CS217 – Algorithm Design and Analysis
Homework 6

Not Strong Enough

June 1, 2020

1

Let $\nu(G)$ denote the size of a maximum matching of G. Obviously, val(MLP(G)) $\geq \nu(G)$ for all graphs. Show that $\nu(G) =$ val(MLP(G)) for all bipartite graphs G. Do this without referring to Kőnig's Theorem.

Solution. Suppose $\mathbf{x}$ is a solution to MLP(G). E_F is the set of edges with fractional values in $\mathbf{x}$. First do as follows:

(1) If E_F doesn't contain a cycle, terminate.
(2) Find a cycle in E_F with fractional values, denote it $C := \{e_i \in E_F : 1 \leq i \leq n, i \in \mathbb{N}\}$.
(3) Add $x_{e_{2k-1}}$ by ϵ and $x_{e_{2k}}$ by $-\epsilon$ for $\{k \in \mathbb{N} : k \leq n/2\}$.
(4) Increase ϵ until there exists i such that $x_{e_i} = 0$ or 1.
(5) go to step(2) if there exists a cycle in E_F.

Since G is a bipartite, C can only be a even cycle, so step (3) makes sense. If we modify the solution by step(3), obviously the constraints of MLP(G) will still be satisfied, and the target function will remain unchanged. The process will terminate because $|E_F|$ decreases by at least 1 in each iteration.

Then do as follows:

(1) If E_F is empty, terminate.
(2) Choose 2 vertex v_1, v_2 that $|\{x_e \in (0,1) : v_1 \in e\}| = 1$, $|\{x_e \in (0,1) : v_2 \in e\}| = 1$ and there's a path between v_1 and v_2 in E_F, denote it $P := \{e'_i \in E_F : 1 \leq i \leq m, i \in \mathbb{N}\}$
(3) Add $x_{e'_{2k-1}}$ by ϵ for $\{k \in \mathbb{N} : 2k-1 \leq m\}$ and $x_{e'_{2k}}$ by $-\epsilon$ for $\{k \in \mathbb{N} : 2k \leq m\}$
(4) Increase ϵ until there exists i such that $x_{e'_i} = 0$ or 1
(5) Go to step (2) if E_F is not empty

Since there's no cycle in E_F, we can definitely choose v_1, v_2 that satisfy the requirements if E_F is not empty. In each iteration, the target function will not decrease. Since v_1, v_2 can only be touched by edges with fractional value or value 0, the constraints of v_1, v_2 can be satisfied. In other words, $\sum_{e \in E : v_1 \in e} x_e = x_{e'_1} \leq 1$, $\sum_{e \in E : v_2 \in e} x_e = x_{e'_m} \leq 1$ The constraints of other vertices v in P will obviously be maintained. The process will termintate because $|E_F|$ decreases by at least 1 in each iteration.

Finally, after 2 processes, the solution becomes integral and the value of target function does not decrease, which means int-val($MLP(G)$) $\geq$ val($MLP(G)$). And we already have int-val($MLP(G)$) $\leq$ val($MLP(G)$). So $\nu(G) =$ int-val($MLP(G)$) $=$ val($MLP(G)$) □

2

We know that $\nu(G) = \tau(G)$ for all bipartite graphs (Kőnig's Theorem) and $\nu(G) \leq \tau(G)$ for all graphs (since every matched edge must be covered by a distinct vertex). Show that $\tau(G) \leq 2\nu(G)$ for all grpahs G.

Proof. Let M be a maximum matching of G. It follows that $|M| = \nu(G)$. Now we choose our vertex set V' to be all the matched vertices in G. So $|V'| = 2\nu(G)$.

Claim that V' is a vertex cover. To see that, assume there exists an edge (u, v) which is not covered by V'. It means that neither u nor v is matched. So we can add edge (u, v) to M, and thus M is not maximum, which leads to a contradiction.

So the size of minimum vertex cover $\tau(G) \leq |V'| = 2\nu(G)$. □

3

Show that $\tau(G) \leq 2\text{opt}(\text{VCLP}(G))$ for all graphs G (including non-bipartite graphs).

