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Laurea Magistrale in Fisica

ALGEBRAIC QUANTIZATION OF JACOBI FIELDS *and* GEOMETRIC APPROACH TO PEIERLS BRACKETS

Relatore Interno: Prof. Livio Pizzocchero
Relatore Esterno: Dott. Claudio Dappiaggi

Candidato:
Antonio Michele Miti
Matricola n° 841205

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Abstract

The aim of this thesis is to extend and modernize the construction, originally proposed by Peierls in 1952, of the pre-symplectic structure for a classical fields theory. The identification of a symplectic space for fields is a crucial ingredient in the scheme of algebraic quantization as it directly reflects on the definition of the canonical commutation rules on the $*-$algebra of quantum observables. Considering a suitable class of abstract dynamical systems, it is possible to reformulate more rigorously the Peierls' algorithm in every step extending it also to non-linear systems. In the course of our work we have highlighted how this procedure intervenes in the realization of algebraic quantization, showing furthermore how this can be related to the *initial data quantization*. At last we compute the Peierls brackets constructing the pre-quantum symplectic space for the case of Jacobi fields, the linearization of the geodesic motion on a pseudo-Riemannian manifold, regarding it as a particular mechanical system.

Sommario

Lo scopo di questa tesi e di estendere e adattare al linguaggio moderno la costruzione, orginariamente proposta da Peierls nel 1952, della struttrura presimplettica associata ad una teoria di campo classica. L'identificazione di uno spazio simplettico per i campi è un ingrediente cruciale nello schema di quantizzazione algebrica in quanto si riflette direttamente nella definizione delle regole di commutazione canonica sulla *-algebra degli osservabili quantistici. Considerando una opportuna classe di sistemi dinamici astratti, è possibile riformulare in modo più rigoroso l'algoritmo di Peierls in ogni suo passo estendendolo inoltre a sistemi non lineari. Nel corso del nostro lavoro abbiamo sottilineato come questa preocedura interviene nella realizzazione della quantizzazione algebrica, mostrando inoltre come questa possa essere collegata alla *quantizzazione per dati iniziali*. In fine abbiamo calcolato le parentesi di Peierls e realizzato lo spazio simplettico pre-quantistico per il caso dei campi di Jacobi, linearizzazione del moto geodetico su varietà pseudo-Riemanniane, riguardandandolo come un particolare sistema meccanico.

List of Symbols

$E = (E, \pi, M; Q)$	Fiber Bundle $\pi : E \to M$ with typical fiber Q.
$\Gamma^{\infty}(E)$	Smooth sections on the bundle E.
$\mathbf{J}^{\pm}_{M}(\mathrm{p})$	Causal future and past of event p.
$\mathbf{I}^{\pm}_{M}(\mathrm{p})$	Chronological future and past of event p.
$\mathscr{P}_{C}(M)$	Set of all the Cauchy surfaces contained in spacetime M.
$\mathtt{C}$	Kinematic configurations set.
$\mathtt{Sol}$	Dynamic configurations set.
Lag	Set of Lagrangian densities.
$\mathscr{L}$	Lagrangian density of the considered system.
Data	Inital data set.
$\mathbf{s}$	Function that maps a fixed initial data to the correspondent unique solution.
$\mathscr{M}$	Phase space.
A	Quantum observables algebra.
$\mathbf{D}^{\pm}_{\chi}$	Peierls' *effect operator* of a Lagrangian density χ on a target functional.

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Introduction

Quantum Field Theory (QFT) is the synthesis of Quantum Mechanics and Special Relativity and it is the general framework for the description of the physics of relativistic quantum systems. Its most direct applications, quantum electrodynamics and the standard model of particles, are both been experimentally verified to an outstanding degree of precision and they allowed us to have an almost fully satisfactory and unified description of the electro-weak forces.

In any case it is by no mean a definitive theory. It is clear that the intrinsic quantum description of elementary particles clashes with the the structure of a deterministic theory, as general relativity. It is almost unanimously accepted that a quantum theory of gravity is needed in order to reconcile general relativity with the principles of quantum mechanics. Yet, despite countless efforts, a quantum theory of the gravitational interaction remains an open problem

While quantum field theory has been a well established topic for the past 50 years, the quest of finding a *theory of everything* has often lead the community to neglect the role of two important aspects concerning the QFT.
The first is the existence of an intermediate regime, going under the name of *QFT in curved background*, which is expected to provide an accurate description of quantum phenomena in a regions where the effects of curved spacetime may be significant, but effects of quantum gravity itself may be neglected. Many successful application of this idea can be found in context of the theory of cosmological inflation or black holes thermodynamic.
The second is the construction of a mathematically rigorous description of quantum fields, in particular of their non-perturbative aspects, based on a sound and shared set of first principles. In other words an *axiomatic foundation of QFT*.

At the moment, *algebraic quantum field theory* (AQFT) is a way to complete the picture of the quantum theory of fields regarding the above two aspects. Its aim is to reach a general and mathematical rigorous description of the foundations of quantum fields on a sufficiently large class of curved, but fixed, backgrounds.

The algebraic approach, as the majority of contemporary quantum field theory, is developed as a quantization of classical fields. As a starting point, a mathematically rigorous *classical field theory* is thus a necessary step towards the understanding of the foundations of the theory. It has to be noted that classical fields such the force

fields of analytical mechanics or the stress tensor in fluid dynamics are not of much interest insofar QFT is concerned. What is essential to determine is a proper definition of the (pre-quantum) classical analogue of the "fundamental" fields. Particularly, it is crucial the identification of the field-theoretic equivalent of the geometric structures underlying the canonical formalism of classical mechanics, namely the *phase space* and an *algebra of classical observables*. From an abstract point of view the first is a symplectic manifold, namely a smooth manifold endowed with a non-degenerate 2-form, while the second is a Poisson algebra constituted by functionals on the phase space.

It is important to emphasize that the algebraic quantization is not a unique and well-defined algorithm that reads in a system of classical mechanics and returns a corresponding quantum mechanical system. Rather it should be seen as a *quantization scheme* which can be realized by several specific procedures. All of them are based on a set set of first principles, essentially proposed by Dimock [16] as an extension of the axioms of Haag and Kastler formulated on Minkowski spacetime, by prescribing the mathematical structure of the quantum observables algebra. What in which such realizations of the algebraic scheme essentially differ, is in the different identification of the suitable symplectic manifold to be associated with the prequantum version of the field under examination.

The most common way to building these structures requires an explicit choice of *Cauchy surface* in the underlying spacetime. This leads to a realization of the algebraic quantization known as *quantization via initial data*.

Of much greater interest is the, fully covariant, construction based on the so-called *covariant phase space* and on the *Peierls brackets*. The first is defined as the *space of dynamical configurations*, *i.e.* the infinite-dimensional space of solutions of the equations of motion, while the second is a particular choice of Poisson brackets.

The construction of such Poisson brackets is achieved following what we call the *Peierls' algorithm*, a procedure originally proposed by Rudolf E. Peierls in a seminal paper dated 1952 [36]. This is an effective, but rather convoluted, "recipe" to prescribe a pre-symplectic structure on the space of dynamical configurations. Browsing through the literature, it is clear that the Peierls' construction never received particular attention since its formulation. This can be ascribed mainly to the lack of a convincing geometric interpretation which had the effect of limiting its reception often relegating its role to that of a mere "mathematical trick".

The aim of this thesis is to review the original Peierls' procedure in every single aspect adapting it to a more rigorous and modern mathematical formalism. To take a step closer to the comprehension of this object we study the well-known geodesic problem regarding it as a special case of a field-like system. Essentially this example is noteworthy from two aspects:

1. It is a system with discrete degrees of freedom. Its "field configurations" are parametrized curves on a Riemannian manifold and in this sense it represents an example complementary to the basic real scalar field.

2. This system is dynamically ruled by the well-known *geodesic equation*. A typical

realization of the algebraic scheme requires to go through the linearization of these, highly non-linear, equations of motion which take the name of *Jacobi equations*. The solutions of these linearized equations, named *Jacobi fields*, are extensively studied from the point of view of differential geometry (where they are introduced as a tangent field over a geodesic variation) but they are rarely analysed as a field-like dynamical system.

Since the Jacobi fields can be quantized both according to the *Peierls procedure* and according the *initial data procedure*, we hope that the comparison between the two symplectic spaces thus obtained, allows us to give a new geometric insight on the original Peierls' method.

We briefly summarize the content of the thesis.

The first chapter is devoted to reviewing the mathematical framework underlying the rigorous formulation of continuous classical systems, starting point of every realizations of the algebraic quantization scheme. We begin by introducing the notion of *smooth bundles*, these are the suitable objects to encode the kinematical structure of a generic field system. Subsequently we define the notion of globally hyperbolic spacetime as the natural arena for the mathematical theory of hyperbolic (systems of) partial differential equations, for which the Cauchy problem is well posed. Eventually we outline the theory of Green hyperbolic operators, the class of linear differential operators on a vector bundle to which the construction of Peierls brackets, as well as the corresponding quantization procedure, applies.

Chapter 2 is dedicated to introducing the procedure for constructing the Peierls brackets. In the first part we make use of the mathematical language developed above in order to formalize the correct abstract mechanical system for which the Peierls procedure is well defined. Furthermore we will show how the most familiar mechanical systems, namely the point particle and the free fields over a spacetime, can be treated in a unified way as special cases of the aforementioned abstract system.
In the end we propose an extended version of the original Peierls' algorithm obtained combining the construction proposed in his paper[36] with some recent references, mainly [32][15][18][39][27]. Instead of limiting ourselves only to the case of a scalar field theory, we extend the step-by-step procedure proposed by Peierls to a large class of abstact mechanical systems, not necessarily linear.

In order to pursue the study of the system under investigation it is necessary to introduce the algebraic scheme of quantization. To this end, the third chapter is focused on presenting two realizations of the algebraic quantization scheme. The two are distinguished by the different construction of the pre-quantum symplectic space associated to the classical system.
The first one is based on the restriction of the Peierls brackets to the class of *classical observables*, the resulting symplectic form is sometimes prescribed axiomatically [15][17][7].
The second exploits the hyperbolicity of the equations of motion of the system and it is known as *quantization* via the *initial data*[42].

In the last chapter we will apply all the formalism developed so far to the case

of the geodesic field. We construct the Peierls brackets for such system and, as a concrete example, we carry out all calculations for the specific case of a FRW space-time with flat spatial sections. Then we will construct two equivalent pre-quantum symplectic spaces related to the geodesic system following the step-by-step algorithm presented in chapter 3 . At last we propose a geometric interpretation of the whole Peierls' construction.

Chapter 1

Mathematical Preliminaries

The interaction between mathematics and Quantum Field Theory are complex and highly not trivial. Since contemporary quantum field theory is mainly developed as a quantization of classical fields, the mathematical foundation of classical field theory represents a necessary step towards the understanding of the foundations of field theory.

Goal of this section is to introduce the three building blocks of *"pre-quantum" field theory* , by this we mean the theory that describes the classical analogue of quantum theory that, in turn, describes the elementary particles. Namely these are the *vector bundles*, *globally hyperbolic spacestimes* and *Green hyperbolic operators*. Given the purpose of this thesis, we shall not dwell on the structures typical of the quantum framework (such *-algebras). We assume that the reader is familiar with the basic notions of differential geometry, external calculus and, to a minor extent, of general relativity.

1.1 Fiber Bundles

Fiber bundles are the stage for the kinematics of classical and quantum fields. Its main role is to encode the kinematic configurations of an arbitrary field theory through the concept of *sections*.

1.1.1 Formal Definition

Although it would be possible to present the concept of *bundle* in a more general way through the language of categories, for the sake of this work it will be sufficient to consider only the case of *smooth bundles*.

Definition 1.1 (Fiber Bundle)**.** We call a *Smooth (Fiber) Bundle* a quadruple (E, M, Q, π) where:

- E, M, Q : smooth manifolds called respectively *Total Space, Base Space, Typical Fiber*.

- $\pi : E \to M$ smooth function (called *Bundle Projection*)

Endowed with a *Local Trivialization*, that is:
$\forall x \in M \quad \exists$ a pair (U, χ) (called *local trivialization*) where:

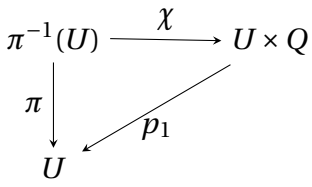
· U : neighbourhood of x

· $\chi : \pi^{-1}(U) \to U \times Q$: diffeomorphism [1] [2]

such that the natural projection $p_1 : U \times F \to U$ satisfies the following equation:

$$p_1 \cdot \chi = \pi|_{\pi^{-1}(p)}$$

i.e.: the following graph commutes:



It is customary to refer to a vector bundle specifying only its total space:

$$E = (E, \pi, M; Q)$$

In the following we adopt this convention whenever this does not lead to misunderstandings.

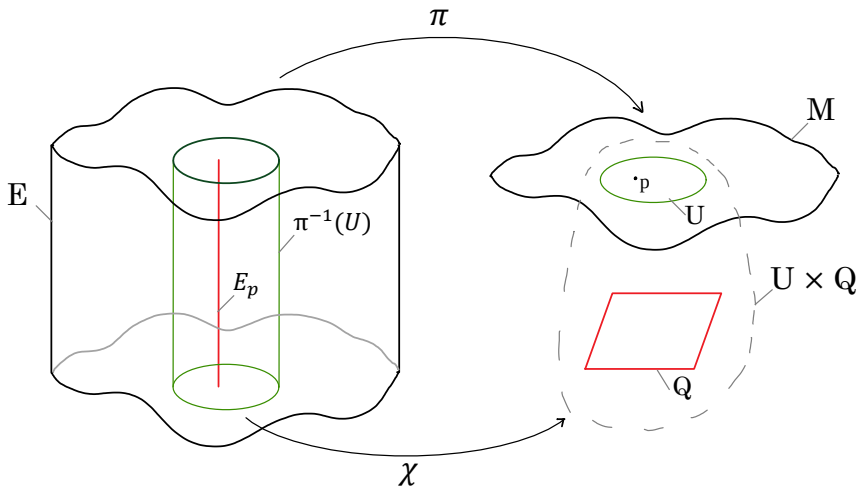


Figure 1.1: The complete fiber bundle Structure.

For all $p \in M$ we refer to the submanifold $E_p := \pi_{-1}(p) \subset E$ as *fiber over the point p*.

[1] surjectivity $\Rightarrow \pi^{-1}(U) \neq \emptyset$.

[2] cartesian product of topological space is a topological space with the direct product topology.

Every fiber E_p is diffeomorphic to the typical fiber F through the local trivialization charts.

Since Definition 1.1 prescribes the existence of local trivializations only, we say that a smooth bundle E is *(globally) trivial* if $E \simeq M \times Q$, *i.e.* there exists a trivialization of E which is defined everywhere.

When a smooth fiber bundle $(E, \pi, M; Q)$ is considered, in addition to the typical functions of the bundle (π, χ_α) also the local charts $(U_{\alpha_k}, \phi_{\alpha_k})_{k=E,M,Q}$, provided by the atlases of E, M and Q, should be provided.

Definition 1.2 (Bundle atlas)**.** We call *Bundle Atlas* a collection of local charts which trivializes E.
I.e. a set of triples $(U_\alpha, \psi_\alpha, \chi_\alpha)$ where:

- U_α is a open set in M such that $\bigcup_\alpha U_\alpha \supseteq M$.
- χ_α is a local trivialization.
- (U_α, ψ_α) is a local chart on M.

Notice that when are given a local chart $(U_\alpha, \psi_\alpha^{(M)})$ on M and a local chart $(U_\alpha^{(Q)}, \psi_\alpha^{(Q)})$ on the fiber manifold it is possible to construct a chart on the total space:

$$\psi_\alpha^{(E)} = \psi_\alpha^{(M)} \times \psi_\alpha^{(Q)} \circ \chi_\alpha$$

Endowing the bundle manifolds with other additional structures, we can introduce important subclasses of smooth bundles:

Definition 1.3 (Vector Bundle)**.** We call *Vector Bundle* a smooth bundle $E = (E, \pi, M; V)$ such that:

- The typical fiber V is a finite dimensional vector space.
- All trivializations χ_α are diffeomorphisms such that:

$$\chi_\alpha\big|_{\pi^{-1}(p)} \in \mathbb{GL}(n, \mathbb{R})$$

1.1.2 Cross Sections

Sections represent the natural mathematical object to encode a $Q-$valued classical field over the space M:

Definition 1.4 (Smooth Section)**.** We call *Smooth Section* any $\phi \in C^\infty(M; E)$ such that

$$\pi \circ \phi = id|_M$$

We refer to:

- *Global section* $\Leftrightarrow \mathrm{dom}(\phi) = M$

- *Local section* $\Leftrightarrow \mathrm{dom}(\phi) \subset M$

We denote the set of all global smooth sections of the bundle E as:

$$\Gamma^{\infty}(E)$$

generally this space is an infinite dimensional manifold. In case of vector bundles it is also a linear Frechet space[29].

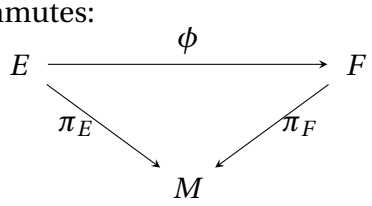
1.1.3 Mapping between Bundles

Consider two smooth bundles $E = (E, \pi, M; Q)$ and $E' = (E', \pi', M; Q')$ on the same base space M.

Definition 1.5 (Bundle map (*Fiber Preserving map*))**.** We call *bundle map* a smooth function $\phi : E \to E'$ such that:

$$\phi(E_x) = F_x \qquad \forall x \in M.$$

i.e. the following graph commutes:



Definition 1.5 is a special case of *bundle-morphism.* (see for example [38])

Consider a smooth manifold N, a (smooth) fiber bundle $E = (E, \pi, M; Q)$, and a smooth function $f : N \to M$. it is possible to induce[25] a bundle structure from M to N:

Definition 1.6 (Pull-Back Bundle)**.** We call *pull-back bundle* of E a triple $f^*(E) = (f^*(E), \pi^*, N)$ such that:

- $f^*(E) = \{(b', e) \in N \times E \quad | \, f(b') = \pi(e)\}$
- $\pi^* : f^*(E) \to N$ such that $\pi^*(b', e) = \mathrm{pr}_1(b', e) = b'$

where $p_1 : U \times F \to U$ is the natural projection on the first entry.

Proposition 1.1.1. *$f^*(E) = (f^*(E) =, \pi^*, N)$ is a smooth bundle with typical fiber Q.*

Proof. To complete the fiber bundle structure it is sufficient to provide a local trivialization atlas.
$\forall (U, \phi)$ local trivialization on (E, π, M) consider $\psi : f^*E \to N \times Q$ such that $\psi(b', e) = \Big(b', pr_2\big(\phi(e)\big)\Big)$.
Then $(f^{-1}(U), \psi)$ is a local trivialization of the pull-back bundle and the fiber of f^*E over a point $b' \in B'$ is just the fiber of E over $f(b')$. □

It is also noteworthy that, given any two vector bundles $E = (E, \pi, M, Q)$ and $E = (E', \pi', M', Q')$, we can construct naturally a third fiber bundle.
Consider $\mathrm{Hom}(E, E')$ the set of all the fiber preserving map between the two bundles:

Definition 1.7 (Bundle of homomorphisms)**.** We call *bundle of homomorphisms* a fiber bundle $\mathrm{Hom}(E, E')$ over the base space M such that the fiber over a base point $p \in M$ is the infinite dimensional manifold $\mathrm{Hom}(E_p, E'_p)$ isomorphic to $\mathrm{Hom}(Q, Q')$.

We shall write $End(E)$ for $\mathrm{Hom}(E, E)$ and we call it bundle of endomorphism, whose typical fiber is $\mathrm{End}(Q)$.

Remark 1. If F, F' are vector bundle then the fiber of $\mathrm{Hom}(F, F')$ over a base point $p \in M$ is $\mathrm{Hom}(F_p, F'_p)$, which is a vector space isomorphic to the vector space $\mathrm{Hom}(V, V')$ of linear applications from V to V'

1.1.4 Tangent Bundles

The *tangent* bundle is a natural structure defined on any smooth manifold, it represents the canonical example of non-trivial vector bundle.

Definition 1.8 (Tangent Bundle)**.** We call *tangent bundle of M* the smooth vector bundle $TM = (TM, \tau, M; \mathbb{R}^m)$ such that:

· The total space is the union of all tangent spaces to

$$TM := \bigsqcup_{p \in M} T_pM \equiv \bigcup_{x \in M} x \times T_xM$$

· The bundle projection maps each tangent vector $v \in T_pM$ to the correspondent base point p;

$$\tau : (p, v_p) \mapsto p$$

In a similar fashion it is possible to construct a structure dual to TM:

Definition 1.9 (Cotangent Bundle)**.** We call *cotangent bundle* the vector bundle T^*M builded by disjoint union of the dual tangent space T^*_pM.

Furthermore, this construction can be repeated for any tensor product of the tangent and cotangent spaces, *i.e.*:

- The *Tensor Bundle* $T^{(k,l)}M$ is build by disjoint unions of the tensor product of tangent space with itself:

$$T^{(k,l)}_p M = \underbrace{T^*_pM \otimes \cdots \otimes T^*_pM}_{\text{k-times}} \otimes \underbrace{T_pM \otimes \cdots \otimes T_pM}_{\text{l-times}}$$

- The *k-form Bundle* $\bigwedge^m(T^*M)$ is build by disjoint unions of the antisimmetrized tensor product of the dual tangent space with itself.

Tautological one-form and symplectic form.

Remark 2. In the context of Classical mechanics is customary to refer to the cotangent bundle T^*Q over the smooth manifold Q - called *Configuration Space* - as *Phase Space*.

Since TQ and T^*Q are diffeomorphic , it might seem that there is no particular reason in treating these two spaces separately, but this is not the case. There are certain geometrical objects that live naturally on T^*Q , not on TQ. Of greatest interest are the Poincaré forms[20].

Consider a smooth manifold Q and call $\mathscr{M} = T^*Q$ the corresponding cotangent bundle.

Definition 1.10 (Tautological (Poincaré) 1-form)**.** We call *tautological form* the 1-form over $\mathscr{M}$:

$$\theta_0 \in \Gamma^\infty(T^*\mathscr{M})$$

such that the action on a generic point $\omega_{\alpha_p} \in T_{\alpha_p}M$ (in the fiber of α_p, which in turn is a one-form on the fiber of $p \in Q$) is given by:

$$\theta_0(\alpha_p) : T_{\alpha_p}\mathscr{M} \to \mathbb{R} \qquad : \omega_{\alpha_p} \mapsto \alpha_q \circ T\tau_Q^*(\omega_{\alpha_p})$$

where T is the *tangent map*, namely the bundle-morphism which act on each fiber as the differential map $d(\tau_Q^*)$.

In the context of classical mechanics we define a special set of coordinates on the cotangent bundle of a manifold called *canonical coordinates*. They are usually written as a set of (q^i, p_j) where q_i are denoting the coordinates on the underlying manifold and the p_j are denoting the conjugate momenta, which are decompositions of 1-forms in T_p^*M on the dual natural basis dq^j computed in the cotangent space at the point q on the manifold.

In canonical coordinates the tautological one-form assumes the famous expression:

$$\theta_0 = \sum_{i=1}^{n} p_i dq^i$$

(note that dq^i is a 1-form on T^*M calculated with respect to the coordinate on the bundle. One should not confuse it with the 1-natural form $dq^i \in T_p^*M$.)

This allows us to define a natural symplectic structure on any phase space:

Definition 1.11 (Canonical (Poincarè) symplectic form)**.** We call *Canonical (Poincarè)* form the symplectic:

$$\omega_0 := -d\theta_0$$

In canonical coordinates it assumes the renown expression:

$$\omega_0 := \sum_{i=1}^{n} dq^i \wedge dp_i$$

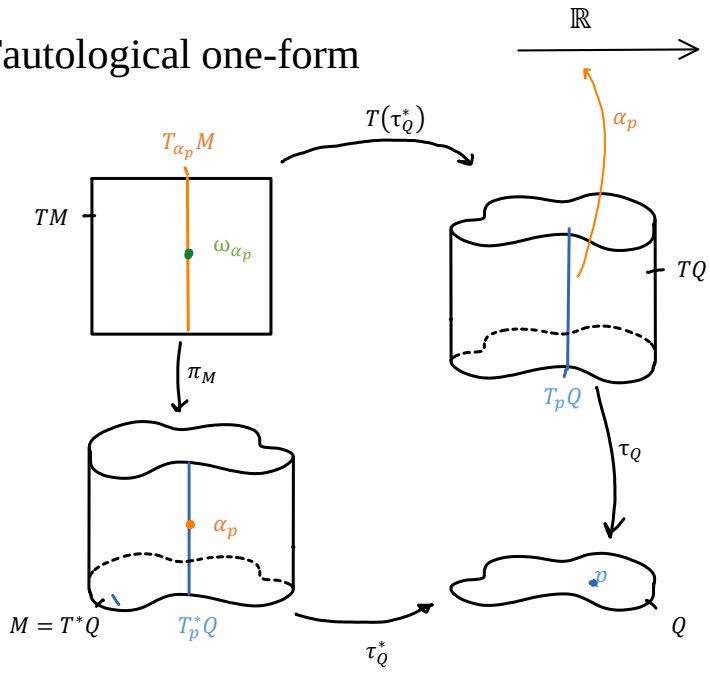


Figure 1.2: The definition of tautological 1-form is achieved exploiting the concept of *Tangent map* and remembering that $\alpha_p : T_p\mathscr{M} \to \mathscr{M}$ is a linear functional.

1.1.5 Jet Bundles

The jet bundle is a construction that makes a new smooth fiber bundle out of a given bundle. The first step is to identify the typical fiber for this construction.

Suppose M is an m-dimensional manifold and that $E = (E, \pi, M, Q)$ is a smooth fiber bundle. Consider the set of all the local sections whose domain contains p:

$$\Gamma^\infty(p) := \{\sigma \in \Gamma^\infty(E) \quad | \, p \in dom(\sigma)\}$$

We define an equivalence relation between such section *up to r-th order*:

Definition 1.12 (r-jet equivalence)**.** Two sections $\sigma, \eta \in \Gamma^\infty(p)$ have the same *r-jet* at p ($\sigma \sim \eta$) iff:

$$\left.\frac{\partial^{|I|}\sigma^\alpha}{\partial x^I}\right|_p = \left.\frac{\partial^{|I|}\eta^\alpha}{\partial x^I}\right|_p \qquad \forall I \in \mathbb{N}_0^m \,|\, 0 \le |I| \le r.$$

where I is a *multi-index* ,see Remark 3.

Remark 3. (multi-index notation)
A multi-index is a finite dimensional vector $I = (i_1, i_2, \ldots, i_m) \in \mathbb{N}_0^m$ with $m < \infty$.
On $\mathbb{R}^n$ a differential operator can be identified by a multi-index:

$$\frac{\partial^{|I|}}{\partial x^I} := \prod_{i=1}^{m} \left(\frac{\partial}{\partial x^i}\right)^{I(i)}$$

(Whenever the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|I| := \sum_{i=1}^{m} I(i)$$

We define the *r-th Jet in p* as the equivalence class under this relation.

Definition 1.13 (Space of r-th Jet in p)**.** We call *space of the r-th jet in p* the set:

$$J^r_p(E) := \frac{\Gamma^\infty(p)}{\sim}$$

where $\sim$ is the r-Jet equivalence.

Definition 1.14. r-jet with representative σ is denoted as $j^r_p\sigma$.
The integer r is also called the order of the jet, p is its source and $\sigma(p)$ is its target.

Glueing together all the jet fibers $J^r_p(E)$ for all the base points $p \in M$, as done for the tangent bundle, we obtain the desired bundle:

Definition 1.15 (r-th Jet Bundle of E)**.** We call *r-th Jet Bundle of E* the smooth bundle $(J^r(E), \pi_r, M)$ where:

- $J^r(E) := \bigsqcup_{p\in M} J^r_p(E) \equiv \left\{ j^r_p\sigma \quad | \, p \in M,\, \sigma \in \Gamma^\infty(p) \right\}$
- $\pi_r : J^r(E) \to M$ such that $j^r_p\sigma \mapsto p$

1.2 Globally Hyperbolic Spacetimes

Rigorously speaking, configurations of a field system are encoded by sections based on a *spacetime* manifold. From a physical point of view, we are interested in those spacetimes which allow to set a well-posed initial value problem for hyperbolic partial differential equations, such as the wave equation. In particular we need to ensure that the spacetime we consider possess at least one distinguished codimension 1 hypersurface on which we can assign the initial data needed to construct a solution of such an equation.
For this purpose we have to introduce *globally hyperbolic spacetimes*.

1.2.1 Reprise in Lorentzian Geometry

Consider a differential manifold M.

Definition 1.16 ((Pseudo-Riemannian) Metric)**.** We call *(Pseudo-Riemannian) Metric* a map on the bundle product of TM with itself:

$$g : TM \times_M TM \to \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a non-degenerate bilinear form.

Definition 1.17 (Pseudo-Riemannian Manifold)**.** We call *Pseudo-Riemannian manifold* a pair (M, g) such that:

- M is a n-dimensional ($n \geq 2$), Hausdorff, second countable, connected, orientable smooth manifold.
- g is a Lorentzian metric.

A Pseudo-Riemannian manifold (M, g) is called:

- *Riemannian* if the sign of g is positive definite.
- *Lorentzian* if the signature is $(+, -, \ldots, -)$ or equivalently $(-, +, \ldots, +)$.

1.2.2 Time Orientation

If a smooth manifold is endowed with a Lorentzian metric (with signature $(-, + + +)$), then the tangent vectors at each point in the manifold can be classified into three different types.

Definition 1.18. $\forall p \in M, \quad \forall X \in T_pM$, we call a vector:

- *time-like* if $g(X, X) < 0$.
- *light-like* if $g(X, X) = 0$.
- *space-like* if $g(X, X) > 0$.

Remark 4. $\forall p \in M$ the timelike tangent vectors in p can be divided into two equivalence classes taking

$$X \sim Y \text{ iff } g(X, Y) > 0 \qquad \forall X, Y \in T_p^{\text{time-like}} M :$$

We can (arbitrarily) call one of these equivalence classes "future-directed" and call the other "past-directed". Physically the designation of the two classes of future- and past-directed timelike vectors corresponds to a choice of an arrow of time at the point. The future- and past-directed designations can be extended to null vectors at a point by continuity.

Definition 1.19 (Time-orientation)**.** We call *time-orientation* a global tangent vector field $\mathfrak{t} \in \Gamma^\infty(TM)$ over the Lorenzian manifold M such that:

- $\text{supp}(\mathfrak{t}) = M$
- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Can be noted that fixing of a time-orientation is equivalent to a consistent smooth choice of a local time-direction.

Definition 1.20 (Spacetime)**.** We call *spacetime* a quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ such that:

- (M, g) is a time-orientable[3] n-dimensional manifold ($n > 2$)
- $\mathfrak{o}$ is a choice of orientation

[3] Manifold for which such *time-orientation* exists.

