Chapter 1

Lagrangian Systems and Peierls Brackets

> ⚠ .. introduzione dedichiamo sforzo alla definizione dei sistemi lagrangiani astratti per dare un punto di vista unificato ai sistemi a gradi libertÃă continui (continui macroscopici, fluidi, campi) e a gradi di libertÃă discreti Li vediamo come sotto classi dei sistemi lagrangiani notiamo che in entrambi i casi c'ÃÍ un possibile dato di cauchy usiamo questo ingrediente per definire l'algoritmo di peierls.

1.1 Abstract Mechanical Systems

It's possible to state a mathematical definition sufficiently broad to include all the systems in ordinary analytical mechanics regardless of the cardinality of degrees of freedom in a unified way.

Definition 1: Abstract Evolutive System

Pair (E, P) composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on manifold M called *"configuration bundle"*.

- $P : \Gamma^\infty(E) \to \Gamma^\infty(E)$
  operator called *"motion operator"*

This formultation is still very distant from the physical interpretation but has the benift to highlight the minimal mathematical objects which must be fixed in order to specify a mechanical systems.

Kinematics The configuration bundle encompass all the kinematical structure of the system, the pivotal role is played by the smooth sections which are to be understood as all the possible conformation of the system.

> **Notation fixing**
>
> $$\mathtt{C} := \Gamma^\infty(M, E)$$
>
> Space of kinematic configurations.

A section is not a statical configuration, equivalent to a specific point in the configuration space of ordinary classical systems, but has to be seen as a specific realization of the kinematics in the sense of a complete description of a possible motion. At this level of abstraction, since no space-time structure has been specified, terms like stasis and motion must be taken with care .The natural physical interpretation should be clearly manifested through the concrete realization of systems with discrete and continuous degree of freedom.

> **Observation 1: Mathematical structure**
>
> Mathematically speaking this set should be regarded as an infinite dimensional Manifold.
> This framework provides a geometric characterization of the notion of variations as tangent vectors on the the space of kinematic configurations .[**?**]

> **Observation 2: Coordinate Representation**
>
> The choice of a chart atlas $\mathscr{A}(M)$ on the base space M and $\mathscr{A}(E)$ on the total space E provides a correspondence between each configuration $\gamma \in \mathtt{C}$ and family of smooth real functions $\{f_{\alpha\beta} : A_\alpha \subset \mathbb{R}^m \to \mathbb{R}^q\}$. The process is trivial:
>
> $$\gamma \in \mathtt{C} \mapsto \{f_{A,U} = \psi_U \circ \gamma \circ \psi_A^{-1} \,|\, (A, \psi_A) \in \mathscr{A}(M), (U, \psi_U) \in \mathscr{A}(E)\}$$
>
> Since the whole section as a global object is quite difficult to handle is customary in field theory to work in the more practical local representation.

> **Observation 3: Further specification of the system's kinematics**
>
> The general formalism doesn't require any other structure to be carried forward. Additional structure on the fiber , the base or the whole bundle are to be prescribed in order to specify a precise physical model, e.g. the spin structure on E for the Dirac Field.[**?**]

Dynamics The operator P is the object that contains all the information about the dynamic evolution of the system. It has the role to select the dinamically compatible configuration among all the admissible kinematic configurations of $\mathtt{C}$, exactly as it

happens in analytical mechanics where the dynamic equations shape the natural motions.
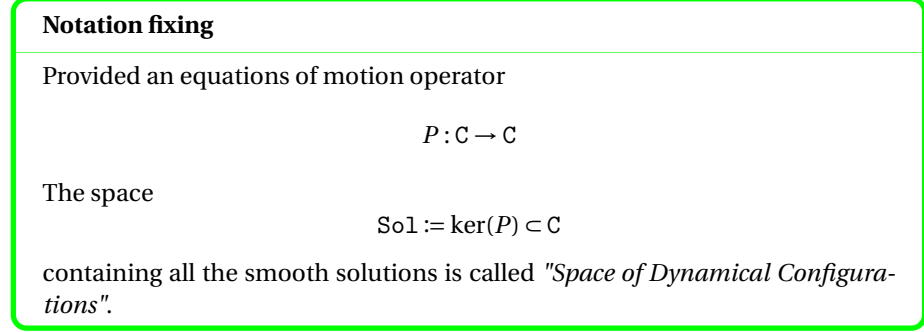

Notation fixing

Provided an equations of motion operator

$$P : \texttt{C} \to \texttt{C}$$

The space

$$\texttt{Sol} := \ker(P) \subset \texttt{C}$$

containing all the smooth solutions is called *"Space of Dynamical Configurations"*.

Figure 1.1: Geometric picture of the basic mechanical system's structure.
⚠immagine

1.1.1 Lagrangian Dynamics

Lagrangian systems constitute a subclass of the abstact mechanical systems of more practical interest:

Definition 2: Lagrangian System

Pair $(E, \mathscr{L})$ composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on the oriented manifold $(M, \mathfrak{o})$ called *"configuration bundle"*.

