AQFT mathematical preliminaries

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Contents

(sono ripetizioni inutili per la tesi, sono informazioni che si ritrovano ovunque... sono informazioni adatte al "knowledge base")

Fin qui ho basato tutto sugli articoli del mio relatore: [2][1] Molte fonti estendono questi temi dal punto di vista matematico, e.g [4]

1 Globally Hyperbolic SpaceTimes

Recurring definitions in general Relativity (excluding the general smooth manifold prolegomena).

Definition 1: SpaceTime

A quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ such that:

- (M, g) is a time-orientable n-dimensional manifold $(n > 2)$
- $\mathfrak{o}$ is a choice of orientation
- $\mathfrak{t}$ is a choice of time-orientation

Definition 2: Lorentzian Manifold

A pair (M, g) such that:

- M is a n-dimensional $(n \geq 2)$, Hausdorff, second countable, connected, orientable smooth manifold.
- g is a Lorentzian metric.

Definition 3: Metric

A function on the bundle product of TM with itself:

$$g : TM \times_M TM \to \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a non-degenerate bilinear form.

Notation fixing

A Pseudo-riemmanian manifold (M, g) is called:

- *Riemmanian* if the sign of g is positive definite.
- *Lorentzian* if the signature is $(+, -, \dots, -)$ or equivalently $(-, +, \dots, +)$.

Observation 1: Causal Structure

If a smooth manifold is endowed with a Lorentzian metric of signature $(+,-,\dots,-)$ then the tangent vectors at each point in the manifold can be classed into three different types.

Notation fixing

$\forall p \in M, \quad \forall X \in T_p M$, the vector is:

- *time-like* if $g(X,X) > 0$.
- *light-like* if $g(X,X) = 0$.
- *space-like* if $g(X,X) < 0$.

Observation 2: Local Time Orientability

$\forall p \in M$ the timelike tangent vectors in p can be divided into two equivalence classes taking

$$X \sim Y \text{ iff } g(X,Y) > 0 \qquad \forall X, Y \in T_p^{\text{time-like}} M:$$

We can (arbitrarily) call one of these equivalence classes "future-directed" and call the other "past-directed". Physically this designation of the two classes of future- and past-directed timelike vectors corresponds to a choice of an arrow of time at the point.
The future- and past-directed designations can be extended to null vectors at a point by continuity.

Definition 4: Time-orientation

A global tangent vector field $\mathfrak{t} \in \Gamma^\infty(TM)$ over the Lorenzian manifold M such that:

- $\mathrm{supp}(\mathfrak{t}) = M$
- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Observation 3

The fixing of a time-orientation is equivalent to a consistent smooth choice of a local time-direction.

Definition 5: Time-Orientable Lorentzian Manifold

A Lorentzian Manifold (M,g) such that exist at least one time-orientation $\mathfrak{t} \in \Gamma^\infty(TM)$.

Notation fixing

Consider a piece-wise smooth curve $\gamma : \mathbb{R} \supset I \to M$ is called:

- *time-like* (resp. light-like, space-like) iff $\dot{\gamma}(p)$ is time-like (resp. light-like, space-like) $\forall p \in M$.
- *causal* iff $\dot{\gamma}(p)$ is nowhere spacelike.
- *future directed* (resp. past directed) iff is causal and $\dot{\gamma}(p)$ is future (resp. past) directed $\forall p \in M$.

Definition 6: Chronological $\substack{\text{future}\\\text{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{I}^{\pm}_{M}(p) := \left\{ q \in M \middle| \, \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ time-like } \substack{\text{future}\\\text{past}} - \text{directed } : \gamma(0) = p, \, \gamma(1) = q \right\}$$

Definition 7: Causal $\substack{\text{future}\\\text{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{J}^{\pm}_{M}(p) := \left\{ q \in M \middle| \, \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ causal } \substack{\text{future}\\\text{past}} - \text{directed } : \gamma(0) = p, \, \gamma(1) = q \right\}$$

Notation fixing

Former concept can be naturally extended to subset $A \subset M$:

- $\mathbf{I}^{\pm}_{M}(A) = \bigcup_{p \in A} \mathbf{I}^{\pm}_{M}(p)$
- $\mathbf{J}^{\pm}_{M}(A) = \bigcup_{p \in A} \mathbf{J}^{\pm}_{M}(p)$

Definition 8: Achronal Set

Subset $\Sigma \subset M$ such that every inextensible timelike curve intersect Σ at most once.

