- Cryptology
  - Merkle-Hellman knapsack cryptosystem
    - Merkle-Hellman additive knapsack cryptosystem
    - Merkle-Hellman multiplicative knapsack cryptosystem
    - Merkle-Hellman multipy-iterated knapsack cryptosystem
  - Advanced knapsack cryptosystems

Additional Research Topics

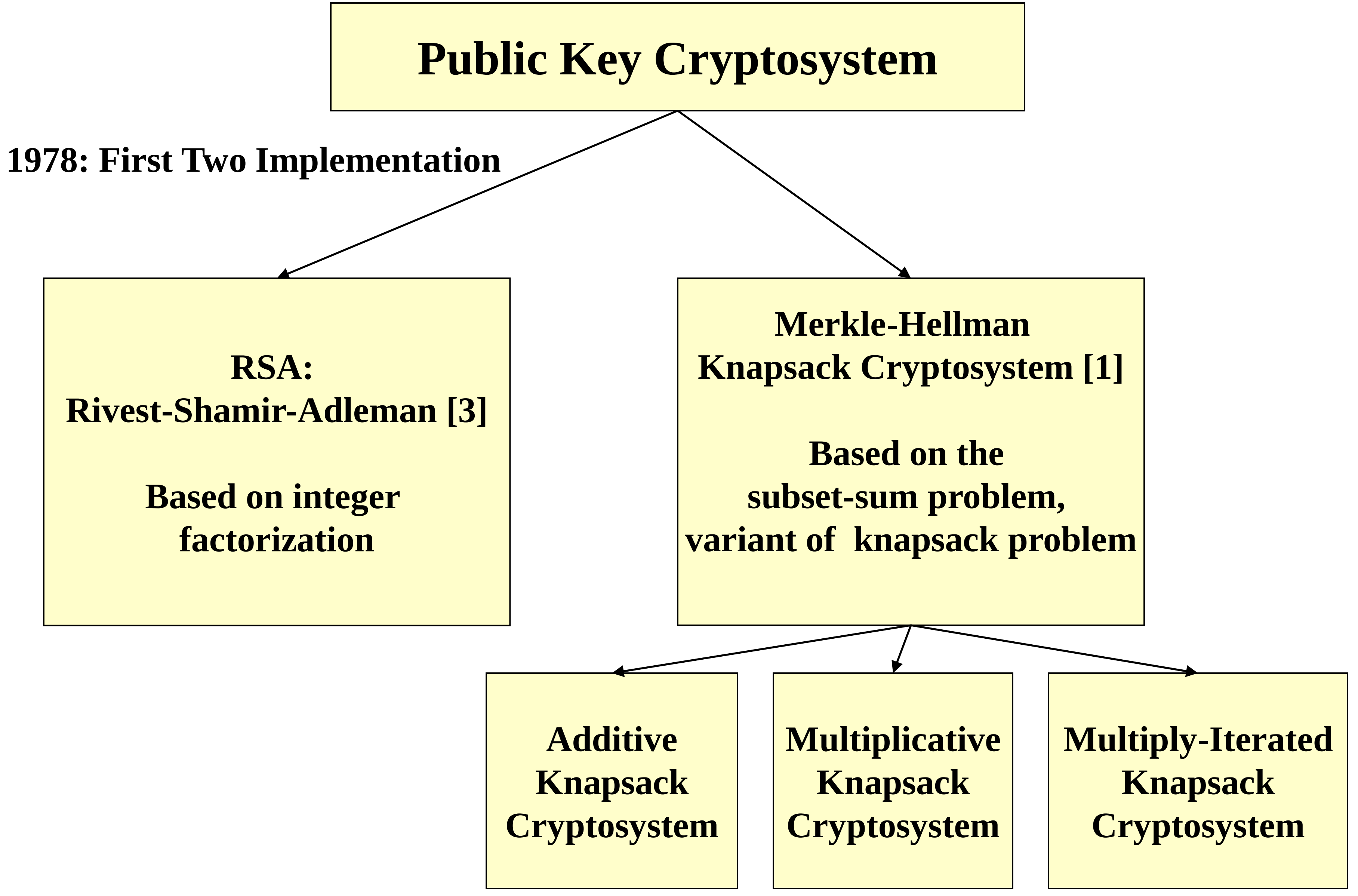
- Data Structures and Algorithms
  - Dynamic Programming Technique
    - Bioinformatics Algorithms.
    - Visualization.
  - Visualization of the Advanced Data Structures and Graph Algorithms
  - Exploring Advanced Sorting Algorithms.
    - Visualization

Public Key Cryptosystem

- In Symmetric or Private Key cryptosystems the encryption and decryption keys are either the same or can be easily found from each other.
- Public Key Cryptosystem (PKC) was introduced in 1976 by Diffie and Hellman [2]. In PKC different keys are used for encryption and decryption.

