Prolegomena on (some) Abstact Algebra

Algebraic structures zoology

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(molto basato su wiki)

Contents

1 Single Operation structures (Magmas)

Consider two set M, A and a binary operation, i.e. a function:

$$\otimes : M \times M \to A$$

Operation $\otimes$ is said:

- *Closed* if $ran(\otimes) \subset M$

$$(x \otimes y) \in M \qquad \forall x, y \in M \tag{1}$$

- *Associative*

$$(x \otimes y) \otimes z = x \otimes (y \otimes z) \qquad \forall x, y, z \in M \tag{2}$$

- *Unital*

$$\exists e \in M \quad : \quad e \otimes x = x \otimes e = x \quad \forall x \in M \tag{3}$$

  Such an element is unique, and thus one speaks of *identity element.*

- *Exists Inverse:*

$$\forall a \in M \quad \exists b \in M \quad : \quad a \otimes b = b \otimes a = e \tag{4}$$

  where e is the identity element and thus one speaks of the identity element.

- *Commutative*

$$x \otimes y = y \otimes x \qquad \forall x, y \in M \tag{5}$$

Depending on the properties satisfied by $\otimes$, the pair $(M, \otimes)$ is called:

Magma	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

SemiGroup	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

Monoid	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

Commutative Monoid	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

Group	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

Abelian Group	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

Observation 1

Another recurring magma is the *Quasi-Group*. `https://it.wikipedia.org/wiki/Quasigruppo`

2 Double Operation Structure (Rings)

Consider two set R, A and two binary operation as above:

$$\otimes : R \times R \to A \qquad \oplus : R \times R \to A$$

Are defined some basic compatibility relations:

- $\otimes$ is *Left distributivite* with respect to $\oplus$

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \qquad \forall a, b, c \in R \tag{6}$$

- $\otimes$ is *Right distributivite* with respect to $\oplus$

$$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \qquad \forall a, b, c \in R \tag{7}$$

Notation fixing

Mimicking the numeric fields we denote

	Identity	Inverse
$\otimes$	$\mathbb{1}$	$(\cdot)^{-1}$
$\oplus$	$\mathbb{0}$	$-(\cdot)$

Depending on the properties satisfied by $\otimes, \oplus$, the triple $(M, \otimes, \oplus)$ is called:

Field	
Closure Associativity Identity Inverse Commutativity	Abelian Group ($\oplus$)
Closure Associativity Identity Inverse Commutativity	Abelian Group ($\otimes$)
Left Distributivity Right Distributivity	Distributivity ($\otimes$) w.r.t. ($\oplus$)

Commutative Ring	
Closure Associativity Identity Inverse Commutativity	Abelian Group ($\oplus$)
Closure Associativity Identity Inverse Commutativity	Commutative Monoid ($\otimes$)
Left Distributivity Right Distributivity	Distributivity ($\otimes$) w.r.t. ($\oplus$)

Ring	
Closure Associativity Identity Inverse Commutativity	Abelian Group ($\oplus$)
Closure Associativity Identity Inverse Commutativity	Monoid ($\otimes$)
Left Distributivity Right Distributivity	Distributivity ($\otimes$) w.r.t. ($\oplus$)

Semi-Ring	
Closure Associativity Identity Inverse Commutativity	Commutative Monoid($\oplus$)
Closure Associativity Identity Inverse Commutativity	Monoid($\otimes$)
Left Distributivity Right Distributivity	Distributivity ($\otimes$) w.r.t. ($\oplus$)

Pseudo-Ring	
Closure Associativity Identity Inverse Commutativity	Abelian Group ($\oplus$)
Closure Associativity Identity Inverse Commutativity	SemiGroup($\otimes$)
Left Distributivity Right Distributivity	Distributivity ($\otimes$) w.r.t. ($\oplus$)

Commutative Semi-Ring	
Closure Associativity Identity Inverse Commutativity	Commutative Monoid($\oplus$)
Closure Associativity Identity Inverse Commutativity	Commutative Monoid($\otimes$)
Left Distributivity Right Distributivity	Distributivity ($\otimes$) w.r.t. ($\oplus$)

