimum among all edges in G. Since $f(u, v) \neq 0$ is minimum, we can always find a path from s to u and a path from v to t such that the flows of all the edges in both paths are at least $f(u, v)$ (e.g., using DFS or BFS). Let the two paths be P_u and P_v. Connecting P_u, P_v with (u, v) we get a new path $P = (P_u, (u, v), P_v)$, with the flows of all edges on this path being at least $f(u, v)$ and the flow contribution of this path being exactly $f(u, v)$.

So we can "extract" this path from G by decreasing the flow of edges in this path by $f(u, v)$. Note that edge (u, v) now has flow 0. After extraction, we get a new graph G' and a new "flow function" $f^{*\prime}$. We can validate that $f^{*\prime}$ is still a real flow by checking the flow-conservative contraint: the inflow decrease and outflow decrease of every vertices on path P are always the same, and the extraction does not influence the inflow and outflow of vertices not on P. So $f^{*\prime}$ is a real flow.

Repeat the "find minimum and then extract" process on the new graph. Note that each time the number of edges with non-negative flow is decreased by 1, and the total number of edges in a graph is finite, so the process will surely terminate. Finally we get many single-path-flows. We claim that among these single-path-flows, at least one path P^* satisfies that $\sum_{e \in P^*} c(e) < d$. Otherwise every single-path-flow has sum of $c(e)$ along the path being at least d, and it follows that $\mathrm{val}(f) \geqslant d$, which contradicts with our assumption. However, $\sum_{e \in P^*} c(e) < d$ is impossible since d is the shortest minimum path distance, which also contradicts with this premise. So we have $\mathrm{opt}(MCF) \geqslant d$.

Combining $\mathrm{opt}(MCF) \leqslant d$ and $\mathrm{opt}(MCF) \geqslant d$ we can finally conclude that $\mathrm{opt}(MCF) = d$. □

3

Write down the dual of MCF.

Proof. First, we write down MCF in terms of inequality:

$$\begin{aligned}
\text{minimize } & \sum_{e\in E} c(e)f(e) \\
\text{subject to } & \sum_{e\in E:e=(u,t)} f(e) \geq 1 && \text{(corresponds to } y_t^+ \text{ in dual form)} \\
& \sum_{e\in E:e=(u,t)} -f(e) \geq -1 && \text{(corresponds to } y_t^- \text{ in dual form)} \\
& \sum_{e\in E:e=(u,v)} f(e) - \sum_{e\in E:e=(v,w)} f(e) \geq 0, \forall v \in V \setminus \{s,t\} && \text{(corresponds to } y_v^+ \text{ in dual form)} \\
& \sum_{e\in E:e=(v,w)} f(e) - \sum_{e\in E:e=(u,v)} f(e) \geq 0, \forall v \in V \setminus \{s,t\} && \text{(corresponds to } y_v^- \text{ in dual form)} \\
& f(e) \geq 0, \forall e \in E.
\end{aligned}$$

Then write down its dual LP: (Let $S = \{e \in E : e = (s,v), v \in V\}$)

$$\begin{aligned}
\text{maximize } & y_t^+ - y_t^- \\
\text{subject to } & (y_v^+ - y_v^-) - (y_u^+ - y_u^-) \leq c(e), \forall e = (u,v) \in E \setminus S \\
& y_v^+ - y_v^- \leq c(e), \forall e = (s,v) \in S \\
& y_v^+, y_v^- \geq 0, \forall v \in V \setminus \{s,t\} \\
& y_t^+, y_t^- \geq 0.
\end{aligned}$$

Now let $z_u = y_u^+ - y_u^-$ for all vertices u, it turns to be:

$$\begin{aligned}
\text{maximize } & z_t \\
\text{subject to } & z_v - z_u \leq c(e), \forall e = (u,v) \in E \setminus S \\
& z_v \leq c(e), \forall e = (s,v) \in S \\
& z_v \in \mathbf{R}, \forall v v \in V \setminus \{s,t\} \\
& z_t \in \mathbf{R}.
\end{aligned}$$

Let $z'_v = z_v + z'_s$, for all vertices u except for s and the problem turns to:

$$\begin{aligned}
\text{maximize } & z'_t - z'_s \\
\text{subject to } & z'_v - z'_u \leq c(e), \forall e = (u,v) \in E \\
& z'_v \in \mathbf{R}, \forall v \in V.
\end{aligned}$$

□

4

Interpret the dual. Show that it is the LP formulation of a “natural” maximization problem on G

Solution. Consider that each vertex has a potential, and for each edge $e = (u,v)$, the potential of the terminal vertex v is no greater than the potential of the start vertex u plus $c(u,v)$, and our goal is to maximize the potential of t. □

Solution. Let $c(e)$ be lengths of edges and

$$d(u) := \begin{cases} 0, u = s \\ \text{the length of the shortest path from } s \text{ to } u, \text{ otherwise} \end{cases}$$

Without loss of generality, we fix $z'_s = 0$. We claim that $z'_u = d(u)$ is an optimal solution.

- $\forall v \in V$, $z'_v = d(v) \in \mathbf{R}$.
- $\forall e = (u, v) \in E$, assume that $d(v) - d(u) > c(e)$. This suggests that the length of the shortest path from s to t with length $d(u) + c(e)$, which is a contradiction to the definition.
- $z'_t - z'_s$ is maximized by letting $z'_v = d(v)$:
  Assume that $z'_t - z'_s = z'_t > d(t)$.
  Let the shortest path from s to t be $v_1, v_2, \cdots, v_r$ where $v_1 = s$ and $v_r = t$. We have $d(v_{i+1}) = d(v_i) + c((v_i, v_{i+1}))$.
  We next prove that $\forall i \in [r], z'_{v_i} > d(v_i)$ by induction from r to 1.
  - Base: $z'_{v_r} > d(v_r)$ by our previous assumption.
  - Inductive step: By inductive hypothesis, we have $z'_{v_{i+1}} > d(v_{i+1})$.
    Then $z'_{v_i} + c(e_i) \geq z'_{v_{i+1}} > d(v_{i+1}) = d(v_i) + c(e_i) \implies z'_{v_i} > d(v_i)$.

  Hence, $z'_s > d(s) = 0$, which is a contradiction.

Therefore, $z'_u = d(u)$ is an optimal solution to the dual program. $\square$