· $\mathfrak{t}$ is a choice of time-orientation

In a spacetime M it is quite important to identify particular classes of subsets. The main tool are the *parametrized curves*:

Definition 1.21. A piece-wise smooth curve $\gamma : \mathbb{R} \supset I \to M$ is called:

- · *time-like* (resp. light-like, space-like) iff $\dot{\gamma}(p)$ is time-like (resp. light-like, space-like) $\forall\, p \in M$.
- · *causal* iff $\dot{\gamma}(p)$ is nowhere spacelike.
- · *future directed* (resp. past directed) iff it is causal and $\dot{\gamma}(p)$ is future (resp. past) directed $\forall\, p \in M$.

Definition 1.22 (Chronological $\overset{\text{future}}{\text{past}}$ of a point)**.** We call *Chronological* $\overset{future}{past}$ of a point p the two subsets :

$$\mathbf{I}^{\pm}_{M}(p) := \left\{ q \in M \middle| \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ time-like } \overset{\text{future}}{\text{past}} - \text{directed} : \gamma(0) = p,\, \gamma(1) = q \right\}$$

Definition 1.23 (Causal $\overset{\text{future}}{\text{past}}$ of a point)**.** We call *Causal* $\overset{future}{past}$ of a point p the two subsets :

$$\mathbf{J}^{\pm}_{M}(p) := \left\{ q \in M \middle| \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ causal } \overset{\text{future}}{\text{past}} - \text{directed} : \gamma(0) = p,\, \gamma(1) = q \right\}$$

These concepts can be naturally extended to any subset $A \subset M$:

- $\mathbf{I}^{\pm}_{M}(A) = \bigcup_{p \in A} \mathbf{I}^{\pm}_{M}(p)$
- $\mathbf{J}^{\pm}_{M}(A) = \bigcup_{p \in A} \mathbf{J}^{\pm}_{M}(p)$

Definition 1.24 (Achronal Set)**.** We call *achronal set* a subset $\Sigma \subset M$ such that every inextensible timelike curve intersects Σ at most once.

Definition 1.25 ($\overset{\text{future}}{\text{past}}$ Domain of dependence of an Achronal set)**.** We call $\overset{future}{past}$ *domain of dependence* of an achronal set $\Sigma \subset M$,the two subset:

$$\mathbf{D}^{\pm}_{M}(\Sigma) := \left\{ q \in M \middle| \forall \gamma \overset{\text{past}}{\text{future}} \text{inextensible causal curve passing through } q : \gamma(I) \cap \Sigma \neq \emptyset \right\}$$

The union of the two domain of dependence:

$$\mathbf{D}_{M}(\Sigma) := \mathbf{D}^{+}_{M}(\Sigma) \cup \mathbf{D}^{-}_{M}(\Sigma)$$

is called *total domain of dependence.*

1.2.3 Globally Hyperbolicity

Finally we come to the key concept of our treatment:

Definition 1.26 (Cauchy Surface)**.** We call *Cauchy surface* a closed, achronal subset $\Sigma \subset M$ such that:

$$\mathbf{D}_{M}(\Sigma) \equiv M$$

We denote the set of all the Cauchy surfaces as $\mathscr{P}_C(M)$.

Definition 1.27 (Globally-Hyperbolic SpaceTime)**.** We call a spacetime M *globally hyperbolic* if it contains at least one *Cauchy Surface*.

According to Definition 1.27, only the existence of a single Cauchy hypersurface is guaranteed. This is slightly disturbing since there is no reason a priori why an initial value hypersurface for a certain partial differential equation should be distinguished. This quandary has been overcome proving (see [6][section 1.3]) that, if a spacetime (M,g) is globally hyperbolic, there exists a foliation of M by Cauchy surfaces:

Theorem 1.2.1 (Globally hyperbolic space characterization)**.** *Let (M,g) be any time-oriented spacetime. The following two statements are equivalent:*

- *(M,g) is globally hyperbolic.*
- *(M,g) is isometric to $\mathbb{R}\times\Sigma$ endowed with the line element $ds^2=\beta dt^2-h_t$ where $t:\mathbb{R}\times\Sigma\to\mathbb{R}$ is the projection on the first factor, β is a smooth and strictly positive function on $\mathbb{R}\times\Sigma$ and $t\mapsto h_t$, $t\in\mathbb{R}$, yields a one-parameter family of smooth Riemannian metrics.*
  Furthermore, for all $t\in\mathbb{R}$, $\{t\}\times\Sigma$ is an (n-1)-dimensional, spacelike, smooth Cauchy surface in M.

The class of globally hyperbolic spacetimes includes most of the physically interesting examples, *e.g.*: Minkowski spacetime, Friedman-Robertson-Walker spacetime and Kerr spacetimes, the two-parameter family of rotating black holes, solutions to the vacuum Einstein's equations.
In what follows we will make primary use of the most trivial example:

Example 1. Trivially, the real line $\mathbb{R}$ is a globally hyperbolic manifold.
Each point $x\in\mathbb{R}$ represent a proper Cauchy surfaces which realize the trivial foliation $\mathbb{R}\simeq p\times\mathbb{R}$ required by theorem 1.2.1

To conclude this section, we introduce some terms which will be often used in the following in order to specify the support properties of the sections of a vector bundle with base a globally hyperbolic spacetime.

Definition 1.28. Let M be a globally hyperbolic spacetime and $E=(E,\pi,M;V)$ a vector bundle of typical fiber V.
We denote:

- $\Gamma_0(E)$ the space of *compactly supported* smooth sections.
- $\Gamma_{sc}(E)$ the space of *spacelike compact* smooth sections.
  $\big(f\in\Gamma_{sc}(E)$ if there exists a compact subset $K\subset M$ such that $\operatorname{supp} f\subset \mathbf{J}_M(K).\big)$
- $\Gamma_{fc}(E)$ the space of *future- compact* smooth sections.
  $\big(f\in\Gamma_{fc}(E)$ if $\operatorname{supp}(f)\cap\mathbf{J}^+_M(K)$ is compact for all $p\in M.\big)$
- $\Gamma_{pc}(E)$ the space of *past- compact* smooth sections.
  $\big(f\in\Gamma_{pc}(E)$ if $\operatorname{supp}(f)\cap\mathbf{J}^-_M(K)$ is compact for all $p\in M.\big)$
- $\Gamma_{tc}(E):=\Gamma_{pc}(E)\cap\Gamma_{fc}(E)$ the space of *timelike compact* smooth sections.

1.3 Green Hyperbolic Operators

Green Hyperbolic Operators are the suitable object to represent a *wave-like propagation* dynamics.

Consider $E = (E, \pi, M; V), E' = (E', \pi', M; V')$ two linear vector bundles over M (with different typical fiber), we define:

Definition 1.29 (Linear Partial Differential operator (of order at most $s \in \mathbb{N}_0$))**.** We call *linear partial differential operator* a linear map $L : \Gamma(E) \to \Gamma(E')$ such that $\forall p \in M$ there exists:

- $U \ni p$ open set rigged with:
  - (U, φ) local chart on M.
  - (U, χ) local trivialization of F
  - (U, χ') local trivialization of F'
- $\{A_\alpha : U \to \mathrm{Hom}(V, V') \mid \alpha \in \mathbb{N}_0^n, |\alpha| \leq s\}$ a collection of smooth maps labeled by multi-indices where s is a fixed integer said *order of the operator*.

which allows to express L locally:

$$\chi' \circ (L\sigma) \circ \varphi^{-1} = \sum_{|\alpha| \leq s} A_\alpha \partial^\alpha (\chi \circ \sigma \circ \varphi^{-1}) \qquad \forall \sigma \in dom(L) \subset \Gamma(E)$$

(where we have make use of the multi-index notation3)

Remark 5. Notice that linear partial differential operators cannot enlarge the support of a section.

Definition 1.29 accounts for a large class of operators, most of which are not typically used in the framework of field theory.
In order to characterize a relevant subset we introduce two auxiliary concepts:

Consider a Linear differential operator $L : \Gamma(E) \to \Gamma(E')$:

Definition 1.30 (Principal Symbol)**.** We call *principal symbol* the map $\sigma_L : T^*M \to \mathrm{Hom}(E, E')$locally defined as follows:
For a given $p \in M$, consider a coordinate chart (U, x^i) around p and local trivializations of E and of E' (as precribed in Definition 1.29).
For all $\xi = \xi_i dx^i \in T_p^* M$ set:

$$\sigma_L(\xi) = \sum_{|\alpha| = s} \xi^\alpha A_\alpha(p)$$

where $\xi^\alpha = \prod_{\mu=0}^{m-1} \xi^{\alpha_\mu}$

Definition 1.31 (Formal Dual Operator (of L))**.** We call *formal dual operator* of L the unique linear partial differential operator $L^\star : \Gamma(G^*) \to \Gamma(E^*)$ such that:

$$< L^\star g', f > = < g', Lf >$$

$\forall f \in \Gamma(E),\ g' \in \Gamma(G^*)$ which have supports with compact overlap.
($< \cdot, \cdot >$ denotes the 1-form evaluation: $< \alpha, v >= \alpha(v) \quad \forall v \in E_p, \alpha \in E_p^*$.)

N.B. : From now on we will consider only bundles with globally-hyperbolic spacetimes as base spaces.

1.3.1 Green Hyperbolic Operators

Let M be a globally hyperbolic spacetime, consider a vector bundle E over M and a L.p.d.o. $L:\Gamma(E)\to\Gamma(E)$:

Definition 1.32 ($\substack{\text{retarded}\\\text{advanced}}(\pm)$ Green Operators)**.** We call $\substack{retarded\\advanced}(\pm)$ *Green Operator* of L a l.p.d.o. $G^{\pm}:\Gamma(E)\to\Gamma(E)$ such that:

- $dom(G^{+})=\Gamma_{pc}(E)\qquad dom(G^{-})=\Gamma_{fc}(E)$
- $LG^{\pm}f=G^{\pm}Lf=f\qquad \forall f\in dom(G^{\pm})$
- $\text{supp}(G^{\pm}f)\subset \mathbf{J}^{\pm}_{M}(\text{supp}(f))\qquad \forall f\in dom(G^{\pm})$

In others words we can say that $G^{\pm}$ is the left-right inverse of the restriction of L to $dom(G^{\pm})$.

Definition 1.33. We call *Advanced minus Retarded operator* or *Causal Propagator*[8] the operator:

$$E:=G^{-}-G^{+}:\Gamma_{tc}(E)\to\Gamma(E)$$

Green operators are not necessarily unique. For this we introduce the following definition:

Definition 1.34 (Green hyperbolic operator)**.** We call *Green hyperbolic* a linear partial differential operator P such that P and $P^{\star}$ have advanced and retarded Green's operators.

For these operators uniqueness of Green's operators is guaranteed:

Theorem 1.3.1 (Characterization of Green Hyperbolic operators)**.** *Let be:*

- $E=(E,\pi,M)$ *a vector bundle over a globally hyperbolic spacetime* M*.*
- $P:\Gamma(E)\to\Gamma(E)$ *a Green hyperbolic operator,* $G^{\pm}$ *its Green's operators and* $G^{\pm}_{\star}$ *the Green's operators of the dual.*

Then:

- P *possesses a unique retarded* G^{+} *and a unique advanced* G^{-} *Green's operator.*
- $<G^{\pm}_{\star}f',f>=<f',G^{\mp}f>\qquad \forall f\in\Gamma_0(E),\ \forall f'\in\Gamma_0(E^{*})$

Proof. See for example [7][proposition 2] □

In what follows we will make extensive use of the following properties:

Proposition 1.3.2. *Let* M *be a globally hyperbolic spacetime. Consider a vector bundle* E *over* M*, a Green-hyperbolic operator* $P:\Gamma(E)\to\Gamma(E)$ *and a time-compact section* $f\in\Gamma_{tc}(E)$*. Let* $G^{\pm}$ *be the retarded and advanced Green operators for* P *and denote with* E *the corresponding advanced-minus-retarded operator.*
Then:

1. $Ef = 0 \Leftrightarrow \exists h \in \Gamma_{tc}(E) \mid f = Ph.$
2. $Pf = 0 \Leftrightarrow f = 0.$
3. $\forall f \in \Gamma(E) \quad \exists h \in \Gamma(E) \quad | Ph = f.$

Proof.

[Th. 1] Assuming that $h \in \Gamma_{tc} = dom(G^-) \cap dom(G^+)$, the converse implication follows slavishly from the definition of the causal propagator:

$$Ef = (G^- - G^+)Ph = h - h$$

In the other hand, assume that $f \in \Gamma_{tc}(E)$ be such that $Ef = 0$. It implies $G^+ f = G^- f$, while the support properties of the retarded and advanced Green operators entail that

$$\mathrm{supp}(G^+ f) \subset \mathbf{J}^+(\mathrm{supp} f) \cap \mathbf{J}^-(\mathrm{supp} f)$$

In other words $h = G^+ f \in \Gamma_{tc}(E)$. If we apply the operator P, it holds

$$Ph = PG^+ f = f$$

[Th. 2] Suppose now that there exists $f \in \Gamma_{tc}(E)$ such that $Pf = 0$. Since a linear partial differential operator does not enlarge the support (Remark 5), one can apply either the retarded or the advanced Green operators obtaining:

$$h = G^{\pm} Ph = 0$$

The converse implication is trivial.

[Th. 3] Consider a pair of function (partition of unity of M) $\{\chi_{\pm} : M \to [0,1]\}$ such that:

- $\chi_{\pm} = 1$ on a past / future compact region.
- $\chi_+(x) + \chi_-(x) = 1 \qquad \forall x \in M.$

For any $f \in \Gamma(E)$ we define the function:

$$h := G^+\big(\chi_+ f\big) + G^-\big(\chi_- f\big)$$

Then

$$Ph = (\chi_+ + \chi_-) f = f$$

□

Corollary 1.3.3. *Let M be a globally hyperbolic spacetime. Consider a vector bundle E over M, the Green-hyperbolic operator $P : \Gamma(E) \to \Gamma(E)$ and a compactly supported section $f \in \Gamma_0(E)$. Let $G^{\pm}$ be retarded and advanced Green operators for P and denote with E the corresponding advanced-minus-retarded operator.*
Then the following statements hold true:

1. $Ef = 0 \Leftrightarrow \exists h \in \Gamma_0(E) \mid f = Ph.$
2. $Pf = 0 \Leftrightarrow f = 0.$
3. $\forall h \in \Gamma_{sc}(E) \quad \exists h \in \Gamma_{sc}(E) \quad | Ph = f.$

1.3.2 Normally Hyperbolic Operators

Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness[4] of a Cauchy problem. However for the large class of the *Normally-Hyperbolic Operators* hyperbolicity is guaranteed both in PDE and Green sense.

Consider a Lorentzian manifold (M,g) and two vector bundles $E=(E,\pi,M;V)$, $E'=(E',\pi',M;V')$,

Definition 1.35 (Normally Hyperbolic Operators)**.** We call *normally hyperbolic operator* a second order linear partial differential operator $P:\Gamma(E)\to\Gamma(E')$ such that:

$$\sigma_P(\xi)=g(\xi,\xi)\mathbb{1}_{E_p}\qquad\forall p\in M,\xi\in T_p^*M$$

Making explicit the coordinate expression of a normally hyperbolic operator P , one realizes how such operators provide the natural generalization of the usual Wave operator.
Consider a globally hyperbolic operator P for all $p\in M$ a trivializing chart (U,φ,χ) centered in p. There exist a collection $\{A,A^\mu|\mu\in\{0,\dots,m-1\}\}$ of smooth $\mathrm{Hom}(V,V)-$valued maps on U, such that P reads as follows:

$$\chi\circ(P\sigma)\circ\varphi^{-1}=\left(g^{\mu\nu}\mathrm{id}_V\partial_\mu\partial_\nu+A^\mu\partial_\mu+A\right)(\chi\circ\sigma\circ\varphi^{-1})\qquad\forall\sigma\in dom(P)\subset\Gamma(E)\quad(1.1)$$

where both the chart and the vector bundle trivializations are understood. One immediately notices that locally this expression agrees - up to terms of lower order in the derivatives - with the one for the d'Alembert operator acting on sections of E constructed out of a covariant derivative ∇ on E, that is the operator:

$$\Box_\nabla=g^{\mu\nu}\nabla_\mu\nabla_\nu:\Gamma(E)\to\Gamma(E)$$

Definition 1.35 becomes even more important if we assume that the underlying background is globally hyperbolic, since we can associate to each normally hyperbolic operator P an initial value problem and we can talk about Green's operators.

Proposition 1.3.4 (Green operators)**.** *Let P be a normally hyperbolic operator, then:*

- *$P^\star$ is a normally hyperbolic operator.*
- *P is Green hyperbolic.*

Proof. We omit the proof, see for example [6][Corollary 3.4.3]. □

Theorem 1.3.5 (Normally hyperbolic operators properties.)**.** *Let be:*

- *$\Sigma\subset M$ a spacelike Cauchy surface with future-pointing unit normal vector field $\vec{n}$.*

[4] I.e. exists an unique solution.

- *P a normally hyperbolic operator and ∇ a P-compatible*[5] *covariant derivative on E*

Then:

- *The Cauchy problem;*

$$\begin{cases} Pu = J & on\ M \\ u = u_0 & on\ \Sigma \\ \nabla_{\vec{n}} u = u_1 & on\ \Sigma \end{cases}$$

  admits a unique solution $u \in \Gamma(E)$ for any $J \in \Gamma(E)$ and $u_0, u_1 \in \Gamma(\Sigma)$

- *P is Green-hyperbolic.*

Proof. First proposition has been proved in different forms in several books, e.g. [4][Corollary 5]. For the second proposition see[6][Corollary 3.4.3] □

1.3.3 Green Functions

In chapter four we will need the explicit expression of the Green operators in the case of linear *ordinary* differential operators (l.o.d.o.). Let us consider them as a trivial case of l.p.d.o. on $\Gamma(E) = C^\infty(\mathbb{R})$ over the trivial globally hyperbolic manifold $M = \mathbb{R}$.

We call L_x the linear operator associated to the inhomogeneous ordinary differential equation (ODE.)

$$L_x[u] = \left[A_n(x)\frac{d^n}{dx^n} + A_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + A_0(x) \right] u(x) = f(x)$$

of n-th order. For the sake of simplicity we will consider only the case of *regular* operators, such that all the $A_k(x)$ are limited functions and $A_n(x)$ is different from zero on the whole domain.

In this context it is more common to talk about Green *functions* instead of Green *operators*.

Definition 1.36 (Green function)**.** We call *Green function* associated to the l.o.d.o. L_x the solution $G(x|\xi)$ of the fundamental distributional equation:

$$L_x G(x|\xi) = \delta(x - \xi) \tag{1.2}$$

The main reason to introduce this structure is that it provides a criterion to construct a solution of a non-homogeneous ODE. independently of the datum f:

Proposition 1.3.6. *Let $G(x|\xi)$ be a Green function of the l.o.d.o. L_x.*
The function

$$u = \int_{\mathbb{R}} G(x|\xi) f(\xi) d\xi$$

is a solution of non-homogeneus equation $L_x[u] = f$.

[5]There existss a section $A \in \Gamma(\mathrm{End}(E))$ such that $\Box_\nabla + A = P$.

Proof.

$$L_x[u] = \int_{\mathbb{R}} L_x\left[G(x|\xi)\right] f(\xi) d\xi = \int_{\mathbb{R}} \delta(x-\xi) f(\xi) d\xi = f(x)$$

□

Accordingly to this result , we can define the *Green operator* $\hat{G}$ associated to the Green function $G(x|\xi)$ as the *Hilbert-Schmidt* integral operator

$$\hat{G}f = \int_{\mathbb{R}} G(x|\xi) f(\xi) d\xi$$

where the Green function takes the role of the *integral kernel* distribution. Let us recall that this operator is to be understood as a *"formal operator"* until the domain is not specified.

Remark 6. The most general solution of Equation 1.2 can be written as:

$$G(x|\xi) = g(x|\xi) + \sum_{k=1}^{n} C_k(\xi)\varphi_k(x,\xi) \tag{1.3}$$

where $g(x|\xi)$ is a particular solution of 1.2 called *singular part* and the $\varphi_k(x,\xi)$ are n linearly independent solutions of the associated homogeneous equation:

$$L_x\varphi(x,\xi) = 0$$

Here n is the order of L_x as a linear differential operator.

The role of the non-singular part is to take in account the different boundary conditions that can be eventually imposed to the problem.
More precisely, the domain restriction of L_x to the space

$$D = \{f \in C^n\big([a,b]\big) \mid f(a) = \alpha, f(b) = \beta \in \mathbb{R}\}$$

of regular functions satisfying a boundary condition, sets univocally the coefficients $C_k(\xi)$ appearing in Eq. 1.3.

According to Eq. 1.1, all normally hyperbolic ordinary differential operators are of second order:

$$L_x = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$$

with regular coefficients. An useful property of these operators is that they can always be rewritten in "canonical form":

Proposition 1.3.7. *If $u(x)$ is a solution of the non-homogeneous equation:*

$$L_x[u] \equiv a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(x)$$

then $v(x) = \dfrac{u(x)}{\phi(x)}$ *solves:*

$$v''(x) + v(x)V(x) = \Phi(x)$$

where

$$\phi(x) = \exp\Big[-\tfrac{1}{2}\int \tfrac{b(x)}{a(x)}dx\Big] \quad ; \quad V(x) = \tfrac{c(x)}{a(x)} - \Big(\tfrac{b(x)}{2a(x)}\Big)^2 - \Big(\tfrac{b(x)}{2a(x)}\Big)' \quad ; \quad \Phi(x) = \tfrac{f(x)}{a(x)\phi(x)}$$

The problem of the calculation of the generic Green function for an ordinary normally hyperbolic differential operator it is reduced to that of calculating the singular part of the Green function of the differential operator in a canonical form:

$$L_x = \frac{d^2}{dx^2} + V(x) \tag{1.4}$$

We recall that it is not possible to write down a general expression of the solution for both the complete equation $L_x[u(x)] = f(x)$ and the associated homogeneous equation. It is therefore not possible to determine the most general $G(x|\xi)$. On the other hand is it possible to the determine the advanced and retarded Green operator when it is known a pair of linearly independent solutions.

Proposition 1.3.8. *The advanced/retarded Green operators $G^{\pm}$ associated to the operator L_x of Eq. 1.4, are the Hilbert-Schmidt operators*

$$G^{\pm} f(x) = \int_{\mathbb{R}} g^{\pm}(x|\xi) f(\xi) d\xi$$

with kernel functions:

$$g^{\pm}(x|\xi) = \pm \frac{1}{W(x)} \theta\big(\pm(x-\xi)\big)\big[\varphi_1(\xi)\varphi_2(x) - \varphi_1(x)\varphi_1(\xi)\big] \tag{1.5}$$

where $\varphi_1(x), \varphi_2(x)$ are two linearly indipendent solutions of the homogeneous equation $L_x[u] = 0$ and $W(x)$ is the corresponding Wronskian

$$W(\varphi_1, \varphi_2) := \left|\begin{bmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{bmatrix}\right|$$

Proof. The presence of the step-function $\theta(y)$ in 1.5 guarantees the first and third condition in definition (1.32). In this case the past/future-compact condition has to be understood as the requirement of left/right -bounded support and $J^{\pm}(I)$ as the right/left extension of the interval $I \subset \mathbb{R}$.

The crucial point, namely that:

$$\big(g^{\pm}(x|\xi)\big)'' + V(x) g^{\pm}(x|\xi) = \delta(x-\xi)$$

can be checked by direct inspection.

Recall that the Wronskian of two linearly independent solutions $\varphi_i(x)$ is constant:

$$W(\varphi_1, \varphi_2) = \left|\begin{bmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{bmatrix}\right| = \varphi_1(x)\varphi_2'(x) - \varphi_2(x)\varphi_1'(x) = W$$

Taking the distributional derivative we have:

$$\begin{aligned} \frac{d}{dx} g^{\pm}(x|\xi) = \frac{\pm 1}{W}\Big\{ \quad & \pm\delta(x-\xi)\left[\varphi_1(\xi)\varphi_2(x) - \varphi_1(x)\varphi_2(\xi)\right] + \\ & + \theta\big(\pm(x-\xi)\big)\left[\varphi_1(\xi)\varphi_2'(x) - \varphi_1'(x)\varphi_2(\xi)\right]\Big\} \end{aligned} \tag{1.6}$$

where it has been exploited that $\delta(-x) = \delta(x)$. Taking the second derivative concludes the proof

$$\begin{aligned}
\frac{d^2}{dx^2} g^{\pm}(x|\xi) = \quad & \tfrac{\pm 1}{W} \Big\{ \quad \pm\delta'(x-\xi)\left[\varphi_1(\xi)\varphi_2(x) - \varphi_1(x)\varphi_2(\xi)\right] + \\
& \qquad \pm 2\delta(x-\xi)\left[\varphi_1(\xi)\varphi_2'(x) - \varphi_1'(x)\varphi_2(\xi)\right] + \\
& \qquad +\theta\big(\pm(x-\xi)\big)\left[\varphi_1(\xi)\varphi_2''(x) - \varphi_1''(x)\varphi_2(\xi)\right] \Big\} = \\
= \quad & \tfrac{\pm 1}{W} \Big\{ \quad \mp\delta(x-\xi)\left[\varphi_1(\xi)\varphi_2'(x) - \varphi_1'(x)\varphi_2(\xi)\right] + \\
& \qquad \pm 2\delta(x-\xi)\left[W\right] + \\
& \qquad -V(x)\theta\big(\pm(x-\xi)\big)\left[\varphi_1(\xi)\varphi_2(x) - \varphi_1(x)\varphi_2(\xi)\right] \Big\} = \\
= \quad & \tfrac{\pm \cancel{W}}{\cancel{W}} \Big\{ \quad \pm \cancel{W} \Big\} \delta(x-\xi) - V(x) g^{\pm}(x|\xi)
\end{aligned}$$

Notice that in the second equation it has been used the definition of derivative of a distribution:

$$\big(\partial_x \delta(x-y)\big)\left[\chi\right] = -\delta(x-y)\left[\partial_x \chi\right] \qquad \forall \chi \in C_0^{\infty}(\mathbb{R})$$

evaluated on the test function:

$$\chi(x,\xi) = \left[\varphi_1(\xi)\varphi_2(x) - \varphi_1(x)\varphi_2(\xi)\right]$$

□

Chapter 2

Lagrangian Systems and Peierls Brackets

In this chapter we will take advantage of the language developed above in order to formalize precisely each step of the original Peierls' procedure[36] and thus establish the class of applicability of this algorithm.

2.1 Abstract Mechanical Systems

In what follows we will refer to a rather particular class of dynamical systems:

Definition 2.1 (Dynamical System)**.** We call *abstract dynamical system* a pair (E,P) composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on manifold M, called *"configuration bundle"*.
- $P : \Gamma^\infty(E) \to \Gamma^\infty(E)$
  differential operator in the sense that it is represented by a linear combination of partial derivative in every local chart.

This formulation is still very far from the physical interpretation but has the benefit to highlight the minimal mathematical framework.

Kinematics The configuration bundle encompasses all the kinematical structures of the system. A pivotal role is played by the smooth global sections which are to be understood as all the possible conformation of the system.

Definition 2.2 (Space of kinematics (off-shell) configurations)**.** We call:

$$\mathtt{C} := \Gamma^\infty(M,E)$$

the space of kinematic configurations.

A section is not a statical configuration, equivalent to a specific point in the configuration space of ordinary classical systems, but has to be seen as a specific realization of the kinematics in the sense of a complete description of a possible motion.

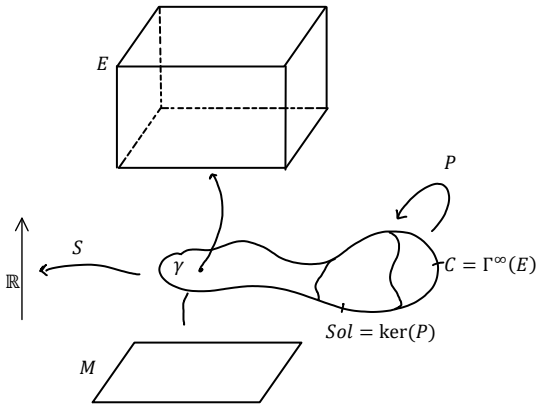


Figure 2.1: Mathematical framework of the mechanics of abstract systems.

The natural physical interpretation should become manifest through the concrete realization of systems with discrete and continuous degree of freedom.

Remark 7. Notice that $\Gamma^\infty(M;E)$ is best understood as an infinite dimensional Manifold.
This framework provides a geometric characterization of the notion of variations as tangent vectors on the the space of kinematic configurations .[18]
In the case where E is a vector bundle and P is a l.p.d.o. $\mathtt{C}$ is a vector space.

The choice of a chart atlas $\mathscr{A}(M)$ on the base space M and $\mathscr{A}(E)$ on the total space E provides a correspondence between each configuration $\gamma \in \mathtt{C}$ and family of smooth real functions $\{f_{\alpha\beta} : A_\alpha \subset \mathbb{R}^m \to \mathbb{R}^q\}$, where m and q are respectively the dimension of Q and M.
The process is trivial:

$$\gamma \in \mathtt{C} \mapsto \{f_{A,U} = \psi_U \circ \gamma \circ \psi_A^{-1} | (A,\psi_A) \in \mathscr{A}(M), (U,\psi_U) \in \mathscr{A}(E)\} \tag{2.1}$$

Since the whole section is quite difficult to handle as a global object, is customary in field theory to work in the more practical local representation.

The general formalism does not require any other structure to be carried forward. Additional structure on the fiber , the base or the whole bundle are to be prescribed in order to specify a precise physical model, e.g. the spin structure on E for the Dirac Field.[7]

Dynamics The operator P is the object that contains all the information about the dynamical evolution of the system. It has the role to select the dinamically compatible configurations among all the admissible kinematic ones, exactly as it happens in analytical mechanics where the dynamic equations shape the natural motions.

Definition 2.3 (Space of Dynamics (on-shell) configurations)**.** We call

$$\mathtt{Sol} := \ker(P) = \left\{ \gamma \in \mathtt{C} \quad \middle| R(P)\left(f\right) = 0 \quad \forall \text{local chart representation} \right\}$$

,where we have denoted respectively as $R(P)$ and f the local chart representation (as in Eq. 2.1) of P and γ on the same chart.
This is the subset of $\mathtt{C}$ containing all the smooth solutions of the motion equations corresponding to the dynamical operator:

$$P : \mathtt{C} \to \mathtt{C}$$

Notice that in case of a linear dynamical system $\ker(P)$ is a proper *kernel* subspace in the sense of liner operator theory.

2.1.1 Lagrangian Dynamics

Lagrangian systems constitute an interesting subclass of the abstract dynamical systems:

Definition 2.4 (Lagrangian System (of r-th order))**.** We call *Lagrangian system* of r-th order the pair $(E, \mathscr{L})$ composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on the oriented pseudo-Riemannian manifold $(M, g, \mathfrak{o})$ called *"configuration bundle"*.
- $\mathscr{L} : J^r E \to \wedge^m T^* M$
  bundle-morphism from the r-th Jet Bundle (see Section 1.1.5) to the top-dimensional form bundle over the base manifold M called *"Lagrangian density"* or simply *"Lagrangian"* of r-th order.

