Advanced Mathematical Techniques
Problem Set 2

1. Find the first four nonzero terms in the Maclaurin series for $\frac{x^3}{e^{2x^2}-1}$. What is the radius of convergence of this expansion?

$$\frac{x^3}{e^{2x^2}-1} = \frac{x^3}{2x^2 + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \frac{(2x^2)^4}{4!} + \cdots} = \frac{x^3}{2x^2 + 2x^4 + \frac{4x^6}{3} + \frac{2x^8}{3} + \frac{4x^{10}}{15} + \cdots}$$
$$= \frac{x^3}{2x^2(1 + x^2 + \frac{2x^4}{3} + \frac{x^6}{3} + \frac{2x^8}{15} + \cdots)} = \frac{x}{2(1 + x^2 + \frac{2x^4}{3} + \frac{x^6}{3} + \frac{2x^8}{15} + \cdots)}$$
$$= \frac{x}{2} \cdot \frac{1}{1 + x^2 + \frac{2x^4}{3} + \frac{x^6}{3} + \frac{2x^8}{15} + \cdots} = \frac{x}{2} \cdot \frac{1}{1 - \left(-x^2 - \frac{2x^4}{3} - \frac{x^6}{3} - \frac{2x^8}{15} - \cdots\right)}$$
$$= \frac{x}{2}\left(1 + \left(-x^2 - \frac{2x^4}{3} - \frac{x^6}{3} - \frac{2x^8}{15} - \cdots\right) + \left(-x^2 - \frac{2x^4}{3} - \frac{x^6}{3} - \frac{2x^8}{15} - \cdots\right)^2 + \cdots\right)$$
$$= \boxed{\frac{x}{2} - \frac{x^3}{2} + \frac{x^5}{6} - \frac{x^9}{90}} + \cdots$$

Radius of convergence will occur when $e^{2x^2} = 1$, or when $2x^2 = 0$. When $x = 0$, this is a removable singularity, so we consider the complex numbers. When $2x^2 = 2n\pi i$, we have a singularity, so the radius of convergence is $\boxed{\sqrt{\pi}}$.

2. Find the first three nonzero terms in the Maclaurin series for $\tan(2x^3)$. What is the radius of convergence of this expansion? Use your result to determine the value of the 9th and 12th derivative of this function at zero.

$$\tan(2x^3) = 2x^3 + \frac{(2x^3)^3}{3} + \frac{2(2x^3)^5}{15} + \cdots = \boxed{2x^3 + \frac{8x^9}{3} + \frac{64x^{15}}{15}} + \cdots$$

Further, we know that the radius of convergence will occur when:

$$|2x^3| < \frac{\pi}{2}$$
$$|x| < \sqrt[3]{\frac{\pi}{4}}$$

The radius of convergence is $\boxed{\sqrt[3]{\frac{\pi}{4}}}$. Also, from the expansion of the series, we know that it has an x^9 term and no x^{12} term, which means the derivatives are nonzero and zero, respectively. At $x = 0$, the 9th derivative is $\boxed{9! \cdot \frac{8}{3}}$, and the 12th derivative is $\boxed{0}$.

3. Determine the value of the integral $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^3} dx$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)^3} dx = 2\int_{0}^{\infty} \frac{x^2}{(x^2+9)^3} dx$$

$$x = 3\tan\theta, dx = 3\sec^2\theta d\theta$$

$$2\int_{0}^{\infty} \frac{x^2}{(x^2+9)^3} dx = 2\int_{0}^{\frac{\pi}{2}} \frac{(9\tan^2\theta)}{(9\tan^2\theta+9)^3} \cdot 3\sec^2\theta d\theta = 2\int_{0}^{\frac{\pi}{2}} \frac{27\tan^2\theta\sec^2\theta}{(9\sec^2\theta)^3} d\theta$$

$$= \frac{54}{729}\int_{0}^{\frac{\pi}{2}} \frac{\tan^2\theta}{\sec^4\theta} d\theta = \frac{2}{27}\int_{0}^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta = \frac{2}{27} \cdot \frac{\pi}{16} = \boxed{\frac{\pi}{216}}$$

