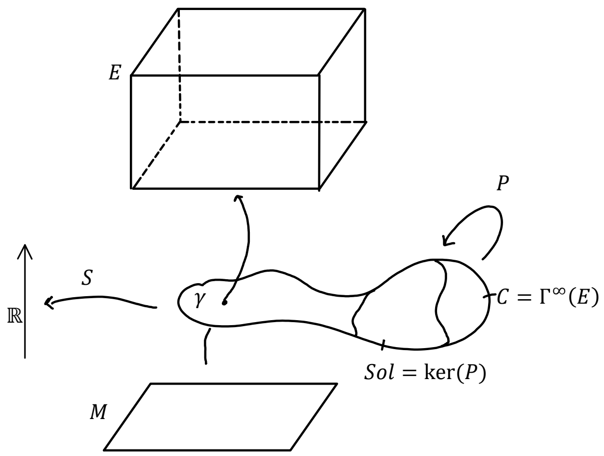
Demystification of Peierels Brackets construction

Definition 1: Dynamical System

We call *abstract dynamical system* a pair (E,P) composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on manifold M, called *"configuration bundle"*.
- $P : \Gamma^\infty(E) \to \Gamma^\infty(E)$
  differential operator in the sense that it is represented by a linear combination of partial derivative in every local chart.



Definition 2: Space of kinematics (off-shell) configurations

We call:

$$\mathtt{C} := \Gamma^\infty(M,E)$$

the space of kinematic configurations.

Definition 3: Space of Dynamics (on-shell) configurations

We call

$$\mathtt{Sol} := \ker(P) = \{\gamma \in \mathtt{C} \quad | R(P)(f) = 0 \quad \forall \text{local chart representation}\}$$

where we have denoted respectively as $R(P)$ and f the local chart representation (as in Eq. **??**) of P and γ on the same chart.
This is the subset of $\mathtt{C}$ containing all the smooth solutions of the motion equations corresponding to the dynamical operator:

$$P : \mathtt{C} \to \mathtt{C}$$

Definition 4: Lagrangian System (of r-th order)

We call *Lagrangian system* of r-th order the pair $(E,\mathscr{L})$ composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on the oriented pseudo-Riemannian manifold $(M,g,\mathfrak{o})$ called *"configuration bundle"*.
- $\mathscr{L} : J^r E \to \wedge^m T^* M$
  bundle-morphism from the r-th Jet Bundle (see Section **??**) to the top-dimensional form bundle over the base manifold M called *"Lagrangian density"* or simply *"Lagrangian"* of r-th order.

Definition 5: Lagrangian Densities on the bundle E

We denote the set of all *Lagrangian densities* (of $r-$th order) on the bundle E as:

$$\mathsf{Lag}^r(E) := \hom\Big(J^r E, \quad \bigwedge^m (T^*M)\Big) \cong \{f : \Gamma^\infty(J^r E) \to \Omega^m(M)\}$$

(where $\Omega^m(M)$ is the common name for $\Gamma^\infty\big(\wedge^m(T^*M)\big)$ in the context of Grassmann algebras.)
The equivalence states the fact that a bundle-morphism induces a map between the sections.

Proposition 0.0.1 $\mathsf{Lag}^r(E)$ *has an vector space structure inherited by the linear structure of* $\Omega(M)$.

Definition 6: Lagrangian functional

We call *Lagrangian functional* a map :

$$\mathcal{O}_{\mathcal{L}} : \mathsf{C} \to D'(M)$$

where $D'(M)$ is the space of regular distribution over M, whose action on any configuration $\phi \in \mathsf{C}$, evaluated on the test-function $f \in C_0^\infty(M)$, it is given by:

$$\mathcal{O}_{\mathcal{L}}[\phi](f) = \int_M \mathcal{L}[\phi] f \, \mathrm{d}\mu$$

Definition 7: Euler-Lagrange operator

We call *Euler-Lagrange operator*, the differential operator

$$Q_\chi : \mathsf{C} \to \mathsf{C}$$

relative to the Lagrangian density $\chi \in \mathsf{Lag}^1(E)$, such that:

$$Q_\chi(\gamma) = \left(\nabla_\mu\left(\frac{\partial\chi}{\partial(\partial_\mu\phi)}\Big|_\gamma\right) - \frac{\partial\chi}{\partial\phi}\Big|_\gamma\right) \qquad \forall\gamma \in \mathsf{C} \tag{1}$$

Where ∇_μ is the covariant derivative corresponding to the background metric g. [a]

[a] $\frac{\partial\chi}{\partial(\partial_\mu\phi)}$ has the be intended as the Lagrangian density constructed differentiating $\chi(\phi, \partial_\mu)$ as an ordinary function, treating its functional entries as an usual scalar variable.

0.0.1 TODO: Peirels constuction lingo

Definition 8: Disturbance

By *"disturbance"* we mean a time-compact supported Lagrangian density $\chi \in \mathsf{Lag}^a$ which acts as a perturbation on the Lagrangian of the system:

$$\mathcal{L} \rightsquigarrow \mathcal{L}' = \mathcal{L} + \epsilon \cdot \chi$$

where ϵ is a real modulation parameter. Recalling that Lag is linear we have that $\mathcal{L}'$ is still a suitable Lagrangian of the system.

