CS217 – Algorithm Design and Analysis
Homework 5

Not Strong Enough

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Consider the induced bipartite subgraph $H_n[L_i \cup L_{i+1}]$, show that for $i < n/2$ the graph has a matching of size $|L_i| = \binom{n}{i}$

Proof. We do this by induction.

1. The initial step is that if $n = 1, 2$, the theorem is correct. This is trivial since in these conditions we only need to consider $i = 0$ and $|L_i| = 1$. Since L_0 and L_1 is connected, there is at least one edge between them.

2. the inductive assumption is that for $n = 1, 2 \ldots k$, the theorem is correct, the inductive step proves that for $n = k+1$ it is correct. In below, write each vertex v in H_{k+1} by its bitcode $b_v = \overline{B_1 B_2 \ldots B_{k+1}}$, where B_i is the coordinate of the vertex in dimension i.

3. Now prove the theorem is true for (n, i),where $n = k+1$. If edge (v, v') is in the matching M, we call v and v' are matched. And each two matched points in H_n has exactly one bit different comparing their bitcodes. More generally, let L_i in H_n be $L_{i,n}$.

We form the matching by:

A. First, consider the collection $L^{(1)}_{i,k+1} = \{v \in L_i : b_v = \overline{b_u 1}\}$, and consider the first k bits. Then the collection equals $L_{i-1,k}$. It is obvious that $i - 1 < k/2$ and by assumption, there is a matching $M_{i-1,k}$ from $L_{i-1,k}$ to $L_{i,k}$. Let the vertex v in $L^{(1)}_{i,k+1}$ matches v' that:

$b_v = \overline{b_t 1}, b_{v'} = \overline{b_{u'} 1}$ where $u \in L_{i-1,k}, u' \in L_{i,k}$ and u, u' are matched by $M_{i-1,k}$. This matching is obviously correct.

Now denote the matched vertices in $L_{i,k}$ by matching $M_{i-1,k}$ be $L^{(1)}_{i,k}$, and the rest in $L_{i,k}$ be $L^{(2)}_{i,k}$.

B. Now consider the rest vertices in $L_{i,k+1}$(denoted by $L^{(2)}_{i,k+1}$). They are in forms of $\{v : b_v = \overline{b_t 0}, u \in L_{i,k}\}$. Consider the subset that: $\{v : b_v = \overline{b_t 0}, u \in L^{(2)}_{i,k}\}$. We matches these points to $L_{i+1,k+1}$ by the following rule:

v is matched by v' if $b_v = \overline{b_t 0}, b_{v'} = \overline{b_t 1}$. Since $u \in L^{(2)}_{i,k}$, all matched v' are not ever matched before.

C. Now consider the last part that $\{v : b_v = \overline{b_t 0}, u \in L^{(1)}_{i,k}\}$.

C1. If $i < k/2$, there is a matching $M_{i,k}$ from $L_{i,k}$ to $L_{i+1,k}$. Let v matches v' that $b_v = \overline{b_t 0}, b_{v'} = \overline{b_{u'} 0}$ and u, u' are matched by $M_{i,k}$.

C2. Otherwise, there is $i \geq k/2, i - 1 < (k+1)/2$ so there is $k = 2u$. And we consider that:

(How to get this??)If $k = 2i$, there is a set of disjoint paths from L_{i-1} to L_{i+1}.(this may be done by some symmetry?) If so, the set of all vertices of $L_{i,k}$ in these paths is our $L^{(1)}_{i,k}$ and has a matching to $L_{i+1,k}$, adding 0 at the end of each bitcode gets what we need. (just like extends $L^{(1)-L_{i+1,k}}_{i,k}$ to $L^{(1)}_{i,k+1} - L_{i+1,k+1}$). ∎

My idea of exercise 5 is to prove the "how to get this" part first. And then by our construction in exercise 4, choose the $L^{(1)}_{i-1}$ part to go to two side.