Interpretazione Geometrica delle parentesi di Peierls nella quantizzazione algebrica del campo Geodetico.

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Abstract

La prima parte della tesi ÃÍ stata rivolta allo studio del framework matematico necessario per dare una formulazione rigorosa dei sistemi classici continui, punto di partenza di ogni schema di quantizzazione algebrica. Nello specifico viene fatta una digressione sui Fibrati Topologici e viene sfruttata la definizione di fibrato liscio per presentare l'approccio geometrico alla meccanica classica sia per sistemi a gradi di libertÃă finiti che continui.

Nella seconda parte viene presentato l'algoritmo di Peierls che rappresenta una âĂIJricettaâĂİ efficace per attribuire una struttura pre-simplettica allo spazio delle configurazioni dinamiche di un sistema qualunque. Dalla ricerca bibliografica ÃÍ evidente come questo strumento a partire dal suo esordio (nel 1952) fino ad oggi non abbia mai ricevuto particolare attenzione. Questo sembra dovuto soprattutto alla mancanza di una convincente interpretazione geometrica.

Per fare un passo verso la comprensione di questo oggetto viene studiato lâĂŹestremamente noto problema della geodetica vedendolo come un sistema campo. Emerge sin da subito come il calcolo delle parentesi Peierls per questo sistema sia legato intrinsecamente al problema del calcolo dei campi di Jacobi lungo una geodetica.

Nella terza parte vengono descritte due realizzazione dello schema di quantizzazione algebrico per i campi bosonici. La prima sfrutta le parentesi di Peierls mentre la seconda interviene sui dati iniziali della dinamica di campo.

Il campo di Jacobi si presta ad essere quantizzato secondo entrambe le prescrizioni. Confrontando le 2 forme simplettiche cosÃň ottenute si cerca di fornire nuovi tasselli per attribuire un'interpretazione geometrica al metodo originale di Peierls.

Contents

Chapter 1

An excursus on Bundle

In this first chapter we will devote a bit of time to present the *Bundles*, a family of algebraic structures of particular importance in modern mathematical-physics.
We will follow a sort of deductive approach.

We start defining the abstract structure of *Fiber bundle* over the category of topological spaces, underlining that they represent the most natural setting for encoding the concept of physicist's *fields* and not forgetting that they form a concrete category per se.

In paragraph 2 we will enrich this abstract object with a so called *G-Structure*, a superstructure that must be necessarily identified if you want to have available a concept of compatibility between overlapping trivializations.

In third paragraph will be further specialize the construct on which is defined the bundle to not being simply a topological spaces but rather a smooth manifold (). [1]

This step provides the possibility to explore the relation between tangent spaces of the two manifolds which constitute the bundle, base and total space. The means for formalizing that will be the operation of *Lift* and *Drop*.

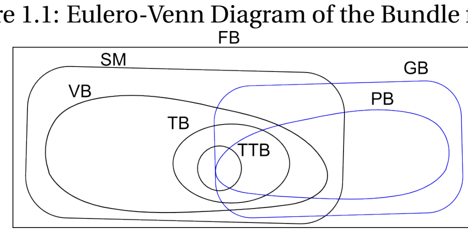
In paragraph 4 will arise for the first time a constraint on the fiber space, namely the prescription that is equipped with a linear space structure. In other words we will talk about *Vector Bundle*[2]. In this less general context we will deal with the problem of establish a bundle structure on a manifold having only a collection of omeomorphic fibers.

At last, in fifth paragraph, will be presented the *Tangent Bundle* the most significant example of smooth vector bundle.

[1] These spaces constitute a subcategory of topological spaces but actually what follows applies to every order of differentiability.

[2] In what follows we only consider *smooth* Vector Bundle.

Figure 1.1: Eulero-Venn Diagram of the Bundle family.



1.1 Fiber Bundle

Roughly speaking a *Fiber Bundle* is a way of attach some set, the so called *fibre*, on every point of another space, called *Base*. The main tool for achieving this "glueing" are the surjective function as we can guess from this observation

Observation 1

$\forall\ \pi : E \to M$ surjective function between generic set with $Dom(\pi) = E$

$$E = \bigsqcup_{p \in M} \pi^{-1}(p) = \bigsqcup_{p \in M} E_p$$

In this extremely simple case the sets $E_p = \pi^{-1}(p)$ take the role of fibers and M the base. In the next section we will see that the space E it's a crucial actor in the formal definition of a fiber bundle to a point that often this *total space* is mistaken with the bundle itself. [8]

1.1.1 Formal Definition

Remark:

In what follows all the set considered are endowed with a topological structure that is are topological space $(X, (top)(X))$.

Definition 1: Fiber Bundle

A *Fiber Bundle* consists in a 4-ple (E, B, π, F) where:

- E : topological space (called *Total Space*)
- B : topological space (called *Base Space*)
- F : topological space (called *Typical Fiber*)
- $\pi : E \to B$ continuous surjective function (called *Bundle Projection*)

Endowed with a *Local Trivialization*:

- $\forall x \in E\ \exists$ a couple (U, χ) (called *local trivialization*)
  - U : neighborhood of x
  - $\chi : \pi^{-1}(U) \to U \times F$: homeomorphism [a] [b]

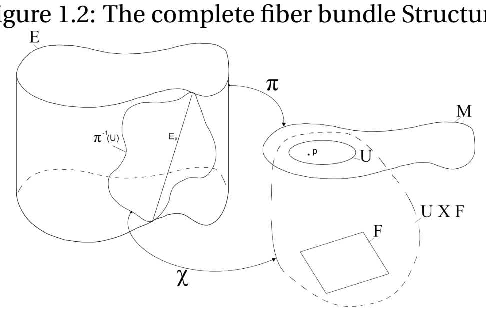
  such that: $p_1 \cdot \chi = \pi|_{\pi^{-1}(p)}$.

  i.e: the following graph commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\chi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "p_1"] \\
U &
\end{tikzcd}$$

[a] surjectivity $\Rightarrow \pi^{-1}(U) \neq \emptyset$.
[b] cartesian product of topological space is a topological space with the direct product topology.

Figure 1.2: The complete fiber bundle Structure.



As said in the introduction, in this aggregate of objects the role of fiber attached to each point of the base space is taken by the counterimage of π. This deserve a proper definition:

Definition 2: Fiber over a point $p \in B$

$E_p := \pi_{-1}(p)$

Ontologically [3] we distinct between "typical fiber" and " fiber over a point" but the axiom of local trivialization assures that topologically they are the same:

Lemma 1.1.1 *The typical fiber F and the fiber upon a point are homeomorphic.*

[3] i.e. element of one it's a different object respect the other.

Th: $F \simeq E_p \; \forall p \in B$

Proof:

For each $p \in B$ is given a local trivialization (U, χ) such that $p \in U$.

Noting that $\forall$ topological space $p \times A \simeq A$, follows from the definition this commutation diagram:

$$
\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\chi"] \arrow[d, "\pi"'] & p \times F \simeq F \arrow[dl, "p_1"] \\
p &
\end{tikzcd}
$$

In conclusion $\chi|_{E_p}$ realizes an homemorphism between F and E_p

□

Notation fixing

It's customary to refer to the fiber bundle (E, B, π, F) indicating only his total space E.

A possible, more heavy, convention is to denote the fiber bundle as a *short sequence*

$$F \to E \xrightarrow{\pi} B$$

1.1.2 Cross Section

The notion of bundle is particular interesting from the perspective of physicist because provides the rigorous description of a $F-$valued field on a space B.

Definition 3: (Cross) Section

Function $\phi : B \to E$ such that:

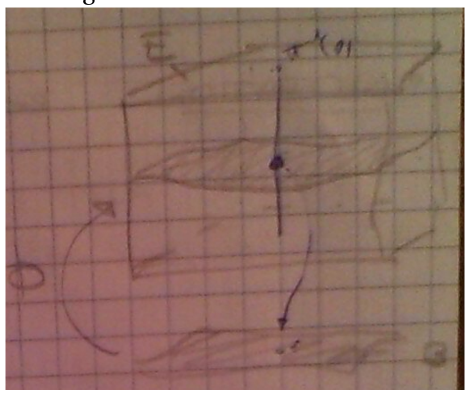
- ϕ continuous.
- $\phi \cdot \pi = \mathrm{Id}_B$

Notation fixing

We refer to:

- *Global section* $\Leftrightarrow \mathrm{dom}(\phi) = B$
- *Local section* $\Leftrightarrow \mathrm{dom}(\phi) \subset B$ [a]

Figure 1.3: Section on a Bundle.



[a]Usually the domain is an open set of B)

Observation 2

The property that essentially makes a section ϕ a good abstraction of a field is the following:

$$\forall p \in B \phi(p) \in \pi^{-1}$$

In other words:

Proposition 1.1.1 *Local section* $\{\phi\}$ *are in a 1:1 correspondence with continuous function* $\{f : B \to F\}$.

Proof:

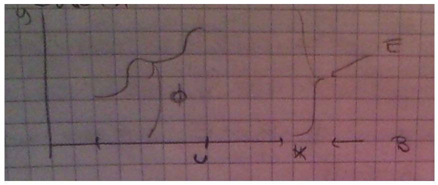
Take $p \in B$ and (U, χ) local trivialization over p.

Define $f : U \to F$ as $f = p_2 \cdot \chi \cdot \phi|_U$, where p_2 is a projection on the second element of a cartesian product space.

Then : $\chi \cdot \phi(p) = (p, f(p))$ (...).

$\square$

Observation 3

The preceding argument give meaning to the claim often presented in geometry books that: " cross section represent an abstract generalization to graph of functions." 

Notation fixing

The *set of all section* is often denoted as:

$$\Gamma(\pi_B)$$

1.1.3 Maps between Fiber Bundles

Consider two fiber bundle (F, E, π, B) and (F', E', π', B').

Definition 4: Bundle Morphism

A pair of map $(\phi_{tot}, \phi_{base})$ where:

- $\phi_{tot} : E \to E'$ continuous.
- $\phi_{base} : B \to B'$ continuous.

Such that

$$\pi' \cdot \phi_{tot} = \phi_{base} \cdot \pi \tag{1.1}$$

, i.e the following graph commutes:

$$\begin{tikzcd} E \arrow[r, "\phi_{tot}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ B \arrow[r, "\phi_{base}"] & B' \end{tikzcd}$$

Observation 4

Restricting the equation (1.1) to act only on a specific fiber,

$$\pi' \cdot \phi_{tot}|_{E_p} = \phi_{base} \cdot \pi(E_p) = \phi_{base}(p) := p'$$

we can see that precedent definition it's equivalent to requirement that ϕ_{tot} is fiber preserving:

$$\forall p \in B \qquad \phi_{tot}(E_p) = E_{\phi_{tot}(p)}$$

Follows that to determine a bundle morphism is sufficient to provide a fiber preserving map between the total spaces. It's then customary to denote a bundle morphism with ϕ_{tot} only.

Proposition 1.1.2 (Fiber Bundle as a Category.)

Hp:

- $\mathfrak{C}$ = *set of all possible fiber bundle.*
- $hom(\mathfrak{C})$ = *set of all bundle morphism.*

Th:

The couple (C, hom(C)) form a concrete category.

Proof:

Technicality (...).

$\square$

Observation 5

Note that the fiber projection of a fiber bundle is a continuous map, than an homomorphism of the topological space category (a particular one, which satisfies the axiom of local triviality).

