

Interpretazione Geometrica delle parentesi di
Peierls nella quantizzazione algebrica del
campo Geodetico.

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Abstract

La prima parte della tesi è stata rivolta allo studio del framework matematico necessario per dare una formulazione rigorosa dei sistemi classici continui, punto di partenza di ogni schema di quantizzazione algebrica. Nello specifico viene fatta una digressione sui Fibrati Topologici e viene sfruttata la definizione di fibrato liscio per presentare l'approccio geometrico alla meccanica classica sia per sistemi a gradi di libertà finiti che continui.

Nella seconda parte viene presentato l'algoritmo di Peierls che rappresenta una ricetta efficace per attribuire una struttura pre-simplettica allo spazio delle configurazioni dinamiche di un sistema qualunque. Dalla ricerca bibliografica è evidente come questo strumento a partire dal suo esordio (nel 1952) fino ad oggi non abbia mai ricevuto particolare attenzione. Questo sembra dovuto soprattutto alla mancanza di una convincente interpretazione geometrica.

Per fare un passo verso la comprensione di questo oggetto viene studiato l'estremamente noto problema della geodetica vedendolo come un sistema campo. Emerge sin da subito come il calcolo delle parentesi Peierls per questo sistema sia legato intrinsecamente al problema del calcolo dei campi di Jacobi lungo una geodetica.

Nella terza parte vengono descritte due realizzazioni dello schema di quantizzazione algebrica per i campi bosonici. La prima sfrutta le parentesi di Peierls mentre la seconda interviene sui dati iniziali della dinamica di campo.

Il campo di Jacobi si presta ad essere quantizzato secondo entrambe le prescrizioni. Confrontando le 2 forme simplettiche così ottenute si cerca di fornire nuovi tasselli per attribuire un'interpretazione geometrica al metodo originale di Peierls.

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Chapter 1

Mathematical Preliminaries

Le interazioni matematiche sono complesse e non triviali (vedi un po' di articoli di introduzione a AQFT per ispirarti)

Tendenzialmente le teorie quantistiche di campi moderne sono di Quantizzazione.. Quindi richiedono di specificare bene la struttura del campo classico (vedi intro di Mangiaratti shardashivly)

Gli strumenti matematici per raccontare la teoria dei campi classici sono essenzialmente 3: Fibrati, S-T G-H, LDOP e GHOP.



IN questo paper non ci soffermeremo sulle strutture del framework puramente quantistico (* algebre e quant'altro). Per un primer vedere articolo di Dappiaggi o Libro Adv aqft.

Diamo per scontato un background di base in Geometria differenziale e derivate esterne (algebre di Grassman? global calculus? non so come chiamarlo!)

Potrei avere la tentazione a provare ad usare un po' di linguaggio basilare delle categorie... la mia fonte \tilde{A} Joy of Cat.



Stile: Intro lapiadaria ai 3 argomenti (bundle cinematica di campo, Glob iper stage per descrivere dinamica di tipo propagativo, Operatori tipo onda). Poi smitragliata di definizioni come faceva Penati.

1.1 Fiber Bundles

1.1.1 ...



Inserire solo i punti salienti del primo capitolo.. Spostare ex primo capitolo spostato nel repository "dispensarium" come dispensa WIP

1.1.2 Some Topics useful in Physics

Jet Bundles


Tautological one-form and symplectic form.

1.2 Globally Hyperbolic Space-times

 Mettere solo le definizioni che uso prese dagli articoli di review delle Fonti

Appunti che mi ero preso scrivendo il secondo capitolo:

This condition is strictly connected to the dynamic behaviour of the system.

 Def di dominio di dipendenza footnote di definizione di spazio tempo
def of cauchy surface Remark causal future past def globally hyperbolic Teo-
rema sulle caratterizzazioni

Notation fixing

We denote the set of all the cauchy surfaces as $\mathcal{P}_C(M)$.

Glom iperbolic determina la fogliazione dello spazio tempo per superfici di cauchy
La superficie di cauchy \tilde{A} questa:


Definition 1: Cauchy surface

questo da la possibilit  della buona posizione dei problemi di cauchy.. fisicamente
 \tilde{A} la condizione minima per definire i dati iniziali dell'evoluzione dinamica. definisco
data...

Rapporto con la condizione sugli operatori...

No! La definizione di green hyperbolicity non garantisce invece l'esistenza
e unicit  del problema di cauchy associata

e non solo, anche l'esistenza degli operatori di green associati che sono
ingrediente fondamentale della costruzione di peierls

 $M \tilde{A}$ glob iper e $P \tilde{A}$ green iper per tener conto del comportamento
propagativo definire sup cauchy definire s-t iperbolico (solo la caratteriz-
zazione di ammettere una sup di cauchy) definire op green iperbolico su spazio
tempo iperbolico (cio  ha delle green ope) Propr di buona definizione es-
istenza e unicit  della soluzione

Di particolare ricorrenza fisica sono gli operatori normally iperbolic espres-
sione in coordinate esempio K-g!



Far notare che minkowski e tanti esempi importanti sono GH

Observation 1

(che serve dopo) lo spazio R è banalmente iperbolico in quanto tutti i punti posso essere visti come superfici di cauchy.

(sono ripetizioni inutili per la tesi, sono informazioni che si ritrovano ovunque... sono informazioni adatta al knowledge base)

Recurring definitions in general Relativity (excluding the general smooth manifold prolegomena).

Definition 2: Space-Time

A quadruple (M, g, o, \mathfrak{t}) such that:

- (M, g) is a time-orientable n -dimensional manifold ($n > 2$)
- o is a choice of orientation
- \mathfrak{t} is a choice of time-orientation

Definition 3: Lorentzian Manifold

A pair (M, g) such that:

- M is a n -dimensional ($n \geq 2$), Hausdorff, second countable, connected, orientable smooth manifold.
- g is a Lorentzian metric.

Definition 4: Metric

A function on the bundle product of TM with itself:

$$g : TM \times_M TM \rightarrow \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_p M \times T_p M \rightarrow \mathbb{R}$$

is a non-degenerate bilinear form.

Notation fixing

- *Riemman* if the sign of g is positive definite, *Pseudo-Riemman* other-

wise.

- *Lorentzian* if the signature is $(+, -, \dots, -)$ or equivalently $(-, +, \dots, +)$.

Observation 2: Causal Structure

If a smooth manifold is endowed with a Lorentzian manifold of signature $(+, -, \dots, -)$ then the tangent vectors at each point in the manifold can be classed into three different types.

Notation fixing

$\forall p \in M, \quad \forall X \in T_p M$, the vector is:

- *time-like* if $g(X, X) > 0$.
- *light-like* if $g(X, X) = 0$.
- *space-like* if $g(X, X) < 0$.

Observation 3: Local Time Orientability

$\forall p \in M$ the timelike tangent vectors in p can be divided into two equivalence classes taking

$$X \sim Y \text{ iff } g(X, Y) > 0 \quad \forall X, Y \in T_p^{\text{time-like}} M :$$

We can (arbitrarily) call one of these equivalence classes "future-directed" and call the other "past-directed". Physically this designation of the two classes of future- and past-directed timelike vectors corresponds to a choice of an arrow of time at the point. The future- and past-directed designations can be extended to null vectors at a point by continuity.

Definition 5: Time-orientation

A global tangent vector field $\mathfrak{t} \in \Gamma^\infty(TM)$ over the Lorentzian manifold M such that:

- $\text{supp}(\mathfrak{t}) = M$
- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Observation 4

The fixing of a time-orientation is equivalent to a consistent smooth choice of a local time-direction.

Definition 6: Time-Orientable Lorentzian Manifold

A Lorentzian Manifold (M, g) such that exist at least one time-orientation $\mathfrak{t} \in \Gamma^\infty(TM)$.

Notation fixing

Consider a piece-wise smooth curve $\gamma : \mathbb{R} \supset I \rightarrow M$ is called:

- *time-like* (resp. light-like, space-like) iff $\dot{\gamma}(p)$ is time-like (resp. light-like, space-like) $\forall p \in M$.
- *causal* iff $\dot{\gamma}(p)$ is nowhere spacelike.
- *future directed* (resp. past directed) iff is causal and $\dot{\gamma}(p)$ is future (resp. past) directed $\forall p \in M$.