N.B. : In what follows all the systems considered will be exclusively of first order. The reason is due to the *Ostrogradsky instability* according to which a non-degenerate Lagrangian dependent on time derivatives of higher than the first corresponds to a linearly unstable dynamics.[33]

In this case, the Lagrangian density is the object containing all the information about the dynamics of the system.

In order to reconstruct the system dynamics from the Lagrangian density has to be understood the mathematical nature of $\mathscr{L}$.
The function $\mathscr{L}$ maps a point q_p on the fiber $J^r_p E$ to a m-form on $T_p M$. Recalling the definition of jet bundles it is clear that for each smooth section on E it is associated a smooth section on $J^r E$:

$$\phi \in \Gamma^\infty(E) \mapsto (\phi, \partial_\mu \phi, \partial_{\mu,\nu} \phi, \dots \partial_{\vec{\alpha}} \phi)$$

where $\vec{\alpha}$ is a multi-index of length r.
The correspondence is not unambiguous since sections equal up to the r-th order define the same jet section. The smoothness of $\mathscr{L}$ ensures that each section of the jet bundle is mapped to a smooth section in the top-forms bundle i.e. the most general integrable object on an orientable manifold.

It should be clear that $\mathscr{L}$ is a specific choice among the vast class of functions suitable to be a good Lagrangian density over the configuration bundle E:

Definition 2.5 (Lagrangian Densities on the bundle E)**.** We denote the set of all *Lagrangian densities* (of $r-$th order) on the bundle E as:

$$\mathsf{Lag}^r(E) := \hom\Big(J^r E, \quad \bigwedge^m (T^*M)\Big) \cong \{f : \Gamma^\infty(J^r E) \to \Omega^m(M)\}$$

(where $\Omega^m(M)$ is the common name for $\Gamma^\infty\big(\bigwedge^m(T^*M)\big)$ in the context of Grassmann algebras.)
The equivalence states the fact that a bundle-morphism induces a map between the sections.

Proposition 2.1.1. $\mathsf{Lag}^r(E)$ *has an vector space structure inherited by the linear structure of* $\Omega(M)$.

Thanks to the correspondence between a section $\phi \in \mathsf{C}$ and his r-th jet, it is possible to consider the Lagrangian as acting directly on the kinematic configurations.
This property suggests the introduction of the class of associated functionals:

Definition 2.6 (Lagrangian functional)**.** We call *Lagrangian functional* a map :

$$\mathscr{O}_{\mathscr{L}} : \mathsf{C} \to D'(M)$$

where $D'(M)$ is the space of regular distribution over M, whose action on any configuration $\phi \in \mathsf{C}$, evaluated on the test-function $f \in C_0^\infty(M)$, it is given by:

$$\mathscr{O}_{\mathscr{L}}[\phi](f) = \int_M \mathscr{L}[\phi] f \, \mathrm{d}\mu$$

Notice that $\mathscr{O}_{\mathscr{L}}[\phi](f)$, as a distribution, is necessarily continuous and linear in the test-functions entry but not in the configurations entry.

The choice of the image of $\mathscr{O}_{\mathscr{L}}$ as a distribution is a necessary precaution to ensure the *convergence* of the functional, whatever is the configuration on which it is evaluated. In fact, despite $\mathscr{L}[\phi]$ being integrable with respect to the measure $\mathrm{d}\mu$, it is not necessary summable if the support of the configuration ϕ becomes arbitrarily large.
This is a simple consequence of the well known sequence of inclusions:

$$\mathscr{L}[\phi] \in C_0^\infty(M) \subset L^1_{\mathrm{loc}}(M,\mu) \supsetneq L^1(M,\mu)$$

Indeed, the functional

$$\mathscr{O}_{\mathscr{L}}[\phi] = \int_{\mathrm{supp}(\phi)} \mathscr{L}[\phi] \mathrm{d}\mu$$

is well defined for all $\mathscr{L} \in \mathsf{Lag}^r(E)$ only over the compactly supported sections. To take account of the global sections it is sufficient to control the integral smearing the integrand with an arbitrary test-function.

The introduction of the Lagrangian density is meaningless without the prescription of a dynamical principle which allows to determine unambiguously a dynamical

operator P on $\mathtt{C}$. This fundamental principle is the *least action principle*. A proper justification of this claim should require the presentation of the differential calculus on the infinite dimensional manifolds $\mathtt{C}$.
Jumping straight to the conclusion we can state this correspondence as a law which assigns to all Lagrangian densities an operator on the kinematic configurations space.

In the case of a first order Lagrangian we define:

Definition 2.7 (Euler-Lagrange operator)**.** We call *Euler-Lagrange operator*, the differential operator

$$Q_\chi : \mathtt{C} \to \mathtt{C}$$

relative to the Lagrangian density $\chi \in \mathtt{Lag}^1(E)$, such that:

$$Q_\chi(\gamma) = \left(\nabla_\mu \left(\frac{\partial \chi}{\partial(\partial_\mu \phi)}\Big|_\gamma \right) - \frac{\partial \chi}{\partial \phi}\Big|_\gamma \right) \qquad \forall \gamma \in \mathtt{C} \tag{2.2}$$

Where ∇_μ is the covariant derivative corresponding to the background metric g. [1]

Remark 8. The whole theory of both Lagrangian densities class and Euler-Lagrange equation could be stated in a more synthetic way in terms of a Grassmann-graded variational bicomplex.[23][37]

2.2 Concrete Realization

In the previous section we stated an abstract definition of Lagrangian systems sufficiently broad to encompass all classical Lagrangian systems with both discrete degrees of freedom, like particles, and continuous degrees of freedom, like fluids or fields. Let us show two of the most significant examples.

2.2.1 Classical Field Theory

A (classical) *Field System* is nothing more than an abstract dynamical system (E, P) (see Def. 2.1) where the base space M is a suitable spacetime manifold[5].
Note that ,at this stage, the question about the Lagrangian nature of the dynamics is purely ancillary.
The idea of taking bundles on a spacetime manifold is physically intuitive, kinematically speaking a field configuration is simply the association of some element of the fiber Q for each point of the spacetime M.

There are, however, a few additional requirements that are often prescribed.

[1] $\frac{\partial \chi}{\partial(\partial_\mu \phi)}$ has the be intended as the Lagrangian density constructed differentiating $\chi(\phi, \partial_\mu)$ as an ordinary function, treating its functional entries as an usual scalar variable.

Linear System Condition

- The configuration bundle $E \xrightarrow{\pi} M$ is a vector bundle.
- The motion operator P is a l.p.d.o..

This condition is a necessary element in case some form of the *superposition principles* has to be taken in account. Obviously this hypothesis is not sufficient to formulate the principle in the strong classical way, *i.e.*:"the response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually" mostly because only free systems can be considered at this stage and any statement about stimulus can make sense.
The first condition, however, ensures that `C` is a vector space, in other words every linear combination of kinematic configurations is still a kinematic configuration. This condition, together with the linearity of operator P, guarantees that also $\texttt{Sol} = \ker(P)$ is a linear subspace.

Wave-like Dynamics Condition

- The base manifold M is a Globally Hyperbolic Spacetime.
- The motion operator P is PDE-hyperbolic[2].

The first condition ensures the existence of Cauchy surfaces and therefore permits to state *Cauchy Problems* assigning an initial data on such submanifolds. Furthermore, PDE-hyperbolicity of the motion operator P guarantees that for every Cauchy surface $\Sigma \subset M$ the corresponding initial data problem is well posed, that is:

$$\begin{cases} Pu = 0 \\ u = u_0 \\ \nabla_{\vec{n}} u = u_1 \\ \vdots \\ \left[\nabla_{\vec{n}}\right]^{k-1} u = u_{k-1} \end{cases} \tag{2.3}$$

where k is the order of the linear differential operator as defined in Def. 1.29, admits a unique solution $u \in \Gamma(E)$ for all

$$(u_0, \ldots, u_{k-1}) \in \underbrace{\Gamma(\Sigma) \times \ldots \times \Gamma(\Sigma)}_{k \text{ times}}$$

This suggests the following definition:

Definition 2.8 (Initial Data Set)**.** Let be P a l.d.p.o of order k, we call Data the set of all the smooth initial data which can be given on the Cauchy Surface Σ. That is:

$$\mathsf{Data}(\Sigma) := \left\{ (f_0, \ldots f_{k-1}) \,\middle|\, f_i \in \Gamma^\infty(\Sigma) \right\} \equiv \underbrace{\Gamma^\infty(\Sigma) \times \ldots \times \Gamma^\infty(\Sigma)}_{k \text{ times}}$$

[2]Hence it is implied that the configuration bundle is linear.

Remark 9. $\mathtt{Data}(\Sigma)$ inherit the linear structure of its component $\Gamma^{\infty}(\Sigma)$.

In this term the well-posedness of the Cauchy problem can be stated as follow:

Proposition 2.2.1. *The map*

$$\mathbf{s} : \mathtt{Data}(\Sigma) \to \mathtt{Sol}$$

which assigns to $(u_0, \dots, u_{k-1}) \in \mathtt{Data}(\Sigma)$ *the unique solution of the Cauchy problem 2.3 is linear and bijective.*

Since any solution, when restricted to a generic Cauchy surface Σ', determines another n-tuple of initial data, i.e.:

$$\phi \equiv \mathbf{s}\big(\phi|_{\Sigma'}, \nabla_{\vec{n}'}\phi|_{\Sigma'}, \dots, \big[\nabla_{\vec{n}'}\big]^{(k-1)}\phi|_{\Sigma'}\big) \quad \forall \phi \in \mathtt{Sol}$$

we can define the set of initial data regardless of the particular Cauchy surface:

N.B. : Embracing the principle of Ostrogradsky, can be neglected operators of order beyond the second. Without loss of generality we denote the element of Data as a pair (f_0, f_1).

Definition 2.9 (Set of smooth initial Data)**.** We denote the set of all smooth initial data as:

$$\mathtt{Data} := \frac{\bigsqcup_{\Sigma \in \mathscr{P}_C(M)} \mathtt{Data}(\Sigma)}{\sim}$$

where $\sim$ is such that:

$$(f_0, f_1)|_{\Sigma} \sim (g_0, g_1)|_{\Sigma'} \Leftrightarrow \mathbf{s}(f_0, f_1) = \mathbf{s}(g_0, g_1)$$

I.e.: two initial data, associated with different surface, are equivalent if they lead to the same solution.

Proposition 2.2.2. Data *is a vector space.*

Proof. it is sufficient to show that:

$$\big[\phi_a + \phi_b\big] = \big[\phi_a\big] + \big[\phi_b\big]$$

where $[\phi] = \big\{(\phi|_{\Sigma}, \nabla_{\vec{n}}\phi|_{\Sigma}) \big| \Sigma \in \mathscr{P}_C\big\}$.
In fact:

$$\begin{aligned}\mathbf{s}_{\Sigma'}\big([(a',b')] + [(c',d')]\big) = \mathbf{s}_{\Sigma}\big([(a,b)] + [(c,d)]\big) = \mathbf{s}_{\Sigma}\big([(a,b)]\big) + \mathbf{s}_{\Sigma}\big([(c,d)]\big) = \\ = \mathbf{s}_{\Sigma'}\big([(a',b')]\big) + \mathbf{s}_{\Sigma'}\big([(c',d')]\big) = \mathbf{s}_{\Sigma'}\big([(a',b')] + [(c',d')]\big)\end{aligned}$$

□

Corollary 2.2.3. *The function* $\mathbf{s} : \mathtt{Data}(\Sigma) \to \mathtt{Sol}$ *which maps every equivalence class to the associated solution is linear and bijective.*

Remark 10. The hyperbolicity condition is the main ingredient in the *initial data quantization* procedure[42].

Green-Hyperbolic Dynamic Condition

- The dynamic is generated by a Lagrangian, i.e. $P = Q_{\mathscr{L}}$.
- The motion operator P is a Green Hyperbolic L.P.D.O.

We remark that Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense, therefore the last two conditions are not equivalent.

Remark 11. The Green-hyperbolicity condition is a necessary hypothesis to carry on the Peierls construction of the brackets underlying to the definition of the classical symplectic structure.

Example 2. One can formulate a *Real Scalar Field on curved backgrounds* within this abstract picture:

Configuration bundle	real scalar field: $Q \equiv \mathbb{R}$ curved background: M is globally hyperbolic
Kinematic Configurations	$\mathtt{C} = C^\infty(M, \mathbb{R})$
Motion Operator	$P = \Box_M + m^2 + \xi R$ normally hyperbolic operator

Where P is the Klein-Gordon Operator: ξ and m^2 are two real numbers, R stands for the scalar curvature build out of g and $\Box_M := g^a b \nabla_a \nabla_b$ is the d'Alambert operator associated to the Levi-Civita connection.
Note that this system satisfies all the above three condition.

2.2.2 Finite Degrees systems as a Field

With a little more effort it is possible to realize every ordinary mechanic system -with discrete degrees of freedom- as a special case of classical field.

Consider a Lagrangian system (Q, L) with configuration space Q and Lagrangian function $L : TQ \to \mathbb{R}$.
Remembering the intuitive meaning of Q as the set of all the statical conformation of the system, it is natural to read as kinematic configuration all and only the trajectories compatible with the constraints. In other words the space of kinematic configurations is the set of all the parametrized smooth curves on Q:

$$\mathtt{C} = C^\infty(\mathbb{R}, Q)$$

Therefore, the configuration bundle of this system is the trivial smooth bundle:

$$E = (\mathbb{R} \times Q, \pi, \mathbb{R})$$

of typical fiber Q, since the corresponding space of sections coincides to the required configuration space:

$$\Gamma^\infty(E) \equiv C^\infty(\mathbb{R}, Q)$$

Also the "field theoretic" picture of the dynamics is not difficult to achieve. It is enough to recall how the Euler-Lagrange equation of an ordinary mechanical system can be derived from the *least action principle*:

Least Action Principle:
The motions of the mechanical system (Q, L) are the stationary points of the action functional:

$$S[\gamma] = \int_{\mathbb{R}} L(t, \gamma^i, \dot{\gamma}^i)\, dt$$

constructed from the Lagrangian of the system.

Remembering that we have interpreted the space $\mathbb{R}$ of the parameter t as the base manifold of the configuration bundle E, it is immediate to see S as the *Total Lagrangian* related to the Lagrangian density:

$$\mathscr{L}[\gamma] := L \circ (\gamma)^{\uparrow}\, dt$$

obtained evaluating the ordinary Lagrangian on the lifted curve $\gamma^{\uparrow}$ (lift of $\gamma \in \mathtt{C}$ from Q to TQ).

Summing up, the mathematical structure of such mechanical system is enconded as follows:

Configuration bundle	Trivial bundle $E = Q \times \mathbb{R}$ Base manifold: $M = \mathbb{R}$
Kinematic Configurations	$\mathtt{C} = C^{\infty}(\mathbb{R}, Q)$
Lagrangian Density	$\mathscr{L}[\gamma] := \left(L \circ \gamma^{\uparrow}\right) dt = L(t, \gamma^i, \dot{\gamma}^i)\, dt$
Motion Operator	$P = Q_{\mathscr{L}} \equiv \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^i} L\right) - \frac{\partial}{\partial x^i} L$ Green hyperbolic operator

Considering that the space M is trivially globally hyperbolic, since every point $t \in \mathbb{R}$ is a genuine "Cauchy Surface", it is evident how this system can be seen as a special case of field theory: a "field of curves".
Notice that ,unless the configuration space is a vector space, the corresponding field theory cannot be linear.

2.3 Hamiltonian Mechanics of Finite Degrees systems

The picture of the systems with discrete degrees of freedom as a field brings out only a part of the typical mathematical ingredients proper of the *Geometric* approach to the classical mechanics

For example we have a avoided to mention the Hamiltonian formalism in the case of abstract fields. Even though it should be possible, in the spirit of what has been done within the Lagrangian picture, to extend the canonical treatment to include systems with continuous degrees of freedom (see for example [24]),we will not expand this topic since the protagonist of this thesis is essentially a classical point particle system.

To fulfill these quantization procedures we will need to draw inspiration from their correspondent classical versions. Let us briefly review them in the finite dimensional case,

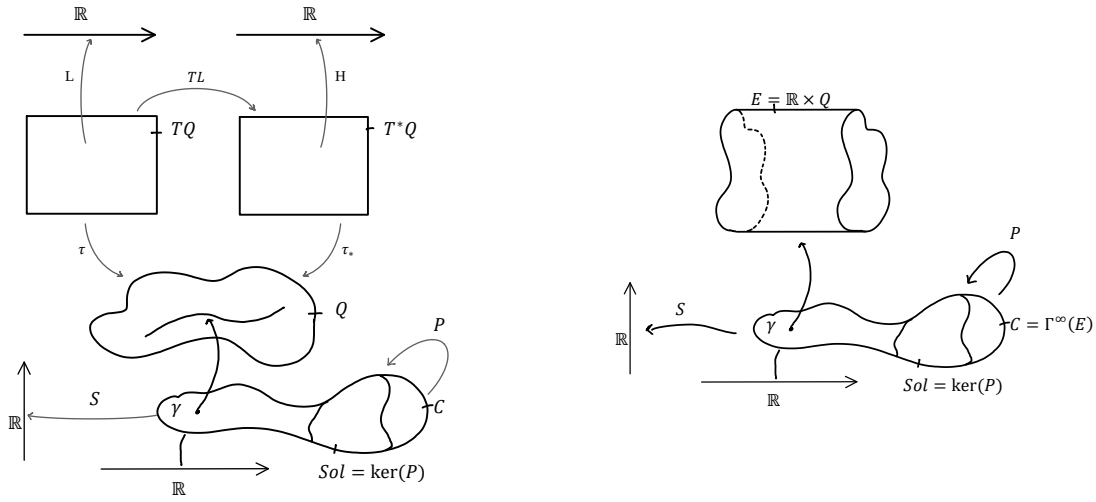


Figure 2.2: An "impressionistic comparison" between the mathematical framework of geometrical mechanics and the field-theoretic picture

Phase Space We recalled in chapter 1 the definition of *Phase Space* in ordinary classical mechanics as the cotangent bundle T^*Q of the classical configuration space Q. We showed that every classical phase space is symplectic trough the natural Poincaré form. Nevertheless every quantization procedure requires a modification of this standard symplectic form in order to implement the canonical commutation relations.

This leads us to make use of the abstract formulation of Hamiltonian systems[2]:

Definition 2.10 ((Ordinary) Hamiltonian System)**.** We call *(ordinary) Hamiltonian system* a triple $(\mathscr{M},\omega,H)$ composed of:

- $(\mathscr{M},\omega)$
  finite dimensional symplectic manifold, *i.e.* a 2n-dimensional smooth manifold $\mathscr{M}$ endowed with a non degenerate closed two-form ω, called *"Phase space"*.
- $H:\mathscr{M}\to\mathbb{R}$
  smooth function called *"Hamiltonian"*

Remark 12. In classical mechanics Hamiltonian systems could be seen as a subset of Lagrangian systems.
The key is the definition of the Legendre Map $TL:TQ\to T^*Q$, in the case that the Lagrangian L is *hyperegular* (*i.e.* TL is a diffeomorphism),it is possible to push-forward L to give a proper Hamiltonian on $\mathscr{M}=T^*Q$. (see for example [2])

Recall that the Darboux theorem states that, at least locally, every symplectic form can be represented in the canonical form:

Theorem 2.3.1 (Darboux)**.** *Let be $(\mathscr{M},\omega)$ a symplectic manifold, then $\forall\, m\in\mathscr{M}$, $\exists$ a local chart (U,φ) (where $\varphi(u)=\left(x^1(u),\dots,x^n(u);y^1(u),\dots,y^n(u)\right)$) such that:*

- $\varphi(m)=0$

- $\omega\big|_U = \sum_{i=1}^{n} dx^i \wedge dy^i$

Proof. We omit the proof which can be found in [2][Th. 3.2.2]. □

Classical Observables Observables in classical mechanics are represented by real valued smooth functions on $\mathscr{M}$:

Definition 2.11 (Space of Classical Observables)**.** The *Space of Classical Observables* is denoted as:

$$\mathscr{E} \equiv C^\infty(\mathscr{M}, \mathbb{R})$$

The space $C^\infty(\mathscr{M}, \mathbb{R})$ of smooth real valued function on $\mathscr{M}$, inherits the structure of commutive algebra over $\mathbb{R}$ from its codomain $\mathbb{R}$.

The symplectic structure on $\mathscr{M}$ gives rise to a second algebraic structure on the vector space of observables. At first it is necessary to introduce the Hamiltonian fields:

Definition 2.12 (Hamiltonian field with energy function $H \in \mathscr{E}$)**.** We call *Hamiltonian field* with energy function $H \in \mathscr{E}$ the vector field $\mathbf{X}_H$ determined by the condition:

$$\omega(\mathbf{X}_H, \cdot) \equiv dH(\cdot)$$

Nondegeneracy of ω guarantees that $\mathbf{X}_H$ exists for all classical observables $H \in \mathscr{E}$. From that follows the definition of the bracket:

Definition 2.13 (Poisson Brackets)**.** We call *Poisson brackets* the bilinear function $\{\cdot,\cdot\} : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ such that:

$$\{f, g\} \coloneqq \omega\big(\mathbf{X}_f, \mathbf{X}_g\big) = df(\mathbf{X}_g) \tag{2.4}$$

Proposition 2.3.2. *The real vector space $\mathscr{E} = C^\infty(M, \mathbb{R})$, together with the Poisson brackets $\{\cdot,\cdot\}$, form a Lie algebra.*
I.e.: $\{\cdot,\cdot\}$ satisfies the following axioms:

- *bilinearity*

$$\{ax+by, z\} = a\{x,z\}+b\{y,z\}, \quad \{z, ax+by\} = a\{z,x\}+b\{z,y\} \qquad \forall a, b \in \mathbb{C}, \forall x, y, z \in \mathscr{E}$$

- *alternativity*

$$\{x, x\} = 0 \qquad \forall x \in \mathscr{E}$$

- *The Jacobi equation:*

$$\{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} = 0 \qquad \forall x, y, z \in \mathscr{E}$$

Proof. We omit the proof, see for example Ref. [2][Prop. 3.3.17]. □

In the canonical coordinates, provided by Darboux theorem, the Poisson bracket assumes the typical expression:

Proposition 2.3.3 (Symplectic coordinate representation)**.** *In canonical coordinates* $(q^1,\dots,q^i,p_1,\dots,p_n)$ *we have:*

$$\{f,g\}=\sum_{i=1}^{n}\Big(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\Big)\qquad \forall f,g\in\mathscr{E} \tag{2.5}$$

Proof. We omit the proof which can be found in [2][Corol. 3.3.14]. □

Solution Space In this framework the Hamilton's equations of motions can be written in terms of the Hamiltonian Fields:

> **Hamilton dynamics principle** The dynamically allowed configurations for $(\mathscr{M},H)$ corresponds to the *integral curves*[2] of the Hamiltonian vector field $X_H\in\Gamma(T\mathscr{M})$

It follows that the specification of a point in the phase space is an appropriate initial data for determining a solution of Hamilton's equations of motion, *i.e.*, each point $y\in\mathscr{M}$ gives rise to a unique solution of the dynamical evolution. Therefore:

$$\mathscr{M}\cong \mathsf{Data}\cong \mathtt{Sol} \tag{2.6}$$

2.3.1 Linear dynamical systems

Most of the physical systems that are encountered in the theory of fields are linear. It is possible to come across linear dynamical systems also in ordinary mechanics.

Remark 13. **Linear Hamiltonian System**

- $\mathscr{M}$ has a natural structure of vector space.
- H is a quadratic function on $\mathscr{M}$.[3]

In this case the difference between the underlying geometric entities tends to fade out as a consequence of the flatness of the configuration space.
The key consequence of the vector structure of $\mathscr{M}$ is that it allows us to identify the tangent space at any point $y\in\mathscr{M}$ with the Phase space $\mathscr{M}$ itself.
Under this identification, the symplectic form becomes a bilinear function $\Omega:\mathscr{M}\times\mathscr{M}\to\mathbb{R}$ on $\mathscr{M}$, *i.e.*, it can be seen as acting directly on the points of the phase space rather than on tangent vectors. In this way, the phase space of a linear dynamical system, may be viewed as a *symplectic vector space* $(\mathscr{M},\Omega)$.

Due to the identification of the phase space and the solution space, follow that the symplectic structure Ω is directly transferred from $\mathscr{M}$ to $\mathtt{Sol}$. This symplectic vector space structure $(\mathtt{Sol},\Omega)$ of the manifold of solutions for a linear dynamical system is the fundamental classical structure that underlies the construction of the *Inital Data* quantization procedure.

[3] Equations of motion are then linear on the affine canonical coordinates. Dynamics is thus simply a collection of coupled harmonic oscillators.

In light of the linear structure the *Linear Observables* take a primary role:

$$\mathscr{E}_{\text{lin}} = \mathscr{M}^*$$

This set is a vector space and every choice of linear canonical coordinates $\psi^\alpha = (q^a, p^b)$ on $\mathscr{M}$ constitutes a basis on $\mathscr{E}_{\text{lin}}$:

$$T\mathscr{M} \simeq \mathscr{M} \;\Rightarrow\; d\psi^\alpha \equiv \psi^\alpha$$

$\mathscr{M}_{\text{lin}}$ forms a Poisson subalgebra.
The presence of the non-degenerate bilinear form provide the usual identification $\mathscr{M} \simeq \mathscr{M}^*$ and therefore the symplectic form on $\mathscr{M}$ can directly reproduced on $\mathscr{E}_{\text{lin}}$:

$$\{\Omega(y_1,\cdot),\Omega(y_2,\cdot)\} = -\Omega(y_1,y_2) \qquad \forall\, y_i \in \mathscr{M} \Rightarrow \Omega(y_1,\cdot) \in \mathscr{E}_{\text{lin}}$$

In conclusion, the essential aspects that characterize the geometry of linear systems are the following:

- The symplectic form of $\mathscr{M}$ is directly defined on the points of the Phase space.
- Since the points of the phase space can be put in correspondence with the solutions (can be considered as the *inital data*) the symplectic form is directly transported to `Sol`.
- The same symplectic structure can be reproduced on the space $\mathscr{E}_{\text{lin}}$ and coincides with the reduction of the Poisson bracket to this subspace.

2.4 Peierls Brackets

Purpose of Peierls' procedure is to provide a bilinear form on the space of Lagrangian densities with time-compact support. This form induces a pre-symplectic structure on suitable subspaces of functionals to which can be recognized the role of *classical observables* of the theory.

Intuitively we can say that the Peierls Brackets implement a sorts of "comparison relation" between two observables similar to the Poisson brackets in ordinary Hamiltonian mechanics.

As we will see there are important differences between the two definitions:

- The Poisson brackets determines how one "quantity" b changes another "quantity" a when it acts as the generator (tipically the Hamiltonian) of the dynamical evolution or vice-versa. [39]
  The Peierls brackets, on the other hand, determines how one "quantity" b when added to the system dynamics (usually the Lagrangian or the total action) with an infinitesimal coupling constant λ affects changes in another "quantity" a and vice-versa.

In other words the Peierls brackets are related to the change in an observable when the trajectory on which it is evaluated gets shifted. The shift is not arbitrary, is determined by an infinitesimal change in the Lagrangian of the system through the addition of a second Lagrangian density.

- While the Poisson brackets between two observables a and b is defined on the whole phase space and is not dependent on the existence of a Hamiltonian, the Peierls brackets refers to a specific trajectory determined by a given Lagrangian.

A rigorous treatment of the notions of *observable* and *Phase Space* should require some further specification depending on which is the considered brackets. We can read the "observable" as an object with a twofold nature. Essentialy it can act both as the generator of the dynamics and as a quantity which can be evaluated on the system configurations.

In this section we present more extensively the original Peierls' construction. Please note that we are not trying to provide the state of the art on the Peierls brackets (see for example [27] for the treatment in presence of gauge freedom and constraints) but only to expand and modernize the first approach given by Peierls. Instead of considering only a scalar theory we extend the algorithm to the broader class of "abstract" mechanical systems defined above.

2.4.1 Peierls' construction.

The Peierls' construction algorithm is well defined for a specific class of systems:

(A) Linear field theory: $E = (E, \pi, M; Q)$ is a vector bundle.

(B) $P = Q_{\mathscr{L}}$ is a Green-hyperbolic linear partial differential operator.

(C) M is a globally hyperbolic spacetime.

The procedure can be summarized in a few steps:

1. Consider a *disturbance* χ that is a time-compact Lagrangian density .
2. Construct the *perturbation of a solution under the action of* χ.
3. Define the *effect of the disturbance* on a second Lagrangian functional.
4. Assemble the mutual effects of two different Lagrangian densities to give a *brackets.*

Let us review each step in details.

Disturbance and Disturbed motion operator

By *"disturbance"* we mean a time-compact supported Lagrangian density $\chi \in \mathtt{Lag}$[4] which acts as a perturbation on the Lagrangian of the system:

$$\mathscr{L} \rightsquigarrow \mathscr{L}' = \mathscr{L} + \epsilon \cdot \chi$$

where ϵ is a real modulation parameter. The support condition is required in order to take in account only perturbations which affect the dynamic for a definite time interval. The ruling operator of the perturbed dynamics is:

$$P_\epsilon = \left[\nabla_\mu\Big(\frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\Big) - \frac{\partial \mathscr{L}}{\partial\phi}\right] + \epsilon\left[\nabla_\mu\Big(\frac{\partial \chi}{\partial(\partial_\mu\phi)}\Big) - \frac{\partial \chi}{\partial\phi}\right] = P + \epsilon Q_\chi \tag{2.7}$$

Remark 14. P_ϵ is not necessary linear, hypothesis (B) guarantees the linearity only for P.

Solution of the disturbed motion

The second ingredient of the Peierls' procedure is the calculus of the *perturbed solutions* under the considered *disturbance*.
These are obtainable by a infinitesimal linear perturbation of any, but fixed, solution $\phi \in \mathtt{Sol}$. The good definition of linear superposition is guaranteed by the hypothesis (A).
More precisely, one looks for to be seek a configuration:

$$\phi'(x) = \phi(x) + \epsilon\eta(x) \in \mathtt{C}$$

such that:

$$\begin{cases} P_\epsilon\phi'(x) & = o(\epsilon) \\ P\phi(x) & = 0 \end{cases}$$

In other word the following equation has to be satisfied:

$$\big[P_\epsilon\big]\phi'(x) = \big[P + \epsilon Q_\chi\big](\phi(x) + \epsilon\eta(x)) = \epsilon\Big(\big[P\big]\eta(x) + \big[Q_\chi\big](\phi(x) + \epsilon\eta(x))\Big) \stackrel{!}{=} o(\epsilon)$$

The linearity condition for operator P does not hold true in general for Q_χ. We can work around this problem considering the linearization[27, pag. 31] of Q_χ around the unperturbed solution $\phi(x)$. The linearization of Q_χ is the unique linear operator $\big[Q^{lin}_\chi(\phi)\big]$ such that:

$$\big[Q_\chi\big](\phi(x) + \epsilon\eta(x)\big) = \big[Q_\chi\big](\phi(x)) + \epsilon\big[Q^{lin}_\chi(\phi)\big](\eta(x)) + o(\epsilon)$$

which can be seen as the first term of a *formal* Taylor expansion of operator Q_χ around ϕ[5] This is reflected in a condition on the perturbation $\eta \in \mathtt{C}_{tc}$:

$$\big[P_\epsilon\big]\phi'(x) = \epsilon\Big(\big[P\big]\eta(x) + \big[Q_\chi\phi(x)\big]\Big) + \epsilon^2\big[Q^{lin}_\chi(\phi)\big]\eta(x) \stackrel{!}{=} o(\epsilon)$$

[4] I.e. the top form $\chi(\phi)$ is time-compact supported for all $\phi \in \mathtt{C}$.