4. Determine the value of the integral $\int_{-\infty}^{\infty} \frac{x^4}{(x^2+1)^6} dx$

$$\int_{-\infty}^{\infty} \frac{x^4}{(x^2+1)^6} dx = 2\int_{0}^{\infty} \frac{x^4}{(x^2+1)^6} dx$$

$$x = \tan\theta, dx = \sec^2\theta d\theta$$

$$2\int_{0}^{\infty} \frac{x^4}{(x^2+1)^6} dx = 2\int_{0}^{\frac{\pi}{2}} \frac{\tan^4\theta}{\sec^{10}\theta} d\theta$$

$$= 2\int_{0}^{\frac{\pi}{2}} \sin^4\theta\cos^6\theta d\theta = 2\int_{0}^{\frac{\pi}{2}} \left(\frac{1-\cos 2\theta}{2}\right)^2 \left(\frac{1+\cos 2\theta}{2}\right)^3 = 2\int_{0}^{\frac{\pi}{2}} \frac{1}{32}(1-\cos 2\theta)^2(1+\cos 2\theta)^3 d\theta$$

$$= \frac{1}{16}\int_{0}^{\frac{\pi}{2}} (1+\cos 2\theta - 2\cos^2 2\theta - 2\cos^3 2\theta + \cos^4 2\theta + \cos^5 2\theta) d\theta = \boxed{\frac{3\pi}{256}}$$

5. Determine the value of the integral $\int_0^1 x^7 \ln^4 x dx$

$$u = -\ln x, x = e^{-u}, dx = -e^{-u} du$$

$$\int_0^1 x^7 \ln^4 x dx = \int_{\infty}^{0} (e^{-u})^7 u^4 (-e^{-u}) du = \int_0^{\infty} e^{-8u} u^4 du$$

$$t = 8u, dt = 8du, du = \frac{1}{8} dt$$

$$\int_0^{\infty} e^{-8u} u^4 du = \int_0^{\infty} e^{-t} \left(\frac{t}{8}\right)^4 \cdot \frac{1}{8} dt = \frac{1}{8^5}\int_0^{\infty} e^{-t} t^4 dt = \frac{1}{8^5} \cdot 4! = \boxed{\frac{3}{4096}}$$

6. Determine the value of the integral $\int_0^1 x\sqrt{x} \ln^2 x dx$

$$\int_0^1 x\sqrt{x} \ln^2 x dx = \int_0^1 x^{3/2} \ln^2 x dx$$

$$u = -\ln x, x = e^{-u}, dx = -e^{-u} du$$

$$\int_0^1 x^{3/2} \ln^2 x dx = \int_{\infty}^{0} (e^{-u})^{3/2} u^2 (-e^{-u}) du = \int_0^{\infty} e^{-5u/2} u^2 du$$

$$t = \frac{5u}{2}, dt = \frac{5}{2}du, du = \frac{2}{5}dt$$

$$\int_0^\infty e^{-5u/2}u^2 du = \int_0^\infty e^{-t}\left(\frac{2t}{5}\right)^2 \cdot \frac{2}{5}dt = \frac{8}{125}\int_0^\infty e^{-t}t^2 dt = \frac{8}{125}\cdot 2! = \boxed{\frac{16}{125}}$$

7. Determine the value of the integral $\int_0^\infty x^6 e^{-3x^2}$

$$u = 3x^2, du = 6xdx, x^2 = \frac{u}{3}$$

$$\int_0^\infty x^6 e^{-3x^2}dx = \int_0^\infty = \int_0^\infty \left(\sqrt{\frac{u}{3}}\right)^6 \cdot e^{-u} \cdot \left(\frac{\sqrt{3}}{6\sqrt{u}}\right) du = \frac{\sqrt{3}}{162}\int_0^\infty u^{\frac{5}{2}} \cdot e^{-u}$$

$$= \frac{\sqrt{3}}{162}\cdot \Gamma\left(\frac{7}{2}\right) = \frac{\sqrt{3}}{162}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \boxed{\frac{5\sqrt{3}\pi}{432}}$$

8. The infinite product expansion for the sine function, $\frac{\sin x}{x} = \prod_{n=1}^\infty \left(1 - \frac{x^2}{n^2\pi^2}\right)$, can be used to determine the value of many interesting infinite products.

