Chapter 1

Mathematical Preliminaries

The interaction between mathematics and Quantum Field Theory are complex and highly not trivial. Since contemporary quantum field theory is mainly developed ~~as~~ quantization of classical fields, the mathematical foundation of classical field theory represent a necessary step towards the ~~comprehension~~ of ~~field theory foundations~~.

Goal of this section is to introduce the three building blocks of classical field theory, namely *vector bundles*, *globally hyperbolic spacestimes* and *Green hyperbolic operators*. Given the purpose of this thesis, ~~we'll~~ not dwell on the structures typical of the quantum framework (such *-algebras). We assume that the reader is familiar with the basic notions of differential geometry, external calculus and, to a minor extent, of general relativity.

1.1 Fiber Bundles

Fiber bundles are the stage for ~~classical and quantum field kinematic~~. Its main role is to encode the kinematic configuration of an arbitrary field theory through the concept of *sections*.

1.1.1 Formal Definition

Although it would be possible to present the ~~concept *bundle*~~ in a more general way through the language of categories, ~~for our argument will~~ be sufficient to consider only the case of *smooth bundles*.

Definition 1: Fiber Bundle

We call a *Smooth (Fiber) Bundle* a ~~4-ple~~ (E, M, Q, π) where:

- E, M, Q : smooth manifolds called respectively *Total Space*, *Base Space*, *Typical Fiber*.
- $\pi : E \to M$ continuous smooth function (called *Bundle Projection*)

Endowed with a *Local Trivialization*:

- $\forall x \in M\ \exists$ a couple (U, χ) (called *local trivialization*)
  - U : neighbourhood of x
  - $\chi : \pi^{-1}(U) \to U \times Q$: diffeomorphism [a] [b]

  such that: $p_1 \cdot \chi = \pi|_{\pi^{-1}(p)}$.

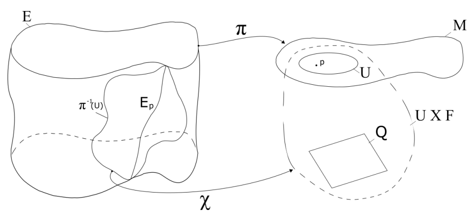
  ~~i.e:~~ the following graph commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\chi"] \arrow[d, "\pi"'] & U \times Q \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

[a] surjectivity $\Rightarrow \pi^{-1}(U) \neq \emptyset$.

[b] cartesian product of topological space is a topological space with the direct product topology.

Figure 1.1: The complete fiber bundle Structure.



Notation fixing

It is customary to refer to a vector bundle specifying only its total space:

$$E = (E, \pi, M; Q)$$

In the following we adopt this convention whenever this does not lead to misunderstandings.

Observation 1

For all $p \in M$ we refer to the submanifold $E_p := \pi_{-1}(p) \subset E$ ~~as *fiber*~~ *over the point p*.
Every fiber E_p is diffeomorphic to the typical fiber F through the local trivialization charts.

Notation fixing

We say that a smooth bundle E is *(globally) trivial* if $E \simeq M \times Q$ i.e there exists a trivialization of E which is defined everywhere.
Note that ~~definition~~ 1 prescribes the existence of local trivializations only.

When a smooth fiber bundle $(E, \pi, M; Q)$ is considered, in addition to the typical functions of the bundle (π, χ_α) ~~should be taken in account also the local charts~~ $\cancel{(U_{\alpha_k}, \phi_{\alpha_k})_{k=E,M,Q}}$ ~~provided by the atlases of E, M and Q;~~

Definition 2: Bundle atlas

We calla a *Bundle Atlas* a collection of local chart which trivializes E. I.e. triples $(U_\alpha, \psi_\alpha, \chi_\alpha)$ where:

- U_α ~~open~~ set in M such that $\bigcup_\alpha U_\alpha \supseteq M$.
- χ_α is a local trivialization.
- (U_α, ψ_α) ~~local~~ chart on M.

