CS217 – Algorithm Design and Analysis
Homework 6

Not Strong Enough

June 1, 2020

1

Let $\nu(G)$ denote the size of a maximum matching of G. Obviously, $\text{val}(\text{MLP}(G)) \geq \nu(G)$ for all graphs. Show that $\nu(G) = \text{val}(\text{MLP}(G))$ for all bipartite graphs G. Do this without referring to Kőnig's Theorem.

Solution. Suppose $\mathbf{x}$ is a solution to $\text{MLP}(G)$. E_F is the set of edges with fractional values in $\mathbf{x}$. First do as follows:

(1) If E_F doesn't contain a cycle, terminate.
(2) Find a cycle in E_F with fractional values, denote it $C := \{e_i \in E_F : 1 \leq i \leq n, i \in \mathbb{N}\}$.
(3) Add $x_{e_{2k-1}}$ by ϵ and $x_{e_{2k}}$ by $-\epsilon$ for $\{k \in \mathbb{N} : k \leq n/2\}$.
(4) Increase ϵ until there exists i such that $x_{e_i} = 0$ or 1.
(5) go to step(2) if there exists a cycle in E_F.

Since G is a bipartite, C can only be a even cycle, so step (3) makes sense. If we modify the solution by step(3), obviously the constraints of $\text{MLP}(G)$ will still be satisfied, and the target function will remain unchanged. The process will terminate because $|E_F|$ decreases by at least 1 in each iteration.

Then do as follows:

(1) If E_F is empty, terminate.
(2) Choose 2 vertex v_1, v_2 that $|\{x_e \in (0,1) : v_1 \in e\}| = 1$, $|\{x_e \in (0,1) : v_2 \in e\}| = 1$ and there's a path between v_1 and v_2 in E_F, denote it $P := \{e'_i \in E_F : 1 \leq i \leq m, i \in \mathbb{N}\}$
(3) Add $x_{e'_{2k-1}}$ by ϵ for $\{k \in \mathbb{N} : 2k-1 \leq m\}$ and $x_{e'_{2k}}$ by $-\epsilon$ for $\{k \in \mathbb{N} : 2k \leq m\}$
(4) Increase ϵ until there exists i such that $x_{e'_i} = 0$ or 1
(5) Go to step (2) if E_F is not empty

Since there's no cycle in E_F, we can definitely choose v_1, v_2 that satisfy the requirements if E_F is not empty. In each iteration, the target function will not decrease. Since v_1, v_2 can only be touched by edges with fractional value or value 0, the constraints of v_1, v_2 can be satisfied. In other words, $\sum_{e \in E : v_1 \in e} x_e = x_{e'_1} \leq 1$, $\sum_{e \in E : v_2 \in e} x_e = x_{e'_m} \leq 1$ The constraints of other vertices v in P will obviously be maintained. The process will termintate because $|E_F|$ decreases by at least 1 in each iteration.

Finally, after 2 processes, the solution becomes integral and the value of target function does not decrease, which means $\text{int-val}(MLP(G)) \geq \text{val}(MLP(G))$. And we already have $\text{int-val}(MLP(G)) \leq \text{val}(MLP(G))$. So $\nu(G) = \text{int-val}(MLP(G)) = \text{val}(MLP(G))$ □

2

We know that $\nu(G) = \tau(G)$ for all bipartite graphs (Kőnig's Theorem) and $\nu(G) \leq \tau(G)$ for all graphs (since every matched edge must be covered by a distinct vertex). Show that $\tau(G) \leq 2\nu(G)$ for all grpahs G.

Proof. Let M be a maximum matching of G. It follows that $|M| = \nu(G)$. Now we choose our vertex set V' to be all the matched vertices in G. So $|V'| = 2\nu(G)$.

Claim that V' is a vertex cover. To see that, assume there exists an edge (u, v) which is not covered by V'. It means that neither u nor v is matched. So we can add edge (u, v) to M, and thus M is not maximum, which leads to a contradiction.

