Capitolo 2: (versione preliminari)

Geodesic Fields

Tony

January 13, 2015

Contents

1 Reprise in Riemannian Geometry

In what follows we present a brief review of the most important result in Riemannian geometry necessary for a better understanding of the geodesic problem.

1.1 Definition of (pseudo)-Riemannian manifold

Definition 1: (Pseudo-)Riemannian manifold

Notation fixing

Metric signature Lorenz manifold.

Theorem 1.1 *$\forall M$ is Riemannianizable.*

1.2 Riemannian manifolds as a category.

Definition 2: Local isometry

Definition 3: Isometry

Definition 4: Killing fields

1.3 Riemannian as a metric space.

Definition 5: Riemannian volume form

Theorem 1.2 *$\forall M$ orientable $\exists 1!$ Riemannian volume form.*

Observation 1

For an insight on the connection between volume form and measure theory see for example [2].

1.4 Tangent bundle of a Riemannian manifold.

Observation 2

g could be seen as a 2-forms (section $\in \Gamma(T^2_0(M))$)

Definition 6: $\flat$ $\sharp$ operator

Theorem 1.3 *On Riemannian manifold M TM is a structure manifold of structure group $G = O(d)$.*
If M is also orientable $G = SO(d)$.

Proof:

See [3] Lemma 1.5.2 and 1.5.3.

□

1.5 Riemannian as a metric space.

See [1] pag $383 - 385$ and [3] pag $15 - 17$.

1.6 Connection structure on a Riemannian manifold.

Connection is a rather general concept definable on any smooth bundle. [1]

On vector bundle we can identify a special kind of connection structure compatible with the vector space structure.[2] There are several equivalent presentation of this concept, each of them stress the importance of one of the many devices carried by this superstructure, for example:

- Derivative of section.
- Parallelism and parallel transportation.
- Specification of an unique horizontal lift among all.

Regarding the Riemannian manifolds we're not interested in connections on general vector bundle but instead to those on the tangent bundle, called *Linear Connection*. There's an infinity of such connection but on (pseudo-)Riemannian manifold it's possible to find a natural prescription that allows us to identify only one among these, called *Levi-Civita Connection*.

Consider (M, g) pseudo-Riemannian manifold.

Definition 7: Linear Connection

[1] In this abstract context connection takes the name of *Erhesmann's connection*.

[2] which takes its name from *Koszul* for distinguish it from the above.

Map $\nabla : \Gamma^\infty(\tau_M) \times \Gamma^\infty(\tau_M) \to \Gamma^\infty(\tau_M)$, we write $(X, Y) \mapsto \nabla_X Y \qquad \forall X, Y \in \Gamma^\infty(\tau_M)$. Such that:

(a) $\nabla_X Y$ is $C^\infty(M)-$linear in X variable.

$$\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y \qquad \forall f, g \in C^\infty(M)$$

(b)

(c)

1.7 Curvature on Riemannian manifold.

2 Geodesic

2.1 Common approach to the Geodesic

3 Review of physics application of geodesic problem.

Essentially [2]. A lot of mechanics systems can be regard as geodesic problem.

3.1 Preliminary remarks: Geometrical encoding of classical mechanics.

sistemi hamiltoniani
sistemi lagrangiani

3.2 Particle on Riemannian manifold under a position dependant potential.

fomm pag 226-228 + 231-233 teo 3.71

3.3 Relativistic particle.

[3]

[3] For an extension of this process to costrained, dissipitative or ergodic systems see fom cap 3.7

4 Jacobi Fields

4.1 Preliminary remarks: Variation of curve.

Let $\sigma : [a, b] \to M$ a piecewise regular curve on smooth manifold M.

Definition 8: Variation of Curve

Variation of curve σ is a continuous application $\Sigma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ such that

- $\sigma_s = \Sigma(s, \cdot)$ is a piecewise regular curve $\forall s \in (-\varepsilon, \varepsilon)$.
- $\sigma_0 = \sigma$.
- $\exists$ a partion $a = t_0 < t_1 < \ldots < t_k = b$ of $[a, b]$ such that

$$\Sigma\Big|_{(-\varepsilon,\varepsilon)\times[t_{j-1},t_j]} \in \mathcal{C}^\infty(\mathbb{R}^2; M)$$

.

Notation fixing

Regarding one entry as a variable and the other as a parameter we can see that Σ determine two family of curves:

- $\sigma_s(\cdot) = \Sigma(s, \cdot)$ is a family of piecewise regular curves called *principal curves*.
- $\sigma^t(\cdot) = \Sigma(\cdot, t)$ is a family of regular curves called *transverse curves*.

Curves in a family have a common parametrization.

Notation fixing

A variation is called *proper* if the endpoints stay fixed, i.e.

$$\sigma_s(a) = \sigma(a) \ \wedge \ \sigma_s(b) = \sigma(b) \qquad \forall s \in (-\varepsilon, \varepsilon)$$

Fields over a variation Σ of a curve σ are defined as follows:

Definition 9: Vector field along a variation

Is a collection $X = \{X_j\}$ of smooth applications $X_j : (-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \to TM$ [a] such that:

$$X_j(s, t) \in T_{\Sigma(s,t)}M \qquad \forall (s, t) \in (-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \quad \forall j = 1, \ldots, k$$

[a] Associate to a subdivision of $a = t_0 < t_1 < \ldots < t_k = b$ of $[a, b]$.

