Chapter 1

Geodesic Fields

In the context of differential geometry, *geodesic curves* are a generalization of *straight lines* in the sense of self-parallel curves. Considering a differential manifold M endowed with an affine connection ∇ we define:

Definition 1: Geodesic

A smooth curve $\gamma : [a,b] \to M$ such that:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \tag{1.1}$$

where $\dot{\gamma}^\mu := \frac{d\gamma^\mu}{dt}$ is the tangent vector to the curve.

In local chart the previous equation assume the well-known expression:

$$\ddot{\gamma}^i + \Gamma^i{}_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0 \tag{1.2}$$

,where $\Gamma^i{}_{jk}$ is the coordinate representation of the Christoffel symbols of the connection, ~~that admit~~ a well-posed Cauchy problem.

Theorem 1.0.1 *Let M be a smooth manifold, $p \in M$, $v \in T_pM$. Then there exist $\epsilon > 0$ and precisely one geodesic*

$$c : [0, \epsilon] \to M$$

with $c(0) = p, \dot{c}(0) = v$.
In addition, c depends smoothly on p and v.

Proof:

Equation 1.2 is a system of second order ODE, and the Picard-Lindelof theorem yields the local existence and uniqueness of a solution with prescribed initial values and derivatives, and this solution depends smoothly on the data.

$\square$

In presence of a pseudo-Riemannian metric ~~is~~ possible to present the geodesic ~~in a metric sense i.e.~~ as the curve extremizing the *Energy Functional*[1]:

Definition 2: Energy functional

$$E : C^1([a,b], Q) \to \mathbb{R} \qquad E(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|^2 dt \tag{1.3}$$

Considering only the proper variation (that keep the end-point fixed), the extremum condition corresponds to equation 1.2 where ∇ is the unique Levi-Civita connection (torsion-free and metric-compatible).

In general relativity the problem of the ~~geodesic equation linearization, named~~ *Jacobi equations* takes a central role. [2]

Definition 3: Jacobi Field

We call a *Jacobi field* along the geodesic γ the tangent vector field over the sub-manifold $\gamma(t,\tau)$, determined by a smooth one parameter family of geodesicsγ_τ (with $\gamma_0 = \gamma$), in respect to theτ coordinate. *i.e.*:

$$J = \left. \frac{\partial \gamma_\tau(t)}{\partial \tau} \right|_{\tau=0}$$

In local charts, a Jacobi ~~fields~~ along the geodesic γ is solution of a linear O.D.E.:

$$\left(X''\right)^\mu + R^\mu_{i\alpha j} T^i X^\alpha T^j = 0 \tag{1.4}$$

where:

- $\left(X'\right)^\mu := \left(\nabla_{\dot\gamma(t)} X\right)^\mu$ is the covariant derivative along the curve γ.
- $T \equiv \dot\gamma(t)$ stands for the tangent vector to ~~the curve~~ γ.
- $R^\mu_{i\alpha j}$ ~~is~~ the ~~components representations~~ of the Riemann curvature tensor,

The rest of this chapter will be devoted to ~~the presentation of the approach to geodesic and Jacobi problem~~ as a physical system.

[1] Remember that for arc-length parametrized curves the Energy functional coincide with the length functional.[**?**, Lemma 1.4.2]

[2] Usually in this context ~~takes~~ the name of *Geodesic deviation* problem[**?**, pag. 46] inasmuch Jacobi field describes the difference between the geodesic and an "infinitesimally close" geodesic.

1.1 Geodesic Problem as a Mechanical Systems

The basic idea is very simple, ~~portray~~ the geodesic curve as the natural motion of a free point particle constrained on the Pseudo-Riemannian manifold Q.

Remark:

In terms of general relativity this problem can be instantly recognized as the derivation of the ~~free falling particles motion~~.

However, there is no lack of alternative viewpoints . The framework of the classical *geometric mechanics* taught us to picture the "static" configurations of a constrained, complex, classical system as a point on the *Configuration space* ~~manifold~~. According to that, the geodesic motion can be seen as a realization of a particular dynamics on a mechanical system with a pseudo-Riemannian configuration space[a].