Proof. From (2) we know that $\tau(G) \leq 2\nu(G)$. Since $\nu(G) \leq \text{opt}(\text{MLP}(G)) \leq \text{opt}(\text{VCLP}(G))$, it follows that $\tau(G) \leq 2\nu(G) \leq 2\text{opt}(\text{VCLP}(G))$. □

4

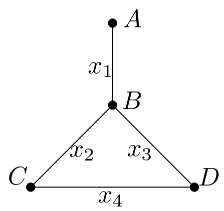
For a graph $G = (V, E)$, let $\tau(G)$ denote the size of a minimum vertex cover, and $\nu(G)$ the size of a maximum matching. Recall the two linear programs VCLP and MLP. Let $\tau_f(G) := \text{opt}(\text{VCLP}(G))$ and $\nu_f(G) := \text{opt}(\text{MLP}(G))$. Note that

$$\nu(G) \le \nu_f(G) = \tau_f(G) \le \tau(G),$$

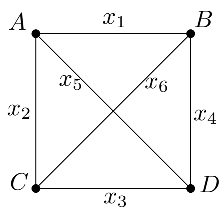
where the equality in the middle follows from Strong LP Duality. Also, if G is bipartite, then the equality holds throughout in (1). Let us say a graph G is *VCLP exact* if $\tau(G) = \tau_f(G)$, and *MLP exact* if $\nu(G) = \nu_f(G)$. As we already know, a bipartite grpah G is both VCLP exact and MLP exact.

From now on, suppose that G is *not* bipartite but $\tau(G) = \tau_f(G)$.

1. Give an example of such a graph G that is not bipartite but still VCLP exact.
2. Give an example of a graph G that is MLP exact but not VCLP exact.
3. Suppose G is VCLP exact. Let $Y \subseteq V(G)$ be a minimum vertex cover. Let $\mathbf{x}$ be an optimal solution of MLP(G). Show that $x_e = 0$ if $e \subseteq Y$ (i.e., if both endpoints of e are in the cover).
4. Show that such a graph G has a matching of size $|Y|$, and thus is MLP exact, too.



G_1



K_4

Solution to (1) and (2).

1. The example is G_1 in the upper left. It is not bipartite since B, C, D constitute an odd cycle.
   The minimum vertex cover of G_1 can be $\{B, C\}$ or $\{B, D\}$, with size 2. So $\tau(G_1) = 2$.
   VCLP(G_1) is
   $$\begin{array}{ll} \text{minimize} & y_A + y_B + y_C + y_D \\ \text{subject to} & y_A + y_B \ge 1 \\ & y_B + y_C \ge 1 \\ & y_B + y_D \ge 1 \\ & y_C + y_D \ge 1 \\ & y_A, y_B, y_C, y_D \ge 0 \end{array}$$
   Note that if we add up the constraints $y_A + y_B \ge 1$ and $y_C + y_D \ge 1$, we get $y_A + y_B + y_C + y_D \ge 2$, which gives a lower bound of the target function.
   Since $\tau(G_1) = 2$, it follows that the lower bound is tight. Hence $\tau_f(G_1) = \tau(G_1) = 2$.
   So G_1 is an example which is not bipartite but still VCLP exact.
2. The example is K_4 in the upper right. The minimum vertex cover can be $\{A, B, C\}, \{A, B, D\}, \{A, C, D\}$ and $\{B, C, D\}$, with size 3. So $\tau(K_4) = 3$.
   However, by setting the value of each vertex to 0.5, we find that all edges are exactly covered ($y_u + y_v = 1$ for edge (u, v)). So $\tau_f(K_4) \le 2$, and thus K_4 is not VCLP exact.
   Now consider the maximum matching and MLP. Obviously we can only match 2 pairs of vertices. So $\nu(K_4) = 2$, and $\nu_f(K_4) \ge 2$ follows. Since we already know that $\tau_f(K_4) \le 2$ and $\nu_f(K_4) = \tau_f(K_4)$ by Strong LP Duality, we can conclude that $\nu(K_4) = \nu_f(K_4) = \tau_f(K_4) = 2$. Therefore K_4 is MLP exact.

So K_4 is an example which is MLP exact but not VCLP exact.

