

Prolegomena on (some) Riemannian Geometry

Riemannian manifolds, geodesic, Jacobi fields...

Tony

July 27, 2015

Cerco di scrivere la teoria di geometria differenziale minimale necessaria per la tesi. Al momento non c'è un ordine logico per stabilito (parlo di geodetiche in senso variazionale prima di dire cosa sono le variazioni....)

Fonti Principali di ispirazione sono :

- Jost, J., 2005. Riemannian Geometry and Geometric Analysis, Berlin/Heidelberg: Springer-Verlag. Available at: <http://link.springer.com/10.1007/3-540-28891-0> [Accessed January 10, 2015].
- Abate, M., Tivena, F., 2011. Geometria Differenziale, Milano: Springer Milan. Available at: <http://link.springer.com/10.1007/978-88-470-1920-1> [Accessed January 10, 2015].
- Abraham, R. et al., 1978. Foundations of mechanics II., Available at: <http://www.ams.org/publications> [Accessed January 10, 2015].

Contents

1	Reprise in Riemannian Geometry	3
1.1	Definition of (pseudo)-Riemannian manifold	3
1.2	Riemannian manifolds as a category.	3
1.3	Riemannian as a measure space.	3
1.4	Tangent bundle of a Riemannian manifold.	4
1.5	Riemannian as a metric space.	4
1.6	Connection structure on a Riemannian manifold.	6
1.7	Curvature on Riemannian manifold.	6
2	Geodesic	7
2.1	Common approach to the Geodesic	7

3	Review of physics application of geodesic problem.	8
3.1	Preliminary remarks: Geometrical encoding of classical mechanics.	8
3.2	Particle on Riemannian manifold under a position dependant potential. . . .	8
3.3	Relativistic particle.	8
4	Jacobi Fields	9
4.1	Preliminary remarks: Variation of curve.	9
4.2	Formal Definition	11
5	Closing Thoughts	13
5.1	Eliminata	13

1 Reprise in Riemannian Geometry

In what follows we present a brief review of the most important result in Riemannian geometry necessary for a better understanding of the geodesic problem.

1.1 Definition of (pseudo)-Riemannian manifold

Definition 1: (Pseudo-)Riemannian manifold

Notation fixing

Metric signature Lorenz manifold.

Theorem 1.1 $\forall M$ is Riemannianizable.

1.2 Riemannian manifolds as a category.

Definition 2: Local isometry

Definition 3: Isometry

Definition 4: Killing fields

1.3 Riemannian as a measure space.

Definition 5: Riemannian volume form

Theorem 1.2 $\forall M$ orientable $\exists!$ Riemannian volume form.

Observation 1

For an insight on the connection between volume form and measure theory see for

example [?].

1.4 Tangent bundle of a Riemannian manifold.

Observation 2

g could be seen as a 2-forms (section $\in \Gamma(T_0^2(M))$)

Definition 6: \flat operator

Theorem 1.3 *On Riemannian manifold M TM is a structure manifold of structure group $G = O(d)$.*

If M is also orientable $G = SO(d)$.

Proof:

See [?] Lemma 1.5.2 and 1.5.3.

□

1.5 Riemannian as a metric space.

See [?] pag 383 – 385 and [?] pag 15 – 17.

Consider the space $C^\infty([a, b], M)$ of the smooth parametrized curves from a closed interval of the real line. We define the following functional:

Definition 7: Length functional

$$L(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\| dt$$

Definition 8: Energy functional

$$E(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|^2 dt$$

Are define them together because are linked trough an inequality:

Proposition 1.1 $\forall \gamma : [a, b] \rightarrow \mathbb{R}$ smooth parametrized curve:

$$L(\gamma)^2 \leq 2(b-a)E(\gamma) \quad (1)$$

equality holds iff $\left\| \frac{d\gamma}{dt}(t) \right\| \equiv \text{const}$

Proof:

See [?, Lemma 1.4.2].

□

Observation 3

Former concepts are extended slavishly to every piecewise smooth curve.

Definition 9: Distance between two points

Function $d : M \times M \rightarrow \mathbb{R}$

$$d(p, q) := \inf \{ L(\gamma) \mid \gamma : [a, b] \rightarrow \mathbb{R} \text{ piecewise smooth, } \gamma(a) = p, \gamma(b) = q \}$$

Observation 4

Distance is well defined for all pair of point iff the manifold is connected.

Proposition 1.2 *The distance function satisfies the axioms of metric:*

- non-negative: $d(p, q) \geq 0 \forall p, q \in M$ $d(p, q) > 0 \forall p \neq q$
- symmetric: $d(p, q) = d(q, p)$
- triangle inequality: $d(p, q) \leq d(p, r) + d(r, q) \forall p, q, r \in M$

Proof:

See [?, Lemma 1.4.1].

□

Corollary 1.1 *The "balls" topology of M induced by the distance function d coincides with the original manifold topology of M .*

Proof:

See [?, corollary 1.4.2].

