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Show that the three versions of Farkas Lemma presented in class are all equivalent:

$$(\neg\exists x : Ax \le b) \iff (\exists y \ge 0 : y^T A = 0, y^T b < 0) \tag{1}$$

$$(\neg\exists x \ge 0 : Ax \le b) \iff (\exists y \ge 0 : y^T A \ge 0, y^T b < 0) \tag{2}$$

$$(\neg\exists x \ge 0 : Ax = b) \iff (\exists y : y^T A \ge 0, y^T b < 0) \tag{3}$$

Proof. (1) $\Rightarrow$ (3):

Let $A' = \begin{bmatrix} A \\ -A \\ -I \end{bmatrix}, b' = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$. Then

$$\begin{aligned}
(\neg\exists x \ge 0 : Ax = b) &\iff (\neg\exists x : Ax \le b, -Ax \le -b, -x \le 0) \\
&\iff (\neg\exists x : A'x \le b') \\
&\iff (\exists y' \ge 0 : y'^T A' = 0, y'^T b' < 0) && \text{(derived from(1))} \\
&\iff (\exists y1, y2, z \ge 0 : (y_1^T - y_2^T)A - z = 0, (y_1^T - y_2^T)b < 0) && (\text{Let } y' = \begin{bmatrix} y_1 \\ y_2 \\ z \end{bmatrix}) \\
&\iff (\exists y, z \ge 0 : y^T A = z, y^T b < 0) && (y = y_1 - y_2) \\
&\iff (\exists y : y^T A \ge 0, y^T b < 0)
\end{aligned}$$

(3) $\Rightarrow$ (2):

let $A' = \begin{bmatrix} A & I \end{bmatrix}$.

$$\begin{aligned}
(\neg\exists x \ge 0 : Ax \le b) &\iff (\neg\exists x, z \ge 0 : Ax + z = b) && (Ax \le b \iff \exists z \ge 0, Ax + z = b) \\
&\iff (\neg\exists x' \ge 0 : A'x' = b) && (\text{let } x' = \begin{bmatrix} x \\ z \end{bmatrix}) \\
&\iff (\exists y : y^T A' \ge 0, y^T b < 0) && \text{(derived from(3))} \\
&\iff (\exists y : y^T A \ge 0, y \ge 0, y^T b < 0) \\
&\iff (\exists y \ge 0 : y^T A \ge 0, y^T b < 0)
\end{aligned}$$

(2) $\Rightarrow$ (1):

Let $A' = \begin{bmatrix} A & -A \end{bmatrix}$.

$$\begin{aligned}
(\neg\exists x : Ax \le b) &\iff (\neg\exists x_1, x_2 \ge 0 : A(x_1 - x_2) \le b) \\
&\iff (\neg\exists x' \ge 0 : A'x' \le b) && (\text{let } x' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) \\
&\iff (\exists y \ge 0 : y^T A' \ge 0, y^T b < 0) && \text{(derived from(2))} \\
&\iff (\exists y \ge 0 : y^T A \ge 0, -y^T A \ge 0, y^T b < 0) \\
&\iff (\exists y \ge 0 : y^T A = 0, y^T b < 0)
\end{aligned}$$

$\square$