

# CS217 – Algorithm Design and Analysis

## Homework 6

Not Strong Enough

June 1, 2020

┌ 1

Let  $\nu(G)$  denote the size of a maximum matching of  $G$ . Obviously,  $\text{val}(\text{MLP}(G)) \geq \nu(G)$  for all graphs. Show that  $\nu(G) = \text{val}(\text{MLP}(G))$  for all bipartite graphs  $G$ . Do this without referring to König's Theorem.

*Solution.* Suppose  $\mathbf{x}$  is a solution to  $\text{MLP}(G)$ .  $E_F$  is the set of edges with fractional values in  $\mathbf{x}$ . First do as follows:

- (1) If  $E_F$  doesn't contain a cycle, terminate.
- (2) Find a cycle in  $E_F$  with fractional values, denote it  $C := \{e_i \in E_F : 1 \leq i \leq n, i \in \mathbb{N}\}$ .
- (3) Add  $x_{e_{2k-1}}$  by  $\epsilon$  and  $x_{e_{2k}}$  by  $-\epsilon$  for  $\{k \in \mathbb{N} : k \leq n/2\}$ .
- (4) Increase  $\epsilon$  until there exists  $i$  such that  $x_{e_i} = 0$  or  $1$ .
- (5) go to step(2) if there exists a cycle in  $E_F$ .

Since  $G$  is a bipartite,  $C$  can only be a even cycle, so step (3) makes sense. If we modify the solution by step(3), obviously the constraints of  $\text{MLP}(G)$  will still be satisfied, and the target function will remain unchanged. The process will terminate because  $|E_F|$  decreases by at least 1 in each iteration.

Then do as follows:

- (1) If  $E_F$  is empty, terminate.
- (2) Choose 2 vertex  $v_1, v_2$  that  $|\{x_e \in (0, 1) : v_1 \in e\}| = 1$ ,  $|\{x_e \in (0, 1) : v_2 \in e\}| = 1$  and there's a path between  $v_1$  and  $v_2$  in  $E_F$ , denote it  $P := \{e'_i \in E_F : 1 \leq i \leq m, i \in \mathbb{N}\}$
- (3) Add  $x_{e'_{2k-1}}$  by  $\epsilon$  for  $\{k \in \mathbb{N} : 2k - 1 \leq m\}$  and  $x_{e'_{2k}}$  by  $-\epsilon$  for  $\{k \in \mathbb{N} : 2k \leq m\}$
- (4) Increase  $\epsilon$  until there exists  $i$  such that  $x_{e'_i} = 0$  or  $1$
- (5) Go to step (2) if  $E_F$  is not empty

Since there's no cycle in  $E_F$ , we can definitely choose  $v_1, v_2$  that satisfy the requirements if  $E_F$  is not empty. In each iteration, the target function will not decrease. Since  $v_1, v_2$  can only be touched by edges with fractional value or value 0, the constraints of  $v_1, v_2$  can be satisfied. In other words,  $\sum_{e \in E: v_1 \in e} x_e = x_{e'_1} \leq 1$ ,  $\sum_{e \in E: v_2 \in e} x_e = x_{e'_m} \leq 1$  The constraints of other vertices  $v$  in  $P$  will obviously be maintained. The process will terminate because  $|E_F|$  decreases by at least 1 in each iteration.

Finally, after 2 processes, the solution becomes integral and the value of target function does not decrease, which means  $\text{int-val}(\text{MLP}(G)) \geq \text{val}(\text{MLP}(G))$ . And we already have  $\text{int-val}(\text{MLP}(G)) \leq \text{val}(\text{MLP}(G))$ . So  $\nu(G) = \text{int-val}(\text{MLP}(G)) = \text{val}(\text{MLP}(G))$  □

┌ **2**

We know that  $\nu(G) = \tau(G)$  for all bipartite graphs (König's Theorem) and  $\nu(G) \leq \tau(G)$  for all graphs (since every matched edge must be covered by a distinct vertex). Show that  $\tau(G) \leq 2\nu(G)$  for all graphs  $G$ .