Definition 7: Chronological $\begin{smallmatrix} \text{future} \\ \text{past} \end{smallmatrix}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{I}_M^\pm(p) := \{q \in M \mid \exists \gamma \in C^\infty((0, 1), M) \text{ time-like } \begin{smallmatrix} \text{future} \\ \text{past} \end{smallmatrix} \text{ - directed} : \gamma(0) = p, \gamma(1) = q\}$$

Definition 8: Causal $\begin{smallmatrix} \text{future} \\ \text{past} \end{smallmatrix}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{J}_M^\pm(p) := \{q \in M \mid \exists \gamma \in C^\infty((0, 1), M) \text{ causal } \begin{smallmatrix} \text{future} \\ \text{past} \end{smallmatrix} \text{ - directed} : \gamma(0) = p, \gamma(1) = q\}$$

Notation fixing

Former concept can be naturally extended to subset $A \subset M$:

- $\mathbf{I}_M^\pm(A) = \bigcup_{p \in A} \mathbf{I}_M^\pm(p)$
- $\mathbf{J}_M^\pm(A) = \bigcup_{p \in A} \mathbf{J}_M^\pm(p)$

Definition 9: Achronal Set

Subset $\Sigma \subset M$ such that every inextendible timelike curve intersect Σ at most once.

Definition 10: $\text{future}_{\text{past}}$ Domain of dependence of an Achronal set

The two subset related to the generic achronal set $\Sigma \subset M$:

$$\mathbf{D}_M^\pm(\Sigma) := \{q \in M \mid \forall \gamma_{\text{future}}^{\text{past}} \text{ inextendible causal curve passing through } q : \gamma(I) \cap \Sigma \neq \emptyset\}$$

Notation fixing

$\mathbf{D}_M(\Sigma) := \mathbf{D}_M^+(\Sigma) \cup \mathbf{D}_M^-(\Sigma)$ is called *total domain of dependence*.

Definition 11: Cauchy Surface

Is a subset $\Sigma \subset M$ such that:

- closed
- achronal
- $\mathbf{D}_M(\Sigma) \equiv M$

1.3 Green Hyperbolic Operators

 Mettere solo le definizioni che uso prese dagli articoli di review delle Fonti

Pensavo di utilizzare la definizione di Green hyperbolic data da Bar che si avvale del concetto di formally dual (che non richiede la presenza del pairing) invece di quella usata in Advances AQFT che richiede solo che ammetta almeno un G^\pm per poi dimostrare tramite teorema che se $\tilde{\Delta}$ anche autoaggiunto vale l'unicità. Si tratta solo di una piccola sfumatura.. Deve essere chiarito che in tutto ciò che faccio interessano che

$$\forall P \exists ! G^\pm$$

. Che poi questa condizione derivi da GH secondo Bar o Gh secondo dap+selfadj $\tilde{\Delta}$ una di quelle questioni propriamente matematiche che poco interessa ai fisici della commissione.



Devo richiedere che il green operator sia unico? sia negli schemi di quantizzazione che nella definizione di peierls faccio largo uso dell'unicità. Per provare questa unicità si passa per la definizione di una forma bilineare che permette di parlare di aggiunto formale e quindi avvalersi del teorema.

Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. [?] [2]

(sono ripetizioni inutili per la tesi, sono informazioni che si ritrovano ovunque... sono informazioni adatta al knowledge base) Basic Definition in L.P.D.O. on smooth vector sections.

Consider $F = F(M, \pi, V)$, $F' = F'(M, \pi', V')$ two linear vector bundle over M with different typical fiber

Definition 12: Linear Partial Differential operator (of order at most $s \in \mathbb{N}_0$)

Linear map $L: \Gamma(F) \rightarrow \Gamma(F')$ such that:

$\forall p \in M$ exists:

- (U, ϕ) local chart on M .
- (U, χ) local trivialization of F
- (U, χ') local trivialization of F'

for which:

$$L(\sigma|_U) = \sum_{|\alpha| \leq s} A_\alpha \partial^\alpha \sigma \quad \forall \sigma \in \Gamma(M)$$

Remark:

(multi-index notation)

A multi-index is a natural valued finite dimensional vector $\alpha = (\alpha_0, \dots, \alpha_{n-1}) \in \mathbb{N}_0^n$ with $n < \infty$.

On \mathbb{R}^n a general differential operator can be identified by a multi-index:

$$\partial^\alpha = \prod_{\mu=0}^{n-1} \partial_\mu^{\alpha_\mu}$$

(Until the Schwartz theorem holds, the order of derivation is irrelevant.)

The order of the multi-index is defined as:

$$|\alpha| := \sum_{\mu=0}^{n-1} \alpha_\mu$$

????????????????

Hp:

Proposition 1.3.1 (Existence and uniqueness for the Cauchy Problem)

$M = (M, g, o, t)$ a globally hyperbolic space-time.

- $\Sigma \subset M$ a spacelike cauchy surface with future-pointing unit normal vector field \vec{n} .

Th:

Observation 5

"Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. " [?] [2]

However the existence and uniqueness can be proved for the large class of the *Normally-Hyperbolic Operators*.

Chapter 2

Lagrangian Systems and Peierls Brackets

⚠ .. introduzione dedichiamo sforzo alla definizione dei sistemi lagrangiani astratti per dare un punto di vista unificato ai sistemi a gradi di libert  continui (continui macroscopici, fluidi, campi) e a gradi di libert  discreti. Li vediamo come sotto classi dei sistemi lagrangiani notiamo che in entrambi i casi c'  un possibile dato di cauchy usiamo questo ingrediente per definire l'algoritmo di peierls.

2.1 Abstract Mechanical Systems

It's possible to state a mathematical definition sufficiently broad to include all the systems in ordinary analytical mechanics regardless of the cardinality of degrees of freedom in a unified way.

Definition 13: Abstract Evolutive System

Pair (E, P) composed of:

- $E \xrightarrow{\pi} M$
smooth fiber bundle of typical fiber Q on manifold M called "*configuration bundle*".
- $P : \Gamma^\infty(E) \rightarrow \Gamma^\infty(E)$
operator called "*motion operator*"

This formulation is still very distant from the physical interpretation but has the benefit to highlight the minimal mathematical objects which must be fixed in order to specify a mechanical systems.

Kinematics The configuration bundle encompass all the kinematical structure of the system, the pivotal role is played by the smooth sections which are to be understood as all the possible conformation of the system.

Notation fixing

$$\mathcal{C} := \Gamma^\infty(M, E)$$

Space of kinematic configurations.

A section is not a statical configuration, equivalent to a specific point in the configuration space of ordinary classical systems, but has to be seen as a specific realization of the kinematics in the sense of a complete description of a possible motion. At this level of abstraction, since no space-time structure has been specified, terms like stasis and motion must be taken with care. The natural physical interpretation should be clearly manifested through the concrete realization of systems with discrete and continuous degree of freedom.

Observation 6: Mathematical structure

Mathematically speaking this set should be regarded as an infinite dimensional Manifold.

This framework provides a geometric characterization of the notion of variations as tangent vectors on the the space of kinematic configurations .[6]

Observation 7: Coordinate Representation

The choice of a chart atlas $\mathcal{A}(M)$ on the base space M and $\mathcal{A}(E)$ on the total space E provides a correspondence between each configuration $\gamma \in \mathcal{C}$ and family of smooth real functions $\{f_{\alpha\beta} : A_\alpha \subset \mathbb{R}^m \rightarrow \mathbb{R}^q\}$. The process is trivial:

$$\gamma \in \mathcal{C} \mapsto \{f_{A,U} = \psi_U \circ \gamma \circ \psi_A^{-1} | (A, \psi_A) \in \mathcal{A}(M), (U, \psi_U) \in \mathcal{A}(E)\}$$

Since the whole section as a global object is quite difficult to handle is customary in field theory to work in the more practical local representation.

Observation 8: Further specification of the system's kinematics

The general formalism doesn't require any other structure to be carried forward. Additional structure on the fiber , the base or the whole bundle are to be prescribed in order to specify a precise physical model, e.g. the spin structure on E for the Dirac Field.[4]

Dynamics The operator P is the object that contains all the information about the dynamic evolution of the system. It has the role to select the dynamically compatible configuration among all the admissible kinematic configurations of \mathcal{C} , exactly as it

happens in analytical mechanics where the dynamic equations shape the natural motions.