Observation 2

Given a local chart $(U_\alpha, \psi_\alpha^{(M)}$ on M and a local chart $(U_\alpha^{(Q)}, \psi_\alpha^{(Q)}$ on the fiber manifold it is possible to construct a chart on the total space:

$$\psi_\alpha^{(E)} = \psi_\alpha^{(M)} \times \psi_\alpha^{(Q)} \circ \chi_\alpha$$

Endowing the ~~bundles~~ manifolds with other additional structures, ~~can be introduced~~ important subclasses of smooth bundles:

Definition 3: Vector Bundle

We call *Vector Bundle* a smooth bundle $E = (E, \pi, M; V)$ such that:

- The typical fiber V is a finite dimensional vector space.
- All ~~the~~ trivialization χ_α are diffeomorphism such that:

$$\chi_\alpha|_{\pi^{-1}(p)} \in \mathbb{GL}(n, \mathbb{R})$$

1.1.2 Cross Sections

Sections represent the natural mathematical object to encode a $Q-$ valued classical field over the space M:

Definition 4: Smooth Section

We call *Smooth Section* a function $\phi : M \to E$ such that:

- ϕ smooth.
- $\phi \cdot \pi = \mathrm{id}_M$

Notation fixing

We refer to:

- *Global section* $\Leftrightarrow$ $\mathrm{dom}(\phi) = M$
- *Local section* $\Leftrightarrow$ $\mathrm{dom}(\phi) \subset M$

We denote the set of all the smooth sections of the bundle E as:

$$\Gamma^{\infty}(E)$$

Observation 3

In general, $\Gamma^{\infty}(E)$ is an infinite dimensional manifold~~s~~.
In case of ~~vector bundle is also~~ a linear Frechet space[**?**], and the section are called "vector fields".

1.1.3 Mapping between Bundles

Consider two smooth bundles $E = (E, \pi, M; Q)$ and $E' = (E', \pi', M; Q')$ on the same base space M.

Definition 5: Bundle map (*Fiber Preserving map*)

We call *bundle map* a smooth function $\phi : E \to E'$ such that:

$$\phi(E_x) = F_x \qquad \forall x \in M.$$

i.e.:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

Observation 4

Definition 5 is a special case of ~~B~~*undle-morphism.* (see for example [3])

Consider a smooth manifold N, a (smooth) fiber bundle $E = (E, \pi, M; Q)$, and a smooth function $f : N \to M$. it is possible to induce[**?**] a bundle structure from ~~the manifold~~ M to N:

Definition 6: Pull-Back Bundle

We call *pull-back* ~~*bundleof*~~ E a ~~3-ple~~ $f^*(E) = (f^*(E) =, \pi^*, N)$ such that:

- $f^*(E) = \{(b', e) \in N \times E \;\; | \; f(b') = \pi(e)\}$
- $\pi^* : f^*(E) \to N$ such that $\pi * (b', e) = \mathrm{pr}_1(b', e) = b'$

Proposition 1.1.1 $f^*(E) = (f^*(E) =, \pi^*, N)$ ~~*constitutes*~~ *a smooth bundle* ~~*of*~~ *typical fiber Q.*

Proof:

To complete the fiber bundle structure ~~is~~ sufficient to provide a local trivialization atlas.
$\forall (U, \phi)$ local trivialization on (E, π, M) consider $\psi : f^*E \to N \times Q$ such that $\psi(b', e) = \Big(b', pr_2\big(\phi(e)\big)\Big)$.
Then $(f^{-1}(U), \psi)$ is a local trivialization of the pull-back bundle and the fiber of f^*E over a point $b' \in B'$ is just the fiber of E over $f(b')$.

$\square$

It is also noteworthy that, given any two vector bundles $E = (E\pi, M, Q)$ and $E = (E'\pi, M', Q')$, we can construct naturally a third fiber bundle.
Consider $\mathrm{Hom}(E, E')$ the set of all the fiber preserving map between the two bundles:

Definition 7: Bundle of homomorphisms

We call *bundle of homomorphism* a fiber bundle $\mathrm{Hom}(E, E')$ over the base space M such that the fiber over a base point $p \in M$ is the infinite dimensional manifold $\mathrm{Hom}(E_p, E'_p)$ isomorphic to $\mathrm{Hom}(Q, Q')$.

Notation fixing

We shall write $End(E)$ for $\mathrm{Hom}(E, E)$ ~~and call~~ it bundle of endomorphism, whose typical fiber is $\mathrm{End}(Q)$.

