Definition 1: Fiber Bundle

We call a *Smooth (Fiber) Bundle* a quadruple $E = (E, M, Q, \pi)$ where:

- E, M, Q : smooth manifolds called respectively *Total Space, Base Space, Typical Fiber.*
- $\pi : E \to M$ smooth, everywhere defined, surjective function (called *Bundle Projection*)

Such that $\forall x \in M \quad \exists$ a *local trivialization*) (U, χ).

Definition 2: Local Trivialization of the Fiber Bundle E

A Pair (U, χ) where:

· U : neighbourhood of $x \in M$

· $\chi : \pi^{-1}(U) \to U \times Q$: diffeomorphism [a] [b]

such that the natural projection $p_1 : U \times F \to U$ satisfies the following equation:

$$p_1 \cdot \chi = \pi|_{\pi^{-1}(p)}$$

i.e.: the following graph commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\chi"] \arrow[d, "\pi"'] & U \times Q \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

[a] surjectivity $\Rightarrow \pi^{-1}(U) \neq \emptyset$.

[b] cartesian product of topological space is a topological space with the direct product topology.

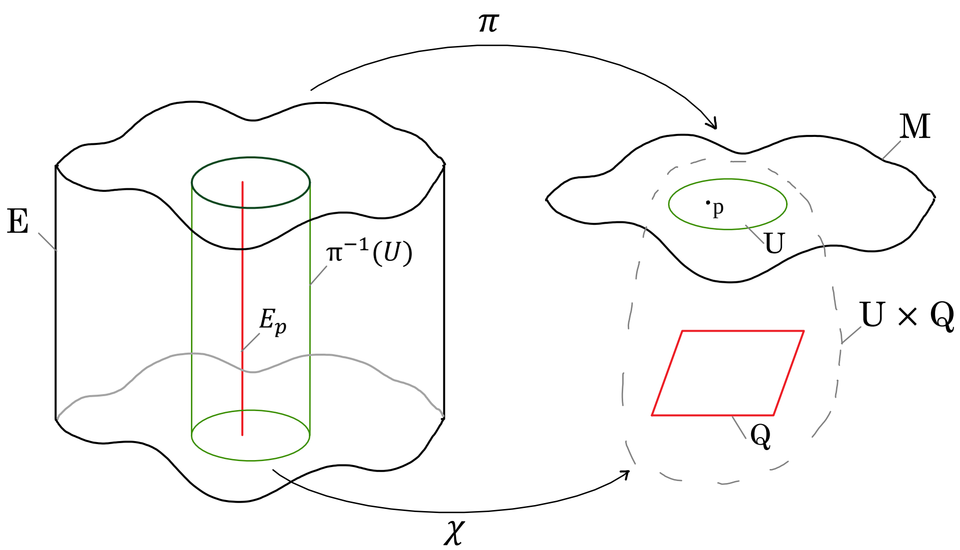

Figure 1: The complete fiber bundle Structure.

Definition 3: Vector Bundle

We call *Vector Bundle* a smooth bundle $E = (E, \pi, M; V)$ such that:

· The typical fiber V is a finite dimensional vector space.

· All trivializations χ_α are diffeomorphisms such that:

$$\chi_\alpha\big|_{\pi^{-1}(p)} \in \mathbb{GL}(n,\mathbb{R}) \quad : \pi^{-1}(p) \to \{p\} \times V \simeq V$$

Definition 4: Smooth (cross) Section

We call *Smooth (cross) Section* a smooth right-inverse function of π.
I.e. any $\phi \in C^\infty(M;E)$ such that: $$\pi \circ \phi = id|_M$$

Definition 5: Bundle map (*Fiber Preserving map*)

We call *bundle map* a smooth function $\phi : E \to E'$ such that:
$$\phi(E_x) = F_x \qquad \forall x \in M.$$

i.e. the following graph commutes:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

Definition 6: Bundle of homomorphisms

We call *bundle of homomorphisms* a fiber bundle $\mathrm{Hom}(E,E')$ over the base space M such that the fiber over a base point $p \in M$ is the infinite dimensional manifold $\mathrm{Hom}(E_p, E'_p)$ isomorphic to $\mathrm{Hom}(Q,Q')$.

