# CS217 – Algorithm Design and Analysis
# Homework 5

Not Strong Enough

May 8, 2020

**1: Exercise 1**

Let $G = (V, c)$ be a flow network. Prove that flow is "transitive" in the following sense: if $r, s, t$ are vertices, and there is an $r$-$s$-flow of value $k$ and an $s$-$t$-flow of value $k$, then there is an $r$-$t$-flow of value $k$.

*Proof.* Note that there is an $r$-$s$-flow of value $k$ means that the value of the maximum $r$-$s$-flow is at least $k$, which also means that the value of the minimum $r$-$s$-cut is at least $k$. Similarly, the value of the minimum $s$-$t$-cut is also at least $k$.

Now consider an $r$-$t$-cut. It is either an $r$-$s$-cut (if $s$ is not in the cut) or an $s$-$t$-cut (if $s$ is in the cut). So the capacity of the minimum $r$-$t$-cut is at least $k$. It follows that the value of the maximum $r$-$t$-flow is at least $k$, and thus there is an $r$-$t$-flow of value $k$. $\square$

## 2: Exercise 3

Prove Menger's Theorem. You have to prove two things: first, not both cases above can occur (this is rather easy); second, one of them must occur (this requires a tool from the lecture).

*Proof.* Let $V(G) = \{v_1, \cdots\}$ and $s = v_p, t = v_q$.

We would like to construct a flow network $(V', s', t', c)$ where $V' = \{v_1, v_1', v_2, v_2', \cdots\}$ and

$$c(u,v) = \begin{cases} 1, \text{ if } \exists i, (u,v) = (v_i, v_i') \\ \infty, \text{ if } \exists i, j, (u,v) = (v_i', v_j) \wedge (v_i, v_j) \in E \\ 0, \text{ otherwise} \end{cases}.$$

And finally let $s' = v_p', t' = v_q$. Then:

- There are $k$ vertex disjoint paths $p_1, \cdots, p_k$ in $G$ iff there is a flow $f$ in $(V', s', t', c)$ with $\mathrm{val}(f) = k$, iff $\max \mathrm{val}(f) \geq k$.
- There are $k-1$ vertices $v_{i_1}, \cdots, v_{ik-1}$ in $V \setminus \{s,t\}$ such that $G - \{v_{i_1}, \cdots, v_{i_{k-1}}\}$ contains no $s-t$ path iff there is a cut $S$ in $(V', s', t', c)$ with $\mathrm{cap}(S) = k-1$ by making $v_{i_j} \in S$ and $v_{i_j}' \notin S$ for all $j$, iff $\min \mathrm{cap}(S) < k$.

By *Max-Flow Min-Cut Theorem*, let $\max \mathrm{val}(f) = \min \mathrm{cap}(S) = l$. Then:

- Either $l \geq k$, resulting in 1 holds while 2 does not.
- Or $l < k$, resulting in 2 holds while 1 does not.

Therefore, *exactly one* of the two statements is true. $\square$

In the two exercises below, let $\Gamma(X)$ be the neighbors of $X$.

**3: Exercise 4**

Consider the induced bipartite subgraph $H_n[L_i \cup L_{i+1}]$, show that for $i < n/2$ the graph has a matching of size $|L_i| = \binom{n}{i}$

*Proof.* Use Hall's Theorem, the size of maximum matching equals $\min_{X \subseteq L_i} |L_i| - |X| + |\Gamma(X)|$.

Since in $H_n[L_i \cup L_{i+1}]$ the degree of each vertex in $L_i$ is $n - i$, and that of each vertex in $L_{i+1}$ is $i + 1$, there is $|X|(n-i) \leq |\Gamma(X)|(i+1)$. As $i < n/2$, $|X| \leq |\Gamma(x)|\frac{i+1}{n-i} \leq |\Gamma(X)|$, and only if $|X| = 0$ can the equality be achieved.

So there is $\min_{X \subseteq L_i}(|L_i| - |X| + |\Gamma(X)|) = |L_i| = \binom{n}{i}$. □

### 4: Exercise 5

Show that there are $\binom{n}{i}$ paths in $H_n$ starting at $L_i$ ending in $L_{n-i}$ and are disjoint.

*Proof.* Construct the flow in following steps:

1. Split each vertex $v$ in $\bigcup_{i\leq k\leq n-i} L_k$ into two, $v_{in}$ and $v_{out}$, there is $(v_{in}, v_{out})$ with capacity 1 in the flow. If there is an edge $(u, v)$ in $H_n$, then there is an edge $(u_out, v_in)$ with capacity $\infty$ in the flow.

2. Set a new point $s$ connected to $v_{i,in}$ for each $v_i \in L_i$ in the flow, the capacity is 1. Symmetrically, for each $v_{n-i} \in L_{n-i}$, there is an edge $(v_{n-i,out}, t)$ with capacity 1 in the flow.

Then there is a flow that, for edges from $s$ or end to $t$ the flow takes 1; for edges in form of $(v_{k,out}, v_{k+1,in})$, the flow takes $\frac{\binom{n}{i}}{\binom{n}{k}(n-k)}$; for edges in form of $(v_{k,in}, v_{k,out})$, the flow takes $\frac{\binom{n}{i}}{\binom{n}{k}}$.

