Chapter 1

Lagrangian Systems and Peierls Brackets

⚠ ... introduzione dedichiamo sforzo alla definizione dei sistemi lagrangiani astratti per dare un punto di vista unificato ai sistemi a gradi libertÃă continui (continui macroscopici, fluidi, campi) e a gradi di libertÃă discreti Li vediamo come sotto classi dei sistemi lagrangiani notiamo che in entrambi i casi c'ÃÍ un possibile dato di cauchy usiamo questo ingrediente per definire l'algoritmo di peierls. ⚠

1.1 Abstract Mechanical Systems

It's possible to state a mathematical definition sufficiently broad to include all the systems in ordinary analytical mechanics regardless of the cardinality of degrees of freedom in a unified way.

Definition 1: Abstract Evolutive System

Pair (E, P) composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on manifold M called *"configuration bundle"*.

- $P : \Gamma^\infty(E) \to \Gamma^\infty(E)$
  operator called *"motion operator"*

This formultation is still very distant from the physical interpretation but has the benift to highlight the minimal mathematical objects which must be fixed in order to specify a mechanical systems.

Kinematics   The configuration bundle encompass all the kinematical structure of the system, the pivotal role is played by the smooth sections which are to be under-

stood as all the possible conformation of the system.

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Notation fixing

$$\mathtt{C} := \Gamma^\infty(M, E)$$

Space of kinematic configurations.

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A section is not a statical configuration, equivalent to a specific point in the configuration space of ordinary classical systems, but has to be seen as a specific realization of the kinematics in the sense of a complete description of a possible motion. At this level of abstraction, since no space-time structure has been specified, terms like stasis and motion must be taken with care .The natural physical interpretation should be clearly manifested through the concrete realization of systems with discrete and continuous degree of freedom.

Observation 1: Mathematical structure

Mathematically speaking this set should be regarded as an infinite dimensional Manifold.
This framework provides a geometric characterization of the notion of variations as tangent vectors on the the space of kinematic configurations .[?]

Observation 2: Coordinate Representation

The choice of a chart atlas $\mathscr{A}(M)$ on the base space M and $\mathscr{A}(E)$ on the total space E provides a correspondence between each configuration $\gamma \in \mathtt{C}$ and family of smooth real functions $\{f_{\alpha\beta} : A_\alpha \subset \mathbb{R}^m \to \mathbb{R}^q\}$. The process is trivial:

$$\gamma \in \mathtt{C} \mapsto \{f_{A,U} = \psi_U \circ \gamma \circ \psi_A^{-1} | (A, \psi_A) \in \mathscr{A}(M), (U, \psi_U) \in \mathscr{A}(E)\}$$

Since the whole section as a global object is quite difficult to handle is customary in field theory to work in the more practical local representation.

Observation 3: Further specification of the system's kinematics

The general formalism doesn't require any other structure to be carried forward. Additional structure on the fiber , the base or the whole bundle are to be prescribed in order to specify a precise physical model, e.g. the spin structure on E for the Dirac Field.[?]

Dynamics The operator P is the object that contains all the information about the dynamic evolution of the system. It has the role to select the dinamically compatible configuration among all the admissible kinematic configurations of $\mathtt{C}$, exactly as it happens in analytical mechanics where the dynamic equations shape the natural motions.
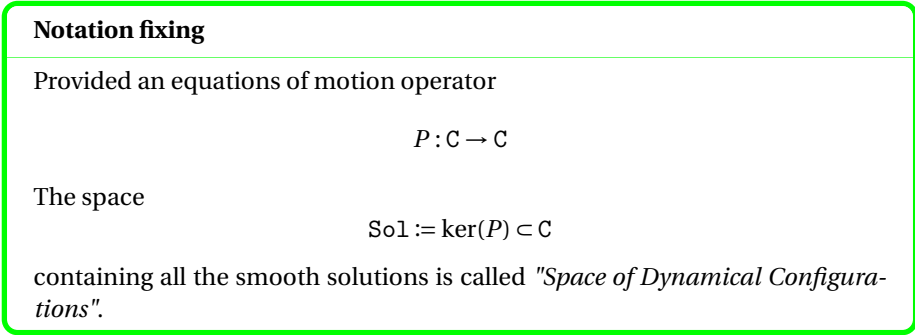
> **Notation fixing**
>
> Provided an equations of motion operator
>
> $$P : \mathtt{C} \to \mathtt{C}$$
>
> The space
>
> $$\mathtt{Sol} := \ker(P) \subset \mathtt{C}$$
>
> containing all the smooth solutions is called *"Space of Dynamical Configurations"*.

Figure 1.1: Geometric picture of the basic mechanical system's structure.
⚠immagine

1.1.1 Lagrangian Dynamics

Lagrangian systems constitute a subclass of the abstact mechanical systems of more practical interest:

> **Definition 2: Lagrangian System**
>
> Pair $(E, \mathscr{L})$ composed of:
>
> - $E \xrightarrow{\pi} M$
>   smooth fiber bundle of typical fiber Q on the oriented manifold $(M, \mathfrak{o})$ called *"configuration bundle"*.
>
> - $\mathscr{L} : J^r E \to \wedge^m T^* M$
>   bundle-morphism from the r-th Jet Bundle to the top-dimensionial forms bundle over the base manifold M called *"Lagrangian density"* or simply *"Lagrangian"* of r-th order.

N.B. : In what follows all the systems considered will be exclusively of first order.

In this case is the Lagrangian density the object containing all the information about the dynamic evolution of the system.