Definition 7: Tangent Bundle

We call *tangent bundle of* M the smooth vector bundle $TM = (TM, \tau, M; \mathbb{R}^m)$ such that:

· The total space is the (disjoint) union of all tangent spaces to M :

$$TM := \bigsqcup_{p\in M} T_pM \equiv \bigcup_{x\in M} x \times T_xM$$

· The bundle projection maps each tangent vector $v \in T_pM$ to the correspondent base point p;

$$\tau : (p, v_p) \mapsto p$$

- The *Cotangent Bundle* T^*M is the vector bundle T^*M builded by disjoint union of the dual tangent space T^*_pM.
- The *Tensor Bundle* $T^{(k,l)}M$ is build by disjoint unions of the tensor product of tangent space with itself:

$$T^{(k,l)}_pM = \underbrace{T^*_pM \otimes \cdots \otimes T^*_pM}_{\text{k-times}} \otimes \underbrace{T_pM \otimes \cdots \otimes T_pM}_{\text{l-times}}$$

- The *k-form Bundle* $\bigwedge^m(T^*M)$ is build by disjoint unions of the antisimmetrized tensor product of the dual tangent space with itself.

Definition 8: Tautological (Poincaré) 1-form

We call *tautological form* the 1-form over $\mathscr{M} = T^*Q$:

$$\theta_0 \in \Gamma^\infty(T^*\mathscr{M})$$

such that the action on a generic point $\omega_{\alpha_p} \in T_{\alpha_p}M$ (in the fiber of α_p, which in turn is a one-form on the fiber of $p \in Q$) is given by:

$$\theta_0(\alpha_p) : T_{\alpha_p}\mathscr{M} \to \mathbb{R} \qquad : \omega_{\alpha_p} \mapsto \alpha_q \circ T\tau^*_Q(\omega_{\alpha_p})$$

where T is the *tangent map*, namely the bundle-morphism which act on each fiber as the differential map $d(\tau^*_Q)$.

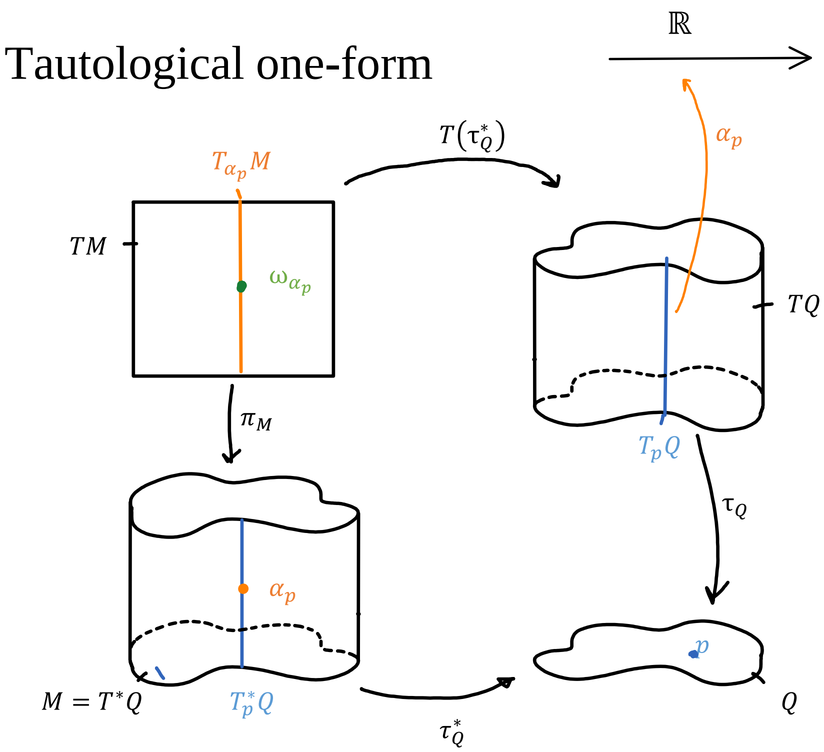

Figure 2: The definition of tautological 1-form is achieved exploiting the concept of *Tangent map* and remembering that $\alpha_p : T_p\mathscr{M} \to \mathscr{M}$ is a linear functional.

Definition 9: Canonical (Poincarè) symplectic form

We call *Canonical (Poincarè)* form the symplectic:

$$\omega_0 := -d\theta_0$$

In canonical coordinates it assumes the renown expression:

$$\omega_0 := \sum_{i=1}^{n} dq^i \wedge dp_i$$

0.0.1 Jet Bundles

The jet bundle is a construction that makes a new smooth fiber bundle out of a given bundle.

Definition 10: r-jet equivalence

Two sections $\sigma, \eta \in \Gamma^\infty(p)$ have the same *r-jet* at p ($\sigma \sim \eta$) iff:

$$\left.\frac{\partial^{|I|}\sigma^\alpha}{\partial x^I}\right|_p = \left.\frac{\partial^{|I|}\eta^\alpha}{\partial x^I}\right|_p \qquad \forall I \in \mathbb{N}_0^m \,|\, 0 \le |I| \le r.$$

where I is a *multi-index* ,see Remark 0.0.1.