1.6 Connection structure on a Riemannian manifold.

Connection is a rather general concept definable on any smooth bundle.¹

On vector bundle we can identify a special kind of connection structure compatible with the vector space structure.² There are several equivalent presentation of this concept, each of them stress the importance of one of the many devices carried by this superstructure, for example:

- Derivative of section.
- Parallelism and parallel transportation.
- Specification of an unique horizontal lift among all.

Regarding the Riemannian manifolds we're not interested in connections on general vector bundle but instead to those on the tangent bundle, called *Linear Connection*. There's an infinity of such connection but on (pseudo-)Riemannian manifold it's possible to find a natural prescription that allows us to identify only one among these, called *Levi-Civita Connection*.

Consider (M, g) pseudo-Riemannian manifold.

Definition 10: Linear Connection

Map $\nabla : \Gamma^\infty(\tau_M) \times \Gamma^\infty(\tau_M) \rightarrow \Gamma^\infty(\tau_M)$, we write $(X, Y) \mapsto \nabla_X Y \quad \forall X, Y \in \Gamma^\infty(\tau_M)$. Such that:

- (a) $\nabla_X Y$ is $C^\infty(M)$ -linear in X variable.

$$\nabla_{fX_1 + gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y \quad \forall f, g \in C^\infty(M)$$

- (b)

- (c)

1.7 Curvature on Riemannian manifold.

¹In this abstract context connection takes the name of *Erhesmann's connection*.

²which takes its name from *Koszul* for distinguish it from the above.

2 Geodesic

2.1 Common approach to the Geodesic

On a manifold endowed with a affine connection a geodesic is defined as a curve "everywhere parallel to itself" providing a generalization of *straight line*.

Definition 11: Geodesic

A curve $\triangle^a \gamma : [a, b] \rightarrow M$ such that:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \quad (2)$$

where $\dot{\gamma}^\mu := \frac{d\gamma^\mu}{dt}$ is the tangent vector to the curve.

^aDevo dire smooth o piecewise?

Notation fixing

In local chart the previous equation assume the popular expression:

$$\ddot{\gamma}^i + \Gamma_{jk}^i \dot{\gamma}^j \dot{\gamma}^k = 0 \quad (3)$$

Where Γ_{jk}^i is the coordinate representation of the Christoffel symbols of the connection.

3 Review of physics application of geodesic problem.

Essentially [?]. A lot of mechanics systems can be regard as geodesic problem.

3.1 Preliminary remarks: Geometrical encoding of classical mechanics.

sistemi hamiltoniani
sistemi lagrangiani

3.2 Particle on Riemannian manifold under a position dependant potential.

fomm pag 226-228 + 231-233 teo 3.71

3.3 Relativistic particle.

3

³For an extension of this process to constrained, dissipative or ergodic systems see fom cap 3.7

4 Jacobi Fields

4.1 Preliminary remarks: Variation of curve.

Let $\sigma : [a, b] \rightarrow M$ a piecewise regular curve on smooth manifold M .

Definition 12: Variation of Curve

Variation of curve σ is a continuous application $\Sigma : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M$ such that

- $\sigma_s = \Sigma(s, \cdot)$ is a piecewise regular curve $\forall s \in (-\varepsilon, \varepsilon)$.
- $\sigma_0 = \sigma$.
- \exists a partition $a = t_0 < t_1 < \dots < t_k = b$ of $[a, b]$ such that

$$\Sigma|_{(-\varepsilon, \varepsilon) \times [t_{j-1}, t_j]} \in \mathcal{C}^\infty(\mathbb{R}^2; M)$$

Notation fixing

Regarding one entry as a variable and the other as a parameter we can see that Σ determine two family of curves:

- $\sigma_s(\cdot) = \Sigma(s, \cdot)$ is a family of piecewise regular curves called *principal curves*.
- $\sigma^t(\cdot) = \Sigma(\cdot, t)$ is a family of regular curves called *transverse curves*.

Curves in a family have a common parametrization.

Notation fixing

A variation is called *proper* if the endpoints stay fixed, i.e.

$$\sigma_s(a) = \sigma(a) \wedge \sigma_s(b) = \sigma(b) \quad \forall s \in (-\varepsilon, \varepsilon)$$

Fields over a variation Σ of a curve σ are defined as follows:

Definition 13: Vector field along a variation

Is a collection $X = \{X_j\}$ of smooth applications $X_j : (-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \rightarrow TM^a$ such that:

$$X_j(s, t) \in T_{\Sigma(s, t)}M \quad \forall (s, t) \in (-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \quad \forall j = 1, \dots, k$$

^aAssociate to a subdivision of $a = t_0 < t_1 < \dots < t_k = b$ of $[a, b]$.