Remark:

If F, F' are vector bundle then the fiber of $\mathrm{Hom}(F, F')$ over a base point $p \in M$ is $\mathrm{Hom}(F_p, F'_p)$, which is a vector space isomorphic to the vector space $\mathrm{Hom}(V, V')$ of linear applications from V to V'

1.1.4 Tangent Bundles

The *tangent* bundle is a natural structure defined on any smooth manifold, ~~represent~~ the canonical example of non-trivial vector bundle.

Definition 8: Tangent Bundle

We call *tangent bundle of M* the smooth vector bundle $TM = (TM, \tau, M; \mathbb{R}^m)$ such that:

- The total space is the union of all tangent spaces to

$$\cancel{M : TM} := \bigsqcup_{p \in M} T_p M \equiv \bigcup_{x \in M} x \times T_x M$$

- The bundle projection maps each tangent vector $v \in T_p M$ to the correspondent base point p;

$$\tau : (p, v_p) \longmapsto p$$

Observation 5

In a similar fashion it is possible to construct a vector bundle relatively to any tensor product of the tangent spaces. i.e.:

- *Cotangent Bundle* T^*M is build by disjoint union of the dual tangent space:

$$T_p^* M$$

- *Tensor Bundle* $T^{(k,l)}M$ is build by disjoint union of the tensor product of tangent space with itself:

$$T_p^{(k,l)} M = \underbrace{T_p^* M \otimes \cdots \otimes T_p^* M}_{\text{k-times}} \otimes \underbrace{T_p M \otimes \cdots \otimes T_p M}_{\text{l-times}}$$

- ~~*k-forms*~~ *Bundle* $\bigwedge^m (T^* M)$ is build by disjoint union of the antisimmetrized tensor product of the dual tangent space with itself.

Tautological one-form and ~~simplectic~~ form.

Notation fixing

In the context of Classical mechanics is customary to refer to the cotangent bundle T^*Q over the smooth manifold Q - called *Configuration Space* - as *Phase Space*.

Since TQ and T^*Q are diffeomorphic , it might seem that there is no particular reason in treating ~~this~~ two spaces separately, ~~but it is not so~~. There are certain geometrical objects that live naturally on T^*Q , not on TQ.
Of greatest interest ~~in mathematical physics~~ are the Poincaré forms[**?**].

Consider a smooth manifold Q and call $\mathscr{M} = T^*Q$ the corresponding cotangent bundle.

Definition 9: Tautological (Poincaré) 1-form

We call *tautological form* the 1-form over $\mathscr{M}$:

$$\theta_0 \in \Gamma^\infty(T^*\mathscr{M})$$

such that the action on a generic point $\omega_{\alpha_p} \in T_{\alpha_p}M$ (in the fiber of α_p, which in turn is a one-form on the fiber of $p \in Q$) is given by:

$$\theta_0(\alpha_p) : T_{\alpha_p}\mathscr{M} \to \mathbb{R} \qquad : \omega_{\alpha_p} \mapsto \alpha_q \circ T\tau_Q^*(\omega_{\alpha_p})$$

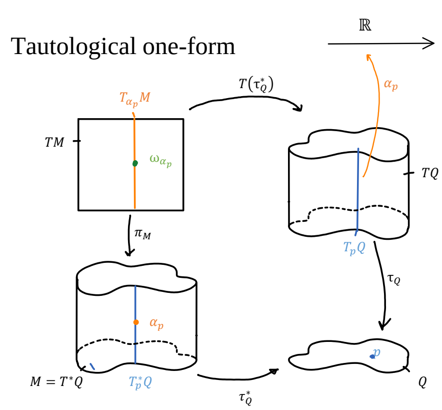


Figure 1.2: The definition of tautological 1-form is achieved exploiting the concept of *Tangent map* and remembering that $\alpha_p : T_p\mathscr{M} \to \mathscr{M}$ is a linear functional.

Notation fixing

Canonical coordinates are defined as a special set of coordinates on the cotangent bundle of a manifold. They are usually written as a set of (q^i, p_j) where q_i are denoting the coordinates on the underlying manifold and the p_j are denoting the conjugate ~~momentum~~, which are decomposition of 1-forms in $T_p^* M$ on the dual natural basis dq^J in the cotangent bundle ~~at point~~ q in the manifold.

