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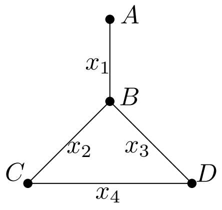
For a graph $G = (V, E)$, let $\tau(G)$ denote the size of a minimum vertex cover, and $\nu(G)$ the size of a maximum matching. Recall the two linear programs VCLP and MLP. Let $\tau_f(G) := \mathrm{opt}(\mathrm{VCLP}(G))$ and $\nu_f(G) := \mathrm{opt}(\mathrm{MLP}(G))$. Note that

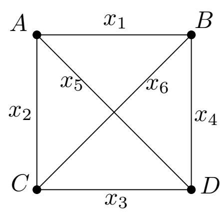
$$\nu(G) \le \nu_f(G) = \tau_f(G) \le \tau(G),$$

where the equality in the middle follows from Strong LP Duality. Also, if G is bipartite, then the equality holds throughout in (1). Let us say a graph G is *VCLP exact* if $\tau(G) = \tau_f(G)$, and *MLP exact* if $\nu(G) = \nu_f(G)$. As we already know, a bipartite grpah G is both VCLP exact and MLP exact.
From now on, suppose that G is *not* bipartite but $\tau(G) = \tau_f(G)$.

1. Give an example of such a graph G that is not bipartite but still VCLP exact.
2. Give an example of a graph G that is MLP exact but not VCLP exact.
3. Suppose G is VCLP exact. Let $Y \subseteq V(G)$ be a minimum vertex cover. Let $\mathbf{x}$ be an optimal solution of MLP(G). Show that $x_e = 0$ if $e \subseteq Y$ (i.e., if both endpoints of e are in the cover).
4. Show that such a graph G has a matching of size $|Y|$, and thus is MLP exact, too.



G_1



K_4

Solution.

1. The example is G_1 in the upper left. It is not bipartite since B, C, D constitute an odd cycle. The minimum vertex cover of G_1 can be $\{B, C\}$ or $\{B, D\}$, with size 2. So $\tau(G_1) = 2$.
   VCLP(G_1) is

$$\begin{array}{ll} \text{minimize} & y_A + y_B + y_C + y_D \\ \text{subject to} & y_A + y_B \ge 1 \\ & y_B + y_C \ge 1 \\ & y_B + y_D \ge 1 \\ & y_C + y_D \ge 1 \\ & y_A, y_B, y_C, y_D \ge 0 \end{array}$$

   Note that if we add up the constraints $y_A + y_B \ge 1$ and $y_C + y_D \ge 1$, we get $y_A + y_B + y_C + y_D \ge 2$, which gives a lower bound of the target function.
   Since $\tau(G_1) = 2$, it follows that the lower bound is tight. Hence $\tau_f(G_1) = \tau(G_1) = 2$.
   So G_1 is an example which is not bipartite but still VCLP exact.
2. The example is K_4 in the upper right. The minimum vertex cover can be $\{A, B, C\}, \{A, B, D\}, \{A, C, D\}$ and $\{B, C, D\}$, with size 3. So $\tau(K_4) = 3$.
   However, by setting the value of each vertex to 0.5, we find that all edges are exactly covered ($y_u + y_v = 1$ for edge (u, v)). So $\tau_f(K_4) \le 2$, and thus K_4 is not VCLP exact.
   Now consider the maximum matching and MLP. Obviously we can only match 2 pairs of vertices. So $\nu(K_4) = 2$, and $\nu_f(K_4) \ge 2$ follows. Since we already know that $\tau_f(K_4) \le 2$ and $\nu_f(K_4) = \tau_f(K_4)$ by Strong LP Duality, we can conclude that $\nu(K_4) = \nu_f(K_4) = \tau_f(K_4) = 2$. Therefore K_4 is MLP exact.
   So K_4 is an example which is MLP exact but not VCLP exact.

□

To solve (3) and (4), define a function $N(\cdot)$ relative to a VCLP exact graph $G = (V, E)$ and a minimum vertex cover Y that:

$$N(A) = \begin{cases} \{v : v \text{ is the neighbor of some } u \in A\} \cap V \backslash Y, \text{if } A \subseteq Y \\ \{v : v \text{ is the neighbor of some } u \in A\} \cap Y, \text{if } A \subseteq V \backslash Y \end{cases}$$

2: Lemma

If $A \subseteq Y$, then $|A| \leq |N(A)|$. Vise versa, if $A \subseteq V \backslash Y$, then $|A| \geq |N(A)|$.