Principal and transverse curves define two special Vector fields along the variation:

Definition 14: Tangent fields of the variation

$$S(s, t) = (\sigma^t)'(s) = d\Sigma_{(s,t)}\left(\frac{\partial}{\partial s}\right) = \frac{\partial \Sigma}{\partial s}(s, t)$$

for all $(s, t) \in (-\varepsilon, \varepsilon) \times [a, b]$.

$$T(s, t) = (\sigma_s)'(t) = d\Sigma_{(s,t)}\left(\frac{\partial}{\partial t}\right) = \frac{\partial \Sigma}{\partial t}(s, t)$$

for all $(s, t) \in (-\varepsilon, \varepsilon) \times [t_{j-1}, t_j]$ and $j = 1, \dots, k-1$ where we have choose a subdivision $a = t_0 < t_1 < \dots < t_k = b$ associated to Σ .

Notation fixing

$V = S(0, \cdot) \in \mathfrak{X}(\sigma)$ takes the special name of *variation field* of Σ .

There's an importation relation between continuous field on a curve and variation:

Proposition 4.1 *For all continuous field V along a piecewise regular curve σ can be found a variation Σ with variation field V .^a*

^aVice versa follows from the continuity of the variation field.

Proof:

See [?] Lemma 7.2.12 .

□

Let now M be a d -dimensional Riemannian manifold with Levi-Civita connection ∇ . The tangent fields of a variation are strictly connected to the curvature of M . We need a lemma:

Lemma 4.1 *For all rectangle $(-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \in \mathbb{R}^2$ on which Σ is \mathcal{C}^∞ we have:*

$$D_S T = D_T S$$

where D_S is the covariant derivative along the transverse curves and D_T over the principal curves.

Proof:

See [?] Lemma 7.2.13 .

□

The crucial result is what follows:

Proposition 4.2 For all vector field V along a variation Σ we have:

$$D_S D_T V - D_T D_S V = R(S, T)V$$

for all rectangle $(-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \in \mathbb{R}^2$ on which Σ is \mathcal{C}^∞ .^a

^a $R(S, T)$ is the curvature endomorphism evaluated on the tangent vector fields on the variation.

Proof:

See [?] Lemma 8.2.3 .

□

(References: [?] page 386-387 + 420-421 ; [?] page 171)

4.2 Formal Definition

The concept of *Jacobi Field* is closely related to variations of geodesic curves.

Let $\gamma : [a, b] \rightarrow M$ be a geodesic of the Riemannian manifold M . We can consider a special class of variations:

Definition 15: Geodesic variation

Is a smooth variation $\Sigma : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M$ such that all the principal curves $\gamma_s(\cdot) = \Sigma(s, \cdot)$ are also geodesic.^a

^aIn other words Σ determines a smoothly variable family of geodesic.

Proposition 4.3 Fixing two tangent vector over a point $p = \gamma(a)$ on the geodesic γ univocally determines a geodesic variation of γ .

Proof:

See [?] Lemma 8.2.5 or [?] Lemma 4.2.3.

□

Definition 16: Jacobi Fields

Is a field $J \in \mathfrak{X}(\gamma)$ over a geodesic γ such that:

$\exists \Sigma$ geodesic variation such that $J = V$ represent its variation field^a.

^aAs defined under (def 14).

The following proposition determines an equivalent (analytical) definition of Jacobi field:

Proposition 4.4

$J \in \mathfrak{X}(\dot{\gamma})$ is a jacobi field iff:

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} J + R(X, \dot{\gamma}) \gamma' = 0$$

Notation fixing

The vector space of all Jacobi fields on the geodesic γ is denoted $\mathcal{J}(\gamma)$.

Notation fixing

$J \in \mathcal{J}(\gamma)$ is called *proper* if $J_0(t) \perp (\dot{\gamma})(t)$.

$\mathcal{J}(\gamma)$ indicates the vector space of all proper Jacobi fields.

Proposition 4.5 *Every killing field X on M is a Jacobi Field along any geodesic in M .*

Proof:

See [?] Corollary 4.2.1.

□

5 Closing Thoughts

5.1 Eliminata

- non messa la definizione dei campi continui e l'osservazione che S è sempre continuo mentre T può non esserlo ([?] pag 420).
- sono stato ambiguo quando parlo di campi lungo la curva.. sulla continuità o meno (vedere abate pag 387)
- non mi è ancora chiaro l'utilità dei jacobi fields... Vediamo le possibilità:
 - Dice Abate a pag. 411 i Jacobi sono lo strumento principale per stabilire una relazione fra curvatura e topologia.
 - Dice Jost a pag. 183 che le Jacobi equation sono una linearizzazione dell'equazione delle geodetiche.
 - Jost a pag 183 – 186 esplora il legame tra J e le mappe esponenziali.
 - Jost nel capitolo 4.3 e Abate a pag 424 + 433 – 435 parlano del legame con i punti coniugati e morse theory.
- Discorso della index form come azione le cui equazioni eulero lagrange determinano l'equazione geodetica. (fonte Jost pag 177 – 179).
- Discorso Decomposizione dei Jacobi field in campi orizzontali e verticali (fonte Jost pag 180 – 181, http://en.wikipedia.org/wiki/Jacobi_field).