Electrodynamics
Worksheet PS.10

(Chapter 3)
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1. Changing field

Consider an electron. For this problem, take the classical Larmour precession rate as $\omega = eB/m$ where e is the elementary charge, B is the magnetic field strength, and m is the mass of the electron.

a. At time $t = 0$, the observable S_x is measured with the result $+\hbar/2$. What is the state vector $|\psi_0\rangle$ immediately after this measurement?

Since the electron's state was measured the electron must now be in the state of the output of said measurement - thus, $\boxed{|\psi_0\rangle = |+\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle_z + |-\rangle_z\right)}$

b. Immediately after the measurement, a magnetic field $\mathbf{B} = B\hat{k}$ is applied and the particle is allowed to evolve for a time T. What is the state of the system at time $t = T$?

Let the Hamiltonian be:

$$\hat{H} = -\vec{\mu} \cdot \vec{B} = \frac{\omega\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus, with a little bit of the 5 step method that I'm too lazy to show, we get:

$$\boxed{|\psi(T)\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}e^{i\omega T}|-z\rangle}$$

c. At $t = T$, the magnetic field is instantaneously switched to $\mathbf{B} = B\hat{\jmath}$. After another time T, a measurement of S_x is carried out once more. What is the probability that a value of $+\hbar/2$ is found?
The new Hamiltonian for this setup is as follows:

$$\hat{H} = \frac{\omega\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Moreover, the rotation operator about the y-axis is:

$$\hat{R}_y = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

so multiplying our initial state by this gives us:

$$|\psi(2T)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta - e^{i2\theta}\sin\theta \\ \sin\theta + e^{i2\theta}\cos\theta \end{pmatrix}$$

from which the square root of the probability of measuring spin up is:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \times \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta - e^{i2\theta}\sin\theta \\ \sin\theta + e^{i2\theta}\cos\theta \end{pmatrix} = \frac{1}{2}\left(\cos\theta - e^{i2\theta}\sin\theta + \sin\theta + e^{i2\theta}\cos\theta\right)$$

$$= e^{i\theta}\left(\cos^2\theta - i\sin^2\theta\right)$$

making the probability of measuring spin up:

$$\left(e^{i\theta}\left(\cos^2\theta - i\sin^2\theta\right)\right)^2 = \left(\cos^4\theta + \sin^4\theta\right) = \boxed{1 - \frac{1}{2}\sin(\omega T)}$$

assuming we measure at time $t = 2T$.

2. A 3 State System Evolution

Consider a three-dimensional Hilbert space spanned by the orthonormal basis states

$$\{|1\rangle, |2\rangle, |3\rangle\}.$$

The Hamiltonian of the system is given by

$$\hat{H} = \hbar\Omega \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where Ω is a real positive constant with units of angular frequency.
At time $t = 0$, the system is prepared in the state

$$|\psi(0)\rangle = |1\rangle.$$

a. Find the eigenenergies for this system.
It's a 3 state system, meaning the eigenenergies must necessarily be the eigenvalues of the matrix:

$$\boxed{\pm\frac{1}{\sqrt{2}}\Omega\hbar, 0}$$

b. The eigenvectors of $\hat{H}$ are:

$$|E_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad |E_2\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \quad |E_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Rewrite the initial state in the energy basis.
The initial state is $|\psi(0)\rangle = |1\rangle = \sum_n |E_n\rangle \langle E_n\rangle \psi(0)$. Calculating the coefficients:

$$\begin{aligned} c_1 &= \langle E_1|1\rangle = \frac{1}{\sqrt{2}}(1) + 0 + 0 = \frac{1}{\sqrt{2}} \\ c_2 &= \langle E_2|1\rangle = \frac{1}{2}(1) + 0 + 0 = \frac{1}{2} \\ c_3 &= \langle E_3|1\rangle = \frac{1}{2}(1) + 0 + 0 = \frac{1}{2} \end{aligned}$$

Thus, the state in the energy basis is:

$$\boxed{|\psi(0)\rangle = \frac{1}{\sqrt{2}}|E_1\rangle + \frac{1}{2}|E_2\rangle + \frac{1}{2}|E_3\rangle}$$

c. Write the time evolved state $|\psi(t)\rangle$, making sure to factor out a global phase from the $|1\rangle$ basis state.
The time evolved state is as follows:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}|E_1\rangle + \frac{1}{2}e^{-i\sqrt{2}\Omega t}|E_2\rangle + \frac{1}{2}e^{i\sqrt{2}\Omega t}|E_3\rangle$$

Expanding back into the provided basis:

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{4}e^{-i\sqrt{2}\Omega t}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} + \frac{1}{4}e^{i\sqrt{2}\Omega t}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} + \frac{1}{2}\cos(\sqrt{2}\Omega t) \\ -\frac{i}{\sqrt{2}}\sin(\sqrt{2}\Omega t) \\ -\frac{1}{2} + \frac{1}{2}\cos(\sqrt{2}\Omega t) \end{pmatrix} \end{aligned}$$

Substituting angle identities:

$$\boxed{|\psi(t)\rangle = \cos^2\left(\frac{\sqrt{2}\Omega t}{2}\right)|1\rangle - \frac{i}{\sqrt{2}}\sin(\sqrt{2}\Omega t)\,|2\rangle - \sin^2\left(\frac{\sqrt{2}\Omega t}{2}\right)|3\rangle}$$

d. Determine the time-dependent measurement probabilities:

i. $P_1(t) = |\langle 1|\psi(t)\rangle|^2 = \boxed{\cos^4\left(\frac{\sqrt{2}\Omega t}{2}\right)}$

ii. $P_2(t) = |\langle 2|\psi(t)\rangle|^2 = \left|-\frac{i}{\sqrt{2}}\sin(\sqrt{2}\Omega t)\right|^2 = \boxed{\frac{1}{2}\sin^2(\sqrt{2}\Omega t)}$

iii. $P_3(t) = |\langle 3|\psi(t)\rangle|^2 = \left|-\sin^2\left(\frac{\sqrt{2}\Omega t}{2}\right)\right|^2 = \boxed{\sin^4\left(\frac{\sqrt{2}\Omega t}{2}\right)}$

e. We know the system starts in the state $|1\rangle$, based on your answer from part (d), which state does it evolve into next - $|2\rangle$ or $|3\rangle$? At what times (in terms of Ω) does the system evolve into each basis state?
The system evolves into $|2\rangle$ next because from $t = 0$ onwards, the probability of getting $|2\rangle$ is squared, compared to that of $|3\rangle$, which is to the fourth power. Since the sine function naturally is less than or equal to 1, this makes $|2\rangle$ have the highest probability of being chosen as the next outcome.

3. Time Dependent Hamiltonian

For this question assume we are driving a spin 1/2 system using a time-dependent field (as described in the book):

$$\hat{H} = \omega_0 \hat{S}_z + \omega_1[\cos(\omega t)\hat{S}_x + \sin(\omega t)\hat{S}_y]$$

or as a matrix

$$\hat{H} = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1 e^{-i\omega t} \\ \omega_1 e^{i\omega t} & -\omega_0 \end{pmatrix}$$

Following the details in the book, complete the following questions.

a. Assuming the resonance condition $\omega = \omega_0$, Solve the differential equation for the coefficients $\alpha_\pm(t)$. Use your results to find the transformed state vectors $|\tilde{\psi}(t)\rangle$ and the state vector $|\psi(t)\rangle$ You should assume the state starts off in the $|+z\rangle$ state.
In the rotating frame, we have:

$$i\begin{pmatrix} \dot{\alpha}_+ \\ \dot{\alpha}_- \end{pmatrix} = \frac{\omega_1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$$

Applying the initial conditions $\alpha_+(0) = 1, \alpha_-(0) = 0$,

$$\boxed{\alpha_+(t) = \cos\left(\frac{\omega_1 t}{2}\right), \quad \alpha_-(t) = -i\sin\left(\frac{\omega_1 t}{2}\right)}$$

$$\boxed{|\tilde{\psi}(t)\rangle = \cos\left(\frac{\omega_1 t}{2}\right)|+z\rangle - i\sin\left(\frac{\omega_1 t}{2}\right)|-z\rangle}$$

The actual state vector in the lab frame is $|\psi(t)\rangle = e^{-i\omega t\hat{S}_z/\hbar}\,|\tilde{\psi}(t)\rangle$:

$$\boxed{|\psi(t)\rangle = e^{-i\omega t/2}\cos\left(\frac{\omega_1 t}{2}\right)|+z\rangle - ie^{i\omega t/2}\sin\left(\frac{\omega_1 t}{2}\right)|-z\rangle}$$

b. Verify that a π-pulse ($\omega_1 t = \pi$) produces a complete spin flip. Calculate both the transformed state vector $|\tilde{\psi}(t)\rangle$ and the actual state vector $|\psi(t)\rangle$ for this pulse.
Set $\omega_1 t = \pi$, meaning $\frac{\omega_1 t}{2} = \frac{\pi}{2}$.

$$|\psi(t)\rangle = \cos(\pi/2)\,|{+z}\rangle - i\sin(\pi/2)\,|{-z}\rangle = \boxed{-i\,|{-z}\rangle}$$

$$|\psi(t)\rangle = \boxed{-ie^{i\omega t/2}\,|{-z}\rangle}$$

This makes the probability of measuring spin-down in the z-direction 1, which affirms that a spin flip has occurred.

All done!