In order to reconstruct the system's dynamic from the Lagrangian density has to be understood the mathematical nature of $\mathscr{L}$. $\mathscr{L}$ maps point q_p on the fiber $J_p^r E$ to a m-form on $T_p M$. Recalling the definition of jet bundles is clear that for each smooth section on E is associated a smooth section on the b$J^r E$:

$$\phi \in \Gamma^\infty(E) \hookrightarrow (\phi, \partial_\mu \phi, \partial_{\mu,\nu} \phi, \dots \partial_{\vec{\alpha}} \phi)$$

where $\vec{\alpha}$ is a multi-index of length r. The correspondence is not univocal since sections equal up to the r-th order define the same jet section. The smoothness of $\mathscr{L}$

ensure that each jet bundle section is mapped to a smooth section in the top-forms bundle i.e. the most general integrable object on a orientable manifold.

It should be clear that $\mathscr{L}$ is a specific choice among the vast class of functions suitable to be a good Lagrangian density over the Configuration Bundle E:

Definition 3: Lagrangian Density on the bundle E

$$\mathrm{Lag}^r(E) := \mathrm{hom}\left(J^r E, \ \overset{m}{\bigwedge}(T^* M)\right) \cong \{f : \Gamma^\infty(J^r E) \to \Omega^m(M)\}$$

(where $\Omega^m(M)$ is the common name for $\Gamma^\infty\left(\bigwedge^m(T^* M)\right)$ in the context of Grassmann algebras.) The equivalence states the fact that a bundle-morphism induce a mapping between the sections.

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this choice fix the "Dynamical identity" of the considered system.

Proposition 1.1.1 $\mathrm{Lag}^r(E)$ *has an obvious vector space structure inherited by the linear structure of* $\Omega(M)$.

Thanks to the correspondence between a section $\phi \in \mathtt{C}$ and his r-th jet, it's possible to consider the Lagrangian as directly acting on the kinematic configurations. In layman terms the image $\mathscr{L}[\phi]\mathrm{d}\mu$, where $\mathrm{d}\mu$ is the measure associated to the orientation $\mathfrak{o}$, is something that can be measured over the whole base space.
This property suggests the introduction of the class of associated functionals:

Definition 4: Lagrangian functional

Is a functional on $\mathtt{C}$ with values on regular distribution over M associated to the generic $\mathscr{L} \in \mathrm{Lag}$.

$$\mathscr{O}_{\mathscr{L}} : \mathtt{C} \to \left(C_0^\infty(M)\right)'$$

Such that the lagrangian functional associated to $\mathscr{L}$, valued on the configuration $\phi \in \mathtt{C}$ and tested on the test-function $f \in C_0^\infty(M)$ it's given by:

$$\mathscr{O}_{\mathscr{L}}[\phi](f) = \int_M \mathscr{L}[\phi] f \, \mathrm{d}\mu$$

Proposition 1.1.2 *As a distribution* $\mathscr{O}_{\mathscr{L}}[\phi](f)$ *is necessarily linear in the test-functions entry but not in the configurations entry.*

Observation 4

The choice of the image of $\mathscr{O}_{\mathscr{L}}$ as a distribution it's a necessary precaution to ensure that functional is *"convergent"* whatever is the configuration on which is evaluated. In fact, despite $\mathscr{L}[\phi]$ is integrable with respect to the measure

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dμ, it's not necessary summable if the support of the configuration ϕ becomes arbitrarily large.

This is a simple consequence of the well known sequence of inclusions:

$$\mathscr{L}[\phi] \in C_0^\infty(M) \subset L_{\text{loc}}^1(M,\mu) \supsetneq L^1(M,\mu)$$

of the functional analysis . Indeed, the functional

$$\mathscr{O}_{\mathscr{L}}[\phi] = \int_{\text{supp}(\phi)} \mathscr{L}[\phi]\,\mathrm{d}\mu$$

is well defined for all $\mathscr{L} \in \mathsf{Lag}^r(E)$ only over the compactly supported sections. To take account of the global sections it's sufficient to dampen the integral multiplying the integrand with an arbitrary test-function.

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Notation fixing

When calculated for the specific density of the Lagrangian system $\mathscr{O}_{\mathscr{L}}$ takes the name of *Action* or *Total Lagrangian*.

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The introduction of the Lagrangian density is meaningless without the prescription of a dynamical principle which allows to determine univocally a differential operator P on the kinematics configurations space $\mathbb{C}$. This fundamental principle is the *least action principle*. A proper justification of this claim should require the presentation of the differential calculus on the infinite dimensional manifolds $\mathbb{C}$. Jumping straight to the conclusion we can state this correspondence as a principle in term of a function which assign for all lagrangian densities an operator on the kinematic configurations space. In the case of first order lagrangian we define

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Definition 5: Euler-Lagrange operator

It's the differential operator
$$Q_\chi : \mathbb{C} \to \mathbb{C}$$
relative to the lagrangian density $\chi \in \mathsf{Lag}^1(E)$, such that:

$$Q_\chi(\gamma) = \left(\partial_\mu \left(\frac{\partial \chi}{\partial(\partial_\mu \phi)} \Big|_\gamma \right) - \frac{\partial \chi}{\partial \phi} \Big|_\gamma \right) \qquad \forall \gamma \in \mathbb{C} \tag{1.1}$$

(where $\left(\frac{\partial \chi}{\partial(\partial_\mu \phi)} \right.$ is the be intended as the lagrangian density constructed differentiating $\chi(\phi, \partial_\mu)$ as an ordinary function treating its functional entries as an usual scalar variable.)

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Observation 5

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> **Observation 6**
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> The whole theory of both Lagrangian densities class and Euler-Lagrange equation could be stated in a more syntetic way in terms of the Grassmann-graded variational bicomplex.[?][?]

1.2 Concrete Realization

In the previous section we claim that the abstract definition of Lagrangian systems is broad enough to encompass all the classical lagrangian systems with both discrete degrees of freedom, like particles, and continuous degree of freedom, like fluids or fields. Let' show two of the most significant examples.