[a] Such systems can be depicted as "geodesic" even in presence of a position-dependant potential.[**?**, Cap 3.7]

Proposition 1.1.1 (Geodesic Motion) *The geodesics on the Pseudo-Riemannian manifold* (Q, g) *are the natural motions of the ordinary Lagrangian system* (Q, L) *where:*

$$L(V_q) := \frac{1}{2} g_q(V, V) \tag{1.5}$$

Observation 1

The geodesic system is not simply Lagrangian but also Hamiltonian. This property follows from the hyperregularity[**?**] of L.

As shown in chapter 2, every system with discrete degrees of freedom can be seen as a trivial field system. From that ~~follows~~ the alternative characterization of geodesic as a Lagrangian field:

Corollary 1.1.1 (Geodesic field) *The geodesics on the Pseudo-Riemannian manifold* (Q, g) *can be seen as the* Dynamical Configurations *of the Lagrangian field system* $(E, \mathscr{L})$ *where:*

- $E = (Q \times \mathbb{R}, \pi, \mathbb{R})$ ~~*trivial*~~ *smooth bundle on the real line.*
- $\mathscr{L}[\gamma] = \frac{1}{2} g(\dot{\gamma}, \dot{\gamma})(t) dt$

Proof:

~~Is~~ a simple application of the correspondence seen in chapter **??**.

$\square$

From this perspective ~~is~~ clear that the Energy Functional corresponds to the action of the geodesic field dynamics and equation 1.2 is nothing more than the ~~motion equation~~ according to the *least action principle*.

1.2 Peierls Bracket of the Geodesic field

The local coordinate expression for the Lagrangian density of the geodesic field ~~results:~~

$$\mathscr{L}\left(t,\gamma^i(t),\dot{\gamma}^i(t)\right) := \frac{1}{2} g_{\mu,\nu}\left(\gamma^i(t)\right)\dot{\gamma}^\mu\dot{\gamma}^\nu \tag{1.6}$$

which is highly non-linear. ~~Explicitly is~~ quadratic in the velocity components $\dot{\gamma}^i$ and implicitly, through $g_{\mu\nu}(\gamma^i(t))$, is non-polynomial ~~in curve coordinate~~ γ^i.

As show in section **??**, for this type of systems the calculation ~~of~~ Peierls bracket can be realized only locally around a predetermined solution. Let us repeat the Peierls' procedure for the system under investigation.

As a consequence of our introduction on the geodesic as a field, we can state the unperturbed dynamic as a L.P.D.O :

$$Q_{\mathscr{L}}\left(q^\mu\right) = \left[\ddot{q}^\mu + \Gamma^\mu_{\ ij}\dot{q}^i\dot{q}^j\right] \tag{1.7}$$

where $\dot{q}^\mu = \frac{d}{dt}q^\mu(t) = \dot{q}^i\partial_i q^\mu$.

A linear variation of $q_0^\mu + \epsilon\eta^\mu$ constructed from the coordinate representation q_0^μ of the geodesic $\gamma_0 \in \mathtt{Sol}$, solves the original ~~motion equations~~ when

$$Q_{\mathscr{L}}\left(q_0^\mu + \epsilon\eta^\mu\right) = \frac{d^2}{dt^2}\left(q_0^\mu + \epsilon\eta^\mu\right) + \left[\Gamma^\mu_{\ ij}(\vec{q}_0 + \epsilon\vec{\eta})\right]\left(\dot{q_0}^i + \epsilon\dot{\eta}^i\right)\left(\dot{q_0}^j + \epsilon\dot{\eta}^j\right) = 0 \stackrel{!}{=} o(\epsilon) \tag{1.8}$$

