Advanced Mathematical Techniques (AMT) Notes

1 Today's stuff

Let's try and prove that the alternating harmonic series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges.

We can start by going from 0 to 1 on a number line, then back to 1/2, then up by 1/3, then down by 1/4, and so on until we see that the sequence is constrained and converges between 0 and 1.

Further, we can see that if we choose to add the terms in a different order, such as

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n+1} - \frac{1}{2} \cdots$$

we can get a different number. Therefore, this is **conditionally convergent**, where the order of the terms matters. We define the alternating series test as a test where if we have a sequence of terms that alternate in signs and have decreasing absolute value, we know the series converges.

Let's try this complicated integral.

$$\int_{-\infty}^{\infty} \frac{x^4}{(x^2+4)^6} dx$$

Consider instead a simpler version:

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2+b} = \frac{1}{a} \int_{-\infty}^{\infty} \frac{dx}{x^2 + \frac{b}{a}}$$

$$= \frac{1}{a} \frac{2}{\sqrt{\frac{b}{a}}} \lim_{R\to\infty} \arctan\left(\frac{x}{\sqrt{\frac{b}{a}}}\right)\Big|_0^R$$

$$= \frac{\pi}{\sqrt{ab}}$$

Returning to the original integral, we know that

$$\frac{\partial}{\partial a}\left(\frac{\pi}{\sqrt{ab}}\right) = -\frac{\pi}{2a^{3/2}b^{1/2}}$$

$$\frac{\partial^2}{\partial b^2}\left(\frac{\pi}{\sqrt{ab}}\right) = \frac{3\pi}{4a^{1/2}b^{5/2}}$$

Therefore,

$$\frac{\partial^5}{\partial a^2 \partial b^3}\left(\frac{\pi}{\sqrt{ab}}\right) = -2 \times 3 \times 4 \times 5 \times \int_{-\infty}^{\infty} \frac{x^4}{(x^2+4)^6} dx$$

$$=$$

Also, we'll be expected to know how to do Gaussian integrals.

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

Therefore,

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy$$

$$\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty} e^{-x^2-y^2}dy$$

Switching to polar,

$$\int_{0}^{\infty}rdr\int_{0}^{2\pi} d\theta e^{-r^2} = \pi$$

Therefore,

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

Adding some sort of auxiliary paramter:

$$\int_{-\infty}^{\infty} e^{-ax^2}dx$$

We can just do a u-substitution to get that this is $\sqrt{\frac{\pi}{a}}$. We can also differentiate with respect to a to get:

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \frac{1}{2}\sqrt{\pi}a^{-\frac{1}{2}}$$

Another one with an auxiliary paramter:

$$\int_0^1 x^n dx = \frac{1}{n+1}$$

Taking the partial with respect to n,

$$\frac{\partial}{\partial n}\left(\frac{1}{n+1}\right) = \int_0^1 x^n \ln x dx = -\frac{1}{(n+1)^2}$$

$$\frac{\partial^2}{\partial n^2}\left(\frac{1}{n+1}\right) = \int_0^1 x^n \ln^2 x dx = \frac{2}{(n+1)^3}$$

$$\frac{\partial^3}{\partial n^3}\left(\frac{1}{n+1}\right) = \int_0^1 x^n \ln^3 x dx = -\frac{6}{(n+1)^4}$$

Moving on to something completely different, let's look at what's called an infinite product. Given the sequence $\{a_n\}$ of numbers, we can define $\{P_n\}_{n=1}^{\infty}$ via $P_1 = 1 + a_1$, $P_{n+1} = (1 + a_{n+1})P_n$. If

$$\lim_{n\to\infty} P_n$$

exists and is nonzero, then it is called the infinite product:

$$\prod_{k=1}^{\infty}(1 + a_k)$$

Infinite products cannot converge to 0 - if they do we say they diverge. The reason is that if we allow that to happen, then any of the terms could be 0 and that would rig your entire product. We make it $1 + a_k$ so that we can use some of the series stuff instead of having to reinvent everything. We can convert the product to the series by taking the logarithm:

$$\ln\left(\prod_{k=1}^{\infty}(1 + a_k)\right) = \sum_{k=1}^{\infty}\ln(1 + a_k)$$

Therefore,

$$\prod_{k=1}^{\infty}(1+a_k)$$

converges if and only if

$$\sum_{k=1}^{\infty}\ln(1+a_k)$$

converges. This turns out to be true for:

$$\frac{1}{2}|a_k| < |ln(1+a_k)| < 2|a_k|$$

This gives us a 3-way inequality, which means for absolute convergence, we have a way to relate sequences.