*Proof.* Let  $M$  be a maximum matching of  $G$ . It follows that  $|M| = \nu(G)$ . Now we choose our vertex set  $V'$  to be all the matched vertices in  $G$ . So  $|V'| = 2\nu(G)$ .

Claim that  $V'$  is a vertex cover. To see that, assume there exists an edge  $(u, v)$  which is not covered by  $V'$ . It means that neither  $u$  nor  $v$  is matched. So we can add edge  $(u, v)$  to  $M$ , and thus  $M$  is not maximum, which leads to a contradiction.

So the size of minimum vertex cover  $\tau(G) \leq |V'| = 2\nu(G)$ . □

┌ **3**

Show that  $\tau(G) \leq 2\text{opt}(\text{VCLP}(G))$  for all graphs  $G$  (including non-bipartite graphs).

*Proof.* From (2) we know that  $\tau(G) \leq 2\nu(G)$ . Since  $\nu(G) \leq \text{opt}(\text{MLP}(G)) \leq \text{opt}(\text{VCLP}(G))$ , it follows that  $\tau(G) \leq 2\nu(G) \leq 2\text{opt}(\text{VCLP}(G))$ . □

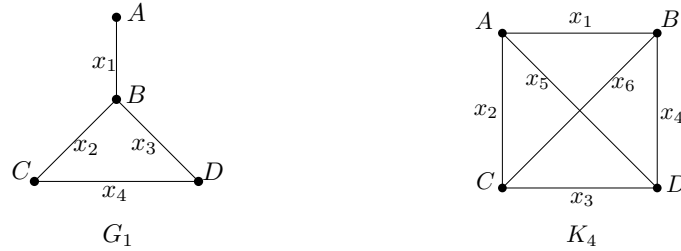
For a graph  $G = (V, E)$ , let  $\tau(G)$  denote the size of a minimum vertex cover, and  $\nu(G)$  the size of a maximum matching. Recall the two linear programs VCLP and MLP. Let  $\tau_f(G) := \text{opt}(\text{VCLP}(G))$  and  $\nu_f(G) := \text{opt}(\text{MLP}(G))$ . Note that

$$\nu(G) \leq \nu_f(G) = \tau_f(G) \leq \tau(G),$$

where the equality in the middle follows from Strong LP Duality. Also, if  $G$  is bipartite, then the equality holds throughout in (1). Let us say a graph  $G$  is *VCLP exact* if  $\tau(G) = \tau_f(G)$ , and *MLP exact* if  $\nu(G) = \nu_f(G)$ . As we already know, a bipartite graph  $G$  is both VCLP exact and MLP exact.

From now on, suppose that  $G$  is *not* bipartite but  $\tau(G) = \tau_f(G)$ .

1. Give an example of such a graph  $G$  that is not bipartite but still VCLP exact.
2. Give an example of a graph  $G$  that is MLP exact but not VCLP exact.
3. Suppose  $G$  is VCLP exact. Let  $Y \subseteq V(G)$  be a minimum vertex cover. Let  $\mathbf{x}$  be an optimal solution of MLP( $G$ ). Show that  $x_e = 0$  if  $e \subseteq Y$  (i.e., if both endpoints of  $e$  are in the cover).
4. Show that such a graph  $G$  has a matching of size  $|Y|$ , and thus is MLP exact, too.



*Solution to (1) and (2).*

1. The example is  $G_1$  in the upper left. It is not bipartite since  $B, C, D$  constitute an odd cycle.

The minimum vertex cover of  $G_1$  can be  $\{B, C\}$  or  $\{B, D\}$ , with size 2. So  $\tau(G_1) = 2$ .

VCLP( $G_1$ ) is

$$\begin{aligned} & \text{minimize} && y_A + y_B + y_C + y_D \\ & \text{subject to} && y_A + y_B \geq 1 \\ & && y_B + y_C \geq 1 \\ & && y_B + y_D \geq 1 \\ & && y_C + y_D \geq 1 \\ & && y_A, y_B, y_C, y_D \geq 0 \end{aligned}$$

Note that if we add up the constraints  $y_A + y_B \geq 1$  and  $y_C + y_D \geq 1$ , we get  $y_A + y_B + y_C + y_D \geq 2$ , which gives a lower bound of the target function.