Observation 6

In canonical coordinate the tautological one-form assumes the ~~famous~~ expression:

$$\theta_0 = \sum_{i=1}^{n} p_i dq^i$$

(note that dq^i is a 1-form on $T^* M$ calculated with respect to the coordinate on the bundle. ~~Has not to be confused~~ with the 1-natural form $dq^i \in T_p^* M$.)

The claim is proved by the following definition:

Definition 10: Canonical (Poincarè) symplectic form

~~Symplectic form:~~

$$\omega_0 := -d\theta_0$$

In canonical coordinates ~~assumes~~ the ~~famous~~ expression:

$$\omega_0 := \sum_{i=1}^{n} dq^i \wedge dp_i$$

1.1.5 Jet Bundles

The jet bundle is a ~~certain~~ construction that makes a new smooth fiber bundle out of a given bundle. The first step is to identify the typical fiber for this construction.

Suppose M is an m-dimensional manifold and that (E, π, M) is a fiber bundle.Consider the set of all the local sections whose domain contains p:

$$\Gamma^\infty(p) := \left\{ \sigma \in \Gamma^\infty(E) \quad \middle| \, p \in dom(\sigma) \right\}$$

We define an equivalence relation between such section *up to r-th order*:

Definition 11: r-jet equivalence

Two section $\sigma, \eta \in \Gamma^\infty(p)$ have the same *r-jet* at p ($\sigma \sim \eta$) iff:

$$\left.\frac{\partial^{|I|}\sigma^\alpha}{\partial x^I}\right|_p = \left.\frac{\partial^{|I|}\eta^\alpha}{\partial x^I}\right|_p \quad \forall I \in \mathbb{N}_0^m \,|\, 0 \leq |I| \leq r.$$

where I is a ~~*Multi-index*1.1.5.~~

Remark:

(multi-index notation)
A multi-index is a ~~natural valued~~ finite dimensional vector $I = (i_1, i_2, \ldots, i_m) \in \mathbb{N}_0^m$ with $m < \infty$.
On $\mathbb{R}^n$ a ~~general~~ differential operator can be identified by a multi-index:

$$\frac{\partial^{|I|}}{\partial x^I} := \prod_{i=1}^m \left(\frac{\partial}{\partial x^i}\right)^{I(i)}$$

(~~Until~~ the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|I| := \sum_{i=1}^m I(i)$$

We define the *r-th Jet in p* as the equivalence class under this relation.

Definition 12: Space of r-th Jet in p

We call *space of the r-th jet in p* the set:

$$J_p^r(E) := \frac{\Gamma^\infty(p)}{\sim}$$

where $\sim$ is the r-Jet equivalence.

Notation fixing

A r-jet with representative σ is denoted as $j_p^r\sigma$.
The integer r is also called the order of the jet, p is its source and $\sigma(p)$ is its target.

~~Glueing~~ all the jet fiber $J_p^r(E)$ ~~together~~ for all the base point $p \in M$, as done for the tangent bundle, we obtain the desired bundle:

Definition 13: r-th Jet Bundle of E

We call *r-th Jet Bundle of E* the smooth bundle $(J^r(E), \pi_r, M)$ where:

- $J^r(E) := \bigsqcup_{p\in M} J^r_p(E) \equiv \{ j^r_p\sigma \mid p \in M,\ \sigma \in \Gamma^\infty(p)\}$
- $\pi_r : J^r(E) \to M$ such that $j^r_p\sigma \mapsto p$

1.2 Globally Hyperbolic Space-times

Rigorously speaking, configurations of a field system are encoded by sections based on a *spacetime* manifold. From a physical point of view, we are interested in those spacetimes which allow to set a well-posed initial value problem for hyperbolic partial differential equations, such as the wave equation. In particular we need to ensure that the spacetime we consider possesses at least one distinguished codimension 1 hypersurface on which we can assign the initial data needed to construct a solution of such an equation.
For this purpose we have to introduce ~~the~~ *globally hyperbolic spacetimes*.

1.2.1 Reprise in Lorentzian Geometry

Consider a ~~diffential~~ manifold M.

Definition 14: (Pseudo-Riemmanian) Metric

~~A function~~ on the bundle product of TM with itself:

$$g : TM \times_M TM \to \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a non-degenerate bilinear form.