- $\mathscr{L} : J^r E \to \wedge^m T^* M$
  bundle-morphism from the r-th Jet Bundle to the top-dimensional forms bundle over the base manifold M called *"Lagrangian density"* or simply *"Lagrangian"* of r-th order.

N.B. : In what follows all the systems considered will be exclusively of first order.

In this case is the Lagrangian density the object containing all the information about the dynamic evolution of the system.

In order to reconstruct the system's dynamic from the Lagrangian density has to be understood the mathematical nature of $\mathscr{L}$. $\mathscr{L}$ maps point q_p on the fiber $J_p^r E$ to a m-form on $T_p M$. Recalling the definition of jet bundles is clear that for each smooth section on E is associated a smooth section on the b$J^r E$:

$$\phi \in \Gamma^\infty(E) \mapsto (\phi, \partial_\mu \phi, \partial_{\mu,\nu} \phi, \dots \partial_{\tilde{\alpha}} \phi)$$

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where $\tilde{\alpha}$ is a multi-index of length r. The correspondence is not univocal since sections equal up to the r-th order define the same jet section. The smoothness of $\mathscr{L}$ ensure that each jet bundle section is mapped to a smooth section in the top-forms bundle i.e. the most general integrable object on a orientable manifold.

It should be clear that $\mathscr{L}$ is a specific choice among the vast class of functions suitable to be a good Lagrangian density over the Configuration Bundle E:

Definition 3: Lagrangian Density on the bundle E

$$\mathsf{Lag}^r(E) := \hom\left(J^r E, \ \overset{m}{\bigwedge}(T^* M)\right) \cong \left\{f : \Gamma^\infty(J^r E) \to \Omega^m(M)\right\}$$

(where $\Omega^m(M)$ is the common name for $\Gamma^\infty\left(\bigwedge^m(T^* M)\right)$ in the context of Grassmann algebras.) The equivalence states the fact that a bundle-morphism induce a mapping between the sections.

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this choice fix the "Dynamical identity" of the considered system.

Proposition 1.1.1 $\mathsf{Lag}^r(E)$ *has an obvious vector space structure inherited by the linear structure of* $\Omega(M)$.

Thanks to the correspondence between a section $\phi \in \mathsf{C}$ and his r-th jet, it's possible to consider the Lagrangian as directly acting on the kinematic configurations. In layman terms the image $\mathscr{L}[\phi]\mathrm{d}\mu$, where $\mathrm{d}\mu$ is the measure associated to the orientation $\mathfrak{o}$, is something that can be measured over the whole base space.
This property suggests the introduction of the class of associated functionals:

Definition 4: Lagrangian functional

Is a functional on C with values on regular distribution over M associated to the generic $\mathscr{L} \in \mathsf{Lag}$.
$$\mathscr{O}_{\mathscr{L}} : \mathsf{C} \to \left(C_0^\infty(M)\right)'$$

Such that the lagrangian functional associated to $\mathscr{L}$, valued on the configuration $\phi \in \mathsf{C}$ and tested on the test-function $f \in C_0^\infty(M)$ it's given by:

$$\mathscr{O}_{\mathscr{L}}[\phi](f) = \int_M \mathscr{L}[\phi] f \, \mathrm{d}\mu$$

Proposition 1.1.2 *As a distribution* $\mathscr{O}_{\mathscr{L}}[\phi](f)$ *is necessarily linear in the test-functions entry but not in the configurations entry.*

Observation 4

The choice of the image of $\mathscr{O}_{\mathscr{L}}$ as a distribution it's a necessary precaution to

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ensure that functional is *"convergent"* whatever is the configuration on which is evaluated. In fact, despite $\mathscr{L}[\phi]$ is integrable with respect to the measure $\mathrm{d}\mu$, it's not necessary summable if the support of the configuration ϕ becomes arbitrarily large.

This is a simple consequence of the well known sequence of inclusions:

$$\mathscr{L}[\phi] \in C_0^\infty(M) \subset L_{\text{loc}}^1(M,\mu) \supsetneq L^1(M,\mu)$$

of the functional analysis . Indeed, the functional

$$\mathscr{O}_\mathscr{L}[\phi] = \int_{\text{supp}(\phi)} \mathscr{L}[\phi]\mathrm{d}\mu$$

is well defined for all $\mathscr{L} \in \mathsf{Lag}^r(E)$ only over the compactly supported sections. To take account of the global sections it's sufficient to dampen the integral multiplying the integrand with an arbitrary test-function.

Notation fixing

When calculated for the specific density of the Lagrangian system $\mathscr{O}_\mathscr{L}$ takes the name of *Action* or *Total Lagrangian*.