1.2.1 Classical Linear Field over a Space-Time

⚠ cosÃň com'ÃÍ la definizione ÃÍ troppo forte?

The field systems are a subset of the lagrangian systems:

> **Definition 6: Linear Fields on curved Background**
>
> It's a Lagragian system $(E, \mathscr{L})$ such that:
>
> - the configuration bundle $E \xrightarrow{\pi} M$ is a <u>vector bundle</u>.
>
> - the base manifold M is a <u>Globally Hyperbolic Spacetime</u>.
>
> - the Euler-Lagrange operator $P = Q_{\mathscr{L}}$ is a <u>Green Hyperbolic operator</u>.
>
> - For each Cauchy surface $\Sigma \subset M$ can be defined a well-posed Cauchy problem for the motion equation of P.[a]
>
> ---
> [a]Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. [?] [?]

The idea of taking bundles on a space-time manifold it's physically intuitive, kinematically speaking a fields configuration it's not more than an than an association of some element of the fiber Q for each point of the space-time M. But the other three condition are worth a deeper insight:

Vector Bundle Condition   Even if it might make sense to speak of nonlinear fields in some more general context, this condition it's a necessary element in case some form of the *superposition principles* as to be taken in account. Obviously this hypothesis is not sufficient to formulate the principle in the strong classical way, i.e.:"the response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually" mostly because only free systems can be considered at this stage and any statement about

stimulus can make sense.

However It assure that C is a vector space and , in conjunction with the linearity of motion operator P, Sol = ker(P) is a linear subspace. In other words every linear combination of kinematic configuration it's still a kinematic configuration.

Global hyperbolicity condition. This condition is strictly connected to the dynamic behaviour of the system.

⚠Def di dominio di dipendendenza footnote di definizione di spazio tempo def cauchy surface Remark causal future past def globally hyperbolic Teorema sulle caratterizzazioni

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Notation fixing

We denote the set of all the cauchy surfaces as $\mathscr{P}_C(M)$.

Glon iperbolic determina la fogliazione dello spazio tempo per superfici di cauchy La superficie di cauchy ÃÍ questa:

Definition 7: Cauchy surface

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questo da la possibilitÃǎ della buona posizione dei problemi di cauchy.. fisicamente ÃÍ la condizione minima per definire i dati iniziali dell'evoluzione dinamica. definisco data...

No! La definizione di green hyperbolicity garantisce invece l'esistenza e unicitÃǎ del problema di cauchy associata

e non solo, anche l'esistenza degli operatori di green associati che sono ingrediente fondamentale della costruzione di peierls

M ÃÍ glob iper e P ÃÍ green iper per tener conto del comporatamento propagativo definire sup cauchy definire s-t iperbolico (solo la caratterizzazione di ammetre una sup di cauchy) definire op green iperbolico su spazio tempo iperbolico (cioÃÍ ha delle green ope) Propr di buona definizione esistenza e unicita della soluzione

Di particolare ricorrenza fisica sono gli operatori normally iperbolic espressione in coordinate esempio K-g! ⚠

⚠

Secondo bar e ginoux per parlare di campo classico non serve specificare nient'altro... la condizione di ∃1! operatore di green di P insieme a quella di Essere un sistema lagrangiano ÃÍ un requisito minimo per definire senza ambiguitÃǎ le parentesi di peierls.

la condizione di green-hyperbolicity (che garantisce di ∃1! $E^{\mp}$ ma non che ∃1! soluzione del PC) corredata della scelta di un pairing permette di quantizzare secondo lo schema algebrico

La condizione di well-posedness del problema di cauchy da la possibilitÃǎ di quantizzare secondo lo schema dei dati iniziali

in tutti questi casi la candizione di Globally -hyperbolic per lo spazio tempo sottostante ÃÍ necessaria ⚠

Green-Hyperbolicity condition. The third condition ensures the existence of the Green's Operator as follows directly from definition. ⚠**Memento:** Pensavo di utilizzare la definizione di Green hyperbolic data da Bar che si avvale del concetto di formally dual (che non richiede la presenza del pairing) invece di quella usata in Advances AQFT che richiede solo che ammetta almeno un $G^\pm$ per poi dimostrare tramite teorema che se Ãĺ anche autoaggiunto vale l'unicitÃă. Si tratta solo di una piccola sfumatura.. Deve essere chiarito che in tutto ciÃš che faccio interessano che

$$\forall P \, \exists 1! \, G^\pm$$

. Che poi questa condizione derivi da GH secondo bar o Gh secondo dap+selfadj Ãĺ una di quelle questioni propriamente matematiche che poco interessa ai fisici della commissione.

⚠Devo richiedere che il green operator sia unico? sia negli schemi di quantizzazione che nella definizione di peierls faccio largo uso dell'unicitÃă. Per provare questa unicitÃă si passa per la definizione di una forma bilineare che permette di parlare di aggiunto formale e quindi avvalersi del teorema.

Cauchy condition. While the existence of a Cauchy surface allows to assign the data of initial value problems, the forth condition ensure the well -posedness of the problem for on every Cauchy surface Σ. I.e:

$$\begin{cases} Pu = 0 \\ u = u_0 \\ \nabla_{\vec{n}} u = u_1 \end{cases} \tag{1.2}$$

admit a unique solution $u \in \Gamma(E)$ for all $(u_0, u_1) \in \Gamma(\Sigma) \times \Gamma(\Sigma)$.
This suggests the following definition:

> **Notation fixing**
>
> The set of all the smooth initial data which can be given on the Cauchy Surface Σ is: $\mathsf{Data}(\Sigma) := \left\{ (f_0, f_1) \big| f_i \in \Gamma^\infty(\Sigma) \right\} \equiv \Gamma^\infty(\Sigma) \times \Gamma^\infty(\Sigma)$

> **Observation 7**
>
> $\mathsf{Data}(\Sigma)$ inherit the linear structure of its component $\Gamma^\infty(\Sigma)$.

