Combinatorics and Graph Theory

Not Strong Enough

March 7, 2020

1: A Recursive Algorithm for the Binomial Coefficient

Using pseudocode, write a recursive algorithm computing $\binom{n}{k}$. Implement it in python! What is the running time of your algorithm, in terms of n and k? Would you say it is an efficient algorithm? Why or why not?

Algorithm 1 Binom_BF: A Recursive Algorithm for the Binomial Coefficient.

```
if k = 0 or k = n then
    return 1
else
    return Binom_BF(n − 1, k − 1) + Binom_BF(n − 1, k)
end if
```

```python
def binom_BF(n, k):
    if k==0 or k==n:
        return 1
    else:
        return binom_BF(n-1, k-1) + binom_BF(n-1, k)
```

In order to count the number of operations in the recursive algorithm to calculate $\begin{pmatrix} n \\ k \end{pmatrix}$. We shall take a look at how the values are calculated. While recursively calculating Fibonacci number F_n can be represented by a tree, we can show this process by a grid graph. A value $\begin{pmatrix} n \\ k \end{pmatrix}$, in grid (n, k), is obtained by $(n-1, k-1)$ and $(n-1, k)$. This indicates that there are edges from $(n-1, k-1)$ and $(n-1, k)$ to (n, k).

The recursive algorithm actually tries all possible paths to (n, k), starting at (i, j) where $j = 0$ or $j = i$. That is to say, the time that value is merged as $\begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i-1 \\ j-1 \end{pmatrix} + \begin{pmatrix} i-1 \\ j \end{pmatrix}$, is the number of paths going through (i, j), which, by the definition of binomial coefficients, is $\begin{pmatrix} n-i \\ k-j \end{pmatrix}$.

Considering the complexity of integer addition $O(\log\ \text{bitsize})$, we can get the running time of the recursive algorithm:

$$\sum_{i=0}^{n}\sum_{j=1}^{i-1}\binom{n-i}{k-j}\log\binom{i}{j} < \sum_{i=0}^{n}\sum_{j=1}^{i-1}\binom{n-i}{k-j}\log\binom{n}{\lfloor\frac{n}{2}\rfloor}.$$

Using the Stirling Formula, we have

$$O(\log\binom{i}{j}) = O((i+\frac{1}{2})\log i - (i-j+\frac{1}{2})\log(i-j) - (j+\frac{1}{2})\log j) = O(i).$$

The upper bound is

$$O(\sum_{i=0}^{n}\sum_{j=1}^{i-1}\binom{n-i}{k-j}\log\binom{n}{\lfloor\frac{n}{2}\rfloor}) = O(n\binom{n}{k}).$$

And the lower bound is

$$\Omega(\binom{n}{k}).$$