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Show that the three versions of Farkas Lemma presented in class are all equivalent:

$$(\neg \exists \mathbf{x} : \mathbf{Ax} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T \mathbf{A} = \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) \quad (1)$$

$$(\neg \exists \mathbf{x} \geq \mathbf{0} : \mathbf{Ax} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T \mathbf{A} \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) \quad (2)$$

$$(\neg \exists \mathbf{x} \geq \mathbf{0} : \mathbf{Ax} = \mathbf{b}) \iff (\exists \mathbf{y} : \mathbf{y}^T \mathbf{A} \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) \quad (3)$$

Proof. (1) \Rightarrow (3):

$$\text{Let } \mathbf{A}' = \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \\ -\mathbf{I} \end{bmatrix}, \mathbf{b}' = \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \\ \mathbf{0} \end{bmatrix}. \text{ Then}$$

$$\begin{aligned} (\neg \exists \mathbf{x} \geq \mathbf{0} : \mathbf{Ax} = \mathbf{b}) &\iff (\neg \exists \mathbf{x} : \mathbf{Ax} \leq \mathbf{b}, -\mathbf{Ax} \leq -\mathbf{b}, -\mathbf{x} \leq \mathbf{0}) \\ &\iff (\neg \exists \mathbf{x} : \mathbf{A}'\mathbf{x} \leq \mathbf{b}') \\ &\iff (\exists \mathbf{y}' \geq \mathbf{0} : \mathbf{y}'^T \mathbf{A}' = \mathbf{0}, \mathbf{y}'^T \mathbf{b}' < 0) && \text{(derived from (1))} \\ &\iff (\exists \mathbf{y}_1, \mathbf{y}_2, \mathbf{z} \geq \mathbf{0} : (\mathbf{y}_1^T - \mathbf{y}_2^T) \mathbf{A} - \mathbf{z} = \mathbf{0}, (\mathbf{y}_1^T - \mathbf{y}_2^T) \mathbf{b} < 0) && \left(\text{Let } \mathbf{y}' = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{z} \end{bmatrix} \right) \\ &\iff (\exists \mathbf{z} \geq \mathbf{0}, \mathbf{y} : \mathbf{y}^T \mathbf{A} = \mathbf{z}, \mathbf{y}^T \mathbf{b} < 0) && (\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2) \\ &\iff (\exists \mathbf{y} : \mathbf{y}^T \mathbf{A} \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) \end{aligned}$$

(3) \Rightarrow (2):

$$\text{let } \mathbf{A}' = \begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix}.$$

$$\begin{aligned} (\neg \exists \mathbf{x} \geq \mathbf{0} : \mathbf{Ax} \leq \mathbf{b}) &\iff (\neg \exists \mathbf{x}, \mathbf{z} \geq \mathbf{0} : \mathbf{Ax} + \mathbf{z} = \mathbf{b}) && (\mathbf{Ax} \leq \mathbf{b} \iff \exists \mathbf{z} \geq \mathbf{0}, \mathbf{Ax} + \mathbf{z} = \mathbf{b}) \\ &\iff (\neg \exists \mathbf{x}' \geq \mathbf{0} : \mathbf{A}'\mathbf{x}' = \mathbf{b}) && \left(\text{let } \mathbf{x}' = \begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} \right) \\ &\iff (\exists \mathbf{y} : \mathbf{y}^T \mathbf{A}' \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) && \text{(derived from (3))} \\ &\iff (\exists \mathbf{y} : \mathbf{y}^T \mathbf{A} \geq \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) \\ &\iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T \mathbf{A} \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) \end{aligned}$$

(2) \Rightarrow (1):

$$\text{Let } \mathbf{A}' = \begin{bmatrix} \mathbf{A} & -\mathbf{A} \end{bmatrix}.$$

$$\begin{aligned} (\neg \exists \mathbf{x} : \mathbf{Ax} \leq \mathbf{b}) &\iff (\neg \exists \mathbf{x}_1, \mathbf{x}_2 \geq \mathbf{0} : \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) \leq \mathbf{b}) \\ &\iff (\neg \exists \mathbf{x}' \geq \mathbf{0} : \mathbf{A}'\mathbf{x}' \leq \mathbf{b}) && \left(\text{let } \mathbf{x}' = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \right) \\ &\iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T \mathbf{A}' \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) && \text{(derived from (2))} \\ &\iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T \mathbf{A} \geq \mathbf{0}, -\mathbf{y}^T \mathbf{A} \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) \\ &\iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T \mathbf{A} = \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) \end{aligned}$$

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