CS217 – Algorithm Design and Analysis
Homework 5

Not Strong Enough

May 6, 2020

1

Let $v(G)$ denote the size of a maximum matching of $G = (V, E)$. Show that a bipartite graph G has at most $2^{v(G)}$ mimimum vertex covers.

Proof. From the Konig's Theorem, we know that the size of mimimum vertex cover is $v(G)$. Let C be a minimum vertex cover. Then we can construct new minimum vertex cover by choosing vertices from C and $V \setminus C$. In other words, all minimum vertex covers can be represented by $X \cup Y, X \subseteq C, Y \subseteq V \setminus C$. Denote $N(A) = \{b | \exists a \in A, \text{there is an edge between a and b}\}$. For all $X \subseteq C$, to construct a vertex cover, Y must touch all edges touched by $C \setminus X$ but not by X.

- If $N(C \setminus X) \cap (C \setminus X) \neq \emptyset$, there does not exist such a Y.
- If $N(C \setminus X) \cap (C \setminus X) = \emptyset$, then Y must be at least $N(C \setminus X) \cap (V \setminus C)$ to be a vertex cover. To make $X \cap Y$ a minimum vertex cover, Y has to be $N(C \setminus X) \cap (V \setminus C)$.

Since X has $2^{v(G)}$ choices, G has at most $2^{v(G)}$ mimimum vertex covers. □