1

Write down the dual of MCF.

Proof. First, we write down MCF in terms of inequality:

$$
\begin{aligned}
\text{minimize } & \sum_{e\in E} c(e)f(e) \\
\text{subject to } & \sum_{e\in E: e=(u,t)} f(e) \geq 1 && \text{(corresponds to } y_t^+ \text{ in dual form)} \\
& \sum_{e\in E: e=(u,t)} -f(e) \geq -1 && \text{(corresponds to } y_t^- \text{ in dual form)} \\
& \sum_{e\in E: e=(u,v)} f(e) - \sum_{e\in E: e=(v,w)} f(e) \geq 0, \forall v \in V \setminus \{s,t\} && \text{(corresponds to } y_v^+ \text{ in dual form)} \\
& \sum_{e\in E: e=(v,w)} f(e) - \sum_{e\in E: e=(u,v)} f(e) \geq 0, \forall v \in V \setminus \{s,t\} && \text{(corresponds to } y_v^- \text{ in dual form)} \\
& f(e) \geq 0, \forall e \in E.
\end{aligned}
$$

Then write down its dual LP: (Let $S = \{e \in E : e = (s,v), v \in V\}$)

$$
\begin{aligned}
\text{maximize } & y_t^+ - y_t^- \\
\text{subject to } & (y_v^+ - y_v^-) - (y_u^+ - y_u^-) \leq c(e), \forall e = (u,v) \in E \setminus S \\
& y_v^+ - y_v^- \leq c(e), \forall e = (s,v) \in S \\
& y_v^+, y_v^- \geq 0, \forall v \in V \setminus \{s,t\} \\
& y_t^+, y_t^- \geq 0.
\end{aligned}
$$

Now let $z_u = y_u^+ - y_u^-$ for all vertices u, it turns to be:

$$
\begin{aligned}
\text{maximize } & z_t \\
\text{subject to } & z_v - z_u \leq c(e), \forall e = (u,v) \in E \setminus S \\
& z_v \leq c(e), \forall e = (s,v) \in S \\
& z_v \in \mathbf{R}, \forall v v \in V \setminus \{s,t\} \\
& z_t \in \mathbf{R}.
\end{aligned}
$$

Let $z'_v = z_v + z'_s$, for all vertices u except for s and the problem turns to:

$$
\begin{aligned}
\text{maximize } & z'_t - z'_s \\
\text{subject to } & z'_v - z'_u \leq c(e), \forall e = (u,v) \in E \\
& z'_v \in \mathbf{R}, \forall v \in V.
\end{aligned}
$$

□

2

Interpret the dual. Show that it is the LP formulation of a "natural" maximization problem on G

Solution. Consider that each vertex has a potential, and for each edge $e = (u,v)$, the potential of the terminal vertex v is no greater than the potential of the start vertex u plus $c(u,v)$, and our goal is to maximize the potential of t. □