Advanced Mathematical Techniques
Problem Set 1

1. We showed in class that the harmonic sequence defined by $H_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges. We start with the sum of the first thousand terms, $H_{1000} \approx 7.48547$.

   (a) Use the integral comparison inequality to bound the value of $H_{10^{12}}$. How far apart are your bounds?

   We are estimating $H_{10^{12}} = H_{1000} + \sum_{k=1001}^{10^{12}} \frac{1}{k}$. The Integral Comparison Test states that for a decreasing function $f(x) = 1/x$:

$$\int_{N+1}^{M+1} f(x)\,dx < \sum_{k=N+1}^{M} a_k < \int_{N}^{M} f(x)\,dx$$

   Here $N = 1000$ and $M = 10^{12}$.
   Upper Bound:

$$\int_{1000}^{10^{12}} \frac{1}{x}\,dx = \ln\left(10^{12}\right) - \ln(1000) = 12\ln(10) - 3\ln(10) = 9\ln(10)$$

$$\approx 9(2.302585) \approx 20.72327$$

$$H_{10^{12}} < 7.48547 + 20.72327 = 28.20874$$

   Lower Bound:

$$\int_{1001}^{10^{12}+1} \frac{1}{x}\,dx = \ln\left(10^{12}+1\right) - \ln(1001) \approx 27.63102 - 6.90875 = 20.72227$$

$$H_{10^{12}} > 7.48547 + 20.72227 = 28.20774$$

   Difference: The bounds are separated by approximately $\int_{1000}^{1001} \frac{1}{x} dx \approx \frac{1}{1000} = \boxed{0.001}$.

   (b) If one term was added each second since the beginning of the universe (13.7 billion years), bound the resulting sum. Use 3.155×10^7 s/year. How far apart are your bounds?

   Step 1: Calculate N.

$$N = 13.7 \times 10^9 \times 3.155 \times 10^7 = 43.2235 \times 10^{16} = 4.32235 \times 10^{17}$$

   Step 2: Set up bounds for sum starting at k=1. Using $\int_1^{N+1} \frac{1}{x} dx < H_N < 1 + \int_1^{N} \frac{1}{x} dx$:
   Lower Bound:

$$\ln(N+1) \approx \ln\left(4.322 \times 10^{17}\right) = \ln(4.322) + 17\ln(10)$$

$$\approx 1.4638 + 39.1439 = \boxed{40.608}$$

   Upper Bound:

$$1 + \ln(N) \approx 1 + 40.608 = \boxed{41.608}$$

   Difference:

$$\text{Diff} = (1 + \ln N) - \ln(N+1) = 1 - \ln\left(\frac{N+1}{N}\right) = 1 - \ln\left(1 + \frac{1}{N}\right)$$

Since N is very large, $\ln(1 + 1/N) \approx 0$.

$$\text{Diff} \approx \boxed{1}$$

(c) If one *trillion* terms were added each second, bound the resulting sum. How far apart are your bounds?

Step 1: Calculate N'.

$$N' = N \times 10^{12} \approx 4.32235 \times 10^{29}$$

Step 2: Calculate Bounds.

$$\ln(N') = \ln(4.322) + 29\ln(10) \approx 1.4638 + 66.7750 = 68.239$$

Lower Bound: $\ln(N' + 1) \approx \boxed{68.239}$ **Upper Bound:** $1 + \ln(N') \approx \boxed{69.239}$
Difference: As in part (b), the difference is $1 - \ln(1 + 1/N') \approx \boxed{1}$.

(d) Estimate the number of terms for the sum to exceed 100.

Using the approximation $H_n \approx \ln n + \gamma$, where $\gamma \approx 0.577$:

$$\ln n + 0.577 = 100 \implies \ln n = 99.423$$

Convert to base 10 to put in scientific notation:

$$n = e^{99.423}$$

$$\log_{10} n = 99.423 \times \log_{10} e \approx 99.423 \times 0.4343 = 43.179$$
$$n = 10^{0.179} \times 10^{43}$$
$$10^{0.179} \approx 1.510$$

Answer: $\boxed{1.510 \times 10^{43}}$

(e) Estimate the number of terms for the sum to exceed 1000.

$$\ln n \approx 1000$$
$$\log_{10} n = 1000 \times 0.43429 = 434.29$$
$$n = 10^{0.29} \times 10^{434}$$
$$10^{0.29} \approx 1.95$$

Answer: $\boxed{1.95 \times 10^{434}}$

(f) Explain what is meant by the statement that the harmonic series diverges, but it does not do so *quickly*.