Notation fixing

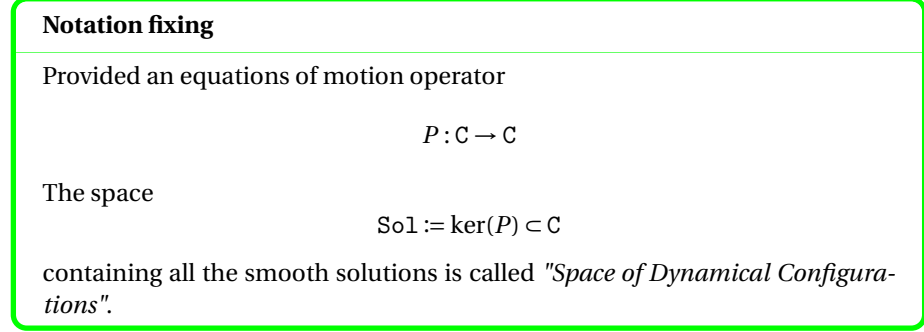
Provided an equations of motion operator

$$P : \mathcal{C} \rightarrow \mathcal{C}$$

The space

$$\text{Sol} := \ker(P) \subset \mathcal{C}$$

containing all the smooth solutions is called "*Space of Dynamical Configurations*".

Figure 2.1: Geometric picture of the basic mechanical system's structure.


2.1.1 Lagrangian Dynamics

Lagrangian systems constitute a subclass of the abstract mechanical systems of more practical interest:

Definition 14: Lagrangian System

Pair (E, \mathcal{L}) composed of:

- $E \xrightarrow{\pi} M$
smooth fiber bundle of typical fiber Q on the oriented manifold (M, σ) called "*configuration bundle*".
- $\mathcal{L} : J^r E \rightarrow \wedge^m T^* M$
bundle-morphism from the r -th Jet Bundle to the top-dimensional forms bundle over the base manifold M called "*Lagrangian density*" or simply "*Lagrangian*" of r -th order.

N.B. : In what follows all the systems considered will be exclusively of first order.

In this case is the Lagrangian density the object containing all the information about the dynamic evolution of the system.

In order to reconstruct the system's dynamic from the Lagrangian density has to be understood the mathematical nature of \mathcal{L} . \mathcal{L} maps point q_p on the fiber $J_p^r E$ to a m -form on $T_p M$. Recalling the definition of jet bundles is clear that for each smooth section on E is associated a smooth section on the $J^r E$:

$$\phi \in \Gamma^\infty(E) \mapsto (\phi, \partial_\mu \phi, \partial_{\mu, \nu} \phi, \dots, \partial_{\bar{\alpha}} \phi)$$

where $\tilde{\alpha}$ is a multi-index of length r . The correspondence is not univocal since sections equal up to the r -th order define the same jet section. The smoothness of \mathcal{L} ensure that each jet bundle section is mapped to a smooth section in the top-forms bundle i.e. the most general integrable object on a orientable manifold.

It should be clear that \mathcal{L} is a specific choice among the vast class of functions suitable to be a good Lagrangian density over the Configuration Bundle E :

Definition 15: Lagrangian Density on the bundle E

$$\text{Lag}^r(E) := \text{hom}\left(J^r E, \bigwedge^m(T^*M)\right) \cong \{f : \Gamma^\infty(J^r E) \rightarrow \Omega^m(M)\}$$

(where $\Omega^m(M)$ is the common name for $\Gamma^\infty(\bigwedge^m(T^*M))$ in the context of Grassmann algebras.) The equivalence states the fact that a bundle-morphism induce a mapping between the sections.

this choice fix the "Dynamical identity" of the considered system.

Proposition 2.1.1 *$\text{Lag}^r(E)$ has an obvious vector space structure inherited by the linear structure of $\Omega(M)$.*

Thanks to the correspondence between a section $\phi \in \mathcal{C}$ and his r -th jet, it's possible to consider the Lagrangian as directly acting on the kinematic configurations. In layman terms the image $\mathcal{L}[\phi]d\mu$, where $d\mu$ is the measure associated to the orientation σ , is something that can be measured over the whole base space. This property suggests the introduction of the class of associated functionals:

Definition 16: Lagrangian functional

Is a functional on \mathcal{C} with values on regular distribution over M associated to the generic $\mathcal{L} \in \text{Lag}$.

$$\mathcal{O}_{\mathcal{L}} : \mathcal{C} \rightarrow (C_0^\infty(M))'$$

Such that the lagrangian functional associated to \mathcal{L} , valued on the configuration $\phi \in \mathcal{C}$ and tested on the test-function $f \in C_0^\infty(M)$ it's given by:

$$\mathcal{O}_{\mathcal{L}}[\phi](f) = \int_M \mathcal{L}[\phi] f d\mu$$

Proposition 2.1.2 *As a distribution $\mathcal{O}_{\mathcal{L}}[\phi](f)$ is necessarily linear in the test-functions entry but not in the configurations entry.*

Observation 9

The choice of the image of $\mathcal{O}_{\mathcal{L}}$ as a distribution it's a necessary precaution to

ensure that functional is "*convergent*" whatever is the configuration on which is evaluated. In fact, despite $\mathcal{L}[\phi]$ is integrable with respect to the measure $d\mu$, it's not necessary summable if the support of the configuration ϕ becomes arbitrarily large.

This is a simple consequence of the well known sequence of inclusions:

$$\mathcal{L}[\phi] \in C_0^\infty(M) \subset L_{\text{loc}}^1(M, \mu) \supsetneq L^1(M, \mu)$$

of the functional analysis . Indeed, the functional

$$\mathcal{O}_{\mathcal{L}}[\phi] = \int_{\text{supp}(\phi)} \mathcal{L}[\phi] d\mu$$

is well defined for all $\mathcal{L} \in \text{Lag}^r(E)$ only over the compactly supported sections. To take account of the global sections it's sufficient to dampen the integral multiplying the integrand with an arbitrary test-function.

Notation fixing

When calculated for the specific density of the Lagrangian system $\mathcal{O}_{\mathcal{L}}$ takes the name of *Action* or *Total Lagrangian*.

The introduction of the Lagrangian density is meaningless without the prescription of a dynamical principle which allows to determine univocally a differential operator P on the kinematics configurations space \mathbb{C} . This fundamental principle is the *least action principle*. A proper justification of this claim should require the presentation of the differential calculus on the infinite dimensional manifolds \mathbb{C} . Jumping straight to the conclusion we can state this correspondence as a principle in term of a function which assign for all lagrangian densities an operator on the kinematic configurations space. In the case of first order lagrangian we define

Definition 17: Euler-Lagrange operator

It's the differential operator

$$Q_{\chi} : \mathbb{C} \rightarrow \mathbb{C}$$

relative to the lagrangian density $\chi \in \text{Lag}^1(E)$, such that:

$$Q_{\chi}(\gamma) = \left(\partial_{\mu} \left(\frac{\partial \chi}{\partial (\partial_{\mu} \phi)} \Big|_{\gamma} \right) - \frac{\partial \chi}{\partial \phi} \Big|_{\gamma} \right) \quad \forall \gamma \in \mathbb{C} \quad (2.1)$$

(where $\left(\frac{\partial \chi}{\partial (\partial_{\mu} \phi)} \right)$ is the be intended as the lagrangian density constructed differentiating $\chi(\phi, \partial_{\mu})$ as an ordinary function treating its functional entries as an usual scalar variable.)

Observation 10

The whole theory of both Lagrangian densities class and Euler-Lagrange equation could be stated in a more syntetic way in terms of the Grassmann-graded variational bicomplex.[7][11]

2.2 Concrete Realization

In the previous section we claim that the abstract definition of Lagrangian systems is broad enough to encompass all the classical lagrangian systems with both discrete degrees of freedom, like particles, and continuous degree of freedom, like fluids or fields. Let' show two of the most significant examples.

2.2.1 Classical Field Theory

Basically a *Fields System* is nothing more than an abstract Field System (E, P) where the base space M is a suitable Spacetime manifold[3]. At this stage the question about the Lagrangian nature of the dynamics is purely ancillary.

The idea of taking bundles on a space-time manifold is physically intuitive, kinematically speaking a fields configuration is simply the association of some element of the fiber Q for each point of the space-time M .

However, there are two more requirements that are often prescribed in commonly studied field theories.

Linear Sistem Condition

- The configuration bundle $E \xrightarrow{\pi} M$ is a vector bundle.

Even if it might make sense to speak of nonlinear fields in some more general context, this condition it's a necessary element in case some form of the *superposition principles* as to be taken in account. Obviously this hypothesis is not sufficient to formulate the principle in the strong classical way, i.e.: "the response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually" mostly because only free systems can be considered at this stage and any statement about stimulus can make sense.

However It assure that C is a vector space and , in conjunction with the linearity of motion operator P , $\text{Sol} = \ker(P)$ is a linear subspace. In other words every linear combination of kinematic configuration it's still a kinematic configuration.

