Capitolo 1: (versione preliminari)

An excursus on Bundle

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August 7, 2015

Contents

In this first chapter we will devote a bit of time to present the *Bundles*, a family of algebraic structures of particular importance in modern mathematical-physics.
We will follow a sort of deductive approach.

We start defining the abstract structure of *Fiber bundle* over the category of topological spaces, underlining that they represent the most natural setting for encoding the concept of physicist's *fields* and not forgetting that they form a concrete category per se.

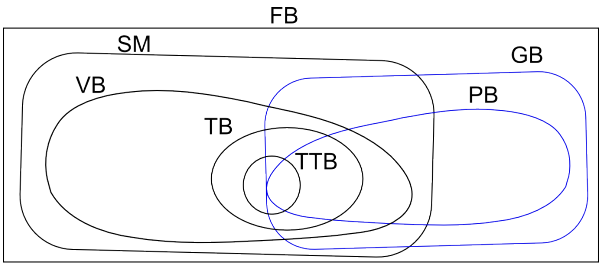
In paragraph 2 we will enrich this abstract object with a so called *G-Structure*, a superstructure that must be necessarily identified if you want to have avaiable a concept of compatibility between overlapping trivializations.

In third paragraph will be further specialize the construct on which is defined the bundle to not being simply a topological spaces but rather a smooth manifold (). [1]
This step provides the possibility to explore the relation between tangent spaces of the two manifolds which constitute the bundle, base and total space. The means for formalizing that will be the operation of *Lift* and *Drop*.

In paragraph 4 will arise for the first time a constraint on the fiber space, namely the prescription that is equipped with a linear space structure. In other words we will talk about *Vector Bundle*[2]. In this less general context we will deal with the problem of establish a bundle structure on a manifold having only a collection of omeomorphic fibers.

At last, in fifth paragraph, will be presented the *Tangent Bundle* the most significant example of smooth vector bundle.

Figure 1: Eulero-Venn Diagram of the Bundle family.



[1] These spaces constitute a subcategory of topological spaces but actually what follows applies to every order of differentiability.

[2] In what follows we only consider *smooth* Vector Bundle.

1 Fiber Bundle

Roughly speaking a *Fiber Bundle* is a way of attach some set, the so called *fibre*, on every point of another space, called *Base*. The main tool for achieving this "glueing" are the surjective function as we can guess from this observation

Observation 1

$\forall\ \pi : E \to M$ surjective function between generic set with $Dom(\pi) = E$

$$E = \bigsqcup_{p \in M} \pi^{-1}(p) = \bigsqcup_{p \in M} E_p$$

In this extremely simple case the sets $E_p = \pi^{-1}(p)$ take the role of fibers and M the base. In the next section we will see that the space E it's a crucial actor in the formal definition of a fiber bundle to a point that often this *total space* is mistaken with the bundle itself. [1]

1.1 Formal Definition

Remark:

In what follows all the set considered are endowed with a topological structure that is are topological space $(X, (top)(X))$.

Definition 1: Fiber Bundle

A *Fiber Bundle* consists in a 4-ple (E, B, π, F) where:

- E : topological space (called *Total Space*)
- B : topological space (called *Base Space*)
- F : topological space (called *Typical Fiber*)
- $\pi : E \to B$ continuous surjective function (called *Bundle Projection*)

Endowed with a *Local Trivialization*:

- $\forall x \in E\ \exists$ a couple (U, χ) (called *local trivialization*)
  - U : neighborhood of x
  - $\chi : \pi^{-1}(U) \to U \times F$: homeomorphism [a] [b]

  such that: $p_1 \cdot \chi = \pi|_{\pi^{-1}(p)}$.

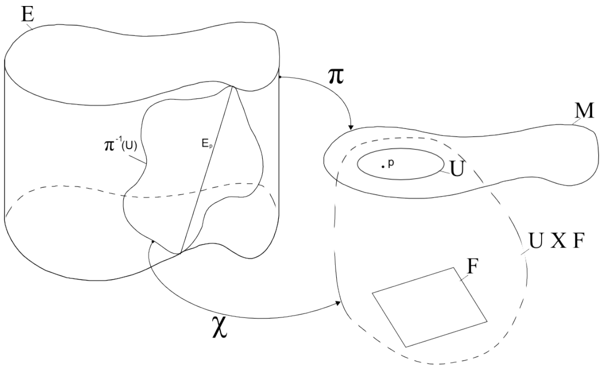
  i.e: the following graph commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\chi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "p_1"] \\
U &
\end{tikzcd}$$

[a] surjectivity $\Rightarrow \pi^{-1}(U) \neq \emptyset$.

[b] cartesian product of topological space is a topological space with the direct product topology.

Figure 2: The complete fiber bundle Structure.



As said in the introduction, in this aggregate of objects the role of fiber attached to each point of the base space is taken by the counterimage of π. This deserve a proper definition:

Definition 2: Fiber over a point $p \in B$

$E_p := \pi_{-1}(p)$

Ontologically[3] we distinct between "typical fiber" and " fiber over a point" but the axiom of local trivialization assures that topologically they are the same:

Lemma 1.1 *The typical fiber F and the fiber upon a point are homeomorphic.*

Th: $F \simeq E_p \; \forall p \in B$

Proof:

For each $p \in B$ is given a local trivialization (U, χ) such that $p \in U$.

Noting that $\forall$ topological space $p \times A \simeq A$, follows from the definition this commutation diagram:

[3] i.e. element of one it's a different object respect the other.

$$
\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\chi"] \arrow[d, "\pi"'] & p \times F \simeq F \arrow[dl, "p_1"] \\
p &
\end{tikzcd}
$$

In conclusion $\chi|_{E_p}$ realizes an homemorphism between F and E_p

□

Notation fixing

It's customary to refer to the fiber bundle $(E, \pi, M; Q)$ indicating only its total space E:

$$E = (E, \pi, M; Q)$$

A possible, more heavy, convention is to denote the fiber bundle as a *short sequence*[**?**]

$$Q \to E \xrightarrow{\pi} M$$

Notation fixing

⚠(da correggere)
We say that a fiber bundle E is *(globally) trivial* if there exists a fiber preserving diffeomorphism from E to the Cartesian product MÃŮQ which is a vector space isomorphism on each fiber. In practice, this corresponds to a trivialization of E which is defined everywhere, to be compared with the notion of a local trivialization per Definition 1.

Given two bundle E and F on the same base M, it's straightforward the definition of some additional structures just performing the algebraic operations fiberwise:

Definition 3: Bundle Restriction

See [5].

⚠: definizione ripetuta!

Definition 4: Cartesian Product Bundle

Fiber bundle $E \times_M F$ over M such that :

$$(E \times_M F)_p = E_p \times F_p$$

a

[a] for every point $p \in M$, we take the tensor product of the fibers, $E_p \times E'_p$.

⚠ List pagg 9 [6]

1.2 Cross Sections

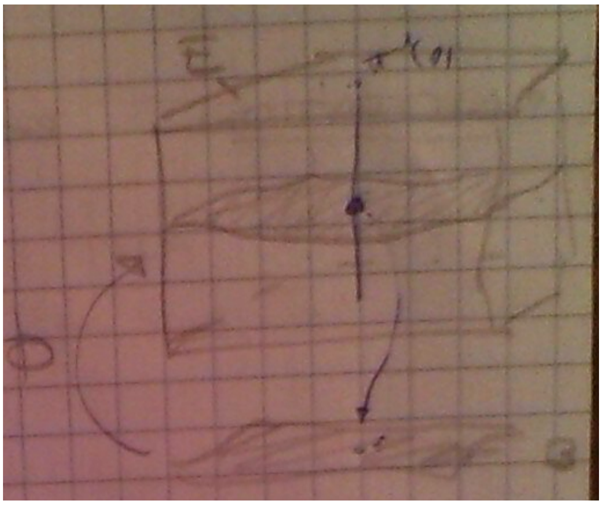
The notion of bundle is particularly interesting from the perspective of physics because provides the rigorous description of a $F-$valued field on the space B.

Definition 5: (Cross) Section

Function $\phi : B \to E$ such that:

- ϕ continuous.
- $\phi \cdot \pi = \text{Id}_B$

Figure 3: Section on a Bundle.



Notation fixing

We refer to:

- *Global section* $\Leftrightarrow \text{dom}(\phi) = B$
- *Local section* $\Leftrightarrow \text{dom}(\phi) \subset B$ [a]

[a] Usually the domain is an open set of B)

Observation 2

The property that essentially makes a section ϕ a good abstraction of a field is the following:

$$\forall p \in B \phi(p) \in \pi^{-1}$$

In other words:

Proposition 1.1 *Local section $\{\phi\}$ are in a 1:1 correspondence with continuous function $\{f : B \to F\}$.*

Proof:

Take $p \in B$ and (U, χ) local trivialization over p.

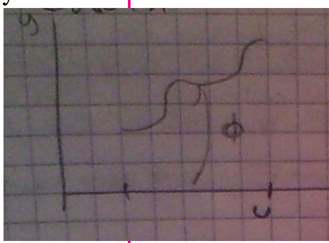
Define $f : U \to F$ as $f = p_2 \cdot \chi \cdot \phi|_U$, where p_2 is a projection on the second element of a cartesian product space.

Then : $\chi \cdot \phi(p) = (p, f(p))$ (...).

□

Observation 3

The preceding argument give meaning to the claim often presented in geometry books



that: " cross section represent an abstract generalization to graph of functions."

Notation fixing

The *set of all section* is often denoted as:

$$\Gamma^{\infty}(E)$$

1.3 Maps between Fiber Bundles

Consider two fiber bundle (F, E, π, B) and (F', E', π', B').

Definition 6: Bundle Morphism

A pair of map $(\phi_{tot}, \phi_{base})$ where:

- $\phi_{tot} : E \to E'$ continuous.

- $\phi_{base} : B \to B'$ continuous.

Such that

$$\pi' \cdot \phi_{tot} = \phi_{base} \cdot \pi \tag{1}$$

, i.e the following graph commutes:

$$\begin{tikzcd} E \arrow[r, "\phi_{tot}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "\phi_{base}"] & B' \end{tikzcd}$$

Observation 4

Restricting the equation (1) to act only on a specific fiber,

$$\pi' \cdot \phi_{tot}|_{E_p} = \phi_{base} \cdot \pi(E_p) = \phi_{base}(p) := p'$$

we can see that precedent definition it's equivalent to requirement that ϕ_{tot} is fiber preserving:

$$\forall p \in B \qquad \phi_{tot}(E_p) = E_{\phi_{tot}(p)}$$

Follows that to determine a bundle morphism is sufficient to provide a fiber preserving map between the total spaces. It's then customary to denote a bundle morphism with ϕ_{tot} only.