The introduction of the Lagrangian density is meaningless without the prescription of a dynamical principle which allows to determine univocally a differential operator P on the kinematics configurations space $\mathbb{C}$. This fundamental principle is the *least action principle*. A proper justification of this claim should require the presentation of the differential calculus on the infinite dimensional manifolds $\mathbb{C}$. Jumping straight to the conclusion we can state this correspondence as a principle in term of a function which assign for all lagrangian densities an operator on the kinematic configurations space. In the case of first order lagrangian we define

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Definition 5: Euler-Lagrange operator

It's the differential operator
$$Q_\chi : \mathbb{C} \to \mathbb{C}$$
relative to the lagrangian density $\chi \in \mathsf{Lag}^1(E)$, such that:

$$Q_\chi(\gamma) = \left(\partial_\mu\left(\frac{\partial\chi}{\partial(\partial_\mu\phi)}\Big|_\gamma\right) - \frac{\partial\chi}{\partial\phi}\Big|_\gamma\right) \qquad \forall\gamma \in \mathbb{C} \tag{1.1}$$

(where $\left(\frac{\partial\chi}{\partial(\partial_\mu\phi)}\right.$ is the be intended as the lagrangian density constructed differentiating $\chi(\phi,\partial_\mu)$ as an ordinary function treating its functional entries as an usual scalar variable.)

> **Observation 5**
>
> The whole theory of both Lagrangian densities class and Euler-Lagrange equation could be stated in a more syntetic way in terms of the Grassmann-graded variational bicomplex.[?][?]

1.2 Concrete Realization

In the previous section we claim that the abstract definition of Lagrangian systems is broad enough to encompass all the classical lagrangian systems with both discrete degrees of freedom, like particles, and continuous degree of freedom, like fluids or fields. Let' show two of the most significant examples.

1.2.1 Classical Field Theory

Basically a *Fields System* is nothing more than an abstract Field System (E, P) where the base space M is a suitable Spacetime manifold[?]. At this stage the question about the Lagrangian nature of the dynamics is purely ancillary.
The idea of taking bundles on a space-time manifold is physically intuitive, kinematically speaking a fields configuration is simply the association of some element of the fiber Q for each point of the space-time M.

However, there are two more requirements that are often prescribed in commonly studied field theories.

Linear Sistem Condition

- The configuration bundle $E \xrightarrow{\pi} M$ is a <u>vector bundle</u>.

Even if it might make sense to speak of nonlinear fields in some more general context, this condition it's a necessary element in case some form of the *superposition principles* as to be taken in account. Obviously this hypothesis is not sufficient to formulate the principle in the strong classical way, i.e.:"the response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually" mostly because only free systems can be considered at this stage and any statement about stimulus can make sense. However It assure that C is a vector space and , in conjunction with the linearity of motion operator P, Sol = ker(P) is a linear subspace. In other words every linear combination of kinematic configuration it's still a kinematic configuration.

Propagative Dynamic Condition

- The base manifold M is a <u>Globally Hyperbolic Spacetime</u>.

- The motion operator P is <u>PDE-hyperbolic</u>.

The first condition ensures the existence of Cauchy surfaces and therefore permits to state *Cauchy Problems* assigning an initial data on such regions. Furthermore, PDE-hyperbolicity of the motion operator P guarantees that for every Cauchy surface $\Sigma \subset M$ the corresponding initial data problem is well posed, that is:

$$\begin{cases} Pu = 0 \\ u = u_0 \\ \nabla_{\vec{n}} u = u_1 \end{cases} \tag{1.2}$$

admit a unique solution $u \in \Gamma(E)$ for all $(u_0, u_1) \in \Gamma(\Sigma) \times \Gamma(\Sigma)$.
This suggests the following definition:

Notation fixing

The set of all the smooth initial data which can be given on the Cauchy Surface Σ is: $\mathsf{Data}(\Sigma) := \left\{ (f_0, f_1) \mid f_i \in \Gamma^\infty(\Sigma) \right\} \equiv \Gamma^\infty(\Sigma) \times \Gamma^\infty(\Sigma)$

Observation 6

$\mathsf{Data}(\Sigma)$ inherit the linear structure of its component $\Gamma^\infty(\Sigma)$.

In this term the well-posedness of the Cauchy problem can be stated as follow:

Proposition 1.2.1 *The map* $\mathbf{s} : \mathsf{Data}(\Sigma) \to \mathsf{Sol}$ *which assign to* $(u_0, u_1) \in \mathsf{Data}(\Sigma)$ *the unique solution of the cauchy problem 1.2 is linear and bijective.*

Since any solution, when restricted to a generic Cauchy surface Σ', determines another pair of initial data, i.e.:

$$\phi \equiv \mathbf{s}(\phi|_{\Sigma'}, \nabla_{\vec{n}'} \phi|_{\Sigma'}) \quad \forall \phi \in \mathsf{Sol}$$

we can define the set of initial data regardless of the particular Cauchy surface:

Definition 6: Set of smooth initial Data

$$\mathsf{Data} := \frac{\bigsqcup\limits_{\Sigma \in \mathscr{P}_C(M)} \mathsf{Data}(\Sigma)}{\sim}$$

where $\sim$ is such that:

$$(f_0, f_1)|_\Sigma \sim (g_0, g_1)|_{\Sigma'} \Leftrightarrow \mathbf{s}(f_0, f_1) = \mathbf{s}(g_0, g_1)$$

Initial data, associated with different surface, are similar if they lead to the same solution.