In this term the well-posedness of the cauchy problem can be stated as follow:

> **Proposition 1.2.1** *The map* $\mathbf{s} : \mathsf{Data}(\Sigma) \to \mathsf{Sol}$ *which assign to* $(u_0, u_1) \in \mathsf{Data}(\Sigma)$ *the unique solution of the cauchy problem 1.2 is linear and bijective.*

Since any solution, when restricted to a generic Cauchy surface Σ', determines another pair of initial data, i.e.:

$$\phi \equiv \mathbf{s}(\phi|_{\Sigma'}, \nabla_{\vec{n}'}\phi|_{\Sigma'}) \quad \forall \phi \in \mathsf{Sol}$$

we can define the set of initial data regardless of the particular Cauchy surface:

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Definition 8: Set of smooth initial Data

$$\text{Data} := \frac{\bigsqcup_{\Sigma \in \mathscr{P}_C(M)} \text{Data}(\Sigma)}{\sim}$$

where $\sim$ is such that:

$$(f_0, f_1)|_\Sigma \sim (g_0, g_1)|_{\Sigma'} \Leftrightarrow \mathbf{s}(f_0, f_1) = \mathbf{s}(g_0, g_1)$$

Initial data, associated with different surface, are similar if they lead to the same solution.

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Proposition 1.2.2 Data *is still a vector space.*

Proof:

⚠ It's sufficient to prove that:

$$[\phi_a + \phi_b] = [\phi_a] + [\phi_b]$$

where $[\phi] = \{(\phi|_\Sigma, \nabla_{\tilde{n}}\phi|_\Sigma)| \Sigma \in \mathscr{P}_C\}$. In fact:

$$\mathbf{s}_{\Sigma'}\big([(a',b')] + [(c',d')]\big) = \mathbf{s}_\Sigma\big([(a,b)] + [(c,d)]\big) = \mathbf{s}_\Sigma\big([(a,b)]\big) + \mathbf{s}_\Sigma\big([(c,d)]\big) =$$
$$= \mathbf{s}_{\Sigma'}\big([(a',b')]\big) + \mathbf{s}_{\Sigma'}\big([(c',d')]\big) = \mathbf{s}_{\Sigma'}\big([(a',b')] + [(c',d')]\big)$$

□

Corollary 1.2.1 *The function* $: \text{Data}(\Sigma) \to \text{Sol}$ *which map every equivalence class to the associated solution is linear and bijective.*

1.2.2 Finite Degree systems

Every system with discrete degrees of freedom can be seen as a trivial field system. The correspondence is easily done:

- Configuration bundle of the system is the trivial $E = Q \times \mathbb{R}$ with base manifold $M = \mathbb{R}$.

- The kinematic configuration are $\mathtt{C} = C^\infty(\mathbb{R}, Q)$ i.e.all the possible parametrized functions on Q.

- The lagrangian density is obtained evaluating the ordinary Lagrangian on the lifted curve:
$$\mathscr{L}[\gamma] := \big(L \circ \gamma^{\text{lift}}\big)dt = \mathscr{L}(t, \gamma^i, \dot{\gamma}^i) \tag{1.3}$$

⚠

1.3   Geometric mechanics of Finite Degree systems

La visione precedente ÃÍ molto generale ma ci sono alcune strutture classiche che voglio replicare sul campo come la forma simplettica, le osservabili e le parentesi di poisson. Mi sembra piÃź chiaro vederle dopo aver raccontato queste, proprio come fa Wald.

1.3.1   Linear dynamical systems

1.4 Peierls Brackets

In this section we present more extensively the original Peierls' construction. Please note that we are not trying to provide the state of the art on the Peierls bracket (see for example [**?**] for the treatment in presence of gauge freedom) but only to expand and modernize the first approach given by Peierls. Instead of considering only scalar theory we extend the algorithm to a broader class of systems.

The Peierls's construction algorithm is well defined for a specific class of systems:

1. Linear field theory: $E = (E, \pi, M)$ is a vector bundle.

2. Linear Lagragian dynamics: $P = Q_{\mathscr{L}}$ is a L.P.D.O.

3. M is a globally Hyperbolic space-time.

4. motion operator P is a green-hyperbolic.

Observation 8: Peierls Bracket vs Poisson Bracket

⚠ [a] Paraphrasing an observation made by Sharan[**?**]:

The Poisson bracket determines how one quantity b(t, q, p) changes another quantity a(t, q, p) when it acts as the Hamiltonian or vice-versa. The Peierls bracket, on the other hand, determines how one quantity b(t, q, p) when added to the system Hamiltonian h with an infinitesimal coefficient Îż affects changes in another quantity a(t, q, p) and vice-versa, i.e. The Peierls bracket is related to the change in an observable when the trajectory on which it is evaluated gets shifted due to an infinitesimal change in the Lagrangian of the system by another Lagragian density.

While the Poisson bracket between two observables a and b is defined on the whole phase space and is not dependent on the existence of a Hamiltonian, the Peierls bracket refers to a specific trajectory determined by a governing Lagrangian.

