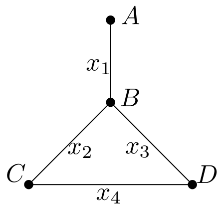
1

For a graph $G = (V, E)$, let $\tau(G)$ denote the size of a minimum vertex cover, and $\nu(G)$ the size of a maximum matching. Recall the two linear programs VCLP and MLP. Let $\tau_f(G) := \text{opt}(\text{VCLP}(G))$ and $\nu_f(G) := \text{opt}(\text{MLP}(G))$. Note that

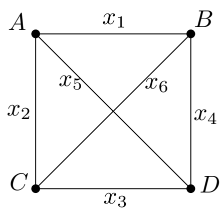
$$\nu(G) \leq \nu_f(G) = \tau_f(G) \leq \tau(G),$$

where the equality in the middle follows from Strong LP Duality. Also, if G is bipartite, then the equality holds throughout in (1). Let us say a graph G is *VCLP exact* if $\tau(G) = \tau_f(G)$, and *MLP exact* if $\nu(G) = \nu_f(G)$. As we already know, a bipartite grpah G is both VCLP exact and MLP exact.
From now on, suppose that G is *not* bipartite but $\tau(G) = \tau_f(G)$.

1. Give an example of such a graph G that is not bipartite but still VCLP exact.
2. Give an example of a graph G that is MLP exact but not VCLP exact.
3. Suppose G is VCLP exact. Let $Y \subseteq V(G)$ be a minimum vertex cover. Let $\mathbf{x}$ be an optimal solution of MLP(G). Show that $x_e = 0$ if $e \subseteq Y$ (i.e., if both endpoints of e are in the cover).
4. Show that such a graph G has a matching of size $|Y|$, and thus is MLP exact, too.



G_1



K_4

Solution. 1. The example is G_1 in the upper left. It is not bipartite since B, C, D constitute an odd cycle.
The minimum vertex cover of G_1 can be $\{B, C\}$ or $\{B, D\}$, with size 2. So $\tau(G_1) = 2$.
VCLP(G_1) is

$$\begin{array}{ll} \text{minimize} & y_A + y_B + y_C + y_D \\ \text{subject to} & y_A + y_B \geq 1 \\ & y_B + y_C \geq 1 \\ & y_B + y_D \geq 1 \\ & y_C + y_D \geq 1 \\ & y_A, y_B, y_C, y_D \geq 0 \end{array}$$

Note that if we add up the constraints $y_A + y_B \geq 1$ and $y_C + y_D \geq 1$, we get $y_A + y_B + y_C + y_D \geq 2$, which gives a lower bound of the target function.
Since $\tau(G_1) = 2$, it follows that the lower bound is tight. Hence $\tau_f(G_1) = \tau(G_1) = 2$.
So G_1 is an example which is not bipartite but still VCLP exact.

2. The example is K_4 in the upper right. The minimum vertex cover can be $\{A, B, C\}, \{A, B, D\}, \{A, C, D\}$ and $\{B, C, D\}$, with size 3. So $\tau(K_4) = 3$.
However, by setting the value of each vertex to 0.5, we find that all edges are exactly covered ($y_u + y_v = 1$ for edge (u, v)). So $\tau_f(K_4) \leq 2$, and thus K_4 is not VCLP exact.
Now consider the maximum matching and MLP. Obviously we can only match 2 pairs of vertices. So $\nu(K_4) = 2$, and $\nu_f(K_4) \geq 2$ follows. Since we already know that $\tau_f(K_4) \leq 2$ and $\nu_f(K_4) = \tau_f(K_4)$ by Strong LP Duality, we can conclude that $\nu(K_4) = \nu_f(K_4) = \tau_f(K_4) = 2$. Therefore K_4 is MLP exact.
So K_4 is an example which is MLP exact but not VCLP exact.

$\square$