Proposition 1.2.2 Data *is still a vector space.*

Proof:

It's sufficient to prove that:

$$[\phi_a + \phi_b] = [\phi_a] + [\phi_b]$$

where $[\phi] = \left\{(\phi|_\Sigma, \nabla_{\vec{n}}\phi|_\Sigma) \middle| \Sigma \in \mathscr{P}_C\right\}$. In fact:

$$\mathbf{s}_{\Sigma'}\big([(a',b')] + [(c',d')]\big) = \mathbf{s}_\Sigma\big([(a,b)] + [(c,d)]\big) = \mathbf{s}_\Sigma\big([(a,b)]\big) + \mathbf{s}_\Sigma\big([(c,d)]\big) =$$
$$= \mathbf{s}_{\Sigma'}\big([(a',b')]\big) + \mathbf{s}_{\Sigma'}\big([(c',d')]\big) = \mathbf{s}_{\Sigma'}\big([(a',b')] + [(c',d')]\big)$$

□

Corollary 1.2.1 *The function* $: \mathrm{Data}(\Sigma) \to \mathrm{Sol}$ *which map every equivalence class to the associated solution is linear and bijective.*

Observation 7

The Propagative dynamic condition is the main ingredient in the *initial data quantization* procedure[**?**].

Green-Hyperbolic Dynamic Condition

- The dynamic is generated by a <u>Lagrangian</u>, i.e. $P = Q_{\mathscr{L}}$.

- The motion operator P is a <u>Green Hyperbolic</u> L.P.D.O.

It's customary in Algebraic quantum field theory to identify quantizable systems through this condition. In fact this condition is a necessary hypothesis to carry on the Peierls construction and the brackets underlie to the definition of the classical symplectic structure.

We remark that Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense, therefore the last two condition are not equivalent.

Example: 1

It's straightforward to frame the well-known *Real Scalar Field on curved background* inside this abstract picture:

Configuration bundle	real scalar field: $F \equiv \mathbb{R}$
	curved background: M is globally hyperbolic
Kinematic Configurations	$\mathtt{C} = C^\infty(M, \mathbb{R})$
Motion Operator	$P = \Box_M + m^2 + \xi R$
	normally hyperbolic operator

Where P is the Klein-Gordon Operator: ξ and m^2 are two real numbers, R stands for the scalar curvature build out of g and $\Box_M := g^a b \nabla_a \nabla_b$ is the d'Alambert operator associated to the Levi-Civita connection.

Note that this system satisfies all the above three condition.

1.2.2 Finite Degree systems as a Field

With a little more effort is possible to see every ordinary mechanic system -with discrete degrees of freedom- as a special case of classical field.

Consider a lagrangian system (Q, L) with configuration space Q and Lagrangian function $L : TQ \rightarrow \mathbb{R}$.

Remembering the intuitive meaning of Q as set of all the statical conformation of the system, it's natural to read as kinematic configuration all and only the trajectories compatible with the constraints. In other words the space of kinematic configurations is the set of all the parametrized smooth curves on Q:

$$\mathbb{C} = C^\infty(\mathbb{R}, Q)$$

Therefore, the configuration bundle of this system is the trivial smooth bundle:

$$E = (\mathbb{R} \times Q, \pi, \mathbb{R})$$

of typical fiber Q, since the corresponding space of sections coincides to the required configuration space:

$$\Gamma^\infty(E) \equiv C^\infty(\mathbb{R}, Q)$$

Also the "field theoretic" picture of the dynamic is not difficult to achieve. It's enough to recall how the Euler-Lagrange of ordinary mechanical system can be derived from the *least action principle*:

Least Action Principle: The natural motions of the mechanical system (Q, L) are the stationary points of the action functional:

$$S[\gamma] = \int_\mathbb{R} L(t, \gamma^i, \dot{\gamma}^i) dt$$

constructed from the Lagrangian of the system.

Remembering that we have interpreted the space $\mathbb{R}$ of the parameter t as the base manifold of the configuration bundle E, it's immediate to see S as the *Total Lagrangian* related to the lagrangian density:

$$\mathcal{L}[\gamma] := L \circ (\gamma)^\uparrow dt$$

obtained evaluating the ordinary Lagrangian on the lifted curve $\gamma^\uparrow$ (lift of $\gamma \in \mathbb{C}$ from Q to TQ).

