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Prove Menger's Theorem. You have to prove two things: first, not both cases above can occur (this is rather easy); second, one of them must occur (this requires a tool from the lecture).

Proof. Let $V(G) = \{v_1, \cdots\}$ and $s = v_p, t = v_q$.

We would like to construct a flow network (V', s', t', c) where $V' = \{v_1, v_1', v_2, v_2', \cdots\}$ and

$$c(u,v) = \begin{cases} 1, \text{ if } \exists i, (u,v) = (v_i, v_i') \\ \infty, \text{ if } \exists i, j, (u,v) = (v_i', v_j) \wedge (v_i, v_j) \in E \\ 0, \text{ otherwise} \end{cases}.$$

And finally let $s' = v_p', t' = v_q$. Then:

- There are k vertex disjoint paths $p_1, \cdots, p_k$ in G iff there is a flow f in (V', s', t', c) with $\operatorname{val}(f) = k$, iff $\max \operatorname{val}(f) \geq k$.
- There are $k-1$ vertices $v_{i_1}, \cdots, v_{ik-1}$ in $V \setminus \{s, t\}$ such that $G - \{v_{i_1}, \cdots, v_{i_{k-1}}\}$ contains no $s-t$ path iff there is a cut S in (V', s', t', c) with $\operatorname{cap}(S) = k-1$ by making $v_{i_j} \in S$ and $v_{i_j}' \notin S$ for all j, iff $\min \operatorname{cap}(S) < k$.

By *Max-Flow Min-Cut Theorem*, let $\max \operatorname{val}(f) = \min \operatorname{cap}(S) = l$. Then:

- Either $l \geq k$, resulting in 1 holds while 2 does not.
- Or $l < k$, resulting in 2 holds while 1 does not.

Therefore, *exactly one* of the two statements is true. $\square$