# Prolegomena on (some) Abstact Algebra

**Algebraic structures zoology**

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(molto basato su wiki)

## Contents

# 1 Single Operation structures (Magmas)

Consider two set $M, A$ and a binary operation, i.e. a function:

$$\otimes : M \times M \to A$$

Operation $\otimes$ is said:

- *Closed* if $ran(\otimes) \subset M$

$$(x \otimes y) \in M \qquad \forall x, y \in M \tag{1}$$

- *Associative*

$$(x \otimes y) \otimes z = x \otimes (y \otimes z) \qquad \forall x, y, z \in M \tag{2}$$

- *Unital*

$$\exists e \in M \quad : \quad e \otimes x = x \otimes e = x \quad \forall x \in M \tag{3}$$

   Such an element is unique, and thus one speaks of *identity element*.

- *Exists Inverse:*

$$\forall a \in M \quad \exists b \in M \quad : \quad a \otimes b = b \otimes a = e \tag{4}$$

   where e is the identity element and thus one speaks of the identity element.

- *Commutative*

$$x \otimes y = y \otimes x \qquad \forall x, y \in M \tag{5}$$

Depending on the properties satisfied by $\otimes$, the pair $(M, \otimes)$ is called:

| Magma | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

| SemiGroup | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

| Monoid | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

| Commutative Monoid | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

| Group | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

| Abelian Group | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

**Observation 1**

Another recurring magma is the *Quasi-Group.* `https://it.wikipedia.org/wiki/Quasigruppo`

# 2  Double Operation Structure (Rings)

Consider two set $R, A$ and two binary operation as above:

$$\otimes : R \times R \to A \qquad \oplus : R \times R \to A$$

Are defined some basic compatibility relations:

- $\otimes$ is *Left distributivite* with respect to $\oplus$

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \qquad \forall a, b, c \in R \tag{6}$$

- $\otimes$ is *Right distributivite* with respect to $\oplus$

$$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \qquad \forall a, b, c \in R \tag{7}$$

**Notation fixing**

Mimicking the numeric fields we denote

|          | Identity | Inverse      |
|----------|----------|--------------|
| $\otimes$ | $\mathbb{1}$ | $(\cdot)^{-1}$ |
| $\oplus$  | $\mathbb{0}$ | $-(\cdot)$   |

Depending on the properties satisfied by $\otimes, \oplus$, the triple $(M, \otimes, \oplus)$ is called:

## Field

| Properties | Structure |
|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group ( ⊕ ) |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group ( ⊗ ) |
| Left Distributivity<br>Right Distributivity | Distributivity<br>( ⊗ ) w.r.t. ( ⊕ ) |

## Commutative Ring

| Properties | Structure |
|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group ( ⊕ ) |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Commutative Monoid ( ⊗ ) |
| Left Distributivity<br>Right Distributivity | Distributivity<br>( ⊗ ) w.r.t. ( ⊕ ) |

## Ring

| Properties | Structure |
|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group ( ⊕ ) |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Monoid ( ⊗ ) |
| Left Distributivity<br>Right Distributivity | Distributivity<br>( ⊗ ) w.r.t. ( ⊕ ) |

## Semi-Ring

| Properties | Structure |
|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Commutative Monoid( ⊕ ) |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Monoid( ⊗ ) |
| Left Distributivity<br>Right Distributivity | Distributivity<br>( ⊗ ) w.r.t. ( ⊕ ) |

## Pseudo-Ring

| Properties | Structure |
|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group ( ⊕ ) |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | SemiGroup( ⊗ ) |
| Left Distributivity<br>Right Distributivity | Distributivity<br>( ⊗ ) w.r.t. ( ⊕ ) |

## Commutative Semi-Ring

| Properties | Structure |
|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Commutative Monoid( ⊕ ) |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Commutative Monoid( ⊗ ) |
| Left Distributivity<br>Right Distributivity | Distributivity<br>( ⊗ ) w.r.t. ( ⊕ ) |

# 3 Structures over a Field

Be $\mathbb{K}$ a field, we call a *scalar multiplication* over the set $V$ the function:

$$\cdot : \mathbb{K} \cdot V \to V \qquad (\lambda, \mathbf{v}) \mapsto \lambda \mathbf{v} \quad \forall \lambda \in \mathbb{K}, \ \forall \mathbf{v} \in V$$

Let be $(V, \clubsuit)$ a magma endowed with a scalar multiplication over field $\mathbb{K}$. Are defined some basic compatibility Relation:

- *Compatibility of scalar multiplication* :

$$a(b\mathbf{v}) = ab(\mathbf{v}) \qquad \forall a, b \in \mathbb{K}, \ \forall \mathbf{v} \in V \tag{8}$$

- *Compatibility of scalar multiplication* w.r.t. $\otimes$:

$$(a\mathbf{v}) \otimes (b\mathbf{u}) = (ab)(\mathbf{v} \otimes \mathbf{u}) \qquad \forall a, b \in \mathbb{K}, \ \forall \mathbf{v}, \mathbf{u} \in V \tag{9}$$

- *Identity element of scalar multiplication*:

$$1\mathbf{v} = \mathbf{v} \qquad 1 \text{identity element in } \mathbb{K} \tag{10}$$

- *Distributivity of scalar multiplication* w.r.t $\oplus$:

$$a(\mathbf{u} \oplus \mathbf{v}) = a\mathbf{u} \oplus a\mathbf{v} \qquad \forall a \in \mathbb{K}, \ \forall \mathbf{u}, \mathbf{v} \in V \tag{11}$$

- *Distributivity of scalar multiplication with respect to field addition*:

$$(a + b)\mathbf{v} = a\mathbf{v} \oplus b\mathbf{v} \tag{12}$$

Depending on the properties satisfied by $\otimes, \oplus$, the triple $(M, \otimes, \oplus)$ is called:

| Vector Space | | |
|---|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group($\oplus$) | |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | No ($\otimes$) | |
| Left Distributivity<br>Right Distributivity | Distributivity<br>($\otimes$) w.r.t. ($\oplus$) | |
| SP Multiplication (**??**)<br>SP Identity(**??**) | Scalar Product<br>Compatibility | |
| Distributivity w.r.t + (**??**)<br>Distributivity w.r.t $\oplus$ (**??**) | Scalar Product<br>distributivity w.r.t. ($\oplus$) | |
| Scalar Product Compatibility w.r.t ($\otimes$) (**??**) | | |

| Algebra over Field $\mathbb{K}$ | | |
|---|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group($\oplus$) | |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Magma ($\otimes$) | |
| Left Distributivity<br>Right Distributivity | Distributivity<br>($\otimes$) w.r.t. ($\oplus$) | |
| SP Multiplication (**??**)<br>SP Identity(**??**) | Scalar Product<br>Compatibility | |
| Distributivity w.r.t + (**??**)<br>Distributivity w.r.t $\oplus$ (**??**) | Scalar Product<br>distributivity w.r.t. ($\oplus$) | |
| Scalar Product Compatibility w.r.t ($\otimes$) (**??**) | | |

| Associative Algebra over $\mathbb{K}$ | | |
|---|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group($\oplus$) | |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | SemiGroup ($\otimes$) | |
| Left Distributivity<br>Right Distributivity | Distributivity<br>($\otimes$) w.r.t. ($\oplus$) | |
| SP Multiplication (**??**)<br>SP Identity(**??**) | Scalar Product<br>Compatibility | |
| Distributivity w.r.t + (**??**)<br>Distributivity w.r.t $\oplus$ (**??**) | Scalar Product<br>distributivity w.r.t. ($\oplus$) | |
| Scalar Product Compatibility w.r.t ($\otimes$) (**??**) | | |

| Unital Algebra over $\mathbb{K}$ | | |
|---|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group($\oplus$) | |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Monoid ($\otimes$) | |
| Left Distributivity<br>Right Distributivity | Distributivity<br>($\otimes$) w.r.t. ($\oplus$) | |
| SP Multiplication (**??**)<br>SP Identity(**??**) | Scalar Product<br>Compatibility | |
| Distributivity w.r.t + (**??**)<br>Distributivity w.r.t $\oplus$ (**??**) | Scalar Product<br>distributivity w.r.t. ($\oplus$) | |
| Scalar Product Compatibility w.r.t ($\otimes$) (**??**) | | |