[5] If $\mathtt{C}$ is a Frechet manifold the expansion could be stated rigorously by defining $\big[Q^{lin}_\chi(\phi_0).\big] = \big[\frac{\partial Q_\chi}{\partial\phi}(\phi_0)\big]$ in terms of the Gateaux derivative.

$$\Rightarrow P\eta = -Q_\chi\phi(x) \tag{2.8}$$

called *Jacobi Equation*. This equation is a non homogeneous PDE with inhomogeneous term $(-Q_\chi\phi(x))$ fixed by the solution $\phi \in \texttt{Sol}$ to be perturbed.

It follows from the definition of Green hyperbolicity that the domain restrictions of P to $\Gamma^\infty_{\text{pc}}$ or $\Gamma^\infty_{\text{fc}}$ admit a unique inverse G^+ and G^- respectively. Therefore, equation 2.8 admits a unique past compact solution η_+, called retarded perturbation of $\phi \in \texttt{Sol}$, and a unique future compact solution η_-, called advanced perturbation:

$$\eta_\pm = G^\pm\big(-Q_\chi\phi\big) \tag{2.9}$$

Note that the time-compact support condition on χ guarantees that $Q_\chi\phi \in dom(G^+)\cap dom(G^-)$. Expression 2.9 reflects perfectly the original Peierls' notation where $\eta_\pm$ were noted as functions of the unperturbed solution: $\eta_+ \equiv \mathrm{D}_\chi\phi$ and $\eta_- \equiv \mathrm{\Gamma}_\chi\phi$.

In most of the practical cases it is possible to give a more "down to earth" characterization of $\eta_\pm$ in terms of a Cauchy problem.
It has to be stressed that this approach is not possible in general since Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense, *i.e.*, the well-posedness of the Cauchy problem is not guaranteed on any Cauchy surface. [5, pag 1] [3, remark 3.18][27, remark 2.1]
Consider a motion operator P which is also hyperbolic. Taking in account the time-compact support condition of χ, is possible to pick up two Cauchy surfaces $\Sigma_\pm$ ("+" stands for "subsequent" to the perturbation while "−" stands for antecedent to the perturbation) such that:

$$J^\mp(\Sigma_\pm) \supset \mathrm{supp}(\chi)$$

for all time-slice foliations of the globally hyperbolic spacetime.

For each of these two surfaces a Cauchy problem can be posed:

$$\begin{cases} P\eta = -Q_\chi\phi \\ (\eta, \nabla_n\eta)\big|_{\Sigma_\pm} = (0,0) \end{cases} \tag{2.10}$$

which , according to the well-posedness of the Cauchy problem, admits an unique solution. The link with the previous presentation is that past/future -compact supported configurations trivially meet the initial data condition for some future/past Cauchy surface.

In conclusion, for any but fixed $\phi \in \texttt{Sol}$ and perturbation χ, there exist two uniquely perturbed solutions:

$$\phi^\pm_\epsilon = \phi + \epsilon\eta_\pm \tag{2.11}$$

such that:

retarded perturbation	$\eta_+ \in \Gamma^\infty_{pc}$	$(\eta_+, \nabla_n\eta_+)\big\vert_{\Sigma_-} = (0,0)$	"propagating forward"
advanced perturbation	$\eta_- \in \Gamma^\infty_{fc}$	$(\eta_-, \nabla_n\eta_-)\big\vert_{\Sigma_+} = (0,0)$	"propagating backward"

Effect Operator

Considering an arbitrary continuous[6] functional $B : \mathtt{C} \to \mathbb{R}$ (not necessarily linear) we can define the effect of a perturbation on the values of B[32, pag. 5] as a map:

$$\mathbf{D}^{\pm}_{\chi} : C^1(\mathtt{C}, \mathbb{R}) \to C^1(\mathtt{C}, \mathbb{R})$$

$$\mathbf{D}^{\pm}_{\chi} B(\phi_0) \coloneqq \lim_{\epsilon \to 0} \Big(\frac{B(\phi^{\pm}_{\epsilon}) - B(\phi_0)}{\epsilon} \Big) \tag{2.12}$$

The advanced and retarded effects of χ on B are then defined by comparing the original system with a new system defined by the same kinematic configuration space $\mathtt{C}$ but with perturbed Lagrangian.
Has to be noted that expression 2.12 is clearly a special case of Gateaux derivative.[10]
The former expression simplifies in case of a linear functional:

$$\mathbf{D}^{\pm}_{\chi} B(\phi_0) = B(\eta_{\pm}) \tag{2.13}$$

The Brackets

Remembering that every Lagrangian density define a continuous functional (Def. 2.6). From that is possible to build a binary function:

$$\{\cdot, \cdot\} : \mathsf{Lag}_{\mathrm{tc}} \times \mathsf{Lag}_{\mathrm{tc}} \to \mathbb{R}$$

as follow:

$$\{\chi, \omega\}(\phi_0) \coloneqq \mathbf{D}^{-}_{\chi} \mathscr{O}_{\omega}(\phi_0) - \mathbf{D}^{+}_{\chi} \mathscr{O}_{\omega}(\phi_0) \tag{2.14}$$

Proposition 2.4.1 (Bilinearity)**.** *When restricted to Linear Lagrangian densities* $\{\cdot, \cdot\}$ *is a bilinear form*

Proof. Linearity in the first entry follows from equation2.9 and the linearity of the Euler-Lagrange operator $Q_{\cdot}$ over Lag.
Linearity in the second entry is guaranteed only for Lagrangian densities ω which provide a linear Lagrangian Functional F_{ω}. □

We are not interested in addressing the cases for which the symplectic property is met on this general ground. In the next chapter we will face the problem to determine symmetry and non-degeneracy properties for the case of *classical observable functionals*, a subclass of Lagrangian functionals of most practical use in the quantization schemes.

2.4.2 Extension to non-linear theories

In the previous construction the Green-hyperbolicity of P plays a primary role. The problem of searching perturbed solution to the disturbed dynamic can be stated even

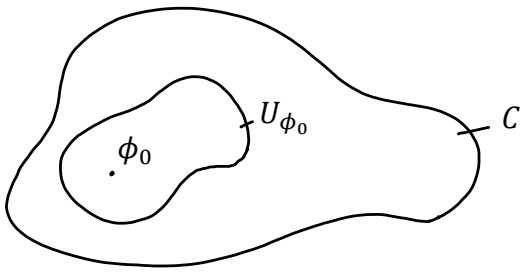


Figure 2.3: Intrinsically, searching a variation of a solution $\gamma_0 \in \texttt{Sol}$ which solve the disturbed motion equation is equivalent to find the intersection of the perturbed solution with a local neighbourhood of Γ_0 : $U_{\gamma_0} \cap \ker(P_\epsilon)$.

in presence of non-linear fields when the configuration bundle is not necessary a vector bundle or the operator ruling the dynamics is not linear.

The crucial point of the Peierls' procedure is to select among all the possible solutions of the perturbed motion P_ϵ that configuration which can be constructed by a variation of a given solution of the non-perturbed dynamics $\gamma_0 \in \texttt{Sol}$. In this sense the problem results in a *"linearization"* inasmuch the search of such solution is restricted to a local neighbourhood of the "point" $\gamma_0 \in \texttt{Sol}$.

Previously the choice to consider only the linear variation was quite natural but in the general case this preferential restriction is no longer possible. Anyway, it is possible to recover a notation similar to Eq. 2.11 by working patchwise, under the choice of a particular coordinate representation.
Fixed a solution $\gamma_0 \in \texttt{Sol}$ and a local trivializing chart (A, ϕ_A) such that $A \cap ran(\gamma_0) \neq \emptyset$ we can define a local infinitesimal variation by acting on his components:

$$\gamma^i_\lambda(x) = \gamma^i_0(x) + \lambda\eta^i(x) \qquad \forall x \in \pi(A)$$

where γ^i_0 are the component of the unperturbed solution in the open set A and $\eta^i \in \mathbb{R}^q$ is a generic real q-ple (q is the dimension of the typical fiber). λ is a real parameter that has to be "sufficiently small" in order to guarantee that the range of γ_λ is properly contained in A.
In other words the construction of the linear variation, that for linear field theories could be done in a global way, in the general case can be recovered only locally varying the components.

Therefore it is possible to define the effect of a disturbance locally, searching a local section $\gamma^i_\epsilon = \gamma^i_0 + \epsilon\eta^i$ which solves the perturbed equation up to the first order in ϵ, *i.e.* :

$$\left[P_\epsilon\right]\gamma^i_\epsilon = o(\epsilon)$$

where $\left[P_\epsilon\right]$ has to be intended as the coordinate representation of the operator with domain restricted to the local sections $\Gamma^\infty(A)$.

[6]The precise notion of continuity require the specification of a (infinite dimensional) manifold structure on C.

Without loss of generality has been taken the same scalar ϵ to modulate both the perturbation γ_ϵ and the perturbation on the operator ruling the dynamics
Consider two different parameters is moot since, in that case, only the smaller one should be taken in account.

From the explicit equation of the perturbed solution:

$$\Big([P]+\epsilon[Q_\chi]\Big)(\gamma_0^i+\epsilon\eta^i)=o(\epsilon)$$

it follows an equation on the components of the local perturbation. In this case one has to deal with the problem of non-linearity not only for Euler-Lagrange operator Q_χ but also for P. Arresting the expansion to the first order in ϵ results in:

$$\Big[P^{lin}_{\gamma_0}\Big]\eta^i(x)=-\Big(Q_\chi(\gamma_0)\Big)(x) \tag{2.15}$$

the *Jacobi equation* on the unperturbed solution $\gamma_0 \in \mathtt{Sol}$.

Has to be stressed that we switched from an operator P defined on $\mathtt{C}$ to an operator $P^{lin}_{\gamma_0}$ defined on the space of variations. From a global point of view this variation can be seen as a tangent vector $\eta \in T_{\gamma_0}\mathtt{C}$.
In the case of a linear system this passage was unnecessary. The Jacobi equation was directly defined on $\mathtt{C}$ since, for linear systems, any section could be seen as a generator of an infinitesimal variation.
This behaviour mimics perfectly what happens in ordinary classical mechanics where the configuration space of a linear system is a vector space, *i.e,* a "flat" manifold[7], which is isomorphic to its tangent space in every point.

Provided that the operator ruling the dynamics (which is now properly a linear partial differential operator) is Green-Hyperbolic, the Peierels' construction can continue as before.
Notice that now the advanced/retarded perturbation are formally identical to equation 2.9:

$$\eta^i_\pm = G^\pm\big(-Q_\chi\gamma_0^j\big)$$

with the important difference that $G^\pm$ are now the Green operators of the linearized equation of motion and they depend strictly on the fixed solution γ_0^j.

In conclusion the perturbed solution:

$$\gamma_\chi^{\pm\, i}=\gamma_0^i\pm G^\pm\big(-Q_\chi\gamma_0^j\big)$$

has to be meant as the "glueing" of all the local chart representations covering the chosen solution.

Example: Finite Dimensional Case

As an example of such process we can consider a *field of curves, i.e.*, an ordinary classical mechanical system in the field theoretic picture. We have shown in section 2.2.2

[7] In sense that admits a global coordinate chart.

that such systems are generally non linear: the configuration bundle is not a vector one and a linear P cannot be defined .

The base manifold is very simple. Indeed $M = \mathbb{R}$ can be seen as a trivial globally-hyperbolic spacetime where every point $t \in M$ is a Cauchy surface. This allows us to specify the above equations in a more intuitive way:

- γ_0^j is a simple local chart representation of the curve.
- for a suitable small ϵ, γ_0^j and $\gamma_\chi^{\pm\, i}$ can be depicted in the same local chart.
- the variation $\eta_\pm$ is then the field over the unperturbed curve whose components compute the separation between γ_0 an $\gamma_\chi^\pm$. In other words:

$$\gamma_\chi^{\pm\, i} = \gamma_0^i \pm G^\pm(-Q_\chi \gamma_0^i)$$

where $G^\pm$ are the right-left inverse of $\left[P^{lin}_{\gamma_0}\right]$ whose existence and uniqueness is guaranteed by the global hyperbolicity of the linearization of P.

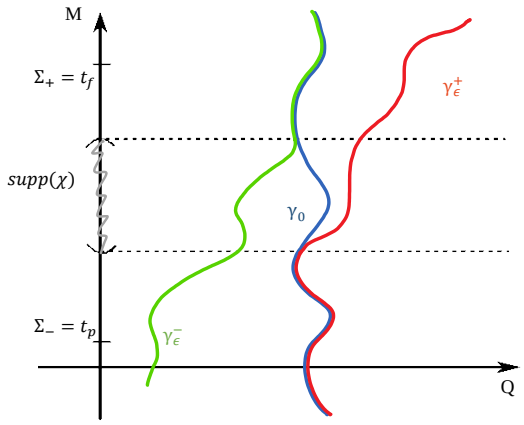


Figure 2.4: Picture of the perturbed solution in case of a finite dimensional system.

Once seen as the perturbation $\epsilon\chi$ disturbs the solution $\gamma_0 \in \texttt{Sol}$, it is possible to compute the effect of a perturbing density χ on the values of the functional F_ω associated to as second Lagrangian density ω as shown in equation 2.12.
By a Taylor expansion arrested to the first order we can say that:

$$\begin{aligned} F_\omega\left(\gamma_\epsilon^\pm\right)[f] &= \int f(t)\omega(t, \gamma_0^i + \epsilon\eta_\pm^i, \dot\gamma_0^i + \epsilon\dot\eta_\pm^i)\,dt = \\ &= F_\omega(\gamma_0)[f] + \epsilon \int f(t) \left[\left.\frac{\partial\omega}{\partial q^i}\right|_{\gamma_0} \eta_\pm^i + \left.\frac{\partial\omega}{\partial \dot q^i}\right|_{\gamma_0} \dot\eta_\pm^i \right] dt \end{aligned}$$

for any test function f.
Substituting this result in the definition of effect operator:

$$\mathbf{D}_\chi^\pm F_\omega(\gamma_0)[f] = \int f(t) \left[\left.\frac{\partial\omega}{\partial q^i}\right|_{\gamma_0} \eta_\pm^i + \left.\frac{\partial\omega}{\partial \dot q^i}\right|_{\gamma_0} \dot\eta_\pm^i \right] dt$$

we get an explicit expression for the Peierls brackets:

$$\{\chi,\omega\}(\gamma_0)[f] = \int f(t)\left[\left(\frac{\partial\omega}{\partial q^i}\right)\Big|_{\gamma_0}\left(\eta_- - \eta_+\right)^i + \left(\frac{\partial\omega}{\partial\dot{q}^i}\right)\Big|_{\gamma_0}\left(\dot{\eta}_- - \dot{\eta}_+\right)^i\right]dt =$$
$$= \int f(t)\left[\left(\frac{\partial\omega}{\partial q^i}\right)\Big|_{\gamma_0} + \left(\frac{\partial\omega}{\partial\dot{q}^i}\right)\Big|_{\gamma_0}\partial_t\right]\left[\left(G^- - G^+\right)\left(-Q_\chi\gamma_0\right)\right]^i dt \qquad (2.16)$$

Chapter 3

Algebraic Quantization

In order to proceed to the quantization of the geodesic system it is necessary to devote a chapter to the description of the *algebraic quantization scheme*. We will show two realizations of the scheme applicable to a class of systems sufficiently broad to encompass the case under examination.

3.1 Overview on the Algebraic Quantization Scheme

Contemporary quantum field theory is mainly developed as a quantization of classical fields. The *"Quantization process"* has to be considered as an algorithm, in the sense of self-containing succession of instructions, that has to be performed in order to establish a correspondence between a classical field theory and its quantum counterpart.

On this basis the axiomatic theory of quantum fields, especially the extension of the Haag and Kastler axioms to curved background proposed by Dimock[16] in (1980), takes the role of "validity check". It provides a set of conditions that must be met in order to establish whether the result can be considered a proper quantum field theory. Basically there are no physical/philosophical principles which justify "a priori" the relation between these mathematical objects (*e.g.* the classical state versus quantum states) individually. The scheme can only be ratified " a posteriori" as a whole.

This is, however, by no means different from what it is discussed in ordinary quantum mechanics where there are essentially two levels: the basic formalism of quantum mechanics, which is substantially axiomatic and permits to define an abstract quantum mechanical system, and the quantization process that determines how to construct the quantum analogous of a classical system realizing the basic axioms.

We refer to the algebraic quantization as a *scheme of quantization* because it is not a single specific procedure but rather a class of algorithms. These algorithms are the same concerning the quantization step per se (construction of the $*-$algebra of classical observable) but they differ in the choice of the classical objects (essentially the classical observables and the bilinear form) to be subjected to the procedure.

Basically an algebraic quantization is achieved in three steps:

A) **Classical Step**
Identify all the mathematical structures necessary to define the field, *i.e.*,the pair (E, P).
In general every quantization process exploits some conditions on the quantum field structure that has to be met.

B) **Pre-Quantum Step**
It consists of the implementation of additional mathematical superstructure on the classical framework. The aim is to establish the specific objects which will be used in the quantization process in the next step.
Generally these objects do not posses any classical meaning, their only purpose is to represent the classical analogue of the crucial structures of the quantum framework. For this reason these structures are said *Pre-Quantum*, their introduction does not have a proper *a priori* explanation but it has to be treated as an ansatz and justified *a posteriori* within the quantum treatment.

Essentially a suitable space of *classical observables* has to be chosen and this space has to be rigged with a well-behaved bilinear form.

The ordinary quantum mechanics equivalent step is the choice of a particular Poisson bracket on $C^\infty(T^*Q)$, which typically implements the *canonical commutation relations* $\{q, p\} = i\hbar$, among all the possible Poisson structures. This is "pre-quantum" in the sense that has to be chosen an alternative symplectic structure different from the natural form (Def. 1.11).

C) **Quantization**
Finally we introduce the rules which realize the correspondence between the chosen classical objects and their quantum analogues. The algebraic approach characterizes the quantization of any field theory as a two-step procedure.
In the first, one assigns to a physical system a suitable $*-$algebra A of observables, the central structure of the algebraic theory, which encodes all structural relations between observables.
The second step consists of selecting a so called *algebraic state* which allows us to recover the interpretation of the elements of A as linear operators on a suitable Hilbert space.

Further conditions are necessary in order to select the physically meaningful states among all possible ones, namely has to be met the *Hadamard condition*.

In the next sections we review two of the possible realizations of the algebraic quantization scheme.

3.2 Quantization with Peierls Brackets

We are going to present a quantization procedure strictly defined for the class of classical theories for which the Peierls' construction make sense, *i.e.*:

1. Linear fields.

2. Lagrangian dynamics.

3. Based on globally-hyperbolic spacetime.

4. Dynamics ruled by a Green-hyperbolic, self-dual operator.

Notable examples falling in this category are the Klein-Gordon and the Proca Field Theory[7].

A Classical Step

The starting point for the realization of any quantum theory is always to provide a precise mathematical formalization of the corresponding classical theory. This step deals with the question of whether the procedure of quantization is applicable to the theory under examination.

A.a Kinematics It is encoded in the configuration bundle of the classical field.

1. One has to specify the base manifold M.
   Has to be a Globally-Hyperbolic spacetime.

2. Specify the fiber.
   The total Space E has to be at least a vector bundle. [1]

A.b Dynamics It is encoded in the motion operator $P : \Gamma^\infty(E) = \mathtt{C} \to \mathtt{C}$.
P must meet the following properties in order to carry out the procedure:

1. P has to be Green-hyperbolic.

2. Is P derived from a Lagrangian: $P = Q_{\mathscr{L}}$.

B PreQuantum Step

B.a Pairing Within the algebraic quantization scheme the choice of the *pairing* takes a crucial role. Basically this structure is a bilinear form on the space of kinematical configurations realized by assigning a fibrewise inner product.

[1] In the case where it is specified a non-zero spin structure, has to be imposed a compatibility condition for the configuration bundle of the system under examination. An explicit example for the Dirac Field can be found in [13].

B.a.I Assignment of a Inner Product The choice of the bundle inner product $< \cdot, \cdot >$ on E is the only discretionary condition and it is the basis of the entire procedure.
Even if its expression is generally suggested by the auxiliary structures defying the configuration bundle [2], for all practical purposes it can be considered as a free parameter. However the choice of a bilinear form is not completely arbitrary, the condition that must be met is the self-adjointness of operator P with respect to the correspondent pairing. Together with Green-hyperbolicity this condition guarantees that $\exists! E$ causal propagator and $E^{\dagger} = -E$.

Definition 3.1. We call *inner product* of the vector bundle E a smooth map:

$$< \cdot, \cdot >: E \times_M E \to \mathbb{R}$$

such that the restriction of $< \cdot, \cdot >$ to any fiber $E_p \times E_p$ is a non-degenerate bilinear form.

The symmetry properties determine the Bosonic/Fermionic character of the quantized theory:

Pairing	Observables linear form	Quantum Theory
symmetric	anti-symmetric	Bosonic
anti-symmetric	symmetric	Fermionic

B.a.II Pairing Definition The *pairing* between two sections is defined as:

$$(X, Y) = \int_M < X, Y >_x \, d\mu(x) \tag{3.1}$$

where $d\mu = d\mathrm{Vol}_\mu$ is the volume form induced by the metric and the orientation on M under the additional constraint:

$$dom\big((\cdot, \cdot)\big) = \big\{(X, Y) \in \Gamma^\infty(E) \times \Gamma^\infty(E) \;\big|\; < X, Y >_x \in L^1(M, \mu)\big\}$$

Some subdomains are of greater practical interest:

$$dom\big((\cdot, \cdot)\big) \supset \{(X, Y) \in \Gamma^\infty(E) \times \Gamma^\infty(E) \,|\, \mathrm{supp} X \cap \mathrm{supp} Y \text{ compact}\} \supset \Gamma_0^\infty(E) \times \Gamma^\infty(E)$$

In particular the pairing between compact supported sections and kinematic configurations is always well-defined.

Proposition 3.2.1. *The pairing between sections with compact support intersection is a non-degenerate bilinear form.*

Proof. Bilinearity of $(\cdot, \cdot)$ follows slavishly from that of the inner product $< \cdot, \cdot >$ and linearity of the Lebesgue integral.

As far as non-degeneracy is concerned, consider a section $\sigma \in \mathtt{C}$ such that

$$(\sigma, \tau) = 0 \quad \forall \tau \in \mathtt{C}$$

Then $<\sigma,\tau>_x$ vanishes almost everywhere on M. Yet σ and τ are smooth then :

$$<\sigma,\tau>_x = 0 \Leftrightarrow \mathrm{supp}(\sigma)\cap\mathrm{supp}(\tau)=\emptyset \qquad \forall\tau\in\mathtt{C}$$

in other words:

$$\mathrm{supp}(\sigma)=\emptyset \quad \Rightarrow \quad \sigma = 0$$

□

In order to carry out the procedure one must check that P is formally self-adjoint in respect to this pairing.

B.b Classical Observables The pairing constitutes the main ingredient to define a set $\mathscr{E}$ of suitable *classical observables*.

Remark 15. A good class of classical observables must be:

- A collection of linear functionals on `Sol`.
- This set must be in a one-to-one correspondence with a linear subspace of `C`.
- It must be sufficiently rich to separate the space of solutions:
  - There are sufficiently many observables to detect any information from any on-shell configuration.
  - Two on-shell configurations are the same if and only if every outcome under all the possible observables is the same.
  - The set contains enough functionals to represent the minimum number of measure processes necessary to distinguish every possible physical configuration.

The concrete construction is achieved in three steps.

B.b.I PreObservables They are defined as a class of *"off-shell"* functionals on `C`:

$$\mathscr{E}_0 \coloneqq \left\{F_f : \mathtt{C}\to\mathbb{R}; F_f(\phi) = (f,\phi)\forall\phi\in\mathtt{C} \,|\, f\in\Gamma_0^\infty(E)\right\}$$

This can be seen as the range of the linear map:

$$F : \Gamma_0^\infty \to \mathscr{E}_0$$

which associates to any section $f\in\Gamma_0^\infty(E)$ the linear functional $F_f(\cdot)=(f,\cdot):\mathtt{C}\to\mathbb{R}$. In other words:

$$\mathtt{C}\supset\Gamma_0^\infty(E)\ni f \xmapsto{F} F_f(\cdot):\mathtt{C}\to\mathbb{R}$$

Proposition 3.2.2. *The class of pre-observables satisfies the following properties:*

1. *$\mathscr{E}_0$ is a faithful representation of the linear space, i.e. the map $F:\Gamma_0^\infty\to\mathscr{E}_0$ is bijective.*

2. *$\mathscr{E}_0$ satisfies the separability condition, i.e. the class is rich enough to distinguish different off-shell configurations:*

$$\forall \phi, \psi \in \mathtt{C}\ \exists f \in \Gamma_0 \quad \textit{such that:} \quad F_f(\phi) \neq F_f(\psi)$$

Proof.

[Th. 1] Surjectivity is guaranteed by definition, every functional in $\mathscr{E}$ is constructed through the pairing with a compactly supported section.
Injectivity is proved ad absurdum. Consider two distinct sections $s, h \in \Gamma_0$ such that $F_s = F_h$. Then

$$(s, \phi) = F_s(\phi) = F_h(\phi) = (h, \phi) \quad \forall \phi \in \mathtt{C}$$

From the linearity of the pairing we have

$$(s - h, \phi) = 0 \quad \forall \phi \in \mathtt{C}$$

it follows from the non-degeneracy of the pairing that $s = h$.

[Th. 2] Ad absurdum again. Consider a pair $\phi, \psi \in \mathtt{C}$ of "inseparable" configurations:

$$(f, \phi) = (f, \psi) \qquad \forall f \in \Gamma_0^\infty(E)$$

From the linearity of the pairing we have

$$(f, \phi - \psi) = 0 \qquad \forall f \in \Gamma_0^\infty(E)$$

from the non-degeneracy of the pairing follow that $\phi = \psi$.

□

This proposition justifies the correspondence between classical pre-observables and compactly supported sections.

B.b.II Domain restriction of the Pre-Observables Consider now the domain restriction of the functionals in $\mathscr{E}_0$ from $\mathtt{C}$ to $\mathtt{Sol}$:

$$\mathscr{E}_0^{\mathtt{Sol}} := \left\{ F_f\big|_{\mathtt{Sol}} : \mathtt{Sol} \to \mathbb{R} \quad \middle| \quad F_f \in \mathscr{E}_0 \right\}$$

Call $r^{\mathtt{Sol}} : \mathscr{E}_0 \to \mathscr{E}_0^{\mathtt{Sol}}$ the map realizing the domain restriction on the elements of $\mathscr{E}_0$, *i.e.* :

$$r^{\mathtt{Sol}} : F_f \mapsto F_f\big|_{\mathtt{Sol}}$$

The map $F^{\mathtt{Sol}} := r^{\mathtt{Sol}} \circ F : \mathscr{E}_0 \mapsto \mathscr{E}_0^{\mathtt{Sol}}$ realizes a correspondence between $\Gamma_0^\infty(E)$ and a linear functional on $\mathtt{Sol}$.
We can conclude that:

$$\mathscr{E}_0^{\mathtt{Sol}} = F^{\mathtt{Sol}}\left(\Gamma_0^\infty(E)\right) = r^{\mathtt{Sol}} \circ F\left(\Gamma_0^\infty(E)\right)$$

Since $\mathtt{Sol} \subset \mathtt{C}$, this space continues to meet the separability condition but the correspondence with $\Gamma_0^\infty(E)$ is no longer injective:

Proposition 3.2.3.

$$\ker\left(F^{\mathtt{Sol}}\right) = P\left(\Gamma_0^\infty(E)\right) \coloneqq N$$

Proof.

[Th: 1] $\left(\ker(F^{\mathtt{Sol}}) \supseteq N\right)$
Since a l.p.d.o. can not enlarge the domain support, $P\tau \in dom(F) \quad \forall \tau \in \Gamma_0^\infty$ then the thesis is well-posed. Exploiting the definition and the self-adjointness of P we have:

$$F_{P\tau}(\sigma) = (P\tau,\sigma) = (\tau,P\sigma) = F_\tau(P\sigma) = F_\tau(0) = 0 \qquad \forall \sigma \in \mathtt{Sol}, \forall \tau \in \mathtt{C}$$

[Th: 2] $\left(\ker(F^{\mathtt{Sol}}) \subseteq N\right)$
Let be $\tau \in N$, then $F_\tau(\sigma) = 0 \ \forall \sigma \in \mathtt{Sol}$. Take $E = \left(G^- - G^+\right)$ the unique causal propagator of P. Then:

$$(E\tau,\sigma) = -(\tau,E\sigma) = 0 \qquad \forall \sigma \in \Gamma_0$$

From the non-degeneracy of the pairing it follows that $E\tau = 0 \Rightarrow \tau \in \ker(E)$. Considering 1.3.3 we have $\ker\left(E\big|_{\Gamma_0}\right) \equiv P\Gamma_0$

□

B.b.III Classical Observable class Due to the degeneracy of the map $F^{\mathtt{Sol}}$, $\mathscr{E}_0^{\mathtt{Sol}}$ can not be a good set of classical observables.
Being the kernel known, we can identify all the elements that posses the same corresponding functional:

$$[f] = \left\{f + Pg \quad | \, g \in \Gamma_0\right\}$$

It is natural then to define the classical observables as the quotient space:

$$\mathscr{E} \coloneqq \frac{\mathscr{E}_0^{\mathtt{Sol}}}{N}$$

Finally, the mapping between these equivalence classes can be easily defined : $\forall [f] \in \frac{\Gamma_0}{P\Gamma_0}$ we build the functional $F_{[f]} : \mathtt{Sol} \to \mathbb{R}$ such that:

$$F_{[f]}(\phi) = F_f(\phi) \qquad \forall \phi \in \mathtt{Sol}, \forall f \in [f]$$

This functional is well-defined, *i.e.* the expression is independent from the choice of the representative, only on $\mathtt{Sol}$. The reason is that if $\phi \in \mathtt{C}\backslash\mathtt{Sol}$, then $F_f(\phi)$ is different for each choice of the representative $f \in [f]$. This construction is said to " implement the on-shell condition at the level of functionals".