Definition 5: Bundle isomorphism

A bundle morphism $(\phi_{tot}, \phi_{base})$ such that $\phi_.$ are homeomorphism.

Notation fixing

It's frequent to refer at the bundle morphism between fiber bundle over the same base ($\phi_{base} = Id_B$) as *Fiber Preserving map.*

$\phi : E \to F$ continuous such that:

$$\phi(E_x) = F_x \qquad \forall x \in M.$$

i.e.:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd}$$

Definition 6: Pull-Back Bundle

Consider a fiber bundle (E,π,M), a topological space N and a continuous function $\phi : N \to M$. Are defined:

$$f^*E = \{(b',e) \in B \times E | f(b') = \pi(e)\}$$

$\pi' : f^*E \to N$ such that $\pi'(b',e) = b'$

$$\begin{tikzcd} f^*E \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

Proposition 1.1.3 *$F \to f^*E \xrightarrow{\pi'} N$ is a fiber bundle of F typical fiber.*

Proof:

If we want to complete the fiber bundle structure we have to provide a local trivialization atlas. $\forall (U,\phi)$ local trivialization on (E,π,M) consider $\psi : f^*E \to N \times F$ such that $\psi(b',e) = \Big(b', p_2\big(\phi(e)\big)\Big)$.

Then $(f^{-1}(U),\psi)$ is a local trivialization of the pull-back bundle.

□

Observation 6

Consider this situation:

$$\begin{tikzcd} & E \arrow[d, "\pi"'] \\ N \arrow[r, "f"] & M \arrow[u, bend right, "s"'] \end{tikzcd}$$

where $s \in \Gamma(\pi_M)$. Pull-Back of Section is easily obtained as follow:

$$f^*s = s \cdot f \in \Gamma(f^*E)$$

1.2 Structure Group and transition Function

From what we have seen seems legit to consider the local trivialization of a fiber bundle as the analogous of a a local chart on a smooth manifold.

That make sense to the idea of fiber bundle (thought as its total space) as a space which is locally a product space link. (But globally may have a different structure, Not all bundle are trivial).

However in the definition of Bundle is required the existence of at least one trivialization chart for each point but no notion of *compatibility* is explicitly required.

1.2.1 The problem of *Overlapping Trivialization.*

Consider two local trivializations (with $i = \alpha, \beta$):

$$\chi_i : \pi^{-1}(U_i) \to U_i \times F$$

overlapping, that is $U_\alpha \cap U_\beta \neq \emptyset$.

Definition 7: Transition Function (from α to β)

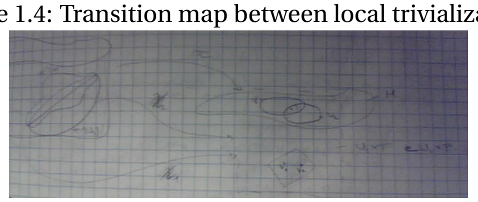
$$g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{aut}(F)$$

[a] given by

$$\chi_\beta \cdot \chi_\alpha^{-1}(p, V_\alpha) = \big(p, g_{\beta\alpha}[p](V_\alpha)\big) = \big(p, V_\beta\big) \qquad \forall p \in U_1 \cap U_2, \forall V_\alpha \in F$$

[a] In the category of topological spaces $\mathrm{aut}(F)$ consists of homeomorphism from F to itself.

Figure 1.4: Transition map between local trivialization.



Notation fixing

It's common to refer to the transition map as the well defined homeomorphism:

$$\chi_\beta \cdot \chi_\alpha^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

instead of the function $g_{\beta\alpha}$ which realizes the transformation.

In analogy with the atlas of chart on a manifold also the collection of all the local trivialization supplied to the Bundle structure takes a specific name:

Definition 8: Bundle (Trivialization) Atlas

Is a collection of local trivialization which cover the entire base space:

$$\{(U_\alpha, \chi_\alpha) \big| \bigcup_\alpha U_\alpha \supseteq M\}$$

Since for each pair of overlapping map is defined a transition function, every bundle atlas carries with itself a collection of such maps:

$$\{g_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{aut}(F) \big| U_\alpha \cap U_\beta \neq \emptyset\}$$

its cardinality is determined by the number of overlapping open set in the atlas.

Proposition 1.2.1 *The transition maps relating to a specific atlas always meet the following properties:*

$$g_{\alpha\alpha}(p) = \mathbb{1}_F \qquad \forall p \in U_\alpha \tag{1.2}$$

$$g_{\beta\alpha}(p) = g^{-1}_{\alpha\beta}(p) \qquad \forall p \in U_\alpha \cap U_\beta \tag{1.3}$$

$$g_{\beta\gamma}(p) g_{\gamma\alpha}(p) = g_{\beta\alpha}(p) \qquad \forall p \in U_\alpha \cap U_\beta \cap U_\gamma{}^a \tag{1.4}$$

[a] *cocycle condition*

Proof:

- (1.2) follows from the composition rule:

$$\chi_\alpha \chi_\alpha^{-1} = \mathbb{1}_{U_\alpha \times F}$$

- (1.3) follows from:

$$\left(\chi_\alpha \cdot \chi_\beta^{-1}\right)^{-1} = \chi_\beta \cdot \chi_\alpha^{-1}$$

- (1.4) follows from:

$$\left(p, g_{\beta\alpha}(p)V\right) = \left[\chi_\beta \cdot \chi_\alpha^{-1}\right](p, V) = \left[\chi_\beta \cdot \chi_\gamma^{-1}\right]\left[\chi_\gamma \cdot \chi_\alpha^{-1}\right](p, V) = \left(p, g_{\beta\gamma}(p) g_{\gamma\alpha}(p) V\right)$$

□

1.2.2 Structure Group

From the definition is clear that the transformation maps are valued in a group (the group of automorphism $\mathrm{aut}(F)$) but in general the set $\{g_{\alpha\beta}[p]\}$ for a fixed p don't form a subgroup [4].

[4] Or , equivalently, the map $\{g_{\alpha\beta}\}$ is not the action of some group.

Example:

Being a group would means that fixed four overlapping trivialization$\alpha, \beta, \gamma, \delta$ must exists another couple of trivialization θ, η such that:

$$g_{\alpha\beta} \cdot g_{\gamma\delta} = g_{\theta\eta}$$

obviusly there's no natural way of construct such composition from the cocycle condition only.

For this reason, the following definition arise spontaneously:

Definition 9: G-Atlas

It's a trivialization atlas $\{(U_i, \chi_i)\}$ such that the corresponding transition maps constitutes a group left-action of the abstact group G on the fiber space F.

Notation fixing

It's common to use the following names when referring to a G-structered fiber bundle:

- *G-Bundle* : fiber bundle rigged with a G-atlas of trivialization.
- *Structure group* : the abstract group G whose actions realize the transition maps.

The choose of such solemn name for the structure group is justified by the following theorem:

Theorem 1.2.1 *[8] Fixing a typical fiber F, a base space M and a G-action $g_{\alpha\beta}$ which map the transition function is sufficient to determine the G-Bundle (F, E, π, M, G).*

Hp:

1. *M, F topological spaces.*
2. *$\{U_\alpha\}_{\alpha \in I}$ open cover of M.*
3. *is given a family $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to aut(F)\}$ such that:*
   - $g_{\alpha\beta} : (p, f) \mapsto g_{\alpha\beta}(f)$ *is an homeomorphism.*
   - $g_{\alpha\alpha}(p) = \mathbb{1}_F \qquad \forall p \in U_\alpha$
   - $g_{\beta\alpha}(p) \cdot g_{\alpha\beta}(p) = \mathbb{1}_F \qquad \forall p \in U_\alpha \cap U_\beta$
   - $g_{\beta\gamma}(p) \cdot g_{\gamma\alpha}(p) \cdot g_{\alpha\beta}(p) = \mathbb{1}_F \qquad \forall p \in U_\alpha \cap U_\beta \cap U_\gamma$

Th:

1. *The quotient space* $E = \dfrac{\bigcup_{\alpha \in I}(U_\alpha \times F)}{\sim}$ *with :*

$$\left(p_\alpha, f\right) \sim \left(p_\beta, g_{\beta,\alpha}(f)\right) \qquad \forall\, p_\alpha = p_\beta \in U_\alpha \cap U_\beta \;\forall f \in F$$

endowed with the quotient topology is a topological space.

2. *The projections on the first argument* $p_1 : U_\alpha \times F \to U_\alpha$ *fitted together defines a good bundle projection* $\pi : E \to M$*, i.e.:*
   - π *injective*
   - $\forall\, p\; \pi^{-1}(p)$ *homemorphic to* F

Proof:

See theorem 1.4.1 for the demostration in a rather simpler case.

□

1.2.3 A glance on Principal Bundle

Imposing a further prescription on the properties of a G-structure on a G-Fiber Bundle we can identify a particular structure often used in mathematical physics.

Definition 10: Principal Bundle

Is a G-bundle such that the transition maps $t_{\alpha\beta}$ as an action of the group G is:

- *free* : $\forall g \in G \setminus \{\mathbb{1}\} \quad : \quad t_{[g]} \cdot s \neq s \quad \forall s \in F$
- *transitive* : $\forall x, y \in F, \exists g \in G \setminus \{\mathbb{1}\}$ such that : $t_{[g]}\, x = y$.

Observation 7

A such action permit a complete identification of F with the group G.

For this reason is common place in the literature to present this further structured bundles as fiber bundle where the typical fiber F is endowed with a Lie group structure and such that the local trivialization functions are Lie group isomorphism when restricted on a fiber.

1.2.4 Toward other type of bundle.

Roughly speaking a fiber bundle is an agglomerate of fiber space over a different space called base. Fiber and Base could have different structure, a sort of compatibility between structure is guaranteed by the properties of π and χ. Category theory provides the appropriate language to treat various bundle structure in a unified way.

Consider two construct $\mathbf{C}_1$, $\mathbf{C}_2$ subcategory of **Top**, concrete category of all topological spaces [5], such that:

$$\mathbf{Top} \supseteq \mathbf{C}_1 \subseteq \mathbf{C}_2$$

Definition 11: $(\mathbf{C}_1)-$**Bundle of** $(\mathbf{C}_2)-$**fiber**

It's a 4-ple (E, M, π, F) where:

- $E, M \in \mathrm{obj}(\mathbf{C}_1)$: (called *Total Space* and *Base Space*)
- $F \in \mathrm{obj}(\mathbf{C}_2)$: (called *Typical Fiber*)
- $\pi \in \mathrm{mor}(E, B) \subset \mathrm{Mor}(\mathbf{C}_1)$: (called *Bundle Projection*)

Such that:

- π surjective.
- $\pi^{-1}(p) \in \mathrm{obj}(\mathbf{C}_2) \quad \forall p \in M$
- $\forall p \in M\, \exists (\chi, U)$ (local trivialization) such that:
  - U is a neighbourhood of p
  - $\chi \in \mathrm{iso}(E, U \times F) \subset \mathrm{Iso}(\mathbf{C}_1)$
  - $\chi|_{\pi^{-1}(p)} \in \mathrm{iso}(\pi^{-1}(p), \{p\} \times F) \subset \mathrm{Iso}(\mathbf{C}_2)$

the main categories of interest are as follows:

Category	Obj	Mor	Iso
Top	topological spaces	continuous functions	homeomorphism
Smooth	smooth manifold	differentiable functions	diffeomorphism
GLie	Lie groups	homomorphism	group isomorphism
Vec	Vector spaces	linear operators	$\mathbb{GL}-$operators

[5] Set of the object must be considered together with a cartesian product operator $\times : \mathrm{obj} \times \mathrm{obj} \to \mathrm{obj}$.