So the size of minimum vertex cover $\tau(G) \leq |V'| = 2\nu(G)$. $\square$

3

Show that $\tau(G) \leq 2\text{opt}(\text{VCLP}(G))$ for all graphs G (including non-bipartite graphs).

Proof. From (2) we know that $\tau(G) \leq 2\nu(G)$. Since $\nu(G) \leq \text{opt}(\text{MLP}(G)) \leq \text{opt}(\text{VCLP}(G))$, it follows that $\tau(G) \leq 2\nu(G) \leq 2\text{opt}(\text{VCLP}(G))$. $\square$

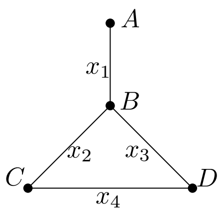
4

For a graph $G = (V, E)$, let $\tau(G)$ denote the size of a minimum vertex cover, and $\nu(G)$ the size of a maximum matching. Recall the two linear programs VCLP and MLP. Let $\tau_f(G) := \mathrm{opt}(\mathrm{VCLP}(G))$ and $\nu_f(G) := \mathrm{opt}(\mathrm{MLP}(G))$. Note that

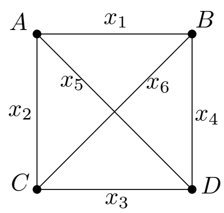
$$\nu(G) \leq \nu_f(G) = \tau_f(G) \leq \tau(G),$$

where the equality in the middle follows from Strong LP Duality. Also, if G is bipartite, then the equality holds throughout in (1). Let us say a graph G is *VCLP exact* if $\tau(G) = \tau_f(G)$, and *MLP exact* if $\nu(G) = \nu_f(G)$. As we already know, a bipartite grpah G is both VCLP exact and MLP exact.
From now on, suppose that G is *not* bipartite but $\tau(G) = \tau_f(G)$.

1. Give an example of such a graph G that is not bipartite but still VCLP exact.
2. Give an example of a graph G that is MLP exact but not VCLP exact.
3. Suppose G is VCLP exact. Let $Y \subseteq V(G)$ be a minimum vertex cover. Let $\mathbf{x}$ be an optimal solution of MLP(G). Show that $x_e = 0$ if $e \subseteq Y$ (i.e., if both endpoints of e are in the cover).
4. Show that such a graph G has a matching of size $|Y|$, and thus is MLP exact, too.



G_1



K_4

Solution to (1) and (2).

1. The example is G_1 in the upper left. It is not bipartite since B, C, D constitute an odd cycle.
   The minimum vertex cover of G_1 can be $\{B, C\}$ or $\{B, D\}$, with size 2. So $\tau(G_1) = 2$.
   VCLP(G_1) is

$$\begin{array}{ll} \text{minimize} & y_A + y_B + y_C + y_D \\ \text{subject to} & y_A + y_B \geq 1 \\ & y_B + y_C \geq 1 \\ & y_B + y_D \geq 1 \\ & y_C + y_D \geq 1 \\ & y_A, y_B, y_C, y_D \geq 0 \end{array}$$

   Note that if we add up the constraints $y_A + y_B \geq 1$ and $y_C + y_D \geq 1$, we get $y_A + y_B + y_C + y_D \geq 2$, which gives a lower bound of the target function.
   Since $\tau(G_1) = 2$, it follows that the lower bound is tight. Hence $\tau_f(G_1) = \tau(G_1) = 2$.
   So G_1 is an example which is not bipartite but still VCLP exact.
2. The example is K_4 in the upper right. The minimum vertex cover can be $\{A, B, C\}, \{A, B, D\}, \{A, C, D\}$ and $\{B, C, D\}$, with size 3. So $\tau(K_4) = 3$.
   However, by setting the value of each vertex to 0.5, we find that all edges are exactly covered ($y_u + y_v = 1$ for edge (u, v)). So $\tau_f(K_4) \leq 2$, and thus K_4 is not VCLP exact.
   Now consider the maximum matching and MLP. Obviously we can only match 2 pairs of vertices. So $\nu(K_4) = 2$, and $\nu_f(K_4) \geq 2$ follows. Since we already know that $\tau_f(K_4) \leq 2$ and $\nu_f(K_4) = \tau_f(K_4)$ by Strong LP Duality, we can conclude that $\nu(K_4) = \nu_f(K_4) = \tau_f(K_4) = 2$. Therefore K_4 is MLP exact.