   (a) Determine the value of $\prod_{k=1}^\infty \frac{4k^2-1}{4k^2}$

$$\prod_{k=1}^\infty \frac{4k^2-1}{4k^2} = \prod_{k=1}^\infty \left(1 - \frac{1}{4k^2}\right) = \prod_{k=1}^\infty \left(1 - \frac{1}{(2k)^2}\right)$$

   Here, $x^2 = \frac{\pi^2}{4}$, $x = \pm\frac{\pi}{2}$. For both of them, we see that

$$\frac{\sin(x/2)}{x/2} = \boxed{\frac{2}{\pi}}$$

   (b) Determine the value of $\prod_{k=1}^\infty \frac{9k^2-1}{9k^2}$

$$\prod_{k=1}^\infty \frac{9k^2-1}{9k^2} = \prod_{k=1}^\infty \left(1 - \frac{1}{9k^2}\right)$$

   Here, $x^2 = \frac{\pi^2}{9}$, $x = \pm\frac{\pi}{3}$. For both of them, we see that:

$$\frac{\sin(x/3)}{x/3} = \boxed{\frac{3\sqrt{3}}{2\pi}}$$

   (c) Explain why the infinite product $\prod_{k=1}^\infty \frac{k^2-1}{k^2}$ is obviously 0. Then, use a limit process with the sine function's infinite product to determine $\prod_{k=2}^\infty \frac{k^2-1}{k^2}$.

   It's obviously 0 since the first term is $\frac{0}{1} = 0$, and anything multiplied by 0 is 0. We can evaluate considering the limit as x approaches π:

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\cdot\prod_{k=2}^\infty \left(1 - \frac{x^2}{k^2\pi^2}\right)$$

$$\prod_{k=2}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right) = \lim_{x\to\pi}\frac{\sin x}{x\left(1-\frac{x^2}{\pi^2}\right)} = \lim_{x\to\pi}\frac{\pi^2\sin x}{x(\pi-x)(\pi+x)}$$
$$= \lim_{x\to\pi}\left[\frac{\pi^2}{x(\pi+x)}\cdot\frac{\sin x}{\pi-x}\right] \xrightarrow{\text{L'Hôpital's rule}} \frac{\pi^2}{\pi\cdot 2\pi} = \boxed{\frac{1}{2}}$$

(d) Use complex numbers along with the infinite product representation of the sine function to determine the value of $\prod_{k=2}^{\infty}\frac{k^2+1}{k^2}$

$$\frac{k^2+1}{k^2} = 1+\frac{1}{k^2} = 1-\frac{(i\pi)^2}{k^2\pi^2} = 1-\frac{x^2}{k^2\pi^2}, x=\pm i\pi$$
$$\prod_{k=2}^{\infty}\frac{k^2+1}{k^2} = \frac{\sin x}{2x}\bigg|_{x=\pm i\pi} = \boxed{\frac{\sin(i\pi)}{2i\pi}}$$

9. Use complex numbers along with the infinite product representation of the sine function to determine the value of $\prod_{k=1}^{\infty}\frac{4k^2+1}{4k^2}$

$$\frac{4k^2+1}{4k^2} = 1+\frac{1}{4k^2} = 1-\frac{(i\pi/2)^2}{k^2\pi^2} = 1-\frac{x^2}{k^2\pi^2}, x=\pm i\frac{\pi}{2}$$
$$\frac{\sin x}{x} = \frac{\sin\left(i\frac{\pi}{2}\right)}{i\frac{\pi}{2}} = \boxed{\frac{2\sin\left(i\frac{\pi}{2}\right)}{i\pi}}$$