Propagative Dynamic Condition

- The base manifold M is a Globally Hyperbolic Spacetime.
- The motion operator P is PDE-hyperbolic.

The first condition ensures the existence of Cauchy surfaces and therefore permits to state *Cauchy Problems* assigning an initial data on such regions. Furthermore, PDE-hyperbolicity of the motion operator P guarantees that for every Cauchy surface $\Sigma \subset M$ the corresponding initial data problem is well posed, that is:

$$\begin{cases} Pu = 0 \\ u = u_0 \\ \nabla_{\vec{n}} u = u_1 \end{cases} \quad (2.2)$$

admit a unique solution $u \in \Gamma(E)$ for all $(u_0, u_1) \in \Gamma(\Sigma) \times \Gamma(\Sigma)$.

This suggests the following definition:

Notation fixing

The set of all the smooth initial data which can be given on the Cauchy Surface Σ is: $\text{Data}(\Sigma) := \{(f_0, f_1) \mid f_i \in \Gamma^\infty(\Sigma)\} \equiv \Gamma^\infty(\Sigma) \times \Gamma^\infty(\Sigma)$

Observation 11

$\text{Data}(\Sigma)$ inherit the linear structure of its component $\Gamma^\infty(\Sigma)$.

In this term the well-posedness of the Cauchy problem can be stated as follow:

Proposition 2.2.1 *The maps $\mathbf{s} : \text{Data}(\Sigma) \rightarrow \text{Sol}$ which assign to $(u_0, u_1) \in \text{Data}(\Sigma)$ the unique solution of the cauchy problem 2.2 is linear and bijective.*

Since any solution, when restricted to a generic Cauchy surface Σ' , determines another pair of initial data, i.e.:

$$\phi \equiv \mathbf{s}(\phi|_{\Sigma'}, \nabla_{\vec{n}'} \phi|_{\Sigma'}) \quad \forall \phi \in \text{Sol}$$

we can define the set of initial data regardless of the particular Cauchy surface:

Definition 18: Set of smooth initial Data

$$\text{Data} := \frac{\bigsqcup_{\Sigma \in \mathcal{P}_C(M)} \text{Data}(\Sigma)}{\sim}$$

where \sim is such that:

$$(f_0, f_1)|_{\Sigma} \sim (g_0, g_1)|_{\Sigma'} \Leftrightarrow \mathbf{s}(f_0, f_1) = \mathbf{s}(g_0, g_1)$$

Initial data, associated with different surface, are similar if they lead to the same solution.

Proposition 2.2.2 *Data is still a vector space.*

Proof:

It's sufficient to prove that:

$$[\phi_a + \phi_b] = [\phi_a] + [\phi_b]$$

where $[\phi] = \{(\phi|_\Sigma, \nabla_{\vec{n}} \phi|_\Sigma) \mid \Sigma \in \mathcal{P}_C\}$. In fact:

$$\begin{aligned} \mathbf{s}_{\Sigma'}([(\phi_a', b')]) + [(\phi_b', d')] &= \mathbf{s}_\Sigma([(\phi_a, b)] + [(\phi_b, d)]) = \mathbf{s}_\Sigma([(\phi_a, b)]) + \mathbf{s}_\Sigma([(\phi_b, d)]) = \\ &= \mathbf{s}_{\Sigma'}([(\phi_a', b')]) + \mathbf{s}_{\Sigma'}([(\phi_b', d')]) = \mathbf{s}_{\Sigma'}([(\phi_a', b') + [(\phi_b', d')]) \end{aligned}$$

□

Corollary 2.2.1 *The function $\text{Data} : \text{Data}(\Sigma) \rightarrow \text{Sol}$ which map every equivalence class to the associated solution is linear and bijective.*

Observation 12

The Propagative dynamic condition is the main ingredient in the *initial data quantization* procedure[13].

Green-Hyperbolic Dynamic Condition

- The dynamic is generated by a Lagrangian, i.e. $P = Q_{\mathcal{L}}$.
- The motion operator P is a Green Hyperbolic L.P.D.O.

It's customary in Algebraic quantum field theory to identify quantizable systems through this condition. In fact this condition is a necessary hypothesis to carry on the Peierls construction and the brackets underlie to the definition of the classical symplectic structure.

We remark that Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense, therefore the last two condition are not equivalent.

Example: 1

It's straightforward to frame the well-known *Real Scalar Field on curved background* inside this abstract picture:

Configuration bundle	real scalar field: $F \equiv \mathbb{R}$ curved background: M is globally hyperbolic
Kinematic Configurations	$\mathcal{C} = C^\infty(M, \mathbb{R})$
Motion Operator	$P = \square_M + m^2 + \xi R$ normally hyperbolic operator

Where P is the Klein-Gordon Operator: ξ and m^2 are two real numbers, R stands for the scalar curvature build out of g and $\square_M := g^a b \nabla_a \nabla_b$ is the d'Alambert operator associated to the Levi-Civita connection.

Note that this system satisfies all the above three condition.

2.2.2 Finite Degree systems as a Field

With a little more effort is possible to see every ordinary mechanic system -with discrete degrees of freedom- as a special case of classical field.

Consider a lagrangian system (Q, L) with configuration space Q and Lagrangian function $L: TQ \rightarrow \mathbb{R}$.

Remembering the intuitive meaning of Q as set of all the statical conformation of the system, it's natural to read as kinematic configuration all and only the trajectories compatible with the constraints. In other words the space of kinematic configurations is the set of all the parametrized smooth curves on Q :

$$\mathcal{C} = C^\infty(\mathbb{R}, Q)$$

Therefore, the configuration bundle of this system is the trivial smooth bundle:

$$E = (\mathbb{R} \times Q, \pi, \mathbb{R})$$

of typical fiber Q , since the corresponding space of sections coincides to the required configuration space:

$$\Gamma^\infty(E) \equiv C^\infty(\mathbb{R}, Q)$$

Also the "field theoretic" picture of the dynamic is not difficult to achieve. It's enough to recall how the Euler-Lagrange of ordinary mechanical system can be derived from the *least action principle*:

Least Action Principle: The natural motions of the mechanical system (Q, L) are the stationary points of the action functional:

$$S[\gamma] = \int_{\mathbb{R}} L(t, \gamma^i, \dot{\gamma}^i) dt$$

constructed from the Lagrangian of the system.

Remembering that we have interpreted the space \mathbb{R} of the parameter t as the base manifold of the configuration bundle E , it's immediate to see S as the *Total Lagrangian* related to the lagrangian density:

$$\mathcal{L}[\gamma] := L \circ (\gamma)^\uparrow dt$$

obtained evaluating the ordinary Lagrangian on the lifted curve γ^\uparrow (lift of $\gamma \in \mathcal{C}$ from Q to TQ).

Summing up, the mathematical structure of such mechanical system is encoded as follows:

Configuration bundle	Trivial bundle $E = Q \times \mathbb{R}$ Base manifold: $M = \mathbb{R}$
Kinematic Configurations	$\mathcal{C} = C^\infty(\mathbb{R}, Q)$
Lagrangian Density	$\mathcal{L}[\gamma] := (L \circ \gamma^!) dt = L(t, \gamma^i, \dot{\gamma}^i) dt$
Motion Operator	$P = Q_{\mathcal{L}} \equiv \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}^i} L - \frac{\partial}{\partial x^i} L \right)$ normally hyperbolic operator (\triangle ?)

Considering that the space M is trivially globally hyperbolic, since every point $t \in \mathbb{R}$ is a genuine "Cauchy Surface", it's evident how this system can be seen as a special case of field theory: a "field of curves".

Observation 13

Unless the configuration space is a vector space, the corresponding field theory cannot be linear.

2.3 Geometric mechanics of Finite Degree systems

La visione precedente Ą molto generale ma ci sono alcune strutture classiche che voglio replicare sul campo come la forma symplettica, le osservabili e le parentesi di poisson. Mi sembra piĄ chiaro vederle dopo aver raccontato queste.

Mi atterrei all'approccio rapido che segue Wald (tralasciando il ponte con la meccanica analitica dei corsi standard e concentrandomi sull'approccio geometrico)

Quindi devo parlare un po' di meccanica geometrica, di



- Spazio delle Fasi
- tautological 1-form
- symplectic form
- canonical coordinate and darboux theorem
- observable as smooth scalar field on the phase space
- poisson structure

2.3.1 Linear dynamical systems

Most of the physical systems that are encountered in the theory of fields are linear. Of course is possible to come across linear systems even in ordinary mechanics. In that case the the difference between the underlying geometric entities tend to fade out as a consequence of the flatness of the configuration space.



