Advanced Mathematical Techniques
Problem Set 1

1. We showed in class that the harmonic sequence defined by $H_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges. We start with the sum of the first thousand terms, $H_{1000} \approx 7.48547$.

   (a) Use the integral comparison inequality to bound the value of $H_{10^{12}}$. How far apart are your bounds?

   We estimate the total sum by adding the tail to the known partial sum:

$$H_{10^{12}} = H_{1000} + \sum_{k=1001}^{10^{12}} \frac{1}{k}$$

   Using the Integral Comparison Test for the sum from $M = 1000$ to $N = 10^{12}$ with $f(x) = \frac{1}{x}$:

$$H_{1000} + \int_{1001}^{10^{12}+1} \frac{1}{x} dx < H_{10^{12}} < H_{1000} + \int_{1000}^{10^{12}} \frac{1}{x} dx$$

   Upper Bound:

$$\text{Upper} = 7.48547 + [\ln x]_{1000}^{10^{12}} = 7.48547 + \ln\left(10^{12}\right) - \ln(1000)$$

$$= 7.48547 + 12\ln(10) - 3\ln(10) = 7.48547 + 9(2.302585) \approx \boxed{28.20874}$$

   Lower Bound:

$$\text{Lower} = 7.48547 + [\ln x]_{1001}^{10^{12}+1} = 7.48547 + \ln\left(10^{12}+1\right) - \ln(1001)$$

$$\approx 7.48547 + 27.63102 - 6.90875 \approx \boxed{28.20774}$$

   Difference: The difference between the bounds is app roximately the area of the first rectangle of the tail:

$$\text{Diff} \approx \int_{1000}^{1001} \frac{1}{x} dx \approx \frac{1}{1000} = \boxed{0.001}$$

   (b) If one term was added each second since the beginning of the universe (13.7 billion years), use the integral comparison inequality to bound the resulting sum. Use 3.155×10^7 s/year. How far apart are your bounds?

$$N = 13.7 \times 10^9 \times 3.155 \times 10^7 = 4.32235 \times 10^{17}$$

   We treat this as $H_N = H_{1000} + \sum_{1001}^{N} \frac{1}{k}$.

$$H_{1000} + \int_{1001}^{N+1} \frac{dx}{x} < H_N < H_{1000} + \int_{1000}^{N} \frac{dx}{x}$$

   Upper Bound:

$$\text{Upper} = 7.48547 + \ln\left(4.32235 \times 10^{17}\right) - \ln(1000)$$

$$= 7.48547 + 40.6083 - 6.9078 \approx \boxed{41.186}$$

Lower Bound:

$$\text{Lower} = 7.48547 + \ln\bigl(4.32235 \times 10^{17} + 1\bigr) - \ln(1001)$$
$$= 7.48547 + 40.6083 - 6.9088 \approx \boxed{41.185}$$

Difference: Because we started our integration at 1000, the bounds are very tight:

$$\text{Diff} \approx \frac{1}{1000} = \boxed{0.001}$$

(c) If one *trillion* terms were added each second, bound the resulting sum. How far apart are your bounds?

$$N' = N \times 10^{12} \approx 4.32235 \times 10^{29}$$

Upper Bound:

$$\text{Upper} = 7.48547 + \ln(N') - \ln(1000)$$
$$\ln(N') = \ln\bigl(4.322 \times 10^{29}\bigr) \approx 68.239$$
$$\text{Upper} = 7.48547 + 68.239 - 6.9078 \approx \boxed{68.817}$$

Lower Bound:

$$\text{Lower} = 7.48547 + \ln(N' + 1) - \ln(1001)$$
$$= 7.48547 + 68.239 - 6.9088 \approx \boxed{68.816}$$

Difference:

$$\text{Diff} \approx \boxed{0.001}$$

(d) Use the integral comparison inequality to estimate the number of terms that must be added for the sum to exceed 100.

Using the simplified integral estimation $\ln(n+1) \approx 100$:

$$n \approx e^{100} - 1$$
$$n \approx 2.688 \times 10^{43}$$

Answer: $\boxed{2.688 \times 10^{43}}$

(e) Use the integral comparison inequality to estimate the number of terms that must be added for the sum to exceed 1000.

