Theory of Probability

Not Strong Enough

March 7, 2020

1 Homework

1: Problem 22

Let f be a monotone function on $\mathbb{R}$ and let A be the set of real numbers x at which f is discontinuous. Show that $|A| \leq |\mathbb{N}|$

Proof. Without loss of generality, assume f increases monotonically. Then for any $x \in A$, there should be $f_+(x) > f_-(x)$. Because f is increasing, for any $x_1 < x_2$, $f_+(x_1) < f_-(x_2)$. Then we can construct a set $E = \{(f_-(x), f_+(x))|x \in A\}$, whose elements are intervals that do not intersect with each other. Obviously, $|E| = |A|$. Due to the density of rational numbers, there are infinite rational numbers in each interval. Using the axiom of choice, we can bind each interval with a rational number in the interval, which forms a injection from E to $\mathbb{Q}$. So $|A| = |E| \leq |\mathbb{Q}| = |\mathbb{N}|$. $\square$