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Let d be the shortest path distance from s to t in the directed graph G , where distance means sum of the $c(e)$ along the path. Show that $\text{opt}(MCF) = d$.

Proof. First we show that $\text{opt}(MCF) \leq d$. This is quite easy. Suppose (v_1, \dots, v_k) where $k \geq 1$ and $(v_i, v_{i+1}) \in E, v_1 = s, v_k = t$ is a shortest path in graph G . Then we can write a feasible solution to MCF by setting all $f(v_i, v_{i+1}) = 1$ and setting the flow of edges which are not in the shortest path to 0. (Obviously the two constraints of MCF are both satisfied.) Note that the value of this solution is exactly d , since $\sum_{e \in E} c(e)f(e) = \sum_{i=1}^{k-1} (c(v_i, v_{i+1}) \cdot 1) = d$. Hence $\text{opt}(MCF) \leq d$.

Next we show that $\text{opt}(MCF) \geq d$. We prove by contradiction. Assume that $\text{opt}(MCF) < d$. Suppose we have a feasible solution f^* with $\text{val}(f^*) < d$. The idea is first to split the flow f^* into some *single-path-flows*. And then we will show that among such $s - t$ -paths, at least one path P^* satisfies that $\sum_{e \in P^*} c(e) < d$. So d is not the shortest path distance from s to t , and it leads to a contradiction.

Now we focus on how to split f^* into multiple single-path-flows. We first choose an edge (u, v) such that $f(u, v)$ is non-negative and minimum among all edges in G . Since $f(u, v) \neq 0$ is minimum, we can always find a path from s to u and a path from v to t such that the flows of all the edges in both paths are at least $f(u, v)$ (e.g., using DFS or BFS). Let the two paths be P_u and P_v . Connecting P_u, P_v with (u, v) we get a new path $P = (P_u, (u, v), P_v)$, with the flows of all edges on this path being at least $f(u, v)$ and the flow contribution of this path being exactly $f(u, v)$.

So we can “extract” this path from G by decreasing the flow of edges in this path by $f(u, v)$. Note that edge (u, v) now has flow 0. After extraction, we get a new graph G' and a new “flow function” $f^{*'}$. We can validate that $f^{*'}$ is still a real flow by checking the flow-conservative constraint: the inflow decrease and outflow decrease of every vertices on path P are always the same, and the extraction does not influence the inflow and outflow of vertices not on P . So $f^{*'}$ is a real flow.

Repeat the “find minimum and then extract” process on the new graph. Note that each time the number of edges with non-negative flow is decreased by 1, and the total number of edges in a graph is finite, so the process will surely terminate. Finally we get many single-path-flows. We claim that among these single-path-flows, at least one path P^* satisfies that $\sum_{e \in P^*} c(e) < d$. Otherwise every single-path-flow has sum of $c(e)$ along the path being at least d , and it follows that $\text{val}(f) \geq d$, which contradicts with our assumption. However, $\sum_{e \in P^*} c(e) < d$ is impossible since d is the shortest minimum path distance, which also contradicts with this premise. So we have $\text{opt}(MCF) \geq d$.

Combining $\text{opt}(MCF) \leq d$ and $\text{opt}(MCF) \geq d$ we can finally conclude that $\text{opt}(MCF) = d$. □