Definition 9: $\substack{\text{future}\\\text{past}}$ Domain of dependence of an Achronal set

The two subset related to the generic achornal set $\Sigma \subset M$:

$$\mathbf{D}^{\pm}_{M}(\Sigma) := \left\{ q \in M \middle| \, \forall \gamma \, \substack{\text{past}\\\text{future}} \text{inextensible causal curve passing through } q : \, \gamma(I) \cap \Sigma \neq \emptyset \right\}$$

Notation fixing

$\mathbf{D}_M(\Sigma) := \mathbf{D}^+_M(\Sigma) \cup \mathbf{D}^-_M(\Sigma)$ is called *total domain of dependence.*

Definition 10: Cauchy Surface

Is a subset $\Sigma \subset M$ such that:

- closed
- achronal
- $\mathbf{D}_M(\Sigma) \equiv M$

The term "hypersurface" is not used by chance:

Proposition 1.1 *Every Cauchy surface Σ is a three dimensional, embedded, C^0 submanifold of M*

Proof:

See Wald (general relativity) teo 8.3.1

□

⚠ ⚠copiato da [2] From a physical point of view, we are interested in those spacetimes which allow to set a well-posed initial value problem for hyperbolic partial differential equations, such as the scalar DâĂŹAlambert wave equation, to quote the simplest, yet most important example. In particular we need to ensure that the spacetime we consider possesses at least one distinguished codimension 1 hypersurface on which we can assign the initial data needed to construct a solution of such an equation.

Definition 11: Globally-Hyperbolic SpaceTime

Spacetime M such that there exists at least one *Cauchy Surface*

According to Definition 11, only the existence of a single Cauchy hypersurface is guaranteed. This is slightly disturbing since there is no reason a priori why an initial value hypersurface for a certain partial differential equation should be distinguished. This quandary has been overcome proving that, if a spacetime (M, g) is globally hyperbolic, then there exists a foliation of M by Cauchy surfaces:

Theorem 1.1 (Globally hyperbolic space characterization) *Let* (M,g) *be any time-oriented spacetime. The following two statements are equivalent:*

- (M,g) *is globally hyperbolic.*
- (M,g) *is isometric to* $\mathbb{R}\times\Sigma$ *endowed with the line element* $ds^2 = \beta dt^2 - h_t$ *where* $t:\mathbb{R}\times\sigma\to\mathbb{R}$ *is the projection on the first factor,* β *is a smooth and strictly positive function on* $\mathbb{R}\times\Sigma$ *and* $t\mapsto h_t$*,* $t\in\mathbb{R}$*, yields a one-parameter family of smooth Riemmanian metrics.*
  Furthermore, for all $t\in\mathbb{R}$*,* $\{t\}\times\Sigma$ *is an (nâĹŠ1)-dimensional, spacelike, smooth Cauchy surface in M.*

Proof:

Teo 3.17 in J. K. Beem, P. E. Ehrlich and K. L. Easley, Global Lorentzian Geometry.
section 1.3 in [3]

□

To conclude this section, we introduce some terms which will be often used in the following in order to specify the support properties of the sections of a vector bundle with base a globally hyperbolic spacetime.

Notation fixing

Let M be a globally hyperbolic spacetime and $E=(E,\pi,M;V)$ a vector bundle of typical fiber V. We denote:

- $\Gamma_0(E)$ the space of *compactly supported* smooth sections.
- $\Gamma_{sc}(E)$ the space of *spacelike compact* smooth sections.
  $\left(f\in\Gamma_{sc}(E) \text{ if there exists a compact subset } K\subset M \text{ such that } \operatorname{supp} f\subset \mathbf{J}_M(K).\right)$
- $\Gamma_{fc}(E)$ the space of *future- compact* smooth sections.
  $\left(f\in\Gamma_{fc}(E) \text{ if } \operatorname{supp}(f)\cap\mathbf{J}^+_M(K) \text{ is compact for all } p\in M.\right)$
- $\Gamma_{pc}(E)$ the space of *past- compact* smooth sections.
  $\left(f\in\Gamma_{pc}(E) \text{ if } \operatorname{supp}(f)\cap\mathbf{J}^-_M(K) \text{ is compact for all } p\in M.\right)$
- $\Gamma_{tc}(E):=\Gamma_{pc}(E)\cap\Gamma_{fc}(E)$ the space of *timelike compact* smooth sections.

