Chapter 1

Lagrangian Systems and Pierels Brackets

⚠ .. introduzione

1.1 Abstract Lagragian Systems

⚠ introduzione

Definition 1: Lagrangian System

Pair $(E, \mathscr{L})$ composed of:

- $E \xrightarrow{\pi} M$ smooth fiber bundle of typical fiber Q on the oriented manifold $(M, \mathfrak{o})$ called *"configuration bundle"*.

- $\mathscr{L} : J^r E \to \wedge^m T^* M$ bundle-morphism from the r-th Jet Bundle to the top-dimensionial forms bundle over the base manifold M called *"Lagrangian density"* or simply *"Lagrangian"* of r-th order.

This definition is broad enough to include all the systems in ordinary analytical mechanics regardless of the cardinality of degrees of freedom in a unified way.

1.1.1 Kinematics

The configuration bundle encompass all the kinematical structure of the system, the pivotal role is played by the smooth sections which are to be understood as all the possible conformation of the system.

Notation fixing

$$\mathtt{C} := \Gamma^{\infty}(M, E)$$

Space of kinematic configurations.

A section is not a statical configuration, equivalent to a specific point in the configuration space of ordinary classical systems, but has to be seen as a specific realization of the kinematics in the sense of a complete description of a possible motion. At this level of abstraction, since no space-time structure has been specified, terms like stasis and motion must be taken with care .The natural physical interpretation should be clearly manifested through the concrete realization of systems with discrete and continuous degree of freedom.

Observation 1: Mathematical structure

Mathematically speaking this set should be regarded as an infinite dimensional Manifold.

This framework provides a geometric characterization of the notion of variations as tangent vectors on the the space of kinematic configurations .[**?**]

Observation 2: Coordinate Representation

The choice of a chart atlas $\mathscr{A}(M)$ on the base space M and $\mathscr{A}(E)$ on the total space E provides a correspondence between each configuration $\gamma \in \mathtt{C}$ and family of smooth real functions $\{f_{\alpha\beta} : A_\alpha \subset \mathbb{R}^m \to \mathbb{R}^q\}$. The process is trivial:

$$\gamma \in \mathtt{C} \mapsto \{f_{A,U} = \psi_U \circ \gamma \circ \psi_A^{-1} | (A, \psi_A) \in \mathscr{A}(M), (U, \psi_U) \in \mathscr{A}(E)\}$$

Since the whole section as a global object is quite difficult to handle is customary in field theory to work in the more practical local representation.

Observation 3: Further specification of the system's kinematics

The general formalism doesn't require any other structure to be carried forward. Additional structure on the fiber , the base or the whole bundle are to be prescribed in order to specify a precise physical model, e.g. the spin structure on E for the Dirac Field.[**?**]

Figure 1.1: Geometric picture of the basic kinematic structure.
⚠immagine

1.1.2 Dynamics

The Lagrangian density is the object that contains all the information about the dynamic evolution of the system. In fact, when a dynamical principle is provided, $\mathscr{L}$

determines univocally a differential operator P on the kinematics configurations space C. This operator is named *"equations of motion"* operator and has the role to select the dinamically compatible configuration among all the admissible kinematic configurations of C, exactly as it happens in analytical mechanics where the dynamic equations shape the natural motions.

Notation fixing

Provided an equations of motion operator

$$P : \mathtt{C} \to \mathtt{C}$$

The space
$$\mathtt{Sol} := \ker(P) \subset \mathtt{C}$$

is called *"Space of Dynamical Configurations"*.

In order to reconstruct the system's dynamic from the Lagrangian density we have to understand the mathematical nature of $\mathscr{L}$. $\mathscr{L}$ maps point q_p on the fiber $J_p^r E$ to a m-form on $T_p M$. Recalling the definition of jet bundles is clear that for each smooth section on E is associated a smooth section on the b$J^r E$:

$$\phi \in \Gamma^\infty(E) \mapsto (\phi, \partial_\mu \phi, \partial_{\mu,\nu} \phi, \dots \partial_{\tilde{\alpha}} \phi)$$

where $\tilde{\alpha}$ is a multi-index of length r. The correspondence is not univocal since sections equal up to the r-th order define the same jet section. The smoothness of $\mathscr{L}$ ensure that each jet bundle section is mapped to a smooth section in the top-forms bundle i.e. the most general integrable object on a orientable manifold.

