Prolegomena on (some) Abstact Algebra

Algebraic structures zoology

Tony

December 2, 2017

(molto basato su wiki)

Contents

1 Single Operation structures (Magmas)

Consider two set M, A and a binary operation, i.e. a function:

$$\otimes : M \times M \to A$$

Operation $\otimes$ is said:

- *Closed* if $ran(\otimes) \subset M$
$$(x \otimes y) \in M \qquad \forall x, y \in M \tag{1}$$
- *Associative*
$$(x \otimes y) \otimes z = x \otimes (y \otimes z) \qquad \forall x, y, z \in M \tag{2}$$
- *Unital*
$$\exists e \in M \quad : \quad e \otimes x = x \otimes e = x \quad \forall x \in M \tag{3}$$
Such an element is unique, and thus one speaks of *identity element.*
- *Exists Inverse:*
$$\forall a \in M \quad \exists b \in M \quad : \quad a \otimes b = b \otimes a = e \tag{4}$$
where e is the identity element and thus one speaks of the identity element.
- *Commutative*
$$x \otimes y = y \otimes x \qquad \forall x, y \in M \tag{5}$$

Depending on the properties satisfied by $\otimes$, the pair $(M, \otimes)$ is called:

Magma	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

SemiGroup	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

Monoid	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

Commutative Monoid	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

Group	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

Abelian Group	
Closure	
Associativity	
Identity	
Inverse	
Commutativity	

Observation 1

Another recurring magma is the *Quasi-Group*. https://it.wikipedia.org/wiki/Quasigruppo

2 Double Operation Structure (Rings)

Consider two set R, A and two binary operation as above:

$$\otimes : R \times R \to A \qquad \oplus : R \times R \to A$$

Are defined some basic compatibility relations:

- $\otimes$ is *Left distributivite* with respect to $\oplus$

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \qquad \forall a, b, c \in R \tag{6}$$

- $\otimes$ is *Right distributivite* with respect to $\oplus$

$$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \qquad \forall a, b, c \in R \tag{7}$$

Notation fixing

Mimicking the numeric fields we denote

	Identity	Inverse
$\otimes$	$\mathbb{1}$	$(\cdot)^{-1}$
$\oplus$	$\mathbb{0}$	$-(\cdot)$

Depending on the properties satisfied by $\otimes, \oplus$, the triple $(M, \otimes, \oplus)$ is called:

Field	
Closure Associativity Identity Inverse Commutativity	Abelian Group $(\oplus)$
Closure Associativity Identity Inverse Commutativity	Abelian Group $(\otimes)$
Left Distributivity Right Distributivity	Distributivity $(\otimes)$ w.r.t. $(\oplus)$

Commutative Ring	
Closure Associativity Identity Inverse Commutativity	Abelian Group $(\oplus)$
Closure Associativity Identity Inverse Commutativity	Commutative Monoid $(\otimes)$
Left Distributivity Right Distributivity	Distributivity $(\otimes)$ w.r.t. $(\oplus)$

Ring	
Closure Associativity Identity Inverse Commutativity	Abelian Group $(\oplus)$
Closure Associativity Identity Inverse Commutativity	Monoid $(\otimes)$
Left Distributivity Right Distributivity	Distributivity $(\otimes)$ w.r.t. $(\oplus)$

Semi-Ring	
Closure Associativity Identity Inverse Commutativity	Commutative Monoid$(\oplus)$
Closure Associativity Identity Inverse Commutativity	Monoid$(\otimes)$
Left Distributivity Right Distributivity	Distributivity $(\otimes)$ w.r.t. $(\oplus)$

Pseudo-Ring	
Closure Associativity Identity Inverse Commutativity	Abelian Group $(\oplus)$
Closure Associativity Identity Inverse Commutativity	SemiGroup$(\otimes)$
Left Distributivity Right Distributivity	Distributivity $(\otimes)$ w.r.t. $(\oplus)$

Commutative Semi-Ring	
Closure Associativity Identity Inverse Commutativity	Commutative Monoid$(\oplus)$
Closure Associativity Identity Inverse Commutativity	Commutative Monoid$(\otimes)$
Left Distributivity Right Distributivity	Distributivity $(\otimes)$ w.r.t. $(\oplus)$

3 Structures over a Field

Be $\mathbb{K}$ a field, we call a *scalar multiplication* over the set V the function:

$$\cdot : \mathbb{K} \cdot V \to V \qquad (\lambda, \mathbf{v}) \mapsto \lambda \mathbf{v} \quad \forall \lambda \in \mathbb{K},\ \forall \mathbf{v} \in V$$

Let be $(V, \clubsuit)$ a magma endowed with a scalar multiplication over field $\mathbb{K}$. Are defined some basic compatibility Relation:

- *Compatibility of scalar multiplication* :

$$a(b\mathbf{v}) = ab(\mathbf{v}) \qquad \forall a, b \in \mathbb{K},\ \forall \mathbf{v} \in V \tag{8}$$

