# Advanced Mathematical Techniques (AMT)
# Worksheet 1

1. Consider the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = \frac{(-1)^n n}{n+1}$

   (a) Write out the first 5 terms of the sequence. What general behavior can you deduce from these terms?

   The first 5 terms are:
   $$a_1 = -\frac{1}{2}, a_2 = \frac{2}{3}, a_3 = -\frac{3}{4}, a_4 = \frac{4}{5}, a_5 = -\frac{5}{6}$$
   From this, it looks like the absolute value of the sequence approaches 1, with each term's sign alternating.

   (b) Is this sequence monotonic? Is it bounded? Would you say that the sequence converges or diverges? If it diverges, would you say that it contains a convergent subsequence?

   The sequence is **not monotonic**, as shown above because $a_2 > a_1$, but $a_3 < a_2$. It **is bounded**, however, because the absolute value of the sequence can never exceed 1, since $\frac{n}{n+1}$ can never exceed 1. I would say **the sequence diverges**, since it will end up oscillating between numbers close to 1 and -1 for all eternity, but it **does contain 2 convergent subsequences** - one of all the positive numbers, going to 1, and one of all the negative ones, going to -1.

   (c) Determine the limit points of this sequence.

2. Consider the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_1 = 3$ and $a_{n+1} = 7 - \frac{2}{a_n}$.

   (a) Write out the first 5 terms of the sequence. What general behavior of the terms do your results suggest? Does it appear to be monotonic? Does it appear to be bounded?

   The first 5 terms are:
   $$a_1 = 3, a_2 = \frac{19}{3}, a_3 = \frac{127}{19}, a_4 = \frac{851}{127}, a_5 = \frac{5703}{851}$$
   The terms seem to **converge** to a number just over 7. The sequence appears to be **monotonic** and bounded, below by 3 and above by 7.

   (b) Show that this sequence is bounded, $2 < a_n < 7$ for all $n$, by induction (assume that $2 < a_k < 7$ for some value of $k$, then use the recurrence relation to show that $2 < a_{k+1} < 7$).

   $$2 < a_k < 7$$
   $$\frac{1}{7} < \frac{1}{a_k} < \frac{1}{2}$$
   $$-1 < -\frac{2}{a_k} < -\frac{2}{7}$$
   $$6 < 7 - \frac{2}{a_k} < \frac{47}{7}$$

$$6 < a_{k+1} < \frac{47}{7}$$

$$2 < 6 < a_{k+1} < \frac{47}{7}$$

$$\boxed{2 < a_{k+1} < \frac{47}{7}}$$

(c) The sequence is indeed monotonic increasing. Prove this by induction. Your proof will be similar to that found in part (b).

For the base case, where $n = 1$:

$$a_1 < a_2 \rightarrow 3 < \frac{19}{3} \text{ is true.}$$

Next, let:

$$a_k < a_{k+1}$$

$$\frac{1}{a_k} > \frac{1}{a_k + 1}$$

$$-\frac{2}{a_k} < -\frac{2}{a_k + 1}$$

$$7 - \frac{2}{a_k} < 7 - \frac{2}{a_k + 1}$$

$$\boxed{a_{k+1} < a_{k+2}}$$

(d) Based on your analysis in (b) and (c), can you say for sure that this sequence converges? If so, what number (exactly) does it converge to?

Since this sequence is bounded, it has at least one limit point, and since it's also monotonically increasing, it **does converge.** It converges to:

$$L = 7 - \frac{2}{L}$$

$$L^2 = 7L - 2$$

$$L^2 - 7L + 2 = 0$$

$$L = \frac{7 \pm \sqrt{41}}{2}$$

Since $\frac{7-\sqrt{41}}{2} < 3$, we know that the sequence converges to:

$$\boxed{\frac{7 + \sqrt{41}}{2}}$$

3. Consider the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_1 = 3$ and $a_{n+1} = 7 + \frac{2}{a_n}$.

(a) Write out the first 5 terms of the sequence. What general behavior of the terms do your results suggest? Does it appear to be monotonic? Does it appear to be bounded?

The first 5 terms are:

$$a_1 = 3, a_2 = \frac{23}{3}, a_3 = \frac{167}{23}, a_4 = \frac{1215}{167}, a_5 = \frac{8839}{1215}$$

The terms seem to **converge** to a number just over 7. The sequence is **not monotonic**, since $a_4 > a_3$, but $a_3 < a_2$. It does appear to be bounded, below by 3 and above by 12.

(b) This sequence is not monotonic, but it is bounded. Show using induction that the sequence is bounded by 2 from below and 12 from above. What convergence property can you state definitively from this result?

$$2 < a_n < 12$$
$$\frac{1}{12} < \frac{1}{a_n} < \frac{1}{2}$$
$$\frac{1}{6} < \frac{2}{a_n} < 1$$
$$\frac{43}{6} < 7 + \frac{2}{a_n} < 8$$
$$\frac{43}{6} < a_{n+1} < 8$$
$$2 < \frac{43}{6} < a_{n+1} < 8 < 12$$
$$\boxed{2 < a_{n+1} < 12}$$

We can't state any convergence property from this because again, the sequence isn't monotonic, so there could be two limit points, and no convergence.