1.3 Smooth Bundle

In the context of mathematical physics is more frequent referring to smooth fiber bundle instead of only topological ones. From 11 follows the following definition:

Definition 12: Smooth Fiber Bundle

Is a fiber bundle (F, E, π, M) such that:

- E, F, M are not only topological but smooth manifold.
- π is a smooth surjective function.
- χ_α is a diffeomorphism $\forall \alpha$.

Since all differentiable manifolds are, in first instance, topological spaces all the statement above remain valid with the exception of consider all function *differentiable* instead of *continuous* only. (e.g. in this framework the section are also differentiable, some texts use the symbol $\Gamma^\infty(\pi_M)$ to stress this fact.)
The few more peculiarity in considering this additional smooth structure on the spaces constituting the bundles essentially come from the presence of the local charts and the tangent spaces.

1.3.1 Relation between local charts and local trivializations.

When a smooth fiber bundle (F, E, π, M) is considered, in addition to the typical functions of the bundle (π, χ_α) are to be taken in account all the collection of local chart for the three manifold : $(U_{\alpha_k}, \phi_{\alpha_k})_{k=E,M,F}$. The context require to not confuse the chart with the trivialization even if there is a relationship between them:

Proposition 1.3.1 *Atlas on M and F induce an atlas on E through the local trivialization.*

Proof:

Consider (U, ϕ_M) and (V, ϕ_F) local charts on M and F respectively.
Every local trivialization (U_α, χ_α) such that $U_\alpha \supseteq U$ is a diffeomorphism, therefore $\chi^{-1} : (U \times V) \mapsto W \in \mathcal{T}(E)$ maps open set in open set, thus

$$\left(\chi^{-1}(U \times V), (\phi_M \times \phi_F) \cdot \chi\right)$$

is a local chart on the manifold E.
Since such local trivialization exist for all point in M with this process is possible to map each fiber and then consitute a whole atlas on E.

$\square$

Proposition 1.3.2 (vice versa) *An atlas on E induce an atlas on M and F through the local trivialization.*

Proof:

Consider (W, ϕ_E) local chart on E and a local trivialization (U_α, χ_α) on M.
Take an open set $U' \subset U_\alpha$ in M , π is continuous then $W' = W \cap \pi_{-1}(U')$ is an open set in E. Moreover $V' = p_2 \cdot \chi_\alpha(W')$ is an open set in F.
In conclusion $\phi_E \cdot \chi_\alpha^{-1}$ constitute a chart on $U' \times V'$ and, by projection on components of the cartesian product, on the manifold M and F.

□

Furthermore could be useful defining a patch on M which map the base spaces and trivializes the bundle in the same time:

Definition 13: Local chart (of M) trivializing (E)

Triple (U, ϕ, χ) such that:

- U open set in M.
- $\phi : U \to \mathbb{R}^{\dim(M)}$ diffeomorphism.
- $\chi : \pi^{-1}(U) \to U \times F$ trivialization.

Observation 8

At this point we can see a source of confusion that comes from the identification of the whole fiber bundle (F, E, π, M) with the total space E only:

Bundle Atlas ≠ Atlas of charts on the manifold E

That suggests to aggregate the two concepts in an unique definition:

Definition 14: Trivializing Atlas of charts

Collection of local charts of M which trivilizes E such that:

$$\{(U_\alpha, \phi_\alpha, \chi_\alpha | \bigcup_\alpha U_\alpha \supseteq M\}$$

Notation fixing

Is customary to consider such atlas of trivializing charts as the proper *bundle atlas* of a smooth bundle.

1.3.2 Lifting objects from the base space to the complete space

There're basically two idea under the concept of *lift* and *drop* in a smooth fiber bundle.

- E and M are smooth manifold, then it's perfectly legit to consider the tangent spaces on both of them.
- π is a smooth map, then are well defined the notion of pull-back and push-forward (through the differential $\mathrm{d}\pi$.

Drop and *Lift* are only two different name, introduced for this context, for the mapping trough the differential of the projection function.
Consider a parametrized curve $\gamma : \mathbb{R} \to E$ on the total space:

Definition 15: Drop of curves

Parametrized curve $\gamma^D : \mathbb{R} \to E$, such that:

$$\gamma^D = \pi \cdot \gamma$$

Regarding the tangent vectors as velocity vectors of equivalence classes of curves follows easly the next definition:

Definition 16: Drop of vectors

$$\forall v \in T_{e_p}E \quad v^D := \mathrm{d}\pi \cdot v = V_* \in T_pM$$

where e_p is a point of E in the fiber over p.

Definition 17: Lift of 1-forms

$$\forall \alpha \in T_p^*M \quad \alpha^L := \alpha^* \in T_{e_p}^*E$$

where e_p is a point of E in the fiber over p.

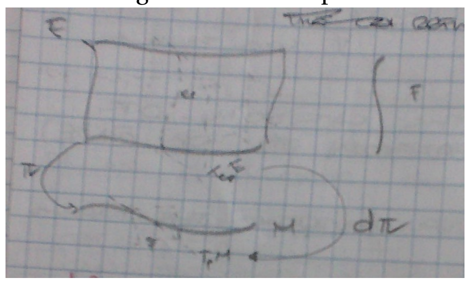
Observation 9

The former operation are naturally implemented by the presence of the special smooth function π, on the contrary their inverse are not natural (π is not invertible) and require some additional structure like the choose of a cross-section.

1.3.3 Decomposition in vertical and horizontal tangent space.

On the total space of a bundle is naturally identified a special class of curves:

Figure 1.5: Lift Drop1.

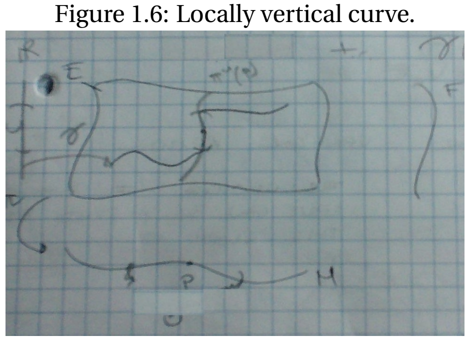


Definition 18: locally vertical curves

$$\gamma : \mathbb{R} \to E \text{ such that: } \exists U \subseteq \mathbb{R} : \pi(\gamma)|_U = p$$

i.e. are curves of which at least a portion of them lies entirely on a fiber.

Figure 1.6: Locally vertical curve.



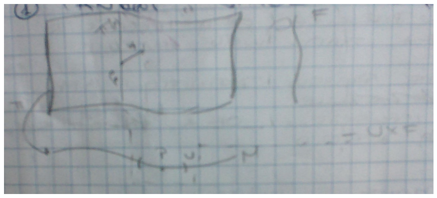
Follows the concept of vertical vectors:

Definition 19: Vertical Vector

$v \in T_e E$ is vertical if $(\mathrm{d}\pi)(v) = 0$

Observation 10

The drop of a vertical curve can be seen as the motion of a particle which remain still in p for an interval U of time in his parameter space.

 Take M, N manifold and $\phi : M \to N$ smooth.
Be $d\phi_p(\dot{\gamma}) = 0 \; \forall p \in \gamma(U)$.
Then $\gamma' = \phi \cdot \gamma$ is the trajectory of a point which remain still for $t \in U \subset \mathbb{R}$.

Definition 20: Vertical Tangent SubSpace

$$V_e E = \ker(d\pi) \subset T_e E$$

Observation 11

$V_e E$ coincides with the tangent space to the submanifold $\pi^{-1}(p) \subset E$ in the point e_p.

Definition 21: horizontal Tangent SubSpace

Complementary subspace[a] $H_e E \subset T_e E$. i.e. such that $T_e E = H_e E \oplus V_e E$

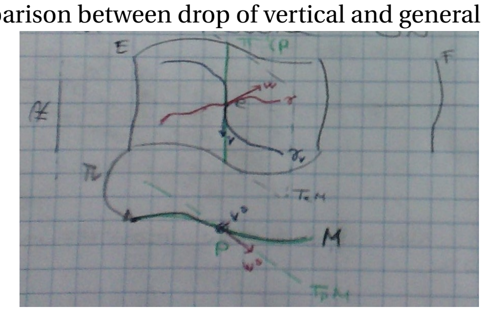
[a] one out of many.

Observation 12

Where the vertical subspace is univocally determined by π his complementary , the horizontal subspace, is not unique in general.

The choice of such names can be argued from figure 1.7.

Figure 1.7: Comparison between drop of vertical and general curve (or vectors).



Observation 13: First take on the concept of Fiber Connection

We have just seen that the concept of *vertical component* of a tangent vector is coupled with the drop of vectors and is univocally determined by the fiber projection π present on the bundle.
This is not true for the opposite concept of *horizontal component*. In general there is not a natural way of selecting a fixed complementary space but additional structure is needed (e.g. a condition of orthogonality provided by a riemmanian metric).
The specification of an horizontal subspace for every point in E is an additional structure called *Fiber Bundle Connection*

1.4 Vector Bundle

Specializing further the smooth fiber bundle imposing the linear space structure leads us to define the *vector bundle*.

Definition 22: Vector Bundle

Is a smooth fiber bundle (V, E, π, M) such that:

- V, typical fiber space, is a vector space.
- All the trivialization χ_α are diffeomorphism such that:

$$\chi_\alpha|_{\pi^{-1}(p)} \in \mathbb{GL}(n, \mathbb{R})$$

Observation 14

It's frequent in literature to present the vector bundle as a smooth bundle with typical fiber $\mathbb{R}^n$.
If we just consider finite dimensional fiber vector space the difference is totally irrelevant in virtue of the well known natural[a] isomorphism $V \simeq \mathbb{R}^n$ of vector decomposition in components on a base.

[a] In the sense that is not dependent by the chosen basis

To encompass this two slightly different point of view we make a little revision of the definition of *trivialization* in the context of vector bundle:

Definition 23: Local chart (of M) trivializing (E)

Triple (U, ϕ, χ)[a] such that:

- U open set in M.
- $\phi : U \to \mathbb{R}^{\dim(M)}$ diffeomorphism.
- $\chi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ trivialization chart.

[a] It's a standard trivializing chart with the extra feature of defining implicitly a decomposition of V on a basis.

Observation 15

If we consider a whole atlas of such chart will follows that the transition maps will be $\mathbb{GL}(n, \mathbb{R})$ valued, in other words the $g_{\alpha\beta}$ will be change of basis matrix.

1.4.1 Construction of a Vector Bundle.

The next theorem represent a criteria to establish when a collection of isomorphic vector spaces constitutes a vector bundles.

Theorem 1.4.1 *Given an "almost"[a] vector bundle it's sufficient to provide a collection of transition functions to complete the structure.*

Hp:

1. *M = smooth manifold*
   E = simple set (not a manifold)
   $\pi : E \to M$ = surjective function (not smooth)

2. *Endowed with an "almost" open trivialization atlas:*
   $\mathscr{A} = \{(U_\alpha, \chi_\alpha)\}$ such that

   - *$\{U_\alpha\}$ it's an open cover of M.*
   - *$\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ bijective (not diffeomorphism) and $p_1 \cdot \chi_\alpha = \pi$.*

3. *Is provided a chart atlas (U_α, ϕ_α) on the precedent open cover together with all the* transition map *$g_{\alpha\beta}$, i.e:*

   $$\forall(\alpha, \beta) : U_\alpha \cap U_\beta \neq \emptyset \; \exists g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{GL}(n, \mathbb{R}) \text{ diffeomorphism}$$

   such that:

   $$\chi_\alpha \cdot \chi_\beta^{-1}(p, \vec{v}) = (p, g_{\alpha\beta}(p)\vec{v})$$

[a] Similar to the case presented in observation 1.

