# Prolegomena on (some) Abstact Algebra

**Algebraic structures zoology**

Tony

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(molto basato su wiki)

## Contents

# 1 Single Operation structures (Magmas)

Consider two set $M, A$ and a binary operation, i.e. a function:

$$\otimes : M \times M \to A$$

Operation $\otimes$ is said:

- *Closed* if $ran(\otimes) \subset M$
$$(x \otimes y) \in M \qquad \forall x, y \in M \tag{1}$$
- *Associative*
$$(x \otimes y) \otimes z = x \otimes (y \otimes z) \qquad \forall x, y, z \in M \tag{2}$$
- *Unital*
$$\exists e \in M \quad : \quad e \otimes x = x \otimes e = x \quad \forall x \in M \tag{3}$$
Such an element is unique, and thus one speaks of the identity element.
- *Exists Inverse:*
$$\forall a \in M \quad \exists b \in M \quad : \quad a \otimes b = b \otimes a = e \tag{4}$$
where e is the identity element and thus one speaks of the identity element.
- *Commutative*
$$x \otimes y = y \otimes x \qquad \forall x, y \in M \tag{5}$$

Depending on the properties satisfied by $\otimes$, the pair $(M, \otimes)$ is called:

| Magma | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

| SemiGroup | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

| Monoid | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

| Commutative Monoid | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

| Group | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

| Abelian Group | |
|---|---|
| Closure | |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |

## 2 Double Operation Structure (Rings)

Consider two set $R, A$ and two binary operation as above:

$$\otimes : R \times R \to A \qquad \oplus : R \times R \to A$$

Are defined some basic compatibility relations:

- $\otimes$ is *Left distributivite* with respect to $\oplus$

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c) \qquad \forall a, b, c \in R \tag{6}$$

- $\otimes$ is *Right distributivite* with respect to $\oplus$

$$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \qquad \forall a, b, c \in R \tag{7}$$

Depending on the properties satisfied by $\otimes, \oplus$, the triple $(M, \otimes, \oplus)$ is called:

| Field | |
|---|---|
| Closure | Abelian Group ( $\oplus$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Closure | Abelian Group ( $\otimes$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Left Distributivity | Distributivity |
| Right Distributivity | ( $\otimes$ ) w.r.t. ( $\oplus$ ) |

| Commutative Ring | |
|---|---|
| Closure | Abelian Group ( $\oplus$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Closure | Commutative Monoid ( $\otimes$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Left Distributivity | Distributivity |
| Right Distributivity | ( $\otimes$ ) w.r.t. ( $\oplus$ ) |

| Ring | |
|---|---|
| Closure | Abelian Group ( $\oplus$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Closure | Monoid ( $\otimes$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Left Distributivity | Distributivity |
| Right Distributivity | ( $\otimes$ ) w.r.t. ( $\oplus$ ) |

| Semi-Ring | |
|---|---|
| Closure | Commutative Monoid( $\oplus$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Closure | Monoid( $\otimes$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Left Distributivity | Distributivity |
| Right Distributivity | ( $\otimes$ ) w.r.t. ( $\oplus$ ) |

| Pseudo-Ring | |
|---|---|
| Closure | Abelian Group ( $\oplus$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Closure | SemiGroup( $\otimes$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Left Distributivity | Distributivity |
| Right Distributivity | ( $\otimes$ ) w.r.t. ( $\oplus$ ) |

| Commutative Semi-Ring | |
|---|---|
| Closure | Commutative Monoid( $\oplus$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Closure | Commutative Monoid( $\otimes$ ) |
| Associativity | |
| Identity | |
| Inverse | |
| Commutativity | |
| Left Distributivity | Distributivity |
| Right Distributivity | ( $\otimes$ ) w.r.t. ( $\oplus$ ) |

**Notation fixing**

Mimicking the numeric fields we denote

| | Identity | Inverse |
|---|---|---|
| $\otimes$ | $\mathbb{1}$ | $(\cdot)^{-1}$ |
| $\oplus$ | $\mathbb{0}$ | $-(\cdot)$ |