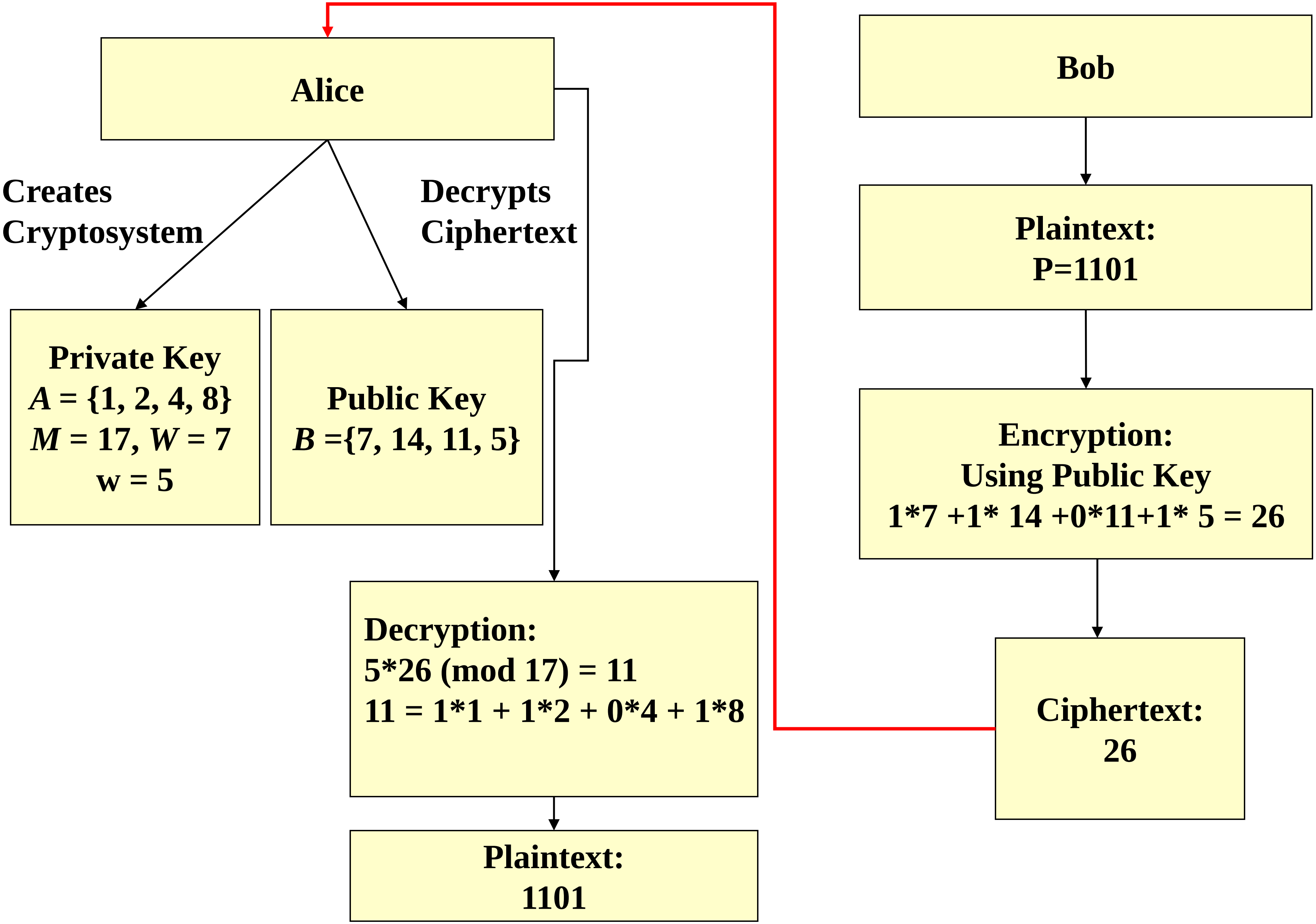
Alice:
1. **Chooses secret (private) key**
2. **Create and publishes public key**
3. **Receives ciphertext**
4. **Decrypts ciphertext using secret key to recover the plaintext – original message**

Bob
1. **Uses Public Key to encrypt the message**
2. **Sends ciphertext – encrypted message to Alice**


