1 Ideal (Semigroups)

Consider a semigroup $(S, \otimes)$ and two subsets A and B.

Notation fixing

$$AB := \{a \otimes b \mid a \in A; b \in B\}$$

Notation fixing

Left coset of $a \in A \subset S$:

$$[a]_L = aS$$

Right coset of $a \in A \subset S$:

$$[a]_R = Sa$$

Definition 1: SubSemigroup

A is a *semigroup of* S iff:

$$AA \subset A$$

Definition 2: Normal SubSemigroup

A is a *normal semigroup of* S, $A \lhd S$ iff:

$$aS = Sa \qquad \forall a \in A$$

⚠ Con i gruppi si legge come coniugazione `https://en.wikipedia.org/wiki/Normal_subgroup`

Definition 3: Right Ideal

A is a *right ideal of* S iff:

$$SA \subset A$$

Definition 4: Left Ideal

A is a *left ideal of* S iff:

$$SA \subset A$$

Definition 5: Two-sided (Bilateral) Ideal

A is a *bilateral ideal of* S if it is both a left and a right ideal. *i.e.:*

$$SAS \subset A$$

Proposition 1.1 *If S is a monoid (unital semigroup) and $A \subset S$ a submonoid, then*

$$SAS \subset A \quad \Leftrightarrow \quad A \text{ two-sided ideal}$$

Proof:

Existence of unital element in S guarantees that both $SA = SA1$ and $AS = 1AS$ are subsets of SAS.
The converse hold true for every semigroups.

□

2 Ideal (Rings)

Consider an arbitrary ring $(R, \oplus, \otimes)$, let $(R, \oplus)$ be its *additive group*.

Definition 6: Right Ideal

Subset $I \subset R$ such that:

1. $(I, \oplus)$ is an additive subgroup of $(R, \oplus)$.
2. I absorbs multiplication on the right:

$$(x \otimes r) \in I \qquad \forall x \in I, \forall r \in R$$

Definition 7: Left Ideal

Subset $I \subset R$ such that:

1. $(I, \oplus)$ is an additive subgroup of $(R, \oplus)$.
2. I absorbs multiplication on the left:

$$(r \otimes x) \in I \qquad \forall x \in I, \forall r \in R$$

Definition 8: Two-sided (Bilateral) Ideal

A is a *bilateral ideal of* S if it is both a left and a right ideal.

Consider a generic subset $I' \subset R$:

Definition 9: Generated (Two-sided) Ideal

The two-sided ideal generated by I' is the ideal:

$$I = \text{span}\left(a \otimes x \otimes b \in A \mid a, b \in R;\ x \in I'\right)$$

Definition 9: Generated (Two-sided) Ideal

⚠ Span non Ãĺ termine giusto!

⚠ ...

Consider an arbitrary ring $(R, \oplus, \otimes)$ and a two-sided ideal $I \subset R$:

Proposition 2.1 (a congruent b modulo I) *Consider $a, b \in R$:*

$$a \sim b \qquad \Leftrightarrow (a - b) \in I$$

is a Congruence Relation

https://en.wikipedia.org/wiki/Quotient_ring

Definition 10: Ring quotient by an ideal

Is the set

$$R/I := \{[a]\}$$

where:

$$[a] = a \oplus I = \{a + r \mid r \in I\}$$

endowed with operations:

- $(a + I) + (b + I) = (a + b) + I$
- $(a + I)(b + I) = (ab) + I$