Divergence means the partial sums H_n grow without bound ($H_n \to \infty$ as $n \to \infty$). However, the rate of growth is logarithmic. As demonstrated in parts (b) and (c), even increasing the number of terms by a factor of a trillion (10^{12}) only adds a constant amount (≈ 27.6) to the sum. It takes exponentially more time to reach higher sums; reaching 100 takes 10^{43} terms, while reaching 1000 takes 10^{434} terms.

(g) Glacial sequence $G_n = \sum_{k=2}^{n} \frac{1}{k \ln k}$. Given $G_{1000} = 2.7274$, bound $G_{10^{12}}$.

Let $f(x) = \frac{1}{x \ln x}$. Use substitution $u = \ln x, du = \frac{1}{x}dx$.

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{u} du = \ln|u| = \ln(\ln x)$$

Bounds for the sum from 1001 to 10^{12}:

$$\text{Upper} = 2.7274 + [\ln(\ln x)]_{1000}^{10^{12}}$$
$$= 2.7274 + \ln\left(\ln 10^{12}\right) - \ln(\ln 1000) = 2.7274 + \ln(12 \ln 10) - \ln(3 \ln 10)$$
$$= 2.7274 + \ln\left(\frac{12}{3}\right) = 2.7274 + \ln 4 \approx 2.7274 + 1.3863 = \boxed{4.1137}$$

The lower bound integrates from 1001 to $10^{12} + 1$, which yields virtually the same number to 4 decimal places (4.1136).

(h) Bound sum for 1 term/sec ($N \approx 4.32 \times 10^{17}$). How far apart are bounds?

Bounds for sum starting at $k = 2$:

$$\int_2^{N+1} f(x)dx < G_N < a_2 + \int_2^{N} f(x)dx$$

Estimate using approximation $G_N \approx \ln(\ln N)$.

$$G_N \approx \ln\left(\ln\left(4.32 \times 10^{17}\right)\right) = \ln(40.6) \approx \boxed{3.704}$$

Difference: The difference between the standard integral test bounds is $a_2 - \int_N^{N+1} f(x)dx$. Since $f(N)$ is negligible, the difference is approximately $a_2 = \frac{1}{2 \ln 2} \approx \boxed{0.721}$.

(i) Bound sum for 1 trillion terms/sec ($N' \approx 4.32 \times 10^{29}$). How far apart are bounds?

$$G_{N'} \approx \ln\left(\ln\left(4.32 \times 10^{29}\right)\right) = \ln(68.2) \approx \boxed{4.223}$$

Difference: The gap between the bounds relies on the first term a_2, so it remains approximately $\boxed{0.721}$.

(j) Terms to exceed 100. Format $10^{a \times 10^n}$.

$$\ln(\ln x) \approx 100 \implies \ln x = e^{100} \implies x = e^{e^{100}}$$

Taking $\log_{10}$ twice:

$$\log_{10} x = e^{100} \log_{10} e$$
$$\log_{10}(\log_{10} x) = \log_{10}(e^{100} \log_{10} e) = 100 \log_{10} e + \log_{10}(\log_{10} e)$$
$$= 43.4294 + \log_{10}(0.4343) = 43.4294 - 0.3622 = 43.067$$

So $\log_{10} x = 10^{43.067} = 10^{0.067} \times 10^{43} \approx 1.167 \times 10^{43}$.

$$x = \boxed{10^{1.167 \times 10^{43}}}$$

2. The integral comparison inequality used to estimate convergent series.

(a) Given $\sum_{k=2}^{100} \frac{\ln^3 k}{k^2} = 4.0558...$, show convergence and bound value. How far apart are bounds?