Summing up, the mathematical structure of such mechanical system is encoded as follows:

Configuration bundle	Trivial bundle $E = Q \times \mathbb{R}$
	Base manifold: $M = \mathbb{R}$
Kinematic Configurations	$\mathbb{C} = C^\infty(\mathbb{R}, Q)$
Lagrangian Density	$\mathscr{L}[\gamma] := \left(L \circ \gamma^\uparrow \right) dt = L(t, \gamma^i, \dot{\gamma}^i) dt$
Motion Operator	$P = Q_{\mathscr{L}} \equiv \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^i} L - \frac{\partial}{\partial x^i} L \right.$
	normally hyperbolic operator ($\triangle$?)

Considering that the space M is trivially globally hyperbolic, since every point $t \in \mathbb{R}$ is a genuine "Cauchy Surface", it's evident how this system can be seen as a special case of field theory: a "field of curves".

Observation 8

Unless the configuration space is a vector space, the corresponding field theory cannot be linear.

1.3   Geometric mechanics of Finite Degree systems

La visione precedente ÃÍ molto generale ma ci sono alcune strutture classiche che voglio replicare sul campo come la forma simplettica, le osservabili e le parentesi di poisson. Mi sembra piÃź chiaro vederle dopo aver raccontato queste.

Mi atterrei all'approccio rapido che segue Wald (tralasciando il ponte con la meccanica analitica dei corsi standard e concentrandomi sull'approccio geometrico)

Quindi devo parlare un po' di meccanica geometrica, di

- Spazio delle Fasi

- tautological 1-form

- simplectic form

- canonical coordinate and darboux theorem

- observable as smooth scalar field on the phase space

- poisson structure

1.3.1   Linear dynamical systems

Most of the physical systems that are encountered in the theory of fields are linear. Of course is possible to come across linear systems even in ordinary mechanics. In that case the the difference between the underlying geometric entities tend to fade out as a consequence of the flatness of the configuration space.

Da riempire: devo dire che

- Q ÃÍ piatto

- TQ ÃÍ fibrato vettoriale

- Q si identifica con il tangente quindi la forma simplettica ÃÍ definita
  direttamente sulle configurazioni

- ecc vedere primo capitolo wald

1.4 Peierls Brackets

In this section we present more extensively the original Peierls' construction. Please note that we are not trying to provide the state of the art on the Peierls bracket (see for example [**?**] for the treatment in presence of gauge freedom) but only to expand and modernize the first approach given by Peierls. Instead of considering only scalar theory we extend the algorithm to a broader class of systems.

Observation 9: Peierls Bracket vs Poisson Bracket

Paraphrasing an observation made by Sharan[**?**]:

The Poisson bracket determines how one quantity b(t, q, p) changes another quantity a(t, q, p) when it acts as the Hamiltonian or vice-versa. The Peierls bracket, on the other hand, determines how one quantity b(t, q, p) when added to the system Hamiltonian h with an infinitesimal coefficient λ affects changes in another quantity a(t, q, p) and vice-versa, i.e. The Peierls bracket is related to the change in an observable when the trajectory on which it is evaluated gets shifted due to an infinitesimal change in the Lagrangian of the system by another Lagragian density.

While the Poisson bracket between two observables a and b is defined on the whole phase space and is not dependent on the existence of a Hamiltonian, the Peierls bracket refers to a specific trajectory determined by a governing Lagrangian.

Purpose of the Peierls' procedure is to provide a bilinear form on the space of Lagrangian densities with time-compact support. This form induces a pre-symplectic structure on suitable subspaces of functionals to which can be recognized the role of *classical observables* of the theory.

⚠ Aggiungere altre chiacchiere e marketing riguardo le PB, vedere nelle fonti cosa dicono i sapienti

1.4.1 Peierls' construction.

The Peierls's construction algorithm is well defined for a specific class of systems:

1. Linear field theory: $E = (E, \pi, M)$ is a vector bundle.

2. Linear Lagragian dynamics: $P = Q_{\mathscr{L}}$ is a L.P.D.O.

3. M is a globally Hyperbolic space-time.

4. Motion operator P is a green-hyperbolic.

The procedure can be summarized in a few steps:

1. Consider a *disturbance* χ that is a time-compact Lagrangian density .

2. Construct the *perturbation of a solution under the disturbance.*

3. Define the *effect of the disturbance* on a second Lagrangian functional.

4. Assemble the mutual effects of two different Lagrangian densities to give a *bracket.*

Let's review each step more carefully.

Disturbance and Disturbed motion operator

By *"disturbance"* we mean a time-compact supported lagrangian density $\chi \in \mathsf{Lag}^1$ which act as a perturbation on the system's lagrangian:

$$\mathscr{L} \rightsquigarrow \mathscr{L}' = \mathscr{L} + \epsilon \cdot \chi$$

where ϵ is a modulation parameter. The support condition is required in order to take in account only perturbations which affect the dynamic for a definite time interval. The motion operator of the disturbed dynamics results:

$$P_\epsilon = \left[\partial_\mu \Big(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \Big) - \frac{\partial \mathscr{L}}{\partial \phi} \right] + \epsilon \left[\partial_\mu \Big(\frac{\partial \chi}{\partial(\partial_\mu \phi)} \Big) - \frac{\partial \chi}{\partial \phi} \right] = P + \epsilon Q_\chi \tag{1.3}$$

Observation 10

P_ϵ is not necessary linear, the second Hypothesis guarantees the linearity only for P.