It should be clear that $\mathscr{L}$ is a specific choice among the vast class of functions suitable to be a good Lagrangian density over the Configuration Bundle E:

Definition 2: Lagrangian Density on the bundle E

$$\mathsf{Lag}^r(E) := \hom\left(J^r E, \quad \overset{m}{\bigwedge}(T^* M) \right) \cong \left\{ f : \Gamma^\infty(J^r E) \to \Omega^m(M) \right\}$$

(where $\Omega^m(M)$ is the common name for $\Gamma^\infty()$ in the context of Grassmann algebras.) The equivalence states the fact that a bundle-morphism induce a mapping between the sections.

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this choice fix the "Dynamical identity" of the considered system.

Proposition 1.1.1 $\mathsf{Lag}^r(E)$ *has an obvious vector space structure inherited by the linear structure of* $\Omega(M)$.

Thanks to the correspondence between a section $\phi \in \mathtt{C}$ and his r-th jet, it's possible to consider the Lagrangian as directly acting on the kinematic configurations. The image $\mathscr{L}[\phi]\mathrm{d}\mu$, where $\mathrm{d}\mu$ is the measure associated to the orientation $\mathfrak{o}$, is an

object that can be measured on the whole base space, this property suggests the introduction of a class of associated functionals:

Definition 3: Lagrangian functional

Is a functional on C with values on regular distribution over M associated to the generic $\mathscr{L} \in \mathsf{Lag}$.
$$\mathscr{O}_{\mathscr{L}} : \mathsf{C} \to \left(C_0^\infty(M) \right)'$$

Such that the lagrangian functional associated to $\mathscr{L}$, valued on the configuration $\phi \in \mathsf{C}$ and tested on the test-function $f \in C_0^\infty(M)$ it's given by:

$$\mathscr{O}_{\mathscr{L}}[\phi](f) = \int_M \mathscr{L}[\phi] f \, \mathrm{d}\mu$$

Observation 4

The choice of the image of $\mathscr{O}_{\mathscr{L}}$ as a distribution it's a necessary precaution to ensure that functional is *"convergent"* whatever is the configuration on which is evaluated. In fact, despite $\mathscr{L}[\phi]$ is integrable with respect to the measure $\mathrm{d}\mu$, it's not necessary summable if the support of the configuration ϕ becomes arbitrarily large.
This is a simple consequence of the well known functional analysis sequence of inclusions:
$$\mathscr{L}[\phi] \in C_0^\infty(M) \subset L_{\mathrm{loc}}^1(M, \mu) \supsetneq \nsupseteq L^1(M, \mu)$$

Indeed, the functional

$$\mathscr{O}_{\mathscr{L}}[\phi] = \int_{\mathrm{supp}(\phi)} \mathscr{L}[\phi] \mathrm{d}\mu$$

is well defined for all $\mathscr{L} \in \mathsf{Lag}^r(E)$ only over the compactly supported sections. To take account of the global sections it's sufficient it's enough to dampen the integral multiplying the integrand for a test-function.

1.1.3   Generalization

From the preceding analysis is clear that the central object of the dynamics is the operator P *motion equation operator* fa le veci delle equazioni della dinamica

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Definition 4: Evolutive System

1.2 Concrete Realization

1.2.1 Fields on curved Background

1.2.2 Finite Degree systems

1.3   Geometric mechanics of Finite Degree systems

1.3.1   Linear dynamical systems

1.4 Peierls Brackets

1.4.1 Finite Dimensional case

1.5   Eliminata

- quando parlo della cinematica mi piacerebbe dare indicazioni sulla struttura matematica dello spazio delle configurazioni cinematiche:

  1. costituisce una frechet manifold (gli unici risultati che ho trovato sono quelli di Palais di "non linear global analysis"
  2. le curve parametrizzate sono le variazioni
  3. classi di equivalenza definiscono delle variazioni infinitesime che costituiscono lo spazio tangente allo spazio delle configurazioni cinematiche
  4. questo spazio tangente ÃÍ isomorfo allo spazio delle sezioni del pullback rispetto alla sezione $\phi \in C$ del verical bundle (vedere forger romero)
  5. il problema dell'atlante e della rappresentazione delle sezioni in carta locale (da scegliere sia sul total space E che sul base space M)