Th:

E admit an unique vector bundle structure in which χ_α are local trivialization.

The hypothesized structure lacks the following properties in order to form a vector bundle:

a) The fiber upon a point has to be isomorphic to the typical fiber, i.e. $E_p \simeq \mathbb{R}^n \quad \forall p$.

b) E has to be a smooth manifold.

c) χ has to be a diffeomorphism

d) π has to be differentiable.

Proof:

a) Using Hp.2 is possible to associates $\forall V \in E$ $\vec{V} \in \mathbb{R}^n$ biunivocally:

$$\chi_\alpha|_{E_p} : V_p \in E_p \leftrightarrow \vec{V} \in \{p\} \times \mathbb{R}^n \simeq \mathbb{R}^n$$

Then endow E_p with a natural vector bundle structure:

$$u_1 + \lambda u_2 = \chi_\alpha^{-1}\big(p, \vec{u}_1 + \lambda \vec{u}_2 \qquad \forall u_1, u_2 \in E_p \; \forall \lambda \in \mathbb{R}$$

In other words all local trivialization containing p induce a vector space

structure, this is is well defined if the linear composition defined through χ_α is the same as the structure defined through χ_β $\forall p \in U_\alpha \cap U_\beta$.
Take $u \in E_p$ and define $\vec{v}, \vec{u}$ such that $\chi_\alpha(u) = \big(p, \vec{v}\big)$ and $\chi_\beta(u) = \big(p, \vec{w}\big)$.
By Hp.3 $(p, \vec{V_\alpha}) = \chi_\alpha \circ \chi_\beta^{-1}(p, \vec{V_\beta}) = (p, [g_{\alpha\beta}](p)\vec{V_\beta})$ So the good definition is assured by:

$$(u_1 + \lambda u_2)^{(\alpha)} = \chi_\alpha^{-1}(p, \vec{v}_1 + \lambda \vec{v}_2 = \chi_\alpha^{-1}(p, g_{\alpha\beta}\vec{w}_1 + \lambda g_{\alpha\beta}\vec{w}_2 = \tag{1.5}$$

$$= \chi_\alpha^{-1}(p, g_{\alpha\beta}(\vec{w}_1 + \lambda \vec{w}_2 = \cancel{\chi_\alpha^{-1} \circ \chi_\alpha} \circ \chi_\beta^{-1}(p, \vec{w}_1 + \lambda \vec{w}_2 = (u_1 + \lambda u_2)^{(\beta)} \tag{1.6}$$

b) It's possible to endow E with an atlas of compatible charts.
$\{\pi^{-1}(A) | A \in \mathrm{top}(M)\}$ constitutes a topology on E.
Surjectivity of $pi \Rightarrow \{\pi^{-1}(U_\alpha) = \tilde{U_\alpha}\}$ it's an open cover of E.
$\tilde{\chi_\alpha} : \pi^{-1}(U_\alpha) \to \mathbb{R}^{\dim(M)} \times \mathbb{R}^n$ such that $tilde\chi_\alpha = (\phi_\alpha \times \mathbb{1}) \circ \chi_\alpha$ consitutes a chart on E $\forall \phi_\alpha$ chart on M.
Transition chart are smooth because composition of two smooth function:

$$\tilde{\chi_\alpha} \circ \tilde{\chi_\beta^{-1}} = (\phi_\alpha \circ \phi_\beta^{-1}, g_{\alpha\beta}) = (\phi_\alpha \circ \phi_\beta^{-1}) \times (g_{\alpha\beta})$$

Then $\tilde{\mathscr{A}} = \{(\tilde{U_\alpha}, \tilde{\chi_\alpha})\}$ constitutes an atlas of $C^\infty-$compatible charts.

c) $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ it's smooth if $(\phi_\alpha \times \mathbb{1}) \circ \chi_\alpha \circ \tilde{\chi_\alpha}^{-1} : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ is also smooth , that's guaranteed by its definition:

$$(\phi_\alpha \times \mathbb{1}) \circ \chi_\alpha \circ \tilde{\chi_\alpha}^{-1} = (\phi_\alpha \times \mathbb{1}) \circ \chi_\alpha \circ \big((\phi_\alpha \times \mathbb{1}) \circ \chi_\alpha\big)^{-1} = \mathbb{1}$$

d) $\pi : \pi^{-1}(U_\alpha) \to U_\alpha$ is smooth if $\phi_\alpha \circ \pi \circ \tilde{\chi_\alpha}^{-1}$ is also smooth, that's guaranteed by Hp.2:

$$\phi_\alpha \circ \pi \circ \tilde{\chi_\alpha}^{-1} = \phi_\alpha \circ \pi \circ \chi_\alpha^{-1} \circ (\phi_\alpha \times \mathbb{1})^{-1} = \phi_\alpha \circ p_1 \circ (\phi_\alpha^{-1} \times \mathbb{1})\mathbb{1}$$

□

Theorem 1.4.2 *It's possible to reconstruct a vector bundle only from the transition maps.*

Hp:

1. *Be M a smooth manifold and $\mathscr{A} = \{(U_\alpha, \phi_\alpha\}$ atlas of local charts.*
2. *$\forall$ couple $U_\alpha, U_\beta \in \mathscr{A}$ is given a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{GL}(n, \mathbb{R})$ such that:*
   (a) $g_{\alpha\alpha}(p) = \mathbb{1}_F \qquad \forall p \in U_\alpha$
   (b) $g_{\beta\alpha}(p) = g^{-1}_{\alpha\beta}(p) \qquad \forall p \in U_\alpha \cap U_\beta$
   (c) $g_{\beta\gamma}(p) g_{\gamma\alpha}(p) = g_{\beta\alpha}(p) \qquad \forall p \in U_\alpha \cap U_\beta \cap U_\gamma$

Th:

1. *It's defined a vector bundle on base space M with $g_{\alpha\beta}$ transition maps.*
2. *Such bundle is unique up to isomorphisms.*

Proof:

See for example Abate[1], page 137.

□

We can state the content of previous demostration as follow:

Hp:

Corollary 1.4.1 *Provided the hypothesis of theorem **??**.*

Th:

1. $E = \dfrac{\bigsqcup_{\alpha \in \mathscr{A}} U_\alpha \times V}{\sim}$
   with $(x, v) \sim (y, w) \Leftrightarrow (x = y) \wedge w = g_{\beta\alpha}(x) v$
   where $x \in U_\alpha$; $y \in U_\beta$; $v, w \in V$
   consitutes a smooth manifold.
2. *Taken* $\pi : E \to M$ *such that* $\pi(x, v) = x$
   then (V, E, π, M) *consitute a vector bundle.*

1.4.2 Vector fields and References.

There are few more feature than the abstract section:

1. $\Gamma(\pi_M)$ of a vector bundle inherit the linear properties from F defining sum and product by a scalar pointwise:

$$\forall s_i \in \Gamma(\pi_M) \qquad \begin{array}{l} \big(s_1 + s_2\big)(p) = \big(s_1\big)(p) + \big(s_2\big)(p) \\ \big(\lambda s_1\big)(p) = \big(\lambda s_1(p)\big) \end{array}$$

2. There's a special cross-section called *null section*:

$$O_E \in \Gamma(\pi_M) \quad \text{such that:} \quad O_E(p) = 0\big|_{E_p} \forall p \in M$$

3. It's possible to extend the concept of basis from F to $\Gamma(M)$.

Local reference.

Consider a vector bundle (F, E, π, E) of finite dimension $dim(F) = f < \infty$

Definition 24: Local reference

r-ple $\{\sigma_1, \ldots, \sigma_r\}$ of sections $\sigma_i \in \Gamma(U)$ on $U \subset E$ open set, such that $\{\sigma_1(p), \ldots, \sigma_r(p)\}$ constitues a basis in $E_p \forall p \in U$.

Proposition 1.4.1 *Giving a local reference is equivalent to give a bundle atlas*

Proof:

$\Leftarrow$ Giving a reference through a local trivialization is rather simple.
Chosen a basis $\{e_j\}$ in F,

$$\Gamma(U) \ni \sigma_j(p) = \chi^{-1}(p, e_j)$$

is a local reference.

$\Rightarrow$ Vice versa $\forall$ local reference $\{\sigma_1, \ldots, \sigma_r\}$ on U we can define:

$$\xi : U \times \mathbb{R}^r \to \pi^{-1}(U) \quad \text{such that: } \xi(p, \vec{w}) = w^i \sigma_i(p)$$

- ξ is bijective, follows from the definition of reference.
- xi is smooth from linearity in w^i variable and smoothness of section in p variable.
- follows from the definition that $\chi := \xi^{-1}$ trivializes E.
- Smoothness of χ follows from the following argument:
  Consider a second local trivialization $\tilde{\chi}$ on U and call $\{\tilde{\sigma_1}, \ldots, \tilde{\sigma_r}\}$ the associated local reference.
  $\forall e_p \in E$ call $\tilde{\chi}_0(e) = (c^1, \ldots, c^r)$, such that: $\tilde{\chi}(e_p) = (p, \tilde{\chi}_0(e)$ and where $c^i \sigma_i(p) = e_p$.
  Applying this decomposition to the set $\{\sigma_1, \ldots, \sigma_r\}$ i.e.:

  $$\tilde{\chi_0}(\sigma_j) = (a^1_j, \ldots, a^r_j)$$

  we obtain the matrix $A = a^i_j(p)$
  Follows from the smoothness of the section that $a^i_j(p)$ are smooth

function in p.
A is invertible because represent a change of basis and its inverse is a matrix $B = b^i_j$ with smooth elements also.
In conclusion χ is a composition of smooth functions:

$$\chi(e_p) = (\mathbb{1} \times B)(p, \tilde{\chi}_0(v)) = (\mathbb{1} \times B)\tilde{\chi}(e_p)$$

$\square$

Observation 16

There's a relation between transition function and change of references between overlapping trivialization.
Given two overlapping trivialization (χ_i, U_i), be $\{\sigma_{1,i}, \ldots, \sigma_{r,i}\}$ the associated local reference, with $i = \alpha, \beta$.
The change of basis matrix

$$\sigma_{j,\beta} = \sum_k (g_{\beta\alpha})^k_j \sigma_{k,\alpha}$$

are exactly the transition map:

$$\chi_\alpha \circ \chi_\beta^{-1}(p, e_{j,\beta}) = \chi_\alpha(\sigma_{j,\beta}) = \chi_\alpha\Big(\sum_k (g_{\beta\alpha})^k_j \sigma_{k,\alpha}\Big) = (p, \sum_k (g_{\beta\alpha})^k_j e_{k,\alpha})$$

In general $\forall \sigma \in \Gamma(U)$

$$\sigma = \sum_j a^j_\alpha \sigma_{j,\alpha} = \sum_k a^k_\beta \sigma_{k,\beta} \quad \text{with: } a^j_\alpha = \sum_h (g_{\alpha\beta})^j_h \, a^h_\beta$$

