1: t9

Derive a formula for $\mathbb{E}_\pi[C(\pi, k)]$, up to additive terms of order $o(n)$. You might want to introduce ???? $= k/n$.

- case $i < k < j$:

$$\begin{aligned}\sum_{i=1}^{k-1}\sum_{j=k+1}^{n}\frac{1}{j-i+1} &< \sum_{i=1}^{k-1}\int_{k+1}^{n+1}\frac{1}{j-i}\mathrm{d}j \\ &= \sum_{i=1}^{k-1}\ln(n+1-i)-\ln(k+1-i) \\ &< \int_{i=0}^{k-1}\ln(n+1-i)-\ln(k-i)\mathrm{d}i \\ &= (n+1)\ln(n+1)-k\ln k-(n-k+2)\ln(n-k+2).\end{aligned}$$

On the other hand, similarly we can have

$$\sum_{i=1}^{k-1}\sum_{j=k+1}^{n}\frac{1}{j-i+1} > -(-k+n+2)\ln(-k+n+2)-(k+2)\ln(k+2)+(n+1)\ln(n+1)+3\ln(3).$$

The difference between the upper bound and the lower bound is $o(n)$, thus the inaccuracy is $o(n)$.
To write this formula more simply, we have

$$\sum_{i=1}^{k-1}\sum_{j=k+1}^{n}\frac{1}{j-i+1} = n\ln n - k\ln k - (n-k)\ln(n-k) + o(n).$$

Summing three terms up,

$$\mathbb{E}_\pi[C(\pi, k)] = n-1-H_k-H_{n-k+1}+n\ln n-k\ln k-(n-k)\ln(n-k)+o(n).$$