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[a]da aggiustare

Purpose of the Peierls' procedure is to provide a bilinear form on the space of Lagrangian densities with time-compact support. This form induces a pre-symplectic structure on suitable spaces Functionals which can be easily reduced to the subclass of suitable classical observable. The algorithm can be summarized in a few steps

1. Consider a *disturbance* χ that is a time-compact lagrangian density .

2. Construct the perturbation of a solution under the disturbance.

3. Define the effect of the disturbance on a second lagrangian functional.

4. Assemble two Effect operators to give a bracket.

1.4.1 Disturbance and Disturbed motion operator

By *"disturbance"* we mean a time-compact supported lagrangian density $\chi \in \mathsf{Lag}$[1] which act as a perturbation on the system's lagrangian:

$$\mathscr{L} \rightsquigarrow \mathscr{L}' = \mathscr{L} + \epsilon \cdot \chi$$

where ϵ is a modulation parameter. The support condition is required in order to take in account only perturbations which affect the system dynamics only for a limited time interval. The motion operator of the disturbed dynamics results:

$$P_\epsilon = \left[\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathscr{L}}{\partial \phi} \right] + \epsilon \left[\partial_\mu \left(\frac{\partial \chi}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \chi}{\partial \phi} \right] = P + \epsilon Q_\chi \qquad (1.4)$$

Observation 9

P_ϵ is not necessary linear, the second Hypothesis guarantees the linearity only for P.

1.4.2 Solution of the disturbed motion

The second ingredient of the Peierls' procedure are the *perturbed a solution under the disturbance* which are solution $\phi' \in \mathsf{C}$ of P_ϵ obtainable by a linear infinitesimal perturbation of a fixed solution $\phi \in \mathsf{Sol}$. More precisely, has to be seek a configuration:

$$\phi'(x) = \phi(x) + \epsilon \eta(x) \in \mathsf{C}$$

(the linear superposition is well-defined according to hypothesis 1) such that:

$$P_\epsilon \phi'(x) = o(\epsilon)$$
$$P\phi(x) = 0$$

In other word has to be satisfied the following equation:

$$\left[P_\epsilon \right] \phi'(x) = \left[P + \epsilon Q_\chi \right] (\phi(x) + \epsilon \eta(x)) = \epsilon \Big(\left[P \right] \eta(x) + \left[Q_\chi \right] (\phi(x) + \epsilon \eta(x)) \Big) \stackrel{!}{=} o(\epsilon)$$

The condition of linearity for operator P doesn't hold for Q_χ in general. We can work around this problem taking into account the linearization[**?**, pag. 31] of operator Q_χ around solution $\phi(x)$. That is the unique linear operator $big[Q_\chi^{lin}(\phi)]$ such that:

$$\left[Q_\chi \right] (\phi(x) + \epsilon \eta(x)) = \left[Q_\chi \right] (\phi(x)) + \epsilon \left[Q_\chi^{lin}(\phi) \right] (\eta(x)) + o(eta)$$

which can be seen as the first term of a *formal* Taylor expansion of operator Q_χ around ϕ. $\triangle$[2] This is reflected in a condition on the perturbation $\eta \in \mathsf{C}_{tc}$:

$$\left[P_\epsilon \right] \phi'(x) = \epsilon \Big(\left[P \right] \eta(x) + \left[Q_\chi \phi(x) \right] \Big) + \epsilon^2 \left[Q_\chi^{lin}(\phi) \right] \eta(x) \stackrel{!}{=} o(\epsilon)$$

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[1]I.e. the top form $\chi(\phi)$ is time-compact supported for all $phi \in \mathsf{C}$.

[2]If C is a Frechet manifold the expansion could be made rigorous defining $\left[Q_\chi^{lin}(\phi_0) \right] = \left[\frac{\partial Q_\chi}{\partial \phi}(\phi_0) \right]$ in term of the Gateux derivative.

$$\Rightarrow P\eta = -Q_\chi\phi(x) \tag{1.5}$$

called *Jacobi Equation.* This equation is a non homogeneous P.D.E. with inhomogeneous term $(-Q_\chi\phi(x))$ fixed by the solution $\phi \in \mathtt{Sol}$ to be perturbed.

Follows from the definition of green hyperbolicity that the domain restriction of P to $\Gamma^\infty_{\mathrm{pc}}$ or $\Gamma^\infty_{\mathrm{fc}}$ admit a unique inverse G^+ and G^- respectively. Therefore, equation **??** admits a unique past compact solution η^+, called retarded perturbation of $\phi \in \mathtt{Sol}$, and a unique future compact solution η^-, called advanced perturbation:

$$\eta^\pm = G^\pm\big(-Q_\chi\phi\big) \tag{1.6}$$

Note that the time-compact support condition on χ guarantees that $Q_\chi\phi \in \mathrm{dom}(G^+)\cap \mathrm{dom}(G^-)$. Expression 1.6 reflects perfectly the orginal Peierls' notation where $\eta^\pm$ were noted as functions of the unperturbed solution: $\eta^+ \equiv \mathrm{D}_\chi\phi$ and $\eta^- \equiv \mathrm{G}_\chi\phi$.

Observation 10

In most practical case it's possible to give a more basic characterization of $\eta^\pm$ in term of a Cauchy problem. Has to be stressed that this approach is not possible in general since Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense i.e. the well-posedness of the Cauchy problem is not guaranteed on any Cauchy surface. [**?**, pag 1] [**?**, remark 3.18] [**?**, remark 2.1]
Consider a motion operator P which is also hyperbolic. Taking in account the time-compact support condition of χ, is possible to pick up two Cauchy surfaces $\Sigma_\pm$ (+ is after the perturbation while $-$ stands for prior to the perturbation) such that:
$$J^\mp(\Sigma_\pm) \supset \mathrm{supp}(\chi)$$
for all time-slice foliation of the globally hyperbolic space-time.

For each of this two surfaces can be posed a Cauchy problem:

$$\begin{cases} P\eta = -Q_\chi\phi \\ (\eta, \nabla_n\eta)\big|_{\Sigma_\pm} = (0,0) \end{cases} \tag{1.7}$$

which , according to the well-posedness of the Cauchy problem, admits an unique solution. The link with the first presentation is that past/future -compact supported configuration always meet the initial data condition for some future/past Cauchy surface.

