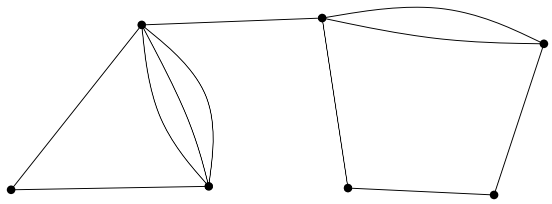
A *multigraph* is a graph that can have multiple edges, called "parallel edges". Without defining it formally, we illustrate it:



A multigraph.

All other definitions, like connected components and spanning trees are the same as for normal (simple) graphs. However, when two spanning trees use different parallel edges, we consider them different:

The same multigraph with two different spanning trees.

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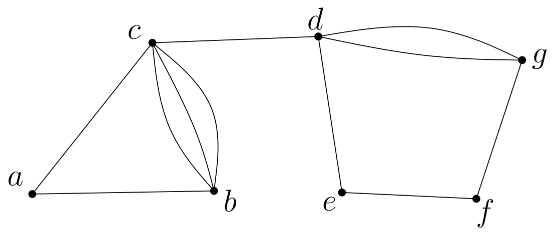
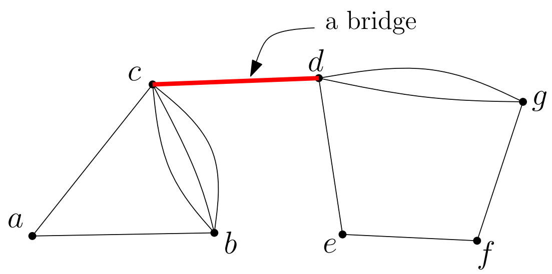
How many spanning trees does the above multigraph on 7 vertices have? Justify your answer!

Solution. Firstly, we label the 7 vertices from a to g as below.



Obviously, as a bridge, the edge $\{c,d\}$ is contained in every spanning tree. So we only need to count the number of spanning trees of vertices $V_1 = \{a,b,c\}$ and $V_2 = \{d,e,f,g\}$ respectively, and then multiplying the two numbers we get the number of spanning trees of the original graph.



Consider the sub-multigraph with vertices a, b and c. If we select none of the edges between b and c, there are only 1 spanning tree. Otherwise we select one of the edges between b and c, and we can select another edge which is either $\{a,c\}$ or $\{a,b\}$. So there are $cnt_l = 1 + 2 \times 3 = 7$ spanning trees in the left sub-multigraph.

Then we consider the right sub-multigraph with vertices d to g. Similarly, there is only one spanning tree which doesn't contain an edge between d and g. And if we select an edge between d and g, the other two edges can be any two among $\{d,e\}$, $\{e,f\}$ and $\{f,g\}$. So there are $cnt_r = 1 + 3 \times 2 = 7$ spanning trees in the right sub-multigraph.

Finally we multiply cnt_l and cnt_r. So the total number of spanning trees in the given multigraph is $cnt_l \times cnt_r = 7 \times 7 = 49$. $\square$