AQFT mathematical preliminaries

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Contents

(sono ripetizioni inutili per la tesi, sono informazioni che si ritrovano ovunque... sono informazioni adatta al knowledge base)

1 Globally Hyperbolic SpaceTimes

Recurring definitions in general Relativity (excluding the general smooth manifold prolegomena).

Definition 1: Space-Time

A quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ such that:

- (M, g) is a time-orientable n-dimensional manifold $(n > 2)$
- $\mathfrak{o}$ is a choice of orientation
- $\mathfrak{t}$ is a choice of time-orientation

Definition 2: Lorentzian Manifold

A pair (M, g) such that:

- M is a n-dimensional $(n \geq 2)$, Hausdorff, second countable, connected, orientable smooth manifold.
- g is a Lorentzian metric.

Definition 3: Metric

A function on the bundle product of TM with itself:

$$g : TM \times_M TM \to \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a non-degenerate bilinear form.

Notation fixing

- *Riemman* if the sign of g is positive definite, *Pseudo-Riemman* otherwise.
- *Lorentzian* if the signature is $(+, -, \ldots, -)$ or equivalently $(-, +, \ldots, +)$.

Observation 1: Causal Structure

If a smooth manifold is endowed with a Lorentzian manifold of signature $(+, -, \ldots, -)$ then the tangent vectors at each point in the manifold can be classed into three different types.

Notation fixing

$\forall p \in M, \quad \forall X \in T_p M$, the vector is:

- *time-like* if $g(X,X) > 0$.
- *light-like* if $g(X,X) = 0$.
- *space-like* if $g(X,X) < 0$.

Observation 2: Local Time Orientability

$\forall p \in M$ the timelike tangent vectors in p can be divided into two equivalence classes taking

$$X \sim Y \text{ iff } g(X,Y) > 0 \qquad \forall X, Y \in T_p^{\text{time-like}} M:$$

We can (arbitrarily) call one of these equivalence classes "future-directed" and call the other "past-directed". Physically this designation of the two classes of future- and past-directed timelike vectors corresponds to a choice of an arrow of time at the point. The future- and past-directed designations can be extended to null vectors at a point by continuity.

Definition 4: Time-orientation

A global tangent vector field $\mathfrak{t} \in \Gamma^\infty(TM)$ over the Lorenzian manifold M such that:

- $\mathrm{supp}(\mathfrak{t}) = M$
- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Observation 3

The fixing of a time-orientation is equivalent to a consistent smooth choice of a local time-direction.

Definition 5: Time-Orientable Lorentzian Manifold

A Lorentzian Manifold (M,g) such that exist at least one time-orientation $\mathfrak{t} \in \Gamma^\infty(TM)$.

Notation fixing

Consider a piece-wise smooth curve $\gamma : \mathbb{R} \supset I \to M$ is called:

- *time-like* (resp. light-like, space-like) iff $\dot{\gamma}(p)$ is time-like (resp. light-like, spacelike) $\forall p \in M$.

- *causal* iff $\dot{\gamma}(p)$ is nowhere spacelike.
- *future directed* (resp. past directed) iff is causal and $\dot{\gamma}(p)$ is future (resp. past) directed $\forall p \in M$.

Definition 6: Chronological $\substack{\text{future}\\ \text{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{I}^{\pm}_M(p) := \left\{q \in M \middle| \, \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ time-like } \substack{\text{future}\\ \text{past}} - \text{directed} \, : \, \gamma(0) = p, \, \gamma(1) = q\right\}$$

Definition 7: Causal $\substack{\text{future}\\ \text{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{J}^{\pm}_M(p) := \left\{q \in M \middle| \, \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ causal } \substack{\text{future}\\ \text{past}} - \text{directed} \, : \, \gamma(0) = p, \, \gamma(1) = q\right\}$$

Notation fixing

Former concept can be naturally extended to subset $A \subset M$:

- $\mathbf{I}^{\pm}_M(A) = \bigcup_{p \in A} \mathbf{I}^{\pm}_M(p)$
- $\mathbf{J}^{\pm}_M(A) = \bigcup_{p \in A} \mathbf{J}^{\pm}_M(p)$

Definition 8: Achronal Set

Subset $\Sigma \subset M$ such that every inextensible timelike curve intersect Σ at most once.

Definition 9: $\substack{\text{future}\\ \text{past}}$ Domain of dependence of an Achronal set

The two subset related to the generic achornal set $\Sigma \subset M$:

$$\mathbf{D}^{\pm}_M(\Sigma) := \left\{q \in M \middle| \, \forall \gamma \, \substack{\text{past}\\ \text{future}} \text{inextensible causal curve passing through } q : \, \gamma(I) \cap \Sigma \neq \emptyset\right\}$$

Notation fixing

$\mathbf{D}_M(\Sigma) :- \mathbf{D}^{+}_M(\Sigma) \cup \mathbf{D}^{-}_M(\Sigma)$ is called *total domain of dependence.*

Definition 10: Cauchy Surface

Is a subset $\Sigma \subset M$ such that:

- closed
- achronal
- $\mathbf{D}_M(\Sigma) \equiv M$

2 Linear Differential Operator

Basic Definition in L.P.D.O. on smooth vector sections.
Consider $F = F(M, \pi, V), F' = F'(M, \pi', V')$ two linear vector bundle over M with different typical fiber

Definition 11: Linear Partial Differential operator (**of order at most** $s \in \mathbb{N}_0$)

Linear map $L : \Gamma(F) \to \Gamma(F')$ such that:
$\forall p \in M$ exists:

- (U, ϕ) local chart on M.
- (U, χ) local trivialization of F
- (U, χ') local trivialization of F'

for which:

$$L(\sigma\big|_U) = \sum_{|\alpha| \le s} A_\alpha \partial^\alpha \sigma \qquad \forall \sigma \in \Gamma(M)$$

Remark:

(multi-index notation)
A multi-index is a natural valued finite dimensional vector $\alpha = (\alpha_0, \dots, \alpha_n - 1) \in \mathbb{N}_0^n$ with $n < \infty$.
On $\mathbb{R}^n$ a general differential operator can be identified by a multi-index:

$$\partial^\alpha = \prod_{\mu=0}^{n-1} \partial_\mu^{\alpha_\mu}$$

(Until the Schwartz theorem holds, the order of derivation is irrelevant.)

The order of the multi-index is defined as:

$$|\alpha| := \sum_{\mu=0}^{n-1} \alpha_\mu$$

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Hp:

Proposition 2.1 (Existence and uniqueness for the Cauchy Problem)

- **M** $= (M, g, \mathfrak{o}, \mathfrak{t})$ *a globally hyperbolic space-time.*
- $\Sigma \subset M$ *a spacelike cauchy surface with future-pointing unit normal vector field* $\vec{n}$.
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Th:

Observation 4

"Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. " [**?**] [**?**]
However the existence and uniqueness can be proved for the large class of the *Normally-Hyperbolic Operators.*