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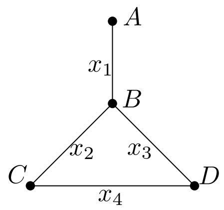
For a graph $G = (V, E)$, let $\tau(G)$ denote the size of a minimum vertex cover, and $\nu(G)$ the size of a maximum matching. Recall the two linear programs VCLP and MLP. Let $\tau_f(G) := \text{opt}(\text{VCLP}(G))$ and $\nu_f(G) := \text{opt}(\text{MLP}(G))$. Note that

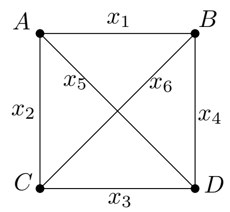
$$\nu(G) \leq \nu_f(G) = \tau_f(G) \leq \tau(G),$$

where the equality in the middle follows from Strong LP Duality. Also, if G is bipartite, then the equality holds throughout in (1). Let us say a graph G is *VCLP exact* if $\tau(G) = \tau_f(G)$, and *MLP exact* if $\nu(G) = \nu_f(G)$. As we already know, a bipartite grpah G is both VCLP exact and MLP exact.
From now on, suppose that G is *not* bipartite but $\tau(G) = \tau_f(G)$.

1. Give an example of such a graph G that is not bipartite but still VCLP exact.
2. Give an example of a graph G that is MLP exact but not VCLP exact.
3. Suppose G is VCLP exact. Let $Y \subseteq V(G)$ be a minimum vertex cover. Let $\mathbf{x}$ be an optimal solution of MLP(G). Show that $x_e = 0$ if $e \subseteq Y$ (i.e., if both endpoints of e are in the cover).
4. Show that such a graph G has a matching of size $|Y|$, and thus is MLP exact, too.



G_1



K_4

Solution. 1. The example is G_1 in the upper left. It is not bipartite since B, C, D constitute an odd cycle.
The minimum vertex cover of G_1 can be $\{B, C\}$ or $\{B, D\}$, with size 2. So $\tau(G_1) = 2$.
VCLP(G_1) is

$$\begin{array}{ll} \text{minimize} & y_A + y_B + y_C + y_D \\ \text{subject to} & y_A + y_B \geq 1 \\ & y_B + y_C \geq 1 \\ & y_B + y_D \geq 1 \\ & y_C + y_D \geq 1 \\ & y_A, y_B, y_C, y_D \geq 0 \end{array}$$

Note that if we add up the constraints $y_A + y_B \geq 1$ and $y_C + y_D \geq 1$, we get $y_A + y_B + y_C + y_D \geq 2$, which gives a lower bound of the target function.
Since $\tau(G_1) = 2$, it follows that the lower bound is tight. Hence $\tau_f(G_1) = \tau(G_1) = 2$.
So G_1 is an example which is not bipartite but still VCLP exact.

2. The example is K_4 in the upper right. The minimum vertex cover can be $\{A, B, C\}, \{A, B, D\}, \{A, C, D\}$ and $\{B, C, D\}$, with size 3. So $\tau(K_4) = 3$.
However, by setting the value of each vertex to 0.5, we find that all edges are exactly covered ($y_u + y_v = 1$ for edge (u, v)). So $\tau_f(K_4) \leq 2$, and thus K_4 is not VCLP exact.
Now consider the maximum matching and MLP. Obviously we can only match 2 pairs of vertices. So $\nu(K_4) = 2$, and $\nu_f(K_4) \geq 2$ follows. Since we already know that $\tau_f(K_4) \leq 2$ and $\nu_f(K_4) = \tau_f(K_4)$ by Strong LP Duality, we can conclude that $\nu(K_4) = \nu_f(K_4) = \tau_f(K_4) = 2$. Therefore K_4 is MLP exact.
So K_4 is an example which is MLP exact but not VCLP exact.

$\square$

To solve (3) and (4), define a function $N(\cdot)$ relative to a VCLP exact graph $G=(V,E)$ and a minimum vertex cover Y that:

$$N(A)=\begin{cases}\{v: v \text{ is the neighbor of some } u\in A\}\cap V\backslash Y, \text{if } A\subseteq Y\\ \{v: v \text{ is the neighbor of some } u\in A\}\cap Y, \text{if } A\subseteq V\backslash Y\end{cases}$$

2: Lemma

If $A\subseteq Y$, then $|A|\le|N(A)|$. Vise versa, if $A\subseteq V\backslash Y$, then $|A|\ge|N(A)|$.