In conclusion the mapping:

$$\frac{\Gamma_0}{P\Gamma_0} \xrightarrow{\quad F \quad} \mathscr{E} = F\left(\frac{\Gamma_0}{P\Gamma_0}\right)$$

,between suitable equivalence classes and linear functionals on $\mathtt{Sol}$, guarantees:

- a faithful representation since F is bijective.
- the separability condition, a fortiori of separability properties of $\mathscr{E}_0^{\mathtt{Sol}}$.

From now on we will identify these two spaces:

$$\mathscr{E} \simeq \frac{\Gamma_0}{P\Gamma_0}$$

in view of the bijectivity of F.

B.c Symplectic structure Endow the space $\mathscr{E}$ just defined with a bilinear form τ constructed restricting the Peierls form.

B.c.I General Peierls Bracket Construction The starting point is the definition of the Peierls Brackets between any pair of Lagrangian densities.
We recall the main consequences ot the Peierls' argument:

- From the Lagragian Densities $\mathtt{Lag}(E)$ (see Def. 2.5) the Lagrangian functionals on $\mathtt{C}$ are defined as a regular distribution (see Def. 2.6).
- Considering a domain restriction from $\mathtt{C}$ to $\Gamma_0(E)$ this functional takes a simpler expression:
$$\mathscr{O}_{\mathscr{L}}(\phi_0) = \int_M \mathscr{L}(\phi_0)\, d\mu$$
- The effect of a Lagrangian density is defined on a smooth functional $B : \mathtt{C} \to \mathbb{R}$ as :
$$\mathbf{D}^{\pm}_{\chi} B(\phi_0) = \lim_{\epsilon \to 0} \Big(\frac{B(\phi^{\pm}_{\epsilon}) - B(\phi_0)}{\epsilon} \Big)$$
- Finally, for each pair of Lagrangian densities, the Peierls brackets (see Eq. (2.14)) are defined as:
$$\{\chi, \omega\}(\phi_0) \coloneqq \mathbf{D}^{+}_{\chi} F_{\omega}(\phi_0) - \mathbf{D}^{-}_{\chi} F_{\omega}(\phi_0)$$

B.c.II Brackets restriction to the Pre-Observables Once these brackets are restricted from the class of Lagrangian functionals to the classical observables $\mathscr{E}_0$, it assumes a very simple expression:

- Notice that each $\phi \in \Gamma_0^{\infty}(E)$ defines a regular Lagrangian density through the inner product:
$$\mathscr{L}_{\phi}[\cdot](x) = < \phi, \cdot >_x$$
- The associated Lagrangian functional is simply:
$$\mathscr{O}_{\mathscr{L}_{\phi}}(\cdot) = \int_M < \phi, \cdot >_x \, d\mu(x) = (\phi, \cdot) = F_{\phi}(\cdot)$$

- The corresponding Euler-Lagrange operator is:

$$Q_{\mathscr{L}_\phi} = \Big(\nabla_\mu\big(\frac{\partial \mathscr{L}}{\partial(\partial_\mu\phi)}\big) - \frac{\partial \mathscr{L}}{\partial\varphi}\Big) = -\frac{\partial \mathscr{L}_\varphi}{\partial\varphi} = -\frac{\partial}{\partial\varphi}(\phi,\varphi) = -\phi$$

  i.e. the operator maps the whole space $\mathtt{C}$ to $\{-\phi\}$.

- The effect operator between a pair of such Lagrangians $\mathscr{L}_\alpha, \mathscr{L}_\beta$ results:

$$\mathbf{D}^{\pm}_{\mathscr{L}_\alpha}\big(\mathscr{O}_{\mathscr{L}_\beta}\big)(\phi_0) = \mathbf{D}^{\pm}_{\mathscr{L}_\alpha}\big(F_\beta\big)(\phi_0) = \lim_{\epsilon\to 0}\Big(\frac{1}{\epsilon}\big(F_\beta(\phi^{\pm}_{\epsilon\mathscr{L}_\alpha}) - F_\beta(\phi_0)\big)\Big) = \tag{3.2}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}F_\beta(\phi^{\pm}_{\epsilon\mathscr{L}_\alpha} - \phi_0) = \big(\beta, \lim_{\epsilon\to 0}\frac{\phi_{\epsilon\mathscr{L}_\alpha} - \phi_0}{\epsilon}\big) = \tag{3.3}$$

$$= (\beta,\eta_\pm) = \big(\beta, -G^{\pm}(Q_{\mathscr{L}_\alpha}\phi_0)\big) = (\beta, G^{\pm}\alpha) \tag{3.4}$$

  $\forall\phi_0 \in \mathtt{Sol}$. In the second row we have exploited respectively the linearity and continuity of the functional. In the third row we have used the Green hyperbolicity and the explicit expression of $Q_{\mathscr{L}_\alpha}$. Notice that the dependence on the test solution ϕ_0 has disappeared.

- Finally the Peierls brackets expression can be stated as :

$$\tau\{[f],[h]\} = \{\mathscr{L}_f, \mathscr{L}_h\} = \big(f, (G^- - G^+)h\big) = \big(f, Eh\big) \qquad \forall f,h\,\Gamma_0^\infty(E) \tag{3.5}$$

The conditions of Green-hyperbolicity and formally self-adjoint are sufficient to guarantee the good definitions of τ.
Notice that the Lagrangian condition is ancillary. This has the purpose to justify the shape of the symplectic form on the space of classical observables as consequent from the Peierls bracket.
It is costumary in the recent literature (*e.g.* [15][7][17]) to overlook the origin of this object and to jump directly to the expression 3.5 assuming it as a definition. The benefit is that this object is expressed only in terms of the Green's operators and no longer presents any direct link to the disturbing Lagrangians. Therefore it can be extended to any Green-hyperbolic theory.

B.c.III Symplectic form on the Classical Observables Has to be noted that the brackets τ are degenerate on $\mathscr{E}_0$. If $f = Ph \in N$ we have:

$$\big(f, Eg\big) = (Ph, Eg) = (h, PEg) = (h,0) = 0 \qquad \forall g \in \Gamma_0^\infty$$

where in the last equality has been used proposition 1.3.2.
τ descends naturally to the equivalence classes of $\mathscr{E}$:

$$\tau\big([\phi],[\psi]\big) := \tau(\phi,\psi) \tag{3.6}$$

Proposition 3.2.4. *The brackets τ satisfy the following properties:*

1. *Definition (Eq. 3.6) does not depend from the representative of the class.*

2. *It is a symplectic form (bilinear, antisymmetric, non-degenerate) in the Bosonic case while it is a scalar product in the Fermionic case.*

Proof.

[Th. 1] τ is degenerate on $\mathscr{E}_0$ since:

$$\big(f, Es\big) = (Ph, Es) = (h, PEs) = (h, 0) = 0 \qquad \forall s \in \Gamma_0^\infty\,,\ \forall f = Ph \in N$$

Then, fixed $g \in \mathscr{E}_0$, we have that the value of $\tau(f,s)$ is the same for each representative $f \in [f]$.

[Th. 2] Bilinearity is guaranteed by the linearity of the pairing and of the Green's operators.
Non-degeneracy of τ follows from that of the pairing:

$$(f, Eh) = 0\, \forall f \in \mathscr{E}_0 \Leftrightarrow Eh = 0 \Leftrightarrow h = Pf$$

but from definition of $\mathscr{E}$ we have $[Pf] = [0]$.
Antisymmetry/symmetry of τ follows from symmetry (resp. antisymmetry) of the Bosonic (resp. Fermionic) bilinear form $< \cdot, \cdot >$:

$$(f, Eh) = \big(f, (G^- - G^+)h\big) = \big((G^+ - G^-)f, h\big) = -(Ef, h) = \mp(h, Ef)$$

□

The pair $(\mathscr{E}, \tau)$ is the symplectic space of observables describing a classical field theory on the globally hyperbolic spacetime M and it is the starting point for the quantization scheme that we shall discuss in the next section. This structure meets two remarkable physical properties:

Theorem 3.2.5. *Consider a globally hyperbolic spacetime M and let $(\mathscr{E}, \tau)$ be the symplectic space of classical observables defined above.*
The following properties hold true:

1. ***Causality axiom*** *The symplectic structure vanishes on pairs of observables localized in causally disjoint regions:*

$$\tau\big([f], [h]\big) = 0 \qquad \forall f, h \in \Gamma_0(E) \;\big|\; supp(f) \cap \boldsymbol{J}_M\big(supp(h)\big) = \emptyset = \boldsymbol{J}_M\big(supp(f)\big) \cap supp(h)$$

2. ***Time-Slice axiom*** *For all $O \subset M$ globally hyperbolic open neighbourhood of a spacelike Cauchy surface Σ for M [2], the map $L : \mathscr{E}(O) \to \mathscr{E}(M)$ which associates to an equivalence class $f \in \Gamma_0(0)/P\Gamma_0(O)$ an equivalence class of its extension by 0 to the whole spacetime:*

$$L[f] = [f] \qquad \forall f \in \Gamma_0(O)$$

is an isomorphism between symplectic spaces.

[2] Namely O is: an open subset providing a globally hyperbolic spacetime $O = (O, \mathfrak{g}|_O, \mathfrak{o}|_O, \mathfrak{t}|_O)$ and a neighbourhood of $\Sigma \in \mathscr{P}_C$ containing all causal curves for M whose endpoints lie in O.

Proof.

[Ax. 1] Consider a pair $f, h \in \Gamma_0(E)$, we have:

$$\tau\big([f],[h]\big) = \big(f, Eh\big) = F_f\big(Eh\big)$$

from Definition1.32 of Green operators it follows:

$$\mathrm{supp}(Eh) \subseteq \mathbf{J}_M(\mathrm{supp}(h)$$

Thus if the two pre-observables are localized in causally disjoint regions $\tau\big([f],[h]\big) = 0$ since it correspond to the pairing of two sections with disjoint supports.

[Ax. 2] The same construction applied to M and to O yields the symplectic spaces $(\mathscr{E}(M), \tau_M)$ and respectively $(\mathscr{E}(O), \tau_O)$. Any $f \in C_0^\infty(O)$ can be extended as 0 to $M \setminus O$, hence identifying a compactly supported, smooth function on M, which we indicate still with f.
These observations entail that the map $L : \mathscr{E}_0 \to \mathscr{E}_M$ specified by $L[f] = [f] \forall f \in \mathscr{E}_0$ is well-defined.

Note that L is linear and that it preserves the symplectic form. In fact, given $[f], [h] \in \mathscr{E}_0$, one has:

$$\tau_M\big(L[f], L[h]\big) = \int_M < f, Eh >_x d\mathrm{vol}_M = \int_O < f, Eh >_x d\mathrm{vol}_O = \tau_O\big([f],[h]\big)$$

where the restriction from M to O in the domain of integration is motivated by the fact that, per construction, $f = 0$ outside O.

Being a symplectic map, L is automatically injective.
In fact, given $[f] \in \mathscr{E}_O$ such that $L[f] = 0$, one has

$$\tau_O([f],[h]) = \tau_M(L[f], L[h]) = 0 \qquad \forall\, [h] \in \mathscr{E}_0$$

and the non-degeneracy of τ_O entails that $[f] = 0$.

The symplectic map L is also surjective.
For each $f \in \Gamma_0(M)$ we look for$f' \in \Gamma_0(M)$ with support inside O such that $[f'] = [f]$ in $\mathscr{E}_M$. Recalling that O is an open neighborhood of the spacelike Cauchy surface Σ and exploiting the usual spacetime decomposition of M, see Theorem 1.2.1 , one finds two spacelike Cauchy surfaces Σ_+, Σ_- for M included in O lying respectively in the future and in the past of Σ.
Let $\{\chi^+, \chi^-\}$ be a partition of unity subordinated to the open cover $\{\mathbf{I}^+_M(\Sigma_-), \mathbf{I}^-_M(\Sigma_+)\}$ of M. By construction the intersection of the supports of χ^+ and of χ^- is a time-like compact region both of O and of M. Since $PEf = 0$, $\chi^+ + \chi^- = 1$ on M and recalling the support properties of E, it follows that

$$f' = P\big(\chi^- Ef\big) = -P\big(\chi^+ Ef\big)$$

is a smooth function with compact support inside O. Furthermore, recalling also the identity $PG^- f = f$, one finds

$$f' - f = P(\chi^- G^- f) - P(\chi^- G^+ f) - P(\chi^+ G^- f) - P(\chi^- G^- f) = -P(\chi^- G^+ f - \chi^+ G^- f)$$

The support properties of both the retarded and advanced Green operators $G^{\pm}$ entail that $-\chi^{-}G^{+}f-\chi^{+}G^{-}f$ is a smooth function with compact support on M. In fact $\mathrm{supp}\left(\chi^{\mp}\right)\cap\mathrm{supp}\left(G^{\pm}f\right)$ is a closed subset of $\mathbf{J}^{m}_{m}p(\Sigma_{\pm})\cap\mathbf{J}^{\pm}_{m}(\mathrm{supp}(f))$, which is compact.
This shows that $\left(f'-f\right)\in P\left(\Gamma_0(M)\right)\subset N$, as proved in proposition 3.2.3. Therefore we found $\left[f'\big|_{O}\right]\in\mathscr{E}(O)$such that $L\left[f'\big|_{O}\right]=[f]$ showing that the symplectic map L is also surjective.

□

C Second Quantization Step

The next step is to construct a quantum field theory out of the classical one, the content of which is encoded the symplectic space $(\mathscr{E},\tau)$. The so-called algebraic approach can be seen as a two-step quantization scheme: in the first one identifies a suitable unital $*-$algebra encoding the structural relations between the observables, such as causality and locality, while, in the second, one selects a state, that is a positive, normalized, linear functional on the algebra which allows us to recover the standard probabilistic interpretation of quantum theories via the Gelfand-Neimark-Segal (GNS) theorem.

C.a Quantum Observables Algebra The crux of the algebraic quantization scheme it is the assignment of a suitable algebras of quantum observables.
Axiomatically we require for the set of quantum observables the following structure:

Definition 3.2 (Quantum Algebra)**.** We call *algebra of quantum observables* associated to the classical field system $(\mathscr{E},\tau)$ the unital $*-$algebra $A=\left((A,\mathbb{C}),\cdot,*\right)$ generated over $\mathbb{C}$ by the symbols

$$\{\mathbb{I}\}\bigcup\{\Phi\big([f]\big)\;|\;[f]\in\mathscr{E}\}$$

such that:

1. The generators are linearly independent:

$$\Phi\big(a[f]+b[s]\big)=a\Phi([f])+b\Phi([s])\qquad\forall[f],[s]\in\mathscr{E},\ \forall a,b\in\mathbb{R}\tag{3.7}$$

2. The generators are *formally self-adjoint* in the sense that:

$$\big(\Phi([f])\big)^{*}=\Phi([f])\qquad\forall[f]\in\mathscr{E}\tag{3.8}$$

3. The (anti-) commutation relations extrapolated from the classical τ are replicated on A:

$$\big[\Phi([f]),\Phi([g])\big]_{\mp}=\Phi([f])\cdot\Phi([g])\mp\Phi([g])\cdot\Phi([f])=i\tau\big([f],[g]\big)\mathbb{I}\tag{3.9}$$

where the sign $\mp$ depend respectively on the anti-symmetry and symmetry of the form τ.

A concrete realization is achieved in four steps.

C.a.I Construction of the generated vector space A It is generated by the symbols $\{\mathbb{1}\} \cup \{\Phi([f])\}_{[f]\in\mathscr{E}}$ namely A can be obtained as a $\mathbb{C}$-linear combination of $\mathbb{1}$ and a finite number of products by elements like $\Phi([f])$.
I.e.:

$$A = \mathrm{span}\Big(\{\mathbb{I}\} \bigcup_{n<\infty} D_n\{\Phi\big([f]\big) \,\big|\, [f] \in \mathscr{E}\}\Big)$$

where $D_n(I)$ is the of the *ordered product with repetition* of n elements picked from set I. In other words every element of A can be expressed as a polynomial in the generators with coefficients in $\mathbb{C}$, it is implied that elements $\mathbb{I}$ acts as unital element in the algebra.
This is nothing more than the *Free space* generated by $\mathbb{I}$ and $\{\mathscr{E}^k = \underbrace{\mathscr{E}\times\ldots\times\mathscr{E}}_{\text{k times}}\}_{k<+\infty}$, in this term the symbol Φ can be intended as the map which associates to any ordered selection (with repetition) of elements in set $\mathscr{E}$ a linear generator of a suitable vector space.
Recalling that the free space is the main ingredient in the definition of tensor product of vector spaces, we can concretely realize the vector space underlying the algebra mimicking what is done for the tensor space. Through the correspondence:

$$\Phi([f])\cdot\Phi([g])\cdot\ldots \quad \leftrightsquigarrow \quad [f]\otimes[g]\otimes\ldots$$

we define the *Universal Tensor Algebra*:

$$A := \bigoplus_{k\in\mathbb{N}_0} \mathscr{E}_{\mathbb{C}}^{\otimes k}$$

direct sum, for all k natural finite numbers, of $\mathscr{E}_{\mathbb{C}}^{\otimes k}$ the k-fold tensor power of the complexification $\mathscr{E}_{\mathbb{C}}$ of the space of classical observables. We have set $\mathscr{E}_{\mathbb{C}}^{\otimes 0} = \mathbb{C}$.
Thus, the elements of the space A are explicitly the sequences $\{V_k \in \mathscr{E}_{\mathbb{C}}^{\otimes k}\}_{k\in\mathbb{N}_0}$ with only a finite number of non-zero entries. Every entry V_k is a linear combination with complex coefficients of elements in the form $[f_1]\otimes\ldots\otimes[f_k]$ with $[f_i]\in\mathscr{E}$.
i.e. $V_k \in \mathrm{span}\{[f_1]\otimes\ldots\otimes[f_k] \,|\, [f_i]\in\mathscr{E}\}$.

C.a.II Endow A with a product This space can be equipped with the structure of an algebra structure as follows:

concretely the symbol Φ giving the generator of A can be seen as the operator:

$$\Phi : \mathscr{E} \to A \qquad : \qquad \Phi([f]) = \{0, [f], 0, \ldots\}$$

and $\mathbb{I} = \{1, 0, \ldots\}$.

Thus a product operator $\cdot : A\times A \to A$, can be given through the action on each element in the sequence:

$$\{u_k\}\times\{v_k\} \mapsto \{w_k = \sum_{i+j=k} u_i \otimes v_j\}$$

The definition is well-posed since:

$$\{0,0,[f]\otimes[g],0,\ldots\} = \Phi([f])\cdot\Phi([g]) = \{0,[f],0,\ldots\}\cdot\{0,[g],0,\ldots\} = \{0,0,[f]\otimes[g],0,\ldots\}$$

The first equivalence follows from the concrete construction of A, the second by definition of Φ and the third by definition of $\cdot$.

This construction satisfies automatically condition 3.7 in definition 3.2:

$$\Phi(a[f]+b[g]) = \{0, a[f]+b[g], 0, \ldots\} = a\{0,[f],0,\ldots\}+b\{0,[g],0,\ldots\} = a\Phi([f])+b\Phi([g])$$

We conclude that $(A,\cdot)$ constitutes an algebra over $\mathbb{C}$ appropriate to our purpose.

C.a.III Construction of the Involution map $*$ It can easily be defined an operation of involution $* : A \to A$ stating the action on the generators:

$$\{\underbrace{0,\ldots,0}_{k\text{times}},[f_1]\otimes[f_2]\otimes\ldots\otimes[f_k],0,\ldots\} \overset{*}{\mapsto} \{\underbrace{0,\ldots,0}_{k\text{times}},[f_k]\otimes[f_{k-1}]\otimes\ldots\otimes[f_1],0,\ldots\}$$

for all $[f_1],\ldots,[f_k] \in \mathscr{E}$, and extending it by anti-linearity to the whole of A:

$$\big(\alpha x + y\big)^* := \bar{\alpha} x^* + y^* \qquad \forall x, y \in A\, \forall \alpha \in \mathbb{C}$$

Involution $*$ implements the $*-$algebra properties:

$$\big(\Phi([f])\cdot\Phi([g])\big)^* = \{0,0,[f]\otimes[g],0,\ldots\}^* = \{0,0,[g]\otimes[f],0,\ldots\}^* = \big(\Phi([g])\cdot\Phi([f])\big) \tag{3.10}$$

where the identity element is represented by $\mathbb{I}$:

$$\mathbb{I}^* = \{1,0,\ldots\}^* = \{1,0,\ldots\} = \mathbb{I} \tag{3.11}$$

hence A is a unital $*-$algebra, implementing relation 3.8 too:

$$\big(\Phi([f])\big)^* = \{0,[f],0,\ldots\}^* = \{0,[f],0,\ldots\} = \Phi([f])$$

C.a.IV CCR implementation The $*-$algebra A already "knows" of the dynamics of the field since this is already encoded in $\mathscr{E}$, however, the canonical commutation relations (CCR) 3.9 are still missing. A way to implement this relation is through the definition of an equivalence relation:

$$\Phi([f])\cdot\Phi([g]) \simeq \pm\Phi([g])\cdot\Phi([f]) + i\tau\big([f],[g]\big)\mathbb{I} \tag{3.12}$$

for all $[f],[g] \in \mathscr{E}$.
Practically this can be implemented via the quotient space relative to the *two-sided ideal of* A generated by

$$I' = \big\{\Phi([f])\cdot\Phi([g]) \mp \Phi([g])\cdot\Phi([f]) - i\tau\big([f],[g]\big)\mathbb{I} \quad \big| \quad [f],[g]\in A\big\}$$

i.e. the subalgebra

$$I = \mathrm{span}\big(\{a\cdot x\cdot b \;\big|\; a,b\in A; x\in I'\}\big)$$

The quotient algebra $A := A/I$ is a set consisting of equivalence classes such that $[0] = I$. Thus it is a proper quantum algebra according to definition 3.2.

Remark 16. Under suitable conditions(namely after the choice of a Poincaré vacuum state) our quantization procedure perfectly agrees with the standard textbook quantization involving creation and annihilation operators (see for example [31]). In fact, assuming that M is Minkowski spacetime, one can relate directly our algebraic approach to the one more commonly used by means of an expansion in Fourier modes of the fundamental quantum fields $\Phi([f])$, which generate the algebra A.

The properties of the classical obsevables presented in theorem 3.2.5 have counterparts at the quantum level as shown by the following theorem:

Theorem 3.2.6. *Consider a globally hyperbolic spacetime M and let A be the unital $*-$algebra of quantum observables defined above.*
The following properties hold:

- ***Causality axiom*** *Elements of the algebra A localized in causally disjoint regions commute:*

$$\Phi([f])\cdot\Phi([g]) = \Phi([g])\cdot\Phi([f]) \qquad \forall f,h\in\Gamma_0(E)\;\big|\; supp(f)\cap \boldsymbol{J}_M\big(supp(h)\big) = \emptyset$$

- ***Time-Slice axiom*** *For all $O\subset M$ globally hyperbolic open neighborhood of a spacelike Cauchy surface Σ for M denote with A_M and with A_O the unital $*-$algebras of observables for the field system respectively over M and over O. Then the unit-preserving *-homomorphism $\Phi(L) : A_O \to A_M$, $\Phi([f]) \mapsto \Phi(L[f])$ is an isomorphism of $*-$algebras, where L denotes the symplectic isomorphism introduced in Theorem 2.*

Proof. We omit the proof, see for example [8][Theorem 4] □

C.b Algebraic State and Hilbert state representation The power of the algebraic approach lies in its ability to separate the algebraic relations of quantum fields from the Hilbert space representations of these relations and thus in some sense to treat all possible Hilbert space representations at once.
The usual interpretation of the physical observables as linear operators on a suitable Hilbert space is, however, not lost. We can recover the probabilistic interpretation of a quantum theory through the introduction of the notion of *algebraic state* taking advantage of the GNS theorem. Further conditions are necessary in order to select the physically meaningful states among all possible ones .

C.b.I GNS reconstruction For any unital $*-$algebra A, which is not necessarily the field algebra of Definition3.2, we generalize the notion of (quantum) state as follows:

Definition 3.3 (Algebraic State)**.** We call an *(algebraic) state* over A the $\mathbb{C}-$linear functional $\omega : A \to \mathbb{C}$ such that:

· it is positive:

$$\omega(a^* a) \geq 0 \qquad \forall a\in A$$

· it is normalized:

$$\omega(\mathbb{I}) = 1$$

Selecting an algebraic state allows us to recover, via the GNS theorem, the standard probabilistic interpretation of quantum theories, *i.e.* the representation of elements of A as linear operators on a suitable Hilbert space.

To make this statement clear we have to recall the definition of a representation of a $*$-algebra:

Definition 3.4. We call $*-$*representation of* A on the Hilbert space H a linear map $\pi : A \to \mathscr{L}(D)$, where $D \subset H$ is a linear dense subspace and $\mathscr{L}(D)$ is the space of linear operator, such that:

· it is *product preserving* :

$$\pi(a)\pi(b) = \pi(ab) \qquad \forall\, a, b \in A$$

· it preserves the unit element:

$$\pi(\mathbb{I}) = \mathrm{id}_D$$

· it preserves the star involution:

$$\pi(a)^\dagger\Big|_D = \pi(a^*) \qquad \forall\, a \in A$$

where † denotes the Hermitian adjoint operation in H.

A representation is said to be *faithful*, if the map π is bijective.

As a preliminary result, we can show that, whenever we represent a $*-$algebra on a Hilbert space via linear bounded operators, we can automatically associate an algebraic state to any unit vector in H[3]

Proposition 3.2.7. *Let A be any topological unital $*-$algebra and let $\pi : A \to \mathscr{B}(H)$ be a faithful, strongly continuous $*$-representation, where $\mathscr{B}(H)$ is the space of bounded operator on H.*
Then for any $\psi \in H$ of unit norm, the functional $\omega_\psi : A \to \mathbb{C}$ defined by:

$$\omega_\psi(a) \coloneqq \langle \psi | \pi(a) \psi \rangle$$

is an algebraic state on A.

Proof. Per construction ω_ψ is linear and continuous since π is linear and strongly continuous.
$\omega_\psi(\mathbb{I}) = 1$ follows from $\|\psi\|_H = 1$ and $\pi(\mathbb{I}) = \mathrm{id}_D$, being π a representation. To conclude we notice that:

$$\omega_\psi(a^* a) \coloneqq \langle \psi | \pi(a^* a) \psi \rangle = \langle \psi | \pi(a)^* \pi(a) \psi \rangle = \|\pi(a)\psi\|_H^2 \geq 0$$

exploiting the $*-$algebra product preserving properties of π. □

[3] We denote the scalar product on the Hilbert space as $\langle \cdot | \rangle$.

A key role is played by the following theorem:

Theorem 3.2.8 (GNS construction.)**.** *Let A be a complex unital $*-$algebra and $\omega : A \to \mathbb{C}$ is a state, then:*

1. *There exists a triple $(D_\omega, \pi_\omega, \Psi_\omega)$, where:*

   - *$D_\omega \subset H_\omega$ is a dense linear subspace of a complex Hilbert space.*
   - *$\pi_\omega : A \to \mathscr{L}(D_\omega)$ is a $*-$representation of A on H_ω with domain D_ω.*
   - *$\Psi_\omega \in D_\omega$ is a unit vector.*

   such that:

   (a) $\Psi_\omega(A) = \{\pi_\omega(a)\Psi_\omega \mid a \in A\}$ *is dense in* H_ω.

   (b) $\omega(a) = \langle \Psi_\omega | \pi_\omega(a)\Psi_\omega \rangle \qquad \forall a \in A$

2. *The GNS triple is determined up to an equivalence,*
   namely, if $\left(D'_\omega, \pi'_\omega, \Psi'_\omega\right)$ is a second quadruple satisfying the preceding conditions, there exists a surjective isometry $U : H_\omega \to H'_\omega$ such that:

   (a) $U\Psi_\omega = \Psi'_\omega$

   (b) $UD_\omega = D'_\omega$

   (c) $U\pi_\omega(a)U^{-1} = \pi'_\omega(a) \qquad \forall a \in A$

Proof. We omit the proof, see for example [28][Theorem 1] or [8][Theorem 4.4] for the stronger version of the theorem regarding the case where A is a C^*-algebra, *i.e.* a $*-$algebra endowed with a norm compatible with multiplication. □

C.b.II Hadamard condition The definition of an algebraic state reviewed in the previous subsection is too general. In fact most of the algebraic states cannot be considered physically acceptable as they do not allow a proper definition of the Wick polynomials, which constitute the basic objects of perturbation theory.

It is possible to identify , therefore, a distinguished subclass of states, called *Hadamard states*, which avoid these pathologies and are unanimously recognized as being the only physically significant ones.
The Hadamard condition seems to be the natural generalisation of the energy-positivity condition of the Minkowski vacuum state which encodes the UV properties of physical states in QFT. Is then a good selection criterion for physical states in quantum field theory in order to discuss perturbatively interacting theories.

Considering the scope of this thesis, we will not discuss further the details of this important subclass of physical states. For a brief account on the formulation of this property in terms of *microlocal analysis*, see for example Ref. [8] [28].

3.3 Quantization by Initial Data

The presentation of this quantization procedure is essentially based on the book of Wald (Ref. [42]).
Unlike the previous algorithm, in this case we do not exploit the condition of existence and uniqueness of Green's Operator (Green hyperbolicity) but rather the well-posedness of the initial data problem on any Cauchy surface (PDE hyperbolicity) . This quantization procedure is then well defined only for the class of classical theories for which the Cauchy problem makes sense, *i.e.*:

1. Linear fields.[4]

2. Based on a globally-hyperbolic spacetime.

3. The dynamics is ruled by a PDE hyperbolic operator.

There are many systems which are quantizable according to both procedures, the most canonical example is again the Klein-Gordon scalar field [42].

A Classical Step

The procedure is slightly different from the previous one. The starting point is still the identification of the proper mathematical formulation of the field system under consideration, *i.e.* the identification of the pair (E, P).
This time the main role is attributed to the space of the initial data Data and to the map **s** providing the corresponding unique solution, as presented in section 2.2.1.

N.B. : For the sake of simplicity we restrict ourselves to dynamical operators only of the second order.

B PreQuantum Step

The step in which, essentially, all the procedures in the scheme of algebraic quantization differ radically is in the assignment on the PreQuantum structures. In this case the strategy consists of mimicking the geometric mechanics picture for a linear point particle system (see Section 2.3.1).

B.a Pairing Even if in the two procedures are involved different PreQuantum structures, the *pairing* plays again a key role. The construction is the same as the one in Paragraph B.a and therefore we shall not repeat it in details.

[4] Not necessarily Lagrangian.

B.b Classical Phase Space construction We define the phase space as a subset of Data:

Definition 3.5 (Classical Phase Space)**.** We call *Classical Phase Space* the vector subspace of Data composed of compactly supported smooth initial dataj:

$$\mathscr{M}(\Sigma) := \Gamma_0^\infty(\Sigma) \times \Sigma_0^\infty(\Sigma) \subset \mathsf{Data}(\Sigma)$$

As pointed by Wald in [42] this choice is essentially justified a posteriori as it allows to define a good symplectic form and a well-defined observable space in a minimal way.