This class of manifolds includes most of the physically interesting examples, e.g.: Minkowski spacetimes, Friedman-Robertson-Walker solutions, Kerr family. [1]
A trivial example:

Example: 1

Trivially, the real line $\mathbb{R}$ is a globally hyperbolic manifold.

Each point $x \in \mathbb{R}$ represent a proper Cauchy surfaces which realize the trivial foliation $\mathbb{R} \simeq 1 \times \mathbb{R}$ required by theorem 1.1

2 Linear Differential Operator

Basic Definition in L.P.D.O. on smooth vector sections.

⚠ (ADVANCES)
Globally hyperbolic spacetimes play a pivotal role, not only because they do not allow for pathological situations, such as closed causal curves, but also because they are the natural playground for classical and quantum fields on curved backgrounds. More precisely, the dynamics of most (if not all) models, we are interested in, is either ruled by or closely related to wave-like equations. Also motivated by physics, we want to construct the associated space of solutions by solving an initial value problem. To this end we need to be able to select both an hypersurface on which to assign initial data and to identify an evolution direction. In view of Theorem 1, globally hyperbolic spacetimes appear to be indeed a natural choice. Goal of this section will be to summarize the main definitions and the key properties of the class of partial differential equations, useful to discuss the models that we shall introduce in the next sections. Since this is an overkilled topic, we do not wish to make any claim of being complete and we recommend to an interested reader to consult more specialized books and papers for more details.

2.1 L.P.D.O

Consider $E = (E, \pi, M; V), E' = (E', \pi', M; V')$ two linear vector bundle over M (with different typical fiber), we define:

Definition 12: Linear Partial Differential operator **(of order at most $s \in \mathbb{N}_0$)**

Linear map $L : \Gamma(E) \to \Gamma(E')$ such that $\forall p \in M$ exists:

- $U \ni p$ open set rigged with:
  - (U, φ) local chart on M.
  - (U, χ) local trivialization of F
  - (U, χ') local trivialization of F'
- $\{A_\alpha : U \to \mathrm{Hom}(V, V') \mid \alpha \in \mathbb{N}_0^n, |\alpha| \leq s\}$ collection of smooth maps labeled by multi-indices.

which allows to express L locally:

$$\chi' \circ (L\sigma) \circ \varphi^{-1} = \sum_{|\alpha| \leq s} A_\alpha \partial^\alpha (\chi \circ \sigma \circ \varphi^{-1}) \qquad \forall \sigma \in dom(L) \subset \Gamma(E)$$

Notation fixing

Using implicitly the coordinate charts [1]:

$$L(\sigma\big|_U) = \sum_{|\alpha|\leq s} A_\alpha \partial^\alpha \sigma \qquad \forall \sigma \in \Gamma(E)$$

Remark:

(multi-index notation)
A multi-index is a natural valued finite dimensional vector $\alpha = (\alpha_0, \ldots, \alpha_n - 1) \in \mathbb{N}_0^n$ with $n < \infty$.
On $\mathbb{R}^n$ a general differential operator can be identified by a multi-index:

$$\partial^\alpha = \prod_{\mu=0}^{n-1} \partial_\mu^{\alpha_\mu}$$

(Until the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|\alpha| := \sum_{\mu=0}^{n-1} \alpha_\mu$$

Observation 4

Notice that linear partial differential operators cannot enlarge the support of a section.

Notation fixing

We say that L is exactly of order k if it is of order at most k, but not of order at most $k-1$.

Observation 5

Notice that, as a consequence of their definition, linear partial differential operators cannot enlarge the support of a section.