Da riempire: devo dire che

- $Q \cong \mathbb{A}^1$ piatto
- $TQ \cong \mathbb{A}^1$ fibrato vettoriale
- Q si identifica con il tangente quindi la forma simplettica ω definita direttamente sulle configurazioni
- ecc vedere primo capitolo wald

2.4 Peierls Brackets

In this section we present more extensively the original Peierls' construction. Please note that we are not trying to provide the state of the art on the Peierls bracket (see for example [9] for the treatment in presence of gauge freedom) but only to expand and modernize the first approach given by Peierls. Instead of considering only scalar theory we extend the algorithm to a broader class of systems.

Observation 14: Peierls Bracket vs Poisson Bracket

Observation 14: Peierls Bracket vs Poisson Bracket

Paraphrasing an observation made by Sharan[12]:

The Poisson bracket determines how one quantity $b(t, q, p)$ changes another quantity $a(t, q, p)$ when it acts as the Hamiltonian or vice-versa. The Peierls bracket, on the other hand, determines how one quantity $b(t, q, p)$ when added to the system Hamiltonian h with an infinitesimal coefficient $\hat{\lambda}$ affects changes in another quantity $a(t, q, p)$ and vice-versa, i.e. The Peierls bracket is related to the change in an observable when the trajectory on which it is evaluated gets shifted due to an infinitesimal change in the Lagrangian of the system by another Lagrangian density.

While the Poisson bracket between two observables a and b is defined on the whole phase space and is not dependent on the existence of a Hamiltonian, the Peierls bracket refers to a specific trajectory determined by a governing Lagrangian.

Purpose of the Peierls' procedure is to provide a bilinear form on the space of Lagrangian densities with time-compact support. This form induces a pre-symplectic structure on suitable subspaces of functionals to which can be recognized the role of *classical observables* of the theory.

⚠ Aggiungere altre chiacchiere e marketing riguardo le PB, vedere nelle fonti cosa dicono i sapienti

2.4.1 Peierls' construction.

The Peierls's construction algorithm is well defined for a specific class of systems:

1. Linear field theory: $E = (E, \pi, M)$ is a vector bundle.
2. Linear Lagrangian dynamics: $P = Q_{\mathcal{L}}$ is a L.P.D.O.
3. M is a globally Hyperbolic space-time.
4. Motion operator P is a green-hyperbolic.

The procedure can be summarized in a few steps:

1. Consider a *disturbance* χ that is a time-compact Lagrangian density .
2. Construct the *perturbation of a solution under the disturbance*.
3. Define the *effect of the disturbance* on a second Lagrangian functional.
4. Assemble the mutual effects of two different Lagrangian densities to give a *bracket*.

Let's review each step more carefully.

Disturbance and Disturbed motion operator

By "*disturbance*" we mean a time-compact supported lagrangian density $\chi \in \text{Lag}^1$ which act as a perturbation on the system's lagrangian:

$$\mathcal{L} \rightsquigarrow \mathcal{L}' = \mathcal{L} + \epsilon \cdot \chi$$

where ϵ is a modulation parameter. The support condition is required in order to take in account only perturbations which affect the dynamic for a definite time interval. The motion operator of the disturbed dynamics results:

$$P_\epsilon = \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} \right] + \epsilon \left[\partial_\mu \left(\frac{\partial \chi}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \chi}{\partial \phi} \right] = P + \epsilon Q_\chi \quad (2.3)$$

Observation 15

P_ϵ is not necessary linear, the second Hypothesis guarantees the linearity only for P .

Solution of the disturbed motion

The second ingredient of the Peierls' procedure is the calculus of the *perturbed solutions* under the considered *disturbance*. These are the solutions $\phi' \in \mathbb{C}$ of P_ϵ obtainable by a infinitesimal linear perturbation of a fixed solution $\phi \in \text{Sol}$. The good definition of linear superposition is guaranteed by the hypothesis 1). More precisely, has to be seek a configuration:

$$\phi'(x) = \phi(x) + \epsilon \eta(x) \in \mathbb{C}$$

such that:

$$\begin{aligned} P_\epsilon \phi'(x) &= o(\epsilon) \\ P \phi(x) &= 0 \end{aligned}$$

¹I.e. the top form $\chi(\phi)$ is time-compact supported for all $\phi \in \mathbb{C}$.

In other word has to be satisfied the following equation:

$$[P_\epsilon]\phi'(x) = [P + \epsilon Q_\chi](\phi(x) + \epsilon\eta(x)) = \epsilon([P]\eta(x) + [Q_\chi](\phi(x) + \epsilon\eta(x))) \stackrel{!}{=} o(\epsilon)$$

The condition of linearity for operator P doesn't hold for Q_χ in general. We can work around this problem taking into account the linearization[9, pag. 31] of operator Q_χ around the unperturbed solution $\phi(x)$. The linearization of Q_χ is the unique linear operator $[Q_\chi^{lin}(\phi)]$ such that:

$$[Q_\chi](\phi(x) + \epsilon\eta(x)) = [Q_\chi](\phi(x)) + \epsilon[Q_\chi^{lin}(\phi)](\eta(x)) + o(\epsilon)$$

which can be seen as the first term of a *formal* Taylor expansion of operator Q_χ around ϕ . $\underline{\Delta}^2$ This is reflected in a condition on the perturbation $\eta \in \mathbb{C}_{tc}$:

$$\begin{aligned} [P_\epsilon]\phi'(x) &= \epsilon([P]\eta(x) + [Q_\chi\phi(x)]) + \epsilon^2[Q_\chi^{lin}(\phi)]\eta(x) \stackrel{!}{=} o(\epsilon) \\ \Rightarrow P\eta &= -Q_\chi\phi(x) \end{aligned} \quad (2.4)$$

called *Jacobi Equation*. This equation is a non homogeneous P.D.E. with inhomogeneous term $(-Q_\chi\phi(x))$ fixed by the solution $\phi \in \text{Sol}$ to be perturbed.

Follows from the definition of green hyperbolicity that the domain restrictions of P to Γ_{pc}^∞ or Γ_{fc}^∞ admit a unique inverse G^+ and G^- respectively. Therefore, equation ?? admits a unique past compact solution η^+ , called retarded perturbation of $\phi \in \text{Sol}$, and a unique future compact solution η^- , called advanced perturbation:

$$\eta^\pm = G^\pm(-Q_\chi\phi) \quad (2.5)$$

Note that the time-compact support condition on χ guarantees that $Q_\chi\phi \in \text{dom}(G^+) \cap \text{dom}(G^-)$. Expression 2.5 reflects perfectly the original Peierls' notation where η^\pm were noted as functions of the unperturbed solution: $\eta^+ \equiv D_\chi\phi$ and $\eta^- \equiv \bar{D}_\chi\phi$.

Observation 16

In most practical case it's possible to give a more basic characterization of η^\pm in term of a Cauchy problem. Has to be stressed that this approach is not possible in general since Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense i.e. the well-posedness of the Cauchy problem is not guaranteed on any Cauchy surface. [3, pag 1] [2, remark 3.18][9, remark 2.1]

Consider a motion operator P which is also hyperbolic. Taking in account the time-compact support condition of χ , is possible to pick up two Cauchy surfaces Σ_\pm (+ is after the perturbation while - stands for prior to the perturbation) such that:

$$J^\mp(\Sigma_\pm) \supset \text{supp}(\chi)$$

for all time-slice foliation of the globally hyperbolic space-time.

²If \mathbb{C} is a Frechet manifold the expansion could be made rigorous defining $[Q_\chi^{lin}(\phi_0)] = [\frac{\partial Q_\chi}{\partial \phi}(\phi_0)]$ in term of the Gateux derivative.

For each of this two surfaces can be posed a Cauchy problem:

$$\begin{cases} P\eta = -Q_\chi\phi \\ (\eta, \nabla_n \eta)|_{\Sigma_\pm} = (0, 0) \end{cases} \quad (2.6)$$

which, according to the well-posedness of the Cauchy problem, admits a unique solution. The link with the first presentation is that past/future -compact supported configuration always meet the initial data condition for some future/past Cauchy surface.