1.4.3 Tensor Vector Bundle.

Consider two fiber bundle (F_1, E_1, π_1, M_1) and (F_2, E_2, π_2, M_2)

Definition 25: Fiber Product of Fiber Bundle

$$(F_1, E_1, \pi_1, M_1) \times (F_2, E_2, \pi_2, M_2) = (F, E_1 \times_M E_2, \pi, M)$$

where:

$$E_1 \times_M E_2 = \big\{ f = (e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2) \big\} \qquad \textit{fiber product set}$$

$$\pi(f) = \pi_1(e_1) = \pi_2(e_2)$$

Theorem 1.4.3 *Fiber product of 2 bundle is a fiber bundle of typical fiber $F_1 \times F_2$.*

Hp: *Consider a fiber bundle product as definition (25) .*

Th:

1. *$E_1 \times_M E_2$ is a submanifold of $E_1 \times E_2$.*
2. *π is a smooth bijection*
3. *From every couple of local trivialization one on E_1 and another on E_2 exists a trivialization on $E_1 \times_M E_2$ of typical fiber $F = F_1 \times F_2$.*

Proof:

1) : see abate p187
2) follows from the differentiability of π_1:

$$\pi(p_1, p_2) = \pi_1(p_1) = \pi_2(p_2)$$

3) we have to show how to construct a trivialization on the bundle product starting by a trivialization on each factor.
Consider two bundle atlas $\{(U_\alpha, \chi^j_\alpha)\}$ on E^j (where $j = 1, 2$) defined on the same open cover of M.
Define $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times (F_1 \times F_2)$ such that:

$$\chi_\alpha(x_1, x_2) = \Big(\pi_1(x_1), \big(p_2 \cdot \chi^1_\alpha(x_1), p_2 \cdot \chi^2_\alpha(x_2)\big)\Big)$$

χ_α are diffeomorphism with:

$$(\chi_\alpha)^{-1}(p, s_1, s_2) = \big((\chi^1_\alpha)^{-1}(p, s1), (\chi^2_\alpha)^{-1}(p, s_2)\big)$$

$\{(U_\alpha, \chi_\alpha)\}$ is a bundle atlas on $E_1 \times_M E_2$. A way to show that is to exhibit that inherit the good properties from the transition map of the spaces product:

$$\chi_\alpha \chi^{-1}_\beta(p, s_1, s_2) = (p, g^1_{\alpha\beta}(p)(s_1), g^2_{\alpha\beta}(p)(s_2))$$

□

What said can be encoded in the following definition:

Definition 26: Cartesian Product Bundle of (E_1, π_1, M) and (E_2, π_2, M)

Fiber bundle $(F = F_1 \times F_2,\ E = E_1 \times_M E_2,\ \pi,\ M)$ where:

$$E_1 \times_M E_2 = \{f = (e_1, e_2) \in E_1 \times E_2 \mid \pi_1(e_1) = \pi_2(e_2)\}$$

$$\pi : E \longrightarrow M || \pi(f) = \pi_1(e_1) = \pi_2(e_2)$$

Endowed with a product bundle chart $(\phi_\alpha \times \psi_\beta, U_\alpha \cap U_\beta)$, where (ϕ_α, U_α) local trivialization of E_1, (ψ_β, U_β) local trivialization of E_2 and :

$$\big(\phi_\alpha \times \psi_\beta\big)\big(x, (v, w)\big) := \big(\phi_\alpha(x, v), \psi_\beta(x, w)\big) \qquad \forall\, v \in \pi_1^{-1}(x),\ w \in \pi_2^{-1}(x)$$

From the notion of product bundle can be derived the notion of *direct sum* and *tensor product of bundle*:

Observation 17

Obviuosly the definition of × for set applies to vector spaces. Endowing that set with specified $\cdot, +$ linear operation we get the so called *direct sum* and *tensor product* spaces.

$$F_1 \oplus F_2 = \begin{pmatrix} F_1 \times F_2 \qquad \text{with:} \\ \cdot : \lambda(v, w) = (\lambda v, \lambda w) \\ + : (v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \end{pmatrix}$$

$$F_1 \otimes F_2 = \frac{F_1 \times F_2}{\sim} = \begin{pmatrix} \text{span}(v \otimes w) \qquad \text{such that:} \\ \lambda(v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w) \\ v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w \\ v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2) \end{pmatrix}$$

Consider two vector bundle (F_1, E_1, π_1, M) and (F_2, E_2, π_2, M) on the same base space and an atlas $\mathscr{A} = (U_\alpha, \chi_\alpha)$ of chart which trivializes E_1 and E_2 with transition function $g_{\alpha\beta}, h_{\alpha\beta}$ respectively.

Definition 27: Direct sum of vector bundles.

The only (from theorem 1.4.2) vector bundle $\big((E_1 \oplus E_2, \pi, M\big)$, such that:

- $(E_1 \oplus E_2)_p = (E_1)_p \oplus (E_2)_p \qquad \forall p \in M$
- transition function respect $\mathscr{A}$ are $g_{\alpha\beta} \times h_{\alpha\beta}$.

Definition 28: Direct product of vector bundles.

The only (from theorem 1.4.2) vector bundle $\big((E_1 \otimes E_2, \pi, M\big)$, such that:

- $(E_1 \oplus E_2)_p = (E_1)_p \oplus (E_2)_p \qquad \forall p \in M$
- transition function respect $\mathscr{A}$ are $g_{\alpha\beta} \otimes h_{\alpha\beta}$ [a].

[a] In finite dimension we can identify $E_1 \otimes E_2 \simeq \mathbb{R}^{n_1 + n_2}$ and $g_{\alpha\beta} \otimes h_{\alpha\beta}$ as the kronecker matrix product

Another useful construction over a vector bundle is the *dual vector bundle*. Recalling that for all vector space V is defined the dual vector space V^* of all linear functional over V endowed with a suitable linear structure follows:

Definition 29: Dual vector bundles.

The only (from theorem 1.4.2) vector bundle $\big((E^*, \varpi, M\big)$, such that:

- $(E*)_p = \big((E)_p\big)^* \qquad \forall p \in M$
- transition function respect $\mathscr{A}$, are on the same open set, $g^*_{\alpha\beta} = \big(g^T_{\alpha\beta}\big)^{-1}$.

Observation 18

The transition function relation are derived from what follows:
Consider a linear operator $A : v \mapsto \tilde{v}$, that is $\tilde{v}^j = A^j_i = v^i$ in coordinate.
The dual of this linear operator $\tilde{A} : \eta \in V^* \mapsto \tilde{v}$ is defined by the following relation: $\tilde{\eta}_i \tilde{V}^i = \eta_j v^j$, that is:

$$\tilde{\eta_i} A^i_k = \eta_k$$

I.e. :

$$\tilde{\eta_i} = \eta_k [A^{-1}]^k_i = [A^{-1}]^T \eta^T$$

Observation 19

Extending pointwise the local properties from T_pM to $\Gamma(\pi_M)$ follows that:

- $\Gamma(\varpi_M) = \Gamma^*(\pi_M)$, i.e. sections of the dual vector bundle (E^*, ϖ, M) are linear functional on sections of (E, π, M).
- For all reference (σ_i) on $\pi : E \to M$ is defined the dual reference (η^j) such that $\eta^j(p)\big(\sigma_i(p)\big) = \delta^j_i$.

1.5 Tangent Bundle

The tangent bundle is a natural structure on every smooth manifold and is also the most important example of vector bundles.

As a set the tangent bundle is defined as the union of all tangent spaces:

Definition 30

$$TM := \bigsqcup_{p \in M} T_pM = \{(p,v) \mid p \in M, v \in T_pM\}$$

Corollary 1.5.1 *(TM, π, M) with $\pi : T_pM \mapsto p$ it's a vector bundle of typical fiber $\mathbb{R}^n$.*

Proof:

The thesis follows from 1.4.1 providing a surjective projection function $\pi : T_pM \mapsto p$ and a "*almost* bundle atlas" $\chi : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ imposing $\chi_\alpha\left(\sum_{j=1}^n V^j \frac{\partial}{\partial x_\alpha^j}\big|_p\right) = (p, V)$. From the definition element of TM are element of T_pM with p not fixed. Then:

- $T_pM \simeq \mathbb{R}^n$ choosing the natural basis of the chart atlas.
- Bijectivity of χ is granted by uniqueness of decomposition on a basis.
- $p_1 \cdot \chi = \pi$ follows directly from the definition of χ_α.
- $g_{\alpha\beta} = \frac{\partial x_\alpha}{\partial x_\beta}$ is a good transition map:

$$\chi_\alpha \cdot \chi_\beta^{-1}(p, V) = \chi_\alpha\left(\sum_{j=1}^n V^j \frac{\partial}{\partial x_\alpha^j}\Big|_p\right) = \chi_\alpha\left(\sum_{h=1}^n \left[\sum_{j=1}^n \frac{\partial x_\alpha^h}{\partial x_\beta^j}(p) V^j\right] \frac{\partial}{\partial x_\alpha^h}\Big|_p\right) = \left(p, \left[\frac{\partial x_\alpha}{\partial x_\beta}\right](p) V\right)$$

□

Take Away Message

Tangent bundle is the unique vector bundle of M such that:

- have for typical fiber $E_p = T_pM \simeq \mathbb{R}^n$
- transition maps between trivialization chart $(U_\alpha, \phi_\alpha, \chi_\alpha)$ is the jacobian matrix of the coordinate transition function $g_{\beta\alpha} = \frac{\partial x_\alpha}{\partial x_\beta}$.

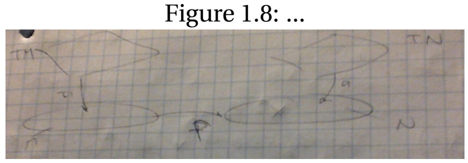
Observation 20

The tangent bundle is a vector bundle of rank n on a n dimensional manifold. Then TM is a $2n-$dimensional manifold.

1.5.1 Tangent Map.

Given 2 manifolds M, N and a differentiable function $F : M \to N$.

Figure 1.8: ...



Definition 31: Tangent Map

Is the map $\mathrm{T}f := \mathrm{d}f : TM \to TN$ such that:

$$\mathrm{T}f : V_p \in T_pM \mapsto \left[\left(f_*(p)\right)V\right] \in T_{f(p)}N$$

Observation 21

In differential geometry it's usual to give different name to objects that are essentially the same in order to emphasize some *"flavour"*.
In this case, we have:

- $\mathrm{d}f(p)$: " Differential of f" is the linear operator between $T_pM \to T_{f(p)}N$ for a fixed $p \in M$.
- $f_*(p)V$: " Push-Forward through f of a tangent vector V" is the image of $\mathrm{d}f(p)$ on $V \in T_pM$.
- $\mathrm{T}f$: " Tangent map of f" is the vector-bundle-morphism which act on every fiber like the differential operator.

1.5.2 Vector fields and natural references.

Notation fixing

The section of TM are called *vector fields* on the manifold M. The reason is straightforward:
Fixing a point in TM is equivalent to pick a point in M and a vector in $\mathbb{R}^n$, thus it represent a tangent vector of base point p.

Known that a vector field could be easily seen as a map $V : M \to TM$ satisfying the section condition $\pi \cdot V = \mathbb{1}_M$.