Proposition 1.2 (Fiber Bundle as a Category.)

Hp:

- $\mathfrak{C}$ = *set of all possible fiber bundle.*
- $hom(\mathfrak{C})$ = *set of all bundle morphism.*

Th: *The couple (C, hom(C)) form a concrete category.*

Proof:

Technicality (...).

□

Observation 5

Note that the fiber projection of a fiber bundle is a continuous map, than an homomorphism of the topological space category (a particular one, which satisfies the axiom of local triviality).

Definition 7: Bundle isomorphism

A bundle morphism $(\phi_{tot}, \phi_{base})$ such that $\phi_{\cdot}$ are homeomorphism.

Notation fixing

It's frequent to refer at the bundle morphism between fiber bundle over the same base ($\phi_{base} = Id_B$) as *Fiber Preserving map.*
$\phi : E \to F$ continuous such that:

$$\phi(E_x) = F_x \qquad \forall x \in M.$$

i.e.:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

⚠Fonte wiki Consider a topological space N, a fiber bundle $E = (E, \pi, M; Q)$, and a continuous function $f : N \to M$. It's possible to induce[**?**] a bundle structure from the manifold M to N:

Definition 8: Pull-Back Bundle

3-ple $f^*(E) = (f^*(E) =, \pi^*, N)$ such that:

- $f^*(E) = \left\{(b', e) \in N \times E \;\middle|\; f(b') = \pi(e)\right\}$
- $\pi^* : f^*(E) \to N$ such that $\pi * (b', e) = \mathrm{pr}_1(b', e) = b'$

$$\begin{tikzcd} f^*E \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

Proposition 1.3 $f^*(E) = (f^*(E) =, \pi^*, N)$ *consitute a fiber bundle of typical fiber* Q.

Proof:

If we want to complete the fiber bundle structure we have to provide a local trivialization atlas. $\forall (U, \phi)$ local trivialization on (E, π, M) consider $\psi : f^*E \to N \times Q$ such that $\psi(b', e) = \Big(b', pr_2\big(\phi(e)\big)\Big)$.

Then $(f^{-1}(U), \psi)$ is a local trivialization of the pull-back bundle and the fiber of f^*E over a point bâĂš $\in B'$ is just the fiber of E over $f(b$âĂš$)$.

□

Observation 6

Consider this situation:

$$\begin{tikzcd} & E \arrow[d, "\pi"'] \\ N \arrow[r, "f"] & M \arrow[u, bend right, "s"'] \end{tikzcd}$$

where $s \in \Gamma(\pi_M)$. Pull-Back of Section is easily obtained as follow:

$$f^* s = s \cdot f \in \Gamma(f^* E)$$

It is also noteworthy that, given any two vector bundles $E = (E\pi, M, Q)$ and $E = (E'\pi, M', Q')$, we can construct naturally a third fiber bundle. Consider $\hom(E, E')$ the set of all the fiber preserving map between the two bundles:

Definition 9: Bundle of morphisms

Fiber bundle $\hom(E, E')$ over the base space M such that the fiber over a base point $p \in M$ is the infinite dimensional manifold $\hom(E_p, E'_p)$ isomorphic to $\hom(Q, Q')$.

Notation fixing

We shall write $End(F)$ for $\hom(E, E)$ and call it bundle of endomorphism, whose typical fiber is $\mathrm{End}(Q)$.

Remark:

If F, F' are vector bundle then the fiber of $\hom(F, F')$ over a base point $p \in M$ is $\hom(F_p, F'_p)$, which is a vector space isomorphic to the vector space $\hom(V, V')$ of linear applications from V to V'

1.4 Toward other type of bundle.

Roughly speaking a fiber bundle is an agglomerate of fiber space over a different space called base. Fiber and Base could have different structure, a sort of compatibility between structure is guaranteed by the properties of π and χ. Category theory provides the appropriate language to treat various bundle structure in a unified way.

Consider two construct $\mathbf{C}_1$, $\mathbf{C}_2$ subcategory of **Top**, concrete category of all topological spaces [4], such that:

$$\mathbf{Top} \supseteq \mathbf{C}_1 \subseteq \mathbf{C}_2$$

Definition 10: $(\mathbf{C}_1)-$**Bundle of** $(\mathbf{C}_2)-$**fiber**

It's a 4-ple (E, M, π, F) where:

- $E, M \in Ob(\mathbf{C}_1)$: (called *Total Space* and *Base Space*)
- $F \in Ob(\mathbf{C}_2)$: (called *Typical Fiber*)
- $\pi \in \hom(E, B) \subset Mor(\mathbf{C}_1)$: (called *Bundle Projection*)

Such that:

- π surjective.
- $\pi^{-1}(p) \in Ob(\mathbf{C}_2) \quad \forall p \in M$
- $\forall p \in M \, \exists (\chi, U)$ (local trivialization) such that:
  - U is a neighbourhood of p
  - $\chi \in \mathrm{iso}(E, U \times F) \subset Iso(\mathbf{C}_1)$
  - $\chi|_{\pi^{-1}(p)} \in \mathrm{iso}(\pi^{-1}(p), \{p\} \times F) \subset Iso(\mathbf{C}_2)$

the main categories of interest are the following:

Category	Obj	Mor	Iso
Top	topological spaces	continuous functions	homeomorphism
Smooth	smooth manifold	differentiable functions	diffeomorphism
GLie	Lie groups	homomorphism	group isomorphism
Vec	Vector spaces	linear operators	$\mathbb{GL}-$operators

[4]Set of the object must be considered together with a cartesian product operator $\times : \mathrm{obj} \times \mathrm{obj} \to \mathrm{obj}$.

2 Structure Group and transition Function

From what we have seen seems legit to consider the local trivialization of a fiber bundle as the analogous of a a local chart on a smooth manifold.

That make sense to the idea of fiber bundle (thought as its total space) as a space which is locally a product space link. (But globally may have a different structure, Not all bundle are trivial).

However in the definition of Bundle is required the existence of at least one trivialization chart for each point but no notion of *compatibility* is explicitly required.

2.1 The problem of *Overlapping Trivialization*.

Consider two local trivializations (with $i = \alpha, \beta$):

$$\chi_i : \pi^{-1}(U_i) \to U_i \times F$$

overlapping, that is $U_\alpha \cap U_\beta \neq \emptyset$.

Definition 11: Transition Function (from α to β)

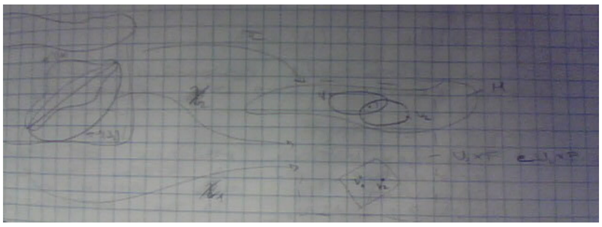
$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{aut}(F)$$

[a] given by

$$\chi_\beta \cdot \chi_\alpha^{-1}\big(p, V_\alpha\big) = \big(p, g_{\beta\alpha}[p](V_\alpha)\big) = \big(p, V_\beta\big) \qquad \forall p \in U_1 \cap U_2, \forall V_\alpha \in F$$

[a] In the category of topological spaces $\mathrm{aut}(F)$ consists of homeomorphism from F to itself.

Figure 4: Transition map between local trivialization.



Notation fixing

It's common to refer to the transition map as the well defined homeomorphism:

$$\chi_\beta \cdot \chi_\alpha^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

instead of the function $g_{\beta\alpha}$ which realizes the transformation.

In analogy with the atlas of chart on a manifold also the collection of all the local trivialization supplied to the Bundle structure takes a specific name:

Definition 12: Bundle (Trivialization) Atlas

Is a collection of local trivialization which cover the entire base space:

$$\{(U_\alpha, \chi_\alpha) \big| \bigcup_\alpha U_\alpha \supseteq M\}$$

Since for each pair of overlapping map is defined a transition function, every bundle atlas carries with itself a collection of such maps:

$$\{g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{aut}(F) \big| U_\alpha \cap U_\beta \neq \emptyset\}$$

its cardinality is determined by the number of overlapping open set in the atlas.

Proposition 2.1 *The transition maps relating to a specific atlas always meet the following properties:*

$$g_{\alpha\alpha}(p) = \mathbb{1}_F \qquad \forall p \in U_\alpha \tag{2}$$

$$g_{\beta\alpha}(p) = g_{\alpha\beta}^{-1}(p) \qquad \forall p \in U_\alpha \cap U_\beta \tag{3}$$

$$g_{\beta\gamma}(p) g_{\gamma\alpha}(p) = g_{\beta\alpha}(p) \qquad \forall p \in U_\alpha \cap U_\beta \cap U_\gamma{}^a \tag{4}$$

[a] *cocycle condition*

Proof:

- (2) follows from the composition rule:

$$\chi_\alpha \chi_\alpha^{-1} = \mathbb{1}_{U_\alpha \times F}$$

- (3) follows from:

$$\left(\chi_\alpha \cdot \chi_\beta^{-1}\right)^{-1} = \chi_\beta \cdot \chi_\alpha^{-1}$$

- (4) follows from:

$$\left(p, g_{\beta\alpha}(p)V\right) = \left[\chi_\beta \cdot \chi_\alpha^{-1}\right](p, V) = \left[\chi_\beta \cdot \chi_\gamma^{-1}\right]\left[\chi_\gamma \cdot \chi_\alpha^{-1}\right](p, V) = \left(p, g_{\beta\gamma}(p) g_{\gamma\alpha}(p) V\right)$$

□

2.2 Structure Group

From the definition is clear that the transformation maps are valued in a group (the group of automorphism $\mathrm{aut}(F)$) but in general the set $\{g_{\alpha\beta}[p]\}$ for a fixed p don't form a subgroup [5].

[5] Or , equivalently, the map $\{g_{\alpha\beta}\}$ is not the action of some group.