Remark:

(multi-index notation)
A multi-index is a finite dimensional vector $I = (i_1, i_2, \dots, i_m) \in \mathbb{N}_0^m$ with $m < \infty$.
On $\mathbb{R}^n$ a differential operator can be identified by a multi-index:

$$\frac{\partial^{|I|}}{\partial x^I} := \prod_{i=1}^{m} \left(\frac{\partial}{\partial x^i} \right)^{I(i)}$$

(Whenever the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|I| := \sum_{i=1}^{m} I(i)$$

Definition 11: Space of r-th Jet in p

We call *space of the r-th jet in p* the set of the equivalence class under the jet equivalence relation.

$$J_p^r(E) := \frac{\Gamma^\infty(p)}{\sim}$$

where $\sim$ is the r-Jet equivalence.

Notation fixing

A r-jet with representative σ is denoted as $j_p^r \sigma$.
The integer r is also called the order of the jet, p is its source and $\sigma(p)$ is its target.

Definition 12: r-th Jet Bundle of E

We call *r-th Jet Bundle of E* the smooth bundle $(J^r(E), \pi_r, M)$ where:

- $J^r(E) := \bigsqcup_{p \in M} J_p^r(E) \equiv \{ j_p^r \sigma \quad | \, p \in M, \, \sigma \in \Gamma^\infty(p) \}$
- $\pi_r : J^r(E) \to M$ such that $j_p^r \sigma \mapsto p$

Definition 13: (Pseudo-Riemannian) Metric

We call *(Pseudo-Riemannian) Metric* a map on the bundle product of TM with itself:

$$g : TM \times_M TM \to \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a non-degenerate bilinear form.

Definition 14: Pseudo-Riemannian Manifold

We call *Pseudo-Riemannian manifold* a pair (M, g) such that:

- M is a n-dimensional $(n \geq 2)$, Hausdorff, second countable, connected, orientable smooth manifold.
- g is a Pseudo-Riemannian metric.

Definition 15: Time-orientation

We call *time-orientation* a global tangent vector field $\mathfrak{t} \in \Gamma^\infty(TM)$ over the Lorenzian manifold M such that:

- $\mathrm{supp}(\mathfrak{t}) = M$

- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Definition 16: Spacetime

We call *spacetime* a quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ such that:

- (M, g) is a time-orientable[a] n-dimensional Lorentzian manifold $(n > 2)$
- $\mathfrak{o}$ is a choice of orientation
- $\mathfrak{t}$ is a choice of time-orientation

[a]Manifold for which such *time-orientation* exists.

Definition 17: Achronal Set

We call *achronal set* a subset $\Sigma \subset M$ such that every inextensible timelike curve intersects Σ at most once.

Definition 18: $\substack{\text{future} \\ \text{past}}$ Domain of dependence of an Achronal set

We call $\substack{\textit{future} \\ \textit{past}}$ *domain of dependence* of an achronal set $\Sigma \subset M$, the two subset:

$$\mathbf{D}^{\pm}_{M}(\Sigma) := \left\{ q \in M \middle| \, \forall \gamma \; \substack{\text{past} \\ \text{future}} \text{inextensible causal curve passing through } q : \; \gamma(I) \cap \Sigma \neq \emptyset \right\}$$

Definition 19: Cauchy Surface

We call *Cauchy surface* a closed, achronal subset $\Sigma \subset M$ such that:

$$\mathbf{D}_M(\Sigma) \equiv M$$

Notation fixing

We denote the set of all the Cauchy surfaces as $\mathscr{P}_C(M)$.

Definition 20: Globally-Hyperbolic SpaceTime

We call a spacetime M *globally hyperbolic* if it contains at least one *Cauchy Surface*.

Notation fixing

Let M be a globally hyperbolic spacetime and $E = (E, \pi, M; V)$ a vector bundle of typical fiber V.
We denote:

- $\Gamma_0(E)$ the space of *compactly supported* smooth sections.
- $\Gamma_{sc}(E)$ the space of *spacelike compact* smooth sections.
  $\big(f \in \Gamma_{sc}(E)$ if there exists a compact subset $K \subset M$ such that $\operatorname{supp} f \subset \mathbf{J}_M(K)$. $\big)$
- $\Gamma_{fc}(E)$ the space of *future- compact* smooth sections.
  $\big(f \in \Gamma_{fc}(E)$ if $\operatorname{supp}(f) \cap \mathbf{J}^{+}_{M}(K)$ is compact for all $p \in M$.$\big)$
- $\Gamma_{pc}(E)$ the space of *past- compact* smooth sections.
  $\big(f \in \Gamma_{pc}(E)$ if $\operatorname{supp}(f) \cap \mathbf{J}^{-}_{M}(K)$ is compact for all $p \in M$.$\big)$
- $\Gamma_{tc}(E) := \Gamma_{pc}(E) \cap \Gamma_{fc}(E)$ the space of *timelike compact* smooth sections.