In conclusion,fixed a solution $\phi \in \mathtt{Sol}$ and a perturbation χ, are uniquely determined two perturbed solution:

$$\phi^\pm_\epsilon = \phi + \epsilon\eta^\pm \tag{1.8}$$

such that:

retarded pertubation	$\eta^+ \in \Gamma^\infty_{pc}$	$(\eta^+, \nabla_n\eta^+)\big|_{\Sigma_-} = (0,0)$	"propagating forward"
advanced pertubation	$\eta^- \in \Gamma^\infty_{fc}$	$(\eta^-, \nabla_n\eta^-)\big|_{\Sigma_+} = (0,0)$	"propagating backward"

1.4.3 Effect Operator

Considering an arbitrary continuous $\triangle^3$ functional $B : \mathtt{Sol} \to \mathbb{R}$ (not necessarily linear) we can define the effect of a perturbation on the values of B[**?**, pag. 5] as a map:

$$\mathbf{E}_\chi^\pm : C^1(\mathtt{Sol}, \mathbb{R}) \to C^1(\mathtt{Sol}, \mathbb{R})$$

$$\mathbf{E}_\chi^\pm B(\phi_0) := \lim_{\epsilon \to 0} \left(\frac{B(\phi_\epsilon^\pm) - B(\phi_0)}{\epsilon} \right)$$

The advanced and retarded effects of χ on B are then defined by comparing the original system with a new system defined by the same kinematic configuration space $\mathtt{C}$ but with perturbed lagrangian.

The former expression appear quite simpler in case of a linear functional:

$$\mathbf{E}_\chi^\pm B(\phi_0) = B(\eta^\pm) \tag{1.9}$$

1.4.4 The Bracket

Remembering that every lagrangian density define a continuous functional (Action). From that is possible to build a binary function:

$$\{\cdot, \cdot\} : \mathsf{Lag}_{\mathrm{tc}} \times \mathsf{Lag}_{\mathrm{tc}} \to \mathbb{R}$$

as follow:

$$\{\chi, \omega\}(\phi_0) := E_\chi^+ F_\omega(\phi_0) - E_\chi^- F_\omega(\phi_0) \tag{1.10}$$

Proposition 1.4.1 (Bilinearity) *When restricted to Linear Lagrangian densities* $\{\cdot, \cdot\}$ *is a bilinear form*

Proof:

Linearity in the first entry follows from equation [**?**] and the linearity of the Euler-Lagrange operator $Q.$ over Lag.
Linearity in the second entry is guaranteed only for lagrangian densities ω which provide a linear Lagrangian Functional F_ω.

$\square$

We don't prove the symplectic property for this general object but we will face this problem for the case of *classical observable functional,* a subclass Lagrangian functional of most practical use in algebraic quantization.

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[3]The precise notion of continuity require the specification of the infinite dimensional manifold structure.

1.4.5 Extension to non-linear theories

In the previous costruction the green-hyperbolicity of motion operator P plays a primary role. Anyway the problem of searching perturbed solution of the disturbed dynamic can be stated even in presence of non-linear fields where the configuration bundle is not necessary a vector bundle and the motion operator is not linear.
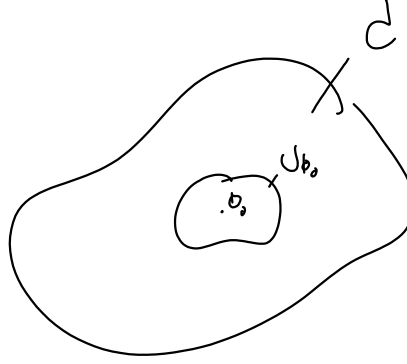
Figure 1.2: Intrinsically, searching a variation of a solution $\gamma_0 \in \texttt{Sol}$ which solve the disturbed motion equation is equivalent to find the intersection of the perturbed solution with a local neighbourhood of Γ_0 : $U_{\gamma_0} \cap \ker(P_\epsilon)$.

The crucial point of the Peierls' procedure is to select among all the possible solution of the perturbed motion P_ϵ that configuration which can be constructed by a variation of some fixed solution of the non-perturbed dynamics $\gamma_0 \in \texttt{Sol}$. In this sense the problem results a *"linearization"* inasmuch the search of such solution is restricted to a local neighbour of the "point" $\gamma_0 \in \texttt{Sol}$.

Previously was natural restrict the choice of possible variations to the linear one only. But in the general case this preferential choice is no longer possible. A way to recover a notation similar to 1.8 is to work patchwise by chosing a coordinate representation. Fixed a solution $\gamma_0 \in \texttt{Sol}$ and a local trivializing chart (A, ϕ_A) such that $A \cap \mathrm{ran}(\gamma_0) \neq \emptyset$ we can define a local infinitesimal variation by acting on his components:

$$\gamma_\lambda^i(x) = \gamma_0^i(x) + \lambda \eta^i(x) \qquad \forall x \in \pi(A)$$

where γ_0^i are the component of the unperturbed solution in the open set A and $\eta^i \in^q$ is a generic real q-ple (q is the dimension of the typical fiber manifold). λ is a real parameter that has to be "sufficiently small" in order to guarantee that the range of γ_λ is properly contained in A. In other words the construction of the linear variation, that for linear field theories could be done in a global way, in the general case can be recovered only locally.

Then is possible to define the effect of the a disturbance locally, searching local

section $\gamma_\epsilon^i = \gamma_0^i + \epsilon\eta^i$ solving the disturbed dynamic equation up to the first order in ϵ i.e.

$$[P_\epsilon]\gamma_\epsilon^i = o(\epsilon)$$

where $[P_\epsilon]$ has to be intended as the coordinate representation of the restriction on the local section $\Gamma^\infty(A)$. ⚠[4]

Observation 11

W.l.o.g has been taken the the same scalar ϵ to modulate both the perturbation γ_ϵ that the disturbance on the motion operator. On the contrary consider two different parameter is immaterial since only the smaller should be taken in account.

