Prolegomena on (some) Riemmanian Geometry

Rieammanian manifolds, geodesic,Jacobi fields...

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Cerco di scrivere la teoria di geometria differenziale minimale necessaria per la tesi. Al momento non c'ÃÍ un ordine logico per stabilito (parlo di geodetiche in senso variazionale prima di dire cosa sono le variazioni....)

Fonti Principali di ispirazione sono :

- Jost, J., 2005. Riemannian Geometry and Geometric Analysis, Berlin/Heidelberg: Springer-Verlag. Available at: http://link.springer.com/10.1007/3-540-28891-0 [Accessed January 10, 2015].
- Abate, M., Tovena, F., 2011. Geometria Differenziale, Milano: Springer Milan. Available at: http://link.springer.com/10.1007/978-88-470-1920-1 [Accessed January 10, 2015].
- Abraham, R. et al., 1978. Foundations of mechanics II., Available at: http://www.ams.org/public pages [Accessed January 10, 2015].

Contents

1 Reprise in Riemannian Geometry

In what follows we present a brief review of the most important result in Riemannian geometry necessary for a better understanding of the geodesic problem.

1.1 Definition of (pseudo)-Riemannian manifold

Definition 1: (Pseudo-)Riemannian manifold

Notation fixing

Metric signature Lorenz manifold.

Theorem 1.1 $\forall M$ *is Riemannianizable.*

1.2 Riemannian manifolds as a category.

Definition 2: Local isometry

Definition 3: Isometry

Definition 4: Killing fields

1.3 Riemannian as a measure space.

Definition 5: Riemannian volume form

Theorem 1.2 $\forall M$ *orientable* $\exists 1!$ *Riemannian volume form.*

Observation 1

For an insight on the connection between volume form and measure theory see for example [?].

1.4 Tangent bundle of a Riemannian manifold.

Observation 2

g could be seen as a 2-forms (section $\in \Gamma(T^2_0(M))$

Definition 6: $\flat$ $\sharp$ operator

(sarebbero abbassamento e innalzamento)

Theorem 1.3 *On Riemannian manifold M TM is a structure manifold of structure group $G = O(d)$.*
If M is also orientable $G = SO(d)$.

Proof:

See [?] Lemma 1.5.2 and 1.5.3.

□

1.5 Riemannian as a metric space.

See [?] pag $383-385$ and [?] pag $15-17$.

Consider the space $C^\infty([a,b], M)$ of the smooth parametrized curves from a closed interval of the real line. We define the following functional:

Definition 7: Length functional

$$L(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\| dt$$

Definition 8: Energy functional

$$E(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|^2 dt$$

Are define them together because are linked trough an inequality:

Proposition 1.1 *$\forall \gamma : [a,b] \to \mathbb{R}$ smooth parametrized curve:*

$$L(\gamma)^2 \leq 2(b-a)E(\gamma) \tag{1}$$

equality holds iff $\left\| \frac{d\gamma}{dt}(t) \right\| \equiv const$

Proof:

See [?, Lemma 1.4.2].

□

Observation 3

Former concepts are extended slavishly to every piecewise smooth curve.

Definition 9: Distance between two points

Function $d : M \times M \to \mathbb{R}$

$$d(p,q) := \inf\left\{ L(\gamma) \middle| \, \gamma : [a,b] \to \mathbb{R} \text{ piecewise smooth, } \gamma(a) = p, \gamma(b) = q \right\}$$

Observation 4

Distance is well defined for all pair of point Iff the manifold is connected.

Proposition 1.2 *The distance function satisfies the axioms of metric:*

- non-negative: $d(p,q) \geq 0 \; \forall p,q \in M \qquad d(p,q) > 0 \; \forall p \neq q$
- simmetric: $d(p,q) = d(q,p)$
- triangle inequality: $d(p,q) \leq d(p,r) + d(r,q) \; \forall p,q,r \in M$

Proof:

See [?, Lemma 1.4.1].

□

Corollary 1.1 *The "balls" topology of M induced by the distance function d coincides with the original manifold topology of M.*

Proof:

See [?, corollary 1.4.2].

□

1.6 Connection structure on a Riemannian manifold.

Connection is a rather general concept definable on any smooth bundle. [1]

On vector bundle we can identify a special kind of connection structure compatible with the vector space structure.[2] There are several equivalent presentation of this concept, each of them stress the importance of one of the many devices carried by this superstructure, for example:

- Derivative of section.
- Parallelism and parallel transportation.
- Specification of an unique horizontal lift among all.

Regarding the Riemannian manifolds we're not interested in connections on general vector bundle but instead to those on the tangent bundle, called *Linear Connection.* There's an infinity of such connection but on (pseudo-)Riemannian manifold it's possible to find a natural prescription that allows us to identify only one among these, called *Levi-Civita Connection.*

Consider (M, g) pseudo-Riemannian manifold.

