CS217 – Algorithm Design and Analysis
Homework 5

Not Strong Enough

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Consider the induced bipartite subgraph $H_n[L_i \cup L_{i+1}]$, show that for $i < n/2$ the graph has a matching of size $|L_i| = \binom{n}{i}$

Proof. We do this by induction.

1. The initial step is that if $n = 1, 2$, the theorem is correct. This is trivial since in these conditions we only need to consider $i = 0$ and $|L_i| = 1$. Since L_0 and L_1 is connected, there is at least one edge between them.

2. the inductive assumption is that for $n = 1, 2 \ldots k$, the theorem is correct, the inductive step proves that for $n = k + 1$ it is correct. In below, write each vertex v in H_{k+1} by its bitcode $b_v = \overline{B_1 B_2 \ldots B_{k+1}}$, where $b_{v,i} = B_i$ is the coordinate of the vertex in dimension i.

3. To prove the theorem is true for (n, i) we use induction again.

1)The initial step is that if $i = 1$, the theorem is correct. This is trivial since the condition is like what in 1. that only one edge is needed.

2)The inductive assumption is that, for $n = k + 1$, $i = 1, 2 \ldots u(u + 1 < n/2)$ the theorem is correct, the inductive step proves that for $n = k + 1, i = u + 1$ it is correct. If such a form exists, it is obvious that all vertex in L_{u+1} has a corresponding vertex in L_{u+2} and only one bit of the bitcodes of the two matched vertex is different. More generally, let L_i in H_n be $L_{i,n}$.

To prove this, we form the matching by:

A. First, consider the collection $L^{(1)}_{u+1,k+1} = \{v \in L_{u+1} : b_{v,u+1} = 1\}$, and consider the first u bits. Then the collection equals $L_{u,k}$. It is obvious that $u < k/2$ and by assumption, there is a matching $M_{u,k}$ from $L_{u,k}$ to $L_{u+1,k}$. Let the vertex v in $L^{(1)}_{u+1,k+1}$ matched v' that:

$b_v = \overline{b_t 1}, b_{v'} = \overline{b_{t'} 1}$ where $t \in L_{u,k}, t' \in L_{u+1,k}$ and t, t' are matched by $M_{u,k}$. This matching is obviously correct.

Now denote the matched vertices in $L_{u+1,k}$ by matching $M_{u,k}$ be $L^{(1)}_{u+1,k}$, and the rest in $L_{u+1,k}$ be $L^{(2)}_{u+1,k}$.

B. Now consider the rest vertices in $L_{u+1,k+1}$(denoted by $L^{(2)}_{u+1,k+1}$). They are in forms of $\{v : b_v = \overline{b_t 0}, t \in L_{u+1,k}\}$. Consider the subset that: $\{v : b_v = \overline{b_t 0}, t \in L^{(2)}_{u+1,k}\}$. We matched these points to $L_{u+2,k+1}$ by the following rule:

v is matched by v' if $b_v = \overline{b_t 0}, b_{v'} = \overline{b_t 1}$. Since $t \in L^{(2)}_{u+1,k}$, all matched v' are not ever matched before.

C. Now consider the last part that $\{v : b_v = \overline{b_t 0}, t \in L^{(1)}_{u+1,k}\}$.

C1. If $u + 1 < k/2$, there is a matching $M_{u+1,k}$ from $L_{u+1,k}$ to $L_{u+2,k}$. Let v matches v' that $b_v = \overline{b_t 0}, b_{v'} = \overline{b_{t'} 0}$ and t, t' are matched by $M_{u+1,k}$.

C2. Otherwise, there is $u + 1 \geq k/2, u < (k + 1)/2$ so there is $k = 2u$. And we consider that:

(How to get this??)If $k = 2u$, there is a set of disjoint paths from L_{u-1} to L_{u+1}.(this may be done by some symmetry?) If so, the set of all vertices of $L_{u+1,k}$ in these paths is our $L^{(1)}_{u+1,k}$ and has a matching to $L_{u+2,k}$, add a 0 at the end of each bitcode is what we need. $\square$

My idea of exercise 5 is to prove the "how to get this" part first. And then by our construction in exercise 4, choose the $L^{(1)}_{u,k}$ part to go to two side.