We must evaluate $\int \frac{\ln^3 x}{x^2} dx$. Use Integration by Parts: $u = (\ln x)^3, dv = x^{-2} dx$.

$$du = 3(\ln x)^2 \frac{1}{x} dx, \quad v = -x^{-1}$$

$$\int = -\frac{\ln^3 x}{x} + \int \frac{3 \ln^2 x}{x^2} dx$$

Repeating this process 3 times yields:

$$F(x) = -\frac{\ln^3 x + 3 \ln^2 x + 6 \ln x + 6}{x}$$

We bound the tail $R_{100} = \sum_{101}^{\infty} a_k$:

$$\int_{101}^{\infty} f(x) dx < R_{100} < \int_{100}^{\infty} f(x) dx$$

Evaluating the upper bound at $x = 100$:

$$\int_{100}^{\infty} = 0 - F(100) = \frac{(\ln 100)^3 + 3(\ln 100)^2 + 6 \ln 100 + 6}{100}$$

Using $\ln 100 \approx 4.60517$:

$$\approx \frac{97.68 + 63.62 + 27.63 + 6}{100} = \frac{194.93}{100} \approx 1.9493$$

Total Sum $\approx 4.0558 + 1.9493 = \boxed{6.0051}$. **Difference:** The bounds differ by $\int_{100}^{101} f(x) dx \approx f(100)$.

$$f(100) = \frac{\ln^3 100}{100^2} \approx \frac{97.68}{10000} \approx \boxed{0.0098}$$

(b) Given $\sum_{k=2}^{100} \frac{1}{k \ln^2 k} = 1.8928...$, show convergence and bound value. How far apart are bounds?

Evaluate $\int_{100}^{\infty} \frac{dx}{x \ln^2 x}$. Let $u = \ln x$.

$$\int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\ln x}$$

$$\text{Tail Upper Bound} = \left[-\frac{1}{\ln x} \right]_{100}^{\infty} = \frac{1}{\ln 100} \approx \frac{1}{4.605} \approx 0.2171$$

Total Sum $\approx 1.8928 + 0.2171 = \boxed{2.110}$. **Difference:** $\approx f(100) = \frac{1}{100(\ln 100)^2} \approx \frac{1}{100(21.2)} \approx \boxed{0.00047}$.

(c) Significance of contributions from $k > 100$.

For part (a), the tail (1.95) is nearly 50% the size of the partial sum (4.05), meaning calculating only 100 terms gives a 33% error relative to the true total. For part (b), the tail (0.217) is about 10% of the total. In both cases, the contributions are **significant**, and the integral comparison is necessary to get a reasonably accurate result without summing millions of terms.

3. Given $f(x) = \int_0^x \frac{1 - \cos(2t^2)}{t^3} dt$.

(a) Write the Taylor expansion of the integrand centered at $t = 0$.

Standard series: $\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \dots$ Substitute $u = 2t^2$:

$$\cos(2t^2) = 1 - \frac{(2t^2)^2}{2} + \frac{(2t^2)^4}{24} - \frac{(2t^2)^6}{720} + \dots$$

$$= 1 - 2t^4 + \frac{16t^8}{24} - \frac{64t^{12}}{720} + \dots = 1 - 2t^4 + \frac{2}{3}t^8 - \frac{4}{45}t^{12} + \dots$$

Now form the integrand $\frac{1-\cos(2t^2)}{t^3}$:

$$\frac{1 - (1 - 2t^4 + \frac{2}{3}t^8 - \frac{4}{45}t^{12})}{t^3} = \frac{2t^4 - \frac{2}{3}t^8 + \frac{4}{45}t^{12}}{t^3}$$

$$\boxed{2t - \frac{2}{3}t^5 + \frac{4}{45}t^9 - \dots}$$

(b) Integrate term-by-term for $f(x)$.

$$f(x) = \int_0^x \left(2t - \frac{2}{3}t^5 + \frac{4}{45}t^9\right) dt$$

$$= \left[t^2 - \frac{2}{3} \cdot \frac{t^6}{6} + \frac{4}{45} \cdot \frac{t^{10}}{10}\right]_0^x$$

$$= \boxed{x^2 - \frac{1}{9}x^6 + \frac{2}{225}x^{10} - \dots}$$

(c) Determine $f^{(10)}(0)$.

The Maclaurin series definition is $\sum \frac{f^{(n)}(0)}{n!}x^n$. The coefficient of x^{10} in our series is $\frac{2}{225}$.

$$\frac{f^{(10)}(0)}{10!} = \frac{2}{225} \implies f^{(10)}(0) = \frac{2 \cdot 10!}{225}$$

$$f^{(10)}(0) = \frac{2 \cdot 3,628,800}{225} = \boxed{32,256}$$

(d) Determine $f^{(20)}(0)$.