- *Compatibility of scalar multiplication* w.r.t. $\otimes$:

$$(a\mathbf{v}) \otimes (b\mathbf{u}) = (ab)(\mathbf{v} \otimes \mathbf{u}) \qquad \forall a, b \in \mathbb{K},\ \forall \mathbf{v}, \mathbf{u} \in V \tag{9}$$

- *Identity element of scalar multiplication*:

$$1\mathbf{v} = \mathbf{v} \qquad 1 \text{identity element in } \mathbb{K} \tag{10}$$

- *Distributivity of scalar multiplication* w.r.t $\oplus$:

$$a(\mathbf{u} \oplus \mathbf{v}) = a\mathbf{u} \oplus a\mathbf{v} \qquad \forall a \in \mathbb{K},\ \forall \mathbf{u}, \mathbf{v} \in V \tag{11}$$

- *Distributivity of scalar multiplication with respect to field addition*:

$$(a + b)\mathbf{v} = a\mathbf{v} \oplus b\mathbf{v} \tag{12}$$

Depending on the properties satisfied by $\otimes, \oplus$, the triple $(M, \otimes, \oplus)$ is called:

Vector Space	
Closure	
Associativity	
Identity	Abelian Group(⊕)
Inverse	
Commutativity	
Closure	
Associativity	
Identity	No (⊗)
Inverse	
Commutativity	
Left Distributivity	Distributivity
Right Distributivity	(⊗) w.r.t. (⊕)
SP Multiplication (8)	Scalar Product
SP Identity(10)	Compatibility
Distributivity w.r.t. ⊗ (11)	Scalar Product
Distributivity w.r.t.⊕ (12)	distributivity w.r.t. (⊕)
Scalar Multiplication Compatibility w.r.t (⊗) (9)	

Algebra over Field $\mathbb{K}$	
Closure	
Associativity	
Identity	Abelian Group(⊕)
Inverse	
Commutativity	
Closure	
Associativity	
Identity	Magma (⊗)
Inverse	
Commutativity	
Left Distributivity	Distributivity
Right Distributivity	(⊗) w.r.t. (⊕)
SP Multiplication (8)	Scalar Product
SP Identity(10)	Compatibility
Distributivity w.r.t. ⊗ (11)	Scalar Product
Distributivity w.r.t.⊕ (12)	distributivity w.r.t. (⊕)
Scalar Multiplication Compatibility w.r.t (⊗) (9)	

Associative Algebra over $\mathbb{K}$	
Closure	
Associativity	
Identity	Abelian Group(⊕)
Inverse	
Commutativity	
Closure	
Associativity	
Identity	SemiGroup (⊗)
Inverse	
Commutativity	
Left Distributivity	Distributivity
Right Distributivity	(⊗) w.r.t. (⊕)
SP Multiplication (8)	Scalar Product
SP Identity(10)	Compatibility
Distributivity w.r.t. ⊗ (11)	Scalar Product
Distributivity w.r.t.⊕ (12)	distributivity w.r.t. (⊕)
Scalar Multiplication Compatibility w.r.t (⊗) (9)	

Unital Algebra over $\mathbb{K}$	
Closure	
Associativity	
Identity	Abelian Group(⊕)
Inverse	
Commutativity	
Closure	
Associativity	
Identity	Monoid (⊗)
Inverse	
Commutativity	
Left Distributivity	Distributivity
Right Distributivity	(⊗) w.r.t. (⊕)
SP Multiplication (8)	Scalar Product
SP Identity(10)	Compatibility
Distributivity w.r.t.⊗ (11)	Scalar Product
Distributivity w.r.t.⊕ (12)	distributivity w.r.t. (⊕)
Scalar Multiplication Compatibility w.r.t (⊗) (9)	

4 TODO

	Totality	Associativity	Identity	Invertibility	Commutativity
Semicategory	unneeded	needed	unneeded	unneeded	unneeded
Category	unneeded	needed	needed	unneeded	unneeded
Groupoid	unneeded	needed	needed	needed	unneeded
Magma	needed	unneeded	unneeded	unneeded	unneeded
Quasigroup	needed	unneeded	unneeded	needed	unneeded
Loop	needed	unneeded	needed	needed	unneeded
Semigroup	needed	needed	unneeded	unneeded	unneeded
Monoid	needed	needed	needed	unneeded	unneeded
Group	needed	needed	needed	needed	unneeded
Abelian Group	needed	needed	needed	needed	needed

- Aggiungere sezione sulle relazioni (vedi materiale braga)
- Trasforma ogni capitolo in tabella landscape
- Aggiungere sezione finale sul punto di vista della universal algebra
- Riguardo ai campi: Fra la compatibilitÃă c'ÃÍ che 0 (identitÃă della somma) sia elemento 0 del prodotto, quindi non ha inverso! dunque l'ipotesi di invertibilitÃă sul gruppo prodotto vale ovunque tranne che su 0!
-