Chapter 1

Mathematical Preliminaries

Le interazioni matematiche sono complesse e non triviali (vedi un po' di articoli di introduzione a AQFT per ispirarti)

Tendenzialmente le teorie quantistiche di campi moderne sono di Quantizzazione.. Quindi richiedono di specificare bene la struttura del campo classico (vedi intro di Mangiaratti shardashivly)

Gli strumenti matematici per raccontare la teoria dei campi classici sono essenzialmente 3: Fibrati, S-T G-H, LDOP e GHOP.

IN questo paper non ci soffermeremo sulle strutture del framework puramente quantistico (* algebre e quant'altro).

Diamo per scontato un background di base in Geometria differenziale e derivate esterne (algebre di Grassman? global calculus? non so come chiamarlo!)

Potrei avere la tentazione a provare ad usare un po' di linguaggio basilare delle categorie... la mia fonte ÃÍ Joy of Cat.

1.1 Fiber Bundles

1.1.1 ...

Inserire solo i punti salienti del primo capitolo.. Spostare ex primo capitolo spostato nel repository "dispensarium" come dispensa WIP

1.1.2 Some Topics useful in Physics

Jet Bundles

Tautological one-form and simplectic form.

1.2 Globally Hyperbolic Space-times

⚠ Mettere solo le definizioni che uso prese dagli articoli di review delle Fonti

(sono ripetizioni inutili per la tesi, sono informazioni che si ritrovano ovunque... sono informazioni adatta al knowledge base)

Recurring definitions in general Relativity (excluding the general smooth manifold prolegomena).

Definition 1: Space-Time

A quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ such that:

- (M, g) is a time-orientable n-dimensional manifold $(n > 2)$
- $\mathfrak{o}$ is a choice of orientation
- $\mathfrak{t}$ is a choice of time-orientation

Definition 2: Lorentzian Manifold

A pair (M, g) such that:

- M is a n-dimensional $(n \geq 2)$, Hausdorff, second countable, connected, orientable smooth manifold.
- g is a Lorentzian metric.

Definition 3: Metric

A function on the bundle product of TM with itself:

$$g : TM \times_M TM \to \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a non-degenerate bilinear form.

Notation fixing

- *Riemman* if the sign of g is positive definite, *Pseudo-Riemman* otherwise.
- *Lorentzian* if the signature is $(+, -, \ldots, -)$ or equivalently $(-, +, \ldots, +)$.

Observation 1: Causal Structure

If a smooth manifold is endowed with a Lorentzian manifold of signature $(+,-,\dots,-)$ then the tangent vectors at each point in the manifold can be classed into three different types.

Notation fixing

$\forall p \in M, \quad \forall X \in T_pM$, the vector is:

- *time-like* if $g(X,X) > 0$.
- *light-like* if $g(X,X) = 0$.
- *space-like* if $g(X,X) < 0$.

Observation 2: Local Time Orientability

$\forall p \in M$ the timelike tangent vectors in p can be divided into two equivalence classes taking

$$X \sim Y \text{ iff } g(X,Y) > 0 \qquad \forall X, Y \in T_p^{\text{time-like}} M:$$

We can (arbitrarily) call one of these equivalence classes "future-directed" and call the other "past-directed". Physically this designation of the two classes of future- and past-directed timelike vectors corresponds to a choice of an arrow of time at the point. The future- and past-directed designations can be extended to null vectors at a point by continuity.

Definition 4: Time-orientation

A global tangent vector field $\mathfrak{t} \in \Gamma^{\infty}(TM)$ over the Lorenzian manifold M such that:

- $\mathrm{supp}(\mathfrak{t}) = M$
- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Observation 3

The fixing of a time-orientation is equivalent to a consistent smooth choice of a local time-direction.

Definition 5: Time-Orientable Lorentzian Manifold

A Lorentzian Manifold (M, g) such that exist at least one time-orientation $\mathfrak{t} \in$

$\Gamma^\infty(TM)$.

Notation fixing

Consider a piece-wise smooth curve $\gamma : \mathbb{R} \supset I \to M$ is called:

- *time-like* (resp. light-like, space-like) iff $\dot\gamma(p)$ is time-like (resp. light-like, space-like) $\forall p \in M$.
- *causal* iff $\dot\gamma(p)$ is nowhere spacelike.
- *future directed* (resp. past directed) iff is causal and $\dot\gamma(p)$ is future (resp. past) directed $\forall p \in M$.