Notation fixing

The collection of all vector fields is a section spaces always carried on every differential manifold, it's often indicated with a particular notation:

$$\Gamma(TM) = \mathfrak{X}(M)$$

In coordiante chart we may read off the components of the vector. Consider a local chart (U, ϕ) over p such that $\phi = (x^1, \dots, x^n)$:

Definition 32: Natural Reference

Sections $(\partial_1, \dots, \partial_n) \subset \mathfrak{X}(M)$ of TM, such that:

$$\partial_j(p) = \frac{\partial}{\partial_j}\Big|_p \in T_pM$$

i.e. are the fields which associate to every point in M a natural tangent vector (tangent to the coordinate curve).

Observation 22

Provided an atlas on M is defined $\forall X \in \mathfrak{X}(M)$ the decomposition:

$$X = \sum_{j=1}^{n} a^j(p)\partial_j(p)$$

Where the component are:

$$a^j(p) = \mathrm{d}\phi_p\big(X(p)\big) \in C^\infty(U) \tag{1.7}$$

ans U is a neighborhood of p.

Observation 23

Obviously a change in local chart induce a transformation in local reference. Consider a second chart $(\tilde{U}, \tilde{\phi}$ with $U \cap \tilde{U} \neq \emptyset$. The change of natural basis on a fixed tangent space is easily extended on the whole natural reference:

$$\tilde{\partial_h} = \sum_{k=1}^{n} \frac{\partial x^k}{\partial \tilde{x}^H}\partial_k \tag{1.8}$$

From that follows $X = \sum_j a^j \partial_j = \sum_k \tilde{a^k} \tilde{\partial_k}$ i.e. the covariant change rule :

$$a^j = \sum_h \frac{\partial x^j}{\partial \tilde{x}^h} \tilde{a}^h$$

is extended to the section.

1.5.3 CoTangent Bundle

Notation fixing

Cotangent Bundle is a specific name for the specialization of 29 to the tangent spaces, i.e. is vector bundle on M with total space $T^* M = \bigsqcup_{p \in M} T_p^* M$ and usual projection π.

Observation 24

This vector bundle is unique, by theorem 1.4.1, providing the following transition functions for a change of trivialization chart.

Recalling that for all local chart $\varphi = (x^1, \dots, x^n)$ are defined:

- $\frac{\partial}{\partial x^h}\big|_p \in T_p M$ = natural basis vector in $T_p M \quad \forall p \in U$.[a]
- $dx^h\big|_p in T_p^* M$ = external derivative of the local chart calculated in p.

follows directly from definition of *external derivative* that:

$$dx_P^j\left(\frac{\partial}{\partial x^h}\big|_p\right) = \frac{\partial x^j}{\partial x^h}(p) = \delta_h^j$$

in other words $\{dx_p^h\}$ are the *natural dual basis* 1-forms.

Recalling also that taken two overlapping chart (U_α, ϕ_α) , (U_β, ϕ_β) on M, we have

$$dx_\beta^k\big|_p = \sum_h \frac{\partial x_\beta^k}{\partial c_\alpha^h} dx_\alpha^h\big|_p$$

i.e. dual coordinate are *covariant* [b].

âĂć Chosing a standard trivialization on the cotangent bundle:

$$\chi_\alpha\left(\sum_j w_j dx_\alpha^j\big|_p\right) = (p, w^T)$$

where $w^T \in \mathbb{R}^n$ is simply the transposition of row vector of 1-form components $(w_1, \dots, w_n)$.

âĂć Follows that:

$$\chi_\alpha \circ \chi_\beta^{-1}(p, w^T) = \chi_\alpha(w_j \mathrm{d}x_\beta^h|_p) = \chi_\alpha([w_j \frac{\partial x_\beta^j}{\partial x_\alpha^h}|_p \mathrm{d}x_\alpha^h|_p) = (p, [\frac{\partial x_\beta}{\partial x_a lpha}(p)]^T w^T)$$

In conclusion:

$$[g_{\alpha\beta}]^{\cdot}_{\cdot} = [\frac{\partial x^{\cdot}_\beta}{\partial^{\cdot}_\alpha}]^T \tag{1.9}$$

are the transition map for the dual bundle [c].

---

[a] Depending on which equivalent presentation of the tangent space is taken into account these can be seen as the tangent vector to the coordinate curve or as a partial derivative operator on $C^\infty(U)$.

[b] To compare to (1.8), the controvariant relation of the natural basis.

[c] To confront with the tangent case in which $[g_{\alpha\beta}]^{\cdot}_{\cdot} = \frac{\partial x^{\cdot}_\alpha}{\partial x^{\cdot}_\beta}$.

Notation fixing

The cross section of the cotangent bundle are called *1-forms* on M

Observation 25

Is possible to review the concept of *external derivative* in the language of tangent bundles and 1-forms [1].
$\forall f \in C^\infty(M)$ $\mathrm{d}f$ is the function $\mathrm{d}f : TM \to T\mathbb{R}$ such that:

$$\mathrm{d}f(V_p) = V(f)\big|_p \qquad \forall V \in \mathfrak{X}(M) \tag{1.10}$$

Observation 26

The dual natural reference is then provided by the external derivative of the local chart and they are the operator $\mathrm{d}x^a(p)$ that returns the component of a vector fields seen in equation (1.7) .

1.5.4 Tensor Bundle

As last effort we can combine all the precedent definition to introduce the tensor bundle:

Definition 33: $(k,l)-$**Tensor Bundle**

Is the unique vector bundle :

$$T^k_l M = \underbrace{T^*M \otimes \cdots \otimes T^*M}_{k times} \otimes \underbrace{TM \otimes \cdots \otimes TM}_{l times}$$

i.e. such that each fiber is in the form $E_p = T^k_l\big(T_p M)\big)$.

Observation 27

Uniqueness follows from definition of $\times$ for vector bundles.
Anyway the transition map for such bundle follows from the transformation of tensor components under change of local charts (see [1]).

Notation fixing

The section of $T^k_l(M)$ are called *tensor fields*.

1.6 Closing Thoughts

1.6.1 Prima stesura dell’introduzione

In questo primo capitolo ci concederemo un po’ di tempo per presentare i *fibrati*, una famiglia di strutture algebriche di particolare importanza per la fisica-matematica moderna.

L’approccio che seguiremo ÃÍ in un certo senso deduttivo.

Partiremo definendo la struttura, particolarmente astratta, di *Fiber Bundle* sopra la categoria degli spazi topologici; sottolineando come questo rappresenti il setting piÃź generale per rappresentare il concetto di campo nell’accezione originata dalla fisica ma senza tralasciare il fatto che la famiglia dei bundle costituisca una categoria concreta (costrutto) di per sÃÍ.

Nel paragrafo 2 arricchiremo questo oggetto astratto con una , cosÃň detta, *G-Structure*. Una soppalcatura che va necessariamente fissata se si vuole disporre di un concetto di compatibilitÃă tra trivializzazioni overlapping.

Nel terzo paragrafo si specializzerÃă la categoria su cui viene definito il fibrato a non essere semplicimente quella degli spazi topologici ma la sottocategoria delle varietÃă smooth [6]. Questo passaggio porta con se la possibilitÃă di esplorare il rapporto tra gli spazi tangenti delle due varietÃă(base e totale) che costituiscono il fibrato. Il mezzo per formalizzare questo rapporto saranno le operazioni di *Lift* e *Drop*, specializzazione a questo contesto delle note operazioni di *Pull-Back* e *Push-Forward* tra varietÃă.

Nel paragrafo 4 si porrÃă per la prima volta un vincolo sulla spazio fibra imponendo che esso sia dotato della struttura di spazio lineare. Si parlerÃă quindi di *Vector Bundle* [7] in questo contesto meno generale ci porremo il problema di dare di stabilire in quali condizioni ÃÍ possibile definire un fibrato su una varietÃă disponendo solo di una collezione di fibre omeomorfe.

Nel capitolo quinto verranno analizzati i *Tangent Bundle* la piÃź importante classe di fibrato vettoriale sulle varietÃă lisci.

(...)

Infine ci si chiederÃă come iterare la struttura di spazio tangente analizzando il rapporto tra i fibrati tangenti sulle due varietÃă costituente un generico bunle smooth.

1.6.2 Eliminata

Questioni di interesse personale che non ho aggiunto al capitolo per esigenze di tempo in quanto non strettamente legati agli argomenti della tesi oppure da spostare secondo consiglio di CD:

- In letteratura si vede spesso riferirsi alla G-struttura del fibrato come *Gauge Group*... PerchÃÍ?

[6] in realtÃă per quanto detto in questo capitolo dovrebbe essere valido per qualsiasi ordine di differenziabilitÃă. (per quanto riguarda hausdorff e second countability?)

[7] In what follows we only consider *smooth* Vector Bundle.

- La questione di chart vs trivialization ÃÍ una riflessione fatta da me superflua per la tesi.. qua ho scritto quasi tutto ma sui miei appunti cartacei ho messo qualche schemino in piÃź.. (quindi li ho scannerizzati e messi nel materiale della tesi!)

- Approccio generale(nel senso di non limitarsi alle varietÃă riemmaniane) alle connessioni con il linguaggio dei fribrati
  la formulazione precisa si fa molto efficacemente disponendo degli spazi doppio tangenti:
  specificare un unico lift
  trasporto parallelo
  curvatura

- Teoremi di costruzione dei vector bundle aggregando insieme un po' di spazi vettoriali. La fonte principale ÃÍ abate. Ma io ho riscritto tutto a pagine 9,10,11 degli appunti della tesi. Non sono superflui, servono essenzialmente per dimostrare che TM e T*M sono dei vector bundle!

- Si parla (per esempio sull'abate capitolo 3 ultimo paragrafo) dei fibrati principali dei riferimenti associati ad un fibrato vettoriale. Salto questo argomento perchÃÍ non serve per la tesi.

- ho visto il fibrato come una tripla ma... immaginiamo di avere una varietÃă M, cose vuol dire "dare un fibrato su di essa?" il secondo teorema del capito sui fibrati vettoriali risponde un po' a questa questione
  inoltre... che vuol dire fissare un punto sul total space ? vuol dire scegliere una coppia (p, v) previa la scelta di una trivializzazione.
  invece nei vettoriali equivale esattamente a dare la coppia (e, v), la scelta di una trivializzazione equivale invece ad una scelta di base in F tale di decomporre il vettore v in una n-pla.

- Abate per definite il $\otimes$ di bundle passa per la definizione di fiber product set (vedi abate e appunti pag 12) .. io ho preferito passare direttamente alla def per i bundle.

- riguardo alla tangent map il FOM appesantisce molto la notazione e i concetti... io sui miei fogli ho seguito un po' la sua strada ma mi sembra tutto superfluo... differenziale, push forward e tangent map sono in fondo la stessa cosa

- la parte sul doppio tangent bundle come setting per la connessione la salto (magari la metto come parte nel capitolo due dove c'ÃÍ la geometria riemmaniana o forse non lo metterÃš mai!).
  Si puÃš perÃš fare un accenno alla ricorsione del tangent bundle come fa qui: Wiki Tangent Bundle

- la questione di rividere il dual tangent space come una varietÃă simplettica ÃÍ da mettere nella parte sulla meccanica classica geometrica. in quanto, dice dappiaggi, ÃÍ solo in questo contesto che si usa questa proprietÃă!

- nel jurgen jost, capitolo sui fibrati, ci sono delle interessanti proposizioni sui fiber bundle su varietÃă riemmaniane, si afferma che in questo caso la G-Structure ÃÍ garantita
- ispirato da Fraenkel ho scritto sui miei appunti la dimostrazione che TM e' una varieta' differenziale. In realtÃă la cosa ÃÍ superflua se si sfrutta il teorema di ricostruzione del vector bundle.

