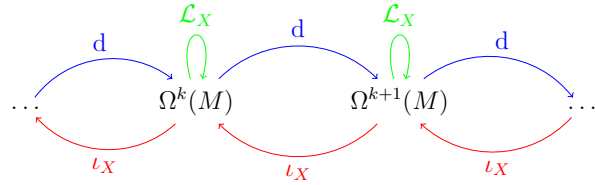


Suppose  $M$  is a smooth manifold, denote by  $\Omega(M)$  the algebra of differential forms on  $M$  and by  $\mathfrak{X}(M)$  the  $C^\infty(M)$ -module of vector fields.

	$f \in C^\infty(M)$	$dx^\mu \in \Omega^1(M)$	$\omega^{(k)} \wedge \beta$	$T_1 \otimes T_2$	$\partial_\mu \in \mathfrak{X}(M)$
$d$	$\left( \frac{\partial f}{\partial x^\nu} \right) dx^\nu$	0	$(d\omega) \wedge \beta + (-)^k \omega \wedge (d\beta)$	-	-
$\mathcal{L}_X$	$X(f) = X^\nu \left( \frac{\partial f}{\partial x^\nu} \right)$	$\mathcal{L}_X dx^\mu = d(X^\mu) = \left( \frac{\partial X^\mu}{\partial x^\nu} \right) dx^\nu$	$(\mathcal{L}_X \omega) \wedge \beta + \omega \wedge (\mathcal{L}_X \beta)$	$(\mathcal{L}_X T_1) \otimes T_2 + T_1 \otimes (\mathcal{L}_X T_2)$	$\mathcal{L}_X \partial_\mu = [X, \partial_\mu]$
$\iota_X$	0	$\iota_X dx^\mu = dx^\mu(X) = X^\mu$	$(\iota_X \omega) \wedge \beta + (-)^k \omega \wedge (\iota_X \beta)$	$(\iota_X T_1) \otimes T_2 + T_1 \otimes (\iota_X T_2)$	0
$g^*$	$g^*(f) = f \circ g$	$g^*(dx^\mu) = d(x^\mu \circ f)$	$g^*(\omega) \wedge g^*(\beta)$	$g^*(T_1) \otimes g^*(T_2)$	$(g^{-1})^* \partial_\mu$ <sup>a</sup>

The *Cartan calculus* consists of the following three *graded derivations* on  $\Omega(M)$

- the *exterior derivative*  $d$ ;
- the space of *Lie derivative operators*  $\mathcal{L}_X$ , where  $X \in \mathfrak{X}(M)$ ;
- the space of *contraction operators*  $\iota_X$ , where  $X \in \mathfrak{X}(M)$ .



Together with the following identities:

$$d^2 = 0, \quad (1)$$

$$d\mathcal{L}_X - \mathcal{L}_X d = 0, \quad (2)$$

$$d\iota_X + \iota_X d = \mathcal{L}_X, \quad (3)$$

$$\mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]}, \quad (4)$$

$$\mathcal{L}_X \iota_Y - \iota_Y \mathcal{L}_X = \iota_{[X,Y]}, \quad (5)$$

$$\iota_X \iota_Y + \iota_Y \iota_X = 0, \quad (6)$$

A *graded derivation* of  $\Omega(M)$  is a degree  $k$  linear operator  $A$  on  $\Omega(M)$  such that

$$A(\omega \wedge \eta) = A(\omega) \wedge \eta + (-1)^{kp} \omega \wedge A(\eta) \quad \forall \omega \in \Omega^k(M), \eta \in \Omega^p(M) \quad (7)$$

$$\Omega(M) = \left( \bigoplus_{k=0}^m \Omega^k(M), \wedge \right) \quad \text{Grassmann Algebra on } M$$

is a graded commutative graded algebra over ring  $C^\infty(M)$ .

$$\mathcal{T}(M) = \left( \bigoplus_{l,k=0}^{\infty} T_l^k(M), \otimes \right) \quad \text{Tensor Algebra on } M$$

is a commutative graded algebra over ring  $C^\infty(M)$ .

<sup>a</sup> $g$  has to be a diffeo

How to compute brackets:

$$[\partial_i, \partial_j] = 0 \quad (8)$$

$$[f\partial_i, g\partial_j] = f \cdot (\partial_i g) \cdot \partial_j - g \cdot (\partial_j f) \cdot \partial_i \quad (9)$$

$$[fX, gY] = f \cdot g \cdot [X, Y] + f \cdot X(g) \cdot Y - g \cdot Y(f) \cdot X \quad (10)$$

$$[X, Y] = (X^i \partial_i Y^j - Y^i \partial_i X^j) \partial_j \quad (11)$$

$(\mathfrak{X}(M), [-, -])$  form a Lie algebra over  $\mathbb{R}$ :

$$[X, Y] = -[Y, X]$$

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

An  $(I)$ -graded vector space  $'V$  is a vector space that can be written as a direct sum of subspaces indexed by elements  $i \in I$ :

$$V = \bigoplus_{i \in I} V_i$$

TODO: glossary:

- Graded vector space
- Graded algebra
- linear operator of degree k
- substitute  $\partial_i f$  with  $\frac{\partial f}{\partial x^i}$  in order to avoid confusion between natural basis vector and usual partial derivative operator acting on function  $f$