Example: 1

Being a group would means that fixed four overlapping trivialization$\alpha, \beta, \gamma, \delta$ must exists another couple of trivialization θ, η such that:

$$g_{\alpha\beta} \cdot g_{\gamma\delta} = g_{\theta\eta}$$

obviusly there's no natural way of construct such composition from the cocycle condition only.

For this reason, the following definition arise spontaneously:

Definition 13: G-Atlas

It's a trivialization atlas $\{(U_i, \chi_i)\}$ such that the corresponding transition maps constitutes a group left-action of the abstact group G on the fiber space F.

Notation fixing

It's common to use the following names when referring to a G-structered fiber bundle:

- *G-Bundle* : fiber bundle rigged with a G-atlas of trivialization.
- *Structure group* : the abstract group G whose actions realize the transition maps.

The choose of such solemn name for the structure group is justified by the following theorem:

Theorem 2.1 *[1] Fixing a typical fiber F, a base space M and a G-action $g_{\alpha\beta}$ which map the transition function is sufficient to determine the G-Bundle (F, E, π, M, G).*

Hp:

1. *M, F topological spaces.*
2. *$\{U_\alpha\}_{\alpha \in I}$ open cover of M.*
3. *is given a family $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to aut(F)\}$ such that:*
    - *$g_{\alpha\beta} : (p, f) \mapsto g_{\alpha\beta}(f)$ is an homeomorphism.*
    - $g_{\alpha\alpha}(p) = \mathbb{1}_F \qquad \forall p \in U_\alpha$
    - $g_{\beta\alpha}(p) \cdot g_{\alpha\beta}(p) = \mathbb{1}_F \qquad \forall p \in U_\alpha \cap U_\beta$
    - $g_{\beta\gamma}(p) \cdot g_{\gamma\alpha}(p) \cdot g_{\alpha\beta}(p) = \mathbb{1}_F \qquad \forall p \in U_\alpha \cap U_\beta \cap U_\gamma$

Th:

1. *The quotient space* $E = \dfrac{\bigcup_{\alpha\in I}(U_\alpha \times F)}{\sim}$ *with :*

$$(p_\alpha, f) \sim (p_\beta, g_{\beta,\alpha}(f)) \qquad \forall p_\alpha = p_\beta \in U_\alpha \cap U_\beta \, \forall f \in F$$

endowed with the quotient topology is a topological space.

2. *The projections on the first argument* $p_1 : U_\alpha \times F \to U_\alpha$ *fitted together defines a good bundle projection* $\pi : E \to M$*, i.e.:*
   - π *injective*
   - $\forall p \, \pi^{-1}(p)$ *homemorphic to* F

Proof:

See theorem 4.1 for the demostration in a rather simpler case.

□

2.3 A glance on Principal Bundle

Imposing a further prescription on the properties of a G-structure on a G-Fiber Bundle we can identify a particular structure often used in mathematical physics.

Definition 14: Principal Bundle

Is a G-bundle such that the transition maps $t_{\alpha\beta}$ as an action of the group G is:

- *free* : $\forall g \in G \setminus \{\mathbb{1}\} \quad : \quad t_{[g]} \cdot s \neq s \quad \forall s \in F$
- *transitive* : $\forall x, y \in F, \exists g \in G \setminus \{\mathbb{1}\}$ such that : $t_{[g]} x = y$.

Observation 7

A such action permit a complete identification of F with the group G.

For this reason is common place in the literature to present this further structured bundles as fiber bundle where the typical fiber F is endowed with a Lie group structure and such that the local trivialization functions are Lie group isomorphism when restricted on a fiber.

3 Smooth Bundle

In the context of mathematical physics is more frequent referring to smooth fiber bundle instead of only topological ones. From 10 follows the following definition:

Definition 15: Smooth Fiber Bundle

Is a fiber bundle (F, E, π, M) such that:

- E, F, M are not only topological but smooth manifold.
- π is a smooth surjective function.
- χ_α is a diffeomorphism $\forall \alpha$.

Since all differentiable manifolds are, in first instance, topological spaces all the statement above remain valid with the exception of consider all function *differentiable* instead of *continuous* only. (e.g. in this framework the section are also differentiable, some texts use the symbol $\Gamma^\infty(\pi_M)$ to stress this fact.)
The few more peculiarity in considering this additional smooth structure on the spaces constituting the bundles essentially come from the presence of the local charts and the tangent spaces.

3.1 Relation between local charts and local trivializations.

When a smooth fiber bundle (F, E, π, M) is considered, in addition to the typical functions of the bundle (π, χ_α) are to be taken in account all the collection of local chart for the three manifold : $(U_{\alpha_k}, \phi_{\alpha_k})_{k=E,M,F}$. The context require to not confuse the chart with the trivialization even if there is a relationship between them:

Proposition 3.1 *Atlas on M and F induce an atlas on E through the local trivialization.*

Proof:

Consider (U, ϕ_M) and (V, ϕ_F) local charts on M and F respectively.
Every local trivialization (U_α, χ_α) such that $U_\alpha \supseteq U$ is a diffeomorphism,
therefore $\chi^{-1} : (U \times V) \longmapsto W \in \mathscr{T}(E)$ maps open set in open set, thus

$$\left(\chi^{-1}(U \times V), (\phi_M \times \phi_F) \cdot \chi\right)$$

is a local chart on the manifold E.
Since such local trivialization exist for all point in M with this process is possible to map each fiber and then consitute a whole atlas on E.

$\square$

Proposition 3.2 (vice versa) *An atlas on E induce an atlas on M and F through the local trivialization.*

Proof:

Consider (W, ϕ_E) local chart on E and a local trivialization (U_α, χ_α) on M.
Take an open set $U' \subset U_\alpha$ in M , π is continuous then $W' = W \cap \pi_{-1}(U')$ is an open set in E. Moreover $V' = p_2 \cdot \chi_\alpha(W')$ is an open set in F.
In conclusion $\phi_E \cdot \chi_\alpha^{-1}$ constitute a chart on $U' \times V'$ and, by projection on components of the cartesian product, on the manifold M and F.

□

Furthermore could be useful defining a patch on M which map the base spaces and trivializes the bundle in the same time:

Definition 16: Local chart (of M) trivializing (E)

Triple (U, ϕ, χ) such that:

- U open set in M.
- $\phi : U \to \mathbb{R}^{\dim(M)}$ diffeomorphism.
- $\chi : \pi^{-1}(U) \to U \times F$ trivialization.

Observation 8

At this point we can see a source of confusion that comes from the identification of the whole fiber bundle (F, E, π, M) with the total space E only:

$$\text{Bundle Atlas} \neq \text{Atlas of charts on the manifold E}$$

That suggests to aggregate the two concepts in an unique definition:

Definition 17: Trivializing Atlas of charts

Collection of local charts of M which trivilizes E such that:

$$\{(U_\alpha, \phi_\alpha, \chi_\alpha | \bigcup_\alpha U_\alpha \supseteq M\}$$

Notation fixing

Is customary to consider such atlas of trivializing charts as the proper *bundle atlas* of a smooth bundle.

3.2 Lifting objects from the base space to the complete space

There're basically two idea under the concept of *lift* and *drop* in a smooth fiber bundle.

- E and M are smooth manifold, then it's perfectly legit to consider the tangent spaces on both of them.
- π is a smooth map, then are well defined the notion of pull-back and push-forward (through the differential $d\pi$.

Drop and *Lift* are only two different name, introduced for this context, for the mapping trough the differential of the projection function.
Consider a parametrized curve $\gamma : \mathbb{R} \to E$ on the total space:

Definition 18: Drop of curves

Parametrized curve $\gamma^D : \mathbb{R} \to E$, such that:

$$\gamma^D = \pi \cdot \gamma$$

Regarding the tangent vectors as velocity vectors of equivalence classes of curves follows easly the next definition:

Definition 19: Drop of vectors

$$\forall v \in T_{e_p}E \quad v^D := d\pi \cdot v = V_* \in T_pM$$

where e_p is a point of E in the fiber over p.

Definition 20: Lift of 1-forms

$$\forall \alpha \in T_p^*M \quad \alpha^L := \alpha^* \in T_{e_p}^*E$$

where e_p is a point of E in the fiber over p.

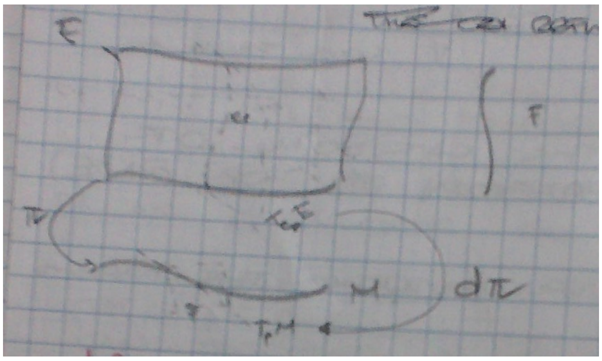
Observation 9

The former operation are naturally implemented by the presence of the special smooth function π, on the contrary their inverse are not natural (π is not invertible) and require some additional structure like the choose of a cross-section.

3.3 Decomposition in vertical and horizontal tangent space.

On the total space of a bundle is naturally identified a special class of curves:

Figure 5: Lift Drop1.

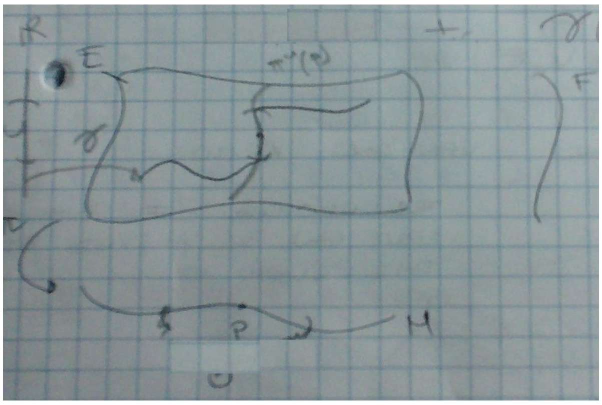


Definition 21: locally vertical curves

$$\gamma : \mathbb{R} \to E \text{ such that: } \exists U \subseteq \mathbb{R} : \pi(\gamma)|_U = p$$

i.e. are curves of which at least a portion of them lies entirely on a fiber.

Figure 6: Locally vertical curve.



