Electrodynamics
Problem Set 9

Magnetic procession

Consider an electron that, at time $t = 0$ is in the state $|\psi(t=0)\rangle = |+\rangle_z = \frac{1}{\sqrt{2}}(|+y\rangle + |-y\rangle)$

Part A

Since the system hasn't been allowed to evolve at all, the state is just the initial, so the probabilities are:

$$\boxed{P(|+x\rangle) = \frac{1}{2} \qquad P(|-x\rangle) = \frac{1}{2}}$$

Just as expected!

Part B

Following the 5 step process:

- Make the Hamiltonian.

$$\hat{H} = \frac{-q}{M_e} B_0 \frac{\hbar}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \frac{\omega\hbar}{2}\hat{\sigma}_y$$

- Determine the eigenstates and eigenenergies of $\hat{H}$.

  Defining $\omega = \frac{eB_0}{m}$,

$$E_1 = \frac{\omega\hbar}{2}, E_2 = -\frac{\omega\hbar}{2}$$
$$|E_1\rangle = |+y\rangle, |E_2\rangle = |-y\rangle$$

- Write $|\psi(0)\rangle$ into the energy basis.

  Since $|E_{1,2}\rangle = |\pm y\rangle$,

$$|\psi_0\rangle = (\langle +y|\psi_0\rangle)|+y\rangle + \langle -y|\psi_0\rangle)|-y\rangle$$
$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|+y\rangle + |-y\rangle)$$
$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle + |E_2\rangle)$$

- Write down the time-evolved state

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|E_1\rangle + \frac{1}{\sqrt{2}}e^{i\omega t}|E_2\rangle$$

- Make measurements! (in this case unnecessary)

Thus, the time-evolved state is just:

$$\boxed{|\psi(t)\rangle = \frac{1}{\sqrt{2}}|E_1\rangle + \frac{1}{\sqrt{2}}e^{i\omega t}|E_2\rangle}$$

Part C

Reforming $|\psi\rangle$,

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}|+z\rangle + \frac{i}{\sqrt{2}}|-z\rangle\right] + \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}|+z\rangle - \frac{i}{\sqrt{2}}|-z\rangle\right]e^{i\omega t}$$
$$\left(\frac{1}{2}+\frac{1}{2}e^{i\omega t}\right)|+z\rangle + i\left(\frac{1}{2}-\frac{1}{2}e^{i\omega t}\right)|-z\rangle$$

Now, finding the actual probability:

$$P_{+x} = |\langle +x|\psi(t)\rangle|^2$$
$$= \left|\left(\frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle\right)|\psi(t)\rangle\right|^2$$
$$= \frac{1}{8}\left|(1+e^{i\omega t}) + i(1-e^{i\omega t})\right|^2$$
$$= \boxed{\frac{1}{2}(1+\sin\omega t)}$$

Thus the probability of spin down is:

$$1-(above) = \boxed{\frac{1}{2}(1-\sin\omega t)}$$

Three state system

Let a Hamiltonian be defined by

$$\hat{H} = \begin{pmatrix} A & 0 & B \\ 0 & C & 0 \\ B & 0 & A \end{pmatrix}$$

where A, B, C are real numbers. Use the following notation for unit vectors:

$$|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}, |2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, |3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Part A

By mass algebra that I don't want to type, we get:

$$\boxed{E_1 = A+B, E_2 = C, E_3 = A-B}$$

$$\boxed{|E_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}, |E_2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}, |E_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}}$$

Part B

This probability is encapsulated by:

$$|\langle 2|\psi(t)\rangle|^2$$

Let:

$$E_1 = |1\rangle, E_2 = |2\rangle, E_3 = |3\rangle$$

Therefore, since we start in state $|2\rangle$,

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ct}|2\rangle$$

The probability of measuring $|2\rangle$ must therefore be:

$$\boxed{|\langle 2|\psi(t)\rangle|^2 = 1}$$

Part C

This probability is encapsulated by:

$$|\langle 3|\psi(t)\rangle|^2$$

But here,

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-\frac{i}{\hbar}(A+B)t}\frac{1}{\sqrt{2}}(|1\rangle + |3\rangle) + e^{-\frac{i}{\hbar}(A-B)t}\frac{1}{\sqrt{2}}(|1\rangle - |3\rangle)\right)$$

Where, if we take our measurement:

$$|\langle 3|\psi(t)\rangle|^2 = \left|\frac{1}{2}\left(e^{-\frac{i}{\hbar}(A+B)t} + e^{-\frac{i}{\hbar}(A-B)t}\right)\right|^2$$

Factoring out and simplifying:

$$|\langle 3|\psi(t)\rangle|^2 = \left|e^{-\frac{i}{\hbar}At}\cos\frac{Bt}{\hbar}\right|^2$$

Thus,

$$\boxed{|\langle 3|\psi(t)\rangle|^2 = \cos^2\frac{Bt}{\hbar}}$$

All done!