Definition 15: Pseudo-Riemmanian Manifold

A pair (M, g) such that:

- M is a n-dimensional ($n \geq 2$), Hausdorff, second countable, connected, orientable smooth manifold.
- g is a Lorentzian metric.

Notation fixing

A Pseudo-riemmanian manifold (M, g) is called:

- *Riemmanian* if the sign of g is positive definite.

- *Lorentzian* if the signature is $(+, -, \ldots, -)$ or equivalently $(-, +, \ldots, +)$.

1.2.2 Time Orientation

If a smooth manifold is endowed with a Lorentzian metric, then the tangent vectors at each point in the manifold can be ~~classed~~ into three different types.

Notation fixing

$\forall p \in M, \quad \forall X \in T_p M$, we call a vector:

- *time-like* if $g(X, X) > 0$.
- *light-like* if $g(X, X) = 0$.
- *space-like* if $g(X, X) < 0$.

Observation 7: Local Time Orientability

$\forall p \in M$ the timelike tangent vectors in p can be divided into two equivalence classes taking

$$X \sim Y \text{ iff } g(X, Y) > 0 \qquad \forall X, Y \in T_p^{\text{time-like}} M :$$

We can (arbitrarily) call one of these equivalence classes "future-directed" and call the other "past-directed". Physically the designation of the two classes of future- and past-directed timelike vectors corresponds to a choice of an arrow of time at the point.
The future- and past-directed designations can be extended to null vectors at a point by continuity.

Definition 16: Time-orientation

We call *time-orientation* a global tangent vector field $\mathfrak{t} \in \Gamma^\infty(TM)$ over the Lorenzian manifold M such that:

- $\mathrm{supp}(\mathfrak{t}) = M$
- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Observation 8

The fixing of a time-orientation is equivalent to a consistent smooth choice of a local time-direction.

Definition 17: Space-Time

We call ~~*Spacetime*~~ a quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ such that:

- (M, g) is a time-orientable[a] n-dimensional manifold $(n > 2)$
- $\mathfrak{o}$ is a choice of orientation
- $\mathfrak{t}$ is a choice of time-orientation

[a]Manifold for which such *time-orientation* exists.

In a spacetime M ~~is quite important~~ to identify particular classes of subsets. The main tool are the *parametrized curves*:

Notation fixing

Consider a piece-wise smooth curve $\gamma : \mathbb{R} \supset I \to M$ ~~is called~~:

- *time-like* (resp. light-like, space-like) iff $\dot{\gamma}(p)$ is time-like (resp. light-like, space-like) $\forall p \in M$.
- *causal* iff $\dot{\gamma}(p)$ is nowhere spacelike.
- *future directed* (resp. past directed) iff ~~is~~ causal and $\dot{\gamma}(p)$ is future (resp. past) directed $\forall p \in M$.

Definition 18: Chronological $\substack{\textbf{future}\\\textbf{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{I}^{\pm}_M(p) := \left\{ q \in M \middle| \, \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ time-like } \substack{\text{future}\\\text{past}} - \text{directed} \, : \, \gamma(0) = p, \, \gamma(1) = q \right\}$$

Definition 19: Causal $\substack{\textbf{future}\\\textbf{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{J}^{\pm}_M(p) := \left\{ q \in M \middle| \, \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ causal } \substack{\text{future}\\\text{past}} - \text{directed} \, : \, \gamma(0) = p, \, \gamma(1) = q \right\}$$

Notation fixing

~~Theese~~ concepts can be naturally extended to any subset $A \subset M$:

- $\mathbf{I}^{\pm}_M(A) = \bigcup_{p \in A} \mathbf{I}^{\pm}_M(p)$
- $\mathbf{J}^{\pm}_M(A) = \bigcup_{p \in A} \mathbf{J}^{\pm}_M(p)$

Definition 20: Achronal Set

We call ~~*Achronal Set*~~ a subset $\Sigma \subset M$ such that every inextensible timelike curve intersect Σ at most once.

