CS217 – Algorithm Design and Analysis
Homework 5

Not Strong Enough

May 8, 2020

In the two exercises below, let $\Gamma(X)$ be the neighbors of X.

1

Consider the induced bipartite subgraph $H_n[L_i \cup L_{i+1}]$, show that for $i < n/2$ the graph has a matching of size $|L_i| = \binom{n}{i}$

Proof. Use Hall's Theorem, the size of maximum matching equals $\min_{X \subseteq L_i} |L_i| - |X| + |\Gamma(X)|$.

Since in $H_n[L_i \cup L_{i+1}]$ the degree of each vertex in L_i is $n-i$, and that of each vertex in L_{i+1} is $i+1$, there is $|X|(n-i) \le |\Gamma(X)|(i+1)$. As $i < n/2$, $|X| \le |\Gamma(x)|\frac{i+1}{n-i} \le |\Gamma(X)|$, and only if $|X| = 0$ can the equality be achieved.

So there is $\min_{X \subseteq L_i}(|L_i| - |X| + |\Gamma(X)|) = |L_i| = \binom{n}{i}$. □

2

Show that there are $\binom{n}{i}$ paths in H_n starting at L_i ending in L_{n-i} and are disjoint.

Proof. We first set a new point s connected to all vertices in L_i and a new point t connected to all vertices in L_{n-i}.

Set the capacity of edges starting from s or ending to t be 1, and the capacity of the rest be ∞. Then there is a flow that, for edges whose capacity is 1 the flow takes 1; for the rest edges, each in form of $(u, v), u \in L_k, v \in L_{k+1}$, the flow takes $\frac{\binom{n}{i}}{\binom{n}{k}(n-k)}$.

It is obvious that the flow is well-defined, that for each vertex (except for s, t) the flow in equals the flow out. And the total flow is $\binom{n}{i}$. Besides, it is the maxflow since the flow out of s is no more than $\binom{n}{i}$.

So there exists a min-cut of the graph that the size of the cut is $\binom{n}{i}$ as well. The collection of the end point of each edge in the cut is a vertex cut, so the size of the minimum vertex cut is $\binom{n}{i}$. By Menger's Theorem, there are $\binom{n}{i}$ disjoint paths from s to t, by removing s and t of these paths, we get $\binom{n}{i}$ disjoint paths from L_i to L_{n-i}

□