[a] I.e. the top form $\chi(\phi)$ is time-compact supported for all $\phi \in \mathsf{C}$.

Definition 9: Disturbed Motion equation under a Disturbance (Jacobi Equation)

$$\Rightarrow P\eta = -Q_\chi\phi(x) \tag{2}$$

called *Jacobi Equation*. It yelds the pertubation to apply to a fixed configuration:

$$\phi'(x) = \phi(x) + \epsilon\eta(x) \in \mathsf{C}$$

such that:

$$\begin{cases} P_\epsilon\phi'(x) & = o(\epsilon) \\ P\phi(x) & = 0 \end{cases}$$

Definition 10: Effect Operator

Considering an arbitrary continuous[a] functional $B : \mathsf{C} \to \mathbb{R}$ (not necessarily linear) we can define the effect of a perturbation on the values of B[**?**, pag. 5] as a map:

$$\mathbf{D}^{\pm}_{\chi} : C^1(\mathsf{C}, \mathbb{R}) \to C^1(\mathsf{C}, \mathbb{R})$$

$$\mathbf{D}^{\pm}_{\chi} B(\phi_0) := \lim_{\epsilon\to 0}\left(\frac{B(\phi^{\pm}_{\epsilon}) - B(\phi_0)}{\epsilon}\right) \tag{3}$$

[a]The precise notion of continuity require the specification of a (infinite dimensional) manifold structure on C.

Definition 11: Peierls Bracket

The binary function

$$\{\cdot,\cdot\} : \mathsf{Lag}_{tc} \times \mathsf{Lag}_{tc} \to \mathbb{R}$$

such that

$$\{\chi, \omega\}(\phi_0) := \mathbf{D}^{-}_{\chi}\mathscr{O}_{\omega}(\phi_0) - \mathbf{D}^{+}_{\chi}\mathscr{O}_{\omega}(\phi_0) \tag{4}$$

0.0.2 TODO: Symplectic space of field-theoretic systems.

Remark:

questo lo devo dire a parole o scriverlo?

- linear system
- green hyperbolic dynamics
- globally hyperbolic spacetime based sections

Definition 12: Configuration Pairing

The *pairing* between two sections is defined as:

$$(X, Y) = \int_M < X, Y >_x d\mu(x) \tag{5}$$

where $d\mu = d\mathrm{Vol}_{\mu}$ is the volume form induced by the metric and the orientation on M under the additional constraint: (The definition is well posed only in:)

$$dom\big((\cdot,\cdot)\big) = \big\{(X,Y) \in \Gamma^{\infty}(E) \times \Gamma^{\infty}(E) \;\big|\; < X, Y >_x \in L^1(M,\mu)\big\}$$

Definition 13: Pre-Observable

They are defined as a class of *"off-shell"* functionals on C:

$$\mathscr{E}_0 := \Big\{ F_f : \mathsf{C} \ni \phi \mapsto (f, \phi) \in \mathbb{R} \quad \Big| \quad f \in \Gamma^{\infty}_0(E) \Big\}$$

This can be seen as the range of the linear map:

$$F : \Gamma_0^\infty \to \mathscr{E}_0$$

which associates to any section $f \in \Gamma_0^\infty(E)$ the linear functional $F_f(\cdot) = (f, \cdot) : \mathtt{C} \to \mathbb{R}$.

Definition 14: Classical Observable

Due to the degeneracy of the map $F^{\mathtt{Sol}}$, $\mathscr{E}_0^{\mathtt{Sol}}$ can not be a good set of classical observables.
Being the kernel known, we can identify all the elements that posses the same corresponding functional:

$$[f] = \{f + Pg \quad | g \in \Gamma_0\}$$

It is natural then to define the classical observables as the quotient space:

$$\mathscr{E} := \frac{\mathscr{E}_0^{\mathtt{Sol}}}{N}$$

Finally, the mapping between these equivalence classes can be easily defined :
$\forall [f] \in \frac{\Gamma_0}{P\Gamma_0}$ we build the functional $F_{[f]} : \mathtt{Sol} \to \mathbb{R}$ such that:

$$F_{[f]}(\phi) = F_f(\phi) \qquad \forall \phi \in \mathtt{Sol}, \forall f \in [f]$$

This functional is well-defined, *i.e.* the expression is independent from the choice of the representative, only on $\mathtt{Sol}$. The reason is that if $\phi \in \mathtt{C} \setminus \mathtt{Sol}$, then $F_f(\phi)$ is different for each choice of the representative $f \in [f]$.
This construction is said to " implement the on-shell condition at the level of functionals".
In conclusion the mapping:

$$\frac{\Gamma_0}{P\Gamma_0} \xmapsto{F} \mathscr{E} = F(\frac{\Gamma_0}{P\Gamma_0})$$

,between suitable equivalence classes and linear functionals on $\mathtt{Sol}$, guarantees:

- a faithful representation since F is bijective.
- the separability condition, a fortiori of separability properties of $\mathscr{E}_0^{\mathtt{Sol}}$.

From now on we will identify these two spaces:

$$\mathscr{E} \simeq \frac{\Gamma_0}{P\Gamma_0}$$

in view of the bijectivity of F.

Definition 15: PreQuantum Symplectic Space

todo...

Proposition 0.0.2 *this structure correspond to a suitable domain restriction of the peierls brackets*