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Show that the three versions of Farkas Lemma presented in class are all equivalent:

$$(\neg \exists \mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T\mathbf{A} = 0, \mathbf{y}^T\mathbf{b} < 0) \tag{1}$$

$$(\neg \exists \mathbf{x} \geq 0 : \mathbf{A}\mathbf{x} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T\mathbf{A} \geq 0, \mathbf{y}^T\mathbf{b} < 0) \tag{2}$$

$$(\neg \exists \mathbf{x} \geq 0 : \mathbf{A}\mathbf{x} = \mathbf{b}) \iff (\exists \mathbf{y} : \mathbf{y}^T\mathbf{A} \geq 0, \mathbf{y}^T\mathbf{b} < 0) \tag{3}$$

Proof. (1) $\Rightarrow$ (3):

Let $\mathbf{A}' = \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \\ -I \end{bmatrix}, \mathbf{b}' = \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \\ 0 \end{bmatrix}$. Then

$$\begin{aligned}
(\neg \exists \mathbf{x} \geq 0 : \mathbf{A}\mathbf{x} = \mathbf{b}) &\iff (\neg \exists \mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}, -\mathbf{A}\mathbf{x} \leq -\mathbf{b}, -\mathbf{x} \leq 0) \\
&\iff (\neg \exists \mathbf{x} : \mathbf{A}'\mathbf{x} \leq \mathbf{b}') \\
&\iff (\exists \mathbf{y}' \geq 0 : \mathbf{y}'^T\mathbf{A}' = 0, \mathbf{y}'^T\mathbf{b}' < 0) && \text{(derived from(1))} \\
&\iff (\exists \mathbf{y}_1, \mathbf{y}_2, z \geq 0 : (\mathbf{y}_1^T - \mathbf{y}_2^T)\mathbf{A} - z = 0, (\mathbf{y}_1^T - \mathbf{y}_2^T)\mathbf{b} < 0) && (\text{Let } \mathbf{y}' = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ z \end{bmatrix}) \\
&\iff (\exists \mathbf{y}, z \geq 0 : \mathbf{y}^T\mathbf{A} = z, \mathbf{y}^T\mathbf{b} < 0) && (\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2) \\
&\iff (\exists \mathbf{y} : \mathbf{y}^T\mathbf{A} \geq 0, \mathbf{y}^T\mathbf{b} < 0)
\end{aligned}$$

(3) $\Rightarrow$ (2):

let $\mathbf{A}' = \begin{bmatrix} \mathbf{A} & I \end{bmatrix}$.

$$\begin{aligned}
(\neg \exists \mathbf{x} \geq 0 : \mathbf{A}\mathbf{x} \leq \mathbf{b}) &\iff (\neg \exists \mathbf{x}, z \geq 0 : \mathbf{A}\mathbf{x} + z = \mathbf{b}) && (\mathbf{A}\mathbf{x} \leq \mathbf{b} \iff \exists z \geq 0, \mathbf{A}\mathbf{x} + z = \mathbf{b}) \\
&\iff (\neg \exists \mathbf{x}' \geq 0 : \mathbf{A}'\mathbf{x}' = \mathbf{b}) && (\text{let } \mathbf{x}' = \begin{bmatrix} \mathbf{x} \\ z \end{bmatrix}) \\
&\iff (\exists \mathbf{y} : \mathbf{y}^T\mathbf{A}' \geq 0, \mathbf{y}^T\mathbf{b} < 0) && \text{(derived from(3))} \\
&\iff (\exists \mathbf{y} : \mathbf{y}^T\mathbf{A} \geq 0, \mathbf{y} \geq 0, \mathbf{y}^T\mathbf{b} < 0) \\
&\iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T\mathbf{A} \geq 0, \mathbf{y}^T\mathbf{b} < 0)
\end{aligned}$$

(2) $\Rightarrow$ (1):

Let $\mathbf{A}' = \begin{bmatrix} \mathbf{A} & -\mathbf{A} \end{bmatrix}$.

$$\begin{aligned}
(\neg \exists \mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}) &\iff (\neg \exists \mathbf{x}_1, \mathbf{x}_2 \geq 0 : \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) \leq \mathbf{b}) \\
&\iff (\neg \exists \mathbf{x}' \geq 0 : \mathbf{A}'\mathbf{x}' \leq \mathbf{b}) && (\text{let } \mathbf{x}' = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}) \\
&\iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T\mathbf{A}' \geq 0, \mathbf{y}^T\mathbf{b} < 0) && \text{(derived from(2))} \\
&\iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T\mathbf{A} \geq 0, -\mathbf{y}^T\mathbf{A} \geq 0, \mathbf{y}^T\mathbf{b} < 0) \\
&\iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T\mathbf{A} = 0, \mathbf{y}^T\mathbf{b} < 0)
\end{aligned}$$

$\square$