Definition 21: Linear Partial Differential operator **(of order at most $s \in \mathbb{N}_0$)**

We call *linear partial differential operator* a linear map $L : \Gamma(E) \to \Gamma(E')$ such that $\forall p \in M$ there exists:

- $U \ni p$ open set rigged with:

– (U,φ) local chart on M.
– (U,χ) local trivialization of F
– (U,χ') local trivialization of F'

- $\{A_\alpha : U \to \mathrm{Hom}(V,V') \mid \alpha \in \mathbb{N}_0^n, |\alpha| \le s\}$ a collection of smooth maps labeled by multi-indices where s is a fixed integer said *order of the operator*.

which allows to express L locally:

$$\chi' \circ (L\sigma) \circ \varphi^{-1} = \sum_{|\alpha| \le s} A_\alpha \partial^\alpha (\chi \circ \sigma \circ \varphi^{-1}) \qquad \forall \sigma \in dom(L) \subset \Gamma(E)$$

(where we have make use of the multi-index notation0.0.1)

Definition 22: Formal Dual Operator (of L)

We call *formal dual operator* of L the unique linear partial differential operator $L^\star : \Gamma(G^*) \to \Gamma(E^*)$ such that:

$$< L^\star g', f >=< g', Lf >$$

$\forall f \in \Gamma(E),\ g' \in \Gamma(G^*)$ which have supports with compact overlap.
($< \cdot, \cdot >$ denotes the 1-form evaluation: $< \alpha, v >= \alpha(v) \quad \forall v \in E_p, \alpha \in E_p^*$.)

Definition 23: $\substack{\text{retarded}\\\text{advanced}}$ ($\pm$) Green Operators

We call $\substack{\textit{retarded}\\\textit{advanced}}$ ($\pm$) *Green Operator* of L a l.p.d.o. $G^\pm : \Gamma(E) \to \Gamma(E)$ such that:
- $dom(G^+) = \Gamma_{pc}(E) \qquad dom(G^-) = \Gamma_{fc}(E)$
- $LG^\pm f = G^\pm L f = f \qquad \forall f \in dom(G^\pm)$
- $\mathrm{supp}(G^\pm f) \subset \mathbf{J}_M^\pm(\mathrm{supp}(f)) \qquad \forall f \in dom(G^\pm)$

In others words we can say that $G^\pm$ is the left-right inverse of the restriction of L to $dom(G^\pm)$.

Notation fixing

We call *Advanced minus Retarded operator* or *Causal Propagator*[**?**] the operator:

$$E := G^- - G^+ : \Gamma_{tc}(E) \to \Gamma(E)$$

Definition 24: Green hyperbolic operator

We call *Green hyperbolic* a linear partial differential operator P such that P and $P^\star$ have advanced and retarded Green's operators.

For these operators uniqueness of Green's operators is guaranteed:

Theorem 0.0.1 (Characterization of Green Hyperbolic operators)

Hp:

- *$E = (E,\pi,M)$ a vector bundle over a globally hyperbolic spacetime M.*
- *$P : \Gamma(E) \to \Gamma(E)$ a Green hyperbolic operator, $G^\pm$ its Green's operators and $G_\star^\pm$ the Green's operators of the dual.*

Th:

- *P possesses a unique retarded G^+ and a unique advanced G^- Green's operator.*
- $< G_\star^\pm f', f >=< f', G^\mp f > \qquad \forall f \in \Gamma_0(E), \forall f' \in \Gamma_0(E^*)$

Definition 25

A Poisson algebra is a Triple $(V, \cdot, \{,\})$ space]] where:

- V is a vector space of field K
- $\cdot : V \times V \to \mathbb{R}$ and $\{,\} : V \times V \to \mathbb{R}$ are bilinear products

such that:

- The product $\cdot$ forms an associative $K-$algebra.
- The product $\{,\}$, called the *Poisson Brackets* is anti-symmetric, and obeys the*Jacobi Identity* (i.e. forms a Lie Algebra)
- The Poisson bracket acts as a derivation of the associative product $\cdot$, i.e. for any three elements x, y, z in the algebra, one has
$$\{x, y \cdot z\} = x, y \cdot z + y \cdot x, z$$