CS217 – Algorithm Design and Analysis
Homework 4

Not Strong Enough

April 4, 2020

1

Let T be a minimum spanning tree of G, and let $c \in \mathbb{R}$. Show that T_c and G_c have exactly the same connected components. (That is, two vertices $u, v \in V$ are connected in T_c if and only if they are connected in G_c). You are encouraged to draw pictures to illustrate your proof.

Solution. Since T_c is a subgraph of G_c, if $u, v \in V$ are connected in T_c, then u, v are connected in G_c.

Now let's assume $u, v \in V$ are not connected in T_c, but are connected in G_c. Then, the path in T from u to v contains an edge y whose weight is greater than c. Also, there exists a path P in G from u to v such that $\forall e \in P, w(e) \leq c$. If we remove y from T, T will be split into 2 connected components where u belongs to one and v belongs to another. So there must be an edge $x \in P$ whose endpoints belongs to different connected components. Adding x into T then will make the 2 components connected again. Let $T' = T + x - y$. Now T' becomes a tree, because it has $|V| - 1$ edges and it's connected. The total weight of T' is $w(T) + w(x) - w(y) < w(T)$, which violates that T is an MST. So u, v are not connected in G_c if they are not connected in T_c.

In conclusion, two vertices $u, v \in V$ are connected in T_c if and only if they are connected in G_c. $\square$