The powers of x in the series are $2, 6, 10, 14, 18, 22 \dots$ (arithmetic progression $4k - 2$). Since 20 is not in this sequence, the coefficient of x^{20} is 0.

$$\boxed{f^{(20)}(0) = 0}$$

4. Given $g(x) = x^2 \int_0^x \frac{\ln(5+2t^3) - \ln 5}{t^3} dt$.

(a) Taylor expansion of integrand.

Simplify the log term: $\ln(5+2t^3) - \ln 5 = \ln\left(\frac{5+2t^3}{5}\right) = \ln\left(1+\frac{2}{5}t^3\right)$. Use series $\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} \dots$ with $u = \frac{2}{5}t^3$.

$$\ln\left(1+\frac{2}{5}t^3\right) = \left(\frac{2}{5}t^3\right) - \frac{1}{2}\left(\frac{2}{5}t^3\right)^2 + \frac{1}{3}\left(\frac{2}{5}t^3\right)^3 \dots$$

$$= \frac{2}{5}t^3 - \frac{2}{25}t^6 + \frac{8}{375}t^9 - \dots$$

Divide by t^3:

$$\text{Integrand} = \boxed{\frac{2}{5} - \frac{2}{25}t^3 + \frac{8}{375}t^6 - \dots}$$

(b) Determine $g(x)$ series.

Integrate the result from (a):

$$\int_0^x \left(\frac{2}{5} - \frac{2}{25}t^3 + \frac{8}{375}t^6\right) dt = \frac{2}{5}x - \frac{2}{25}\frac{x^4}{4} + \frac{8}{375}\frac{x^7}{7}$$

$$= \frac{2}{5}x - \frac{1}{50}x^4 + \frac{8}{2625}x^7$$

Multiply by x^2:

$$g(x) = x^2\left(\frac{2}{5}x - \frac{1}{50}x^4 + \dots\right) = \boxed{\frac{2}{5}x^3 - \frac{1}{50}x^6 + \frac{8}{2625}x^9 - \dots}$$

(c) Determine $g^{(11)}(0)$.

The powers of x are $3, 6, 9, 12 \dots$ (multiples of 3). Since 11 is not a multiple of 3, the coefficient is 0.

$$\boxed{g^{(11)}(0) = 0}$$

(d) Determine $g^{(18)}(0)$.

This term comes from the power x^{18}. Working backwards: x^{18} in $g(x)$ comes from x^{16} in the integral, which comes from t^{15} in the integrand. In the $\ln(1+u)$ expansion, we need the term where $u^k \propto t^{15}$. Since $u \propto t^3$, we need u^5. Wait, $u^6 \propto t^{18}$. Let's re-trace. Log term: $c_k(\frac{2}{5}t^3)^k$. Integrand (divide by t^3): t^{3k-3}. Integral: x^{3k-2}. Times x^2: x^{3k}. We need $3k = 18 \implies k = 6$. The 6th term of $\ln(1+u)$ is $-\frac{u^6}{6}$.

$$\text{Term in Log} = -\frac{1}{6}\left(\frac{2}{5}t^3\right)^6 = -\frac{1}{6}\left(\frac{2}{5}\right)^6 t^{18}$$

$$\text{Integrand} = -\frac{1}{6}\left(\frac{2}{5}\right)^6 t^{15}$$

$$\text{Integral} = -\frac{1}{6}\left(\frac{2}{5}\right)^6 \frac{x^{16}}{16}$$

$$g(x)\text{ term} = -\frac{1}{96}\left(\frac{2}{5}\right)^6 x^{18}$$

Coefficient $C_{18} = -\frac{1}{96}\frac{64}{15625} = -\frac{2}{3\cdot 15625} = -\frac{2}{46875}$.

$$g^{(18)}(0) = 18! \cdot C_{18} = \boxed{-\frac{2\cdot 18!}{46875}}$$

5. Determine first four nonzero terms of $h(x) = \frac{x}{1-x+x^2}$ and radius of convergence.

   Recognize the denominator as part of sum of cubes: $(1+x)(1-x+x^2) = 1+x^3$. Multiply numerator and denominator by $(1+x)$:

   $$h(x) = \frac{x(1+x)}{1+x^3} = (x+x^2)\frac{1}{1-(-x^3)}$$

   Use geometric series $\frac{1}{1-u} = 1+u+u^2+\dots$ where $u = -x^3$.