Definition 21: $\substack{\textbf{future}\\\textbf{past}}$ Domain of dependence of an Achronal set

The two subset related to the generic achornal set $\Sigma \subset M$:

$$\mathbf{D}^{\pm}_M(\Sigma) := \left\{ q \in M \middle| \; \forall \gamma \; {}^{\text{past}}_{\text{future}}\text{inextensible causal curve passing through } q : \; \gamma(I) \cap \Sigma \neq \emptyset \right\}$$

Notation fixing

$\mathbf{D}_M(\Sigma) :- \mathbf{D}^+_M(\Sigma) \cup \mathbf{D}^-_M(\Sigma)$ is called *total domain of dependence*.

1.2.3 Globally Hyperbolicity

Finally we come to the key concept of our treatment:

Definition 22: Cauchy Surface

Subset $\Sigma \subset M$ such that:

- closed
- achronal
- $\mathbf{D}_M(\Sigma) \equiv M$

Notation fixing

We denote the set of all the ~~cauchy~~ surfaces as $\mathscr{P}_C(M)$.

Definition 23: Globally-Hyperbolic SpaceTime

Spacetime M such that there exists at least one *Cauchy Surface*

According to Definition 23, only the existence of a single Cauchy hypersurface is guaranteed. This is slightly disturbing since there is no reason a priori why an initial value hypersurface for a certain partial differential equation should be distinguished. This quandary has been overcome proving (see [2][section 1.3]) that, if a spacetime (M, g) is globally hyperbolic, there exists a foliation of M by Cauchy surfaces:

Theorem 1.2.1 (Globally hyperbolic space characterization) *Let (M,g) be any time-oriented spacetime. The following two statements are equivalent:*

- *(M,g) is globally hyperbolic.*
- *(M,g) is isometric to $\mathbb{R}\times\Sigma$ endowed with the line element $ds^2 = \beta dt^2 - h_t$ where $t : \mathbb{R}\times \cancel{\sigma} \to \mathbb{R}$ is the projection on the first factor, β is a smooth and strictly positive function on $\mathbb{R}\times\Sigma$ and $t \mapsto h_t$, $t\in\mathbb{R}$, yields a one-parameter family of smooth Riemmanian metrics.*
  Furthermore, for all $t\in\mathbb{R}$, $\{t\}\times\Sigma$ is an (n-1)-dimensional, spacelike, smooth Cauchy surface in M.

The class of globally hyperbolic spacetimes includes most of the physically interesting examples, e.g.: Minkowski ~~spacetimes,~~ Friedman-Robertson-Walker ~~solutions,~~ ~~Kerr family,~~ ~~However,~~ in what follows we will make primary use of the most trivial example:

Example: 1

Trivially, the real line $\mathbb{R}$ is a globally hyperbolic manifold.
Each point $x\in\mathbb{R}$ represent a proper Cauchy surfaces which realize the trivial foliation $\mathbb{R}\simeq 1\times\mathbb{R}$ required by theorem 1.2.1

To conclude this section, we introduce some terms which will be often used in the following in order to specify the support properties of the sections of a vector bundle with base a globally hyperbolic spacetime.

Notation fixing

Let M be a globally hyperbolic spacetime and $E = (E,\pi,M;V)$ a vector bundle of typical fiber V.
We denote:

- $\Gamma_0(E)$ the space of *compactly supported* smooth sections.
- $\Gamma_{sc}(E)$ the space of *spacelike compact* smooth sections.
  $\big(f\in\Gamma_{sc}(E)$ if there exists a compact subset $K\subset M$ such that $\operatorname{supp} f \subset \mathbf{J}_M(K).\big)$
- $\Gamma_{fc}(E)$ the space of *future- compact* smooth sections.
  $\big(f\in\Gamma_{fc}(E)$ if $\operatorname{supp}(f)\cap\mathbf{J}^+_M(K)$ is compact for all $p\in M.\big)$
- $\Gamma_{pc}(E)$ the space of *past- compact* smooth sections.
  $\big(f\in\Gamma_{pc}(E)$ if $\operatorname{supp}(f)\cap\mathbf{J}^-_M(K)$ is compact for all $p\in M.\big)$
- $\Gamma_{tc}(E) := \Gamma_{pc}(E)\cap\Gamma_{fc}(E)$ the space of *timelike compact* smooth sections.

1.3 Green Hyperbolic Operators

Green Hyperbolic Operator are the suitable object to represent a *wave-like propagation* dynamic.

