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Describe an optimal solution of the dual program.

Solution. Let $c(e)$ be lengths of edges and

$$d(u) := \begin{cases} 0, u = s \\ \text{the length of the shortest path from } s \text{ to } u, \text{ otherwise} \end{cases}$$

Without loss of generality, we fix $z'_s = 0$. We claim that $z'_u = d(u)$ is an optimal solution.

- $\forall v \in V$, $z'_v = d(v) \in \mathbf{R}$.
- $\forall e = (u, v) \in E$, assume that $d(v) - d(u) > c(e)$. This suggests that the length of the shortest path from s to t with length $d(u) + c(e)$, which is a contradiction to the definition.
- $z'_t - z'_s$ is maximized by letting $z'_v = d(v)$:
  Assume that $z'_t - z'_s = z'_t > d(t)$.
  Let the shortest path from s to t be $v_1, v_2, \cdots, v_r$ where $v_1 = s$ and $v_r = t$. We have $d(v_{i+1}) = d(v_i) + c((v_i, v_{i+1}))$.
  We next prove that $\forall i \in [r], z'_{v_i} > d(v_i)$ by induction from r to 1.
  - Base: $z'_{v_r} > d(v_r)$ by our previous assumption.
  - Inductive step: By inductive hypothesis, we have $z'_{v_{i+1}} > d(v_{i+1})$ for $i \in [r-1]$.
    Then $z'_{v_i} + c(e_i) \geq z'_{v_{i+1}} > d(v_{i+1}) = d(v_i) + c(e_i) \implies z'_{v_i} > d(v_i)$.

  Hence, $z'_s > d(s) = 0$, which is a contradiction.

Therefore, $z'_u = d(u)$ is an optimal solution to the dual program. □