Solution of the disturbed motion

The second ingredient of the Peierls' procedure is the calculus of the *perturbed solutions* under the considered *disturbance*. These are the solutions $\phi' \in \mathsf{C}$ of P_ϵ obtainable by a infinitesimal linear perturbation of a fixed solution $\phi \in \mathsf{Sol}$. The good definition of linear superposition is guaranteed by the hypothesis 1). More precisely, has to be seek a configuration:

$$\phi'(x) = \phi(x) + \epsilon \eta(x) \in \mathsf{C}$$

such that:

$$P_\epsilon \phi'(x) = o(\epsilon)$$
$$P\phi(x) = 0$$

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[1] I.e. the top form $\chi(\phi)$ is time-compact supported for all $\phi \in \mathsf{C}$.

In other word has to be satisfied the following equation:

$$[P_\epsilon]\phi'(x) = [P + \epsilon Q_\chi](\phi(x) + \epsilon\eta(x)) = \epsilon\Big([P]\eta(x) + [Q_\chi](\phi(x) + \epsilon\eta(x))\Big) \overset{!}{=} o(\epsilon)$$

The condition of linearity for operator P doesn't hold for Q_χ in general. We can work around this problem taking into account the linearization[?, pag. 31] of operator Q_χ around the unperturbed solution $\phi(x)$. The linearization of Q_χ is the unique linear operator $[Q_\chi^{lin}(\phi)]$ such that:

$$[Q_\chi](\phi(x) + \epsilon\eta(x)) = [Q_\chi](\phi(x)) + \epsilon[Q_\chi^{lin}(\phi)](\eta(x)) + o(eta)$$

which can be seen as the first term of a *formal* Taylor expansion of operator Q_χ around ϕ. $\triangle^2$ This is reflected in a condition on the perturbation $\eta \in C_{tc}$:

$$[P_\epsilon]\phi'(x) = \epsilon\Big([P]\eta(x) + [Q_\chi\phi(x)]\Big) + \epsilon^2[Q_\chi^{lin}(\phi)]\eta(x) \overset{!}{=} o(\epsilon)$$

$$\Rightarrow P\eta = -Q_\chi\phi(x) \tag{1.4}$$

called *Jacobi Equation.* This equation is a non homogeneous P.D.E. with inhomogeneous term $(-Q_\chi\phi(x))$ fixed by the solution $\phi \in$ Sol to be perturbed.

Follows from the definition of green hyperbolicity that the domain restrictions of P to Γ_{pc}^∞ or Γ_{fc}^∞ admit a unique inverse G^+ and G^- respectively. Therefore, equation **??** admits a unique past compact solution η^+, called retarded perturbation of $\phi \in$ Sol, and a unique future compact solution η^-, called advanced perturbation:

$$\eta^\pm = G^\pm\big(-Q_\chi\phi\big) \tag{1.5}$$

Note that the time-compact support condition on χ guarantees that $Q_\chi\phi \in \text{dom}(G^+) \cap \text{dom}(G^-)$. Expression 1.5 reflects perfectly the orginal Peierls' notation where $\eta^\pm$ were noted as functions of the unperturbed solution: $\eta^+ \equiv D_\chi\phi$ and $\eta^- \equiv \text{C}_\chi\phi$.

Observation 11

In most practical case it's possible to give a more basic characterization of $\eta^\pm$ in term of a Cauchy problem. Has to be stressed that this approach is not possible in general since Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense i.e. the well-posedness of the Cauchy problem is not guaranteed on any Cauchy surface. [?, pag 1] [?, remark 3.18] [?, remark 2.1]
Consider a motion operator P which is also hyperbolic. Taking in account the time-compact support condition of χ, is possible to pick up two Cauchy surfaces $\Sigma_\pm$ (+ is after the perturbation while − stands for prior to the perturbation) such that:

$$J^\mp(\Sigma_\pm) \supset \text{supp}(\chi)$$

for all time-slice foliation of the globally hyperbolic space-time.

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^{2}If C is a Frechet manifold the expansion could be made rigorous defining $[Q_\chi^{lin}(\phi_0)] = [\frac{\partial Q_\chi}{\partial\phi}(\phi_0)]$ in term of the Gateux derivative.

For each of this two surfaces can be posed a Cauchy problem:

$$\begin{cases} P\eta = -Q_\chi\phi \\ (\eta, \nabla_n\eta)\big|_{\Sigma_\pm} = (0,0) \end{cases} \tag{1.6}$$

which , according to the well-posedness of the Cauchy problem, admits an unique solution. The link with the first presentation is that past/future -compact supported configuration always meet the initial data condition for some future/past Cauchy surface.