Proposition 3.3.1. *The map* $\mathbf{s}$ *is a bijection from* $\mathscr{M}(\Sigma)$ *to* $\mathtt{Sol}_{sc}$ *the space of spacelike compact solutions.*

Proof. Consider a pair $f_0, f_1 \in \Gamma_0^\infty(\Sigma)$. From the support condition for the solution of the Cauchy problem:

$$\mathrm{supp}\Big(\mathbf{s}\big([f_0,f_1]\big)\Big) \subseteq \mathbf{J}_M\Big(\mathrm{supp}(f_0) \cup \mathrm{supp}(f_1)\Big) \tag{3.13}$$

it follows that the support of $\mathbf{s}\big([f_0,f_1]\big)$ is included in the domain of dependence of the compact, achronal set $\Big(\mathrm{supp}(f_0) \cup \mathrm{supp}(f_1)\Big)$. Hence it is spacelike compact,
i.e.

$$\mathbf{s}\big(\mathscr{M}(\Sigma)\big) \subseteq \mathtt{Sol}_{sc}$$

The converse is also true since for all $\gamma \in \mathtt{Sol}_{sc}$ and for any Cauchy surface Σ it follows $\mathrm{supp}\big(\gamma\big|_\Sigma\big)$ is compact. □

Therefore, similar to what was concluded in paragraph 2.9, we have: $\mathbf{s}(\mathscr{M}(\Sigma)) = \mathtt{Sol}_{sc} \forall \sigma \in \mathscr{P}_C(M)$ hence

$$\mathtt{Sol}_{sc} \simeq \frac{\bigsqcup\limits_{\Sigma \in \mathscr{P}_C(M)} \mathsf{Data}(\Sigma)}{\sim} := \mathscr{M} \tag{3.14}$$

where $\sim$ is such that:

$$(f_0,f_1)|_\Sigma \sim (g_0,g_1)|_{\Sigma'} \Leftrightarrow \mathbf{s}(f_0,f_1) = \mathbf{s}(g_0,g_1)$$

B.c Symplectic Structure on the Phase Space Remembering that for a classical linear system, with second order dynamical equations, the symplectic structure can be defined globally on the phase space, we define a bilinear form on $\mathscr{M}$ mimicking eq. 2.5:

Definition 3.6 (Initial data symplectic form)**.**

$$\Omega : \mathscr{M}(\Sigma) \times \mathscr{M}(\Sigma) \to \mathbb{C} \qquad : \qquad \Omega\Big\{[f_0,f_1],[g_0,g_1]\Big\} = \int_\Sigma d\Sigma\Big((f_1,g_0) - (f_0,g_1)\Big)$$

where $d\Sigma$ is the volume form naturally induced by the spacetime metric (and corresponding measure $d\mu(x) = d\mathrm{vol}_M$) on the subspace Σ.

Has to be noted that at the basis of this construction there is the choice of a Cauchy surface Σ, in this sense this second procedure is *"non-covariant"* in contrast to the Peierls' algorithm.

Proposition 3.3.2. *Let $\Omega : \mathscr{M}(\Sigma) \times \mathscr{M}(\Sigma) \to \mathbb{C}$ the function defined above, it satisfies the following properties:*

1. *bilinear.*
2. *antisymmetric if $< \cdot, \cdot >$ is symmetric.*
3. *non degenerate:*

$$\Omega\big([f_0, f_1], [h_0, h_1]\big) = 0 \forall [f_0, f_1] \in \mathscr{M}(\Sigma) \Leftrightarrow [h_0, h_1] = [0, 0]$$

Proof.

[Th. 1] Bilinearity follows directly from that of the bundle inner product and of the Lebesgue integral.

[Th. 2] Given the symmetry of the inner product : $< f, h >= \pm < h, f >$ it follows that:

$$< f_1, h_0 > - < f_0, h_1 >= \mp\big(< h_1, f_0 > - < h_0, f_1 > \big)$$

I.e., Ω has opposed symmetry property respect to $< \cdot, \cdot >$.

[Th. 3] Since the null property is valid for every $[f_0, f_1] \in \mathscr{M}(\Sigma)$ it is also valid for data $[h_1, 0]$ and $[0, h_0]$. These lead to:

$$\int_\Sigma \| h_i \|^2 \, d\Sigma = 0 \to h_i = 0$$

for $i \in 0, 1$, thus $[h_0, h_1] = [0, 0]$.

□

Considering the one-to-one correspondence between $\mathscr{M}$ and $\mathtt{Sol}_{sc}$ we can transport this function on the space of spacelike compact solutions:

$$\sigma_\Sigma\{\varphi, \psi\} := \Omega\{[\varphi|_\Sigma, \nabla_n \varphi|_\Sigma], [\psi|_\Sigma, \nabla_n \psi|_\Sigma]\}$$

where n denotes the unit (future directed) normal vector to Σ.

Except for some particular cases this definition is strictly dependant from the chosen Cauchy surface Σ. Generally, for any pair of solutions $\varphi, \psi \in \mathtt{Sol}_{sc}$ and for any pair of Cauchy surfaces $\Sigma, \Sigma' \in \mathscr{P}_C(M)$, we have:

$$\begin{aligned} \sigma_\Sigma(\psi, \phi) &= \int_\Sigma < \nabla_n \phi, \psi > - < \phi, \nabla_n \psi > d\Sigma = \\ &\neq \int_{\Sigma'} < \nabla_n \phi, \psi > - < \phi, \nabla_n \psi > d\Sigma' = \sigma_{\Sigma'}(\psi, \phi) \end{aligned}$$

In this term the phase space $\mathscr{M}(\Sigma)$ is *non-covariant.*

Example 3. The Klein-Gordon scalar field (E,P) where:

$$\begin{aligned} E &= M \times \mathbb{R} \\ P &= \Box_M + m^2 + \xi R \end{aligned} \tag{3.15}$$

is one of such cases where it can be proven the independence of the phase space construction from the choice of Σ.
Let be $\varphi, \psi \in \mathtt{Sol}_{sc}$ two spacelike compact solutions, from these one can construct a "current":

$$J_\mu := \varphi \cdot \nabla_\mu \psi - \psi \cdot \nabla_\mu \varphi$$

that is a tangent vector field on the spacetime manifolds M.
Exploiting the equations of motion, it follows that J_μ is a conserved current:

$$\begin{aligned} \nabla^\mu J_\mu &= \nabla^\mu \varphi \nabla_\mu \psi - \nabla^\mu \psi \nabla_\mu \varphi + \varphi \nabla^\mu \nabla_\mu \psi - \psi \nabla^\mu \nabla_\mu \varphi = \\ &= \varphi\big(P-\kappa\big)\psi - \psi\big(P-\kappa\big)\varphi = 0 \end{aligned} \tag{3.16}$$

where κ is the constant factor in operator P ruling the dynamics.
Now consider two Cauchy surfaces Σ, Σ' such that $\mathbf{J}^+_M(\Sigma) \supset \Sigma'$ and denote I an open set such that:

$$\mathrm{supp}\big(\varphi|_\Sigma\big) \cup \mathrm{supp}\big(\psi|_\Sigma\big) \subset I \subset \Sigma$$

Denote as D the region between I and Σ':

$$D := \big(\mathbf{J}^+_M(I) \cap \mathbf{J}^-_M(\Sigma')\big)$$

In virtue of eq 3.16 follows:

$$0 = \int_D \nabla^\mu J_\mu = \int_{\partial D} n^\mu J_\mu = \big(\int_{\Sigma'} d\Sigma - \int_\Sigma d\Sigma\big)\big(\varphi \cdot \nabla_n \psi - \psi \cdot \nabla_n \varphi\big)$$

where in the second equivalence we applied *Stokes Theorem* and n denotes the outgoing normal vector.
In others words:

$$\sigma_\Sigma(\varphi,\psi) = \sigma_{\Sigma'}(\varphi,\psi) \qquad \forall \varphi,\psi \in \mathtt{Sol}_{sc},\ \forall \Sigma,\Sigma' \in \mathscr{P}_C(M)$$

B.d Poisson space of linear observables Exactly as shown in section 2.3.1 it is possible to define the set of classical observables through the symplectic form on $\mathscr{M}(\Sigma)$. The key role is taken by the linear observables:

$$\mathscr{E}_{Lin} := \Big\{\Omega\big([\varphi,\pi],\cdot\big) : \mathscr{M}(\Sigma) \to \mathbb{R} \quad | \, [\phi,\pi] \in \mathscr{M}(\Sigma)\Big\} \simeq \mathtt{Sol}_{sc}$$

The symplectic form is slavishly transferred from $\mathscr{M}(\Sigma)$ to $\mathscr{E}_{Lin}$:

$$\Big\{\Omega\big([\phi_0,\pi_0],\cdot\big), \Omega\big([\phi_1,\pi_1],\cdot\big)\Big\} := -\Omega\big([\phi_0,\pi_0],[\phi_1,\pi_1]\big)$$

C Second Quantization Step

The pair $(\mathscr{E}_{Lin}, \Omega)$ takes the place of $(\mathscr{E}, \tau)$ in the quantization procedure. Once the classic symplectic manifold is identified, the concrete construction of the quantum algebra is accomplished as before. We shall not repeat the construction it in details.

3.4 Link between the two realizations

To a system compatible with both the quantization procedures, for example in the case when the operator P is normally hyperbolic, we associate two apparently different symplectic spaces : $(\mathscr{E}, \tau)$ and $(\mathscr{E}_{Lin}, \Omega)$.

A crucial result that can be generally proved is that the space of linear functional $\mathscr{E}_{Lin}$ and the space of classical observables $\mathscr{E}$ are isomorphic. According to that the two procedures differ only in the attribution of the corresponding symplectic form.

The linear isomorphism $\mathscr{E} = \Gamma_0^\infty / \big(P(\Gamma_0^\infty)\big) \simeq \mathtt{Sol}_{sc} \simeq \mathscr{E}_{Lin}$ follows directly from the next theorem:

Theorem 3.4.1. *Let M be a globally hyperbolic spacetime. Consider a vector bundle E over M and a Green-hyperbolic operator $P : \Gamma(E) \to \Gamma(E)$. Let $G^\pm$ be retarded and advanced Green operators for P and denote with E the corresponding advanced-minus-retarded operator.*
Then the following statements hold true:

1. *The map:*
$$\Xi : \frac{\Gamma_{tc}(E)}{P\big(\Gamma_{tc}(E)\big)} \to \mathtt{Sol} \qquad \Xi : [f] \mapsto Ef \tag{3.17}$$
, where $\mathtt{Sol}$ *is the space of smooth solutions of P as defined in 2.3, is a well-defined vector space isomorphisms.*

2. *The domain restriction of map Ξ:*
$$\Xi : \mathscr{E} = \frac{\Gamma_0(E)}{P\big(\Gamma_0(E)\big)} \to \mathtt{Sol}_{sc} \qquad \Xi : [f] \mapsto Ef \tag{3.18}$$
, where $\mathtt{Sol}_{sc}$ *is the space of the space-like compact solutions of P, is a well-defined vector space isomorphisms.*

Proof.

[Th. 1] The well-posedness of Ξ follows directly from the definition of the causal propagator:
$$PEf = 0 \ \forall f \in \Gamma_{tc}(E) \qquad \Rightarrow \quad \Xi(\mathscr{E}_0) \subseteq \mathtt{Sol}$$
while the explicit definition of equivalence classes:
$$[f] \equiv \{f + Pg \,|\, g \in \Gamma_{tc}(E)\} \qquad \Rightarrow \quad \Xi[f] \equiv \{Ef + EPg \,|\, g \in \Gamma_{tc}(E)\} = Ef$$

guarantees that the image does not depend on the representative of $[f]$

The map Ξ is injective.
Given $f, f' \in \Gamma_{tc}(E)$ such that $Ef = Ef'$, from the linearity of E it follows:

$$E(f - f') = 0$$

applying proposition 1.3.2, one finds $h \in \Gamma tc(E)$ such that $Ph = f - f'$. In other words f and f' are two representatives of the same equivalence class in $\frac{\Gamma_{tc}(E)}{P(\Gamma_{tc}(E)}$.

The map Ξ is surjective.
Given $u \in \mathtt{Sol}$ and taking into account a partition of unity $\{\chi_+, \chi_-\}$ on M such that $\chi_\pm = 1$ in a past/future compact region, one finds $P(\chi_+ u + \chi_- u) = Pu = 0$, therefore

$$h = P(\chi_- u) = -P(\chi_+ u)$$

is timelike compact.
Exploiting the properties of retarded and advanced Green operators

$$Eh = G^- P(\chi_- u) - G^+ P(\chi_- h) = G^- P(\chi_- u) + G^+ P(\chi_+ h) = \chi_- u + \chi_+ u = u$$

one concludes that $\Xi(\mathscr{E}_0) \supseteq \mathtt{Sol}$.

[Th. 2] The proof follows slavishly from that of proposition 1.3.2 and of [Th. 1], therefore we shall not repeat it in details. One has to keep in mind that E maps sections with compact support to sections with spacelike compact support and that the intersection between a spacelike compact region and a timelike compact one is compact.

□

In some cases the two symplectic spaces coincide completely.

Example 4. Consider the Klein-Gordon scalar field (E, P). We have already proved the independence of the phase space construction from the choice of Σ.
Let be $(\mathscr{E}, \tau)$ and $(\mathscr{E}_{Lin}, \Omega)$ the two classical symplectic spaces according to the initial data quantization and to the Peierls quantization, where

$$\sigma : \mathtt{Sol}_{sc} \times \mathtt{Sol}_{sc} \to \mathbb{R} \qquad \sigma(\psi, \phi) \int_\Sigma \big(< \nabla_n \psi, \phi > - < \psi, \nabla_n \phi > \big) d\Sigma$$

$$\tau : \mathscr{E} \times \mathscr{E} \to \mathbb{R} \qquad \tau\big([f], [g]\big) = \int_M < f, Eg > d\mu$$

are the corresponding symplectic forms.
The isomorphism $\Xi : [f] \mapsto Ef$ between the two underlying vector spaces is a symplectomorphism inasmuch it preserves the symplectic forms:

$$\sigma(\phi, \psi) = \tau\big([f], [g]\big)$$

where $\phi = Ef$ and $\psi = Eg$.
In fact from the definition of τ it follows:

$$\tau\big([f],[g]\big) := \int_M fEh d\mathrm{Vol}_M = \int_{\mathbf{J}^+_M(\Sigma)} f\psi d\mathrm{Vol}_M + \int_{\mathbf{J}^-_M(\Sigma)} f\psi d\mathrm{Vol}_M = \\ = \int_{\mathbf{J}^+_M(\Sigma)} (PG^- f)\psi d\mathrm{Vol}_M + \int_{\mathbf{J}^-_M(\Sigma)} (PG^+ f)\psi d\mathrm{Vol}_M \tag{3.19}$$

where the integral has been decomposed by splitting the domain of integration into two subsets whose intersection has zero measure and we have exploited the properties of the retarded and advanced operators.
Using $G^\pm$ inside the integral over $\mathbf{J}^\pm_M(\Sigma)$ and considering the explicit representation of the Klein-Gordon operator (Eq. 3.15) allows us to integrate by parts twice:

$$\int_{\mathbf{J}^+_M(\Sigma)} (PG^- f)\psi d\mathrm{Vol}_M = \int_{\mathbf{J}^+_M(\Sigma)} (\Box_M G^- f)\psi d\mathrm{Vol}_M + \kappa \int_{\mathbf{J}^+_M(\Sigma)} (G^- f)\psi d\mathrm{Vol}_M = \\ = -\int_\Sigma \big(\nabla_n (G^- f)\big)\psi d\Sigma - \int_{\mathbf{J}^+_M(\Sigma)} (\nabla_n G^- f)\nabla_n \psi d\mathrm{Vol}_M + \kappa \int_{\mathbf{J}^+_M(\Sigma)} (G^- f)\psi d\mathrm{Vol}_M = \\ = -\int_\Sigma \big(\nabla_n (G^- f)\big)\psi d\Sigma + \int_\Sigma \big((G^- f)\big)\nabla_n \psi d\Sigma + \int_{\mathbf{J}^+_M(\Sigma)} (G^- f)(\Box_m + \kappa)\psi d\mathrm{Vol}_M \tag{3.20}$$

where κ is the constant factor in P. Notice that, on account of *Stokes theorem*, the sign of the normal outgoing normal vector is taken in account.
Combining the two preceding equations one concludes that:

$$\tau\big([f],[g]\big) = -\int_\Sigma \big(\nabla_n (G^- f)\big)\psi d\Sigma + \int_\Sigma \big((G^- f)\big)\nabla_n \psi d\Sigma + \\ + \int_\Sigma \big(\nabla_n (G^+ f)\big)\psi d\Sigma - \int_\Sigma \big((G^+ f)\big)\nabla_n \psi d\Sigma = \\ = \int_\Sigma \big(\phi \nabla_n \psi - \psi \nabla_n \phi\big) d\Sigma := \sigma(\phi,\psi) \tag{3.21}$$

Chapter 4

Geodesic Fields

In the context of differential geometry, *geodesic curves* are a generalization of *straight lines* in the sense of self-parallel curves.
Considering a differential manifold M endowed with an affine connection ∇ we define:

Definition 4.1 (Geodesic)**.** A smooth curve $\gamma : [a,b] \to M$ such that:

$$\nabla_{\dot\gamma}\dot\gamma = 0 \tag{4.1}$$

where $\dot\gamma^\mu := \frac{d\gamma^\mu}{dt}$ is the tangent vector to the curve.

In local chart the previous equation assumes the well-known expression:

$$\ddot\gamma^i + \Gamma^i_{jk}\dot\gamma^j\dot\gamma^k = 0 \tag{4.2}$$

where Γ^i_{jk} is the coordinate representation of the Christoffel symbols of the connection.

Equation (4.2) admits a well-posed Cauchy problem.

Theorem 4.0.1. *Let M be a smooth manifold, $p \in M$, $v \in T_pM$. Then there exist $\epsilon > 0$ and precisely one geodesic*

$$c : [0,\epsilon] \to M$$

with $c(0) = p, \dot c(0) = v$.
In addition, c depends smoothly on p and v.

Proof. Equation 4.2 is a system of second order ODE, and the Picard-Lindelof theorem yields the local existence and uniqueness of a solution with prescribed initial values and derivatives, and this solution depends smoothly on the data. □

In presence of a pseudo-Riemannian metric it is possible to present the geodesic as the curve extremizing the *energy Functional*[1]:

[1] Remember that for arc-length parametrized curves the Energy functional coincides with the length functional.[26, Lemma 1.4.2]

Definition 4.2 (Energy functional)**.**

$$H : C^1([a,b], Q) \to \mathbb{R} \qquad H(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|^2 dt \tag{4.3}$$

Considering only the proper variations (that keep the end-point fixed), the extremum condition corresponds to equation 4.2 where ∇ is the unique Levi-Civita connection (torsion-free and metric-compatible).[26]

In general relativity the problem of the linearization of the geodesic equation yields the Jacobi equation which takes a central role. [2]

Definition 4.3 (Jacobi Field)**.** We call a *Jacobi field* along the geodesic γ the tangent vector field over the submanifold $\gamma(t,\tau)$, determined by a smooth one parameter family of geodesics γ_τ (with $\gamma_0 = \gamma$), in respect to the τ coordinate. *i.e.*:

$$J = \left. \frac{\partial \gamma_\tau(t)}{\partial \tau} \right|_{\tau=0}$$

In local charts, a Jacobi field along the geodesic γ is solution of a linear ODE:

$$\left(X''\right)^\mu + R^\mu_{i\alpha j} T^i X^\alpha T^j = 0 \tag{4.4}$$

where:

- $\left(X'\right)^\mu := \left(\nabla_{\dot\gamma(t)} X\right)^\mu$ is the covariant derivative along the curve γ.
- $T \equiv \dot\gamma(t)$ stands for the tangent vector to γ.
- $R^\mu_{i\alpha j}$ are the representation in components of the Riemann curvature tensor,

The rest of this chapter will be devoted to discussing geodesic and Jacobi fields as a physical system.

4.1 Geodesic Problem as a Mechanical Systems

The basic idea is very simple, to portray the geodesic curve as the natural motion of a free point particle constrained on the Pseudo-Riemannian manifold Q.

Remark 17. In terms of general relativity this problem can be instantly recognized as the derivation of the motion of free-falling particles.

However, there is no lack of alternative viewpoints . The framework of the classical *geometric mechanics* taught us to picture the "static" configurations of a constrained, complex, classical system as a point on the *Configuration space*.According to that, the geodesic motion can be seen as a realization of a particular dynamics on a mechanical system with a pseudo-Riemannian configuration space[3].

[2] Usually in this context takes the name of *Geodesic deviation* problem[43, pag. 46] inasmuch Jacobi field describes the difference between the geodesic and an "infinitesimally close" geodesic.

[3] Such systems can be depicted as "geodesic" even in presence of a position-dependant potential.[2, Cap 3.7]

Proposition 4.1.1 (Geodesic Motion)**.** *The geodesics on the Pseudo-Riemannian manifold* (Q,g) *are the natural motions of the ordinary Lagrangian system* (Q,L) *where:*

$$L(V_q) \coloneqq \frac{1}{2} g_q(V,V) \tag{4.5}$$

Proof. A direct computation of the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial V^i}\right) = \frac{\partial L}{\partial q^i}$$

for the Lagrangian

$$L = \frac{1}{2} g_{ij}(\vec{q}) V^i V^j$$

brings to the following equation motion:

$$g_{kj}\ddot{q}^j = \left(\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial q^k}\dot{q}^i\dot{q}^j\right) - \frac{\partial g_{kj}}{\partial q^i}\dot{q}^i\dot{q}^j\right)$$

Multiplying for the inverse of the metric we find:

$$g^{lk}g_{kj}\ddot{q}^j \equiv \ddot{q}^l = \frac{1}{2}g^{lk}\left(\frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{kj}}{\partial q^i} - \frac{\partial g_{ki}}{\partial q^j}\right)\dot{q}^i\dot{q}^j = -\Gamma^l_{ij}\dot{q}^i\dot{q}^j$$

where in the last equation we have recognized the expression of Christoffel symbols in term of the Riemannian metric. □

Remark 18. The geodesic system is not simply Lagrangian but also Hamiltonian. This property follows from the hyperregularity[2] of L.

As shown in chapter 2, every system with discrete degrees of freedom can be seen as a trivial field system. From that it follows the alternative characterization of geodesics as a Lagrangian field:

Corollary 4.1.2 (Geodesic field)**.** *The geodesics on the Pseudo-Riemannian manifold* (Q,g) *can be seen as the* Dynamical Configurations *of the Lagrangian field system* $(E,\mathscr{L})$ *where:*

- $E = (Q\times\mathbb{R},\pi,\mathbb{R})$ *trivial smooth bundle on the real line.*
- $\mathscr{L}[\gamma] = \frac{1}{2}g(\dot{\gamma},\dot{\gamma})(t)dt$

Proof. It is a simple application of the correspondence seen in chapter 2.2.2. □

From this perspective it is clear that the energy functional corresponds to the action of the geodesic field dynamics and equation 4.2 is nothing more than the equations of motion according to the *least action principle*.

4.2 Peierls Bracket of the Geodesic field

The local coordinate expression for the Lagrangian density of the geodesic field is:

$$\mathscr{L}\big(t,\gamma^i(t),\dot\gamma^i(t)\big) := \frac{1}{2} g_{\mu,\nu}\Big(\gamma^i(t)\Big)\dot\gamma^\mu\dot\gamma^\nu \tag{4.6}$$

which is highly non-linear.
It is explicitly is quadratic in the velocity components $\dot\gamma^i$ and implicitly, through $g_{\mu\nu}(\gamma^i(t))$, is non-polynomial in the coordinates γ^i.

As show in section 2.4.2, for this type of systems the calculation of the Peierls bracket can be realized only locally around a predetermined solution.
Let us repeat the Peierls' procedure for the system under investigation.

As a consequence of our introduction on the geodesic as a field, we can state the unperturbed dynamic as a L.P.D.O :

$$Q_{\mathscr{L}}\big(q^\mu\big) = \Big[\ddot q^\mu + \Gamma^\mu_{ij}\dot q^i\dot q^j\Big] \tag{4.7}$$

where $\dot q^\mu = \frac{d}{dt}q^\mu(t) = \dot q^i\partial_i q^\mu$.

A linear variation of $q_0^\mu + \epsilon\eta^\mu$ constructed from the coordinate representation q_0^μ of the geodesic $\gamma_0 \in \texttt{Sol}$, solves the original equations of motion when

$$Q_{\mathscr{L}}\big(q_0^\mu+\epsilon\eta^\mu\big) = \frac{d^2}{dt^2}\Big(q_0^\mu+\epsilon\eta^\mu\Big)+\Big[\Gamma^\mu_{ij}\big(\vec q_0+\epsilon\vec\eta\big)\Big]\Big(\dot q_0{}^i+\epsilon\dot\eta^i\Big)\Big(\dot q_0{}^j+\epsilon\dot\eta^j\Big) = 0 \stackrel{!}{=} o(\epsilon) \tag{4.8}$$

If we consider only the first order in the parameter ϵ we can expand the expression of the Christoffel symbols:

$$\Big[\Gamma^\mu_{ij}\big(\vec q_0+\epsilon\vec\eta\big)\Big] = \Big[\Gamma^\mu_{ij}(\vec q_0) + \epsilon\eta^\alpha\big(\partial_\alpha\Gamma^\mu_{ij}\big)\Big|_{\vec q_0} + o(\epsilon)\Big]$$

Collecting all the terms in equation 4.8 up to the first order in ϵ it follows a condition on the perturbation:

$$\begin{aligned} 0 &= \ddot\eta^\mu + \eta^\alpha\big(\partial_\alpha\Gamma^\mu_{ij}\big)\Big|_{\vec q_0}\dot q_0{}^i\dot q_0{}^j + \Gamma^\mu_{ij}\big(\dot\eta^i\dot q_0{}^j + \dot q_0{}^i\dot\eta^j\big) = \\ &= \Big\{g^\mu_\alpha\frac{d^2}{dt^2} + \Gamma^\mu_{i\alpha}(\vec q_0)\big[2\dot q_0{}^i\frac{d}{dt}\big] + \big[\partial_\alpha\Gamma^\mu_{ij}(\vec q_0)\dot q_0{}^i\dot q_0{}^j\big]\Big\}\eta^\alpha = P^\mu_\alpha\eta^\alpha \end{aligned} \tag{4.9}$$

where P^μ_α is a linear partial differential operator acting on the *variations ,i.e.,* the components of a field along the geodesic γ_0.
As showed in section 2.4.2, equating the linearized dynamics operator with the term $-\big(Q_\chi(\gamma_0)\big)(x)$ (see equation 2.15) leads to the inhomogeneous *Jacobi operator* from which all the standard construction of the brackets Peierls follows.

Proposition 4.2.1. *The differential equation*

$$P^\mu_\alpha\eta^\alpha = 0$$

corresponding to the l.p.d.o. P defined in equation 4.8 corresponds to equation 4.4 defining the Jacobi fields *along the geodesic γ_0.*

Proof. For convenience, we adopt the following notation:

$$\eta^\mu \partial_\mu \coloneqq X \equiv X^\mu \partial_\mu$$
$$\dot{q_0}^i \partial_i \equiv \dot{\gamma}_0 \coloneqq T \equiv T^i \partial_i$$

We have to show that the equation just found:

$$\ddot{X}^\mu + X^\alpha \left(\partial_\alpha \Gamma^\mu_{ij}\right) T^i T^j + \Gamma^\mu_{\alpha j}\left(2T^j \dot{X}^\alpha\right) = 0 \tag{4.10}$$

where $\dot{X}^\mu = \frac{d}{dt} X^\mu = T^i \partial_i X^\mu$, corresponds to the equation defying the Jacobi field:

$$\left(X''\right)^\mu + R^\mu_{i\alpha j} T^i X^\alpha T^j = 0 \tag{4.11}$$

where $\left(X'\right))^\mu \coloneqq (D_t X)^\mu$ and $D_t = T^i \nabla_i$ is the covariant derivative along the curve. Since:

$$\begin{aligned} X'' = D_t D_t X &= D_t\left(\partial_\mu\left(\dot{X}^\mu + \Gamma^\mu_{i\alpha} T^i X^\alpha\right)\right) \\ &= \left(\ddot{X}^\mu + \frac{d}{dt}\left(\Gamma^\mu_{j\alpha} T^j X^\alpha\right) + \Gamma^\mu_{j\nu} T^j \dot{X}^\nu + T^j \Gamma^\mu_{j\nu} \Gamma^\nu_{i\alpha} T^i X^\alpha\right)\partial_\mu \end{aligned} \tag{4.12}$$

We can write equation 4.10 in term of the covariant derivative as:

$$\begin{aligned} \left(X''\right)^\mu &= -\left(X^\alpha\left(\partial_\alpha \Gamma^\mu_{ij}\right) T^i T^j + \Gamma^\mu_{\alpha i}\left(2T^i \dot{X}^\alpha\right) - \frac{d}{dt}\left(\Gamma^\mu_{i\alpha} T^i X^\alpha\right) - T^j \Gamma^\mu_{js} \Gamma^s_{i\alpha} T^i X^\alpha\right) = \\ &= -\left(X^\alpha\left(\partial_\alpha \Gamma^\mu_{ij}\right) T^i T^j - \dot{\Gamma}^\mu_{i\alpha} T^i X^\alpha - \Gamma^\mu_{i\alpha} \dot{T}^i X^\alpha - T^j \Gamma^\mu_{js} \Gamma^s_{i\alpha} T^i X^\alpha\right) \end{aligned}$$

remembering that the geodesic condition is still to be met:

$$\dot{T}^i = -\Gamma^I_{jk} T^j T^k$$

we can conclude that:

$$\begin{aligned} \left(X''\right)^\mu &= -\left(X^\alpha\left(\partial_\alpha \Gamma^\mu_{ij}\right) T^i T^j - T^j\left(\partial_j \Gamma^\mu_{i\alpha}\right) T^i X^\alpha + X^\alpha \Gamma^\mu_{\alpha s} \Gamma^s_{ij} T^i T^j - T^j \Gamma^\mu_{js} \Gamma^s_{i\alpha} T^i X^\alpha\right) = \\ &= -\left(R^\mu_{i\alpha j} T^i X^\alpha T^j\right) \end{aligned}$$

□

4.2.1 Example: Geodesic field on FRW spacetime

Let us compute the Peierls' bracket for the single special case of geodesic motion on Friedmann–Lemaître–Robertson–Walker spacetimes (FRW).

We recall that the FRW spacetimes are homogeneous and isotropic solutions of Einstein's equations. These manifolds are endowed with a line element of the form:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + sin^2\theta d\phi^2\right) \tag{4.13}$$

where $0 < a(t) \in C^\infty(I)$ is said *scale factor*, $I \subset \mathbb{R}$ is the time domain and k is a constant normalizable to ± 1 or 0.
The assumptions of homogeneity and isotropy alone determine the spacetime metric up to the value of the scale factor a and up to three possibilities of spatial geometry. The simply connected spacetimes compatibile with 4.13 are topologically equivalent (diffeomorphic) to $\mathbb{R} \times \Sigma$ where Σ is a 3D manifold which depends on the value of k:

$k = 1$ $\quad \Sigma \simeq S^3$ (3-sphere)
$k = 0$ $\quad \Sigma \simeq \mathbb{R}^3$ (three-dimensional Euclidean space)
$k = -1$ $\quad \Sigma \simeq H^3$ (three-dimensional hyperboloid)

For the sake of simplicity let us reduce ourselves to the case of *flat* spatial geometry, *i.e.*:

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) = g_{\mu\nu}dx^\mu dx^\nu$$

in the *cosmological* reference frame.