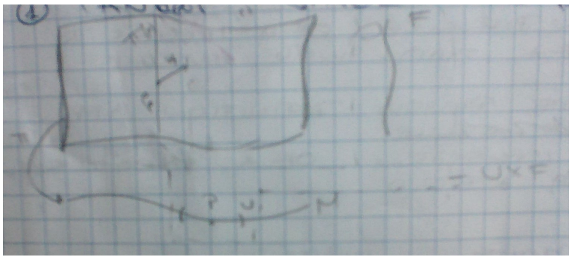
Follows the concept of vertical vectors:

Definition 22: Vertical Vector

$v \in T_e E$ is vertical if $\left(\mathrm{d}\pi\right)(v) = 0$

Observation 10

The drop of a vertical curve can be seen as the motion of a particle which remain still in p for an interval U of time in his parameter space.

 Take M, N manifold and $\phi : M \to N$ smooth.
Be $\mathrm{d}\phi_p(\dot{\gamma}) = 0 \;\forall\, p \in \gamma(U)$.
Then $\gamma' = \phi \cdot \gamma$ is the trajectory of a point which remain still for $t \in U \subset \mathbb{R}$.

Definition 23: Vertical Tangent SubSpace

$$V_e E = \ker(\mathrm{d}\pi) \subset T_e E$$

Observation 11

$V_e E$ coincides with the tangent space to the submanifold $\pi^{-1}(p) \subset E$ in the point e_p.

Definition 24: horizontal Tangent SubSpace

Complementary subspace[a] $H_e E \subset T_e E$. i.e. such that $T_e E = H_e E \oplus V_e E$

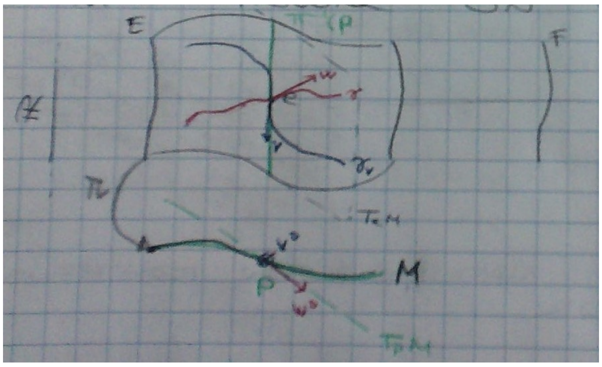
[a] one out of many.

Observation 12

Where the vertical subspace is univocally determined by π his complementary , the horizontal subspace, is not unique in general.

The choice of such names can be argued from figure 7.

Figure 7: Comparison between drop of vertical and general curve (or vectors).



Observation 13: First take on the concept of Fiber Connection

We have just seen that the concept of *vertical component* of a tangent vector is coupled with the drop of vectors and is univocally determined by the fiber projection π present on the bundle.
This is not true for the opposite concept of *horizontal component*. In general there is not a natural way of selecting a fixed complementary space but additional structure is needed (e.g. a condition of orthogonality provided by a riemmanian metric).
The specification of an horizontal subspace for every point in E is an additional structure called *Fiber Bundle Connection*

4 Vector Bundle

Specializing further the smooth fiber bundle imposing the linear space structure leads us to define the *vector bundle*.

Definition 25: Vector Bundle

Is a smooth fiber bundle (V, E, π, M) such that:

- V, typical fiber space, is a vector space.
- All the trivialization χ_α are diffeomorphism such that:

$$\chi_\alpha|_{\pi^{-1}(p)} \in \mathbb{GL}(n, \mathbb{R})$$

Observation 14

It's frequent in literature to present the vector bundle as a smooth bundle with typical fiber $\mathbb{R}^n$.
If we just consider finite dimensional fiber vector space the difference is totally irrelevant in virtue of the well known natural[a] isomorphism $V \simeq \mathbb{R}^n$ of vector decomposition in components on a base.

[a] In the sense that is not dependent by the chosen basis

To encompass this two slightly different point of view we make a little revision of the definition of *trivialization* in the context of vector bundle:

Definition 26: Local chart (of M) trivializing (E)

Triple (U, ϕ, χ)[a] such that:

- U open set in M.
- $\phi : U \to \mathbb{R}^{\dim(M)}$ diffeomorphism.
- $\chi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ trivialization chart.

[a] It's a standard trivializing chart with the extra feature of defining implicitly a decomposition of V on a basis.

Observation 15

If we consider a whole atlas of such chart will follows that the transition maps will be $\mathbb{GL}(n, \mathbb{R})$ valued, in other words the $g_{\alpha\beta}$ will be change of basis matrix.

4.1 Construction of a Vector Bundle.

The next theorem represent a criteria to establish when a collection of isomorphic vector spaces constitutes a vector bundles.

Theorem 4.1 *Given an "almost"*[a] *vector bundle it's sufficient to provide a collection of transition functions to complete the structure.*

Hp:

1. *M = smooth manifold*
   E = simple set (not a manifold)
   $\pi : E \to M$ = surjective function (not smooth)

2. *Endowed with an "almost" open trivialization atlas:*
   $\mathscr{A} = \{(U_\alpha, \chi_\alpha)\}$ such that
   - *$\{U_\alpha\}$ it's an open cover of M.*
   - *$\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ bijective (not diffeomorphism) and $p_1 \cdot \chi_\alpha = \pi$.*

3. *Is provided a chart atlas (U_α, ϕ_α) on the precedent open cover together with all the* transition map *$g_{\alpha\beta}$, i.e:*

$$\forall(\alpha,\beta) : U_\alpha \cap U_\beta \neq \emptyset \; \exists g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{GL}(n,\mathbb{R}) \text{ diffeomorphism}$$

such that:

$$\chi_\alpha \cdot \chi_\beta^{-1}(p, \vec{v}) = (p, g_{\alpha\beta}(p)\vec{v})$$

[a] Similar to the case presented in observation 1.

Th:

E admit an unique vector bundle structure in which χ_α are local trivialization.

The hypothesized structure lacks the following properties in order to form a vector bundle:

a) The fiber upon a point has to be isomorphic to the typical fiber, i.e. $E_p \simeq \mathbb{R}^n \quad \forall p$.

b) E has to be a smooth manifold.

c) χ has to be a diffeomorphism

d) π has to be differentiable.

Proof:

a) Using Hp.2 is possible to associates $\forall V \in E$ $\vec{V} \in \mathbb{R}^n$ biunivocally:

$$\chi_\alpha|_{E_p} : V_p \in E_p \leftrightarrow \vec{V} \in \{p\} \times \mathbb{R}^n \simeq \mathbb{R}^n$$

Then endow E_p with a natural vector bundle structure:

$$u_1 + \lambda u_2 = \chi_\alpha^{-1}\big(p, \vec{u}_1 + \lambda \vec{u}_2 \qquad \forall u_1, u_2 \in E_p \; \forall \lambda \in \mathbb{R}$$

In other words all local trivialization containing p induce a vector space struc-

ture, this is is well defined if the linear composition defined through χ_α is the same as the structure defined through χ_β $\forall p \in U_\alpha \cap U_\beta$.
Take $u \in E_p$ and define $\vec{v}, \vec{u}$ such that $\chi_\alpha(u) = \big(p, \vec{v}\big)$ and $\chi_\beta(u) = \big(p, \vec{w}\big)$.
By Hp.3 $(p, \vec{V}_\alpha) = \chi_\alpha \circ \chi_\beta^{-1}(p, \vec{V}_\beta) = (p, [g_{\alpha\beta}](p)\vec{V}_\beta)$ So the good definition is assured by:

$$(u_1 + \lambda u_2)^{(\alpha)} = \chi_\alpha^{-1}(p, \vec{v}_1 + \lambda \vec{v}_2 = \chi_\alpha^{-1}(p, g_{\alpha\beta}\vec{w}_1 + \lambda g_{\alpha\beta}\vec{w}_2 = \tag{5}$$

$$= \chi_\alpha^{-1}(p, g_{\alpha\beta}(\vec{w}_1 + \lambda \vec{w}_2 = \cancel{\chi_\alpha^{-1} \circ \chi_\alpha} \circ \chi_\beta^{-1}(p, \vec{w}_1 + \lambda \vec{w}_2 = (u_1 + \lambda u_2)^{(\beta)} \tag{6}$$

b) It's possible to endow E with an atlas of compatible charts.
$\{\pi^{-1}(A) | A \in \mathrm{top}(M)\}$ constitutes a topology on E.
Surjectivity of $pi \Rightarrow \{\pi^{-1}(U_\alpha) = \tilde{U_\alpha}\}$ it's an open cover of E.
$\tilde{\chi_\alpha} : \pi^{-1}(U_\alpha) \to \mathbb{R}^{\dim(M)} \times \mathbb{R}^n$ such that $tilde\chi_\alpha = (\phi_\alpha \times \mathbb{1}) \circ \chi_\alpha$ consitutes a chart on E $\forall \phi_\alpha$ chart on M.
Transition chart are smooth because composition of two smooth function:

$$\tilde{\chi_\alpha} \circ \tilde{\chi_\beta^{-1}} = (\phi_\alpha \circ \phi_\beta^{-1}, g_{\alpha\beta}) = (\phi_\alpha \circ \phi_\beta^{-1}) \times (g_{\alpha\beta})$$

Then $\tilde{\mathscr{A}} = \{(\tilde{U_\alpha}, \tilde{\chi_\alpha})\}$ constitutes an atlas of $C^\infty-$compatible charts.

c) $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ it's smooth if $(\phi_\alpha \times \mathbb{1}) \circ \chi_\alpha \circ \tilde{\chi_\alpha}^{-1} : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ is also smooth , that's guaranteed by its definition:

$$(\phi_\alpha \times \mathbb{1}) \circ \chi_\alpha \circ \tilde{\chi_\alpha}^{-1} = (\phi_\alpha \times \mathbb{1}) \circ \chi_\alpha \circ \big((\phi_\alpha \times \mathbb{1}) \circ \chi_\alpha\big)^{-1} = \mathbb{1}$$

d) $\pi : \pi^{-1}(U_\alpha) \to U_\alpha$ is smooth if $\phi_\alpha \circ \pi \circ \tilde{\chi_\alpha}^{-1}$ is also smooth, that's guaranteed by Hp.2:

$$\phi_\alpha \circ \pi \circ \tilde{\chi_\alpha}^{-1} = \phi_\alpha \circ \pi \circ \chi_\alpha^{-1} \circ (\phi_\alpha \times \mathbb{1})^{-1} = \phi_\alpha \circ p_1 \circ (\phi_\alpha^{-1} \times \mathbb{1})\mathbb{1}$$

□

Theorem 4.2 *It's possible to reconstruct a vector bundle only from the transition maps.*

Hp:

1. *Be M a smooth manifold and $\mathscr{A} = \{(U_\alpha, \phi_\alpha\}$ atlas of local charts.*

2. *$\forall$ couple $U_\alpha, U_\beta \in \mathscr{A}$ is given a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{GL}(n, \mathbb{R})$ such that:*

   (a) $g_{\alpha\alpha}(p) = \mathbb{1}_F \qquad \forall p \in U_\alpha$

   (b) $g_{\beta\alpha}(p) = g^{-1}_{\alpha\beta}(p) \qquad \forall p \in U_\alpha \cap U_\beta$

   (c) $g_{\beta\gamma}(p) g_{\gamma\alpha}(p) = g_{\beta\alpha}(p) \qquad \forall p \in U_\alpha \cap U_\beta \cap U_\gamma$

Th:

1. *It's defined a vector bundle on base space M with $g_{\alpha\beta}$ transition maps.*

2. *Such bundle is unique up to isomorphisms.*

Proof:

See for example Abate[**?**], page 137.