Public Key Cryptosystem
1978: First Two Implementation
RSA:
Rivest-Shamir-Adleman [3]
Based on integer factorization
Merkle-Hellman
Knapsack Cryptosystem [1]
Based on the subset-sum problem, variant of knapsack problem
Additive Knapsack Cryptosystem
Multiplicative Knapsack Cryptosystem
Multiply-Iterated Knapsack Cryptosystem

Merkle-Hellman Knapsack Cryptosystem Example

- Alice: Private Key
  - Private Key: $A = \{1, 2, 4, 8\}$, $M = 17$, $W = 7$, w = 5
  - Public Key: $B = \{7, 14, 11, 5\}$
- Bob: Encryption
  - Plaintext 1101
  - Ciphertext = 7 + 14 + 5 = 26
- Alice: Decryption
  - 5*26 (mod 17) = 11
  - 11 = 1*1 + 1*2 +0*4 + 1*8
  - Plaintext: 1101


Alice
Bob
Creates Cryptosystem
Decrypts Ciphertext
Private Key
A = {1, 2, 4, 8}
M = 17, W = 7
w = 5
Public Key
B ={7, 14, 11, 5}
Plaintext:
P=1101
Encryption:
Using Public Key
1*7 +1* 14 +0*11+1* 5 = 26
Ciphertext:
26
Decryption:
5*26 (mod 17) = 11
11 = 1*1 + 1*2 + 0*4 + 1*8
Plaintext:
1101

Merkle-Hellman Knapsack Cryptosystem

- 1982: Single iteration Merkle - Hellman Knapsack Cryptosystem was broken by Adi Shamir [4,5,6]

- 1983: At the CRYPTO ’83 , Adleman used an Apple II computer to demonstrate Shamir’s method [8]

- 1985: Multiple iteration Merkle-Hellman knapsack was broken by Brickell [9], a system of 40 iterations was breaking in about an hour of Cray-1 time

Merkle-Hellman Knapsack Cryptosystem

- History has not been kind to knapsack schemes [11] Lecture Notes on Cryptography, S. Goldwasser, M. Bellare
- Merkle offered $100 award for breaking singly - iterated knapsack
- Singly-iterated Merkle - Hellman KC was broken by Adi Shamir in 1982 [4,5,6] using Hendrik W. Lenstra’s polynomial time algorithm [7] for the integer programming problem when the number of variables is fixed.
- At the CRYPTO ’83 conference, Adleman used an Apple II computer to demonstrate Shamir’s method [8]
- Merkle offered $1000 award for breaking multiply-iterated knapsack
- Multiply-iterated Merkle-Hellman knapsack was broken by Brickell in 1985 [9]

Classical Knapsack Problem

- General 0-1 knapsack problem: given *n* items of different values v_i and weights w_i, find the most valuable subset of the items while the overall weight does not exceed a given capacity W
- The knapsack problem is NP-hard [10]
- The knapsack problem could be solved in pseudo-polynomial time through dynamic programming

Subset-Sum Problem

- Subset – Sum problem is a special case of knapsack problem when a value of each item is equal to its weight
- Input: set of positive integers: $A = \{a_1, a_2, \ldots a_n\}$ and the positive integer S
- Output:
  - TRUE, if there is a subset of A that sums to S and the subset itself
  - FALSE otherwise.
- The subset-sum problem is NP-hard

Easy Knapsack Problem

- An easy knapsack problem is one in which set $A = \{a_1, a_2, \ldots a_n\}$ is a super-increasing sequence
- A super-increasing sequence is one in which the next term of the sequence is greater than the sum of all preceding terms:

  $a_2 > a_1, a_3 > a_1 + a_2, \ldots, a_n > a_1 + a_2 + \ldots + a_{n-1}$
- Example: A= $\{1, 2, 4, 8, \ldots 2^{n-1}\}$ is super-increasing sequence

Polynomial Time Algorithm for Easy Knapsack Problem

- Input: $A = \{a_1, \ldots a_n\}$ is super-increasing sequence, S
- Output: TRUE and P – binary array of n elements, $P[i] = 1$ means: a_i belongs to subset of A that sums to S, $P[0] = 0$ otherwise. The algorithm returns FALSE if the subset doesn't exist

for $i \leftarrow n$ **to** 1

    if $S \geq a_i$

        then $P[i] \leftarrow 1$ **and** $S \leftarrow S - a_i$

    else $P[i] \leftarrow 0$

if S != 0

    then return (FALSE – no solution)

else return ($P[1], P[2], \ldots P[n]$).