Definition 6: Chronological $\substack{\textbf{future}\\ \textbf{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{I}^\pm_M(p) := \left\{q \in M \middle| \, \exists \gamma \in C^\infty\big((0,1), M\big) \text{ time-like } \substack{\text{future}\\ \text{past}} - \text{directed} \, : \, \gamma(0) = p, \, \gamma(1) = q\right\}$$

Definition 7: Causal $\substack{\textbf{future}\\ \textbf{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{J}^\pm_M(p) := \left\{q \in M \middle| \, \exists \gamma \in C^\infty\big((0,1), M\big) \text{ causal } \substack{\text{future}\\ \text{past}} - \text{directed} \, : \, \gamma(0) = p, \, \gamma(1) = q\right\}$$

Notation fixing

Former concept can be naturally extended to subset $A \subset M$:

- $\mathbf{I}^\pm_M(A) = \bigcup_{p\in A} \mathbf{I}^\pm_M(p)$
- $\mathbf{J}^\pm_M(A) = \bigcup_{p\in A} \mathbf{J}^\pm_M(p)$

Definition 8: Achronal Set

Subset $\Sigma \subset M$ such that every inextensible timelike curve intersect Σ at most once.

Definition 9: $\substack{\textbf{future}\\ \textbf{past}}$ Domain of dependence of an Achronal set

The two subset related to the generic achornal set $\Sigma \subset M$:

$\mathbf{D}^{\pm}_{M}(\Sigma) := \left\{ q \in M \middle| \, \forall \gamma \, {}^{\text{past}}_{\text{future}} \text{inextensible causal curve passing through } q : \, \gamma(I) \cap \Sigma \neq \emptyset \right\}$

Notation fixing

$\mathbf{D}_M(\Sigma) :- \mathbf{D}^{+}_{M}(\Sigma) \cup \mathbf{D}^{-}_{M}(\Sigma)$ is called *total domain of dependence.*

Definition 10: Cauchy Surface

Is a subset $\Sigma \subset M$ such that:

- closed
- achronal
- $\mathbf{D}_M(\Sigma) \equiv M$

1.3 Green Hyperbolic Operators

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(sono ripetizioni inutili per la tesi, sono informazioni che si ritrovano ovunque... sono informazioni adatta al knowledge base) Basic Definition in L.P.D.O. on smooth vector sections.
Consider $F = F(M, \pi, V), F' = F'(M, \pi', V')$ two linear vector bundle over M with different typical fiber

Definition 11: Linear Partial Differential operator **(of order at most $s \in \mathbb{N}_0$)**

Linear map $L : \Gamma(F) \to \Gamma(F')$ such that:
$\forall p \in M$ exists:

- (U, ϕ) local chart on M.
- (U, χ) local trivialization of F
- (U, χ') local trivialization of F'

for which:

$$L\left(\sigma\big|_U\right) = \sum_{|\alpha| \leq s} A_\alpha \partial^\alpha \sigma \qquad \forall \sigma \in \Gamma(M)$$

Remark:

(multi-index notation)
A multi-index is a natural valued finite dimensional vector $\alpha = (\alpha_0, \dots, \alpha_n - 1) \in \mathbb{N}_0^n$ with $n < \infty$.
On $\mathbb{R}^n$ a general differential operator can be identified by a multi-index:

$$\partial^\alpha = \prod_{\mu=0}^{n-1} \partial_\mu^{\alpha_\mu}$$

(Until the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|\alpha| := \sum_{\mu=0}^{n-1} \alpha_\mu$$

?????????????????????

Hp:

Proposition 1.3.1 (Existence and uniqueness for the Cauchy Problem)

- $\mathbf{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ *a globally hyperbolic space-time.*
- $\Sigma \subset M$ *a spacelike cauchy surface with future-pointing unit normal vector field* $\vec{n}$.
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Th:

Observation 4

"Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. " [**?**] [**?**]
However the existence and uniqueness can be proved for the large class of the *Normally-Hyperbolic Operators.*