Proof. If $A\subseteq Y$ and there is $|B|<|A|$ where $B=N(A)$. Consider the solution of VCLP relative to Y(that is, if $v\in Y$, $y_v=1$, otherwise $y_v=0$). Let

$$\begin{cases}y'_v=y_v-\epsilon, & v\in A\\ y'_v=y_v+\epsilon, & v\in B\\ y'_v=y_v & otherwise\end{cases}$$

where $\epsilon<\frac{1}{2}$. To prove that (y'_v) is also a solution for VCLP, we only need to check:

a)$y'_u+y'_v\ge1, u,v\in Y$. There is $y'_u+y'_v\ge1-\epsilon+1-\epsilon=2-2\epsilon>1$;

b)$y'_u+y'_v\ge1, u\in A, v\in V\backslash B$. There is $y'_u+y'_v\ge(1-\epsilon)+\epsilon=1$;

The rest constraints remains true since the left hand side is not changed from solution (y_v).

So $(y_v)'$ is a solution to the VCLP too, but it is $\epsilon(|A|-|B|)$ less than solution (y_v), which rises a contradiction that Y is a minimum solution.

Likewise, let $A=N(B)$ and use the same method to construct (y'_v) can prove the 'vice versa' part. □

Now we begin to prove (3) and (4):

Proof. (3) Consider the optimal solution of MLP be $X=(x_e)$, assume there is $y_1,y_2\in Y,(y_1,y_2)\in V, x_{y_1y_2}=x\ge0$. Let $A_0=\{y_1,y_2\}, B_0=N(A_0), C_0=N(B_0)$.

By the lemma above, there is $|B_0|>|C_0|$. Let all edges from C to B be $\{e_1,e_2\dots e_m\}$, there is:

$$\sum_{i=1}^{m}y_e+2x\le|C_0|\le|B_0|$$

By the property of vertex cover, any vertex in B can only connected to vertices in C_0. Let $B=\{b_1,b_2\dots b_s\}$, and $v(b_i)=\sum_{\forall e\in E:b_i\in e}x_e$, then there is:

$$\sum_{i=1}^{s}v(b_i)=\sum_{i=1}^{m}y_e\le|B_0|-2x$$

. So there exists $I=\{v_{i_1},v_{i_2}\dots v_{i_t}\}\subseteq B$, where $\sum_{j=1}^{t}(1-v(b_{i_j}))\ge2x$. We call $1-v(b_i)$ the capacity of vertex b_i.

Let I_1 be all vertices connected to y_1 and in I, and the capacity of I_1, $c(I_1)$ is the sum of capacity of vertices in I_1, I_2 likewise.

Let $A_1=\{y_2\}, B_1=N(A_1), C_1=N(B_1)$, by the same method, we can prove that the $c(B_1)$ is no less than x. And that means the $c(I_2)$ is no less than x, since I_2 contains all those vertices with positive capacity in B_1

Without loss of generality, we can assume that $c(I_1)$ is more than x.

Now we prove that in different cases we can always modify the optimal solution X to get a better solution.

In each case, we do the same thing first: let $x_{y_1y_2}=0$(which gives an decreasement of x in the target). We then define a slack of $(y_i,I_j)(i=1,2;j=1,2,3,4,5)$ be that:

Increase x_e for each edge e from y_i to I_j until the constraint of at least one side of e is strict.

Let $I_3 = I_1 \cap I_2, I_4 = I_1 \backslash I_3, I_5 = I_2 \backslash I_3$,

a)If the $c(I_5) \geq 0$, do a slack to (y_2, I_5). Since the $c(I_5 \geq)0$, the increasement is more than 0. Notice that $c(I_1)$ is not changed, so now we do the slack of (y_1, I_1) and the increasement is not less than x. Not the total increasement is more than x, so the new solution is better.

b)If $c(I_5) = 0$, there is $c(I_2) \geq 0 \to c(I_3) \geq 0$ and $c(I_1) = c(I) \geq 2x$. Now we do slack of (y_2, I_3) but stop as soon as the increasement is x. Now there is $c(I_1) \geq 2x - x = x$, so doing a slack of (y_1, I_1) can get an increasement more than x. In all, the increasement is more than x, so the new solution is better.

Since in each case there is a better solution, there is a contradiction (that (x_e) is already the optimal). So the assumption is incorrect and it is proved.

This part is straightly proved by Hall's Lemma and the lemma above. Remove all edges from Y to Y, then the graph is a bipartite graph: $G = (Y, V \backslash Y)$. By the lemma above, the condition of Hall's Lemma is satisfied. So there is a perfect matching of size $|Y|$. $\square$