

1

Show that the three versions of Farkas Lemma presented in class are all equivalent:

$$(\neg \exists \mathbf{x} : \mathbf{Ax} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T \mathbf{A} = 0, \mathbf{y}^T \mathbf{b} < 0) \quad (1)$$

$$(\neg \exists \mathbf{x} \geq 0 : \mathbf{Ax} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T \mathbf{A} \geq 0, \mathbf{y}^T \mathbf{b} < 0) \quad (2)$$

$$(\neg \exists \mathbf{x} \geq 0 : \mathbf{Ax} = \mathbf{b}) \iff (\exists \mathbf{y} : \mathbf{y}^T \mathbf{A} \geq 0, \mathbf{y}^T \mathbf{b} < 0) \quad (3)$$

Proof. (1) \Rightarrow (3):

$$\text{Let } \mathbf{A}' = \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \\ -I \end{bmatrix}, \mathbf{b}' = \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \\ 0 \end{bmatrix}. \text{ Then}$$

$$\begin{aligned} (\neg \exists \mathbf{x} : \mathbf{Ax} \leq \mathbf{b}) &\iff (\neg \exists \mathbf{x} : \mathbf{Ax} \leq \mathbf{b}, -\mathbf{Ax} \leq -\mathbf{b}, -\mathbf{x} \leq 0) \\ &\iff (\neg \exists \mathbf{x} : \mathbf{A}'\mathbf{x} \leq \mathbf{b}') \\ &\iff (\exists \mathbf{y}' \geq 0 : \mathbf{y}'^T \mathbf{A}' = 0, \mathbf{y}'^T \mathbf{b}' < 0) \quad (\text{derived from (1)}) \\ &\iff (\exists \mathbf{y}_1, \mathbf{y}_2, z \geq 0 : (\mathbf{y}_1^T - \mathbf{y}_2^T) \mathbf{A} - z = 0, (\mathbf{y}_1^T - \mathbf{y}_2^T) \mathbf{b} < 0) \quad (\text{Let } \mathbf{y}' = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ z \end{bmatrix}) \\ &\iff (\exists \mathbf{y}, z \geq 0 : \mathbf{y}^T \mathbf{A} = z, \mathbf{y}^T \mathbf{b} < 0) \quad (\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2) \\ &\iff (\exists \mathbf{y} : \mathbf{y}^T \mathbf{A} \geq 0, \mathbf{y}^T \mathbf{b} < 0) \end{aligned}$$

(3) \Rightarrow (2):

$$\text{let } \mathbf{A}' = \begin{bmatrix} \mathbf{A} & I \end{bmatrix}.$$

$$\begin{aligned} (\neg \exists \mathbf{x} \geq 0 : \mathbf{Ax} \leq \mathbf{b}) &\iff (\neg \exists \mathbf{x}, z \geq 0 : \mathbf{Ax} + z = \mathbf{b}) \quad (\mathbf{Ax} \leq \mathbf{b} \iff \exists z \geq 0, \mathbf{Ax} + z = \mathbf{b}) \\ &\iff (\neg \exists \mathbf{x}' \geq 0 : \mathbf{A}'\mathbf{x}' = \mathbf{b}) \quad (\text{let } \mathbf{x}' = \begin{bmatrix} \mathbf{x} \\ z \end{bmatrix}) \\ &\iff (\exists \mathbf{y} : \mathbf{y}^T \mathbf{A}' \geq 0, \mathbf{y}^T \mathbf{b} < 0) \quad (\text{derived from (3)}) \\ &\iff (\exists \mathbf{y} : \mathbf{y}^T \mathbf{A} \geq 0, \mathbf{y} \geq 0, \mathbf{y}^T \mathbf{b} < 0) \\ &\iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T \mathbf{A} \geq 0, \mathbf{y}^T \mathbf{b} < 0) \end{aligned}$$

(2) \Rightarrow (1):

$$\text{Let } \mathbf{A}' = \begin{bmatrix} \mathbf{A} & -\mathbf{A} \end{bmatrix}.$$

$$\begin{aligned} (\neg \exists \mathbf{x} : \mathbf{Ax} \leq \mathbf{b}) &\iff (\neg \exists \mathbf{x}_1, \mathbf{x}_2 \geq 0 : \mathbf{A}(\mathbf{x}_1 - \mathbf{x}_2) \leq \mathbf{b}) \\ &\iff (\neg \exists \mathbf{x}' \geq 0 : \mathbf{A}'\mathbf{x}' \leq \mathbf{b}) \quad (\text{let } \mathbf{x}' = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}) \\ &\iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T \mathbf{A}' \geq 0, \mathbf{y}^T \mathbf{b} < 0) \quad (\text{derived from (2)}) \\ &\iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T \mathbf{A} \geq 0, -\mathbf{y}^T \mathbf{A} \geq 0, \mathbf{y}^T \mathbf{b} < 0) \\ &\iff (\exists \mathbf{y} \geq 0 : \mathbf{y}^T \mathbf{A} = 0, \mathbf{y}^T \mathbf{b} < 0) \end{aligned}$$

□