In conclusion, fixed a solution $\phi \in \text{Sol}$ and a perturbation χ , are uniquely determined two perturbed solution:

$$\phi_\epsilon^\pm = \phi + \epsilon \eta^\pm \quad (2.7)$$

such that:

<i>retarded perturbation</i>	$\eta^+ \in \Gamma_{pc}^\infty$	$(\eta^+, \nabla_n \eta^+) _{\Sigma_-} = (0, 0)$	"propagating forward"
<i>advanced perturbation</i>	$\eta^- \in \Gamma_{fc}^\infty$	$(\eta^-, \nabla_n \eta^-) _{\Sigma_+} = (0, 0)$	"propagating backward"

Effect Operator

Considering an arbitrary continuous $\hat{\Delta}^3$ functional $B : \text{Sol} \rightarrow \mathbb{R}$ (not necessarily linear) we can define the effect of a perturbation on the values of B [10, pag. 5] as a map:

$$\begin{aligned} \mathbf{E}_\chi^\pm : C^1(\text{Sol}, \mathbb{R}) &\rightarrow C^1(\text{Sol}, \mathbb{R}) \\ \mathbf{E}_\chi^\pm B(\phi_0) &:= \lim_{\epsilon \rightarrow 0} \left(\frac{B(\phi_\epsilon^\pm) - B(\phi_0)}{\epsilon} \right) \end{aligned} \quad (2.8)$$

The advanced and retarded effects of χ on B are then defined by comparing the original system with a new system defined by the same kinematic configuration space C but with perturbed lagrangian.

Observation 17

Expression 2.8 is clearly a special case of Gateaux derivative. [?]

The former expression appear quite simpler in case of a linear functional:

$$\mathbf{E}_\chi^\pm B(\phi_0) = B(\eta^\pm) \quad (2.9)$$

The Bracket

Remembering that every lagrangian density define a continuous functional (Action). From that is possible to build a binary function:

$$\{\cdot, \cdot\} : \text{Lag}_{tc} \times \text{Lag}_{tc} \rightarrow \mathbb{R}$$

³The precise notion of continuity require the specification of the infinite dimensional manifold structure.

as follow:

$$\{\chi, \omega\}(\phi_0) := E_{\chi}^{+} F_{\omega}(\phi_0) - E_{\chi}^{-} F_{\omega}(\phi_0) \quad (2.10)$$

Proposition 2.4.1 (Bilinearity) *When restricted to Linear Lagrangian densities $\{\cdot, \cdot\}$ is a bilinear form*

Proof:

Linearity in the first entry follows from equation [?] and the linearity of the Euler-Lagrange operator Q over Lag.

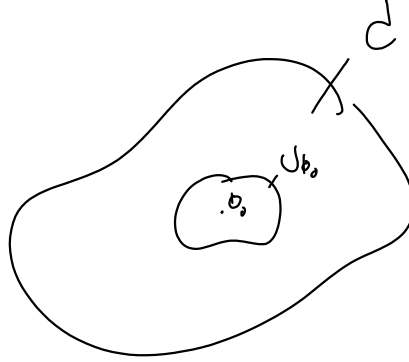
Linearity in the second entry is guaranteed only for lagrangian densities ω which provide a linear Lagrangian Functional F_{ω} .

□

We don't care to probe the symplectic property in this general ground. In the next chapter we will face the problem to determine symmetry and non-degeneracy properties for the case of *classical observable functional*, a subclass of Lagrangian functionals of most practical use in the quantization schemes.

2.4.2 Extension to non-linear theories

In the previous construction the green-hyperbolicity of motion operator P plays a primary role. Anyway the problem of searching perturbed solution of the disturbed dynamic can be stated even in presence of non-linear fields where the configuration bundle is not necessary a vector bundle and the motion operator is not linear.



Nuova sezione 1 Pagina 1

Figure 2.2: Intrinsically, searching a variation of a solution $\gamma_0 \in \text{Sol}$ which solve the disturbed motion equation is equivalent to find the intersection of the perturbed solution with a local neighbourhood of $\Gamma_0 : U_{\gamma_0} \cap \ker(P_{\epsilon})$.

The crucial point of the Peierls' procedure is to select among all the possible solution of the perturbed motion P_ϵ that configuration which can be constructed by a variation of some fixed solution of the non-perturbed dynamics $\gamma_0 \in \text{Sol}$. In this sense the problem results a "*linearization*" inasmuch the search of such solution is restricted to a local neighbourhood of the "point" $\gamma_0 \in \text{Sol}$.

Previously the choice to consider only the linear variation was quite natural but in the general case this preferential restriction is no longer possible. A way to recover a notation similar to 2.7 is to work patchwise choosing a coordinate representation. Fixed a solution $\gamma_0 \in \text{Sol}$ and a local trivializing chart (A, ϕ_A) such that $A \cap \text{ran}(\gamma_0) \neq \emptyset$ we can define a local infinitesimal variation by acting on his components:

$$\gamma_\lambda^i(x) = \gamma_0^i(x) + \lambda \eta^i(x) \quad \forall x \in \pi(A)$$

where γ_0^i are the component of the unperturbed solution in the open set A and $\eta^i \in \mathcal{Q}$ is a generic real q-ple (q is the dimension of the typical fiber manifold). λ is a real parameter that has to be "sufficiently small" in order to guarantee that the range of γ_λ is properly contained in A . In other words the construction of the linear variation, that for linear field theories could be done in a global way, in the general case can be recovered only locally varying the components.

Therefore is possible to define the effect of the a disturbance locally, searching local section $\gamma_\epsilon^i = \gamma_0^i + \epsilon \eta^i$ solving the disturbed dynamic equation up to the first order in ϵ i.e.

$$[P_\epsilon] \gamma_\epsilon^i = o(\epsilon)$$

where $[P_\epsilon]$ has to be intended as the coordinate representation of the restriction on the local section $\Gamma^\infty(A)$.⁴

Observation 18

W.l.o.g has been taken the the same scalar ϵ to modulate both the perturbation γ_ϵ that the disturbance on the motion operator. On the contrary consider two different parameter is immaterial since only the smaller should be taken in account.

From the explicit equation of the perturbed solution:

$$([P] + \epsilon [Q_\chi]) (\gamma_0^i + \epsilon \eta^i) = o(\epsilon)$$

follows an equation on the components of the local perturbation. In this case has to be dealt with the problem of non-linearity not only for Euler-Lagrange operator Q_χ but also for P . Arresting the expansion to the first order in ϵ results:

$$[P_{\gamma_0}^{lin}] \eta^i(x) = - (Q_\chi(\gamma_0))(x) \quad (2.11)$$

the *Jacobi equation* on the unperturbed solution $\gamma_0 \in \text{Sol}$.

⁴non mi   evidente se la rappresentazione in coordinate di un operatore agente sulle sezioni si realizza in modo ovvio, ma non vedo nemmeno ostruzioni! Di sicuro l'operazione   ben definita per gli L.P.D.O visto che la definizione prevede proprio che su ogni carta locale trivizzante l'operatore sia lineare alle derivate parziali

Observation 19

We've moved from an operator P defined on \mathbb{C} to an operator $P_{\gamma_0}^{lin}$ defined on the space of variation. From a global point of view this variation can be seen as the tangent vector i.e. $s \eta \in T_{\gamma_0}\mathbb{C}$. In the case of the linear system this passage was unnecessary, the Jacobi equation was directly defined on \mathbb{C} since, for linear system, any section could be seen as a generator of an infinitesimal variation. This behaviour mimics perfectly what happens in ordinary classical mechanics where the configuration space of a linear system is a vector space i.e a "flat" manifold^a which is isomorphic to his tangent space in every point.

^aIn sense that admits a global coordinate chart.

Provided that the linearized motion operator (which is now properly a linear partial differential operator) is Green-Hyperbolic, the Peierls construction can continue as before. Has to be noted that now the advanced/retarded perturbation are formally identical to the former:

$$\eta^{\pm i} = G^{\pm}(-Q_{\chi}\gamma_0^j)$$

with the important difference that G^{\pm} are now the Green operators of the linearized motion operator and depend on the fixed solution γ_0^j .

In conclusion the perturbed solution:

$$\gamma_{\chi}^{\pm i} = \gamma_0^i \pm G^{\pm}(-Q_{\chi}\gamma_0^j)$$

has to be intended as the "glueing" of all the local chart representations covering the chosen solution.

Example: Finite Dimensional Case

As an example of such process we can consider a *field of curves* i.e. an ordinary classical mechanical system in the field theoretic picture. We have shown in section 2.2.2 that such systems are generally non linear: the configuration bundle is not a vector one and then linearity of P cannot be defined.

However the base manifold is very simple. Indeed $M = \mathbb{R}$ can be seen as a trivial globally-hyperbolic space-time where every real scalar $t \in M$ is a Cauchy surface.