If we consider only the first order in the parameter ϵ we can expand the expression of the Christoffel symbols:

$$\left[\Gamma^\mu_{\ ij}(\vec{q}_0 + \epsilon\vec{\eta})\right] = \left[\Gamma^\mu_{\ ij}(\vec{q}_0) + \epsilon\eta^\alpha\left(\partial_\alpha\Gamma^\mu_{\ ij}\right)\Big|_{\vec{q}_0} + o(\epsilon)\right]$$

Collecting all the terms in equation 1.8 ~~up the~~ first order in ϵ ~~follows~~ a condition on the perturbation:

$$\begin{aligned} 0 &= \ddot{\eta}^\mu + \eta^\alpha\left(\partial_\alpha\Gamma^\mu_{\ ij}\right)\Big|_{\vec{q}_0}\dot{q_0}^i\dot{q_0}^j + \Gamma^\mu_{\ ij}\left(\dot{\eta}^i\dot{q_0}^j + \dot{q_0}^i\dot{\eta}^j\right) = \\ &= \left\{g^\mu_\alpha\frac{d^2}{dt^2} + \Gamma^\mu_{\ i\alpha}(\vec{q}_0)\left[2\dot{q_0}^i\frac{d}{dt}\right] + \left[\partial_\alpha\Gamma^\mu_{\ ij}(\vec{q}_0)\dot{q_0}^i\dot{q_0}^j\right]\right\}\eta^\alpha = P^\mu_\alpha\eta^\alpha \end{aligned} \tag{1.9}$$

where P^μ_α is a linear partial differential operator acting on the *variations*, *,i.e.*, the components of a field along the geodesic γ_0. As showed in section **??**, equating the linearized dynamics operator with the term $-\left(Q_\chi(\gamma_0)\right)(x)$ (see equation **??**) lead to the inhomogeneous *Jacobi operator* from which follows all the standard construction of the brackets Peierls.

Proposition 1.2.1 *The differential equation*

$$P^\mu_\alpha \eta^\alpha = 0$$

corresponding to the l.p.d.o P defined in equation 1.8 corresponds to equation 1.4 defining the Jacobi fields *along the geodesic γ_0.*

Proof:

For convenience, we adopt the following notation:

$$\eta^\mu \partial_\mu := X \equiv X^\mu \partial_m u$$
$$\dot{q_0}^i \partial_i \equiv \dot{\gamma_0} := T \equiv T^i \partial_i$$

We have to show that the equation just found:

$$\ddot{X}^\mu + X^\alpha \left(\partial_\alpha \Gamma^\mu_{ij}\right) T^i T^j + \Gamma^\mu_{\alpha j}\left(2 T^i \dot{X}^\alpha\right) = 0 \tag{1.10}$$

where $\dot{X}^\mu = \frac{d}{dt} X^\mu = T^i \partial_i X^\mu$, correspond to ~~equation~~ defying the Jacobi field:

$$\left(X''\right)^\mu + R^\mu_{i\alpha j} T^i X^\alpha T^j = 0 \tag{1.11}$$

where $\left(X'\right)^\mu := \left(D_t X\right)^\mu$ and $D_t = T^i \nabla_i$ is the covariant derivative along the curve.
Since:

$$\begin{aligned} X'' &= D_t D_t X = D_t \left(\partial_\mu \left(\dot{X}^\mu + \Gamma^\mu_{i\alpha} T^i X^\alpha\right)\right) \\ &= \left(\ddot{X}^\mu + \frac{d}{dt}\left(\Gamma^\mu_{i\alpha} T^j X^\alpha\right) + \Gamma^\mu_{j\nu} T^j \dot{X}^\nu + T^j \Gamma^\mu_{j\nu} \Gamma^\nu_{i\alpha} T^i X^\alpha\right)\partial_\mu \end{aligned} \tag{1.12}$$