In (3) and (4), suppose that Y is a minimum vertex cover of G. Note that there is no edge $(u, v) \in E$ such that both $u, v \in V \setminus Y$. Because $u, v \in V \setminus Y$ means that (u, v) is not covered by Y, which is contradict with Y being a vertex cover.

3. VCLP is to minimize $\mathbf{c}^T\mathbf{y}$ with constraints $A\mathbf{y} \geq \mathbf{b}$, and MLP is to maximize $\mathbf{b}^T\mathbf{x}$ with constraints $A^T\mathbf{x} \leq \mathbf{c}$, where $\mathbf{b} = \mathbf{1}, \mathbf{c} = \mathbf{1}$.

   So there is
$$\mathbf{x}^T\mathbf{b} \leq \mathbf{x}^T A\mathbf{y} \leq \mathbf{c}^T\mathbf{y}$$
   By Strong LP Duality, let an optimal solution be $\mathbf{x^{(0)}}, \mathbf{y^{(0)}}$, and there is $\mathbf{b}^T\mathbf{x^{(0)}} = \mathbf{c}^T\mathbf{y^{(0)}}$.

   Let $\mathbf{b^{(0)}} := A\mathbf{y^{(0)}}$. There is $(\mathbf{b}^T - \mathbf{b^{(0)}}^T)\mathbf{x^{(0)}} = 0$. As $A\mathbf{y} \geq \mathbf{b}$, there is $b_e - b_e^{(0)} \leq 0$ for all edges $e \in E$. Besides, there is $x_e^{(0)} \geq 0$ for all edges $e \in E$.

   So, if $b_e - b_e^{(0)} < 0$, then there must be $x_e^{(0)} = 0$.

   Let $\mathbf{y^{(1)}}$ be the solution corresponding to an minimum vertex cover(by VCLP exact, it is an optimal solution of VCLP). Then there is $b_e - Ay_e^1 = 1 - 2 < 0$ for all $e \in Y$. So $x_e^0 = 0$ for all $e \in Y$ for every optimal solution $\mathbf{x^0}$ of the MLP.

4. Define $N(\cdot)$ be:
$$N(A) = \{v : v \text{ is the neighbor of some } u \in A\} \cap (V \setminus Y)$$
   We claim that if $A \subseteq Y$, then $|A| \leq |N(A)|$.

   Otherwise, if $A \subseteq Y$ and there is $|B| < |A|$ where $B = N(A)$.

   Consider an optimal solution of VCLP relative to Y(that is, if $v \in Y$, $y_v = 1$, otherwise $y_v = 0$). Let
$$y'_v = \begin{cases} y_v - \epsilon & , v \in A \\ y_v + \epsilon & , v \in B \\ y_v & , \text{ otherwise.} \end{cases}$$
   where $\epsilon < \frac{1}{2}$. For edge $(u, v) \in E$:

   a) $y'_u + y'_v \geq 1$ for $u, v \in A$. Since there is $y'_u + y'_v = (1 - \epsilon) + (1 - \epsilon) = 2 - 2\epsilon > 1$;

   b) $y'_u + y'_v \geq 1$ for $u \in A, v \in Y \setminus A$. Since there is $y'_u + y'_v = (1 - \epsilon) + 1 = 2 - \epsilon > 1$;

   c) $y'_u + y'_v \geq 1$ for $u \in A, v \in B$. Since there is $y'_u + y'_v = (1 - \epsilon) + \epsilon = 1$;

   d) $y'_u + y'_v \geq 1$ for $u \in Y \setminus A, v \in B$. Since there is $y'_u + y'_v = 1 + \epsilon > 1$;

   e) The rest of constraints remains holding since the left hand side is not changed from solution (y_v).

   So $(y_v)'$ is a solution to the VCLP, too. But it is $\epsilon(|A| - |B|)$ less than solution (y_v), which rises a contradiction that Y is a minimum solution.

   Hence what we claimed is proved. Remove all edges from Y to Y, the graph is a bipartite graph G with $L = Y$, $R = V \setminus Y$. By what we claimed, there is $|L| \leq |R|$.

   Notice that what we claimed exactly means that the condition of Hall's Theorem is satisfied, so by Hall's Theorem, there is a matching of size $|L| = |Y|$.

□