Da Finire

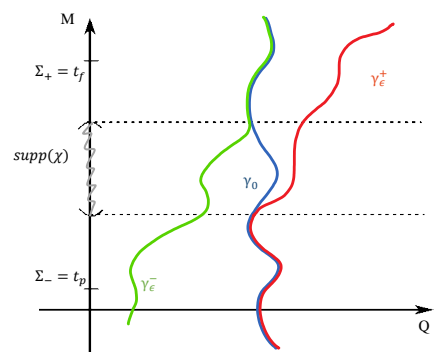


Figure 2.3:

Chapter 3

Algebraic Quantization

The point we want to get, that we will face in the next chapter, is the algebraic quantization of geodesic system. For this purpose it is necessary to devote a chapter to the description of algebraic quantization scheme. We will show two realizations of the scheme applicable to a class of systems sufficiently broad to encompass the system that we want to examine.

3.1 Overview on the Algebraic Quantization Scheme.

Contemporary quantum field theory is mainly developed as quantization of classical fields. Classical field theory thus is a necessary step towards quantum field theory.¹ The "*Quantization process*" has to be considered as an algorithm, in the sense of self-containing succession of instruction, that has to be performed in order to establish a correspondence between a classical field theory and its quantum counterpart.²

On this basis the axiomatic theory of quantum fields takes the role of "validity check". It provide a set of conditions that must be met in order to establish whether the result can be consider a proper quantum field theory. Basically there are no physical/philosophical principles which justifies "a priori" the relation between mathematical objects (e.g the classical state versus quantum states) individually. The scheme can only be ratified "a posteriori" as whole verifying the agreement with the experimental observations.

However this is by no means different from what is discussed in ordinary quantum mechanics where there are essentially two plane: the basic formalism of quantum mechanics, which is substantially axiomatic and permits to define an abstract quantum mechanical system, and the quantization process that determine how to construct the quantum analogous of a classical system realizing the basic axioms.

¹ Cito testualmente Mangiarotti, shardanashivly

² forse l'nlab esprime la cosa meglio di me <http://ncatlab.org/nlab/show/quantization>. Sono d'accordo con il loro approccio ma non voglio usare la loro formulazione perch  in fondo ci sono arrivato anche da solo :P

We refer to the algebraic quantization as a *scheme of quantization* because it's not a single specific procedure but rather a class of algorithms. These algorithms are the same concerning the quantization step per se (costruction of the $*$ -algebra of classical observable) but they differ in the choice of the classical objects (essentially the classical observables and the bilinear form) to be subjected to the procedure.

Basically an algebraic quantization is achieved in three steps:

1. Classical Step

Identify all the mathematical structures necessary to define the field, i.e. the pair (E, P) .

In general every quantization process exploit some conditions on the quantum field structure that has to be met.

2. Pre-Quantum Step

\triangle Are implemented some additional mathematical over-structure on the classic framework. The aim is to establish the specific objects which will be submitted to the quantization process in the next step. Generally these object don't have any a classical meaning, their only purpose is to represent the classical analogous of the crucial structures of the quantum framework. From that we say *Pre-Quantum*, their introduction doesn't have a proper *a priori* explanation but has to be treated as an anstanz and justified *a posteriori* within the quantum treatment.

Essentially has to be chosen a suitable space of *Classical observable* and this space has to be rigged with a well-behaved bilinear form.

The ordinary quantum mechanics equivalent step is the choice of a particular Poisson bracket on $C^\infty(T^*Q)$, which typically implement the *canonical commutation relations* $\{q, p\} = i\hbar$, among all the possible Poisson structure. Note that this is a "pre-quantum" step because in classical Hamiltonian mechanics is considered only the Poisson structure carried from the natural symplectic form [1].

3. Quantization

Finally are introduced the rules which realize the correspondence between the chosen classical objects and their quantum analogues. \triangle ³ The algebraic approach characterizes the quantization of any field theory as a two-step procedure. In the first, one assigns to a physical system a suitable \hat{A} -algebra A of observables, the central structure of the algebraic theory which encodes all structural relations between observables. The second step consists of selecting a so-called *Hadamard state* which allows us to recover the interpretation of the elements of A as linear operators on a suitable Hilbert space.

\triangle ⁴ As said previously, the realization of the Algebraic scheme are many: Fedosov's procedure, by Deformation, Peierls' procedure, by Initial Data etc . In the next section we review the last two.

³ Sto Cito direttamente [5].

⁴ Frase che non mi piace ma voglio far presente che le realizzazioni dello schema algebrico sono molteplici!

3.2 Quantization with Peierls Bracket.

⚠ Temp ⚠ da contestualizzare (e spostare)

Observation 20

In the algebraic quantization scheme the choice of the bundle bilinear form take a pivotal role since it is the basis of the so-called *pairing*. In effect this is the only discretionary parameter of the whole procedure. The prescription on the symmetry properties determine the Bosonic/Fermionic character of the quantized theory:

Pairing	Observables linear form	Quantum Theory
symmetric	anti-symmetric	Bosonic
anti-symmetric	symmetric	Fermionic

Observation 21

What we are going to show is a quantization procedure strictly defined for a specific class of classical theories:

1. Linear Fields.
2. Lagrangian Dynamics.
3. On Globally-Hyperbolic Space-time.
4. with Green-hyperbolic motion Operator.

Fall into this category prominent examples like Klein-Gordon and Proca Field Theory.[4] Has to be noted that the Lagrangian condition is ancillary. This has the purpose to justify the shape of the symplectic form on the classical observables space as consequent from the Peierls bracket. It's customary to overlook to the origin of this object and jump directly to the expression ?? in term of the Green's operator that no longer present any direct link to the Lagrangian and therefore can be extended to any green-hyperbolic theory.

Briefly the procedure can be resumed in few steps:

1. Classical Step
Has to be stated the mathematical structure of the classical theory under examination.
 - (a) Kinematics: is encoded in the configuration bundle of the theory.
 - i. Specify the base manifold M .
Has to be a Globally-Hyperbolic Space-time.
 - ii. Specify the Fiber and the total Space E auxiliary structure, e.g: spin-structure or trasformation laws under diffeomorphism on the base space.
 E has to be at least a vector bundle.

- (b) Dynamics: has to be specified the local coordinate expression of the motion operator $P : \Gamma^\infty(E) = \mathbb{C} \rightarrow \mathbb{C}$.
 - i. Is P Green-hyperbolic?
 - ii. Is P derived from a lagrangian: $P = Q_{\mathcal{L}}$?

2. Pre-Quantum Step

- (a) Pairing: construct a basic bilinear form on the space of kinematic configurations.
 - i. Choose $\langle \cdot, \cdot \rangle$ a bilinear form on the bundle E .
Generally this object is suggested by the m
 - ii.
- (b) Classical Observables
 - i.
 - ii.
- (c) Symplectic structure
 - i.
 - ii.

3. Quantization Step

- (a) Quantum Observables Algebra
A concrete realization is achieved in three step.
 - i.
 - ii.
- (b) Hadamard State
 - i.
 - ii.



Da ricopiare!

3.2.1 Classical Step

Applicability of the procedure.



Da ricopiare!

3.2.2 PreQuantum Step.




Da ricopiare!

3.2.3 Second Quantization Step.

 Da ricopiare!


3.3 Quantization by Initial Data.

 Da ricopiare!


3.3.1 PreQuantum Step.

 Da ricopiare!

3.4 Link between the two realizations

 Intro da Ricopiare

3.4.1 Equivalence of the Classical Observables

 Intro da Ricopiare

3.4.2 Equivalence of the Brackets

 Non completata! vedi email del 9 luglio.

Chapter 4

Geodesic Fields

In the context of differential geometry, *geodesic curves* are a generalization of *straight lines* in the sense of self-parallel curves. Considering a differential manifold M endowed with an affine connection ∇ we define:

Definition 19: Geodesic

A curve $\overset{a}{\Delta} \gamma : [a, b] \rightarrow M$ such that:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \quad (4.1)$$

where $\dot{\gamma}^\mu := \frac{d\gamma^\mu}{dt}$ is the tangent vector to the curve.

^aDevo dire smooth o piecewise?

Notation fixing

In local chart the previous equation assume the popular expression:

$$\ddot{\gamma}^i + \Gamma_{jk}^i \dot{\gamma}^j \dot{\gamma}^k = 0 \quad (4.2)$$

Where Γ_{jk}^i is the coordinate representation of the Christoffel symbols of the connection.