Proof. If $A \subseteq Y$ and there is $|B| < |A|$ where $B = N(A)$. Consider the solution of VCLP relative to Y(that is, if $v \in Y$, $y_v = 1$, otherwise $y_v = 0$). Let

$$y'_v = \begin{cases} y_v - \epsilon & , v \in A \\ y_v + \epsilon & , v \in B \\ y_v & , \text{ otherwise} \end{cases}$$

where $\epsilon < \frac{1}{2}$. To prove that (y'_v) is also a solution for VCLP, we only need to check:

a)$y'_u + y'_v \geq 1, u, v \in Y$. There is $y'_u + y'_v \geq 1 - \epsilon + 1 - \epsilon = 2 - 2\epsilon > 1$;

b)$y'_u + y'_v \geq 1, u \in A, v \in V \backslash B$. There is $y'_u + y'_v \geq (1 - \epsilon) + \epsilon = 1$;

The rest constraints remains true since the left hand side is not changed from solution (y_v).

So $(y_v)'$ is a solution to the VCLP too, but it is $\epsilon(|A| - |B|)$ less than solution (y_v), which rises a contradiction that Y is a minimum solution.

Likewise, let $A = N(B)$ and use the same method to construct (y'_v) can prove the 'vice versa' part. □

Now we begin to prove (3) and (4):

Proof. (3) Consider the optimal solution of MLP be $X = (x_e)$, assume there is $y_1, y_2 \in Y, (y_1, y_2) \in V, x_{y_1 y_2} = x \geq 0$. Let $A_0 = \{y_1, y_2\}, B_0 = N(A_0), C_0 = N(B_0)$.

By the lemma above, there is $|B_0| > |C_0|$. Let all edges from C to B be $\{e_1, e_2 \ldots e_m\}$, there is:

$$\sum_{i=1}^{m} y_e + 2x \leq |C_0| \leq |B_0|$$

By the property of vertex cover, any vertex in B can only connected to vertices in C_0. Let $B = \{b_1, b_2 \ldots b_s\}$, and $v(b_i) = \sum_{\forall e \in E : b_i \in e} x_e$, then there is:

$$\sum_{i=1}^{s} v(b_i) = \sum_{i=1}^{m} y_e \leq |B_0| - 2x$$

. So there exists $I = \{v_{i_1}, v_{i_2} \ldots v_{i_t}\} \subseteq B$, where $\sum_{j=1}^{t} (1 - v(b_{i_j})) \geq 2x$. We call $1 - v(b_i)$ the capacity of vertex b_i.

Let I_1 be all vertices connected to y_1 and in I, and the capacity of I_1, $c(I_1)$ is the sum of capacity of vertices in I_1, I_2 likewise.

Let $A_1 = \{y_2\}, B_1 = N(A_1), C_1 = N(B_1)$, by the same method, we can prove that the $c(B_1)$ is no less than x. And that means the $c(I_2)$ is no less than x, since I_2 contains all those vertices with positive capacity in B_1

Without loss of generality, we can assume that $c(I_1)$ is more than x.

Now we prove that in different cases we can always modify the optimal solution X to get a better solution.

In each case, we do the same thing first: let $x_{y_1 y_2} = 0$(which gives an decreasement of x in the target). We then define a slack of $(y_i, I_j)(i = 1, 2; j = 1, 2, 3, 4, 5)$ be that:

Increase x_e for each edge e from y_i to I_j until the constraint of at least one side of e is strict.

Let $I_3 = I_1 \cap I_2, I_4 = I_1 \backslash I_3, I_5 = I_2 \backslash I_3$,

a)If the $c(I_5) \geq 0$, do a slack to (y_2, I_5). Since the $c(I_5 \geq)0$, the increasement is more than 0. Notice that $c(I_1)$ is not changed, so now we do the slack of (y_1, I_1) and the increasement is not less than x. Not the total increasement is more than x, so the new solution is better.

b)If $c(I_5) = 0$, there is $c(I_2) \geq 0 \rightarrow c(I_3) \geq 0$ and $c(I_1) = c(I) \geq 2x$. Now we do slack of (y_2, I_3) but stop as soon as the increasement is x. Now there is $c(I_1) \geq 2x - x = x$, so doing a slack of (y_1, I_1) can get an increasement more than x. In all, the increasement is more than x, so the new solution is better.

Since in each case there is a better solution, there is a contradiction (that (x_e) is already the optimal). So the assumption is incorrect and it is proved.

This part is straightly proved by Hall's Lemma and the lemma above. Remove all edges from Y to Y, then the graph is a bipartite graph: $G = (Y, V \backslash Y)$. By the lemma above, the condition of Hall's Lemma is satisfied. So there is a perfect matching of size $|Y|$. □