Definition 10: Linear Connection

Map $\nabla : \Gamma^\infty(\tau_M) \times \Gamma^\infty(\tau_M) \to \Gamma^\infty(\tau_M)$, we write $(X, Y) \mapsto \nabla_X Y \qquad \forall X, Y \in \Gamma^\infty(\tau_M)$. Such that:

(a) $\nabla_X Y$ is $C^\infty(M)-$linear in X variable.

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y \qquad \forall f, g \in C^\infty(M)$$

(b)

(c)

1.7 Temp: Memento covariant Derivative

http://www.physicspages.com/2014/01/02/covariant-derivative-of-the-metric-tensor/

$\nabla\mu g_{\alpha\beta} = 0$

[1]In this abstract context connection takes the name of *Erhesmann's connection.*

[2]which takes its name from *Koszul* for distinguish it from the above.

1.8 Curvature on Riemannian manifold.

2 Geodesic

2.1 Common approach to the Geodesic

On a manifold endowed with a affine connection a geodesic is defined as a curve "everywhere parallel to itself" providing a generalization of *straight line*.

Definition 11: Geodesic

A curve ⚠[a] $\gamma : [a,b] \to M$ such that:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \tag{2}$$

where $\dot{\gamma}^\mu := \frac{d\gamma^\mu}{dt}$ is the tangent vector to the curve.

[a]Devo dire smooth o piecewise?

Notation fixing

In local chart the previous equation assume the popular expression:

$$\ddot{\gamma}^i + \Gamma^i{}_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0 \tag{3}$$

Where $\Gamma^i{}_{jk}$ is the coordinate representation of the Christoffel symbols of the connection.

3 Review of physics application of geodesic problem.

Essentially [?]. A lot of mechanics systems can be regard as geodesic problem.

3.1 Preliminary remarks: Geometrical encoding of classical mechanics.

sistemi hamiltoniani
sistemi lagrangiani

3.2 Particle on Riemannian manifold under a position dependant potential.

fomm pag 226-228 + 231-233 teo 3.71

3.3 Relativistic particle.

[3]

[3] For an extension of this process to costrained, dissipitative or ergodic systems see fom cap 3.7

4 Jacobi Fields

4.1 Preliminary remarks: Variation of curve.

Let $\sigma : [a, b] \to M$ a piecewise regular curve on smooth manifold M.

Definition 12: Variation of Curve

Variation of curve σ is a continuous application $\Sigma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ such that

- $\sigma_s = \Sigma(s, \cdot)$ is a piecewise regular curve $\forall s \in (-\varepsilon, \varepsilon)$.
- $\sigma_0 = \sigma$.
- $\exists$ a partion $a = t_0 < t_1 < \ldots < t_k = b$ of $[a, b]$ such that

$$\Sigma\big|_{(-\varepsilon,\varepsilon)\times[t_{j-1},t_j]} \in \mathscr{C}^\infty(\mathbb{R}^2; M)$$

.

Notation fixing

Regarding one entry as a variable and the other as a parameter we can see that Σ determine two family of curves:

- $\sigma_s(\cdot) = \Sigma(s, \cdot)$ is a family of piecewise regular curves called *principal curves*.
- $\sigma^t(\cdot) = \Sigma(\cdot, t)$ is a family of regular curves called *transverse curves*.

Curves in a family have a common parametrization.

Notation fixing

A variation is called *proper* if the endpoints stay fixed, i.e.

$$\sigma_s(a) = \sigma(a) \ \wedge \ \sigma_s(b) = \sigma(b) \qquad \forall s \in (-\varepsilon, \varepsilon)$$

Fields over a variation Σ of a curve σ are defined as follows:

Definition 13: Vector field along a variation

Is a collection $X = \{X_j\}$ of smooth applications $X_j : (-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \to TM$ [a] such that:

$$X_j(s, t) \in T_{\Sigma(s,t)}M \qquad \forall (s, t) \in (-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \quad \forall j = 1, \ldots, k$$

[a] Associate to a subdivision of $a = t_0 < t_1 < \ldots < t_k = b$ of $[a, b]$.

Principal and transverse curves define two special Vector fields along the variation:

Definition 14: Tangent fields of the variation

$$S(s,t) = (\sigma^t)'(s) = \mathrm{d}\Sigma_{(s,t)}\left(\frac{\partial}{\partial s}\right) = \frac{\partial \Sigma}{\partial s}(s,t)$$

for all $(s,t) \in (-\varepsilon, \varepsilon) \times [a,b]$.

$$T(s,t) = (\sigma_s)'(t) = \mathrm{d}\Sigma_{(s,t)}\left(\frac{\partial}{\partial t}\right) = \frac{\partial \Sigma}{\partial t}(s,t)$$

for all $(s,t) \in (-\varepsilon, \varepsilon) \times [t_{j-1}, t_j]$ and $j = 1, \ldots, k-1$ where we have choose a subdivision $a = t_0 < t_1 < \ldots < t_k = b$ associated to Σ.