From the explicit equation of the perturbed solution:

$$\Big([P] + \epsilon[Q_\chi]\Big)(\gamma_0^i + \epsilon\eta^i) = o(\epsilon)$$

follows an equation on the components of the local perturbation. In this case has to be dealt with the problem of non-linearity not only for Euler-Lagrange operator Q_χ but also for P. Arresting the expension to the first order in ϵ results:

$$\Big[P_{\gamma_0}^{lin}\Big]\eta^i(x) = -\Big(Q_\chi(\gamma_0)\Big)(x) \tag{1.11}$$

the *Jacobi equation* on the unperturbed solution $\gamma_0 \in$ Sol.

Observation 12

We've moved from an operator P definited on C to an operator $P_{\gamma_0}^{lin}$ defined on the space of variation. From a global point of view this variation can be seen as the tangent vector i.e. s $\eta \in T_{\gamma_0}$C. In the case of the linear system this passage was unnecessary, the Jacobi equation was directly defined on C since, for linear system, any section could be seen as a generator of an infinitesimal variation. This behaviour mimics perfectly what happens in ordinary classical mechanics where the configuration space of a linear system is a vector space i.e a "flat" manifold[a] which is isomorphic to his tangent space in every point.

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[a]In sense that admits a global coordinate chart.

Provided that the linearized motion operator (which is now properly a linear partial differential operator) is Green-Hyperbolic, the Peierels construction can continue as before. Has to be noted that now the advanced/retarded perturbation are formally identical to the former:

$$\eta^{\pm i} = G^\pm\big(-Q_\chi\gamma_0^j\big)$$

---

[4]non mi ÃÍ evidente se la rappresentazione in coordinate di un operatore agente sulle sezioni si realizza in modo ovvio, ma non vedo nemmeno ostruzioni! Di sicuro l'operazione ÃÍ ben definita per gli L.P.D.O visto che la definizione prevede proprio che su ogni carta locale trivilizzante l'operatore sia lineare alle derivate parziali

with the important difference that $G^{\pm}$ are now the Green operators of the linearized motion operator and depend on the fixed solution γ_0^j.

In conclusion the perturbed solution:

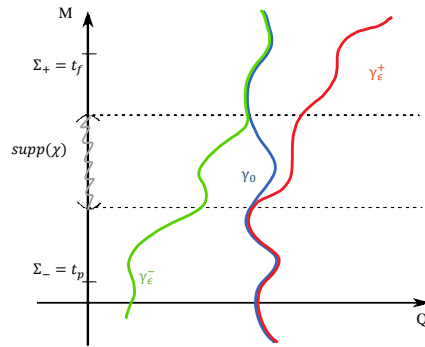$$\gamma_\chi^{\pm\, i} = \gamma_0^i \pm G^{\pm}\big(-Q_\chi \gamma_0^j\big)$$

has to be intended as the "glueing" of all the local chart representations covering the chosen solution.

1.4.6   Finite Dimensional case

⚠

.. c'ÃÍ su i miei appunti ma potrebbe essere superfluo.. No, mi conviene metterlo qui piuttosto che nel capitolo su Jacobi!

⚠

Figure 1.3: Intrinsically, searching a variation of a solution $\gamma_0 \in \mathtt{Sol}$ which solve the disturbed motion equation is equivalent to find the intersection of the perturbed solution with a local neighbourhood of $\Gamma_0 : U_{\gamma_0} \cap \ker(P_\epsilon)$.

17

1.5 Dubbi

- quando parlo della cinematica mi piacerebbe dare indicazioni sulla struttura matematica dello spazio delle configurazioni cinematiche:

    1. costituisce una frechet manifold (gli unici risultati che ho trovato sono quelli di Palais di "non linear global analysis"

    2. le curve parametrizzate sono le variazioni

    3. classi di equivalenza definiscono delle variazioni infinitesime che costituiscono lo spazio tangente allo spazio delle configurazioni cinematiche

    4. questo spazio tangente ÃÍ isomorfo allo spazio delle sezioni del pullback rispetto alla sezione $\phi \in C$ del verical bundle (vedere forger romero)

    5. il problema dell'atlante e della rappresentazione delle sezioni in carta locale (da scegliere sia sul total space E che sul base space M)

- fare riferimento al teorema di Ostrowsky per giustificare il fatto che consideriamo solo il primo ordine. le langrangiana con termini cinetici esotici sono instabili (nel senzo che non ammetto come soluzioni sezioni globali ma solo locali).

- Devo mettere la trafila di definizioni? direi di no, qualcosa va detto qualcosa va messo in footnote

    (sono ripetizioni inutili per la tesi, sono informazioni che si ritrovano ovunque... sono informazioni adatta al knowledge base)

    Recurring definitions in general Relativity (excluding the general smooth manifold prolegomena).

Definition 9: Space-Time

A quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ such that:

- (M, g) is a time-orientable n-dimensional manifold ($n > 2$)
- $\mathfrak{o}$ is a choice of orientation
- $\mathfrak{t}$ is a choice of time-orientation

Definition 10: Lorentzian Manifold

A pair (M, g) such that:

- M is a n-dimensional ($n \geq 2$), Hausdorff, second countable, connected, orientable smooth manifold.
- g is a Lorentzian metric.

Definition 11: Metric

A function on the bundle product of TM with itself:

$$g : TM \times_M TM \to \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_p M \times T_p M \to \mathbb{R}$$

is a non-degenerate bilinear form.

Notation fixing

– *Riemman* if the sign of g is positive definite, *Pseudo-Riemman* otherwise.