In conclusion,fixed a solution $\phi \in \mathtt{Sol}$ and a perturbation χ, are uniquely determined two perturbed solution:

$$\phi_\epsilon^\pm = \phi + \epsilon\eta^\pm \tag{1.7}$$

such that:

retarded pertubation	$\eta^+ \in \Gamma_{pc}^\infty$	$(\eta^+, \nabla_n\eta^+)\big|_{\Sigma_-} = (0,0)$	"propagating forward"
advanced pertubation	$\eta^- \in \Gamma_{fc}^\infty$	$(\eta^-, \nabla_n\eta^-)\big|_{\Sigma_+} = (0,0)$	"propagating backward"

Effect Operator

Considering an arbitrary continuous $\triangle^3$ functional $B : \mathtt{Sol} \to \mathbb{R}$ (not necessarily linear) we can define the effect of a perturbation on the values of B[?, pag. 5] as a map:

$$\mathbf{E}_\chi^\pm : C^1(\mathtt{Sol}, \mathbb{R}) \to C^1(\mathtt{Sol}, \mathbb{R})$$

$$\mathbf{E}_\chi^\pm B(\phi_0) := \lim_{\epsilon \to 0}\left(\frac{B(\phi_\epsilon^\pm) - B(\phi_0)}{\epsilon} \right) \tag{1.8}$$

The advanced and retarded effects of χ on B are then defined by comparing the original system with a new system defined by the same kinematic configuration space $\mathtt{C}$ but with perturbed lagrangian.

Observation 12

Expression 1.8 is clearly a special case of Gateaux derivative.[?]

The former expression appear quite simpler in case of a linear functional:

$$\mathbf{E}_\chi^\pm B(\phi_0) = B(\eta^\pm) \tag{1.9}$$

The Bracket

Remembering that every lagrangian density define a continuous functional (Action). From that is possible to build a binary function:

$$\{\cdot, \cdot\} : \mathsf{Lag}_{tc} \times \mathsf{Lag}_{tc} \to \mathbb{R}$$

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[3]The precise notion of continuity require the specification of the infinite dimensional manifold structure.

as follow:

$$\{\chi, \omega\}(\phi_0) := E_\chi^+ F_\omega(\phi_0) - E_\chi^- F_\omega(\phi_0) \qquad (1.10)$$

Proposition 1.4.1 (Bilinearity) *When restricted to Linear Lagrangian densities* $\{\cdot, \cdot\}$ *is a bilinear form*

Proof:

Linearity in the first entry follows from equation [**?**] and the linearity of the Euler-Lagrange operator Q. over Lag.
Linearity in the second entry is guaranteed only for lagrangian densities ω which provide a linear Lagrangian Functional F_ω.

$\square$

We don't care to probe the simplectic property in this general ground. In the next chapter we will face the problem to determine symmetry and non-degeneracy properties for the case of *classical observable functional,* a subclass of Lagrangian functionals of most practical use in the quantization schemes.

1.4.2   Extension to non-linear theories

In the previous construction the green-hyperbolicity of motion operator P plays a primary role. Anyway the problem of searching perturbed solution of the disturbed dynamic can be stated even in presence of non-linear fields where the configuration bundle is not necessary a vector bundle and the motion operator is not linear.
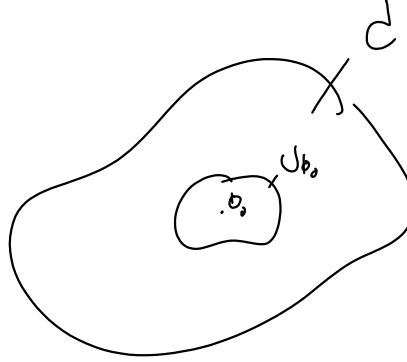
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Figure 1.2: Intrinsically, searching a variation of a solution $\gamma_0 \in \texttt{Sol}$ which solve the disturbed motion equation is equivalent to find the intersection of the perturbed solution with a local neighbourhood of $\Gamma_0 : U_{\gamma_0} \cap \ker(P_\epsilon)$.

The crucial point of the Peierls' procedure is to select among all the possible solution of the perturbed motion P_ϵ that configuration which can be constructed by a variation of some fixed solution of the non-perturbed dynamics $\gamma_0 \in \mathtt{Sol}$. In this sense the problem results a *"linearization"* inasmuch the search of such solution is restricted to a local neighbourhood of the "point" $\gamma_0 \in \mathtt{Sol}$.