# 3 Structures over a Field

Be $\mathbb{K}$ a field, we call a *scalar multiplication* over the set $V$ the function:

$$\cdot : \mathbb{K} \cdot V \to V \qquad (\lambda, \mathbf{v}) \mapsto \lambda \mathbf{v} \quad \forall \lambda \in \mathbb{K},\ \forall \mathbf{v} \in V$$

Let be $(V, \clubsuit)$ a magma endowed with a scalar multiplication over field $\mathbb{K}$. Are defined some basic compatibility Relation:

- *Compatibility of scalar multiplication* :

$$a(b\mathbf{v}) = ab(\mathbf{v}) \qquad \forall a, b \in \mathbb{K},\ \forall \mathbf{v} \in V \tag{8}$$

- *Compatibility of scalar multiplication* w.r.t. $\otimes$:

$$(a\mathbf{v}) \otimes (b\mathbf{u}) = (ab)(\mathbf{v} \otimes \mathbf{u}) \qquad \forall a, b \in \mathbb{K},\ \forall \mathbf{v}, \mathbf{u} \in V \tag{9}$$

- *Identity element of scalar multiplication*:

$$1\mathbf{v} = \mathbf{v} \qquad 1 \text{identity element in } \mathbb{K} \tag{10}$$

- *Distributivity of scalar multiplication* w.r.t $\oplus$:

$$a(\mathbf{u} \oplus \mathbf{v}) = a\mathbf{u} \oplus a\mathbf{v} \qquad \forall a \in \mathbb{K},\ \forall \mathbf{u}, \mathbf{v} \in V \tag{11}$$

- *Distributivity of scalar multiplication with respect to field addition*:

$$(a + b)\mathbf{v} = a\mathbf{v} \oplus b\mathbf{v} \tag{12}$$

Depending on the properties satisfied by $\otimes, \oplus$, the triple $(M, \otimes, \oplus)$ is called:

| Vector Space | |
|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group( ⊕ ) |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | No ( ⊗ ) |
| Left Distributivity<br>Right Distributivity | Distributivity<br>( ⊗ ) w.r.t. ( ⊕ ) |
| SP Multiplication (8)<br>SP Identity(10) | Scalar Product<br>Compatibility |
| Distributivity w.r.t + (11)<br>Distributivity w.r.t ⊕ (12) | Scalar Product<br>distributivity w.r.t. ( ⊕ ) |
| Scalar Product Compatibility w.r.t ( ⊗ ) (9) | |

| Algebra over Field $\mathbb{K}$ | |
|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group( ⊕ ) |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Magma ( ⊗ ) |
| Left Distributivity<br>Right Distributivity | Distributivity<br>( ⊗ ) w.r.t. ( ⊕ ) |
| SP Multiplication (8)<br>SP Identity(10) | Scalar Product<br>Compatibility |
| Distributivity w.r.t + (11)<br>Distributivity w.r.t ⊕ (12) | Scalar Product<br>distributivity w.r.t. ( ⊕ ) |
| Scalar Product Compatibility w.r.t ( ⊗ ) (9) | |

| Associative Algebra over $\mathbb{K}$ | |
|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group( ⊕ ) |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Magma ( ⊗ ) |
| Left Distributivity<br>Right Distributivity | Distributivity<br>( ⊗ ) w.r.t. ( ⊕ ) |
| SP Multiplication (8)<br>SP Identity(10) | Scalar Product<br>Compatibility |
| Distributivity w.r.t + (11)<br>Distributivity w.r.t ⊕ (12) | Scalar Product<br>distributivity w.r.t. ( ⊕ ) |
| Scalar Product Compatibility w.r.t ( ⊗ ) (9) | |

| Unital Algebra over $\mathbb{K}$ | |
|---|---|
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Abelian Group( ⊕ ) |
| Closure<br>Associativity<br>Identity<br>Inverse<br>Commutativity | Magma ( ⊗ ) |
| Left Distributivity<br>Right Distributivity | Distributivity<br>( ⊗ ) w.r.t. ( ⊕ ) |
| SP Multiplication (8)<br>SP Identity(10) | Scalar Product<br>Compatibility |
| Distributivity w.r.t + (11)<br>Distributivity w.r.t ⊕ (12) | Scalar Product<br>distributivity w.r.t. ( ⊕ ) |
| Scalar Product Compatibility w.r.t ( ⊗ ) (9) | |