Chapter 1

Geodesic Fields

Usually, in the context of differential geometry, a *geodesic curve* is characterized as a self-parallel curve in order to generalize the *straight lines*. Considering a differential manifold M endowed with an affine connection ∇ we define:

Definition 1: Geodesic

A curve ⚠[a] $\gamma : [a, b] \to M$ such that:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \tag{1.1}$$

where $\dot{\gamma}^{\mu} := \frac{d\gamma^{\mu}}{dt}$ is the tangent vector to the curve.

[a] Devo dire smooth o piecewise?

Notation fixing

In local chart the previous equation assume the popular expression:

$$\ddot{\gamma}^i + \Gamma^i_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0 \tag{1.2}$$

Where Γ^i_{jk} is the coordinate representation of the Christoffel symbols of the connection.

In presence of a pseudo-Riemannian metric is possible to present the geodesic in a metric sense i.e. as the curve which extremizes the *Energy Functional*[1]:

Definition 2: Energy functional

[1] Remember that for arc-length parametrized curves the Energy functional coincide with the length functional.[**?**, Lemma 1.4.2]

$$E(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|^2 dt \tag{1.3}$$

Considering only the proper variation (that keep the end-point fixed), the extremum condition corresponds to equation where ∇ is the unique Levi-Civita connection (torsion-free and metric-compatible).

In general relativity the problem of the geodesic equation linearization, named *Jacobi equations* takes a central role. [2] ⚠(nel file di ripasso di geometria riemanniana ho scritto gran parte delle definizione conviene vedere cosa mi serve effettivamente... Di certo mi avvalgo della seguente equazione

Notation fixing

In local charts the Jacobi fields along the geodesic γ solve a linear O.D.E.:

$$\left(X''\right)^\mu + R^\mu_{i\alpha j} T^i X^\alpha T^j = 0 \tag{1.4}$$

where:

- $\left(X'\right)^\mu := \left(\nabla_{\dot\gamma(t)} X\right)^\mu$ is the covariant derivative along the curve γ.
- $T \equiv \dot\gamma(t)$ stands for the tangent vector to the curve γ.

The rest of this chapter will be dedicated to presenting the physical approach to the Geodesic.

1.1 Geodesic Problem as a Mechanical Systems

The basic idea is very simple, portray the geodesic curve as the natural motion of a free particle constrained on the Pseudo-Riemannian manifold Q.

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obvious enough this problem can be seen as a generalization of the calculation of the motions of free falling parcticles In terms of general relativity this problem can be instantly recognized as the derivation of the free-falling particles motion.

However, there is no lack of alternative viewpoints . The framework of the classical Geometric Mechanics teach us to picture the "static" configurations of a constrained, complex, classical system as a point on the *Configuration space* manifold. According to that, the geodesic motion can be seen as a realization of a particular dynamics on each mechanical system endowed with a pseudo-Riemannian configuration space[3].

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[2]Usually in this context takes the name of *Geodesic deviation* problem[**?**, pag. 46].

[3]Such systems can be depicted as "geodesic" even in presence of a position-dependant potential.[**?**, Cap 3.7]

Theorem 1.1.1 (Geodesic Motion) *The geodesics on the Pseudo-Riemmanian manifold* (Q, g) *are the natural motions of the ordinary Lagrangian system* (Q, L) *where:*

$$L(V_q) := \frac{1}{2} g_q(V, V) \tag{1.5}$$

Proof:

The Euler-Lagrange equation of L coincides with the geodesic equation 1. ⚠.. ÃÍ sul quaderno non so se metterla

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Observation 1

The geodesic system is not simply Lagrangian but also Hamiltonian. This property follows from the hyperregularity[**?**] of L.

⚠Anyway we will neglect this fact inasmuch in what follows only the Lagrangian character assumes a role.

As shown in chapter 2, every system with discrete degrees of freedom can be seen as the trivial field system. From that follows the alternative characterization of geodesic as a lagrangian field:

Corollary 1.1.1 (Geodesic field) *The geodesics on the Pseudo-Riemmanian manifold* (Q, g) *can be seen as the* Dynamical Configurations *of the lagrangian field system* $(E, \mathscr{L})$ *where:*

- $E = (Q \times \mathbb{R}, \pi, \mathbb{R})$ *trivial smooth bundle on the real line.*
- $\mathscr{L}[\gamma] = \frac{1}{2} g(\dot{\gamma}, \dot{\gamma})(t) dt$

Proof:

Is simple application of the correspondence seen in chapter **??**.

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From this perspective is clear that the Energy Functional can be seen as the action in the geodesic field dynamics and equation 1 is nothing more than the motion equation under the *least action principle*.

1.2 Peierls Bracket of the Geodesic field

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Da dire: espressione in coordinate della lagrangiana, ÃÍ altmente non lineare perchÃÍ implicitamente ÃÍ $g_{\mu\nu}(\gamma^i(t))$ non polinomiale in γ^i ed esplicitamente ÃÍ quadratica, Mostrare esplicitamente che l'equazione di jacobi per il sistema ÃÍ effittivamente l'equjazione di jacobi (questo ÃÍ triviale se vedi come definisce il campo di jacobi jurgen

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1.2.1 Example: Geodesic field on FRW space-time.

1.3 Algebraic quantization of the Geodesic Field

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va ripetuto che la geodetica ÃÍ non lineare quindi ciÃš che effettivamente si quantizza ÃÍ jacobi lungo una prefissata geodetica. questo ÃÍ un campo lineare.

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1.3.1 Peierls Approach

1.3.2 Inital data Approach

1.4 Interpretations??????