1.6.3 Possibile Estensioni

Per capitoli

1. FB
2. GB
3. SB
4. VB
   - Considerazioni sul vector bundle.Che vuol dire in parole provere fissare un punto sul fibrato.
   - (da wiki VB) esempi: trivial bundle, moebius strip
   - (da wiki VB) accenno ai banach bundle
5. TB
   - (da freed) osservazione pag4 > $g_{\beta\alpha} = \mathrm{d}(x_\beta \circ x_\alpha^{-1})$
   - (da fraenkel) TM as set is a manifold.
   - (Abate pag 139) Tangent bundles as sub category, morhism = tangent map
   - (fraenkel e alt) T^*M as a phase space and natural simplectic structure (dapp dice di presentare i concetti meccanici in un altro capitolo)
   - paragrafo di confronto di mappe tra tangent bundle,
tangent map vs pull/push vs differential operator
fiber derivative
   - Tangent bundle over riemmanian manifold (pag 37-39 Jurgen Jost)
   - Vector bundle of p-form (pag 40-41 Jurgen Jost)
6. TTB
   - Presentazione del doppio tangente nello spirtito di wiki (wiki VB pag 4)
   - (Wiki VB) Vertical lift
   - (WIKI TTB)

Possibili fonti da considerare:

- Xavier Gracia, FIBRE DERIVATIVES: SOME APPLICATIONS TO SINGULAR LAGRANGIANS
- http://www.math.toronto.edu/selick/mat1345/notes.pdfLink
- Koszul, Lectures on fiber bundles, http://www.math.tifr.res.in/~publ/ln/tifr20.pdfLink
- jmf, Connections on principal fibre bundles http://empg.maths.ed.ac.uk/Activities/GT/Lect1.pdfLink
- http://personal.maths.surrey.ac.uk/st/T.Bridges/GEOMETRIC-PHASE/Connections_intro.pdfConnectionsIntro
- http://math.stanford.edu/~ralph/fiber.pdfTopology of Fiber Bundles
- https://www.ma.utexas.edu/users/dafr/M392C/Notes/FiberBundles.pdfFreed
- http://www.cjcaesar.ch/fribourg/docfrib/fibre_bundles/fibre_bundles.pdfmichael laurent
- www1.maths.leeds.ac.uk/~ahubery/Fibre-Bundles.pdfLeeds, fiber bundle
- Google in generale,, ovviamente c'e' tanto! https://www.google.it/search?q=FibreBndls.pdf&ie=utf-8&oe=utf-8&rls=org.mozilla:en-US:unofficial&client=iceweasel-a&channel=sb&gws_rd=cr&ei=uQ-tVM_ODMKtygOBvoGwDA#rls=org.mozilla:en-US:unofficial&channel=sb&q=fibre+bundlesgoog
- Connection on fiber bundles http://en.wikipedia.org/wiki/Connection_%28vector_bundle%29wiki

1.6.4 TODO

- Ricopiare dimostrazione 1.4.2
- rifare tutte le illustrazioni con inkscape.

1.6.5 Take away messages.

...

Chapter 2

Lagrangian Systems and Peierls Brackets

⚠.. introduzione dedichiamo sforzo alla definizione dei sistemi lagrangiani astratti per dare un punto di vista unificato ai sistemi a gradi libertÃă continui (continui macroscopici, fluidi, campi) e a gradi di libertÃă discreti Li vediamo come sotto classi dei sistemi lagrangiani notiamo che in entrambi i casi c'ÃÍ un possibile dato di cauchy usiamo questo ingrediente per definire l'algoritmo di peierls. ⚠

2.1 Abstract Mechanical Systems

It's possible to state a mathematical definition sufficiently broad to include all the systems in ordinary analytical mechanics regardless of the cardinality of degrees of freedom in a unified way.

Definition 34: Abstract Evolutive System

Pair (E, P) composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on manifold M called *"configuration bundle"*.
- $P : \Gamma^\infty(E) \to \Gamma^\infty(E)$
  operator called *"motion operator"*

This formultation is still very distant from the physical interpretation but has the benift to highlight the minimal mathematical objects which must be fixed in order to specify a mechanical systems.

Kinematics The configuration bundle encompass all the kinematical structure of the system, the pivotal role is played by the smooth sections which are to be understood as all the possible conformation of the system.

Notation fixing

$$\mathtt{C} := \Gamma^{\infty}(M, E)$$

Space of kinematic configurations.

A section is not a statical configuration, equivalent to a specific point in the configuration space of ordinary classical systems, but has to be seen as a specific realization of the kinematics in the sense of a complete description of a possible motion. At this level of abstraction, since no space-time structure has been specified, terms like stasis and motion must be taken with care .The natural physical interpretation should be clearly manifested through the concrete realization of systems with discrete and continuous degree of freedom.

Observation 28: Mathematical structure

Mathematically speaking this set should be regarded as an infinite dimensional Manifold.
This framework provides a geometric characterization of the notion of variations as tangent vectors on the the space of kinematic configurations .[7]

Observation 29: Coordinate Representation

The choice of a chart atlas $\mathscr{A}(M)$ on the base space M and $\mathscr{A}(E)$ on the total space E provides a correspondence between each configuration $\gamma \in \mathtt{C}$ and family of smooth real functions $\{f_{\alpha\beta} : A_\alpha \subset \mathbb{R}^m \to \mathbb{R}^q\}$. The process is trivial:

$$\gamma \in \mathtt{C} \mapsto \{f_{A,U} = \psi_U \circ \gamma \circ \psi_A^{-1} | (A, \psi_A) \in \mathscr{A}(M), (U, \psi_U) \in \mathscr{A}(E)\}$$

Since the whole section as a global object is quite difficult to handle is customary in field theory to work in the more practical local representation.

Observation 30: Further specification of the system's kinematics

The general formalism doesn't require any other structure to be carried forward. Additional structure on the fiber , the base or the whole bundle are to be prescribed in order to specify a precise physical model, e.g. the spin structure on E for the Dirac Field.[5]

Dynamics The operator P is the object that contains all the information about the dynamic evolution of the system. It has the role to select the dinamically compatible configuration among all the admissible kinematic configurations of $\mathtt{C}$, exactly as it happens in analytical mechanics where the dynamic equations shape the natural motions.

Notation fixing

Provided an equations of motion operator

$$P : \mathtt{C} \to \mathtt{C}$$

The space

$$\mathtt{Sol} := \ker(P) \subset \mathtt{C}$$

containing all the smooth solutions is called *"Space of Dynamical Configurations"*.

Figure 2.1: Geometric picture of the basic mechanical system's structure.
⚠immagine

2.1.1 Lagrangian Dynamics

Lagrangian systems constitute a subclass of the abstact mechanical systems of more practical interest:

Definition 35: Lagrangian System

Pair $(E, \mathscr{L})$ composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on the oriented manifold $(M, \mathfrak{o})$ called *"configuration bundle"*.

- $\mathscr{L} : J^r E \to \wedge^m T^* M$
  bundle-morphism from the r-th Jet Bundle to the top-dimensionial forms bundle over the base manifold M called *"Lagrangian density"* or simply *"Lagrangian"* of r-th order.

N.B. : In what follows all the systems considered will be exclusively of first order.

In this case is the Lagrangian density the object containing all the information about the dynamic evolution of the system.

In order to reconstruct the system's dynamic from the Lagrangian density has to be understood the mathematical nature of $\mathscr{L}$. $\mathscr{L}$ maps point q_p on the fiber $J^r_p E$ to a m-form on $T_p M$. Recalling the definition of jet bundles is clear that for each smooth section on E is associated a smooth section on the b$J^r E$:

$$\phi \in \Gamma^\infty(E) \mapsto (\phi, \partial_\mu \phi, \partial_{\mu,\nu}\phi, \ldots \partial_{\vec{\alpha}}\phi)$$

where $\vec{\alpha}$ is a multi-index of length r. The correspondence is not univocal since sections equal up to the r-th order define the same jet section. The smoothness of $\mathscr{L}$

ensure that each jet bundle section is mapped to a smooth section in the top-forms bundle i.e. the most general integrable object on a orientable manifold.

It should be clear that $\mathscr{L}$ is a specific choice among the vast class of functions suitable to be a good Lagrangian density over the Configuration Bundle E:

Definition 36: Lagrangian Density on the bundle E

$$\mathsf{Lag}^r(E) := \hom\Big(J^r E, \quad \bigwedge^m (T^* M)\Big) \cong \{f : \Gamma^\infty(J^r E) \to \Omega^m(M)\}$$

(where $\Omega^m(M)$ is the common name for $\Gamma^\infty\big(\bigwedge^m(T^*M)\big)$ in the context of Grassmann algebras.) The equivalence states the fact that a bundle-morphism induce a mapping between the sections.

this choice fix the "Dynamical identity" of the considered system.

Proposition 2.1.1 $\mathsf{Lag}^r(E)$ *has an obvious vector space structure inherited by the linear structure of* $\Omega(M)$.

Thanks to the correspondence between a section $\phi \in \mathtt{C}$ and his r-th jet, it's possible to consider the Lagrangian as directly acting on the kinematic configurations. In layman terms the image $\mathscr{L}[\phi]\mathrm{d}\mu$, where $\mathrm{d}\mu$ is the measure associated to the orientation $\mathfrak{o}$, is something that can be measured over the whole base space.
This property suggests the introduction of the class of associated functionals:

Definition 37: Lagrangian functional

Is a functional on $\mathtt{C}$ with values on regular distribution over M associated to the generic $\mathscr{L} \in \mathsf{Lag}$.

$$\mathscr{O}_{\mathscr{L}} : \mathtt{C} \to \big(C_0^\infty(M)\big)'$$

Such that the lagrangian functional associated to $\mathscr{L}$, valued on the configuration $\phi \in \mathtt{C}$ and tested on the test-function $f \in C_0^\infty(M)$ it's given by:

$$\mathscr{O}_{\mathscr{L}}[\phi](f) = \int_M \mathscr{L}[\phi] f \,\mathrm{d}\mu$$

Proposition 2.1.2 *As a distribution $\mathscr{O}_{\mathscr{L}}[\phi](f)$ is necessarily linear in the test-functions entry but not in the configurations entry.*

Observation 31

The choice of the image of $\mathscr{O}_{\mathscr{L}}$ as a distribution it's a necessary precaution to ensure that functional is *"convergent"* whatever is the configuration on which is evaluated. In fact, despite $\mathscr{L}[\phi]$ is integrable with respect to the measure

dμ, it's not necessary summable if the support of the configuration ϕ becomes arbitrarily large.
This is a simple consequence of the well known sequence of inclusions:

$$\mathscr{L}[\phi] \in C_0^{\infty}(M) \subset L^1_{\text{loc}}(M,\mu) \supsetneqq L^1(M,\mu)$$

of the functional analysis . Indeed, the functional

$$\mathscr{O}_{\mathscr{L}}[\phi] = \int_{\text{supp}(\phi)} \mathscr{L}[\phi]\mathrm{d}\mu$$

is well defined for all $\mathscr{L} \in \mathsf{Lag}^r(E)$ only over the compactly supported sections. To take account of the global sections it's sufficient to dampen the integral multiplying the integrand with an arbitrary test-function.

Notation fixing

When calculated for the specific density of the Lagrangian system $\mathscr{O}_{\mathscr{L}}$ takes the name of *Action* or *Total Lagrangian*.