Consider $E = (E, \pi, M; V), E' = (E', \pi', M; V')$ two linear vector bundle over M (with different typical fiber), we define:

Definition 24: Linear Partial Differential operator **(of order at most $s \in \mathbb{N}_0$)**

Linear map $L : \Gamma(E) \to \Gamma(E')$ such that $\forall\, p \in M$ ~~exists:~~

- $U \ni p$ open set rigged with:
  - (U, φ) local chart on M.
  - (U, χ) local trivialization of F
  - (U, χ') local trivialization of F'
- $\{A_\alpha : U \to \mathrm{Hom}(V, V')\,|\,\alpha \in \mathbb{N}_0^n, |\alpha| \leq s\}$ ~~collection~~ of smooth maps labeled by multi-indices.

which allows to express L locally:

$$\chi' \circ (L\sigma) \circ \varphi^{-1} = \sum_{|\alpha| \leq s} A_\alpha \partial^\alpha (\chi \circ \sigma \circ \varphi^{-1}) \qquad \forall \sigma \in dom(L) \subset \Gamma(E)$$

(where we have make use of the multi-index notation1.1.5)

Observation 9

Notice that linear partial differential operators cannot enlarge the support of a section.

Definition 24 accounts for a large class of operators, most of which are not typically used in the framework of field theory.
In order to characterize a relevant subset we introduce two auxiliary concepts:

Consider a Linear differential operator $L : \Gamma(E) \to \Gamma(E')$:

Definition 25: Principal Symbol

We call *principal symbol* the map $\sigma_L : T^*M \to \mathrm{Hom}(E, E')$locally defined as follows:
For a given $p \in M$, consider a coordinate chart (U, x^i) around p and local trivializations of E and of E' (as precribed in Definition 24).
For all $\xi = \xi_i dx^i \in T_p^* M$ set:

$$\sigma_L(\xi) = \sum_{|\alpha| = s} \xi^\alpha A_\alpha(p)$$

where $\xi^\alpha = \prod_{\mu=0}^{m-1} \xi^{\alpha_\mu}$

Definition 26: Formal Dual Operator (of L)

The unique linear partial differential operator $L^\star : \Gamma(G^*) \to \Gamma(E^*)$ such that:

$$< L^\star g', f >=< g', Lf >$$

$\forall f \in \Gamma(E),\ g' \in \Gamma(G^*)$ which have supports with compact overlap.
($< \cdot, \cdot >$ denote the 1-form evaluation: $< \alpha, v >= \alpha(v) \quad \forall v \in E_p, \alpha \in E_p^*$.)

N.B. : From now on we will consider only bundles with globally-hyperbolic space-time ~~base.~~

1.3.1 Green Hyperbolic Operators

Let M be a globally hyperbolic spacetime, consider a vector bundle E over M and a L.p.d.o. $L : \Gamma(E) \to \Gamma(E)$:

Definition 27: $\substack{\text{retarded} \\ \text{advanced}}$ ($\pm$) Green Operators

L.p.d.o. $G^\pm : \Gamma(E) \to \Gamma(E)$ such that:

- $dom(G^+) = \Gamma_{pc}(E) \qquad dom(G^-) = \gamma_{fc}(E)$
- $LG^\pm f = G^\pm Lf = f \qquad \forall f \in dom(G^\pm)$
- $\text{supp}(G^\pm f) \subset \mathbf{J}_M^\pm(\text{supp}(f)) \qquad \forall f \in dom(G^\pm)$

Observation 10

From the definition ~~follows~~ that $G^\pm$ is the left-right inverse of the restriction of L to $dom(G^\pm)$.

Notation fixing

We refer to the operator:

$$E := G^- - G^+ : \Gamma_{tc}(E) \to \Gamma(E)$$

as the *Advanced minus Retarded operator* or *Causal Propagator*[**?**].

Green operators are not necessarily unique. For this we introduce the following definition:

Definition 28: Green hyperbolic operator

The linear partial differential operator P is be called Green hyperbolic if P and $P^\star$ have advanced and retarded Green's operators.

~~for such~~ operators uniqueness of Green's operator is guaranteed:

Theorem 1.3.1 (Characterization of Green Hyperbolic operators)

Hp:

$E = (E, \pi, M)$ a vector ~~bundles~~ over a globally hyperbolic spacetime M.