□

We can state the content of previous demostration as follow:

Hp:

Corollary 4.1 *Provided the hypothesis of theorem* **??**.

Th:

1. $E = \dfrac{\bigsqcup_{\alpha \in \mathscr{A}} U_\alpha \times V}{\sim}$
   with $(x, v) \sim (y, w) \Leftrightarrow (x = y) \wedge w = g_{\beta\alpha}(x) v$
   where $x \in U_\alpha$; $y \in U_\beta$; $v, w \in V$
   consitutes a smooth manifold.

2. *Taken* $\pi : E \to M$ *such that* $\pi(x, v) = x$
   then (V, E, π, M) *consitute a vector bundle.*

4.2 Vector fields and References.

There are few more feature than the abstract section:

1. $\Gamma(\pi_M)$ of a vector bundle inherit the linear properties from F defining sum and product by a scalar pointwise:

$$\forall s_i \in \Gamma(\pi_M) \qquad \begin{array}{l} (s_1 + s_2)(p) = (s_1)(p) + (s_2)(p) \\ (\lambda s_1)(p) = (\lambda s_1(p)) \end{array}$$

2. There's a special cross-section called *null section*:

$$O_E \in \Gamma(\pi_M) \quad \text{such that:} \quad O_E(p) = 0\big|_{E_p} \forall p \in M$$

3. It's possible to extend the concept of basis from F to $\Gamma(M)$.

4.2.1 Local reference.

Consider a vector bundle (F, E, π, E) of finite dimension $dim(F) = f < \infty$

Definition 27: Local reference

r-ple $\{\sigma_1, \ldots, \sigma_r\}$ of sections $\sigma_i \in \Gamma(U)$ on $U \subset E$ open set, such that $\{\sigma_1(p), \ldots, \sigma_r(p)\}$ constitues a basis in $E_p \forall p \in U$.

Proposition 4.1 *Giving a local reference is equivalent to give a bundle atlas*

Proof:

$\Leftarrow$ Giving a reference through a local trivialization is rather simple.
Chosen a basis $\{e_j\}$ in F,

$$\Gamma(U) \ni \sigma_j(p) = \chi^{-1}(p, e_j)$$

is a local reference.

$\Rightarrow$ Vice versa $\forall$ local reference $\{\sigma_1, \ldots, \sigma_r\}$ on U we can define:

$$\xi : U \times \mathbb{R}^r \to \pi^{-1}(U) \quad \text{such that: } \xi(p, \vec{w}) = w^i \sigma_i(p)$$

- ξ is bijective, follows from the definition of reference.
- ξ is smooth from linearity in w^i variable and smoothness of section in p variable.
- follows from the definition that $\chi := \xi^{-1}$ trivializes E.
- Smoothness of χ follows from the following argument:
  Consider a second local trivialization $\tilde{\chi}$ on U and call $\{\tilde{\sigma_1}, \ldots, \tilde{\sigma_r}\}$ the associated local reference.
  $\forall e_p \in E$ call $\tilde{\chi}_0(e) = (c^1, \ldots, c^r)$, such that: $\tilde{\chi}(e_p) = (p, \tilde{\chi}_0(e)$ and where $c^i \sigma_i(p) = e_p$.
  Applying this decomposition to the set $\{\sigma_1, \ldots, \sigma_r\}$ i.e.:

  $$\tilde{\chi}_0(\sigma_j) = (a_j^1, \ldots, a_j^r)$$

  we obtain the matrix $A = a_j^i(p)$
  Follows from the smoothness of the section that $a_j^i(p)$ are smooth function in p.
  A is invertible because represent a change of basis and its inverse is a matrix

$B = b^i_j$ with smooth elements also.
In conclusion χ is a composition of smooth functions:

$$\chi(e_p) = (\mathbb{1} \times B)(p, \tilde{\chi}_0(v)) = (\mathbb{1} \times B)\tilde{\chi}(e_p)$$

□

Observation 16

There's a relation between transition function and change of references between overlapping trivialization.
Given two overlapping trivialization (χ_i, U_i), be $\{\sigma_{1,i}, \dots, \sigma_{r,i}\}$ the associated local reference, with $i = \alpha, \beta$.
The change of basis matrix

$$\sigma_{j,\beta} = \sum_k (g_{\beta\alpha})^k_j \sigma_{k,\alpha}$$

are exactly the transition map:

$$\chi_\alpha \circ \chi_\beta^{-1}(p, e_{j,\beta}) = \chi_\alpha(\sigma_{j,\beta}) = \chi_\alpha\Big(\sum_k (g_{\beta\alpha})^k_j \sigma_{k,\alpha}\Big) = (p, \sum_k (g_{\beta\alpha})^k_j e_{k,\alpha})$$

In general $\forall \sigma \in \Gamma(U)$

$$\sigma = \sum_j a^j_\alpha \sigma_{j,\alpha} = \sum_k a^k_\beta \sigma_{k,\beta} \quad \text{with: } a^j_\alpha = \sum_h (g_{\alpha\beta})^j_h a^h_\beta$$

4.3 Tensor Vector Bundle.

Consider two fiber bundle (F_1, E_1, π_1, M_1) and (F_2, E_2, π_2, M_2)

Definition 28: Fiber Product of Fiber Bundle

$$(F_1, E_1, \pi_1, M_1) \times (F_2, E_2, \pi_2, M_2) = (F, E_1 \times_M E_2, \pi, M)$$

where:

$$E_1 \times_M E_2 = \{f = (e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2)\} \qquad \textit{fiber product set}$$

$$\pi(f) = \pi_1(e_1) = \pi_2(e_2)$$

Theorem 4.3 *Fiber product of 2 bundle is a fiber bundle of typical fiber* $F_1 \times F_2$.

Hp: *Consider a fiber bundle product as definition (28) .*

Th:

1. $E_1 \times_M E_2$ *is a submanifold of* $E_1 \times E_2$.
2. π *is a smooth bijection*
3. *From every couple of local trivialization one on* E_1 *and another on* E_2 *exists a trivialization on* $E_1 \times_M E_2$ *of typical fiber* $F = F_1 \times F_2$.

Proof:

1) : see abate p187
2) follows from the differentiability of π_1:

$$\pi(p_1, p_2) = \pi_1(p_1) = \pi_2(p_2)$$

3) we have to show how to construct a trivialization on the bundle product starting by a trivialization on each factor.
Consider two bundle atlas $\{(U_\alpha, \chi^j_\alpha)\}$ on E^j (where $j = 1,2$) defined on the same open cover of M.
Define $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times (F_1 \times F_2)$ such that:

$$\chi_\alpha(x_1, x_2) = \Big(\pi_1(x_1), \big(p_2 \cdot \chi^1_\alpha(x_1), p_2 \cdot \chi^2_\alpha(x_2)\big)\Big)$$

χ_α are diffeomorphism with:

$$(\chi_\alpha)^{-1}(p, s_1, s_2) = \big((\chi^1_\alpha)^{-1}(p, s1), (\chi^2_\alpha)^{-1}(p, s_2)\big)$$

$\{(U_\alpha, \chi_\alpha)\}$ is a bundle atlas on $E_1 \times_M E_2$. A way to show that is to exhibit that inherit the good properties from the transition map of the spaces product:

$$\chi_\alpha \chi_\beta^{-1}(p, s_1, s_2) = (p, g^1_{\alpha\beta}(p)(s_1), g^2_{\alpha\beta}(p)(s_2))$$

□

What said can be encoded in the following definition:

Definition 29: Cartesian Product Bundle of (E_1, π_1, M) **and** (E_2, π_2, M)

Fiber bundle ($F = F_1 \times F_2$, $E = E_1 \times_M E_2$, π, M) where:

$$E_1 \times_M E_2 = \big\{f = (e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2)\big\}$$

$$\pi : E \to M || \pi(f) = \pi_1(e_1) = \pi_2(e_2)$$

Endowed with a product bundle chart $(\phi_\alpha \times \psi_\beta, U_\alpha \cap U_\beta)$, where (ϕ_α, U_α) local trivialization of E_1, (ψ_β, U_β) local trivialization of E_2 and :

$$\big(\phi_\alpha \times \psi_\beta\big)\big(x,(v,w)\big) := \big(\phi_\alpha(x,v), \psi_\beta(x,w)\big) \qquad \forall\, v \in \pi_1^{-1}(x),\; w \in \pi_2^{-1}(x)$$

From the notion of product bundle can be derived the notion of *direct sum* and *tensor product of bundle*:

Observation 17

Obviuosly the definition of $\times$ for set applies to vector spaces. Endowing that set with specified $\cdot, +$ linear operation we get the so called *direct sum* and *tensor product* spaces.

$$F_1 \oplus F_2 = \begin{pmatrix} F_1 \times F_2 \qquad \text{with:} \\ \cdot : \lambda(v,w) = (\lambda v, \lambda w) \\ + : (v_1,w_1) + (v_2,w_2) = (v_1+v_2, w_1+w_2) \end{pmatrix}$$

$$F_1 \otimes F_2 = \frac{F_1 \times F_2}{\sim} = \begin{pmatrix} \text{span}(v \otimes w) \qquad \text{such that:} \\ \lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w) \\ v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w \\ v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2) \end{pmatrix}$$

Consider two vector bundle (F_1, E_1, π_1, M) and (F_2, E_2, π_2, M) on the same base space and an atlas $\mathscr{A} = (U_\alpha, \chi_\alpha)$ of chart which trivializes E_1 and E_2 with transition function $g_{\alpha\beta}, h_{\alpha\beta}$ respectively.