In presence of a pseudo-Riemannian metric is possible to present the geodesic in a metric sense i.e. as the curve which extremizes the *Energy Functional*¹:

Definition 20: Energy functional

¹Remember that for arc-length parametrized curves the Energy functional coincide with the length functional.[8, Lemma 1.4.2]

$$E(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|^2 dt \quad (4.3)$$

Considering only the proper variation (that keep the end-point fixed), the extremum condition corresponds to equation 4.2 where ∇ is the unique Levi-Civita connection (torsion-free and metric-compatible).

In general relativity the problem of the geodesic equation linearization, named *Jacobi equations* takes a central role.²

(nel file di ripasso di geometria riemanniana ho scritto gran parte delle definizioni conviene vedere cosa mi serve effettivamente... Di certo mi avvalgo della seguente equazione)

Notation fixing

In local charts the Jacobi fields along the geodesic γ solve a linear O.D.E.:

$$(X'')^\mu + R^\mu_{i\alpha j} T^i X^\alpha T^j = 0 \quad (4.4)$$

where:

- $(X')^\mu := (\nabla_{\dot{\gamma}(t)} X)^\mu$ is the covariant derivative along the curve γ .
- $T \equiv \dot{\gamma}(t)$ stands for the tangent vector to the curve γ .

The rest of this chapter will be dedicated to presenting the physical approach to the Geodesic.

4.1 Geodesic Problem as a Mechanical Systems


The basic idea is very simple, portray the geodesic curve as the natural motion of a free particle constrained on the Pseudo-Riemannian manifold Q .

obvious enough this problem can be seen as a generalization of the calculation of the motions of free falling particles. In terms of general relativity this problem can be instantly recognized as the derivation of the free-falling particles motion.

However, there is no lack of alternative viewpoints. The framework of the classical Geometric Mechanics teach us to picture the "static" configurations of a constrained, complex, classical system as a point on the *Configuration space* manifold. According to that, the geodesic motion can be

²Usually in this context takes the name of *Geodesic deviation* problem[?, pag. 46].

seen as a realization of a particular dynamics on each mechanical system endowed with a pseudo-Riemannian configuration space^a.


 ^aSuch systems can be depicted as "geodesic" even in presence of a position-dependant potential.[1, Cap 3.7]

Theorem 4.1.1 (Geodesic Motion) *The geodesics on the Pseudo-Riemmanian manifold (Q, g) are the natural motions of the ordinary Lagrangian system (Q, L) where:*

$$L(V_q) := \frac{1}{2} g_q(V, V) \quad (4.5)$$

Proof:

The Euler-Lagrange equation of L coincides with the geodesic equation 4.2.


.. Ãl sul quaderno non so se metterla

□

Observation 22

The geodesic system is not simply Lagrangian but also Hamiltonian. This property follows from the hyperregularity[1] of L .

Observation 22

 Anyway we will neglect this fact inasmuch in what follows only the Lagrangian character assumes a role.

As shown in chapter 2, every system with discrete degrees of freedom can be seen as the trivial field system. From that follows the alternative characterization of geodesic as a lagrangian field:

Corollary 4.1.1 (Geodesic field) *The geodesics on the Pseudo-Riemmanian manifold (Q, g) can be seen as the Dynamical Configurations of the lagrangian field system (E, \mathcal{L}) where:*

- $E = (Q \times \mathbb{R}, \pi, \mathbb{R})$ trivial smooth bundle on the real line.
- $\mathcal{L}[\gamma] = \frac{1}{2} g(\dot{\gamma}, \dot{\gamma})(t) dt$

Proof:

Is simple application of the correspondence seen in chapter 2.2.2.

□

From this perspective is clear that the Energy Functional can be seen as the action in the geodesic field dynamics and equation 4.2 is nothing more than the motion equation according to the *least action principle*.

Figure 4.1: Impressionistic view of the geometric mechanics structure.


4.2 Peierls Bracket of the Geodesic field

The local coordinate expression of the lagrangian density of the geodesic field results:

$$\mathcal{L}(t, \gamma^i(t), \dot{\gamma}^i(t)) := \frac{1}{2} g_{\mu\nu}(\gamma^i(t)) \dot{\gamma}^\mu \dot{\gamma}^\nu \quad (4.6)$$

this is highly non-linear. Explicitly is quadratic in the velocity components $\dot{\gamma}^i$ and implicitly, through $g_{\mu\nu}(\gamma^i(t))$, is non-polynomial in coordinate γ^i .

As show in section ??, for this type of systems, the calculation of Peierls bracket can be realized only locally around a predetermined solution. Let's repeat the Peierls' procedure for the system under investigation.

 **Introduzione da rivedere, dimostro che l'operatore linearizzato senza termine inhomogeneo corrisponde all'equazione di Jacobi vera e propria mentre con termine inhomogeneo dato dalle E-P del disturbo da l'equazione che definisce la perturbazione ritardata e anticipata.**

As a consequence of our introduction on the geodesic as a field, we can state the unperturbed dynamic as a L.P.D.O :

$$Q_{\mathcal{L}}(q^\mu) = \left[\ddot{q}^\mu + \Gamma_{ij}^\mu \dot{q}^i \dot{q}^j \right] \quad (4.7)$$

where $\dot{q}^\mu = \frac{d}{dt} q^\mu(t) = \dot{q}^i \partial_i q^\mu$.

A linear variation of $q_0^\mu + \epsilon \eta^\mu$ constructed from the coordinate representation q_0^μ of the geodesic $\gamma_0 \in \text{Sol}$, solves the original motion equations when

$$Q_{\mathcal{L}}(q_0^\mu + \epsilon \eta^\mu) = \frac{d^2}{dt^2} (q_0^\mu + \epsilon \eta^\mu) + \left[\Gamma_{ij}^\mu (\vec{q}_0 + \epsilon \vec{\eta}) \right] (\dot{q}_0^i + \epsilon \dot{\eta}^i) (\dot{q}_0^j + \epsilon \dot{\eta}^j) = 0 \stackrel{!}{=} o(\epsilon) \quad (4.8)$$

If we consider only the first order in the parameter ϵ we can expand the expression of the Christoffel symbols:

$$\left[\Gamma_{ij}^\mu (\vec{q}_0 + \epsilon \vec{\eta}) \right] = \left[\Gamma_{ij}^\mu (\vec{q}_0) + \epsilon \eta^\alpha (\partial_\alpha \Gamma_{ij}^\mu) \right]_{\vec{q}_0} + o(\epsilon)$$

Collecting all the terms in equation 4.8 up the first order in ϵ follows a condition on the perturbation:

$$0 = \ddot{\eta}^\mu + \eta^\alpha (\partial_\alpha \Gamma_{ij}^\mu)_{\vec{q}_0} \dot{q}_0^i \dot{q}_0^j + \Gamma_{ij}^\mu (\dot{\eta}^i \dot{q}_0^j + \dot{q}_0^i \dot{\eta}^j) = \left\{ g_{\alpha\mu}^{\mu} \frac{d^2}{dt^2} + \Gamma_{i\alpha}^\mu (\vec{q}_0) \left[2 \dot{q}_0^i \frac{d}{dt} \right] + [\partial_\alpha] \right\} \eta^\alpha$$

Da dire: espressione in coordinate della lagrangiana, \tilde{L} altmente non lineare perch \tilde{L} implicitamente $\tilde{L} g_{\mu\nu}(\gamma^i(t))$ non polinomiale in γ^i ed esplicitamente \tilde{L} quadratica, Mostrare esplicitamente che l'equazione di jacobi per il sistema \tilde{L} effettivamente l'equazione di jacobi (questo \tilde{L} triviale se vedi come definisce il campo di jacobi jürgen

4.2.1 Example: Geodesic field on FRW space-time.

4.3 Algebraic quantization of the Geodesic Field

va ripetuto che la geodetica \tilde{L} non lineare quindi ci \tilde{L} che effettivamente si quantizza \tilde{L} jacobi lungo una prefissata geodetica. questo \tilde{L} un campo lineare.

INtro: paragrafo sul quaderno: "qual' \tilde{L} l'interesse che spinge a quantizzare questo sistema "Campo di Jacobi"?

Disclaimer: Non approfondisco pi \tilde{L} di tanto gli step di quantizzazione vera e propria. il ruolo di peierls \tilde{L} nel prequantistico, definisce il bracket che poi va implementato sull'algebra. Una volta decisa la parentesi la macchinetta procede in automatico.

4.3.1 Peierls Approach

Paragrafo sul Quaderno: ...

4.3.2 Initial data Approach

Ancora su fogli di Brutta!

4.4 Interpretations??????

Speriamo bene.. :S

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