It is obvious that the flow is well-defined, that for each vertex (except for $s, t$) the flow in equals the flow out. And the total flow is $\binom{n}{i}$. Besides, it is the maxflow since the flow out of $s$ is no more than $\binom{n}{i}$.

So there also exists a integer max flow $F$ whose size is $\binom{n}{i}$. Since in the flow network, the capacity of $(v_{in}, v_{out})$ for each $v$ is 1, each path in the flow network is disjoint. (Otherwise, the flow in for some vertex $v_{in}$ is greater than 1). The capacity of each path is 1, so there are $\binom{n}{i}$ paths.

Combine $v_in$ and $v_out$ together and remove $s, t$, these $\binom{n}{i}$ paths turns to be disjoint paths from $L_i$ to $L_{n-i}$. $\square$

### 5: Exercise 6

Let $\nu(G)$ denote the size of a maximum matching of $G = (V, E)$. Show that a bipartite graph $G$ has at most $2^{\nu(G)}$ mimimum vertex covers.

*Proof.* From the König's Theorem, we know that the size of mimimum vertex cover is $\nu(G)$ if $G$ is a bipartite graph. Let $C$ be a minimum vertex cover. Then we can construct new minimum vertex covers by choosing vertices from $C$ and $V \setminus C$. In other words, all minimum vertex covers can be represented by $X \cup Y$, where $X \subseteq C, Y \subseteq V \setminus C$. Denote $N(A) = \{b \mid \exists a \in A, \text{ there is an edge between } a \text{ and } b\}$. For all $X \subseteq C$, to construct a vertex cover, $Y$ must touch all edges touched by $C \setminus X$ but not by $X$.

- If $N(C \setminus X) \cap (C \setminus X) \neq \emptyset$, there does not exist such a $Y$.
- If $N(C \setminus X) \cap (C \setminus X) = \emptyset$, then $Y$ must be at least $N(C \setminus X) \cap (V \setminus C)$ to be a vertex cover. To make $X \cap Y$ a minimum vertex cover, $Y$ has to be $N(C \setminus X) \cap (V \setminus C)$.

Since $X$ has $2^{\nu(G)}$ choices, $G$ has at most $2^{\nu(G)}$ mimimum vertex covers. □

Obviously, this is not true for general (non-bipartite) graphs: the triangle $K_3$ has $\nu(K_3) = 1$ but it has three minimum vertex covers. The five-cycle $C_5$ has $\nu(C_5) = 2$ but has five minimum vertex covers.

**6: Exercise 7**

Is there a function $f\colon \mathbf{N_0} \to \mathbf{N_0}$ such that every graph with $\nu(G) = k$ has at most $f(k)$ minimum vertex covers? How small a function $f$ can you obtain?

*Solution.* Suppose that we have a graph $G = (V, E)$ and one of its maximum matching $M \subseteq E$ with $|M| = \nu(G) = k$.

We have the two following observations:

- For any vertex cover $C \subseteq V$ of $G$, for every edge in $M$ there must be at least one of its endpoint which is in $C$. Otherwise there exists an edge in $M$ such that neither of its endpoints is in $C$, which means that this edge is uncovered and therefore $C$ is not a vertex cover.
- For any vertex $v$ which is not matched, all of its neighbors must be matched, or the edge between $v$ and one of its unmatched neighbors can be added to the maximum matching and therefore $M$ is not maximum.

We now construct a vertex set $C_0 \subseteq V$ such that for every edge $(u, v)$ in $M$, either $u \in C_0$, or $v \in C_0$, or both $u, v \in C_0$. There are $3^k$ possible $C_0$ in total.

For each possible $C_0$, note that it may not be a "vertex cover" by far. So we try to construct another vertex set $C_1$ from $C_0$. Let $C_1$ be an empty set at the beginning. From the second observation above, for every unmatched vertex $v$, there are two cases. If all of its neighbors are in $C_0$, then we do nothing, since every edge connected to $v$ is covered by vertices in $C_0$. Otherwise we add it into $C_1$ to cover the edges which $C_0$ didn't cover. Note that $C_1$ is *uniquely determined by* $C_0$.

Let $\mathcal{C}$ be a family of vertex covers, which is initialized to empty. Now consider $C_0 \cup C_1$. We know that it is also uniquely determined by $C_0$. If it is a vertex cover, we add it to $\mathcal{C}$. There are at most $3^k$ vertex covers in $\mathcal{C}$, since there are $3^k$ possible $C_0$, and for some $C_0$ and its corresponding $C_1$, $C_0 \cup C_1$ may not be a vertex cover.

Claim that any minimum vertex cover $C$ must belong to $\mathcal{C}$. Because from the first observation, we can let the unique $C_0$ be the matched vertices covered by $C$. And then unique $C_1$ can be constructed from $C_0$. $C_0 \cup C_1$ is the minimum vertex cover when $C_0$ is fixed. So $C = C_0 \cup C_1 \in \mathcal{C}$.

So there are at most $3^k$ minimum vertex covers in total, and $f(k) = 3^k$. Also note that this upper bound is *tight*. Just consider the triangle $K_3$ — it has $3 = 3^1 = 3^{\nu(K_3)}$ minimum vertex covers. $\square$