

# Advanced Mathematical Techniques (AMT)

## Notes

### 1 Today's stuff

Let's try and prove that the alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

converges.

We can start by going from 0 to 1 on a number line, then back to 1/2, then up by 1/3, then down by 1/4, and so on until we see that the sequence is constrained and converges between 0 and 1.

Further, we can see that if we choose to add the terms in a different order, such as

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n+1} - \frac{1}{2} \dots$$

we can get a different number. Therefore, this is **conditionally convergent**, where the order of the terms matters. We define the alternating series test as a test where if we have a sequence of terms that alternate in signs and have decreasing absolute value, we know the series converges.

Let's try this complicated integral.

$$\int_{-\infty}^{\infty} \frac{x^4}{(x^2 + 4)^6} dx$$

Consider instead a simpler version:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{ax^2 + b} &= \frac{1}{a} \int_{-\infty}^{\infty} \frac{dx}{x^2 + \frac{b}{a}} \\ &= \frac{1}{a} \frac{2}{\sqrt{\frac{b}{a}}} \lim_{R \rightarrow \infty} \arctan \left( \frac{x}{\sqrt{\frac{b}{a}}} \right) \Big|_0^R \\ &= \frac{\pi}{\sqrt{ab}} \end{aligned}$$

Returning to the original integral, we know that

$$\begin{aligned} \frac{\partial}{\partial a} \left( \frac{\pi}{\sqrt{ab}} \right) &= -\frac{\pi}{2a^{3/2}b^{1/2}} \\ \frac{\partial^2}{\partial b^2} \left( \frac{\pi}{\sqrt{ab}} \right) &= \frac{3\pi}{4a^{1/2}b^{5/2}} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial^5}{\partial a^2 \partial b^3} \left( \frac{\pi}{\sqrt{ab}} \right) &= -2 \times 3 \times 4 \times 5 \times \int_{-\infty}^{\infty} \frac{x^4}{(x^2 + 4)^6} dx \\ &= \end{aligned}$$

Also, we'll be expected to know how to do Gaussian integrals.

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

Therefore,

$$I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy$$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} e^{-x^2-y^2} dy$$

Switching to polar,

$$\int_0^{\infty} r dr \int_0^{2\pi} d\theta e^{-r^2} = \pi$$

Therefore,

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

Adding some sort of auxiliary paramter:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx$$

We can just do a u-substitution to get that this is  $\sqrt{\frac{\pi}{a}}$ . We can also differentiate with respect to  $a$  to get:

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{1}{2} \sqrt{\pi} a^{-\frac{1}{2}}$$

Another one with an auxiliary paramter:

$$\int_0^1 x^n dx = \frac{1}{n+1}$$

Taking the partial with respect to  $n$ ,

$$\frac{\partial}{\partial n} \left( \frac{1}{n+1} \right) = \int_0^1 x^n \ln x dx = -\frac{1}{(n+1)^2}$$

$$\frac{\partial^2}{\partial n^2} \left( \frac{1}{n+1} \right) = \int_0^1 x^n \ln^2 x dx = \frac{2}{(n+1)^3}$$

$$\frac{\partial^3}{\partial n^3} \left( \frac{1}{n+1} \right) = \int_0^1 x^n \ln^3 x dx = -\frac{6}{(n+1)^4}$$

Moving on to something completely different, let's look at what's called an infinite product. Given the sequence  $\{a_n\}$  of numbers, we can define  $\{P_n\}_{n=1}^{\infty}$  via  $P_1 = 1 + a_1$ ,  $P_{n+1} = (1 + a_{n+1})P_n$ . If

$$\lim_{n \rightarrow \infty} P_n$$

exists and is nonzero, then it is called the infinite product:

$$\prod_{k=1}^{\infty} (1 + a_k)$$

Infinite products cannot converge to 0 - if they do we say they diverge. The reason is that if we allow that to happen, then any of the terms could be 0 and that would rig your entire product. We make it  $1 + a_k$  so that we can use some of the series stuff instead of having to reinvent everything. We can convert the product to the series by taking the logarithm:

$$\ln \left( \prod_{k=1}^{\infty} (1 + a_k) \right) = \sum_{k=1}^{\infty} \ln(1 + a_k)$$

Therefore,

$$\prod_{k=1}^{\infty} (1 + a_k)$$

converges if and only if

$$\sum_{k=1}^{\infty} \ln(1 + a_k)$$

converges. This turns out to be true for:

$$\frac{1}{2}|a_k| < |\ln(1 + a_k)| < 2|a_k|$$

This gives us a 3-way inequality, which means for absolute convergence, we have a way to relate sequences.