(c) The sequence is convergent. Determine exactly what number it converges to and verify this result from investigating a few terms in the sequence.

$$L = 7 + \frac{2}{L}$$
$$L^2 - 7L - 2 = 0$$
$$L = \frac{7 \pm \sqrt{57}}{2}$$

Since $\frac{7-\sqrt{57}}{2} < 0$, and we know our sequence is bounded by 2 and 12, we know that the converged value is:

$$\boxed{\frac{7+\sqrt{57}}{2}}$$

4. Consider the sequence $\{S_n\}_{n=1}^{\infty}$, where $S_1 = 1$ and $S_{n+1} = S_n + \frac{1}{n+1}$.

(a) Write out the first 5 terms of the sequence. Does it appear to be monotonic? Does it appear to be bounded?

The first five terms are:

$$S_1 = 1, S_2 = \frac{3}{2}, S_3 = \frac{11}{6}, S_4 = \frac{25}{12}, S_5 = \frac{137}{60}$$

It appears to be monotonically increasing, and it appears to be bounded by 3.

(b) The difference between adjacent terms in this sequence, $S_{n+1} - S_n = \frac{1}{n+1}$, clearly goes to 0 as $n \to \infty$. Does this imply the sequence is convergent? Why or why not?

No. Convergence requires bounds, and even though each term gets closer and closer to the next, that doesn't mean that the sequence is bounded.

(c) Prove that this sequence is monotonic (it is not difficult, and does not require induction). Does this mean it converges?

$$n > 0, \frac{1}{n+1} > 0 \; \forall \; n, \; \therefore \; S_n \text{ is monotonically increasing.}$$

Again, this does not necessarily mean that it converges, because we don't know that there are bounds on the sequence, so it could just increase forever, like how $1, 2, 3$ is monotonically increasing but obviously divergent.

(d) Work through the logic of the following expression:

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}(\text{last 2 sum to } > \frac{1}{2}, < 1)...$$

Can you state conclusively whether or not this sequence converges based on this analysis? If it converges, can you give a bound that its limit is definitely less than?

I can say this doesn't converge because we know that there are infinite of these "groups" in the limit. Each of these groups sums to a number greater than $\frac{1}{2}$. Since there are infinite groups of them, then we know that there are infinite amounts of at least $\frac{1}{2}$, meaning that the sum of this sequence will be greater than the sum of the sequence $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, ...$, which diverges. Thus, this sequence also diverges.

5. Consider the sequence $\{S_n\}_{n=1}^{\infty}$, where $S_1 = 1$ and $S_{n+1} = S_n + \frac{1}{(n+1)^2}$.

(a) Write out the first 5 terms of the sequence. Does it appear to be monotonic? Does it appear to be bounded?

The first five terms are:

$$S_1 = 1, S_2 = \frac{5}{4}, S_3 = \frac{49}{36}, S_4 = \frac{205}{144}, S_5 = \frac{5269}{3600}$$

It appears to be monotonically increasing, and it appears to be bounded below by 1 and above by 2.

(b) The difference between adjacent terms in this sequence, $S_{n+1} - S_n = \frac{1}{(n+1)^2}$, clearly goes to 0 as $n \to \infty$. Does this imply the sequence is convergent? Why or why not?

No. Convergence requires bounds, and even though each term gets closer and closer to the next, that doesn't mean that the sequence is bounded.

(c) Prove that this sequence is monotonic. Does this mean it converges?

$$n > 0, \frac{1}{n+1} > 0, \frac{1}{n+1} > 0,\ \forall\ n,\ \therefore\ S_n \text{ is monotonically increasing.}$$

Again, this does not necessarily mean that it converges, because we don't know that there are bounds on the sequence, so it could just increase forever, like how $1, 2, 3$ is monotonically increasing but obviously divergent. We don't know there is a limit point yet.

(d) Work through the logic of the following expression:

$$S_n = 1 + \frac{1}{2}^2 + \frac{1}{3}^2 + \frac{1}{4^2}(\text{last 2 sum to } > \frac{1}{2}, < 1)...$$

Can you state conclusively whether or not this sequence converges based on this analysis? If it converges, can you give a bound that its limit is definitely less than?

I can say this does converge because we know that there are infinite of these "groups" in the limit. Each of these groups sums to a number less than $\frac{1}{2}^p$, where $p$ represents the "group" number. Since there are infinite groups of them, then we know that there are infinite amounts of at most $\frac{1}{2}^p$, meaning that the sum of this sequence will be less than the sum of the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$, which converges. Thus, this sequence also converges.

6. Consider the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = cos(n^2)$.

(a) Write out the first 5 terms of the sequence (decimal approximations are appropriate). Does it appear to be monotonic? Does it appear to be bounded?

The first 5 terms are:

$$a_1 = 0.5403, a_2 = -0.6536, a_3 = -0.9111, a_4 = -0.9577, a_5 = 0.9912$$

This is not monotonic, since $S_1 > S_2$ but $S_5 > S_4$. It is bounded by -1 and 1 as lower and upper bounds, since those are the bounds of the cosine function.

(b) The difference between adjacent terms in this sequence, $a_{n+1} - a_n = cos((n+1)^2) - cos(n^2)$, clearly doesn't go to 0 as $n \to \infty$. Does this mean that the sequence doesn't converge?

Yes. The limit of the sequence must approach 0 in order for the it to converge.

(c) Prove that the sequence is bounded. What does this imply about its convergence characteristics?

The sequence is bounded because of the inherent bounds on the cosine function, which are $-1 \leq \cos(n) \leq 1\ \forall n$.

(d) Can you prove that this sequence has at least 1 limit point? Can you find a limit point of this sequence? What is the difference between these two statements?

The sequence has at least one limit point because it's an infinite sequence that's bounded, but we don't know what that limit point is because finding the limit points is far more difficult than ascertaining that one exists. The difference between the two is that one is just proving that something is there, while the other asks what it is.