We can write equation 1.10 in term of the covariant derivative as:

$$\begin{aligned} \left(X''\right)^\mu &= -\left(X^\alpha \left(\partial_\alpha \Gamma^\mu_{ij}\right) T^i T^j + \Gamma^\mu_{\alpha i}\left(2 T^i \dot{X}^\alpha\right) - \frac{d}{dt}\left(\Gamma^\mu_{i\alpha} T^i X^\alpha\right) - T^j \Gamma^\mu_{js} \Gamma^s_{i\alpha} T^i X^\alpha\right) = \\ &= -\left(X^\alpha \left(\partial_\alpha \Gamma^\mu_{ij}\right) T^i T^j - \dot{\Gamma}^\mu_{i\alpha} T^i X^\alpha - \Gamma^\mu_{i\alpha} \dot{T}^i X^\alpha - T^j \Gamma^\mu_{js} \Gamma^s_{i\alpha} T^i X^\alpha\right) \end{aligned}$$

remembering that the geodesic condition is still to be met:

$$\dot{T}^i = -\Gamma^I_{jk} T^j T^k$$

we can conclude that:

$$\begin{aligned} \left(X''\right)^\mu &= -\left(X^\alpha \left(\partial_\alpha \Gamma^\mu_{ij}\right) T^i T^j - T^j \left(\partial_j \Gamma^\mu_{i\alpha}\right) T^i X^\alpha + X^\alpha \Gamma^\mu_{\alpha s} \Gamma^s_{ij} T^i T^j - T^j \Gamma^\mu_{js} \Gamma^s_{i\alpha} T^i X^\alpha\right) = \\ &= -\left(R^\mu_{i\alpha j} T^i X^\alpha T^j\right) \end{aligned}$$

1.2.1 Example: Geodesic field on FRW space-time.

l'idea è che le metriche possono essere diagonalizzate risulta un sistema di equazioni del moto ode disaccoppiate di cui posso calcolare la funzione di green come indicato nelle dispense che trattano il calcolo di green per le ode. se ho l'operatore di green posso calcolare in esplicito le soluzioni perturbate e quindi le peierls. Guardare pag 5 advance + pag 6 primer

1.3 Algebraic quantization of the Geodesic Field

The algebraic quantization scheme applies only to ~~linear field systems~~. Since equation1.2 is highly non linear, it is not the geodesic system that can actually be quantized but rather its linearization, the Jacobi field along a fixed geodesic γ_0.

A Classical Framework

The basic idea is that, chosen a geodesic γ_0, the ~~kinematics~~ configurations of the Jacobi fields are tangent fields along the fixed curves.

A.a Kinematics The configuration bundle E corresponds to the *Pull-back bundle* $\gamma_0^*(TQ)$ of the tangent bundle along the geodesic γ_0. Then:

- E is a vector bundle over $\mathbb{R}$.
- The base manifold $\mathbb{R}$ can be considered as a degenerate globally hyperbolic spacetime, $\mathscr{P}_C(\mathbb{R}) = \mathbb{R}$.
- the fibers are $E_p := T_{\gamma_0(p)}Q$
- $\mathtt{C} = \Gamma^\infty(E) = \mathfrak{X}(\gamma_0)$ is constituted by vector fields along the curve γ_0.

A.b Dynamics The coordinate representation of the motion equation is:

$$(PX)^\mu = (X'')^\mu + R^\mu_{i\alpha j} T^i T^j X^\alpha$$

where $X \in \mathtt{C}$ and $T^i = \dot{\gamma_0}^i$. According to equation **??** this operator fall exactly in ~~the~~ *normally hyperbolic* operator ~~class~~ hence ~~is~~ quantizable both by Peierls procedure and by initial data.

1.3.1 PreQuantum Framework

A Peierls approach

A.a Pairing Since Q is a Riemannian manifold and $\mathtt{C}$ is composed by tangent vector fields, it is straightforward to choose as inner product on the configuration bundle E the metric function defined on Q:

$$\langle X,Y\rangle_t := g\left(X\left(\gamma_0(t)\right), Y\left(\gamma_0(t)\right)\right) \qquad \forall X,Y \in E_t \tag{1.13}$$

~~Follows slavishly~~ the definition of pairing:

$$(X,Y) = \int_{\mathbb{R}} \langle X,Y\rangle_t \, dt \tag{1.14}$$

~~well-defined~~ for every pair $X,Y \in \mathtt{C}$ such that $\mathrm{supp}(X) \cap \mathrm{supp}(Y)$ is compact.