Using $\ln(n+1) \approx 1000$:

$$n \approx e^{1000}$$
$$n \approx 10^{434.29} = \boxed{1.970 \times 10^{434}}$$

(f) Explain what is meant by the statement that the harmonic series diverges, but it does not do so *quickly*.

It means the sum grows without bound (diverges), but the rate of growth is logarithmic. As seen above, increasing the number of terms by a factor of a trillion (10^{12}) only increased the sum by ≈ 27.6.

(g) Glacial sequence $G_n = \sum_{k=2}^{n} \frac{1}{k \ln k}$. Given $G_{1000} = 2.7274$, bound $G_{10^{12}}$.

$$G_{1000} + \int_{1001}^{10^{12}+1} f(x)dx < G_{10^{12}} < G_{1000} + \int_{1000}^{10^{12}} f(x)dx$$

Upper Bound:

$$2.7274 + \ln\left(\ln 10^{12}\right) - \ln(\ln 1000)$$

$$= 2.7274 + \ln(12 \ln 10) - \ln(3 \ln 10) = 2.7274 + \ln(4) \approx \boxed{4.1137}$$

Lower Bound:

$$2.7274 + \ln\left(\ln\left(10^{12} + 1\right)\right) - \ln(\ln 1001) \approx \boxed{4.1136}$$

Difference:

$$\text{Diff} \approx f(1000) = \frac{1}{1000 \ln 1000} \approx \frac{1}{6907} \approx \boxed{0.00014}$$

(h) Bound sum for 1 term/sec ($N \approx 4.32 \times 10^{17}$). How far apart are bounds?

Upper Bound:

$$2.7274 + \ln(\ln N) - \ln(\ln 1000)$$

$$\ln(\ln N) \approx \ln(40.608) \approx 3.704$$

$$\ln(\ln 1000) \approx 1.933$$

$$\text{Sum} \approx 2.7274 + 3.704 - 1.933 \approx \boxed{4.498}$$

Difference: Since we start at 1000, the difference is $f(1000)$:

$$\text{Diff} \approx \boxed{0.00014}$$

(i) Bound sum for 1 trillion terms/sec ($N' \approx 4.32 \times 10^{29}$). How far apart are bounds?

Upper Bound:

$$2.7274 + \ln(\ln N') - \ln(\ln 1000)$$

$$\ln(\ln N') \approx \ln(68.239) \approx 4.223$$

$$\text{Sum} \approx 2.7274 + 4.223 - 1.933 \approx \boxed{5.017}$$

Difference:

$$\text{Diff} \approx f(1000) \approx \boxed{0.00014}$$

$$\int_{2}^{n+1} \frac{1}{x \ln x} dx \leq \sum_{k=2}^{n} \frac{1}{k \ln k}$$

$$\ln(\ln(n+1)) - \ln(\ln(2)) \leq 100$$
$$\ln(\ln(n+1)) - (-0.3665) \leq 100$$
$$\ln(\ln(n+1)) \leq 99.6335$$
$$\ln(n+1) \leq e^{99.6335}$$
$$n \approx e^{e^{99.6335}}$$
$$\log_{10}(10^x) = \log_{10}\left(e^{e^{99.6335}}\right)$$
$$x = e^{99.6335} \cdot \log_{10}(e)$$
$$x \approx (2.688 \times 10^{43}) \cdot (0.4343)$$
$$\boxed{x \approx 1.167 \cdot 10^{43}}$$

2. The integral comparison inequality used to estimate convergent series.

   (a) Given $\sum_{k=2}^{100} \frac{\ln^3 k}{k^2} = 4.0558...$, show convergence and bound value. How far apart are bounds?

   (b) Given $\sum_{k=2}^{100} \frac{1}{k \ln^2 k} = 1.8928...$, show convergence and bound value. How far apart are bounds?

   $\int \frac{1}{x \ln^2 x} dx = -\frac{1}{\ln x}$.
   Upper Tail: $\left[-\frac{1}{\ln x}\right]_{100}^{\infty} = \frac{1}{\ln 100} \approx 0.2171$. Total Sum $\approx 1.8928 + 0.2171 = \boxed{2.110}$.
   Difference:
   $$\text{Diff} \approx f(100) = \frac{1}{100(\ln 100)^2} \approx \boxed{0.00047}$$

   (c) Significance of contributions from $k > 100$.