In order to write explicitly the geodesic equation we have to compute the Christoffel symbols. According to the basic definition:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$$

we find that the non-vanishing components of $\Gamma^\lambda_{\mu\nu}$ are :

$$\Gamma^0_{ii} = a\dot{a} \tag{4.14}$$

$$\Gamma^i_{i0} = \Gamma^i_{0i} = \frac{\dot{a}}{a} \tag{4.15}$$

where we have used the usual convention to express with latin indices the spatial coordinates, with 0 the time coordinate and with Greek indices all the spacetime coordinates.

In the cosmological coordinate chart the geodesic equation reads:

$$\frac{d^2}{d\tau^2}\gamma^0 = -a\dot{a}\left(\tfrac{d}{d\tau}\gamma^i\,\tfrac{d}{d\tau}\gamma_i\right) \tag{4.16}$$

$$\frac{d^2}{d\tau^2}\gamma^i = -2\frac{\dot{a}}{a}\left(\tfrac{d}{d\tau}\gamma^0\,\tfrac{d}{d\tau}\gamma^i\right) \tag{4.17}$$

Jacobi fields are dependant on the choice of a base geodesic. Let us consider a simple time-like geodesic, namely the *cosmological* free-falling observer:

$$\gamma^\mu(t) = \begin{bmatrix} a_0 + mt \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad T^\mu = \frac{d}{dt}\gamma^\mu(t) = \begin{bmatrix} m \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Remembering the definition of the components of the Riemann tensor:

$$R^\sigma{}_{\mu\nu\kappa} = \frac{\partial\Gamma^\sigma{}_{\mu\nu}}{\partial x^\kappa} - \frac{\partial\Gamma^\sigma{}_{\mu\kappa}}{\partial x^\nu} + \Gamma^\lambda{}_{\mu\nu}\Gamma^\sigma{}_{\kappa\lambda} - \Gamma^\lambda{}_{\mu\kappa}\Gamma^\sigma{}_{\nu\lambda}$$

we find by direct computation that the non vanishing components are:

$$R^0_{i0i} = -R^0_{ii0} = a\ddot{a} \qquad R^i_{00i} = -R^i_{0i0} = \frac{\ddot{a}}{a} \qquad R^i_{jij} = -R^i_{jji} = \dot{a}^2 \tag{4.18}$$

Since we have parametrized the curve with $x^0 = t$ the covariant derivative in Eq. 4.4 corresponds to an ordinary derivative, then a Jacobi $X^\mu(t)$ along the time-like geodesic $\gamma(t)$ satisfies the following uncoupled homogeneous equations:

$$\frac{d^2}{dt^2}X^0 = 0 \tag{4.19}$$

$$\frac{d^2}{dt^2}X^i = -m^2 R^\mu_{0\nu 0}X^\nu = \left(m^2\frac{\ddot{a}}{a}\right)X^i \tag{4.20}$$

Regarding this problem as a field system, we can say that the operator P ruling the dynamics is diagonal:

$$PX = \begin{bmatrix} \left(\frac{d}{dt}\right)^2 & 0 \\ 0 & \mathbb{1}\left(\left(\frac{d}{dt}\right)^2 - m^2\frac{\ddot{a}}{a}\right) \end{bmatrix}\begin{bmatrix} X^0 \\ X^i \end{bmatrix} = 0$$

The main difficulty to face in the explicit Peierls bracket construction is the computation of the Green operator of P. In this case we can say with certainty that both Green operators are diagonal:

$$G^\pm = \begin{bmatrix} G^\pm_0 & 0 \\ 0 & G^\pm_i \end{bmatrix}$$

where $G^\pm_0$ and $G^\pm_i$ are the Green operators of the only two ODEs involved:

$$P_0 = \left(\frac{d}{dt}\right)^2 \qquad P_i = \left(\frac{d^2}{dt^2} - m^2\frac{\ddot{a}}{a}\right)$$

As proved in Section 1.3.3 these are Hilbert–Schmidt integral operators:

$$G^\pm_\mu f(t) = \int_{\mathbb{R}} g^\pm_\mu(t|\xi) f(\xi)\, d\xi$$

where the integral kernel function is given by Equation 1.5.

Eq. 4.19 is a linear autonomous ordinary differential equation, two explicit linearly independent solutions are

$$\varphi_1(x) = 1 \qquad \varphi_2(t) = t$$

with $W(1,t) = 1$.
The advanced and retarded Green functions corresponding to P_0 can be promptly computed:

$$g^\pm_0(t|\xi) = \pm\theta\big(\pm(t-\xi)\big)\,[t-\xi] \tag{4.21}$$

At the same time, since the coefficient $m^2\frac{\ddot{a}(t)}{a(t)}$ is in general time dependant, Eq. 4.20 is a non autonomous ODE and an analytical expression of its general solution does not exist.
It is possible, however, to express the corresponding Green function as a convergent Dyson expansion[14]. The idea is to consider the non-autonomous term $V(t) = -m^2\frac{\ddot{a}(t)}{a(t)}$ as if it were a perturbing potential.
Remembering that $G_0^{\pm}(t|\xi)$ is the Green function of operator P_0, which can be seen as the time-independent part of differential operator P_i, we can formally express the Green function of P_i as follow:

$$G_i^+(\tau|\tau') = G_0^+(\tau|\tau') + \sum_{n=1}^{\infty}(-)^n \int_{-\infty}^{\tau} dt_1 \cdots \int_{-\infty}^{t_{n-1}} dt_{n-1}\, G_0^+(\tau|t_1)G_0^+(t_1|t_2)\cdots G_0^+(t_{n-1}|t_n)\times \\ \times V(t_1)V(t_2)\cdots V(t_n)\times \\ \times G_0^+(t_n|\tau')$$

similarly G_i^- can be obtained considering the advanced Green function G_0^- and integrating from t to $+\infty$. In both cases the convergence of the series is not guaranteed and should be verified for each explicit expression of the scale function $a(t)$.

In the special case of *De Sitter* spacetime, where $a(t) = \exp(Ht)$ and the coefficient $\ddot{a}/a = H^2$ is a positive constant [43], two linearly independent solutions are given by

$$\varphi_1(x) = e^{-Vt} \qquad \varphi_2(x) = e^{Vt}$$

with $V = mH$ and Wronskian $W = 2V$. The Green functions are

$$g_i^{\pm}(t|\xi) = \pm\frac{1}{2V}\theta\big(\pm(t-\xi)\big)\left[e^{V(t-\xi)} - e^{-V(t-\xi)}\right]$$

Combing all these results we are able to give an explicit expression of the Peierls symplectic form τ- as defined in Eq. 3.5 for the De Sitter spacetime. For any pair (X, Y) of compactly supported fields over $\gamma(t)$ we have:

$$\begin{aligned}
\{X,Y\} &= \int \left\langle X, \left(G^- - G^+\right) Y\right\rangle dt = \\
&= \int_{\mathbb{R}} dt \begin{bmatrix} X^0(t) & \vec{X}(t)\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & a^2(t)\mathbb{1}\end{bmatrix}\begin{bmatrix}(G_0^- - G_0^+) & 0\\ 0 & (G_i^- - G_i^+)\end{bmatrix}\begin{bmatrix}Y^0(t)\\ \vec{Y}(t)\end{bmatrix} = \\
&= \int_{\mathbb{R}} dt \left(-X^0(t)(G_0^- - G_0^+)Y^0(t) + a^2(t)\vec{X}\cdot(G_i^- - G_i^+)\vec{Y}(t)\right) = \\
&= \int_{\mathbb{R}} dt \int_{\mathbb{R}} d\xi \left(-X^0(t)Y^0(\xi)\big(-\theta(\xi-t)-\theta(t-\xi)\big)[t-\xi]\right) + \\
&\quad + \int_{\mathbb{R}} dt \int_{\mathbb{R}} d\xi a^2(t)\left(\vec{X}(t)\cdot\vec{Y}(\xi)\big(-\theta(\xi-t)-\theta(t-\xi)\big)\frac{1}{2V}\left[e^{V(t-\xi)} - e^{-V(t-\xi)}\right]\right) = \\
&= \int_{\mathbb{R}} dt \int_{\mathbb{R}} d\xi \left(X^0(t)Y^0(\xi)[t-\xi] - \vec{X}(t)\cdot\vec{Y}(\xi)\frac{a^2(t)}{2V}\left[e^{V(t-\xi)} - e^{-V(t-\xi)}\right]\right)
\end{aligned}$$

4.3 Algebraic quantization of the Geodesic Field

The algebraic quantization scheme applies only to systems of linear fields. Since equation4.2 is highly non linear, it is not the geodesic system that can actually be quantized but rather its linearization, the Jacobi field along a fixed geodesic γ_0.

A Classical Framework

The basic idea is that, chosen a geodesic γ_0, the kinematical configurations of the Jacobi fields are tangent fields along the fixed curves.

A.a Kinematics The configuration bundle E corresponds to the *Pull-back bundle* $\gamma_0^*(TQ)$ of the tangent bundle along the geodesic γ_0. Then:

- E is a vector bundle over $\mathbb{R}$.
- The base manifold $\mathbb{R}$ can be considered as a degenerate globally hyperbolic spacetime, $\mathscr{P}_C(\mathbb{R}) = \mathbb{R}$.
- the fibers are $E_p \coloneqq T_{\gamma_0(p)}Q$
- $\mathtt{C} = \Gamma^\infty(E) = \mathfrak{X}(\gamma_0)$ is constituted by vector fields along the curve γ_0.

A.b Dynamics The coordinate representation of the motion equation is:

$$(PX)^\mu = (X'')^\mu + R^\mu_{i\alpha j} T^i T^j X^\alpha$$

where $X \in \mathtt{C}$ and $T^i = \dot{\gamma_0}^i$. According to equation 1.1 this operator falls exactly in the class of *normally hyperbolic* operators hence it is quantizable both by means of Peierls procedure and by means of initial data.

4.3.1 PreQuantum Framework

A Peierls approach

A.a Pairing Since Q is a Riemannian manifold and $\mathtt{C}$ is composed by tangent vector fields, it is straightforward to choose as inner product on the configuration bundle E the metric function defined on Q:

$$\langle X, Y\rangle_t \coloneqq g\left(X\left(\gamma_0(t)\right), Y\left(\gamma_0(t)\right)\right) \qquad \forall X, Y \in E_t \tag{4.22}$$

It follows slavishly the definition of a pairing:

$$(X, Y) = \int_{\mathbb{R}} \langle X, Y\rangle_t \, dt \tag{4.23}$$

well-defined for every pair $X, Y \in \mathtt{C}$ such that $\mathrm{supp}(X) \cap \mathrm{supp}(Y)$ is compact.

The operator P ruling the dynamics is formally self-adjoint:

$$(Y, PX) = \int Y_\mu P X^\mu \, dt = \int \left(Y_\mu \ddot{X}^\mu + Y_\mu R^\mu_{i\alpha j} T^i T^j X^\alpha \right) dt =$$
$$= \int \left(\ddot{Y}_\mu X^\mu + X_\mu R^\mu_{i\alpha j} T^i T^j Y^\alpha \right) dt = \int P Y_\mu X^\mu \, dt = (PY, X)$$

where we have integrated by parts twice (the boundary value being null since the integrand is compactly supported) and we have exploited the curvature tensor identity:

$$\langle R(X,T)T, Y \rangle = \langle R(Y,T)T, X \rangle \tag{4.24}$$

A.b Classical Observables Replicating what has been done in the general case, we construct the *pre-observables* as the functionals $F_f : \mathtt{C} \to \mathbb{R}$ for all $f \in \Gamma_0(E)$ compactly supported fields along the geodesic γ_0:

$$F_f(X) = \int_{\mathbb{R}} < X, f >_t dt \qquad \forall X \in \mathtt{C} \tag{4.25}$$

The space of classical observables is then obtained through the usual quotient:

$$\mathscr{E} \simeq \frac{\Gamma_0}{P\Gamma_0}$$

The observables functionals are the maps:

$$F_{[f]}(X) = F_f(X) \qquad \forall X \in \mathtt{Sol}$$

where f is a representative of the equivalence class $[f] \in \mathscr{E}$.

A.c Symplectic Structure The geodesic motion is a particular case of a system with finite degrees of freedom, thus the Peierls brackets between two Lagrangian functionals χ, ω around a geodesic γ_0 tested on a function $f \in C_0^\infty(\mathbb{R})$ are given by Eq. 2.16. Restricting the definition to the simplest Lagrangian functionals constructible from the classical observables:

$$\chi[\phi] := (\chi, \phi) \qquad \chi \in \mathscr{E}, \phi \in \mathtt{Sol}$$

corresponding to Lagrangian densities in the form:

$$\chi(\vec{q}\, i, \dot{\vec{q}}) :=< \chi, \vec{q} >= \chi^i q_i$$

such that $Q_\chi \gamma_0^i = \chi^i$, the Peierls brackets expression reduces to

$$\{\chi, \omega\}(\gamma_0)[f] = \int f(t) \left\langle \chi, \left(G^- - G^+ \right) \omega \right\rangle dt$$

The test-function f can be neglected for regular distributions.

We conclude that,according to the Peierls' procedure, the classical symplectic space is the pair $(\mathscr{E}, \tau)$ where:

$$\tau([\chi],[\omega]) = \{\chi, \omega\} = \int \left\langle \chi, \left(G^- - G^+ \right) \omega \right\rangle dt = (\chi, E\omega) \qquad \forall \chi, \omega \in \Gamma_0(E)$$

B Initial data Approach

B.a Classical Phase Space The base manifold for the configuration bundle under examination is the real line $\mathbb{R}$ that can be seen as a degenerate globally hyperbolic spacetime. Thus each point $p \in \mathbb{R}$ is a Cauchy surfaces and no further support condition can be imposed.
Considering that the operator P is of second order, we have:

$$\mathscr{M}(p) \equiv \mathsf{Data}(p) = \Gamma^{\infty}(p) \times \Gamma^{\infty}(p) = T_{\gamma_0(p)}Q \times T_{\gamma_0(p)}Q$$

and

$$\mathscr{M} \simeq \mathtt{Sol}$$

using the map which yields the unique solution starting from an initial data.

B.b Symplectic Structure on the Phase Space The general definition 3.6 of the symplectic form on the classical phase space reduces to:

$$\Omega : \mathscr{M}(p) \times \mathscr{M}(p) \to \mathbb{C} \qquad : \qquad \Omega\Big\{[V_0, V_1], [W_0, W_1]\Big\} = g(V_1, W_0) - g(V_0, W_1)$$

where g is the inner product on Q.
This formula can be transferred to the space of solutions:

$$\sigma_p : \mathtt{Sol} \times \mathtt{Sol} \to \mathbb{C} \qquad : \qquad \sigma_p\Big\{X, Y\Big\} = \Omega\Big\{[Y(t), D_t Y(t)], [X(t), D_t X(t)]\Big\}$$

Mimicking what has been done in example 3 for the scalar field, the independence of the phase space construction from the particular choice of $p \in \mathbb{R}$ can be proved.
Taken $X, Y \in \mathtt{Sol}$ two Jacobi fields on $\gamma_0(t)$, a scalar field over $\mathbb{R}$ can be defined:

$$J(t) := \Omega\Big\{[Y(t), D_t Y(t)], [X(t), D_t X(t)]\Big\} = X^{\alpha}(t) g_{\alpha\beta} D_t Y^{\beta}(t) - Y^{\alpha}(t) g_{\alpha\beta} D_t X^{\beta}(t)$$

where $D_t = T^{\mu}\nabla_{\mu}$ as usual. This is clearly a conserved current:

$$\begin{aligned} D_t J = \quad & (D_t X)^{\alpha} g_{\alpha\beta} (D_t Y)^{\beta} - (D_t Y)^{\alpha} g_{\alpha\beta} (D_t X)^{\alpha} + X^{\alpha} g_{\alpha\beta} D_t D_t Y^{\beta} - Y^{\alpha} g_{\alpha\beta} D_t D_t X^{\beta} = \\ = \quad & X^{\alpha} g_{\alpha\beta} P Y^{\beta} - Y^{\alpha} g_{\alpha\beta} P X^{\beta} - X_{\beta} R^{\beta}_{i\alpha j} T^i Y^{\alpha} T^j + Y_{\beta} R^{\beta}_{i\alpha j} T^i X^{\alpha} T^j = 0 \end{aligned} \tag{4.26}$$

exploiting the conditions $\nabla_{\mu} g_{\alpha\beta} = 0$, $PX = PY = 0$ and equation 4.24.
Hence:

$$\int_p^{p'} D_t J = J(p) - J(p') = 0$$

In others words:

$$\sigma_{\Sigma}(X, Y) = \sigma_{\Sigma'}(X, Y) \qquad \forall X, Y \in \mathtt{Sol}\ ;\ \forall \Sigma, \Sigma' \in \mathscr{P}_C(M) = \mathbb{R}$$

In conclusion, according to the initial data procedure, the classical symplectic space is the pair $(\mathtt{Sol}, \sigma)$ such that:

$$\sigma(X, Y) = X_{\mu}(\Sigma) \left(D_t Y(\Sigma)\right)^{\nu} - \left(D_t X(\Sigma)\right)_{\mu} Y^{\nu}(\Sigma) \qquad \forall X, Y \in \mathtt{Sol}$$

where Σ is an arbitrary point in $\mathbb{R}$.

4.3.2 Comparisons

The two procedures yield two different classical symplectic spaces: $(\mathscr{E},\tau)$ and $(\mathtt{Sol},\sigma)$. We have proved in section 3.4 (Theorem 3.4.1) that the two vector spaces are isomorphic through the map Ξ realized with the causal propagator E. Furthermore, in this case it can be proved that Ξ preserves the symplectic form.

Once again we mimic what has been done for the case of a scalar field (Ex 4). Consider two compactly supported vector fields $f,h\in\Gamma_0(E)$ along γ_0 and call $X=Ef$, $T=Eh$ the corresponding Jacobi field. In fact from the definition of τ it follows:

$$\begin{aligned}\tau\big([f],[g]\big)=&(f,Eh)=\int_{\mathbb{R}} f^\mu g_{\mu\nu}(Eh)^\nu dt=\int_\Sigma^\infty f^\mu g_{\mu\nu}Y^\nu dt+\int_{-\infty}^\Sigma f^\mu g_{\mu\nu}Y^\nu dt\\ &=\int_\Sigma^\infty (PG^-f)^\mu g_{\mu\nu}Y^\nu dt+\int_{-\infty}^\Sigma (PG^+f)^\mu g_{\mu\nu}Y^\nu dt\end{aligned}\tag{4.27}$$

where the integral has been decomposed by splitting the domain of integration into two subsets whose intersection has zero measure and we have exploited the properties of the retarded and advanced operators.

Considering the explicit representation of the operator P, we can integrate by parts twice:

$$\begin{aligned}&\int_\Sigma^\infty (PG^-f)^\mu g_{\mu\nu}Y^\nu dt=\int_\Sigma^\infty \big(D_t^2G^-f+R(G^-f,T)T\big)^\mu g_{\mu\nu}Y^\nu dt \quad =\\
&=\int_\Sigma^\infty D_t\big(\big(D_tG^-f\big)^\mu g_{\mu\nu}Y^\nu\big)dt-\int_\Sigma^\infty \big(D_tG^-f\big)^\mu g_{\mu\nu}(D_tY)^\nu dt \quad +\\
&\quad+\int_\Sigma^\infty \big(R(G^-f,T)T\big)^\mu g_{\mu\nu}Y^\nu dt \quad =\\
&=-D_t(G^-f)^\mu g_{\mu\nu}Y^\nu\big|_\Sigma-\int_\Sigma^\infty D_t\big(\big(G^-f\big)^\mu g_{\mu\nu}(D_tY)^\nu\big)dt \quad +\\
&\quad+\int_\Sigma^\infty \big((G^-f)^\mu g_{\mu\nu}(D_t^2Y)^\nu+\big(R(G^-f,T)T\big)^\mu g_{\mu\nu}Y^\nu\big)dt \quad =\\
&=-D_t(G^-f)^\mu g_{\mu\nu}Y^\nu\big|_\Sigma+(G^-f)^\mu g_{\mu\nu}(D_tY)^\nu\big|_\Sigma+\int_\Sigma^\infty \big((G^-f)^\mu g_{\mu\nu}(PY)^\nu\big)dt \quad =\\
&=-D_t(G^-f)^\mu g_{\mu\nu}Y^\nu\big|_\Sigma+(G^-f)^\mu g_{\mu\nu}(D_tY)^\nu\big|_\Sigma\end{aligned}\tag{4.28}$$

where *Stokes theorem* and property 4.24 have been used.

Combining the two above equations one concludes that:

$$\begin{aligned}\tau\big([f],[g]\big)=&-D_t(G^-f)^\mu g_{\mu\nu}Y^\nu\big|_\Sigma+(G^-f)^\mu g_{\mu\nu}(D_tY)^\nu\big|_\Sigma \quad +\\
&+D_t(G^+f)^\mu g_{\mu\nu}Y^\nu\big|_\Sigma-(G^+f)^\mu g_{\mu\nu}(D_tY)^\nu\big|_\Sigma \quad =\\
&=\big(Ef\big)^\mu g_{\mu\nu}(D_tY)^\nu\big|_\Sigma-\big(D_tEf\big)^\mu g_{\mu\nu}Y^\nu\big|_\Sigma \quad =\\
&=X_\mu(\Sigma)\,(D_tY(\Sigma))^\mu-(D_tX(\Sigma))_\mu\,Y^\mu(\Sigma) \quad \equiv\sigma(X,Y)\end{aligned}\tag{4.29}$$

$(\mathscr{E},\tau)$ and $(\mathtt{Sol},\sigma)$ are isomorphic not only as vector spaces but also as symplectic spaces.

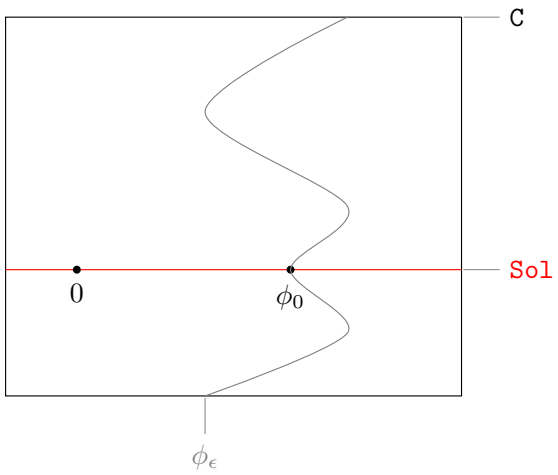
4.4 Geometric approach

4.4.1 Geometric picture of Peierls brackets

Before addressing the formalization as geometric objects of the constituent parts of the Peierls' algorithm, let us provide a geometric visualization of the procedure explained in Section 2.4.

A General Construction on Linear system.

As a starting point we consider a linear field system:

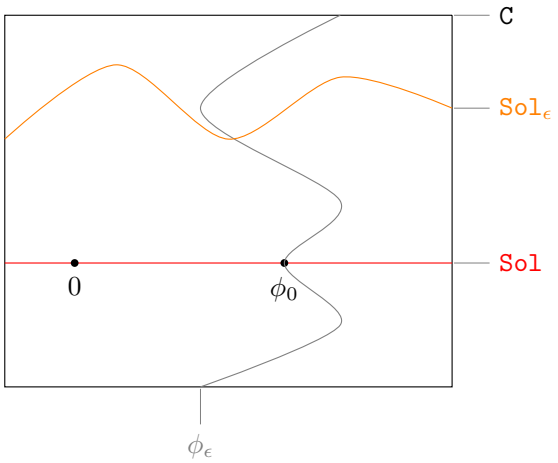


- The set C, space of kinematic configurations, is a linear submanifold. For the sake of simplicity we depict it as a plane, neglecting the complexities related to infinite-dimensional manifods.
- Consequently, the space of solutions Sol is a linear submanifolds, in our picture a line, containing the section 0.

We recall that the main character of the Peierls' procedure are the *Lagrangian densities*, elements in $\mathsf{Lag}(E)$. Abstractly, they are maps from kinematic configurations to volume forms on the spacetime. From a more practical point of view we are interested in its twofold nature:

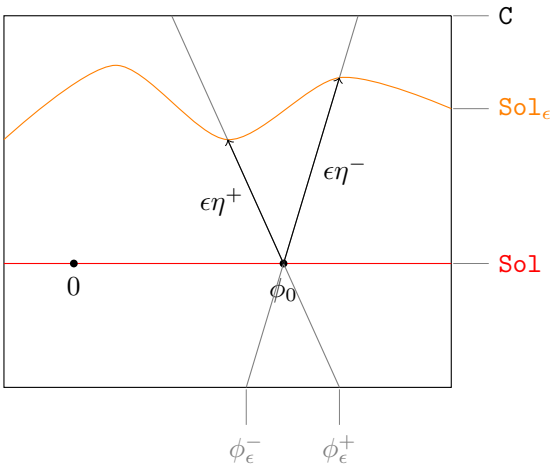
- As the object ruling the dynamics, through the correspondence $\mathscr{L} \mapsto Q_{\mathscr{L}}$, or, eventually, as a disturbance on the fixed unperturbed Lagrangian.
- As a quantity evaluable on the conformations of the system, through the correspondence $\mathscr{L} \mapsto \mathscr{O}_{\mathscr{L}}$.

In the first instance, the Peierl's procedure considers a not necessarily linear fixed Lagrangian density $\chi \in \mathsf{Lag}(E)$. The only constraint is imposed on the support condition of χ.



- Call $\mathtt{Sol}_\epsilon$ the space of solutions of the disturbed dynamics equation $Q_{\mathscr{L}+\epsilon\chi}$. Since Q_χ is generally not linear we depict this space as an arbitrary curve.
  Without any loss of generality we neglect the possibility that $\mathtt{Sol}\cap\mathtt{Sol}_\epsilon \neq \emptyset$ in this picture.
- To an arbitrary variation of a solution ϕ_0 corresponds a generic parametrized curve on the plane.

We have proved in Section 2.4.1 that such choice determines two particular variations of any fixed solution $\phi_0 \in \mathtt{Sol}$.

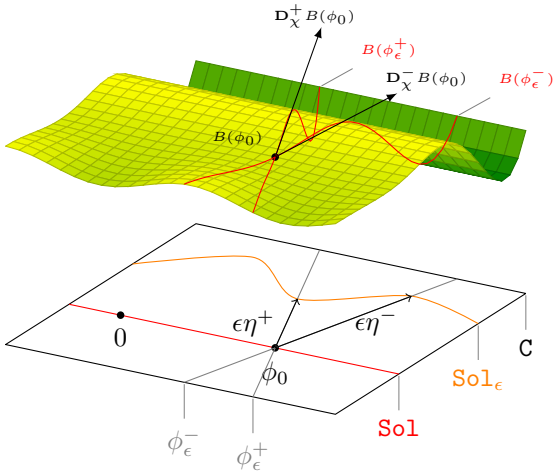


- Among all the possible linear variations of ϕ_0, we consider the two variations
  $$\phi_\epsilon^\pm = \phi_0 + \epsilon\eta_\pm$$
  where $\eta_\pm$ are the unique solution of Eq. 2.8, *i.e.*:
  $$\eta_\pm = G^\pm\left(-Q_\chi\phi_0\right)$$
  determined by the fixed perturbation χ.

In layman terms we can say that the quantity $Q_\chi\phi_0$ quantifies how much ϕ_0 fails to be a solution of the disturbed equations of motion. More explicitly :

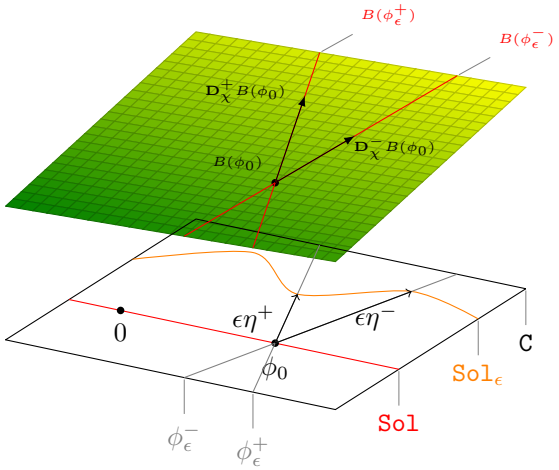
$$P_\epsilon\phi_0 = \cancel{P\phi_0} + \epsilon Q_\chi\phi_0$$

The subsequent step in the Peierls' procedure is the introduction of the effect operator. This object can be seen as a function, associated to the disturbance χ, which maps any continuous fuctional B on $\mathtt{C}$ to a functional $\mathbf{D}^\pm_\chi B$ on $\mathtt{Sol}$.



- A generic continuous functional on the systems can be seen as a continuous function on this $\mathbb{R}^2$ plane.
  We depict them as a surface embedded in $\mathbb{R}^3$.
- Accordingly, `Sol` can be regarded as the locus of the *"total lagrangian"* $\mathscr{O}_{\mathscr{L}}$ extrema.
- Directly from Definition 2.12 , we can see that $\mathbf{D}^\pm_\chi B$, the effect of χ on B evaluated in ϕ_0, is formally a directional derivative in the direction of $\phi^\pm_\epsilon$ calculated in point ϕ_0. We depict such quantity as a tangent vector to the surface B but, more properly, it corresponds to the value of the inclination of this vector.

All of this machinery has a rather simpler picture when the functional B is linear.



- When B is a linear functional the effect takes the simple form:

$$\mathbf{D}^\pm_\chi B(\phi_0) = B(\eta_\pm)$$

  In this case the dependence on the initial solution ϕ_0 is accounted implicitly in the construction of perturbation $\eta_\pm(\phi_0)$.

Finally, considering two Lagrangian densities χ, ω and the corresponding Lagrangian functional $\mathscr{O}_\omega$, we achieve the Peierls brackets as:

$$\{\chi,\omega\}(\phi_0) := \mathbf{D}^-_\chi \mathscr{O}_\omega(\phi_0) - \mathbf{D}^+_\chi \mathscr{O}_\omega(\phi_0)$$

According to our geometric picture we can conclude that the Peierls brackets compute the difference between the slopes of the two function $\mathscr{O}_\omega(\phi^\pm_\epsilon)$, which can be seen as a real function on the single variable ϵ, calculated in $\epsilon = 0$. We stress again that $\phi^\pm_\epsilon$ are not arbitrary but they are two specific variations associated to χ according to the Peierls' procedure.

Thus, the brackets between χ and ω vanish in three cases:

1. The two curves $\mathscr{O}_\omega(\phi_\epsilon^+)$ and $\mathscr{O}_\omega(\phi_\epsilon^-)$ have the same derivative when evaluated in ϕ_0.