Definition 30: Direct sum of vector bundles.

The only (from theorem 4.2) vector bundle $\big((E_1 \oplus E_2, \pi, M\big)$, such that:

- $(E_1 \oplus E_2)_p = (E_1)_p \oplus (E_2)_p \qquad \forall\, p \in M$
- transition function respect $\mathscr{A}$ are $g_{\alpha\beta} \times h_{\alpha\beta}$.

Definition 31: Direct product of vector bundles.

The only (from theorem 4.2) vector bundle $\big((E_1 \otimes E_2, \pi, M\big)$, such that:

- $(E_1 \oplus E_2)_p = (E_1)_p \oplus (E_2)_p \qquad \forall\, p \in M$
- transition function respect $\mathscr{A}$ are $g_{\alpha\beta} \otimes h_{\alpha\beta}$ [a].

[a] In finite dimension we can identify $E_1 \otimes E_2 \simeq \mathbb{R}^{n_1+n_2}$ and $g_{\alpha\beta} \otimes h_{\alpha\beta}$ as the kronecker matrix product

Another useful construction over a vector bundle is the *dual vector bundle*.
Recalling that for all vector space V is defined the dual vector space V^* of all linear functional over V endowed with a suitable linear structure follows:

Definition 32: Dual vector bundles.

The only (from theorem 4.2) vector bundle $\big((E^*,\varpi,M\big)$, such that:

- $(E*)_p = \big((E)_p\big)^* \qquad \forall p \in M$
- transition function respect $\mathscr{A}$, are on the same open set, $g^*_{\alpha\beta} = \big(g^T_{\alpha\beta}\big)^{-1}$.

Observation 18

The transition function relation are derived from what follows:
Consider a linear operator $A : v \mapsto \tilde{v}$, that is $\tilde{v}^j = A^j_i = v^i$ in coordinate.
The dual of this linear operator $A : \eta \in V^* \mapsto \tilde{v}$ is defined by the following relation: $\tilde{\eta}_i \tilde{V}^i = \eta_j v^j$, that is:

$$\tilde{\eta_i} A^i_k = \eta_k$$

I.e. :

$$\tilde{\eta_i} = \eta_k [A^{-1}]^k_i = [A^{-1}]^T \eta^T$$

Observation 19

Extending pointwise the local properties from T_pM to $\Gamma(\pi_M)$ follows that:

- $\Gamma(\varpi_M) = \Gamma^*(\pi_M)$, i.e. sections of the dual vector bundle (E^*,ϖ,M) are linear functional on sections of (E,π,M).
- For all reference (σ_i) on $\pi : E \to M$ is defined the dual reference (η^j) such that $\eta^j(p)\big(\sigma_i(p)\big) = \delta^j_i$.

5 Tangent Bundle

The *tangent* bundle is a natural structure defined on any smooth manifold, represent the canonical example of non-trivial vector bundle.

As a set the tangent bundle is defined as the union of all tangent spaces:

Definition 33

$$TM := \bigsqcup_{p\in M} T_pM \equiv \bigcup_{x\in M} x \times T_xM \equiv \{(p,v) \mid p \in M, v \in T_pM\}$$

Corollary 5.1 *(TM,π,M) with $\pi : T_pM \mapsto p$ it's a vector bundle of typical fiber $\mathbb{R}^n$.*

Proof:

The thesis follows from 4.1 providing a surjective projection function $\pi : T_pM \mapsto p$ and a "*almost* bundle atlas" $\chi : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ imposing $\chi_\alpha\left(\sum_{j=1}^n V^j \frac{\partial}{\partial x_\alpha^j}\big|_p\right) = (p,V)$. From the definition element of TM are element of T_pM with p not fixed. Then:

- $T_pM \simeq \mathbb{R}^n$ choosing the natural basis of the chart atlas.
- Bijectivity of χ is granted by uniqueness of decomposition on a basis.
- $p_1 \cdot \chi = \pi$ follows directly from the definition of χ_α.
- $g_{\alpha\beta} = \frac{\partial x_\alpha}{\partial x_\beta}$ is a good transition map:

$$\chi_\alpha \cdot \chi_\beta^{-1}(p,V) = \chi_\alpha\left(\sum_{j=1}^n V^j \frac{\partial}{\partial x_\alpha^j}\big|_p\right) = \chi_\alpha\left(\sum_{h=1}^n \left[\sum_{j=1}^n \frac{\partial x_\alpha^h}{\partial x_\beta^j}(p) V^j\right] \frac{\partial}{\partial x_\alpha^h}\big|_p\right) = \left(p, \left[\frac{\partial x_\alpha}{\partial x_\beta}\right](p) V\right)$$

□

Take Away Message

Tangent bundle is the unique vector bundle of M such that:

- have for typical fiber $E_p = T_pM \simeq \mathbb{R}^n$
- transition maps between trivialization chart $(U_\alpha, \phi_\alpha, \chi_\alpha)$ is the jacobian matrix of the coordinate transition function $g_{\beta\alpha} = \frac{\partial x_\alpha}{\partial x_\beta}$.

⚠Repetita

Definition 34: Tangent Bundle

The smooth vector bundle $TM = (TM, \tau, M; \mathbb{R}^m)$ such that:

- The total space is the union of all tangent spaces to

$$M : TM := \bigsqcup_{p\in M} T_pM \equiv \bigcup_{x\in M} x \times T_xM$$

- The bundle projection maps each tangent vector $v \in T_pM$ to the correspondent base point p;

$$\tau : (p, v_p) \mapsto p$$

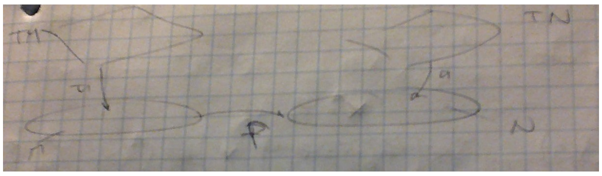
Observation 20

The tangent bundle is a vector bundle of rank n on a n dimensional manifold.
Then TM is a $2n-$dimensional manifold.

5.1 Tangent Map.

Given 2 manifolds M, N and a differentiable function $F : M \to N$.

Figure 8: ...



Definition 35: Tangent Map

Is the map $\mathrm{T}f := \mathrm{d}f \ : \ TM \to TN$ such that:

$$\mathrm{T}f \ : V_p \in T_pM \mapsto \left[\left(f_*(p)\right)V\right] \in T_{f(p)}N$$

Observation 21

In differential geometry it's usual to give different name to objects that are essentially the same in order to emphasize some *"flavour"*.
In this case, we have:

- $\mathrm{d}f(p)$: " Differential of f" is the linear operator between $T_pM \to T_{f(p)}N$ for a fixed $p \in M$.

- $f_*(p)\,V$: " Push-Forward through f of a tangent vector V" is the image of $\mathrm{d}f(p)$ on $V \in T_pM$.
- $\mathrm{T}f$: " Tangent map of f" is the vector-bundle-morphism which act on every fiber like the differential operator.

5.2 Vector fields and natural references.

Notation fixing

The section of TM are called *vector fields* on the manifold M. The reason is straightforward:
Fixing a point in TM is equivalent to pick a point in M and a vector in $\mathbb{R}^n$, thus it represent a tangent vector of base point p.
Known that a vector field could be easily seen as a map $V : M \to TM$ satisfying the section condition $\pi \cdot V = \mathbb{1}_M$.

Notation fixing

The collection of all vector fields is a section spaces always carried on every differential manifold, it's often indicated with a particular notation:

$$\Gamma(TM) = \mathfrak{X}(M)$$

In coordiante chart we may read off the components of the vector.
Consider a local chart (U, ϕ) over p such that $\phi = (x^1, \dots, x^n)$:

Definition 36: Natural Reference

Sections $(\partial_1, \dots, \partial_n) \subset \mathfrak{X}(M)$ of TM, such that:

$$\partial_j(p) = \frac{\partial}{\partial_j}\Big|_p \in T_pM$$

i.e. are the fields which associate to every point in M a natural tangent vector (tangent to the coordinate curve).

Observation 22

Provided an atlas on M is defined $\forall X \in \mathfrak{X}(M)$ the decomposition:

$$X = \sum_{j=1}^{n} a^j(p)\partial_j(p)$$

Where the component are:

$$a^j(p) = \mathrm{d}\phi_p\big(X(p)\big) \in C^\infty(U) \tag{7}$$

ans U is a neighborhood of p.

Observation 23

Obviously a change in local chart induce a transformation in local reference. Consider a second chart $(\tilde{U}, \tilde{\phi}$ with $U \cap \tilde{U} \neq \emptyset$. The change of natural basis on a fixed tangent space is easily extended on the whole natural reference:

$$\tilde{\partial}_h = \sum_{k=1}^{n} \frac{\partial x^k}{\partial \tilde{x}^H} \partial_k \tag{8}$$

From that follows $X = \sum_j a^j \partial_j = \sum_k \tilde{a^k} \tilde{\partial_k}$ i.e. the covariant change rule :

$$a^j = \sum_h \frac{\partial x^j}{\partial \tilde{x}^h} \tilde{a}^h$$

is extended to the section.

5.3 CoTangent Bundle

Notation fixing

Cotangent Bundle is a specific name for the specialization of 32 to the tangent spaces, i.e. is vector bundle on M with total space $T^*M = \bigsqcup_{p\in M} T_p^* M$ and usual projection π.

Observation 24

This vector bundle is unique, by theorem 4.1, providing the following transition functions for a change of trivialization chart.

Recalling that for all local chart $\varphi = (x^1, \dots, x^n)$ are defined:

- $\dfrac{\partial}{\partial x^h}\Big|_p \in T_p M$ = natural basis vector in $T_p M \quad \forall p \in U.$[a]
- $\mathrm{d}x^h\big|_p in T_p^* M$ = external derivative of the local chart calculated in p.

follows directly from definition of *external derivative* that:

$$\mathrm{d}x_P^j\Big(\frac{\partial}{\partial x^h}\Big|_p\Big) = \frac{\partial x^j}{\partial x^h}(p) = \delta_h^j$$

in other words $\{\mathrm{d}x_p^h\}$ are the *natural dual basis* 1-forms.