Since  $\tau(G_1) = 2$ , it follows that the lower bound is tight. Hence  $\tau_f(G_1) = \tau(G_1) = 2$ .

So  $G_1$  is an example which is not bipartite but still VCLP exact.

2. The example is  $K_4$  in the upper right. The minimum vertex cover can be  $\{A, B, C\}$ ,  $\{A, B, D\}$ ,  $\{A, C, D\}$  and  $\{B, C, D\}$ , with size 3. So  $\tau(K_4) = 3$ .

However, by setting the value of each vertex to 0.5, we find that all edges are exactly covered ( $y_u + y_v = 1$  for edge  $(u, v)$ ). So  $\tau_f(K_4) \leq 2$ , and thus  $K_4$  is not VCLP exact.

Now consider the maximum matching and MLP. Obviously we can only match 2 pairs of vertices. So  $\nu(K_4) = 2$ , and  $\nu_f(K_4) \geq 2$  follows. Since we already know that  $\tau_f(K_4) \leq 2$  and  $\nu_f(K_4) = \tau_f(K_4)$  by Strong LP Duality, we can conclude that  $\nu(K_4) = \nu_f(K_4) = \tau_f(K_4) = 2$ . Therefore  $K_4$  is MLP exact.

So  $K_4$  is an example which is MLP exact but not VCLP exact.

□

Suppose that  $Y$  is a minimum vertex cover of  $G$ . Note that there is no edge  $(u, v) \in E$  such that both  $u, v \in V \setminus Y$ . Because  $u, v \in V \setminus Y$  means that  $(u, v)$  is not covered by  $Y$ , which is contradict with  $Y$  being a vertex cover.

To solve (3) and (4), define a function  $N(\cdot)$  relative to a VCLP exact graph  $G = (V, E)$  and a minimum vertex cover  $Y$  that:

$$N(A) = \begin{cases} \{v : v \text{ is the neighbor of some } u \in A\} \cap (V \setminus Y) & , \text{ if } A \subseteq Y \\ \{v : v \text{ is the neighbor of some } u \in A\} \cap Y & , \text{ if } A \subseteq V \setminus Y \end{cases}$$

### 5: Lemma

If  $A \subseteq Y$ , then  $|A| \leq |N(A)|$ . Vise versa, if  $A \subseteq V \setminus Y$ , then  $|A| \geq |N(A)|$ .

*Proof.* If  $A \subseteq Y$  and there is  $|B| < |A|$  where  $B = N(A)$ . Consider an optimal solution of VCLP relative to  $Y$  (that is, if  $v \in Y$ ,  $y_v = 1$ , otherwise  $y_v = 0$ ). Let

$$y'_v = \begin{cases} y_v - \epsilon & , v \in A \\ y_v + \epsilon & , v \in B \\ y_v & , \text{ otherwise.} \end{cases}$$

where  $\epsilon < \frac{1}{2}$ . To prove that  $(y'_v)$  is also a solution for VCLP, we only need to check that for edge  $(u, v) \in E$ :

- a)  $y'_u + y'_v \geq 1$  for  $u, v \in A$ . Since there is  $y'_u + y'_v = (1 - \epsilon) + (1 - \epsilon) = 2 - 2\epsilon > 1$ ;
- b)  $y'_u + y'_v \geq 1$  for  $u \in A, v \in Y \setminus A$ . Since there is  $y'_u + y'_v = (1 - \epsilon) + 1 = 2 - \epsilon > 1$ ;
- c)  $y'_u + y'_v \geq 1$  for  $u \in A, v \in B$ . Since there is  $y'_u + y'_v = (1 - \epsilon) + \epsilon = 1$ ;
- d)  $y'_u + y'_v \geq 1$  for  $u \in Y \setminus A, v \in B$ . Since there is  $y'_u + y'_v = 1 + \epsilon > 1$ ;
- e) The rest of constraints remains holding since the left hand side is not changed from solution  $(y_v)$ .

So  $(y'_v)$  is a solution to the VCLP, too. But it is  $\epsilon(|A| - |B|)$  less than solution  $(y_v)$ , which rises a contradiction that  $Y$  is a minimum solution.