Merkle-Hellman Additive Knapsack Cryptosystem

Alice:

1. Constructs the Knapsack cryptosystem
2. Publishes the public key
3. Receives the ciphertext
4. Decrypts the ciphertext using private key

Bob:

1. **Encrypts the plaintext using public key**
2. **Sends the plaintext to Alice**

Alice
Knapsack Cryptosystem Construction

- Chooses $A = \{a_1, \ldots a_n\}$ super-increasing sequence,
  A is a private (easy) knapsack
  $a_1 + \ldots + a_n = E$
- Chooses M - the next prime larger than E.
- Chooses W that satisfies $2 \leq W < M$ and $(W, M) = 1$
- Computes Public (hard) knapsack $B = \{b_1, \ldots .b_n\}$, where $b_i = Wa_i \pmod{M}$, $1 \leq i \leq n$
- **Keeps Private Key:** A, W, M
- **Publishes Public key**: B

Bob – Encryption Process

- Binary Plaintext P breaks up into sets of n elements long: $P = \{P_1, \ldots P_k\}$
- For each set P_i compute $\sum_{j=1}^{n} P_{ij} b_j = C_i$
- $\boldsymbol{C_i}$ is the ciphertext that corresponds to plaintext P_i
- $\boldsymbol{C = \{C_1, \ldots C_k)}$ is ciphertext that corresponds to the plaintext P
- ***C*** **is sent to Alice**

Alice – Decryption Process

- Computes w, the multiplicative inverse of W mod M:
  $wW \equiv 1 \pmod{M}$
- The connection between easy and hard knapsacks:
  $Wa_i = b_i \pmod{M}$ or $wb_i = a_i \pmod{M}$ $1 \le i \le n$
- For each C_i computes: $S_i = wC_i \pmod{M}$

$$S_i = wC_i = w\sum_{j=1}^{n} P_{ij} b_j = \sum_{j=1}^{n} P_{ij} w b_j = \sum_{j=1}^{n} P_{ij} a_j$$

- Plaintext P_i could be found using polynomial time algorithm for easy knapsack

Example

- Alice Private Key:
  - $A = \{1, 2, 4, 8\}$, $M = 17$, $W = 7$, $2 \le W < 17$, $(7, 17) = 1$
- Public Key:

B={7 mod 17, 14 mod 17, 28 mod 17, 56 mod 17}={7, 14, 11, 5}

- Bob Encryption:
  - Plaintext: 1101
  - Ciphertext = 7 + 14 + 5 = 26
- Alice Decryption:
  - $w = 5$ – multiplicative inverse of 7 (mod 17)
  - 5*26 (mod 17) = 11
  - Plaintext: 1101 (11 = 1*1 + 1*2 +0*4 + 1*8)

Ciphertext Only Cryptanalytic Attack on Merkle-Hellman Knapsack: Dynamic Programming Algorithm

- ***Input:*** $B=\{b_1, b_2, \ldots b_n\}$ – public key, C - ciphertext
- ***Output:*** The binary array P – plaintext
- ***Algorithm:*** Let $Q[i, j]$ be TRUE if there is a subset of first i elements of B that sums to j, $0 \le i \le n$, $0 \le j \le C$

Step 1: Computation of P

$Q[0][0] \leftarrow$ TRUE
for $j = 1$ to C do: $Q[0][j] \leftarrow$ FALSE
for $i = 1$ to n do:
  for $j = 0$ to C do:
    if $(j - B[i] < 0)$: $Q[i][j] = Q[i\text{-}1][j]$

    else: $Q[i][j] = Q[i\text{-}1][j\text{-}B[i]]$ *or* $Q[i\text{-}1][j]$

Step 2: Backtracking

Let P be an array of $n + 1$ elements initialized to 0

$i \leftarrow n$, $j \leftarrow C$

```
while i > 0:
    if (j – B[i]) ≥ 0):
        if  (Q[i-1][j-B[i]] is True):
                P[i] ← P[i] + 1
                j ← j – B[i]
        i ← i – 1

    else:  i ← i – 1
```

Output: array P, elements of P that equal to 1 construct a desired subset of B that sums to C

EXAMPLE
Input: B={1, 4, 5, 2}, C =3

	j = 0	j = 1	j = 2	j = 3
i = 0	TRUE	FALSE	FALSE	FALSE
i = 1 B[1] =1	TRUE	TRUE Element is taken	FALSE	FALSE
i = 2 B[2] = 4	TRUE	TRUE	FALSE	FALSE
i = 3 B[3] = 5	TRUE	TRUE	FALSE	FALSE
i = 4 B[4] = 2	TRUE	TRUE	TRUE	TRUE Element is taken

Q[i-1][j-B[i]] *or* Q[i-1][j]