~~Operator~~ P ruling the dynamics is formally self-adjoint:

$$\begin{aligned}(Y,PX) = \int Y_\mu P X^\mu dt = \int \left(Y_\mu \ddot{X}^\mu + Y_\mu R^\mu_{i\alpha j} T^i T^j X^\alpha\right) dt = \\ = \int \left(\ddot{Y}_\mu X^\mu + X_\mu R^\mu_{i\alpha j} T^i T^j Y^\alpha\right) dt = \int P Y_\mu X^\mu dt = (PY,X)\end{aligned}$$

~~where is been~~ integrated by parts twice (~~border value are~~ null since the integrand is compactly supported) and ~~has been~~ exploited ~~a~~ curvature tensor identity:

$$\langle R(X,T)T,Y\rangle = \langle R(Y,T)T,X\rangle \tag{1.15}$$

A.b Classical Observables Replicating what ~~is~~ done in the general case, we construct the *pre-observables* as the functionals $F_f : \mathtt{C} \to \mathbb{R}$ for all $f \in \Gamma_0(E)$ compactly supported field along ~~geodesic~~ γ_0 ~~as~~:

$$F_f(X) = \int_{\mathbb{R}} < X, f >_t dt \qquad \forall X \in \mathtt{C} \tag{1.16}$$

The ~~classical observables space~~ is then obtained through the usual quotient:

$$\mathscr{E} \simeq \frac{\Gamma_0}{P\Gamma_0}$$

~~observables functionals~~ are the maps:

$$F_{[f]}(X) = F_f(X) \qquad \forall X \in \mathtt{Sol}$$

where f is a representative ~~of equivalence~~ class $[f] \in \mathscr{E}$.

A.c Symplectic Structure ~~Geodesic~~ motion is a particular case of ~~system~~ with finite degrees of freedom, thus the Peierls bracket between two Lagrangian functionals χ, ω around a geodesic γ_0 tested on a function $f \in C_0^\infty(\mathbb{R})$ ~~is~~ given by **??**. Restricting the definition to the simplest Lagrangian functionals constructible from the classical observables:

$$\chi[\phi] := (\chi, \phi) \qquad \chi \in \mathscr{E}, \phi \in \mathtt{Sol}$$

corresponding to Lagrangian densities in the form:

$$\chi(\vec{q}i,\dot{\vec{q}}) := < \chi, \vec{q} > = \chi^i q_i$$

such that $Q_\chi \gamma_0^i = \chi^i$, the Peierls brackets expression reduces to

$$\{\chi,\omega\}(\gamma_0)[f] = \int f(t)\left\langle \chi, \left(G^- - G^+\right)\omega\right\rangle dt$$

~~For regular distribution can be neglected the test-function f.~~

We conclude that,according to the Peierls' procedure, the classical symplectic space is the pair $(\mathscr{E},\tau)$ where:

$$\tau([\chi],[\omega]) = \{\chi,\omega\} = \int \left\langle \chi, \left(G^- - G^+\right)\omega\right\rangle dt = (\chi, E\omega) \qquad \forall \chi,\omega \in \Gamma_0(E)$$

B Initial data Approach

B.a Classical Phase Space The base manifold for the configuration bundle under examination is the real line $\mathbb{R}$ that can be seen as a degenerate globally hyperbolic spacetime. Thus each point $p \in \mathbb{R}$ ~~are Cauchy surfaces~~ and no further support condition can be imposed. Considering that ~~operator~~ P is of second order, we have:

$$\mathscr{M}(p) \equiv \mathsf{Data}(p) = \Gamma^\infty(p) \times \Gamma^\infty(p) = T_{\gamma_0(p)}Q \times T_{\gamma_0(p)}Q$$

and

$$\mathscr{M} \simeq \mathtt{Sol}$$

using the map which yields the unique solution starting from an initial data.