So K_4 is an example which is MLP exact but not VCLP exact.

In (3) and (4), suppose that Y is a minimum vertex cover of G. Note that there is no edge $(u, v) \in E$ such that both $u, v \in V \setminus Y$. Because $u, v \in V \setminus Y$ means that (u, v) is not covered by Y, which is contradict with Y being a vertex cover.

3. VCLP is to minimize $c^T y$ with constraints $A \cdot y \geq b$, and MLP is to maximize $b^T x$ with constraints $A^T x \leq c$, where $b = 1, c = 1$.

   So there is
   $$x^T \cdot b \leq x^T \cdot A \cdot y \leq c^T \cdot y$$
   By strong duality, let optimal solution be $x^{(0)}, y^{(0)}$, there is $b^T * x^{(0)} = c^T * y^{(0)}$

   Let $A \cdot y^{(0)} = b^{(0)}$ there is $(b^T - b_0^T) \cdot x_0 = 0$. As $A \cdot y >= b$, there is $b_e - b_e^{(0)} \leq 0, \forall e \in E$. Besides, there is $x_e^{(0)} \geq 0, \forall e \in E$.

   So if $b_e^T - b_e^{(0)} < 0$ there is $x_e^{(0)} = 0$.

   Let $y^{(0)}$ be the solution corresponding to the vertex cover(by VCLP exact, it is an optimal solution of VCLP), then there is $b_e - b_e^{(0)} = 1 - 2 < 0, \forall e \in Y$, so $x_e^{(0)} = 0, \forall e \in Y$ for each optimal solution $x^{(0)}$ of the MLP.

4. Define $N(\cdot)$ be:
   $$N(A) = \{v : v \text{ is the neighbor of some } u \in A\} \cap (V \setminus Y)$$
   We claim that if $A \subseteq Y$, then $|A| \leq |N(A)|$.

   Otherwise, if $A \subseteq Y$ and there is $|B| < |A|$ where $B = N(A)$.

   Consider an optimal solution of VCLP relative to Y(that is, if $v \in Y$, $y_v = 1$, otherwise $y_v = 0$). Let
   $$y'_v = \begin{cases} y_v - \epsilon & , v \in A \\ y_v + \epsilon & , v \in B \\ y_v & , \text{ otherwise.} \end{cases}$$
   where $\epsilon < \frac{1}{2}$. For edge $(u, v) \in E$:

   a) $y'_u + y'_v \geq 1$ for $u, v \in A$. Since there is $y'_u + y'_v = (1 - \epsilon) + (1 - \epsilon) = 2 - 2\epsilon > 1$;

   b) $y'_u + y'_v \geq 1$ for $u \in A, v \in Y \setminus A$. Since there is $y'_u + y'_v = (1 - \epsilon) + 1 = 2 - \epsilon > 1$;

   c) $y'_u + y'_v \geq 1$ for $u \in A, v \in B$. Since there is $y'_u + y'_v = (1 - \epsilon) + \epsilon = 1$;

   d) $y'_u + y'_v \geq 1$ for $u \in Y \setminus A, v \in B$. Since there is $y'_u + y'_v = 1 + \epsilon > 1$;

   e) The rest of constraints remains holding since the left hand side is not changed from solution (y_v).

   So $(y_v)'$ is a solution to the VCLP, too. But it is $\epsilon(|A| - |B|)$ less than solution (y_v), which rises a contradiction that Y is a minimum solution.

   Hence what we claimed is proved. Remove all edges from Y to Y, the graph is a bipartite graph G with $L = Y$, $R = V \backslash Y$. By what we claimed, there is $|L| \leq |R|$

   Notice that what we claimed exactly means that the condition of Hall's Theorem is satisfied, so by Hall's Theorem, there is a perfect matching of size $|L| = |Y|$.

□