Definition 12 accounts for a large class of operators, most of which are not typically used in the framework of field theory, especially because they cannot be associated to an initial value problem. In order to select a relevant subset we introduce a useful concept (see also [3][pag 172]).

Consider a Linear differential operator $L : \Gamma(E) \to \Gamma(E')$:

Definition 13: Principal Symbol

The map $\sigma_L : T^* M \to \mathrm{Hom}(E, E')$locally defined as follows:
For a given $p \in M$, consider a coordinate chart (U, x^i) around p and local trivializations of E and of E' (as precribed in Definition 12).
For all $\xi = \xi_i dx^i \in T^*_p M$ set:

$$\sigma_L(\xi) = \sum_{|\alpha|=s} \xi^\alpha A_\alpha(p)$$

where $\xi^\alpha = \prod_{\mu=0}^{m-1} \xi^{\alpha_\mu}$

Remembering the definition of Natural pairing (see [6]) we can define a dual operator of a L.P.D.O. $L : \Gamma(E) \to \Gamma(G)$:

Definition 14: Formal Dual Operator

Linear partial differential operator $L^\star : \Gamma(G^*) \to \Gamma(E^*)$ such that:

$$< L^\star g', f > = < g', Lf >$$

$\forall f \in \Gamma(E)$, $g' \in \Gamma(G^*)$ which have supports with compact overlap.

Proposition 2.1 *$\forall P$ linear partial differential operator, $\exists 1! P^\dagger$.*

Similarly , in presence of a bundle inner product $g : E \otimes_M E \to \mathbb{R}$ and h respectively on E and G, can be defined an adjoint operator through the dual space isomorphism:

Definition 15: Formal Adjoint Operator

Linear partial differential operator $L^\dagger : \Gamma(G) \to \Gamma(E)$ such that:

$$g(L^\dagger g, f) = h(g, Lf)$$

$\forall f \in \Gamma(E), g \in \Gamma(G)$ with compact supports intersection.

Notation fixing

In presence of an inner product g , denote $f^* = g(f, \cdot) \in \Gamma^*(E)$ the dual section associated to $f \in \Gamma(E)$ through g. Obviously:

$$< f^*, f' > \equiv g(f, f')$$

Notation fixing

$L:\Gamma(E)\to\Gamma(E)$ is *self-adjoint* whenever $L^\dagger = L$.

2.2 Green Operators

N.B. : rom now on we will consider only bundles with globally-hyperbolic spacetime base.

Let M be a globally hyperbolic spacetime, consider a vector bundle E over M and a L.p.d.o. $L:\Gamma(E)\to\Gamma(E)$:

Definition 16: $\substack{\textbf{retarded}\\\textbf{advanced}}$ $(\pm)$ Green Operators

L.p.d.o. $G^\pm:\Gamma(E)\to\Gamma(E)$ such that:

- $dom(G^+) = \Gamma_{pc}(E) \qquad dom(G^-) = \gamma_{fc}(E)$
- $LG^\pm f = G^\pm Lf = f \qquad \forall f\in dom(G^\pm)$
- $\operatorname{supp}(G^\pm f)\subset \mathbf{J}^\pm_M(\operatorname{supp}(f)) \qquad \forall f\in dom(G^\pm)$

Definition 17: Advanced $(-)$ Green Operator of L

L.p.d.o. $G^-:\Gamma_{fc}(E)\to\Gamma(E)$ such that:

- $dom(G^-) = \Gamma_{fc}(E)$
- $LG^- f = G^- Lf = f \qquad \forall f\in\Gamma_{fc}(E)$
- $\operatorname{supp}(G^- f)\subset \mathbf{J}^-_M(\operatorname{supp}(f)) \qquad \forall f\in\Gamma_{fc}(E)$

Definition 18: Retarded $(+)$ Green Operator of L

L.p.d.o. $G^+:\Gamma_{pc}(E)\to\Gamma(E)$ such that:

- $dom(G^+) = \Gamma_{pc}(E)$
- $LG^+ f = G^+ Lf = f \qquad \forall f\in\Gamma_{pc}(E)$
- $\operatorname{supp}(G^+ f)\subset \mathbf{J}^+_M(\operatorname{supp}(f)) \qquad \forall f\in\Gamma_{pc}(E)$

Observation 6

From the definition follows that $G^\pm$ is the left-right inverse of the restriction of L to

$dom(G^{\pm})$.

