Advanced Mathematical Techniques
Bookwork 1

1. Problem 5.1.1: Show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$.
   Hint: Show by induction that $s_m = \frac{m}{2m+1}$

   When $m = 1$,
   $$s_1 = \frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{3}$$
   When $m = k$,
   $$s_k = \frac{k}{2k+1}$$
   When $m = k+1$,
   $$s_{k+1} = s_k + \frac{1}{(2(k+1)-1)(2(k+1)+1)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$
   $$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2(k+1)+1}$$
   $$\lim_{k\to\infty} \frac{k+1}{2(k+1)+1} = \boxed{\frac{1}{2}}$$

2. Problem 5.2.7: Test the following for convergence.

   (a) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

   By the limit comparison test:
   $$a_n = \frac{1}{n(n+1)}$$
   $$b_n = \frac{1}{n^2}$$
   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n(n+1)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n(n+1)} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$
   $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$), therefore $\sum_{n=1}^{\infty} a_n$ also $\boxed{\text{converges}}$.

   (b) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

   By the integral test:
   $$f(x) = \frac{1}{x \ln x}$$
   For $x \geq 2$, $f(x)$ is positive, continuous and decreasing.
   $$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t\to\infty} \int_2^t \frac{1}{x \ln x} dx = \lim_{t\to\infty} [\ln(\ln x)]_2^t = \lim_{t\to\infty} (\ln(\ln t) - \ln(\ln 2)) = +\infty$$
   Thus the series $\boxed{\text{diverges}}$.

(c) $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

By the direct comparison test:

$$n \geq 1 \text{ (since } n = 1 \text{ is the lower bound of the sum)}$$

$$n \cdot 2^n \geq 2^n$$
$$\frac{1}{n \cdot 2^n} \leq \frac{1}{2^n}$$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, therefore, $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ also $\boxed{\text{converges}}$.

(d) $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$

Using a telescoping series:

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) = \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} [\ln(n+1) - \ln(n)]$$

$$= \ln(2) - \ln(1) + \ln(3) - \ln(2) + \ln(4) - \ln(3) + \cdots = \lim_{n\to\infty} \ln(n+1) = +\infty$$

Thus the series $\boxed{\text{diverges}}$.

(e) $\sum_{n=1}^{\infty} \frac{1}{n \cdot n^{\frac{1}{n}}}$ By the limit comparison test:

$$a_n = \frac{1}{n \cdot n^{\frac{1}{n}}}, b_n = \frac{1}{n}, \ln b_n = \frac{1}{n} \ln n$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$
$$\lim_{n\to\infty} b_n = e^0 = 1$$

$$\therefore\ a_n \text{ behaves like } \frac{1}{n} \text{ for large } n$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, therefore $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{diverges}}$.

3. Problem 5.2.10: Given that a pockeet calculator yields $\sum_{n=1}^{\infty} n^{-3} = 1.202007$, show that:

$$1.202056 \leq \sum_{n=1}^{\infty} n^{-3} \leq 1.202057.$$

Hint: Use integrals to set lower and upper bounds on $\sum_{n=101}^{\infty} n^{-3}$

By the integral test:

$$\int_{101}^{\infty} \frac{1}{x^3} dx \leq \sum_{n=101}^{\infty} \frac{1}{n^3} \leq \int_{100}^{\infty} \frac{1}{x^3} dx$$

$$\int_{101}^{\infty} \frac{1}{x^3} dx = \left[-\frac{1}{2x^2}\right]_{101}^{\infty} = \frac{1}{2 \cdot 101^2} \approx 0.000049$$

$$\int_{100}^{\infty} \frac{1}{x^3} dx = \left[-\frac{1}{2x^2}\right]_{100}^{\infty} = \frac{1}{2 \cdot 100^2} = 0.00005$$

$$\therefore 0.000049 \leq \sum_{n=101}^{\infty} \frac{1}{n^3} \leq 0.00005$$

$$\therefore 1.202007 + 0.000049 \leq \sum_{n=1}^{\infty} n^{-3} \leq 1.202007 + 0.00005$$

$$\boxed{1.202056 \leq \sum_{n=1}^{\infty} n^{-3} \leq 1.202057}$$

4. Problem 5.6.14: The relativistic sum ω of two velocities u and v is given by:

$$\frac{\omega}{c} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}}$$

If:

$$\frac{v}{c} = \frac{u}{c} = 1 - \alpha,$$

where $0 \leq \alpha \leq 1$, find $\frac{\omega}{c}$ in power of α through terms of α^3.

$$\frac{\omega}{c} = \frac{1 - \alpha + 1 - \alpha}{1 + (1-\alpha)^2} = \frac{2(1-\alpha)}{1 + (1-\alpha)^2}$$

$$= 1 - \frac{\alpha^2}{2 - 2\alpha + \alpha^2} = 1 - \frac{1}{2}\alpha^2 \left[\frac{1}{1 - (\alpha - \frac{\alpha^2}{2})}\right]$$

$$= 1 - \frac{1}{2}\alpha^2 \left[1 + \left(\alpha - \frac{\alpha^2}{2}\right) + \left(\alpha - \frac{\alpha^2}{2}\right)^2 + \cdots\right]$$

$$= 1 - \frac{1}{2}\alpha^2 \left[1 + \alpha + \frac{\alpha^2}{2} - \alpha^3 + \cdots\right]$$

Thus $\boxed{\frac{\omega}{c} \approx 1 - \frac{1}{2}\alpha^2 - \frac{1}{2}\alpha^3}$.

5. Problem 5.6.17: In a head-on proton-proton collision, the ratio of the kinetic energy in the center of mass system to the incident kinetic energy is:

$$R = \frac{\sqrt{2mc^2(E_k + 2mc^2)} - 2mc^2}{E_k}$$

Find the value of this ratio of kinetic energies for:

(a) $E_k \ll mc^2$ (nonrelativistic)

$$R = \frac{\sqrt{2mc^2 E_k + 4m^2c^4} - 2mc^2}{E_k} = 2mc^2\sqrt{1 + \frac{E_k}{2mc^2}}$$

$$E_k \ll mc^2 \Rightarrow \frac{E_k}{2mc^2} \text{ is very small.}, \therefore \approx 1 + \frac{E_k}{4mc^2}$$

$$\therefore\ R \approx \frac{2mc^2 + \frac{E_k}{2} - 2mc^2}{E_k} = \boxed{\frac{1}{2}}$$

(b) $E_k \gg mc^2$ (extreme-relativistic)

$$\sqrt{2mc^2 E_k + 4m^2c^4} \approx \sqrt{2mc^2 E_k}$$

$$R \approx \frac{\sqrt{2mc^2 E_k} - 2mc^2}{E_k} \approx \frac{\sqrt{2mc^2 E_k}}{E_k} = \sqrt{\frac{2mc^2}{E_k}}$$

As the disparity between E_k and mc^2 becomes larger, $R \approx -\frac{2mc^2}{E_k} \Rightarrow \boxed{0}$