– *Lorentzian* if the signature is $(+, -, \ldots, -)$ or equivalently $(-, +, \ldots, +)$.

Observation 13: Causal Structure

If a smooth manifold is endowed with a Lorentzian manifold of signature $(+, -, \ldots, -)$ then the tangent vectors at each point in the manifold can be classed into three different types.

Notation fixing

$\forall p \in M, \quad \forall X \in T_p M$, the vector is:

– *time-like* if $g(X, X) > 0$.

– *light-like* if $g(X, X) = 0$.

– *space-like* if $g(X, X) < 0$.

Observation 14: Local Time Orientability

$\forall p \in M$ the timelike tangent vectors in p can be divided into two equivalence classes taking

$$X \sim Y \text{ iff } g(X, Y) > 0 \qquad \forall X, Y \in T_p^{\text{time-like}} M :$$

We can (arbitrarily) call one of these equivalence classes "future-directed" and call the other "past-directed". Physically this designation of the two classes of future- and past-directed timelike vectors corresponds to a choice of an arrow of time at the point. The future- and past-directed designations can be extended to null vectors at a point by continuity.

Definition 12: Time-orientation

A global tangent vector field $\mathfrak{t} \in \Gamma^\infty(TM)$ over the Lorenzian manifold M such that:

- $\text{supp}(\mathfrak{t}) = M$
- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Observation 15

The fixing of a time-orientation is equivalent to a consistent smooth choice of a local time-direction.

Definition 13: Time-Orientable Lorentzian Manifold

A Lorentzian Manifold (M, g) such that exist at least one time-orientation $\mathfrak{t} \in \Gamma^\infty(TM)$.

Notation fixing

Consider a piece-wise smooth curve $\gamma : \mathbb{R} \supset I \to M$ is called:

- *time-like* (resp. light-like, space-like) iff $\dot\gamma(p)$ is time-like (resp. light-like, space-like) $\forall p \in M$.
- *causal* iff $\dot\gamma(p)$ is nowhere spacelike.
- *future directed* (resp. past directed) iff is causal and $\dot\gamma(p)$ is future (resp. past) directed $\forall p \in M$.

Definition 14: Chronological $^{\text{future}}_{\text{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{I}_M^\pm(p) := \left\{ q \in M \,\middle|\, \exists \gamma \in C^\infty\big((0,1), M\big) \text{ time-like } ^{\text{future}}_{\text{past}}-\text{directed} : \gamma(0) = p, \gamma(1) = q \right\}$$

Definition 15: Causal $^{\text{future}}_{\text{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{J}_M^\pm(p) := \left\{ q \in M \,\middle|\, \exists \gamma \in C^\infty\big((0,1), M\big) \text{ causal } ^{\text{future}}_{\text{past}}-\text{directed} : \gamma(0) = p, \gamma(1) = q \right\}$$

Notation fixing

Former concept can be naturally extended to subset $A \subset M$:

- $\mathbf{I}_M^\pm(A) = \bigcup_{p \in A} \mathbf{I}_M^\pm(p)$
- $\mathbf{J}_M^\pm(A) = \bigcup_{p \in A} \mathbf{J}_M^\pm(p)$

Definition 16: Achronal Set

Subset $\Sigma \subset M$ such that every inextensible timelike curve intersect Σ at most once.

Definition 17: $_{\text{past}}^{\text{future}}$ Domain of dependence of an Achronal set

The two subset related to the generic achornal set $\Sigma \subset M$:

$$\mathbf{D}_M^\pm(\Sigma) := \left\{ q \in M \,\middle|\, \forall \gamma\,_{\text{future}}^{\text{past}} \text{inextensible causal curve passing through } q : \gamma(I) \cap \Sigma \neq \emptyset \right.$$

Notation fixing

$\mathbf{D}_M(\Sigma) :- \mathbf{D}_M^+(\Sigma) \cup \mathbf{D}_M^-(\Sigma)$ is called *total domain of dependence.*

Definition 18: Cauchy Surface

Is a subset $\Sigma \subset M$ such that:

- closed
- achronal
- $\mathbf{D}_M(\Sigma) \equiv M$

Basic Definition in L.P.D.O. on smooth vector sections.

Consider $F = F(M, \pi, V), F' = F'(M, \pi', V')$ two linear vector bundle over M with different typical fiber

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Definition 19: Linear Partial Differential operator (of order at most $s \in \mathbb{N}_0$)

Linear map $L : \Gamma(F) \to \Gamma(F')$ such that:
$\forall p \in M$ exists:

    – (U, ϕ) local chart on M.

    – (U, χ) local trivialization of F

    – (U, χ') local trivialization of F'

for which:
$$L\big(\sigma\big|_U\big) = \sum_{|\alpha| \leq s} A_\alpha \partial^\alpha \sigma \qquad \forall \sigma \in \Gamma(M)$$

Remark:

(multi-index notation)

A multi-index is a natural valued finite dimensional vector $\alpha = (\alpha_0, \dots, \alpha_{n-1}) \in \mathbb{N}_0^n$ with $n < \infty$.

On $\mathbb{R}^n$ a general differential operator can be identified by a multi-index:

$$\partial^\alpha = \prod_{\mu=0}^{n-1} \partial_\mu^{\alpha_\mu}$$

(Until the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|\alpha| := \sum_{\mu=0}^{n-1} \alpha_\mu$$

?????????????????????

Proposition 1.5.1 (Existence and uniqueness for the Cauchy Problem)   –
    $\mathbf{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ *a globally hyperbolic space-time.*

    – $\Sigma \subset M$ *a spacelike cauchy surface with future-pointing unit normal vector field* $\vec{n}$.

    –

Observation 16

"Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. " [?] [?]

However the existence and uniqueness can be proved for the large class of the *Normally-Hyperbolic Operators*.