1

Let $v(G)$ denote the size of a maximum matching of G. Obviously, $val(MLP(G)) \geq v(G)$ for all graphs. Show that $v(G) = val(MLP(G))$ for all bipartite graphs G. Do this without referring to Konig's Theorem.

Solution. Suppose X is a solution to $MLP(G)$. E_F is the set of edges with fractional values in X. First do as follows:

(1) If E_F doesn't contain a cycle, terminate.
(2) Find a cycle in E_F with fractional values, denote it $C := \{e_i \in E_F : 1 \leq i \leq n, i \in \mathbb{N}\}$.
(3) Add $x_{e_{2k-1}}$ by ϵ and $x_{e_{2k}}$ by $-\epsilon$ for $\{k \in \mathbb{N} : k <= n/2\}$.
(4) Increase ϵ until there exists i such that $x_{e_i} = 0$ or 1.
(5) go to step(2) if there exists a cycle in E_F.

Since G is a bipartite, C can only be a even cycle, so step (3) makes sense. If we modify the solution by step(3) , obviously the constraints of $MLP(G)$ will still be satisfied, and the target function will remain unchanged. The process will terminate because $|E_F|$ decreases by at least 1 in each iteration.

Then do as follows:

(1) If E_F is empty, terminate.
(2) Choose 2 vertex v_1, v_2 that $|\{x_e \in (0,1) : v_1 \in e\}| = 1$, $|\{x_e \in (0,1) : v_2 \in e\}| = 1$ and there's a path between v_1 and v_2 in E_F, denote it $P := \{e'_i \in E_F : 1 \leq i \leq m, i \in \mathbb{N}\}$
(3) Add $x_{e'_{2k-1}}$ by ϵ for $\{k \in \mathbb{N} : 2k - 1 \leq m\}$ and $x_{e'_{2k}}$ by $-\epsilon$ for $\{k \in \mathbb{N} : 2k \leq m\}$
(4) Increase ϵ until there exists i such that $x_{e'_i} = 0$ or 1
(5) Go to step (2) if E_F is not empty

Since there's no cycle in E_F, we can definitely choose v_1, v_2 that satisfy the requirements if E_F is not empty. In each iteration, the target function will not decrease. Since v_1, v_2 can only be touched by edges with fractional value or value 0, the constraints of v_1, v_2 can be satisfied. In other words, $\sum_{e \in E: v_1 \in e} x_e = x_{e'_1} \leq 1$, $\sum_{e \in E: v_2 \in e} x_e = x_{e'_m} \leq 1$ The constraints of other vertices v in P will obviously be maintained. The process will termintate because $|E_F|$ decreases by at least 1 in each iteration.

Finally, after 2 processes, the solution becomes integral and the value of target function does not decrease, which means int-val$(MLP(G)) \geq val(MLP(G))$. However we have int-val$(MLP(G)) \leq val(MLP(G))$. So $v(G) =$ int-val$(MLP(G)) = val(MLP(G))$ $\square$