Notation fixing

$V = S(0, \cdot) \in \mathfrak{X}(\sigma)$ takes the special name of *variation field of* Σ.

There's an importation relation between continuous field on a curve and variation:

Proposition 4.1 *For all continuous field V along a piecewise regular curve σ can be found a variation Σ with variation field V.* [a]

[a] Vice versa follows from the continuity of the variation field.

Proof:

See [**?**] Lemma 7.2.12 .

□

Let now M be a d-dimensional Riemannian manifold with Levi-Civita connection ∇. The tangent fields of a variation are strictly connected to the curvature of M. We need a lemma:

Lemma 4.1 *For all rectangle $(-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \in \mathbb{R}^2$ on which Σ is $\mathscr{C}^\infty$ we have:*

$$D_S T = D_T S$$

where D_S is the covariant derivative along the transverse curves and D_T over the principal curves.

Proof:

See [**?**] Lemma 7.2.13 .

□

The crucial result is what follows:

Proposition 4.2 *For all vector field V along a variation Σ we have:*

$$D_S D_T V - D_T D_S V = R(S,T)V$$

for all rectangle $(-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \in \mathbb{R}^2$ on which Σ is $\mathscr{C}^\infty$. [a]

[a] $R(S,T)$ is the curvature endomorphism evaluated on the tangent vector fields on the variation.

Proof:

See [?] Lemma 8.2.3 .

□

(References: [?] page 386-387 + 420-421 ; [?] page 171)

4.2 Formal Definition

The concept of *Jacobi Field* is closely related to variations of geodesic curves. In fact it describes the difference between the geodesic and an "infinitesimally close" geodesic. In other words, the Jacobi fields along a geodesic form the tangent space to the geodesic in the space of all geodesics.
Let $\gamma : [a,b] \to M$ be a geodesic of the Riemannian manifold M. We can consider a special class of variations:

Definition 15: Geodesic variation

Is a smooth variation $\Sigma : (-\varepsilon, \varepsilon) \times [a,b] \to M$ such that all the principal curves $\gamma_s(\cdot) = \Sigma(s,\cdot)$ are also geodesic.[a]

[a] In other words Σ determines a smoothly variable family of geodesic.

Proposition 4.3 *Fixing two tangent vector over a point $p = \gamma(a)$ on the geodesic γ univocally determines a geodesic variation of γ.*

Proof:

See [?] Lemma 8.2.5 or [?] Lemma 4.2.3.

□

Definition 16: Jacobi Fields

Is a field $J \in \mathfrak{X}(\gamma)$ over a geodesic γ such that:

$\exists\Sigma$ geodesic variation such that $J = V$ represent its variation field [a].

[a]As defined under (def 14).

The following proposition determines an equivalent (analytical) definition of Jacobi field:

Proposition 4.4

$J \in \mathfrak{X}(\dot{\gamma})$ is a jacobi field iff:

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} J + R(X, \dot{\gamma})\gamma' = 0$$

Notation fixing

The vector space of all Jacobi fields on the geodesic γ is denoted $\mathscr{J}(\gamma)$.

Notation fixing

$J \in J(\gamma)$ is called *proper* if $J_0(t) \perp (\dot{\gamma})(t)$.
$\mathscr{J}(\gamma)$ indicates the vector space of all proper Jacobi fields.

Proposition 4.5 *Every killing field X on M is a Jacobi Field along any geodesic in M.*

Proof:

See [?] Corollary 4.2.1.

$\square$

5 Closing Thoughts

5.1 Eliminata

- non messa la definizione dei campi continui e l'osservazione che S ÃÍ sempre continuo mentre T puÃš non esserlo ([**?**] pag 420).
- sono stato ambiguo quando parlo di campi lungo la curva.. sulla continuitÃă o meno (vedere abate pag 387)
- non mi ÃÍ ancora chiaro l'utilitÃă dei jacobi fields... Vediamo le possibilitÃă:
  - Dice Abate a pag. 411 i Jacobi sono lo strumento principale per stabilire una relazione fra curvatura e topologia.
  - Dice Jost a pag. 183 che le Jacobi equation sono una linearizzazione dell'equazione delle geodetiche.
  - Jost a pag $183 - 186$ esplora il legame tra J e le mappe esponenziali.
  - Jost nel capitolo 4.3 e Abate a pag $424 + 433 - 435$ parlano del legame con i punti coniugati e morse theory.
- Discorso della index form come azione le cui equazioni eulero lagrange determinano l'equazione geodetica. (fonte Jost pag $177 - 179$).
- Discorso Decomposizione dei Jacobi field in campi orizzonatali e verticali (fonte Jost pag $180 - 181$, `http://en.wikipedia.org/wiki/Jacobi_field`.