Obviously, this is not true for general (non-bipartite) graphs: the triangle K_3 has $\nu(K_3) = 1$ but it has three minimum vertex covers. The five-cycle C_5 has $\nu(C_5) = 2$ but has five minimum vertex covers.

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Is there a function $f \colon \mathbf{N_0} \to \mathbf{N_0}$ such that every graph with $\nu(G) = k$ has at most $f(k)$ minimum vertex covers? How small a function f can you obtain?

Solution. Suppose that we have a graph $G = (V, E)$ and one of its maximum matching $M \subseteq E$ with $|M| = \nu(G) = k$.

We have the two following observations:

- For any vertex cover $C \subseteq V$ of G, for every edge in M there must be at least one of its endpoint which is in C. Otherwise there exists an edge in M such that neither of its endpoints is in C, which means that this edge is uncovered and therefore C is not a vertex cover.
- For any vertex v which is not matched, all of its neighbors must be matched, or the edge between v and one of its unmatched neighbors can be added to the maximum matching and therefore M is not maximum.

We now construct a vertex set $C_0 \subseteq V$ such that for every edge (u, v) in M, either $u \in C_0$, or $v \in C_0$, or both $u, v \in C_0$. There are 3^k possible C_0 in total.

For each possible C_0, note that it may not be a "vertex cover" by far. So we try to construct another vertex set C_1 from C_0. Let C_1 be an empty set at the beginning. From the second observation above, for every unmatched vertex v, there are two cases. If all of its neighbors are in C_0, then we do nothing, since every edge connected to v is covered by vertices in C_0. Otherwise we add it into C_1 to cover the edges which C_0 didn't cover. Note that C_1 is *uniquely determined by* C_0.

Let $\mathcal{C}$ be a family of vertex covers, which is initialized to empty. Now consider $C_0 \cup C_1$. We know that it is also uniquely determined by C_0. If it is a vertex cover, we add it to $\mathcal{C}$. There are at most 3^k vertex covers in $\mathcal{C}$, since there are 3^k possible C_0, and for some C_0 and its corresponding C_1, $C_0 \cup C_1$ may not be a vertex cover.

Claim that any minimum vertex cover C must belong to $\mathcal{C}$. Because from the first observation, we can let the unique C_0 be the matched vertices covered by C. And then unique C_1 can be constructed from C_0. $C_0 \cup C_1$ is the minimum vertex cover when C_0 is fixed. So $C = C_0 \cup C_1 \in \mathcal{C}$.

So there are at most 3^k minimum vertex covers in total, and $f(k) = 3^k$. Also note that this upper bound is *tight*. Just consider the triangle K_3 — it has $3 = 3^1 = 3^{\nu(K_3)}$ minimum vertex covers. $\square$