3 Structures over a Field

Be $\mathbb{K}$ a field, we call a *scalar multiplication* over the set V the function:

$$\cdot : \mathbb{K} \cdot V \to V \qquad (\lambda, \mathbf{v}) \mapsto \lambda\mathbf{v} \quad \forall \lambda \in \mathbb{K},\ \forall \mathbf{v} \in V$$

Let be $(V, \clubsuit)$ a magma endowed with a scalar multiplication over field $\mathbb{K}$. Are defined some basic compatibility Relation:

- *Compatibility of scalar multiplication* :

$$a(b\mathbf{v}) = ab(\mathbf{v}) \qquad \forall a, b \in \mathbb{K},\ \forall \mathbf{v} \in V \tag{8}$$

- *Compatibility of scalar multiplication* w.r.t. $\otimes$:

$$(a\mathbf{v}) \otimes (b\mathbf{u}) = (ab)(\mathbf{v} \otimes \mathbf{u}) \qquad \forall a, b \in \mathbb{K},\ \forall \mathbf{v}, \mathbf{u} \in V \tag{9}$$

- *Identity element of scalar multiplication*:

$$1\mathbf{v} = \mathbf{v} \qquad 1 \text{identity element in } \mathbb{K} \tag{10}$$

- *Distributivity of scalar multiplication* w.r.t $\oplus$:

$$a(\mathbf{u} \oplus \mathbf{v}) = a\mathbf{u} \oplus a\mathbf{v} \qquad \forall a \in \mathbb{K},\ \forall \mathbf{u}, \mathbf{v} \in V \tag{11}$$

- *Distributivity of scalar multiplication with respect to field addition*:

$$(a + b)\mathbf{v} = a\mathbf{v} \oplus b\mathbf{v} \tag{12}$$

Depending on the properties satisfied by $\otimes, \oplus$, the triple $(M, \otimes, \oplus)$ is called:

Vector Space	
Closure Associativity Identity Inverse Commutativity	Abelian Group(⊕)
Closure Associativity Identity Inverse Commutativity	No (⊗)
Left Distributivity Right Distributivity	Distributivity (⊗) w.r.t. (⊕)
SP Multiplication (8) SP Identity(10)	Scalar Product Compatibility
Distributivity w.r.t. ⊗ (11) Distributivity w.r.t.⊕ (12)	Scalar Product distributivity w.r.t. (⊕)
Scalar Multiplication Compatibility w.r.t (⊗) (9)	

Algebra over Field $\mathbb{K}$	
Closure Associativity Identity Inverse Commutativity	Abelian Group(⊕)
Closure Associativity Identity Inverse Commutativity	Magma (⊗)
Left Distributivity Right Distributivity	Distributivity (⊗) w.r.t. (⊕)
SP Multiplication (8) SP Identity(10)	Scalar Product Compatibility
Distributivity w.r.t. ⊗ (11) Distributivity w.r.t.⊕ (12)	Scalar Product distributivity w.r.t. (⊕)
Scalar Multiplication Compatibility w.r.t (⊗) (9)	

Associative Algebra over $\mathbb{K}$	
Closure Associativity Identity Inverse Commutativity	Abelian Group(⊕)
Closure Associativity Identity Inverse Commutativity	SemiGroup (⊗)
Left Distributivity Right Distributivity	Distributivity (⊗) w.r.t. (⊕)
SP Multiplication (8) SP Identity(10)	Scalar Product Compatibility
Distributivity w.r.t. ⊗ (11) Distributivity w.r.t.⊕ (12)	Scalar Product distributivity w.r.t. (⊕)
Scalar Multiplication Compatibility w.r.t (⊗) (9)	

Unital Algebra over $\mathbb{K}$	
Closure Associativity Identity Inverse Commutativity	Abelian Group(⊕)
Closure Associativity Identity Inverse Commutativity	Monoid (⊗)
Left Distributivity Right Distributivity	Distributivity (⊗) w.r.t. (⊕)
SP Multiplication (8) SP Identity(10)	Scalar Product Compatibility
Distributivity w.r.t.⊗ (11) Distributivity w.r.t.⊕ (12)	Scalar Product distributivity w.r.t. (⊕)
Scalar Multiplication Compatibility w.r.t (⊗) (9)	