Advanced Mathematical Techniques
Bookwork 2

1. Problem 5.7.4: The analysis of the diffraction pattern of a circular opening involves:

$$\int_0^{2\pi} \cos(c\cos\phi)\,d\phi$$

Expand the integrand in a series and integrate by using:

$$\int_0^{2\pi} \cos^{2n}\phi d\phi = \frac{2\pi\cdot(2n)!}{2^{2n}n!} \qquad \int_0^{2\pi} \cos^{2n+1}\phi d\phi = 0$$

$$\int_0^{2\pi} \cos(c\cos\varphi)\,d\varphi = \int_0^{2\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(c\cos\varphi)^{2n}\,d\varphi = \int_0^{2\pi}\sum_{n=0}^{\infty}\frac{(-1)^n c^{2n}}{(2n)!}\cos^{2n}\varphi\,d\varphi = \sum_{n=0}^{\infty}\frac{(-1)^n c^{2n}}{(2n)!}\int_0^{2\pi}\cos^{2n}\varphi\,d\varphi$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n c^{2n}}{(2n)!}\left[\frac{(2n)!}{2^{2n}(n!)^2}\cdot 2\pi\right] = \sum_{n=0}^{\infty}\frac{(-1)^n c^{2n}}{2^{2n}(n!)^2}\cdot 2\pi = 2\pi\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{c}{2}\right)^{2n} = \boxed{2\pi J_0(c)}$$

2. Problem 5.7.7: Expand the incomplete factorial function:

$$\gamma(n+1,x) \equiv \int_0^x e^{-t}t^n dt$$

in a eries in powers of x. What is the range of convergence of the resulting series?

$$\int_0^x e^{-t}t^n\,dt = \int_0^x\left(\sum_{p=0}^{\infty}\frac{(-t)^p}{p!}\right)t^n\,dt = \int_0^x\sum_{p=0}^{\infty}\frac{(-1)^p t^{n+p}}{p!}\,dt$$

$$= \sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\int_0^x t^{n+p}\,dt = \sum_{p=0}^{\infty}\frac{(-1)^p x^{n+p+1}}{p!(n+p+1)} = \boxed{x^{n+1}\left[\frac{1}{n+1}-\frac{x}{n+2}+\frac{x^2}{2!(n+3)}-\cdots+\frac{(-1)^p x^p}{p!(n+p+1)}+\cdots\right]}$$

$\boxed{\text{Range of convergence: } \infty\text{. The series converges for all } x.}$

3. The Klein-Nisha formula for the scattering of photons by electrons contains a term of the form:

$$f(\varepsilon) = \frac{(1+\varepsilon)}{\varepsilon^2}\left[\frac{2+2\varepsilon}{1+2\varepsilon}-\frac{\ln(1+2\varepsilon)}{\varepsilon}\right]$$

Here, $\varepsilon = \frac{hv}{mc^2}$, the ratio of the photon energy to the electron rest mass energy. Find:

$$\lim_{\varepsilon\to 0} f(\varepsilon)$$

$$\lim_{\varepsilon\to 0}\frac{1+\varepsilon}{\varepsilon^2}\left[\frac{2+2\varepsilon}{1+2\varepsilon}-\frac{\ln(1+2\varepsilon)}{\varepsilon}\right] = \lim_{\varepsilon\to 0}\frac{1+\varepsilon}{\varepsilon^2}\left[\left(1+\frac{1}{1+2\varepsilon}\right)-\frac{1}{\varepsilon}\left(2\varepsilon-\frac{(2\varepsilon)^2}{2}+\frac{(2\varepsilon)^3}{3}-\cdots\right)\right]$$

$$= \lim_{\varepsilon\to 0}\frac{1+\varepsilon}{\varepsilon^2}\left[(1+1-2\varepsilon+4\varepsilon^2-\cdots)-\left(2-2\varepsilon+\frac{8}{3}\varepsilon^2-\cdots\right)\right] = \lim_{\varepsilon\to 0}\frac{1+\varepsilon}{\varepsilon^2}\left[\frac{4}{3}\varepsilon^2+\text{ terms of order }\varepsilon^3\text{ or higher}\right]$$