B.b Symplectic Structure on the Phase Space The general definition **??** of the symplectic form on the classical phase space reduces to:

$$\Omega : \mathscr{M}(p) \times \mathscr{M}(p) \to \mathbb{C} \qquad : \qquad \Omega\Big\{[V_0,V_1],[W_0,W_1]\Big\} = g(V_1,W_0) - g(V_0,W_1)$$

where g is the inner product on Q.
~~Which~~ can be transferred ~~on the solutions space~~:

$$\sigma_p : \mathtt{Sol} \times \mathtt{Sol} \to \mathbb{C} \qquad : \qquad \sigma_p\Big\{X,Y\Big\} = \Omega\Big\{[Y(t),D_tY(t)],[X(t),D_tX(t)]\Big\}$$

Mimicking what ~~is~~ done in example **??** for the scalar field, can be proved the independence of the phase space construction from the particular choice of $p \in \mathbb{R}$. Taken $X, Y \in \mathtt{Sol}$ two Jacobi fields on $\gamma_0(t)$, ~~can be defined a scalar field over $\mathbb{R}$:~~

$$J(t) := \Omega\Big\{[Y(t),D_tY(t)],[X(t),D_tX(t)]\Big\} = X^\alpha(t) g_{\alpha\beta} D_t Y^\beta(t) - Y^\alpha(t) g_{\alpha\beta} D_t X^\beta(t)$$

where $D_t = T^\mu \nabla_\mu$ as usual. This is clearly a conserved current:

$$
\begin{aligned}
D_t J = &\quad (D_t X)^\alpha g_{\alpha\beta} (D_t Y)^\beta - (D_t Y)^\alpha g_{\alpha\beta} (D_t X)^\alpha + X^\alpha g_{\alpha\beta} D_t D_t Y^\beta - Y^\alpha g_{\alpha\beta} D_t D_t X^\beta = \\
= &\qquad X^\alpha g_{\alpha\beta} P Y^\beta - Y^\alpha g_{\alpha\beta} P X^\beta - X_\beta R^\beta_{i\alpha j} T^i Y^\alpha T^j + Y_\beta R^\beta_{i\alpha j} T^i X^\alpha T^j = 0 \qquad (1.17)
\end{aligned}
$$

exploiting the conditions $\nabla_\mu g_{\alpha\beta} = 0$, $PX = PY = 0$ and equation 1.15.
Hence:

$$\int_p^{p'} D_t J = J(p) - J(p') = 0$$

In others words:

$$\sigma_\Sigma(X,Y) = \sigma_{\Sigma'}(X,Y) \qquad \forall X, Y \in \mathtt{Sol}\ \forall \Sigma, \Sigma' \in \mathscr{P}_C(M) = \mathbb{R}$$

In conclusion, according to the initial data procedure, the classical symplectic space is the pair $(\mathtt{Sol}, \sigma)$ such that:

$$\sigma(X,Y) = X_\mu(\Sigma)\,(D_t Y(\Sigma))^\nu - (D_t X(\Sigma))_\mu\, Y^\nu(\Sigma) \qquad \forall X, Y \in \mathtt{Sol}$$

where Σ is an arbitrary point in $\mathbb{R}$.

1.3.2 Comparisons

The two procedures ~~yields~~ two different classical symplectic spaces: $(\mathscr{E}, \tau)$ and $(\mathtt{Sol}, \sigma)$. We have proved in section **??** (Theorem **??**) that the two vector spaces are isomorphic through the map Ξ realized with the causal propagator E. Furthermore, in this case ~~can~~ be proved that Ξ preserves the symplectic form.
Once again we mimic what ~~is~~ done for the ~~scalar field case~~ (ex **??**). Consider two compactly supported vector field $f, h \in \Gamma_0(E)$ along γ_0 and call $X = Ef$, $T = Eh$ the corresponding Jacobi field. In fact from ~~definition of τ follows~~:

$$
\begin{aligned}
\tau\big([f],[g]\big) = (f, Eh) = \int_{\mathbb{R}} f^\mu g_{\mu\nu} (Eh)^\nu dt = \int_\Sigma^\infty f^\mu g_{\mu\nu} Y^\nu dt + \int_{-\infty}^\Sigma f^\mu g_{\mu\nu} Y^\nu dt \\
= \int_\Sigma^\infty (PG^- f)^\mu g_{\mu\nu} Y^\nu dt + \int_{-\infty}^\Sigma (PG^+ f)^\mu g_{\mu\nu} Y^\nu dt \qquad (1.18)
\end{aligned}
$$

where ~~has been decomposed the integral~~ by splitting the domain of integration into two subsets whose intersection has zero measure ~~and are~~ exploited the properties of the retarded and advanced operators.
Considering the explicit representation of ~~operator~~ P ~~allows us to~~ integrate by parts

twice:

$$\int_\Sigma^\infty (PG^- f)^\mu g_{\mu\nu} Y^\nu dt = \int_\Sigma^\infty \left(D_t^2 G^- f + R(G^- f, T)T\right)^\mu g_{\mu\nu} Y^\nu dt =$$
$$= \int_\Sigma^\infty D_t \left((D_t G^- f)^\mu g_{\mu\nu} Y^\nu\right) dt - \int_\Sigma^\infty (D_t G^- f)^\mu g_{\mu\nu} (D_t Y)^\nu dt + \int_\Sigma^\infty \left(R(G^- f, T)T\right)^\mu g_{\mu\nu} Y^\nu dt =$$
$$= - D_t (G^- f)^\mu g_{\mu\nu} Y^\nu \big|_\Sigma - \int_\Sigma^\infty D_t \left((G^- f)^\mu g_{\mu\nu} (D_t Y)^\nu\right) dt + \int_\Sigma^\infty \left((G^- f)^\mu g_{\mu\nu} (D_t^2 Y)^\nu + \left(R(G^- f, T)T\right)^\mu g_{\mu\nu} Y^\nu\right) dt =$$
$$= - D_t (G^- f)^\mu g_{\mu\nu} Y^\nu \big|_\Sigma + (G^- f)^\mu g_{\mu\nu} (D_t Y)^\nu \big|_\Sigma + \int_\Sigma^\infty \left((G^- f)^\mu g_{\mu\nu} (PY)^\nu\right) dt =$$
$$= - D_t (G^- f)^\mu g_{\mu\nu} Y^\nu \big|_\Sigma + (G^- f)^\mu g_{\mu\nu} (D_t Y)^\nu \big|_\Sigma \tag{1.19}$$

where the *Stokes theorem* and property 1.15 ~~are~~ been used.
Combining the two above equations one concludes that:

$$\begin{aligned} \tau\left([f],[g]\right) = & - D_t (G^- f)^\mu g_{\mu\nu} Y^\nu \big|_\Sigma + (G^- f)^\mu g_{\mu\nu} (D_t Y)^\nu \big|_\Sigma + \\ & + D_t (G^+ f)^\mu g_{\mu\nu} Y^\nu \big|_\Sigma - (G^+ f)^\mu g_{\mu\nu} (D_t Y)^\nu \big|_\Sigma = \\ = & \left(Ef\right)^\mu g_{\mu\nu} (D_t Y)^\nu \big|_\Sigma - \left(D_t E f\right)^\mu g_{\mu\nu} Y^\nu \big|_\Sigma = \\ = & X_\mu(\Sigma) \, (D_t Y(\Sigma))^\nu - (D_t X(\Sigma))_\mu \, Y^\nu(\Sigma) \equiv \sigma(X, Y) \end{aligned} \tag{1.20}$$

~~Then~~ $(\mathscr{E}, \tau)$ and $(\mathtt{Sol}, \sigma)$ are isomorphic not only as vector spaces but also as symplectic spaces.

1.4 Interpretations??????

⚠ Speriamo bene.. :S