Recalling also that taken two overlapping chart (U_α, ϕ_α) , (U_β, ϕ_β) on M, we have

$$\mathrm{d}x_\beta^k\big|_p = \sum_h \frac{\partial x_\beta^k}{\partial c_\alpha^h} \mathrm{d}x_\alpha^h\big|_p$$

i.e. dual coordinate are *covariant* [b].

âĂć Chosing a standard trivialization on the cotangent bundle:

$$\chi_\alpha\big(\sum_j w_j \mathrm{d}x_\alpha^j\big|_p\big) = (p, w^T)$$

where $w^T \in \mathbb{R}^n$ is simply the transposition of row vector of 1-form components $(w_1, \ldots, w_n)$.

âĂć Follows that:

$$\chi_\alpha \circ \chi_\beta^{-1}(p, w^T) = \chi_\alpha\big(w_j \mathrm{d}x_\beta^h\big|_p\big) = \chi_\alpha\big([w_j \frac{\partial x_\beta^j}{\partial x_\alpha^h}\big|_p \mathrm{d}x_\alpha^h\big|_p\big) = \big(p, [\frac{\partial x_\beta}{\partial x_alpha}(p)]^T w^T\big)$$

In conclusion:

$$[g_{\alpha\beta}]^{\cdot}_{\cdot} = [\frac{\partial x_\beta^{\cdot}}{\partial_\alpha^{\cdot}}]^T \tag{9}$$

are the transition map for the dual bundle [c].

[a] Depending on which equivalent presentation of the tangent space is taken into account these can be seen as the tangent vector to the coordinate curve or as a partial derivative operator on $C^\infty(U)$.

[b] To compare to (8), the controvariant relation of the natural basis.

[c] To confront with the tangent case in which $[g_{\alpha\beta}]^{\cdot}_{\cdot} = \frac{\partial x_\alpha^{\cdot}}{\partial x_\beta^{\cdot}}$.

Notation fixing

The cross section of the cotangent bundle are called *1-forms* on M

Observation 25

Is possible to review the concept of *external derivative* in the language of tangent bundles and 1-forms [2].
$\forall f \in C^\infty(M)$ $\mathrm{d}f$ is the function $\mathrm{d}f : TM \to T\mathbb{R}$ such that:

$$\mathrm{d}f(V_p) = V(f)\big|_p \qquad \forall V \in \mathfrak{X}(M) \tag{10}$$

Observation 26

The dual natural reference is then provided by the external derivative of the local chart and they are the operator $\mathrm{d}x^a(p)$ that returns the component of a vector fields seen in equation (7) .

5.4 Tensor Bundle

As last effort we can combine all the precedent definition to introduce the tensor bundle:

Definition 37: $(k,l)-$**Tensor Bundle**

Is the unique vector bundle :

$$T_l^k M = \underbrace{T^* M \otimes \cdots \otimes T^* M}_{\text{k-times}} \otimes \underbrace{TM \otimes \cdots \otimes TM}_{\text{l-times}}$$

i.e. such that each fiber is in the form $E_p = T_l^k\big(T_p M\big)$.

Observation 27

Uniqueness follows from definition of $\times$ for vector bundles.
Anyway the transition map for such bundle follows from the transformation of tensor components under change of local charts (see [2]).

Notation fixing

The section of $T_l^k(M)$ are called *tensor fields.*

5.5 Phase Space

Notation fixing

In the context of Classical mechanics is customary to refer to the cotangent bundle T^*Q over the smooth manifold Q - called *Configuration Space* - as *Phase Space.*

Since TQ and T^*Q are diffeomorphic , it might seem that there is no particular reason in treating this two spaces separately, but it is not so. There are certain geometrical objects that live naturally on T^*Q , not on TQ. Of greatest interest in mathematical-physics are the PoincarÃÍ forms[**?**].

Consider a smooth manifold Q and call $\mathscr{M} = T^*Q$ the corresponding cotangent bundle.

Definition 38: Tautological (PoincarÃÍ) 1-form

Is the 1-form over $\mathscr{M}$:

$$\theta_0 \in \Gamma^\infty(T^*\mathscr{M})$$

such that the action on a generic point $\omega_{\alpha_p} \in T_{\alpha_p}M$ (in the fiber of α_p, which in turn is a one-form on the fiber of $p \in Q$) is given by:

$$\theta_0(\alpha_p) : T_{\alpha_p}\mathscr{M} \to \mathbb{R} \qquad : \omega_{\alpha_p} \mapsto \alpha_q \circ T\tau^*_Q(\omega_{\alpha_p})$$

[a]

[a]For a slightly different formulation see : [4] [3] wiki

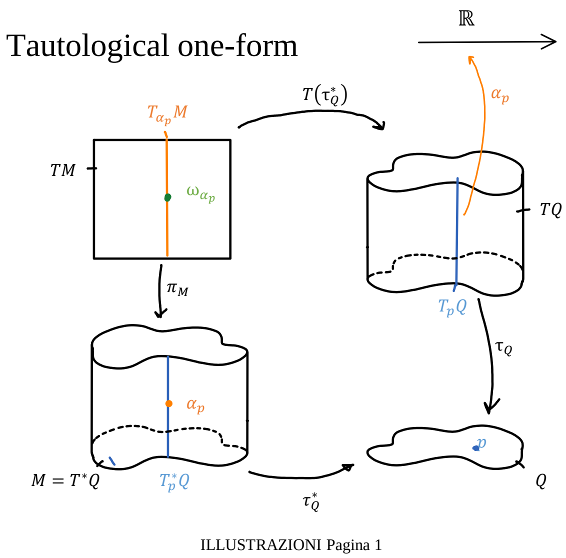


Figure 9: The definition of tautological 1-form is achieved exploiting the concept of *Tangent map* and remembering that $\alpha_p : T_p\mathscr{M} \to \mathscr{M}$ is a linear functional.

Notation fixing

Canonical coordinates are defined as a special set of coordinates on the cotangent bundle of a manifold. They are usually written as a set of (q^i, p_j) where q_i are denoting the coordinates on the underlying manifold and the p_j are denoting the conjugate momentum, which are decomposition of 1-forms in T^*_pM on the dual natural basis dq^J in the cotangent bundle at point q in the manifold.

Proposition 5.1 (Coordinate Representation of θ_0) *In canonical coordinate the tau-*

tological one-form assumes the famous expression:

$$\theta_0 = \sum_{i=1}^{n} p_i dq^i$$

(note that dq^i is a 1-form on T^*M calculated with respect to the coordinate on the bundle. Has not to be confused with the 1-natural form $dq^i \in T_p^*M$.)

Proof:

See [3][pag 179]

□

Definition 39: Canonical (PoincarÃÍ) symplectic form

Symplectic form:

$$\omega_0 := -d\theta_0$$

In canonical coordinates assumes the famous expression:

$$\omega_0 := \sum_{i=1}^{n} dp_i \wedge dq^i$$

Proposition 5.2 (Canonical symplectic form) *$\omega := d\theta_0$ is a symplectic 2-form on M.*

Proof:

Theorem 3.2.10 [3]

□

Proposition 5.3 (Canonical Coordinate Representation)

As a mathematical curiosity, we note that the cotangent bundle of any manifold is orientable. Indeed, it carries a symplectic structure and hence a volume element.

⚠Repetita

The claim is proved by the following definition:

Definition 40: Canonical (PoincarÃÍ) symplectic form

Symplectic form:

$$\omega_0 := -d\theta_0$$

In canonical coordinates assumes the famous expression:

$$\omega_0 := \sum_{i=1}^{n} dq^i \wedge dp_i$$

6 Higher-order tangent bundles

wiki:Higher-order tangent bundles
Since the tangent bundle TM is itself a smooth manifold, the second-order tangent bundle can be defined via repeated application of the tangent bundle construction:

$$T^2M = T(TM)$$

⚠ In general, the k-th order tangent bundle T^kM can be defined recursively as $T(T^{k-1}M)$.

A smooth map $f : M \to N$ has an induced derivative, for which the tangent bundle is the appropriate domain and range $Df : TM \to TN$. Similarly, higher-order tangent bundles provide the domain and range for higher-order derivatives $D^kf : T^kM \mapsto T^kN$.

A distinct but related construction are the jet bundles on a manifold, which are bundles consisting of jets.

7 Jet Bundles

The jet bundle is a certain construction that makes a new smooth fiber bundle out of a given smooth fiber bundle.
The first step is to identify the typical fiber for this construction. Suppose M is an m-dimensional manifold and that (E, π, M) is a fiber bundle.

Consider the set of all the local sections whose domain contains p

$$\Gamma^\infty(p) := \left\{\sigma \in \Gamma^\infty(E) \quad \middle| \, p \in dom(\sigma)\right\}$$

We define an equivalence relation between such section *up to r-th order*:

Definition 41: r-jet equivalence

Two such section $\sigma, \eta \in \Gamma^\infty(p)$ have the <u>same *r-jet* at *p*</u> ($\sigma \sim \eta$) iff:

$$\left.\frac{\partial^{|I|}\sigma^\alpha}{\partial x^I}\right|_p = \left.\frac{\partial^{|I|}\eta^\alpha}{\partial x^I}\right|_p \qquad \forall I \in \mathbb{N}_0^m \,|\, 0 \le |I| \le r.$$

where I is a *Multi-index*.

Remark:

(multi-index notation)
A multi-index is a natural valued finite dimensional vector $I = (i_1, i_2, \dots, i_m) \in \mathbb{N}_0^m$ with $m < \infty$.
On $\mathbb{R}^n$ a general differential operator can be identified by a multi-index:

$$\frac{\partial^{|I|}}{\partial x^I} := \prod_{i=1}^{m} \left(\frac{\partial}{\partial x^i} \right)^{I(i)}$$

(Until the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|I| := \sum_{i=1}^{m} I(i)$$

We define the *r-th Jet in p* as the equivalence class under this relation.

Definition 42: Space of r-th Jet in p

$$J_p^r(E) := \frac{\Gamma^\infty(p)}{\sim}$$

where $\sim$ is the r-Jet equivalence.

Notation fixing

A r-jet with representative σ is denoted as $j_p^r \sigma$.
The integer r is also called the order of the jet, p is its source and $\sigma(p)$ is its target.