2. The starting solution ϕ_0 is also a solution of Q_ω.
   In this case to ϕ_0 corresponds an extremum of $\mathscr{O}_\omega$ thus its derivative vanishes according to each variation passing trough ϕ_0.

3. The starting solution ϕ_0 is also a solution of Q_χ.
   In this case the two perturbation $\phi_\epsilon^\pm$ are degenerate since they correspond to a transformation along a symmetry of the system and $\eta_\pm = 0$.

In conclusion the Peierls brackets compute the quantity:

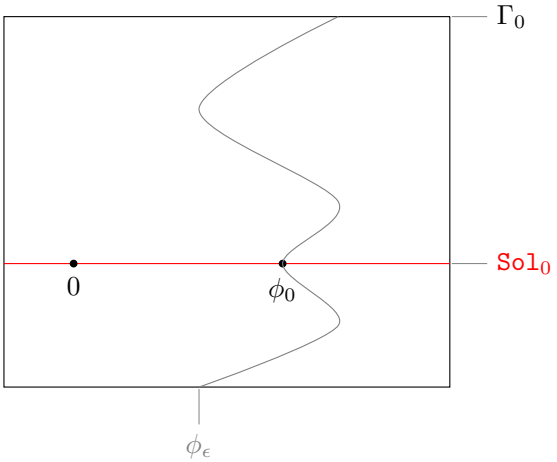
$$\mathscr{O}_\omega(\phi_{\epsilon\chi}^+) - \mathscr{O}_\omega(\phi_{\epsilon\chi}^-) \simeq \{\chi,\omega\}(\phi_0) + o(\epsilon) \tag{4.30}$$

up to the first order, in the expansion parameter ϵ, of a formal Taylor expansion. The section $\phi_{\epsilon\chi}^\pm$ can be seen as the variation of a fixed unperturbed solution ϕ_0 induced by a perturbation term $\epsilon\chi$ in the Lagrangian ruling the dynamics depending on whether such disturbance propagates forward or backward in time. Thus the quantity 4.30 is the difference between the change in the value of the Lagrangian functional $\mathscr{O}_\omega$ evaluated along the perturbation propagating backward and forward.

Remark 19. The above procedure could be generalized to non linear systems regarding it as a "Linearization".
The plane $\mathtt{C}$ depicted above can be seen as a formal tangent space to the non-linear manifold of the kinematic configurations. While in the case of linear fields the manifold $\mathtt{C}$ is flat and essentially coincides with its unique tangent space, for general field systems this correspondence can be admitted only locally.

B Peierls brackets role in second quantization.

The prequantum structure, in the procedure of quantization via Peierls brackets, requires to restrict ourselves from the set $\mathtt{C}$ of all the global sections to the space Γ_0 of compactly supported sections.



- For linear fields Γ_0 is still a linear space, we can keep the same representation of the preceding paragraph.

In our picture, the choice of a pairing

$$(\cdot,\cdot):\Gamma_0\times\Gamma_0\to\mathbb{R}^+$$

(see Section B.a) corresponds to the attribution of a scalar product on the plane Γ_0. Therefore, Γ_0 is a vector space rigged with a scalar product $(\cdot,\cdot)$.

To summarize, to each $f\in\Gamma_0$ we associate:

1. a continuous linear functional [4] on Γ_0:

$$F_f(\cdot)=(f,\cdot)$$

2. a Lagrangian density:

$$f\mapsto\mathscr{L}_f:=\langle f,\cdot\rangle_{(x)}$$

where $<\cdot,\cdot>$ is the inner product on the configuration bundle E and such that $F_f\equiv\mathscr{O}_{\mathscr{L}_f}$.

3. a Euler-Lagrange operator:

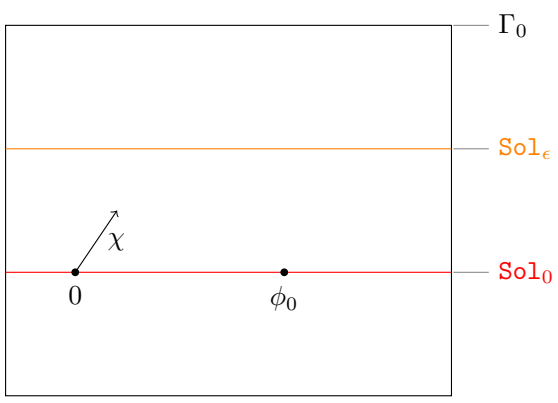
$$Q_f(\gamma)=-f\qquad\gamma\in\mathtt{C}$$

which maps every kinematic configuration to $-f\in\Gamma_0$.

In virtue of these correspondences, the Peierls' procedure allows us to equip the space Γ_0 with a pre-symplectic structure

$$\tau(f,g)=\{\mathscr{L}_f,\mathscr{L}_g\}=\big(f,(G^--G^+)g\big)$$

Let us visualize this construction in details.



- We are considering only Lagrangian densities constructed from elements of Γ_0, these can be depicted as vectors on the plane Γ_0.
- $\mathtt{Sol}_\epsilon$ consists of all the sections $\gamma\in\Gamma_0$ such that:

$$P_\epsilon\gamma=(P-\epsilon Q_\chi)\gamma=P\gamma-\epsilon\chi=0$$

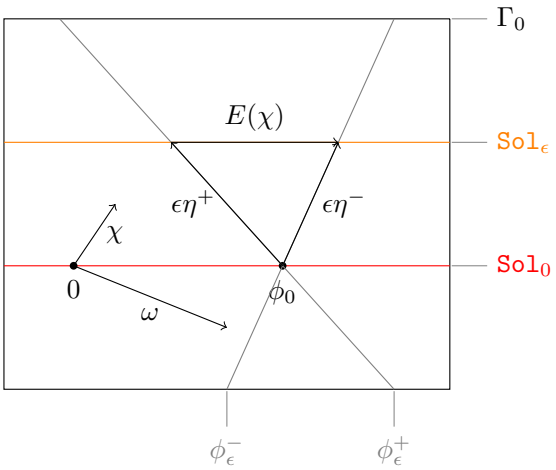
They are solutions of the inhomogeneous equation $P\gamma=\epsilon\chi$.

[4] One should not be misled by the extreme simplification of our geometric visualization. It must kept in mind that all the spaces involved are, at best, manifolds with uncountable dimensions. Important results as the theorem of Riesz-Fréchet can not be taken in account.

Remembering that the solutions of a inhomogeneous differential problem can be built superposing solutions of the homogeneous problem with a "particular solution", we can affirm that

$$\mathtt{Sol}_\epsilon = \{\phi + \epsilon G^\pm \chi \quad | \phi \in \mathtt{Sol} \quad \} = \mathtt{Sol} + \epsilon G^- \chi = \mathtt{Sol} + \epsilon G^+ \chi$$

and we can depict this space as a line parallel to $\mathtt{Sol}$.



- In this case the directions $\eta_\pm$ of the Peierls' variation $\phi_\epsilon^\pm = \phi_0 + \epsilon\eta_\pm$ of a fixed $\phi_0 \in \mathtt{Sol}$ are independent from ϕ_0:

$$\eta_\pm = G^\pm(-Q_\chi \phi_0) = G^\pm \chi$$

- consequently $E\chi = (G^- - G^+)Q_\chi\phi_0$ is always a vector on the line $\mathtt{Sol}_\epsilon \parallel \mathtt{Sol}_0$.
  This has been proved in Theorem: 3.4.1 through the definition of the isomorphism Ξ.

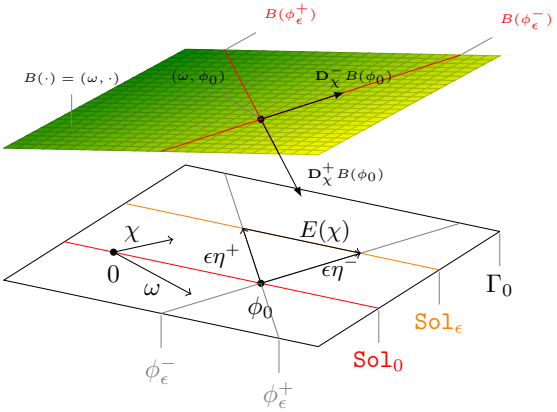
The effect of disturbance χ on an arbitrary continuous linear functional is:

$$\mathbf{D}^\pm_\chi B(\phi_0) = B(\eta_\pm) = B(G^\pm\chi) \qquad \forall \phi_0 \in Sol$$

Considering instead the Lagrangian functional relative to $\omega \in \Gamma_0$ we find

$$\mathbf{D}^\pm_\chi F_\omega(\phi_0) = \left(\omega, G^\pm\chi\right)$$

which leads to the pre-symplectic form τ.



- For the sake of simplicity we depict the pairing relation as the ordinary inner product on the plane Γ_0.
- The Lagrangian functional $B_\omega(\cdot) = (\omega, \cdot)$ correspondent to $\omega \in \Gamma_0$ can be depicted as a plane which intersects plane xy along the normal to ω.

In conclusion, according to our visualization, the brackets between χ and ω vanish when:

$$\omega \perp E\chi \iff \chi \perp E\omega$$

where the perpendicularity condition is meant with respect to the inner product on Γ_0 induced by the pairing 3.1.

Looking at the picture we can deduce that if ω is orthogonal to $\mathtt{Sol}_0$ the brackets are necessary null:

$$\{\omega, \chi\} = -(E\omega, \chi) = (\omega, E\chi) = 0 \qquad \forall \chi \in \Gamma_0$$

Since $E\omega \in \mathtt{Sol}_0$ the equivalence above is met only if:

$$(G^- - G^+)Q_\omega \phi_0 = E\omega = 0$$

This is guaranteed from the condition $\ker(\mathscr{O}_\omega) \supset \mathtt{Sol}$. In fact, when the functional $\mathscr{O}_\omega(\cdot) = (\omega, \cdot) : \mathtt{C} \to \mathbb{R}$ is domain restricted to $\mathtt{Sol}$, it provides degenerate Euler-Lagrange equations: $Q_\omega|_{\mathtt{Sol}} = 0$.

C Peierls Brackets of the Jacobi field

Now we take a step further considering the Peierls brackets in the case of a Jacobi Fields.

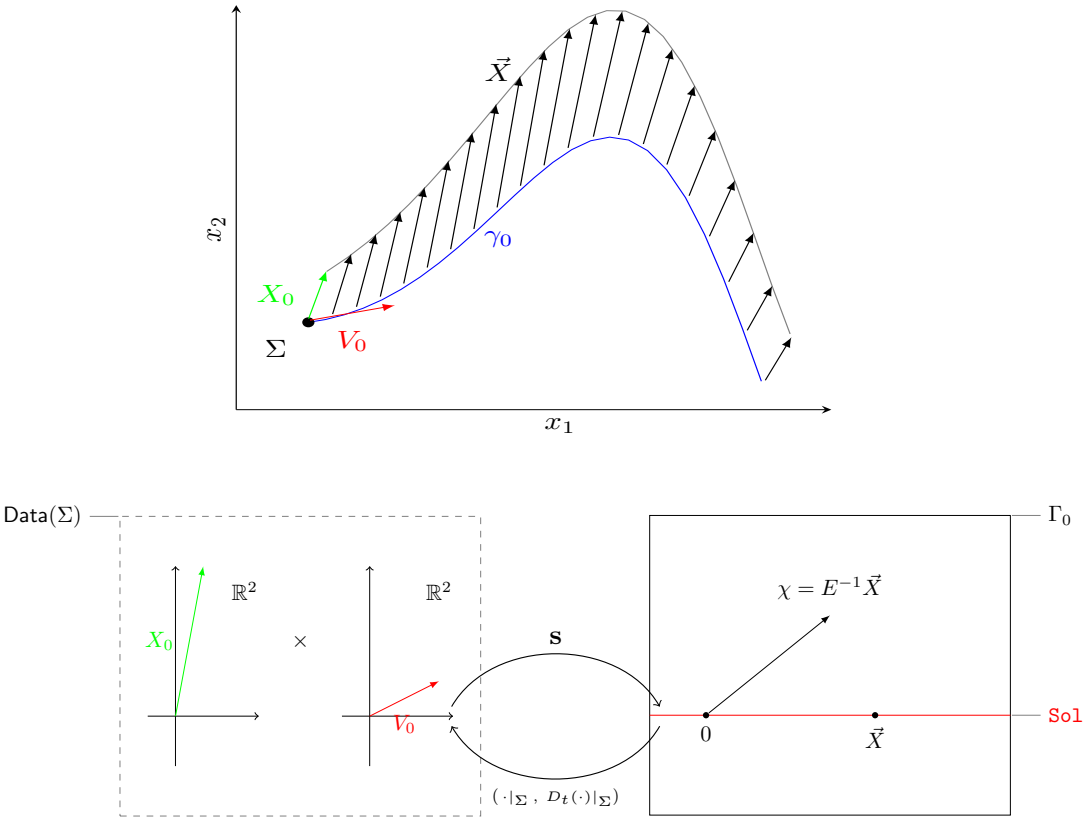


Figure 4.1: Below: the geometric structures encoding the "pre-quantum" Jacobi field. Above: a local chart representation of a Jacobi field $\vec{X}$ along a fixed geodesic γ_0 on a 2-dimensional Riemannian manifold M.

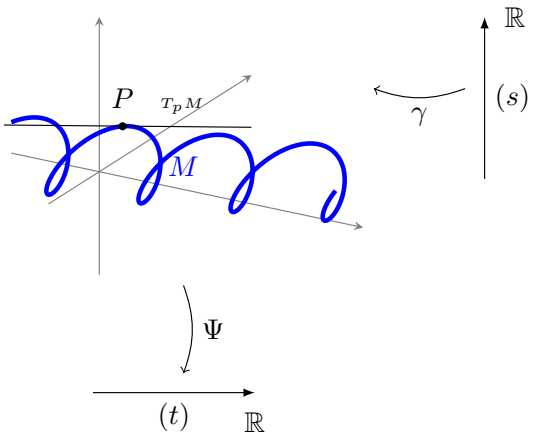
This example provides a testing ground to compare the two pre-quantization procedures showed in Chapter 3.
The crucial point is that the following correspondences:

$$(\chi) \in \Gamma_0 \overset{\Xi}{\mapsto} (E\chi) \in \mathtt{Sol} \overset{|_\Sigma}{\mapsto} \left(E\chi|_\Sigma, D_t\left(E\chi\right)|_\Sigma\right) \in \mathsf{Data}(\Sigma)$$

$$\binom{a}{b} \in \mathsf{Data}(\Sigma) \overset{\mathbf{s}}{\mapsto} \left(\mathbf{s}\binom{a}{b}\right) \in \mathtt{Sol} \overset{E^{-1}}{\longmapsto} \left(E^{-1}\mathbf{s}\binom{a}{b}\right) \in \Gamma_0$$

are symplectomorphism. This allows us to compare the symplectic form τ on $\mathscr{E}$ with the symplectic form Ω on $\mathsf{Data}(\Sigma)$.

In order to obtain a more intuitive representation we limit ourselves to the most simple case of one-dimensional Riemannian manifolds.



- We depict M as an embedded curve in the three dimensional space.
- Local charts are simply functions $\Psi : M \to \mathbb{R}$, we call t the corresponding coordinate.
- Parametrized curves are functions $\gamma : \mathbb{R} \to M$, we call s the parameter.
- The tangent space to T_PM corresponds to the tangent line to M in P.

Briefly, to each smooth curve γ we associate the ordinary smooth function $f(s) = \Psi \cdot \gamma(s)$.
In this case the metric is simply a smooth, non vanishing, function $g : M \to \mathbb{R}$. The inner product between two "tangent vectors" is:

$$\langle a, b\rangle = gab \qquad \forall a, b \in T_PM \simeq \mathbb{R}$$

There is just one Christoffel symbol:

$$\Gamma^0_{00} = \frac{1}{2} g^{-1}\left(\frac{\partial g}{\partial t}\right)$$

The corresponding geodesic equation is:

$$\partial_s^2\gamma + \frac{1}{2}\left[g^{-1}\partial_t g\right]|_{\gamma(s)}\left(\partial_s\gamma\right)^2 = 0$$

Clearly the fields along a fixed geodesic $\gamma(s)$ are scalar functions $X(s) \in \mathbb{R}$ too. Since the Riemann tensor on a one dimensional manifold is null, the single Jacobi equation results:

$$\partial_s^2 X = 0$$

In conclusion the Jacobi Fields along $\gamma(s)$ are all the straight lines:

$$X(s) = a + bs$$

therefore $\texttt{Sol} \simeq \mathbb{R}^2$ is a two dimensional linear subspace of the linear manifold:

$$\Gamma_0 = C_0^\infty(\mathbb{R})$$

From the explicit computation of the Green functions (see Eq. 4.21), it follows, moreover, that the exact expression for the Green operators is:

$$G^\pm \psi(s) = \pm \int_{\mathbb{R}} \theta\big(\pm(s-\xi)\big)\,[s-\xi]\,\psi(\xi)\,d\xi$$

for all $\psi \in \Sigma_0$. The effect of the "advanced minus retarded" operator is:

$$(G^- - G^+)\psi(s) = -\int_{\mathbb{R}} [s-\xi]\,\psi(\xi)\,d\xi = s\left[-\int_{\mathbb{R}} \psi(\xi)\,d\xi\right] + \left[\int_{\mathbb{R}} \xi\psi(\xi)\,d\xi\right]$$

in agreement with the expected linear expression.

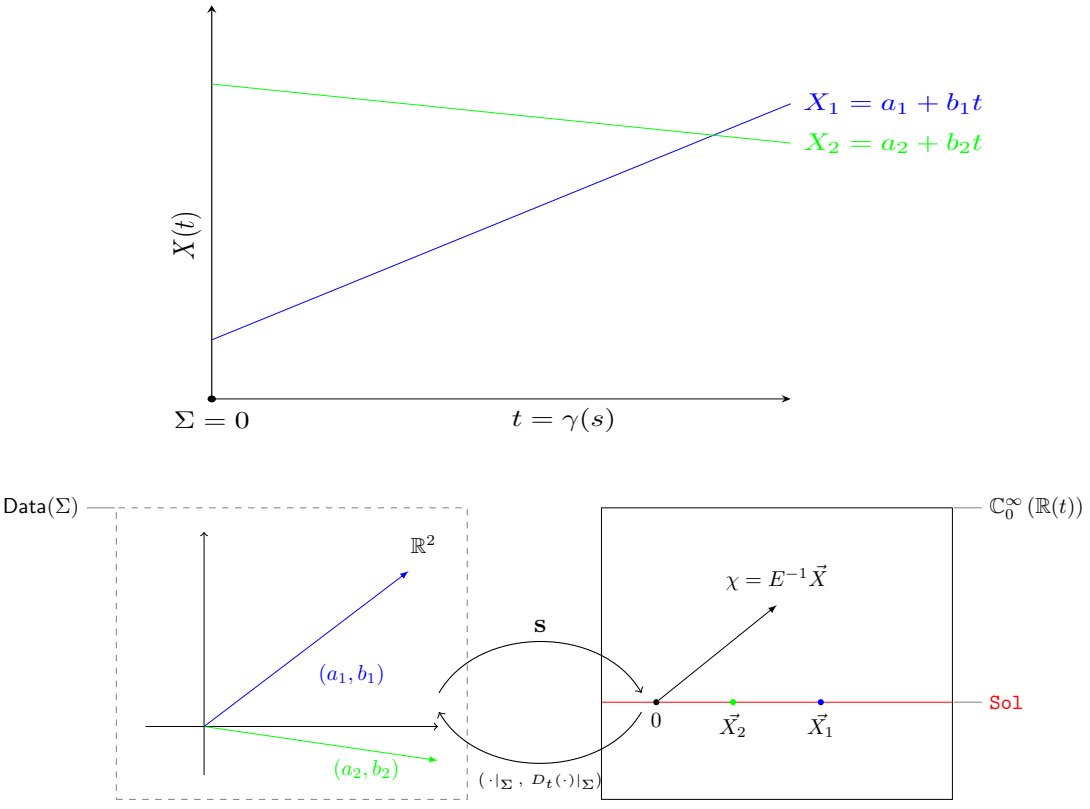


Figure 4.2

4.4.2 Geometric Mechanics of Classical Fields

According to Lessing[30], we call "geometric mechanics" the mathematical discipline which employs modern geometry to describe mechanical systems. It is no secret that geometry is an intrinsic part of mechanics. For example the space of all admissible configurations of a ordinary mechanical system has the natural geometric structure of a manifold. Therefore the bet of geometric mechanics is that is possible to attribute to every mechanical system a *phase space* together with the implicit claim that it is possible to reconstruct from this space each physically relevant quantity.In the spirit of *coordinate-free canonical formalism* the central object, necessary and sufficient, to encode the entire mathematical structure of a classical mechanical system is a *symplectic manifold*.

In view of the quantization schemes this approach has proven to be winning. The reason is that from the symplectic structure descends in a simple way a Poisson structure which can be modified in order to encompass the pre-quantum version of the canonical commutation rules.

Because of the relation of Poisson brackets to quantization, the problem of the quantization of fields required the translation of the canonical formalism from mechanics to field theories (multiple independent variable instead of one). One of the most annoying flaws of the usual canonical formalism in field theory is its lack of manifest covariance, that is its lack of explicit invariance under spacetime coordinate transformations (in the context of general relativity).

This defect is built into the theory from the very beginning, since the usual *Hamiltonian* canonical formalism represents the dynamical variables of classical field theory by functions on some spacelike hypersurface (Cauchy data) and provides differential equations for their time evolution off this hypersurface: it presupposes a splitting of spacetime into space and time, in the form of a foliation of spacetime into Cauchy surfaces. As a result, canonical quantization leads to models of quantum field theory whose covariance is far from obvious and in fact constitutes a formidable problem.

Luckily, the essence of the canonical formalism can be developed in a way that manifestly preserves all the relevant symmetries called "covariant canonical formalism. The key ideas are two:

- for any "sufficently well behaved" system there is a one-to-one correspondence between solutions and initial data.
- the point (q^i, p^i) of the phase of an ordinary Hamiltonian system can be seen as initial data of a motion. To each point in the phase space it corresponds one and only one direction of the Hamiltonian flow.

This leads us to the central concept proposed by Crnkovic and Witten[12]

Definition 4.4. We call *Covariant Phase Space* of a system the space (`Sol`) of all of its solutions to the classical equations of motion, in other words the space of classical trajectories of the system.

Example 5. Consider a non-relativistic particle propagating on a Riemannian manifold X with the usual action functional. A trajectory is uniquely fixed by the position $x_i \in X$ and the momentum $p \in T^*_x X$ of the particle at a given time, the ordinary Phase space is thus constituted by the pair (x_i, p_i). Correspondingly the space of all solutions and hence the (covariant) phase space of the system may be identified with the cotangent bundle T^*X of X.

Similarly, the covariant phase space of the relativistic particle on a (pseudo-) Riemannian manifold X is the space of geodesics of X (in the absence of a background gauge field).

Identifying points of the phase space with whole solutions of the equations of motion requires a conceptual shift in thinking about the time development of the system. In the standard Hamiltonian framework one identifies solutions of the equations of motion with integral curves of the Hamiltonian vector fields. In the covariant phase space that identification becomes untenable.

Since these phase spaces are (locally) naturally parameterized by the suitable boundary conditions or choice of a Cauchy surface (which uniquely determine the corresponding history of the physical system), this construction is strictly related to the pre-symplectic structure proposed in the quantization "by initial data". The “covariant” in “covariant phase space” is to indicate that it comes without any particular choice of Cauchy surface.

The hard part is to find a covariant description of a symplectic structure on the covariant Phase Space . In the last decades there were many attempts to develop a fully covariant formulation of such object for classical field theories. The “covariant functional formalism”, strongly advocated in the 1980’s by Crnkovic,Witten and Zuckerman, is based on the constuction of a symplectic structure starting from a "covariant presymplectic current density"[12][27]. This approach is slightly different from the construction "by initial data". A glimpse of such construction can be found in Example 3 where we have exhibited a conserved current in order to prove that the symplectic definited in 3.6 is independent from the choice of the Cauchy surface

A problem to address is how to frame the Peierls’ construction inside the covariant Phase space formalism. We have seen in Chapter 2 how the covariant phase space $\mathtt{Sol}$ can be embedded into the space of field configurations $\mathtt{C}$. This embedding is characterized as the zero locus of the equations of motion. While the bracket $\{\cdot,\cdot\}$ defined by Wald[42](see Def 3.6) and its covariant version on $\mathtt{Sol}$ (see [27]) can be essentially be view as the *symplectic form* of a phase space, the Peierls algorithm is a recipe which provide a binary form on the space of Lagrangians. Identificating $\mathtt{Lag}$ to the space of the Lagrangian functionals on $\mathtt{C}$ is easy to relate the Lagrangian densities to Hamiltonian functions of ordinary mechanics. In this term is more correct to state that the form constructed by the Peierls’ algorithm is indeed a non-degenerate Poisson structure on the algebra of functions on the covariant phase space is given by the Peierls bracket.

While symplectic and Poisson structures are obviously related, the two covariant formalisms that we have described naturally appear in somewhat different problems. Only recently has been formalized the equivalence conditions for covariant construc-

tions of the symplectic and Poisson structures in an elegant and fully covariant way, thus without using the canonical formalism as an intermediary [18][27].

The main drawback of this geometric point of view is the lack of mathematical rigor, since it is often restricted to the formal extrapolation of techniques from ordinary calculus on manifolds to the infinite dimensional setting. Transforming such formal results into mathematical theorems is a separate problem, often highly complex and difficult. A further investigation from the mathematical point of view and a serious functional analytical effort are still needed to describe the infinite dimensional spaces of field configurations and algebras of observables using infinite dimensional differential geometry.

We want to remark that in the course of this work we focused on the analysis of the steps that make up the Peierls' algorithm. It is costumary to refer to the Peierls brackets as the bilinear form τ acting on $\mathscr{E}$ (see Definition 3.5) but its original procedure is way more general as it is defined on the whole space of Lagrangian densities, modulo a technical condition on the support.

In addition, we acknowledge that the abstract take to mechanical systems proposed in Chapter 2 is by no mean the "state of the art" on the mathematical theory of classical field systems. Basically are required three more aspect to be faced in order to present the most general pre-quantum field theory:

- Gauge freedom.
- Spin structure.
- Constraints.

An earnest approach to this matters would require an important insight on the mathematical foundations of the classical field systems based on the *Variational Bicomplex*.[38, 23, 27]

4.4.3 The quandary of a geometric interpretations

At this point the reader should have realized that the construction proposed by Peierls' is anything but straightforward. It is likely that the author was guided by experience, deducting its general formula from many examples of unequal time Poisson bracket (or rather the quantum commutator) of point fields , computed using the canonical method.

In the course of this section we have provided a geometric visualization of the Peierls construction and identified the role of the corresponding "Poisson form" $(\cdot, E\cdot)$ within the framework of geometric mechanics. However this can not be considered by any means a *geometric interpretation*.

Despite this both the Peierls' construction and the "initial data" construction (including its covariant version proposed by Crnković) appear quite esoteric and out of the blue. Basically these are treated as "black box" which provide a satisfactory symplectic form from the point of view of quantization schemes. They are not justified a

priori and each individual steps of both the constructions lack a geometric and physical interpretation.

At this point one could wonders what we have learned to the application of the Peierls method to the Jacobi Fields. The main idea was to compare the symplectic form generated by this method by the one constructed trough the initial data method in the particular case of a system with finite dimensional phase space *i.e.* such that the underlying symplectic structure is unique according to the Darboux theorem.

Unfortunately, we must note that even in this case the comparison between the two methods is far from being immediate. In order to expand the geometrical understanding of Peierls method, we believe would be necessary not only to formalize the mathematical structure of the manifolds Data and Sol (which are the basis of the canonical phase space formalism), but in particular has is to be determined the geometric structure of the space of Lagrangian densities which underlies the concept of "disturbance" on which the whole method is based.

Conclusions and outlook

We shall now briefly summarize our results.
In Chapter 1 we have reviewed the basic mathematical structures which underlie to the rigorous treatment of any field theory on curved background. The smooth fiber bundles are the natural objects to encompass the kinematics of a system. Imposing that the base manifold is globally hyperbolic permits to depict a deterministic classical dynamics reconstructable by the choice of an appropriate set of initial data. At the same time Green-hyperbolic operators are an abstract general class of equations which obey a wave-like propagation.
In Chapter 2 we have dealt with the problem of the construction of Peierls brackets for theories which are not necessarily linear . In the first instance we have identified a general class of systems for which the original Peierls' algorithm applies without major changes. We have shown that this class is not a mere academic exercise but it includes all classical Lagrangian systems regardless of the cardinality of the degrees of freedom. For such class we revived the Peierls' procedure in details, adapting all its steps to the language currently in use in the modern algebraic theory of fields.
We have devoted Chapter 3 to reviewing the algebraic quantization procedure describing two different realizations. We paid particular attention to the different "prequantum" structures which can be attributed to the classical theory to be quantized. In the first procedure we have stressed how the 2-form on the classical symplectic space, which in most part of our references tends to be postulated, may instead be derived from the more general Peierls brackets. Regarding the second, non covariant, construction of *quantization by initial data*, we showed how it can be compared with the first procedure and how, in rather important cases such as the scalar field theory, it can be proved to be equivalent.
In the last Chapter we have applied the Peierls brackets and the two procedures of the algebraic quantization to the case of geodesic motion. We wrote down the explicit calculation of the Peierls' symplectic form in the special case of Jacobi fields along the isotropic, homogeneous, free-falling observers in De Sitter spacetimes. This simple example made clear that under the rather elegant expression $\{\chi,\omega\} := (\chi, E\omega)$ lie nontrivial calculations, in particular regarding the explicit expression of the Green operators for the ordinary differential equations of Jacobi. At last we proved the equivalence between the algebraic quantization via the Peierls brackets and via the initial data.

To conclude, let us discuss some possible extensions of our work. In this thesis we have tried to keep our mathematical formalization not excessively sophisticated. Currently we are examining further extensions of the Peierls construction to non Lagrangian fields or to systems with Gauge freedom (see for example [27]), all of these are based on the variational bicomplex formalism [38]. We have preferred to keep the level of our discussion to an introductory level since the current lack of bibliography on the theme are hinders the recognition of the role of the Peierls brackets among the schemes of Algebraic quantization.

It may be noticed, browsing the recent literature, a growing interest in the identification of the correct canonical structures in the case of systems with continuous degrees of freedom. Such formalism is currently based on the concept of *"covariant phase space"* which is usually defined as the (infinite-dimensional) space of solutions of the equations of motion. This geometric theory fits neatly into the philosophy underlying the symplectic formalism in general; in particular, it admits a natural definition of the Poisson bracket through the Peierls construction. The main advantage of this approach, would be to make clear the parallelism between the geometrical mechanics of ordinary finite dimensional systems and fields. Its main drawback is the lack of mathematical rigor, since it is often restricted to the formal extrapolation of techniques from ordinary calculus on manifolds to the infinite-dimensional setting: transforming such formal results into mathematical theorems is a separate problem, often highly complex and difficult. The application of the modern results in nonlinear global analysis to this topic are currently not extensively investigated.

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