Notation fixing

We refer to the operator:

$$E := G^{-} - G^{+} : \Gamma_{tc}(E) \to \Gamma(E)$$

as the *Advanced minus Retarded operator* or *Causal Propagator*[2].

Green operators are not necessarily unique. For this we introduce the following definition:

Definition 19: Green hyperbolic operator

The linear partial differential operator P is be called Green hyperbolic if P and $P^{\star}$ have advanced and retarded GreenâĂŹs operators.

for such operators uniqueness of Green's operator comes from free:

Hp:

Theorem 2.1 (Characterization of Green Hyperbolic operators)

$E = (E, \pi, M)$ a vector bundles over a globally hyperbolic spacetime M.

$P : \Gamma(E) \to \Gamma(E)$ a green hyperbolic operator, $G^{\pm}$ its Green's operators and $G^{\pm}_{\star}$ the Green's operators of the dual.

Th:

L posses an unique retarded G^{+} and advanced G^{-} green operator.

$< G^{\pm}_{\star} f', f > = < f', G^{\mp} f > \qquad \forall f \in \Gamma_0(E), \forall f' \in \Gamma_0(E^*)$

Proof:

see [1][proposition 2]

□

The same result can be obtained considering the pairing induced by an inner product:

Corollary 2.1 *see [1][Lemma 1]*

From that follows:

Corollary 2.2 *If L is Green hyperbolic and self-adjoint, then:*

- $\exists 1!\ G^{\pm}$ *Green's operators of L*

- $(G^{\pm})^{\dagger} = G^{\mp}$

2.3 Cauchy problem and Solution Space

The globally-hyperbolic condition property of the base manifold M is what allows to define *Cauchy problems* associated to a linear partial differential operator $P : \Gamma(E) \to \Gamma(E)$:

Remark:

For every cauchy surface $\Sigma \subset M$, for each couple $(u_0, u_1) \in \Gamma(\Sigma) \times \Gamma(\Sigma)$ we can state a *Cauchy Problem*:

$$\begin{cases} Pu = 0 \\ u = u_0 \\ \nabla_{\vec{n}} u = u_1 \end{cases} \tag{1}$$

(Someties we refer to $Pu = 0$ as *Wave equation* of $u \in \Gamma(E)$.)

Definition 20: PDE-hyperbolic operator

L.d.p.o. P such that exists an unique solution for every Cauchy problem 1

Observation 7

"Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. " [**?**] [**?**]
However the existence and uniqueness can be proved for the large class of the *Normally-Hyperbolic Operators.*

2.4 Normally Hyperbolic Operators

Is a class of L:P.D.O. hyperbolic in both PDE and Green sense. Given a Lorentzian manifold (M, g) and two vector bundles $E = (E, \pi, M; V), E' = (E', \pi', M; V')$,

Definition 21: Normally Hyperbolic Operators

Second order linear partial differential operator $P : \Gamma(E) \to \Gamma(E')$ such that:

$$\sigma_P(\xi) = g(\xi, \xi)\mathbb{1}_{E_p} \qquad \forall p \in M, \xi \in T_p^* M$$

Observation 8

Making explicit the coordinate expression of a normally hyperbolic operator P , one realizes how such operators provide the natural generalization of the usual Wave oper-

ator.
Consider a globally hyperbolic operator P for all $p \in M$ a trivializing chart (U, φ, χ) centered in p. There exist a collection $\{A, A^{\mu} | \mu \in \{0, \dots, m-1\}\}$ of smooth $\mathrm{Hom}(V, V)-$valued maps on U such that, P reas as follows:

$$\chi \circ (P\sigma) \circ \varphi^{-1} = \left(g^{\mu\nu} \mathrm{id}_V \partial \mu \partial_n u + A^{\mu} \partial_{\mu} + A\right)(\chi \circ \sigma \circ \varphi^{-1}) \qquad \forall \sigma \in dom(P) \subset \Gamma(E)$$

where both the chart and the vector bundle trivializations are understood. One immediately notices that locally this expression agrees up to terms of lower order in the derivatives with the one for the d'Alembert operator acting on sections of E constructed out of a covariant derivative ∇ on E, that is the operator:

$$\Box_{\nabla} = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} : \Gamma(E) \to \Gamma(E)$$

This definition becomes even more important if we assume, moreover, that the underlying background is globally hyperbolic, since we can associate to each normally hyperbolic operator P an initial value problem and talk about Green's operator.

