

CheatSheet Tensor Algebra

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1 Factoid

(TODO: da capire. tutto ciò vale anche se V è solo un modulo su un anello invece che spazio vettoriale su campo. Inoltre anche la proprietà di algebra è conservata)

The space of all the function valued in a vector space are a vector space itself. Namely, for any set Ω and vector space (V, \mathbb{K}) the set

$$V^\Omega = \{x : \Omega \rightarrow V \mid x(i) = x_i\}$$

endowed with the functions:

$$\begin{aligned} + : V^\Omega \times V^\Omega &\rightarrow V^\Omega \quad s.t. (x+y)_i = x_i + y_i \quad \forall i \in \Omega; \forall x, y \in V^\Omega; \\ \cdot : \mathbb{K} \times V^\Omega &\rightarrow V^\Omega \quad s.t. (\lambda x)_i = \lambda x_i \quad \forall i \in \Omega; \forall x \in V^\Omega; \forall \lambda \in \mathbb{K} \end{aligned}$$

e.g.:

$$\mathbb{R}^\Omega = \mathbb{R}^n \quad \text{with } \Omega = \{i \in \mathbb{N} \mid 1 \leq i \leq n\}$$

Come dicevo prima ho che le funzioni a valori in \mathbb{R} formano un algebra su campo \mathbb{R} .

2 Tensor (Linear) Algebra

Let's see at this stage the *Tensor Calculus* as a machinery able to build up a plethora of other vector spaces starting from a single prototype.

The main blocks to accomplish this construction are two.

Let be V, W two vector space on the same field \mathbb{K}

$$V^* = \{\alpha : V \rightarrow \mathbb{K} \mid \alpha \text{ linear}\} \subset \mathbb{K}^V$$

$$V \otimes_{\mathbb{K}} W = \{\pi : V^* \times W^* \rightarrow \mathbb{K} \mid \pi \text{ bilinear}\} \subset \mathbb{K}^{(V^* \times W^*)}$$

Other than the tensor product of space is possible to multiply covector.

(warning sloppy abused notation

Def ---

$$\otimes : V^* \times W^* \rightarrow V \otimes_{\mathbb{K}} W \quad | \quad (\alpha \otimes \beta)(v, w) = \alpha(v)\beta(w)$$

The last inclusion relations are there to remind us that they are naturally vector spaces according to 1.

From there we define:

$$\mathcal{T}^p(V) = \mathcal{T}_0^p(V) = \{\pi : \underbrace{V^* \times \dots \times V^*}_{p \text{ times}} \rightarrow \mathbb{K} \quad | \quad \text{multilinear}\} = \underbrace{V \otimes \dots \otimes V}_{p \text{ times}}$$

$$\mathcal{T}_q(V) = \mathcal{T}_q^0(V) = \{\pi : \underbrace{V \times \dots \times V}_{q \text{ times}} \rightarrow \mathbb{K} \quad | \quad \text{multilinear}\} = \underbrace{V^* \otimes \dots \otimes V^*}_{q \text{ times}}$$

$$\mathcal{T}_q^p(V) = \mathcal{T}^p(V) \otimes \mathcal{T}_q(V) = \{\pi : \underbrace{V \times \dots \times V}_{q \text{ times}} \times \underbrace{V^* \times \dots \times V^*}_{p \text{ times}} \rightarrow \mathbb{K} \quad | \quad \text{multilinear}\}$$

$$\mathcal{T}_0^0(V) = \mathbb{K}$$

$$\mathcal{T}^\bullet(V) = \bigoplus_{p \geq 0} \mathcal{T}^p(V)$$

$$\mathcal{T}_\bullet(V) = \bigoplus_{q \geq 0} \mathcal{T}_q(V)$$

$$\mathcal{T}(V) = \bigoplus_{p, q \geq 0} \mathcal{T}_q^p(V)$$

The definition of tensor product of forms can be easily extend to Tensors. Therefore: $(\mathcal{T}(V), \otimes)$ forms a *Associative Algebra over* \mathbb{K} .

3 Exterior Algebra

$$V \wedge_{\mathbb{K}} V = \{\alpha : V^* \times V^* \rightarrow \mathbb{K} \quad | \quad \text{multilinear, skewsymmetric}\}$$

--- *questae* ⁻ *giusta* ---

$$\wedge : V^* \times V^* \rightarrow V \wedge_{\mathbb{K}} V \quad | \quad (\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \beta(v)\alpha(w)$$

--- *oss* ---

$$(\omega_1 \wedge \dots \wedge \omega_k)(x_1, \dots, x_k) = \det(\omega_i(x_j))$$

$$\Lambda^k(V^*) = \underbrace{V^* \wedge \dots \wedge V^*}_{k \text{ times}} = \{\alpha : \underbrace{V \times \dots \times V}_{k \text{ times}} \rightarrow \mathbb{K} \quad | \quad \text{multilinear, alternating}\} \subset \mathcal{T}_k^0(V) = \mathcal{T}_0^k(V^*)$$

$$\Lambda(V^*) = \bigoplus_{k=0}^n \Lambda^k(V^*)$$

– – *Remark* – –

A completely similar construction could be achieved defining the symmetrizing product instead of the antisymmetrizing! (per ora non esplicito questa cosa.)

$\Lambda(V^*) \subset \mathcal{T}(V)$ is a linear subspace but not a subalgebra. $(\Lambda(V^*), \wedge)$ constitutes an algebra called exterior or Grassman algebra.

3.1 basis

Fixed a basis $\{e_i\}$ in V it's automatically obtained a basis on the zoo of tensor spaces

$e_i^* = e^i \mid e^i(e_j) = \delta_j^i$ on the dual

$e_{i_1} \otimes \dots \otimes e_{i_p} \otimes e^{j_1} \otimes \dots \otimes e^{j_q}$ on tensors

$e_1 \wedge \dots \wedge e_k$ in $\Lambda^k(V^*)$

– – – *oss* – – –

Ho qualche d'extern in astratto?