Glueing all the jet fiber $J_p^r(E)$ together for all the base point $p \in M$, as done for the tangent bundle, we obtain the wanted bundle:

Definition 43: r-th Jet Bundle of E

The triple $(J^r(E), \pi_r, M)$ where:

- $J^r(E) := \bigsqcup_{p \in M} J_p^r(E) \equiv \{ j_p^r \sigma \quad | \, p \in M, \, \sigma \in \Gamma^\infty(p) \}$
- $\pi_r : J^r(E) \to M$ such that $j_p^r \sigma \mapsto p$

Proposition 7.1 *$J^r(E) = (J^r(E), \pi_r, M)$ is a smooth bundle.*

8 Closing Thoughts

8.1 Prima stesura dell'introduzione

In questo primo capitolo ci concederemo un po' di tempo per presentare i *fibrati*, una famiglia di strutture algebriche di particolare importanza per la fisica-matematica moderna.

L'approccio che seguiremo ÃÍ in un certo senso deduttivo.

Partiremo definendo la struttura, particolarmente astratta, di *Fiber Bundle* sopra la categoria degli spazi topologici; sottolineando come questo rappresenti il setting piÃź generale per rappresentare il concetto di campo nell'accezione originata dalla fisica ma senza tralasciare il fatto che la famiglia dei bundle costituisca una categoria concreta (costrutto) di per sÃÍ.

Nel paragrafo 2 arricchiremo questo oggetto astratto con una , cosÃň detta, *G-Structure*. Una soppalcatura che va necessariamente fissata se si vuole disporre di un concetto di compatibilitÃă tra trivializzazioni overlapping.

Nel terzo paragrafo si specializzerÃă la categoria su cui viene definito il fibrato a non essere semplicemente quella degli spazi topologici ma la sottocategoria delle varietÃă smooth [6]. Questo passaggio porta con se la possibilitÃă di esplorare il rapporto tra gli spazi tangenti delle due varietÃă(base e totale) che costituiscono il fibrato. Il mezzo per formalizzare questo rapporto saranno le operazioni di *Lift* e *Drop*, specializzazione a questo contesto delle note operazioni di *Pull-Back* e *Push-Forward* tra varietÃă.

Nel paragrafo 4 si porrÃă per la prima volta un vincolo sulla spazio fibra imponendo che esso sia dotato della struttura di spazio lineare. Si parlerÃă quindi di *Vector Bundle* [7] in questo contesto meno generale ci porremo il problema di dare di stabilire in quali condizioni ÃÍ possibile definire un fibrato su una varietÃă disponendo solo di una collezione di fibre omeomorfe.

Nel capitolo quinto verranno analizzati i *Tangent Bundle* la piÃź importante classe di fibrato vettoriale sulle varietÃă lisci.

(...)

Infine ci si chiederÃă come iterare la struttura di spazio tangente analizzando il rapporto tra i fibrati tangenti sulle due varietÃă costituente un generico bunle smooth.

8.2 Eliminata

Questioni di interesse personale che non ho aggiunto al capitolo per esigenze di tempo in quanto non strettamente legati agli argomenti della tesi oppure da spostare secondo consiglio di CD:

- In letteratura si vede spesso riferirsi alla G-struttura del fibrato come *Gauge Group*... PerchÃÍ?

[6] in realtÃă per quanto detto in questo capitolo dovrebbe essere valido per qualsiasi ordine di differenziabilitÃă. (per quanto riguarda hausdorff e second countability?)

[7] In what follows we only consider *smooth* Vector Bundle.

- La questione di chart vs trivialization ÃÍ una riflessione fatta da me superflua per la tesi.. qua ho scritto quasi tutto ma sui miei appunti cartacei ho messo qualche schemino in piÃź.. (quindi li ho scannerizzati e messi nel materiale della tesi!)

- Approccio generale(nel senso di non limitarsi alle varietÃă riemmaniane) alle connessioni con il linguaggio dei fribrati
  la formulazione precisa si fa molto efficacemente disponendo degli spazi doppio tangenti:
  specificare un unico lift
  trasporto parallelo
  curvatura

- Teoremi di costruzione dei vector bundle aggregando insieme un po' di spazi vettoriali. La fonte principale ÃÍ abate. Ma io ho riscritto tutto a pagine 9,10,11 degli appunti della tesi. Non sono superflui, servono essenzialmente per dimostrare che TM e T*M sono dei vector bundle!

- Si parla (per esempio sull'abate capitolo 3 ultimo paragrafo) dei fibrati principali dei riferimenti associati ad un fibrato vettoriale. Salto questo argomento perchÃÍ non serve per la tesi.

- ho visto il fibrato come una tripla ma... immaginiamo di avere una varietÃă M, cose vuol dire "dare un fibrato su di essa?" il secondo teorema del capito sui fibrati vettoriali risponde un po' a questa questione
  inoltre... che vuol dire fissare un punto sul total space ? vuol dire scegliere una coppia (p, v) previa la scelta di una trivializzazione.
  invece nei vettoriali equivale esattamente a dare la coppia (e, v), la scelta di una trivializzazione equivale invece ad una scelta di base in F tale di decomporre il vettore v in una n-pla.

- Abate per definite il ⊗ di bundle passa per la definizione di fiber product set (vedi abate e appunti pag 12) .. io ho preferito passare direttamente alla def per i bundle.

- riguardo alla tangent map il FOM appesantisce molto la notazione e i concetti... io sui miei fogli ho seguito un po' la sua strada ma mi sembra tutto superfluo... differenziale, push forward e tangent map sono in fondo la stessa cosa

- la parte sul doppio tangent bundle come setting per la connessione la salto (magari la metto come parte nel capitolo due dove c'ÃÍ la geometria riemmaniana o forse non lo metterÃš mai!).
  Si puÃš perÃš fare un accenno alla ricorsione del tangent bundle come fa qui: Wiki Tangent Bundle

- la questione di rividere il dual tangent space come una varietÃă simplettica ÃÍ da mettere nella parte sulla meccanica classica geometrica. in quanto, dice dappiaggi, ÃÍ solo in questo contesto che si usa questa proprietÃă!

- nel jurgen jost, capitolo sui fibrati, ci sono delle interessanti proposizioni sui fiber bundle su varietÃă riemmaniane, si afferma che in questo caso la G-Structure ÃÍ garantita

- ispirato da Fraenkel ho scritto sui miei appunti la dimostrazione che TM e' una varieta' differenziale. In realtÃă la cosa ÃÍ superflua se si sfrutta il teorema di ricostruzione del vector bundle.

8.3 Possibile Estensioni

Per capitoli

1. FB

2. GB

3. SB

4. VB
   - Considerazioni sul vector bundle.Che vuol dire in parole provere fissare un punto sul fibrato.
   - (da wiki VB) esempi: trivial bundle, moebius strip
   - (da wiki VB) accenno ai banach bundle
   - Nell'ottica della quantizzazione ÃÍ necessario parlare del bundle innerproduct

5. TB
   - (da freed) osservazione pag4 > $g_{\beta\alpha} = \mathrm{d}(x_\beta \circ x_\alpha^{-1})$
   - (da fraenkel) TM as set is a manifold.
   - (Abate pag 139) Tangent bundles as sub category, morhism = tangent map
   - (fraenkel e alt) T^*M as a phase space and natural simplectic structure (dapp dice di presentare i concetti meccanici in un altro capitolo)
   - paragrafo di confronto di mappe tra tangent bundle,
tangent map vs pull/push vs differential operator
fiber derivative
   - Tangent bundle over riemmanian manifold (pag 37-39 Jurgen Jost)
   - Vector bundle of p-form (pag 40-41 Jurgen Jost)

6. TTB
   - Presentazione del doppio tangente nello spirtito di wiki (wiki VB pag 4)
   - (Wiki VB) Vertical lift
   - (WIKI TTB)

Possibili fonti da considerare:

- Xavier Gracia, FIBRE DERIVATIVES: SOME APPLICATIONS TO SINGULAR LAGRANGIANS
- `http://www.math.toronto.edu/selick/mat1345/notes.pdf` Link
- Koszul, Lectures on fiber bundles, `http://www.math.tifr.res.in/~publ/ln/tifr20.pdf` Link
- jmf, Connections on principal fibre bundles `http://empg.maths.ed.ac.uk/Activities/GT/Lect1.pdf` Link
- `http://personal.maths.surrey.ac.uk/st/T.Bridges/GEOMETRIC-PHASE/Connections_intro.pdf` ConnectionsIntro
- `http://math.stanford.edu/~ralph/fiber.pdf` Topology of Fiber Bundles
- `https://www.ma.utexas.edu/users/dafr/M392C/Notes/FiberBundles.pdf` Freed
- `http://www.cjcaesar.ch/fribourg/docfrib/fibre_bundles/fibre_bundles.pdf` michael laurent
- `www1.maths.leeds.ac.uk/~ahubery/Fibre-Bundles.pdf` Leeds, fiber bundle
- Google in generale„ ovviamente c'e' tanto! `https://www.google.it/search?q=FibreBndls.pdf&ie=utf-8&oe=utf-8&rls=org.mozilla:en-US:unofficial&client=iceweasel-a&channel=sb&gws_rd=cr&ei=uQ-tVM_ODMKtygOBvoGwDA#rls=org.mozilla:en-US:unofficial&channel=sb&q=fibre+bundles` goog
- Connection on fiber bundles `http://en.wikipedia.org/wiki/Connection_%28vector_bundle%29` wiki

8.4 TODO

- Ricopiare dimostrazione 4.2
- rifare tutte le illustrazioni con inkscape.

8.5 Take away messages.

...

References

[1] Daniel S. Freed - Fiber Bundles and Vector Bundles - https://www.ma.utexas.edu/users/dafr/M392C/Notes/FiberBundles.pdf

[2] Marco Abate and Francesca Tovena - *Geometria differenziale.*

[3] Ralph Abraham and Jerrold E. Marsden - *Foundations of Mechanics.*

[4] Theodore Frankel - *The geometry of Physics.*

[5] Benini, M. and Dappiaggi, C. in Advances in AQFT 1âĂŞ49

[6] Benini, M., Dappiaggi, C. and Hack, T.-P. Quantum Field Theory on Curved Backgrounds âĂŤ a Primer. Int. J. Mod. Phys. A 28, 1330023 (2013).