   $$h(x) = (x+x^2)(1-x^3+x^6-x^9+\dots)$$

   Expand:

   $$\begin{aligned} &= x(1-x^3+x^6) + x^2(1-x^3+x^6) \\ &= x - x^4 + x^7 + x^2 - x^5 + x^8 \end{aligned}$$

   Reorder by power:

   $$\boxed{x + x^2 - x^4 - x^5 + \dots}$$

   Radius of Convergence: The radius is the distance from the center (0) to the nearest singularity (roots of denominator).

   $$1 - x + x^2 = 0 \implies x = \frac{1\pm\sqrt{1-4}}{2} = \frac{1\pm i\sqrt{3}}{2}$$

   The magnitude is $|x| = \sqrt{(1/2)^2 + (\sqrt{3}/2)^2} = \sqrt{1/4+3/4} = 1$.

   $$\boxed{R = 1}$$

6. Consider sequence $\{a_k\}$ and $S_n = \sum a_k$. Assume S_n diverges.

   (a) Explain why $\{S_n\}$ is monotonic increasing.

   Given that $\{a_k\}$ are positive numbers ($a_k > 0$), the partial sum $S_{n+1} = S_n + a_{n+1}$ must be strictly greater than S_n. Thus, the sequence is monotonic increasing.

   (b) Explain why $\sum \frac{a_n}{S_n}$ diverges.

   Consider a block of terms from $N+1$ to $N+P$.

   $$\begin{aligned} \sum_{n=N+1}^{N+P} \frac{a_n}{S_n} &> \sum_{n=N+1}^{N+P} \frac{a_n}{S_{N+P}} \quad (\text{since } S_n < S_{N+P}) \\ &= \frac{1}{S_{N+P}} \sum_{n=N+1}^{N+P} a_n = \frac{S_{N+P} - S_N}{S_{N+P}} = 1 - \frac{S_N}{S_{N+P}} \end{aligned}$$

   Since S_n diverges to ∞, for any fixed N, we can choose P large enough such that $S_{N+P} \gg S_N$, making the ratio $\frac{S_N}{S_{N+P}}$ close to 0. Thus, the sum of the block is $> 1/2$ (or close to 1). Since the tail does not vanish, the series diverges.

(c) Result for Harmonic sequence?

$a_n = 1/n, S_n \approx \ln n$. The derived series is $\sum \frac{1}{n \ln n}$, which is the Glacial series discussed earlier (divergent).

(d) Explain why there is no 'slowest diverging series'.

For any divergent series $\sum a_n$, we can construct a new series $\sum \frac{a_n}{S_n}$ that also diverges. Because $S_n \to \infty$, the terms $\frac{a_n}{S_n}$ are much smaller than a_n, meaning the new series diverges "slower". We can apply this transformation recursively ($\sum \frac{a_n}{S_n \ln S_n}$, etc.) infinitely many times.

(e) Explain why $\frac{a_n}{S_n^2} < \frac{1}{S_{n-1}} - \frac{1}{S_n}$.

$$\frac{1}{S_{n-1}} - \frac{1}{S_n} = \frac{S_n - S_{n-1}}{S_{n-1}S_n} = \frac{a_n}{S_{n-1}S_n}$$

Since $S_n > S_{n-1}$, we have $S_n^2 > S_{n-1}S_n$. Therefore, $\frac{a_n}{S_n^2} < \frac{a_n}{S_{n-1}S_n}$.

(f) Bound the series tail $\sum \frac{a_n}{S_n^2}$.

$$\sum_{n=N+1}^{\infty} \frac{a_n}{S_n^2} < \sum_{n=N+1}^{\infty} \left(\frac{1}{S_{n-1}} - \frac{1}{S_n} \right)$$

This is a telescoping sum:

$$= \left(\frac{1}{S_N} - \frac{1}{S_{N+1}} \right) + \left(\frac{1}{S_{N+1}} - \frac{1}{S_{N+2}} \right) + \cdots = \frac{1}{S_N} - \lim_{M \to \infty} \frac{1}{S_M}$$

Since $S_M \to \infty$, the limit is 0. The sum is bounded by $1/S_N$, so the series **converges**.

(g) What series do you get if you treat the Harmonic sequence in this way?

$a_n = 1/n, S_n \approx \ln n$. The series is $\sum \frac{1}{n(\ln n)^2}$. As shown in problem 2(b), this series converges.