   In (a), the tail (1.95) is $\approx$ 32% of the total. In (b), the tail (0.217) is $\approx$ 10%. These are significant contributions; the partial sum S_{100} is not a good approximation on its own.

3. Given $f(x) = \int_0^x \frac{1-\cos(2t^2)}{t^3} dt$.

   (a) Taylor expansion of integrand.

   $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n} t^{4n-3}}{(2n)!}$$

   (b) Integrate term-by-term for $f(x)$.

   $$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n}}{(2n)!(4n-2)} x^{4n-2}$$

(c) Determine $f^{(10)}(0)$.

Coefficient of x^{10} is $\frac{2}{225}$. Taylor term is $\frac{f^{(10)}(0)}{10!}x^{10}$.

$$f^{(10)}(0) = \frac{2 \cdot 10!}{225} = \boxed{32,256}$$

(d) Determine $f^{(20)}(0)$.

Powers are $4k-2$ (2, 6, 10, 14, 18, 22...). 20 is not in the series.

$$\boxed{f^{(20)}(0) = 0}$$

4. Given $g(x) = x^2 \int_0^x \frac{\ln\left(5+2t^3\right)-\ln 5}{t^3} dt$.

(a) Taylor expansion of integrand.

$$\frac{\ln\left(1+\frac{2}{5}t^3\right)}{t^3} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{2}{5}\right)^n t^{3n-3}$$

(b) Determine $g(x)$ series.

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(3n-2)} \left(\frac{2}{5}\right)^n x^{3n}$$

(c) Determine $g^{(11)}(0)$.

Powers are multiples of 3. 11 is not a multiple of 3.

$$\boxed{0}$$

(d) Determine $g^{(18)}(0)$.

Corresponds to $n=6$ term in $\ln(1+u)$ expansion ($u^6 \propto t^{18}$).

$$C_{18} = -\frac{1}{6}(\frac{2}{5})^6 \cdot \frac{1}{16} = -\frac{2}{46875}$$

$$g^{(18)}(0) = 18! \cdot C_{18} = \boxed{-\frac{2 \cdot 18!}{46875}}$$

5. Determine first four nonzero terms of $h(x) = \frac{x}{1-x+x^2}$ and radius of convergence.

Use geometric series on $\frac{x(1+x)}{1+x^3} = (x+x^2)(1-x^3+x^6-\dots)$.

$$x + x^2 - x^4 - x^5 + \dots$$

Terms: $\boxed{x, x^2, -x^4, -x^5}$. Roots of denominator $1-x+x^2$ are $e^{\pm i\pi/3}$. Magnitude is 1.

$$\boxed{R = 1}$$

6. Consider sequence $\{a_k\}$ and $S_n = \sum a_k$. Assume S_n diverges.

   (a) Explain why $\{S_n\}$ is monotonic increasing.

   $a_k > 0 \implies S_{n+1} = S_n + a_{n+1} > S_n$.

   (b) Explain why $\sum \frac{a_n}{S_n}$ diverges.

   $\sum_N^{N+P} \frac{a_n}{S_n} > \frac{S_{N+P}-S_N}{S_{N+P}} \to 1$. Fails Cauchy criterion.

   (c) Result for Harmonic sequence?

   $\sum \frac{1}{n \ln n}$. Diverges.

   (d) Explain why there is no 'slowest diverging series'.

   Dividing by partial sum always yields a slower divergent series. Process can be repeated infinitely.

   (e) Explain why $\frac{a_n}{S_n^2} < \frac{1}{S_{n-1}} - \frac{1}{S_n}$.

   $\frac{1}{S_{n-1}} - \frac{1}{S_n} = \frac{a_n}{S_{n-1}S_n}$. Since $S_n > S_{n-1}$, $S_n^2 > S_{n-1}S_n$, so $\frac{a_n}{S_n^2} < \frac{a_n}{S_{n-1}S_n}$.

   (f) Bound the series tail $\sum \frac{a_n}{S_n^2}$.

   Telescoping sum $\sum(\frac{1}{S_{n-1}} - \frac{1}{S_n}) = \frac{1}{S_N} - \frac{1}{S_\infty} = \frac{1}{S_N}$. Series converges.

   (g) What series do you get if you treat the Harmonic sequence in this way?

   $\sum \frac{1}{n(\ln n)^2}$. Converges.