$$= \boxed{\frac{4}{3}}$$

4. Problem 5.7.19 (a): An analysis of the Gibbs phenomenon of Section 14.5 leds to the expression:

$$\frac{2}{\pi}\int_0^{\pi}\frac{\sin\xi}{\xi}d\xi$$

Expand the integrand in a series and integrate term by term. Find the numerical value of this expression to four significant figures.

$$\frac{2}{\pi}\int_0^{\pi}\frac{\sin\xi}{\xi}d\xi = \frac{2}{\pi}\int_0^{\pi}\left(\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n}}{(2n+1)!}\right)d\xi = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^{\pi}\xi^{2n}d\xi = \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{(2n+1)(2n+1)!}$$

$$= 2\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n+1)(2n+1)!} = 2\left(1-\frac{\pi^2}{18}+\frac{\pi^4}{600}-\frac{\pi^6}{35280}+\cdots\right)\approx\boxed{1.179}$$

5. Problem 5.8.1: The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may be represented parametrically by $x=a\sin\theta$, $y=b\cos\theta$. Show that the length of the arc within the first quadrant is:

$$a\int_0^{\frac{\pi}{2}}\sqrt{1-m\sin^2\theta}\,d\theta = aE(m), \text{where } 0\le m=\frac{a^2-b^2}{a^2}\le 1$$

$$\int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}d\theta = \int_0^{\pi/2}\sqrt{(a\cos\theta)^2+(-b\sin\theta)^2}d\theta = \int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}d\theta$$

$$= \int_0^{\pi/2}\sqrt{a^2(1-\sin^2\theta)+b^2\sin^2\theta}d\theta = \int_0^{\pi/2}\sqrt{a^2-(a^2-b^2)\sin^2\theta}d\theta = \int_0^{\pi/2}\sqrt{a^2\left(1-\frac{a^2-b^2}{a^2}\sin^2\theta\right)}d\theta$$

$$= \boxed{a\int_0^{\pi/2}(1-m\sin^2\theta)^{1/2}d\theta = aE(m)}$$

6. Problem 5.11.5: Show that:

$$\prod_{n=2}^{\infty}\left[1-\frac{2}{n(n+1)}\right]=\frac{1}{3}$$

$$\prod_{n=2}^{\infty}\left[1-\frac{2}{n(n+1)}\right] = \lim_{N\to\infty}\prod_{n=2}^{N}\frac{(n-1)(n+2)}{n(n+1)} = \lim_{N\to\infty}\left(\prod_{n=2}^{N}\frac{n-1}{n}\right)\left(\prod_{n=2}^{N}\frac{n+2}{n+1}\right)$$

$$= \lim_{N\to\infty}\left(\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{N-1}{N}\right)\left(\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{N+2}{N+1}\right) = \lim_{N\to\infty}\left(\frac{1}{N}\right)\left(\frac{N+2}{3}\right) = \boxed{\frac{1}{3}}$$

7. Problem 5.11.6: Show that:

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$$

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right) = \lim_{N\to\infty}\prod_{n=2}^{N}\frac{n^2-1}{n^2} = \lim_{N\to\infty}\prod_{n=2}^{N}\frac{(n-1)(n+1)}{n\cdot n} = \lim_{N\to\infty}\left(\prod_{n=2}^{N}\frac{n-1}{n}\right)\left(\prod_{n=2}^{N}\frac{n+1}{n}\right)$$

$$= \lim_{N\to\infty}\left(\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{N-1}{N}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{N+1}{N}\right) = \lim_{N\to\infty}\left(\frac{1}{N}\right)\left(\frac{N+1}{2}\right) = \lim_{N\to\infty}\frac{N+1}{2N} = \boxed{\frac{1}{2}}$$