Likewise, let  $A = N(B)$  and use the same method to construct  $(y'_v)$  can prove the “vice versa” part.

□

Now we begin to prove (3) and (4):

*Proof.* (3) Consider the optimal solution of MLP be  $X = (x_e)$ . Assume there is  $y_1, y_2 \in Y, (y_1, y_2) \in V, x_{y_1 y_2} = x \geq 0$ . Let  $A_0 = \{y_1, y_2\}, B_0 = N(A_0), C_0 = N(B_0)$ .

By the lemma above, there is  $|B_0| > |C_0|$ . Let all edges from  $C$  to  $B$  be  $\{e_1, e_2 \dots e_m\}$ , there is:

$$\sum_{i=1}^m y_e + 2x \leq |C_0| \leq |B_0|$$

By the property of vertex cover, any vertex in  $B$  can only connected to vertices in  $C_0$ . Let  $B = \{b_1, b_2 \dots b_s\}$ , and  $v(b_i) = \sum_{e \in E: b_i \in e} x_e$ , then there is:

$$\sum_{i=1}^s v(b_i) = \sum_{i=1}^m y_e \leq |B_0| - 2x$$

. So there exists  $I = \{v_{i_1}, v_{i_2} \dots v_{i_t}\} \subseteq B$ , where  $\sum_{j=1}^t (1 - v(b_{i_j})) \geq 2x$ . We call  $1 - v(b_i)$  the capacity of vertex  $b_i$ .

Let  $I_1$  be all vertices connected to  $y_1$  and in  $I$ , and the capacity of  $I_1$ ,  $c(I_1)$  is the sum of capacity of vertices in  $I_1$ ,  $I_2$  likewise.

Let  $A_1 = \{y_2\}$ ,  $B_1 = N(A_1)$ ,  $C_1 = N(B_1)$ , by the same method, we can prove that the  $c(B_1)$  is no less than  $x$ . And that means the  $c(I_2)$  is no less than  $x$ , since  $I_2$  contains all those vertices with positive capacity in  $B_1$ .

Without loss of generality, we can assume that  $c(I_1)$  is more than  $x$ .

Now we prove that in different cases we can always modify the optimal solution  $X$  to get a better solution.

In each case, we do the same thing first: let  $x_{y_1 y_2} = 0$  (which gives an decrease of  $x$  in the target). We then define a slack of  $(y_i, I_j)$  ( $i = 1, 2; j = 1, 2, 3, 4, 5$ ) be that:

Increase  $x_e$  for each edge  $e$  from  $y_i$  to  $I_j$  until the constraint of at least one side of  $e$  is strict.

Let  $I_3 = I_1 \cap I_2$ ,  $I_4 = I_1 \setminus I_3$ ,  $I_5 = I_2 \setminus I_3$ ,

a) If the  $c(I_5) \geq 0$ , do a slack to  $(y_2, I_5)$ . Since the  $c(I_5) \geq 0$ , the increasement is more than 0. Notice that  $c(I_1)$  is not changed, so now we do the slack of  $(y_1, I_1)$  and the increasement is not less than  $x$ . Not the total increasement is more than  $x$ , so the new solution is better.

b) If  $c(I_5) = 0$ , there is  $c(I_2) \geq 0 \rightarrow c(I_3) \geq 0$  and  $c(I_1) = c(I) \geq 2x$ . Now we do slack of  $(y_2, I_3)$  but stop as soon as the increasement is  $x$ . Now there is  $c(I_1) \geq 2x - x = x$ , so doing a slack of  $(y_1, I_1)$  can get an increasement more than  $x$ . In all, the increasement is more than  $x$ , so the new solution is better.

Since in each case there is a better solution, there is a contradiction (that  $(x_e)$  is already the optimal). So the assumption is incorrect and it is proved.

(4) This part is straightly proved by Hall's Theorem and the lemma above. Remove all edges from  $Y$  to  $Y$ , then the graph is a bipartite graph  $G$  with  $L = Y$ ,  $R = V \setminus Y$ . By the lemma above, the condition of Hall's Theorem is satisfied. So there is a matching of size  $|Y|$ .  $\square$