Principal and transverse curves define two special Vector fields along the variation:

Definition 10: Tangent fields of the variation

$$S(s,t) = (\sigma^t)'(s) = \mathrm{d}\Sigma_{(s,t)}\big(\frac{\partial}{\partial s}\big) = \frac{\partial \Sigma}{\partial s}(s,t)$$

for all $(s,t) \in (-\varepsilon, \varepsilon) \times [a,b]$.

$$T(s,t) = (\sigma_s)'(t) = \mathrm{d}\Sigma_{(s,t)}\big(\frac{\partial}{\partial t}\big) = \frac{\partial \Sigma}{\partial t}(s,t)$$

for all $(s,t) \in (-\varepsilon, \varepsilon) \times [t_{j-1}, t_j]$ and $j = 1, \ldots, k-1$ where we have choose a subdivision $a = t_0 < t_1 < \ldots < t_k = b$ associated to Σ.

Notation fixing

$V = S(0, \cdot) \in \mathfrak{X}(\sigma)$ takes the special name of *variation field of* Σ.

There's an importation relation between continuous field on a curve and variation:

Proposition 4.1 *For all continuous field V along a piecewise regular curve σ can be found a variation Σ with variation field V.* [a]

[a] Vice versa follows from the continuity of the variation field.

Proof:

See [1] Lemma 7.2.12 .

□

Let now M be a d-dimensional Riemannian manifold with Levi-Civita connections ∇. The tangent fields of a variation are strictly connected to the curvature of M. We need a lemma:

Lemma 4.1 *For all rectangle $(-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \in \mathbb{R}^2$ on which Σ is $\mathcal{C}^\infty$ we have:*

$$D_S T = D_T S$$

where D_S is the covariant derivativa along the transverse curves and D_T over the principal curves.

Proof:

See [1] Lemma 7.2.13 .

□

The crucial result is what follows:

Proposition 4.2 *For all vector field V along a variation Σ we have:*

$$D_S D_T V - D_T D_S V = R(S,T)V$$

for all rectangle $(-\varepsilon, \varepsilon) \times [t_{j-1}, t_j] \in \mathbb{R}^2$ on which Σ is $\mathcal{C}^\infty$. [a]

[a] $R(S,T)$ is the curvature endomorphism evaluated on the tangent vector fields on the variation.

Proof:

See [1] Lemma 8.2.3 .

□

(References: [1] page 386-387 + 420-421 ; [3] page 171)

4.2 Formal Definition

The concept of *Jacobi Field* is closely related to variations of geodesic curves.
Let $\gamma : [a,b] \to M$ be a geodesic of the Riemannian manifold M. We can consider a special class of variations:

Definition 11: Geodesic variation

Is a smooth variation $\Sigma : (-\varepsilon, \varepsilon) \times [a,b] \to M$ such that all the principal curves $\gamma_s(\cdot) = \Sigma(s, \cdot)$ are also geodesic.[a]

[a] In other words Σ determines a smoothly variable family of geodesic.

Proposition 4.3 *Fixing two tangent vector over a point $p = \gamma(a)$ on the geodesic γ univocally determines a geodesic variation of γ.*

Proof:

See [1] Lemma 8.2.5 or [3] Lemma 4.2.3.

□

Definition 12: Jacobi Fields

Is a field $J \in \mathfrak{X}(\gamma)$ over a geodesic γ such that:
$\exists \Sigma$ geodesic variation such that $J = V$ represent its variation field [a].

[a] As defined under (def 10).

The following proposition determines an equivalent (analytical) definition of Jacobi field:

Proposition 4.4

$J \in \mathfrak{X}(\dot{\gamma})$ is a jacobi field iff:

$$\nabla_{\frac{d}{dt}} \nabla_{\frac{d}{dt}} J + R(X, \dot{\gamma})\gamma' = 0$$

Notation fixing

The vector space of all Jacobi fields on the geodesic γ is denoted $\mathcal{J}(\gamma)$.

Notation fixing

$J \in J(\gamma)$ is called *proper* if $J_0(t) \perp \dot{(\gamma)}(t)$.
$\mathcal{J}(\gamma)$ indicates the vector space of all proper Jacobi fields.

Proposition 4.5 *Every killing field X on M is a Jacobi Field along any geodesic in M.*

Proof:

See [3] Corollary 4.2.1.

□

References

[1] Marco Abate and Francesca Tovena. *Geometria Differenziale*. UNITEXT. Springer Milan, Milano, 2011.

[2] Ralph Abraham, Jerrold E. Marsden, Tudor Ratiu, and Richard Cushman. *Foundations of mechanics*. Ii edition, 1978.

[3] Jurgen Jost. *Riemannian Geometry and Geometric Analysis*. Universitext. Springer-Verlag, Berlin/Heidelberg, 2005.

5 Closing Thoughts

5.1 Eliminata

- non messa la definizione dei campi continui e l'osservazione che S sempre continuo mentre T pu non esserlo ([1] pag 420).
- sono stato ambiguo quando parlo di campi lungo la curva.. sulla continuit o meno (vedere abate pag 387)
- non mi ancora chiaro l'utilit dei jacobi fields... Vediamo le possibilit:
  - Dice Abate a pag. 411 i Jacobi sono lo strumento principale per stabilire una relazione fra curvatura e topologia.
  - Dice Jost a pag. 183 che le Jacobi equation sono una linearizzazione dell'equazione delle geodetiche.
  - Jost a pag $183 - 186$ esplora il legame tra J e le mappe esponenziali.
  - Jost nel capitolo 4.3 e Abate a pag $424 + 433 - 435$ parlano del legame con i punti coniugati e morse theory.
- Discorso della index form come azione le cui equazioni eulero lagrange determinano l'equazione geodetica. (fonte Jost pag $177 - 179$).
- Discorso Decomposizione dei Jacobi field in campi orizzonatali e verticali (fonte Jost pag $180 - 181$, `http://en.wikipedia.org/wiki/Jacobi_field`.