$P : \Gamma(E) \to \Gamma(E)$ a Green hyperbolic operator, $G^\pm$ its Green's operators and $G^\pm_\star$ the Green's operators of the dual.

Th:

L ~~posses an~~ unique retarded G^+ and a unique advanced G^- ~~Green~~ operator.

$< G^\pm_\star f', f >=< f', G^\mp f > \qquad \forall f \in \Gamma_0(E), \forall f' \in \Gamma_0(E^*)$

Proof:

See for example [?][proposition 2]

□

1.3.2 Normally Hyperbolic Operators

Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness[1] of a Cauchy problem. However for the large class of the *Normally-Hyperbolic Operators* hyperbolicity is guaranteed both in PDE and Green sense.

Consider a Lorentzian manifold (M, g) and two vector bundles $E = (E, \pi, M; V), E' = (E', \pi', M; V')$,

Definition 29: Normally Hyperbolic Operators

Second order linear partial differential operator $P : \Gamma(E) \to \Gamma(E')$ such that:

$$\sigma_P(\xi) = g(\xi, \xi)\mathbb{1}_{E_p} \qquad \forall p \in M, \xi \in T^*_p M$$

Observation 11

[1] I.e. exists an unique solution.

Making explicit the coordinate expression of a normally hyperbolic operator P , one realizes how such operators provide the natural generalization of the usual ~~Wave~~ operator.
Consider a globally hyperbolic operator P for all $p \in M$ a trivializing chart (U, φ, χ) centered in p. There exist a collection $\left\{A, A^{\mu} | \mu \in \{0, \dots, m-1\}\right\}$ of smooth $\mathrm{Hom}(V, V)-$valued maps on U such that, P ~~reas~~ as follows:

$$\chi \circ (P\sigma) \circ \varphi^{-1} = \left(g^{\mu\nu} \mathrm{id}_V \partial\mu\partial_n u + A^{\mu}\partial_{\mu} + A\right)(\chi \circ \sigma \circ \varphi^{-1}) \qquad \forall \sigma \in dom(P) \subset \Gamma(E)$$

where both the chart and the vector bundle trivializations are ~~understood~~. One ~~immediately~~ notices that locally this expression agrees- up to terms of lower order in the derivative-s with the one for the d'Alembert operator acting on sections of E constructed out of a covariant derivative ∇ on E, that is the operator:

$$\Box_{\nabla} = g^{\mu\nu}\nabla_{\mu}\nabla_{\nu} : \Gamma(E) \to \Gamma(E)$$

This definition becomes even more important if we assume that the underlying background is globally hyperbolic, since we can associate to each normally hyperbolic operator P an initial value problem ~~and talk~~ about Green's operator.

Proposition 1.3.1 (Green operators) *Let P be a normally hyperbolic operator, then:*

- *$P^{\star}$ is a normally hyperbolic operator.*
- *P is Green hyperbolic.*

Proof:

We omit the proof, see for example [**?**][Corollary 3.4.3].

□

Theorem 1.3.2 (Normally hyperbolic operators properties.)

Hp:

- *$\Sigma \subset M$ a spacelike Cauchy surface with future-pointing unit normal vector field $\vec{n}$.*
- *P a normally hyperbolic operator and ∇ a P-compatible[a] covariant derivative on E*

[a] There existss a section $A \in \Gamma(\mathrm{End}(E))$ such that $\Box_{\nabla} + A = P$.

Th:

- *The Cauchy problem;*

$$\begin{cases} Pu = J & \text{on } M \\ u = u_0 & \text{on } \Sigma \\ \nabla_{\vec{n}} u = u_1 & \text{on } \Sigma \end{cases}$$

admit a unique solution $u \in \Gamma(E)$ *for any* $J \in \Gamma(E)$ *and* $u_0, u_1 \in \Gamma(\Sigma)$

- P *is Green-hyperbolic.*

Proof:

First proposition has been proved in different forms in several books, e.g. [1][Corollary 5]. For the second proposition see[2][Corollary 3.4.3]

□

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[3] G. Sardanashvily. Advanced Differential Geometry for Theoreticians. Fiber bundles, jet manifolds and Lagrangian theory. page 221, 2013.