Proposition 2.2 (Green operators) *Be P normally hyperbolic operator, then:*

- *$P^{\star}$ is a normally hyperbolic operator.*
- *P is Green hyperbolic.*

Proof:

[3] [Corollary 3.4.3]

□

Proposition 2.3 (Existence and uniqueness for the Cauchy Problem)

Hp:

- *$E = (E, \pi, M; V)$ a vector bundle on $M = (M, g, \mathfrak{o}, \mathfrak{t})$, globally hyperbolic spacetime.*
- *$\Sigma \subset M$ a spacelike Cauchy surface with future-pointing unit normal vector field $\vec{n}$.*
- *P a normally hyperbolic operator and ∇ a P-compatible[a] covariant derivative on E*

[a] There existss a section $A \in \Gamma(\mathrm{End}(E))$ such that $\Box_{\nabla} + A = P$.

Th:

- *The Cauchy problem;*

$$\begin{cases} Pu = J & on\ M \\ u = u_0 & on\ \Sigma \\ \nabla_{\vec{n}} u = u_1 & on\ \Sigma \end{cases}$$

admit a unique solution $u \in \Gamma(E)$ *for any* $J \in \Gamma(E)$ *and* $u_0, u_1 \in \Gamma(\Sigma)$

- $supp(U) \subset \mathbf{J}_M\big(supp(u_0) \cup supp(u_0) \cup supp(J)\big)$

Proof:

The proof of this proposition has been given in different forms in several books, [3]6, 26] and in [3, Corollary 5]. Notice that equation (2) is not linear since we

□

2.5 Continuity

(...) see p.15-16 [2]

3 *-Algebras

Definition 22: Algebra over a field

A pair $\big((V,\mathbb{K}),\cdot\big)$ which consists of a vector space over field $\mathbb{K}$ and a multiplication operator $\cdot : V\times V \to V$ such that:

1. is bilinear.
2. satisfies left/right distributivity :
$$(x+y)\cdot z = x\cdot z + y\cdot z \qquad x\cdot(y+z) = x\cdot y + x\cdot z \qquad \forall x,y,z\in V$$
3. is scalar compatible:
$$(\alpha x)\cdot(\beta y) = (\alpha\beta)x\cdot y \qquad \forall\alpha,\beta\in\mathbb{K}\ \forall x,y\in V$$

Definition 23: Unital Algebra (over a field)

Algebra $(V,\cdot)$ over field $\mathbb{K}$ such that:

1. $\exists\mathbb{1}\in V$ such that $\mathbb{1}\cdot u = u\cdot\mathbb{1} = u \qquad \forall u\in V$
2. associativity:
$$v\cdot(w\cdot u) = (v\cdot w)\cdot u = v\cdot w\cdot u \qquad \forall u,v,w\in V$$

Definition 24: Algebra (over a field) generate by a subspace $W\subset V$

Is an Algebra $(V,\cdot)$ over field $\mathbb{K}$ such that each elements of V can be obtained as a polynomial in the elements of W:

$$V = \operatorname{span}\{\prod_i W_i \big| \{w_i\}\subset W\}$$

Definition 25: *-algebra over a field

Triple $\big((V,\mathbb{K}),\cdot,*\big)$ where $\big((V,\mathbb{K}),\cdot\big)$ constitutes an unital algebra over field $\mathbb{K}$ and $*: V\to V$ is an *involutive* map :

- $(\alpha x+y)^* = \bar{\alpha}x^* + y^* \qquad \forall\alpha\in\mathbb{K},\ \forall x,y\in V$
- $(x\cdot y)^* = y^*\cdot x^*\forall x,y\in V$
- $\mathbb{1}^* = \mathbb{1}$

- $(x^*)^* = x \qquad \forall x \in V$

References

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