

Advanced Mathematical Techniques (AMT)

Notes

1 Recap

Remember that we can establish convergence of a series if we can make the "tail" (the ending) of that series small. $\sum_n = 1^\infty a_n$ converges if and only if given any $\varepsilon > 0 \in \mathbb{R}$ such that:

$$\left| \sum_{n=N+1}^{N+P} a_n \right| < \varepsilon \forall P \in \mathbb{N}$$

2 Today's stuff

Going back to BC, let's recall that the series cannot converge if the limit of the series is greater than 0. That is, if there's a series which has a limit of 1, say, then it will not converge because it will end up diverging to ∞ . Further recall the definition of absolute convergence, where:

$$c_n \text{ converges absolutely if } \sum_{n=1}^{\infty} c_n \text{ converges.}$$

Clearly, if a series is absolutely convergent, then it is convergent (duh). Recall further the triangle inequality, where for any two complex numbers, $|z| - |w| \leq |z + w| \leq |z| + |w|$. Let's also recall the direct comparison test, where if we have a sequence $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$, where both are sequences of positive numbers with $a_n \geq b_n \forall n$. If $\sum a_n$ converges, then so does $\sum b_n$ - if $\sum b_n$ diverges, then so does $\sum a_n$. If $\sum a_n$ diverges, then you know nothing about $\sum b_n$. You need to have the inequality in the right way to use it. That's why more often, the test used is the limit comparison test, where if $\lim_{n \rightarrow \infty} \left| \frac{a_n}{b_n} \right| = L \neq 0$, then both either converge or diverge. They do whatever they do together. A proof of this follows:

Since $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$, given any $\varepsilon > 0$ in \mathbb{R} such that $L - \varepsilon < \frac{a_n}{b_n} < L + \varepsilon$. This means that $(L - \varepsilon)b_n < a_n < (L + \varepsilon)b_n$, for every $n > N$. Further, for any $M > N$, we have:

$$(L - \varepsilon) \sum_{n=M+1}^{M+P} b_n < \sum_{n=M+1}^{M+P} a_n < (L + \varepsilon) \sum_{n=M+1}^{M+P} b_n$$

Now, if $\sum_{n=1}^{\infty} a_n$ converges, that means that the tail of the a_n can be as small as I want. That means that the term before it must also be (as long as L is not 0) very small. Thus, the series $\sum_{n=1}^{\infty} b_n$ converges. However, if $\sum_{n=1}^{\infty} a_n$ diverges, then you can prove that $\sum_{n=1}^{\infty} b_n$ also diverges by the same logic, just using the right side instead of the left. If you know stuff about b_n rather than a_n , you can just consider the reciprocal of L instead (which we can only do because $L \neq 0$). We could also just use the inequality from the other side.

Now, let's consider the integral comparison test, where, given the positive sequence $\{a_n\}_{n=1}^{\infty}$, if a decreasing, continuous function can be found for which $f(n) = a_n$ for all n , then:

$$\int_{N+1}^{N+P+1} f(x) dx < \sum_{n=N+1}^{N+P} a_n < \int_N^{N+P} f(x) dx$$

In other words, the improper integral and the infinite series either both converge or both diverge. We can only make these "both" statements if we have 3 sides to this inequality. Also, we can do a lot more than just divergence proving - for instance,

$$\int_{n+1}^{m+1} f(x) dx < \sum_{k=n+1}^m a_k < \int_n^m f(x) dx$$

Let's say we want to estimate $H_{1 \text{ trillion}}$, where H_n represents the sum of the harmonic numbers up until that point. In order to estimate $H_{10^{12}}$ using the fact that $H_{10} = 2.92\dots$

$$\int_{11}^{10^{12}+1} \frac{dx}{x} < \sum_{k=11}^{10^{12}} \frac{1}{k} < \int_{10}^{10^{12}} \frac{dx}{x}$$

$$\ln \frac{10^{12}+1}{11} < \sum_{k=11}^{10^{12}} \frac{1}{k} < \ln 10^{11}$$

Using a calculator and dropping the +1,

$$25.233 < \sum_{k=11}^{10^{12}} \frac{1}{k} < 25.438$$

Thus, considering the first estimation of H_{10} as 2.928, we get:

$$28.162 < H_{10^{12}} < 28.2574$$

Turns out, the bounds are always about 0.1 apart in the long run.