Merkle-Hellman Multiplicative Knapsack Cryptosystem

- Alice:
  - Chooses set of relatively prime numbers
    $P = \{p_1, \ldots p_n\}$ – private (easy) knapsack
  - Chooses prime $M > p_1 * \ldots * p_n$
  - Chooses primitive root b mod M
  - Computes the public (hard) knapsack

  $A = \{a_1, \ldots . a_n\}$, where a_i is discrete logarithm of p_i to base b:

  $1 \le a_i <$ M, such that: $\boldsymbol{p_i \equiv b^{a_i} \pmod{M}}$
  - **Private Key: *P*, *M*, *b***
  - **Public Key: *A***

Merkle-Hellman Multiplicative Knapsack Cryptosystem- Encryption

- Binary Plaintext *T* breaks up into sets of *n* elements long: T = $\{T_1, \ldots T_k\}$
- For each set T_i compute $\sum_{j=1}^{n} T_{ij} a_j = C_i$
- $\boldsymbol{C_i}$ is the ciphertext that corresponds to plaintext T_i
- **$C = \{C_1, \ldots C_k)$** is ciphertext that corresponds to the plaintext *T*
- ***C* is sent to Alice**

Merkle-Hellman Multiplicative Knapsack Cryptosystem- Decryption

- For each C_i computes $S_i \equiv b^{C_i} \pmod{M}$
- S_i is a subset product of the easy knapsack:

$$S_i = b^{C_i} = b^{\sum_{j=1}^{n} T_{ij} a_j} = \prod_{j=1}^{n} b^{T_{ij} a_j} = \prod_{j=1}^{n} (b^{a_j})^{T_{ij}} \equiv \prod_{j=1}^{n} p_j^{T_{ij}} \pmod{M}$$

- $T_{ij} = 1$ if and only if p_j divides S_i

Merkle-Hellman Multiplicative Knapsack Example

- Easy (Private) Knapsack: $P = \{2, 3, 5, 7\}$
- $M = 211$, $b = 17$
- Hard (Public) Knapsack: A= {19, 187, 198, 121}

  $2 \equiv 17^{19} (\text{mod } 211)$, $3 \equiv 17^{187} (\text{mod } 211)$,

  $5 \equiv 17^{198} (\text{mod } 211)$, $7 \equiv 17^{121} (\text{mod } 211)$
- Plaintext: T = 1101
- Ciphertext: C = 327 = 19 + 187 + 121
- Decryption: $S = 42 = 17^{327} (\text{mod } 211)$
- $42 = 2^1 * 3^1 * 5^0 * 7^1$
- Plaintext: 1101

Multiply-Iterated Merkle-Hellman Knapsack Cryptosystem

- $A = \{a_1, \ldots a_n\}$ super-increasing sequence,

  A is a private (easy) knapsack, $a_1 + \ldots + a_n = E$
- For the m-times iterated knapsack cryptosystem: set of m multiplier-modulus pairs (w_i, M_i), $1 \le i \le m$
- To construct a public key knapsack: $\boldsymbol{B = \{b_1^m, b_2^m, \ldots, b_n^m\}}$

$$w_1 b_i^1 \equiv a_i (\mathrm{mod}\, M_1),\ 1 \le i \le n,\ M_1 > E$$

$$w_2 b_i^2 \equiv b_i^1 (\mathrm{mod}\, M_2), 1 \le i \le n,\ M_2 > \sum_{i=1}^{n} a_i^1$$

.....

$$w_m b_i^m \equiv b_i^{m-1} (\mathrm{mod}\, M_m), 1 \le i \le n,\ M_m > \sum_{i=1}^{n} a_i^{m-1}$$

Multiply-Iterated Merkle-Hellman Knapsack Cryptosystem Example

- A={1, 2, 4, 8}- super-increasing sequence (easy) knapsack, $m = 3$ (number of iterations)
- 1st iteration: $M_1 = 17$, $W_1 = 7$, $w_1 = 5$

  B^1 ={7 mod 17, 14 mod 17, 28 mod 17, 56 mod 17}={7, 14, 11, 5}
- 2nd iteration: $M_2 = 41$, $W_2 = 18$, $w_2 = 16$

  B^2 ={126 mod 41, 252 mod 41, 198 mod 41, 90 mod 41}={3, 6, 34, 8}
- 3rd iteration: $M_2 = 53$, $W_2 = 25$, $w_2 = 17$

  B^3 ={75 mod 53, 150 mod 53, 850 mod 53, 200 mod 53}={22, 44, 2, 41}
- **Public Key: {22, 44, 2, 41}**

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