1: t8

On expection, how many comparisions will it make to use QUICKSELECT to find the minimum of the array.

Using the analysis in exercise 9(see below)2, we can get that

$$\begin{aligned}
\sum_{1\le i<j\le n} B_{i,j,k} &= \sum_{i=1}^{k-1}\sum_{j=i+1}^{k} B_{i,j,k} + \sum_{i=k}^{n-1}\sum_{j=i+1}^{n} B_{i,j,k} \\
&= k - H_k + n - (k+1) - H_{n-k+1} \\
&= 1 - H_1 + n - 2 - H_n \\
&= n - 2 - H_n \\
&= n - \log(n) - 2 - o(1)
\end{aligned}$$

.

2: t9

Derive a formula for $\mathbb{E}_\pi[C(\pi, k)]$, up to additive terms of order $o(n)$. You might want to introduce $\kappa = k/n$.

- case $i < j \le k$:

$$\begin{aligned}
\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} B_{i,j,k} &= \sum_{i=1}^{k-1} \frac{1}{k-i+1} * (k-i) \\
&= \sum_{i=1}^{k-1}(1 - \frac{1}{k-i+1}) = k - (1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{k}) \\
&= k - H_k.
\end{aligned}$$

- case $k \le i < j$:

$$\begin{aligned}
\sum_{i=k}^{n-1}\sum_{j=i+1}^{n} B_{i,j,k} &= \sum_{j=k+1}^{n}\sum_{i=k}^{j-1} B_{i,j,k} \\
&= \sum_{j=k+1}^{n} \frac{j-k}{j-k+1} \\
&= \sum_{j=k+1}^{n} (1 - \frac{1}{j-k+1}) \\
&= n - k + 1 - (1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-k+1}) \\
&= n - k + 1 - H_{n-k+1}.
\end{aligned}$$

- case $i < k < j$:

$$\begin{aligned}
\sum_{i=1}^{k-1}\sum_{j=k+1}^{n} \frac{1}{j-i+1} &< \sum_{i=1}^{k-1}\int_{k+1}^{n+1} \frac{1}{j-i}\mathrm{d}j \\
&= \sum_{i=1}^{k-1} \ln(n+1-i) - \ln(k+1-i) \\
&< \int_{i=0}^{k-1} \ln(n+1-i) - \ln(k-i)\mathrm{d}i \\
&= (n+1)\ln(n+1) - k\ln k - (n-k+2)\ln(n-k+2).
\end{aligned}$$

On the other hand, similarly we can have

$$\sum_{i=1}^{k-1} \sum_{j=k+1}^{n} \frac{1}{j-i+1} > -(-k+n+2)\ln(-k+n+2) - (k+2)\ln(k+2) + (n+1)\ln(n+1) + 3\ln(3).$$

The difference between the upper bound and the lower bound is $o(n)$, thus the inaccuracy is $o(n)$.
To write this formula more simply, we have

$$\sum_{i=1}^{k-1} \sum_{j=k+1}^{n} \frac{1}{j-i+1} = n\ln n - k\ln k - (n-k)\ln(n-k) + o(n).$$

Summing three terms up, and multiply it by 2(because there are two possible orders between i and j),

$$\begin{aligned}\mathbb{E}_{\pi}[C(\pi,k)] &= 2(n + n\ln n - k\ln k - (n-k)\ln(n-k)) + o(n) \\ &= 2(1 + \ln\frac{1}{1-\kappa} + \kappa\ln\frac{1-\kappa}{\kappa})n + o(n).\end{aligned}$$