Previously the choice to consider only the linear variation was quite natural but in the general case this preferential restriction is no longer possible. A way to recover a notation similar to 1.7 is to work patchwise choosing a coordinate representation. Fixed a solution $\gamma_0 \in \mathtt{Sol}$ and a local trivializing chart (A, ϕ_A) such that $A \cap \mathrm{ran}(\gamma_0) \neq \emptyset$ we can define a local infinitesimal variation by acting on his components:

$$\gamma_\lambda^i(x) = \gamma_0^i(x) + \lambda \eta^i(x) \qquad \forall x \in \pi(A)$$

where γ_0^i are the component of the unperturbed solution in the open set A and $\eta^i \in^q$ is a generic real q-ple (q is the dimension of the typical fiber manifold). λ is a real parameter that has to be "sufficiently small" in order to guarantee that the range of γ_λ is properly contained in A. In other words the construction of the linear variation, that for linear field theories could be done in a global way, in the general case can be recovered only locally variating the components.

Therefore is possible to define the effect of the a disturbance locally, searching local section $\gamma_\epsilon^i = \gamma_0^i + \epsilon \eta^i$ solving the disturbed dynamic equation up to the first order in ϵ i.e.

$$\left[P_\epsilon\right]\gamma_\epsilon^i = o(\epsilon)$$

where $\left[P_\epsilon\right]$ has to be intended as the coordinate representation of the restriction on the local section $\Gamma^\infty(A)$. ⚠[4]

Observation 13

W.l.o.g has been taken the the same scalar ϵ to modulate both the perturbation γ_ϵ that the disturbance on the motion operator. On the contrary consider two different parameter is immaterial since only the smaller should be taken in account.

From the explicit equation of the perturbed solution:

$$\left(\left[P\right] + \epsilon\left[Q_\chi\right]\right)\left(\gamma_0^i + \epsilon\eta^i\right) = o(\epsilon)$$

follows an equation on the components of the local perturbation. In this case has to be dealt with the problem of non-linearity not only for Euler-Lagrange operator Q_χ but also for P. Arresting the expension to the first order in ϵ results:

$$\left[P_{\gamma_0}^{lin}\right]\eta^i(x) = -\left(Q_\chi(\gamma_0)\right)(x) \tag{1.11}$$

the *Jacobi equation* on the unperturbed solution $\gamma_0 \in \mathtt{Sol}$.

---

[4]non mi ÃÍ evidente se la rappresentazione in coordinate di un operatore agente sulle sezioni si realizza in modo ovvio, ma non vedo nemmeno ostruzioni! Di sicuro l'operazione ÃÍ ben definita per gli L.P.D.O visto che la definizione prevede proprio che su ogni carta locale trivializzante l'operatore sia lineare alle derivate parziali

> **Observation 14**
>
> We've moved from an operator P definited on $\mathcal{C}$ to an operator $P_{\gamma_0}^{lin}$ defined on the space of variation. From a global point of view this variation can be seen as the tangent vector i.e. s $\eta \in T_{\gamma_0}\mathcal{C}$. In the case of the linear system this passage was unnecessary, the Jacobi equation was directly defined on $\mathcal{C}$ since, for linear system, any section could be seen as a generator of an infinitesimal variation. This behaviour mimics perfectly what happens in ordinary classical mechanics where the configuration space of a linear system is a vector space i.e a "flat" manifold[a] which is isomorphic to his tangent space in every point.
>
> ---
> [a]In sense that admits a global coordinate chart.

Provided that the linearized motion operator (which is now properly a linear partial differential operator) is Green-Hyperbolic, the Peierels construction can continue as before. Has to be noted that now the advanced/retarded perturbation are formally identical to the former:

$$\eta^{\pm i} = G^\pm \big(- Q_\chi \gamma_0^j \big)$$

with the important difference that $G^\pm$ are now the Green operators of the linearized motion operator and depend on the fixed solution γ_0^j.

In conclusion the perturbed solution:

$$\gamma_\chi^{\pm i} = \gamma_0^i \pm G^\pm \big(- Q_\chi \gamma_0^j \big)$$

has to be intended as the "glueing" of all the local chart representations covering the chosen solution.

Example: Finite Dimensional Case

As an example of such process we can consider a *field of curves* i.e. an ordinary classical mechanical system in the field theoretic picture. We have shown in section 1.2.2 that such systems are generally non linear: the configuration bundle is not a vector one and then linearity of P cannot be defined .

However the base manifold is very simple. Indeed $M = \mathbb{R}$ can be seen as a trivial globally-hyperbolic space-time where every real scalar $t \in M$ is a Cauchy surface.
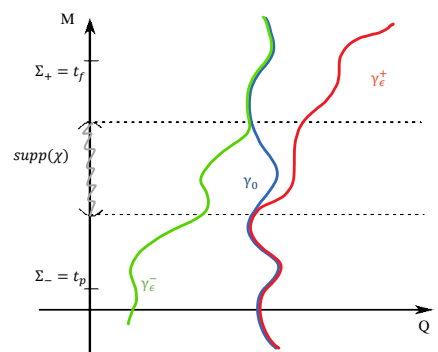
⚠ Da Finire

Figure 1.3: