CheatSheet Tensor Algebra

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1 Factoid

(TODO: da capire. tutto ciò valeanche se V è solo un modulo su una anello invece che spazio vettoriale su campo. Inoltre anche la proprietà di algebra è conservata)

The space of all the function valued in a vector space are a vector space itself. Namely, for any set Ω and vector space $(V, \mathbb{K})$ the set

$$V^\Omega = \{x : \Omega \to V \quad | \quad x(i) = x_i\}$$

endowed with the functions:

$$+ : V^\Omega \times V^\Omega \to V^\Omega \quad s.t.(x+y)_i = x_i + y_i \quad \forall i \in \Omega;\ \forall x, y \in V^\Omega;$$

$$\cdot : \mathbb{K} \times V^\Omega \to V^\Omega \quad s.t.(\lambda x)_i = \lambda x_i \quad \forall i \in \Omega;\ \forall x \in V^\Omega;\ \forall \lambda \in \mathbb{K}$$

e.g.:

$$\mathbb{R}^\Omega = \mathbb{R}^n \quad \text{with } \Omega = \{i \in \mathbb{N} \quad | \quad 1 \leq i \leq n\}$$

Come dicevo prima ho che le funzioni a valori in R formano un algebra su campo R.

2 Tensor (Linear) Algebra

Let's see at this stage the *Tensor Calculus* as a machinery able to build up a plethora of other vector spaces starting from a single prototype.
The main blocks to accomplish this construction are two.

Let be V, W two vector space on the same field $\mathbb{K}$

$$V^* = \{\alpha : V \to \mathbb{K} \quad | \quad \alpha \text{linear}\} \subset \mathbb{K}^V$$

$$V \otimes_{\mathbb{K}} W = \{\pi : V^* \times W^* \to \mathbb{K} \quad | \quad \pi \text{bilinear}\} \subset \mathbb{K}^{(V^* \times W^*)}$$

Other than the tensor product of space is possible to multiplicate covector.

(warning sloppy abused notation

$$Def---$$

$$\otimes : V^* \times W^* \to V \otimes_{\mathbb{K}} W \quad | \quad (\alpha \otimes \beta)(v, w) = \alpha(v)\beta(w)$$

The last inlcusion relations are there to remind us that they are naturally vector spaces according to 1.

From there we define:

$$\mathcal{T}^p(V) = \mathcal{T}^p_0(V) = \{\pi : \underbrace{V^* \times \ldots \times V^*}_{\text{p times}} \to \mathbb{K} \quad | \quad \text{multilinear}\} = \underbrace{V \otimes \ldots \otimes V}_{\text{p times}}$$

$$\mathcal{T}_q(V) = \mathcal{T}^0_q(V) = \{\pi : \underbrace{V \times \ldots \times V}_{\text{q times}} \to \mathbb{K} \quad | \quad \text{multilinear}\} = \underbrace{V^* \otimes \ldots \otimes V^*}_{\text{q times}}$$

$$\mathcal{T}^p_q(V) = \mathcal{T}^p(V) \otimes \mathcal{T}_q(V) = \{\pi : \underbrace{V \times \ldots \times V}_{\text{q times}} \times \underbrace{V^* \times \ldots \times V^*}_{\text{p times}} \to \mathbb{K} \quad | \quad \text{multilinear}\}$$

$$\mathcal{T}^0_0(V) = \mathbb{K}$$

$$\mathcal{T}^\bullet(V) = \bigoplus_{p \geq 0} \mathcal{T}^p(V)$$

$$\mathcal{T}_\bullet(V) = \bigoplus_{q \geq 0} \mathcal{T}_q(V)$$

$$\mathcal{T}(V) = \bigoplus_{p,q \geq 0} \mathcal{T}^p_q(V)$$

The definition of tensor product of forms can be easily extend to Tensors. Therefore: $(\mathcal{T}(V), \otimes)$ forms a *Associative Algebra over* $\mathbb{K}$.

3 Exterior Algebra

$$V \wedge_{\mathbb{K}} V = \{\alpha : V^* \times V^* \to \mathbb{K} \quad | \quad \text{multilinear, skewsymmetric}\}$$

$$---questae^{-}giusta---$$

$$\wedge : V^* \times V^* \to V \wedge_{\mathbb{K}} V \quad | \quad (\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \beta(v)\alpha(w)$$

$$---oss---$$

$$(\omega_1 \wedge \ldots \wedge \omega_k)(x_1, \ldots, x_k) = \det\big(\omega_i(x_j)\big)$$

$$\Lambda^k(V^*) = \underbrace{V^* \wedge \ldots \wedge V^*}_{ktimes} = \{\alpha : \underbrace{V \times \ldots \times V}_{ktimes} \to \mathbb{K} \quad | \quad \text{multilinear, alternating}\}\} \subset \mathcal{T}^0_k(V) = \mathcal{T}^k_0(V^*)$$

$$\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*)$$

$--\ Remark\ --$

A completely similar construction could be achieved defining the symmetrizing product instead of the antisymmetrizing! (per ora non esplicito questa cosa.)

$\Lambda(V^*) \subset \mathcal{T}(V)$ is a linear subspace but not a subalgebra. $(\Lambda(V^*), \wedge)$ constitutes an algebra called exterior or Grassman algebra.

3.1 basis

Fixed a basis $\{e_i\}$ in V it's automatically obtained a basis on the zoo of tensor spaces
$e_i^* = e^i \quad | \quad e^i(e_j) = \delta^i_j$ on the dual
$e_{i_1} \otimes \ldots \otimes e_{i_p} \otimes e^{j_1} \otimes \ldots \otimes e^{j_q}$ on tensors
$e_1 \wedge \ldots \wedge e_k$ in $\Lambda^k(V^*)$

$---oss---$

Ho qualche d extern in astratto?