The introduction of the Lagrangian density is meaningless without the prescription of a dynamical principle which allows to determine univocally a differential operator P on the kinematics configurations space $\mathtt{C}$. This fundamental principle is the *least action principle*. A proper justification of this claim should require the presentation of the differential calculus on the infinite dimensional manifolds $\mathtt{C}$. Jumping straight to the conclusion we can state this correspondence as a principle in term of a function which assign for all lagrangian densities an operator on the kinematic configurations space. In the case of first order lagrangian we define

Definition 38: Euler-Lagrange operator

It's the differential operator

$$Q_{\chi} : \mathtt{C} \to \mathtt{C}$$

relative to the lagrangian density $\chi \in \mathsf{Lag}^1(E)$, such that:

$$Q_{\chi}(\gamma) = \left(\partial_{\mu}\left(\frac{\partial \chi}{\partial(\partial_{\mu}\phi)}\Big|_{\gamma}\right) - \frac{\partial \chi}{\partial \phi}\Big|_{\gamma}\right) \qquad \forall \gamma \in \mathtt{C} \tag{2.1}$$

(where $\left(\frac{\partial \chi}{\partial(\partial_{\mu}\phi)}\right.$ is the be intended as the lagrangian density constructed differentiating $\chi(\phi, \partial_{\mu})$ as an ordinary function treating its functional entries as an usual scalar variable.)

Observation 32

Observation 33

The whole theory of both Lagrangian densities class and Euler-Lagrange equation could be stated in a more syntetic way in terms of the Grassmann-graded variational bicomplex.[9] [13]

2.2 Concrete Realization

In the previous section we claim that the abstract definition of Lagrangian systems is broad enough to encompass all the classical lagrangian systems with both discrete degrees of freedom, like particles, and continuous degree of freedom, like fluids or fields. Let' show two of the most significant examples.

2.2.1 Classical Linear Field over a Space-Time

⚠cosÃň com'ÃÍ la definizione ÃÍ troppo forte?

The field systems are a subset of the lagrangian systems:

Definition 39: Linear Fields on curved Background

It's a Lagragian system $(E, \mathscr{L})$ such that:

- the configuration bundle $E \xrightarrow{\pi} M$ is a vector bundle.
- the base manifold M is a Globally Hyperbolic Spacetime.
- the Euler-Lagrange operator $P = Q_{\mathscr{L}}$ is a Green Hyperbolic operator.
- For each Cauchy surface $\Sigma \subset M$ can be defined a well-posed Cauchy problem for the motion equation of P.[a]

[a] Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. [?] [3]

The idea of taking bundles on a space-time manifold it's physically intuitive, kinematically speaking a fields configuration it's not more than an than an association of some element of the fiber Q for each point of the space-time M. But the other three condition are worth a deeper insight:

Vector Bundle Condition Even if it might make sense to speak of nonlinear fields in some more general context, this condition it's a necessary element in case some form of the *superposition principles* as to be taken in account. Obviously this hypothesis is not sufficient to formulate the principle in the strong classical way, i.e.:"the response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually" mostly because only free systems can be considered at this stage and any statement about

stimulus can make sense.
However It assure that C is a vector space and , in conjunction with the linearity of motion operator P, $\texttt{Sol} = \ker(P)$ is a linear subspace. In other words every linear combination of kinematic configuration it's still a kinematic configuration.

Global hyperbolicity condition. This condition is strictly connected to the dynamic behaviour of the system.

⚠Def di dominio di dipendendenza footnote di definizione di spazio tempo def cauchy surface Remark causal future past def globally hyperbolic Teorema sulle caratterizzazioni

Notation fixing

We denote the set of all the cauchy surfaces as $\mathscr{P}_C(M)$.

Glon iperbolic determina la fogliazione dello spazio tempo per superfici di cauchy La superficie di cauchy ÃÍ questa:

Definition 40: Cauchy surface

questo da la possibilitÃă della buona posizione dei problemi di cauchy.. fisicamente ÃÍ la condizione minima per definire i dati iniziali dell'evoluzione dinamica. definisco data...

No! La definizione di green hyperbolicity garantisce invece l'esistenza e unicitÃă del problema di cauchy associata

e non solo, anche l'esistenza degli operatori di green associati che sono ingrediente fondamentale della costruzione di peierls

M ÃÍ glob iper e P ÃÍ green iper per tener conto del comporatamento propagativo definire sup cauchy definire s-t iperbolico (solo la caratterizzazione di ammetre una sup di cauchy) definire op green iperbolico su spazio tempo iperbolico (cioÃÍ ha delle green ope) Propr di buona definizione esistenza e unicita della soluzione

Di particolare ricorrenza fisica sono gli operatori normally iperbolic espressione in coordinate esempio K-g! ⚠

⚠

Secondo bar e ginoux per parlare di campo classico non serve specificare nient'altro... la condizione di $\exists 1!$ operatore di green di P insieme a quella di Essere un sistema lagrangiano ÃÍ un requisito minimo per definire senza ambiguitÃă le parentesi di peierls.

la condizione di green-hyperbolicity (che garantisce di $\exists 1!$ $E^{\mp}$ ma non che $\exists 1!$ soluzione del PC) corredata della scelta di un pairing permette di quantizzare secondo lo schema algebrico

La condizione di well-posedness del problema di cauchy da la possibilitÃă di quantizzare secondo lo schema dei dati iniziali

in tutti questi casi la candizione di Globally -hyperbolic per lo spazio tempo sottostante ÃÍ necessaria ⚠

Green-Hyperbolicity condition. The third condition ensures the existence of the Green's Operator as follows directly from definition. ⚠**Memento:** Pensavo di utilizzare la definizione di Green hyperbolic data da Bar che si avvale del concetto di formally dual (che non richiede la presenza del pairing) invece di quella usata in Advances AQFT che richiede solo che ammetta almeno un $G^{\pm}$ per poi dimostrare tramite teorema che se ÃÍ anche autoaggiunto vale l'unicitÃă. Si tratta solo di una piccola sfumatura.. Deve essere chiarito che in tutto ciÃš che faccio interessano che

$$\forall P \exists 1! G^{\pm}$$

. Che poi questa condizione derivi da GH secondo bar o Gh secondo dap+selfadj ÃÍ una di quelle questioni propriamente matematiche che poco interessa ai fisici della commissione.

⚠Devo richiedere che il green operator sia unico? sia negli schemi di quantizzazione che nella definizione di peierls faccio largo uso dell'unicitÃă. Per provare questa unicitÃă si passa per la definizione di una forma bilineare che permette di parlare di aggiunto formale e quindi avvalersi del teorema.

Cauchy condition. While the existence of a Cauchy surface allows to assign the data of initial value problems, the forth condition ensure the well -posedness of the problem for on every Cauchy surface Σ. I.e:

$$\begin{cases} Pu = 0 \\ u = u_0 \\ \nabla_{\vec{n}} u = u_1 \end{cases} \tag{2.2}$$

admit a unique solution $u \in \Gamma(E)$ for all $(u_0, u_1) \in \Gamma(\Sigma) \times \Gamma(\Sigma)$.
This suggests the following definition:

Notation fixing

The set of all the smooth initial data which can be given on the Cauchy Surface Σ is: $\mathsf{Data}(\Sigma) := \left\{ (f_0, f_1) \middle| f_i \in \Gamma^\infty(\Sigma) \right\} \equiv \Gamma^\infty(\Sigma) \times \Gamma^\infty(\Sigma)$

Observation 34

$\mathsf{Data}(\Sigma)$ inherit the linear structure of its component $\Gamma^\infty(\Sigma)$.

In this term the well-posedness of the cauchy problem can be stated as follow:

Proposition 2.2.1 *The map* $\mathbf{s} : \mathsf{Data}(\Sigma) \to \mathtt{Sol}$ *which assign to* $(u_0, u_1) \in \mathsf{Data}(\Sigma)$ *the unique solution of the cauchy problem 2.2 is linear and bijective.*

Since any solution, when restricted to a generic Cauchy surface Σ', determines another pair of initial data, i.e.:

$$\phi \equiv \mathbf{s}(\phi|_{\Sigma'}, \nabla_{\vec{n}'} \phi|_{\Sigma'}) \quad \forall \phi \in \mathtt{Sol}$$

we can define the set of initial data regardless of the particular Cauchy surface:

Definition 41: Set of smooth initial Data

$$\mathsf{Data} := \frac{\bigsqcup_{\Sigma \in \mathscr{P}_C(M)} \mathsf{Data}(\Sigma)}{\sim}$$

where $\sim$ is such that:

$$(f_0, f_1)|_\Sigma \sim (g_0, g_1)|_{\Sigma'} \Leftrightarrow \mathbf{s}(f_0, f_1) = \mathbf{s}(g_0, g_1)$$

Initial data, associated with different surface, are similar if they lead to the same solution.

Proposition 2.2.2 Data *is still a vector space.*

Proof:

⚠It's sufficient to prove that:

$$[\phi_a + \phi_b] = [\phi_a] + [\phi_b]$$

where $[\phi] = \{(\phi|_\Sigma, \nabla_{\vec{n}}\phi|_\Sigma) | \Sigma \in \mathscr{P}_C\}$. In fact:

$$\begin{aligned}\mathbf{s}_{\Sigma'}\big([(a',b')] + [(c',d')]\big) = \mathbf{s}_{\Sigma}\big([(a,b)] + [(c,d)]\big) = \mathbf{s}_{\Sigma}\big([(a,b)]\big) + \mathbf{s}_{\Sigma}\big([(c,d)]\big) = \\ = \mathbf{s}_{\Sigma'}\big([(a',b')]\big) + \mathbf{s}_{\Sigma'}\big([(c',d')]\big) = \mathbf{s}_{\Sigma'}\big([(a',b')] + [(c',d')]\big)\end{aligned}$$

□

Corollary 2.2.1 *The function* $: \mathsf{Data}(\Sigma) \to \mathtt{Sol}$ *which map every equivalence class to the associated solution is linear and bijective.*

2.2.2 Finite Degree systems

Every system with discrete degrees of freedom can be seen as a trivial field system. The correspondence is easily done:

- Configuration bundle of the system is the trivial $E = Q \times \mathbb{R}$ with base manifold $M = \mathbb{R}$.
- The kinematic configuration are $\mathtt{C} = C^\infty(\mathbb{R}, Q)$ i.e.all the possible parametrized functions on Q.
- The lagrangian density is obtained evaluating the ordinary Lagrangian on the lifted curve:

$$\mathscr{L}[\gamma] := \left(L \circ \gamma^{\text{lift}}\right) dt = \mathscr{L}(t, \gamma^i, \dot{\gamma}^i) \tag{2.3}$$

⚠

2.3 Geometric mechanics of Finite Degree systems

La visione precedente ÃÍ molto generale ma ci sono alcune strutture classiche che voglio replicare sul campo come la forma simplettica, le osservabili e le parentesi di poisson. Mi sembra piÃź chiaro vederle dopo aver raccontato queste, proprio come fa Wald.

2.3.1 Linear dynamical systems

2.4 Peierls Brackets

In this section we present more extensively the original Peierls' construction. Please note that we are not trying to provide the state of the art on the Peierls bracket (see for example [11] for the treatment in presence of gauge freedom) but only to expand and modernize the first approach given by Peierls. Instead of considering only scalar theory we extend the algorithm to a broader class of systems.

The Peierls's construction algorithm is well defined for a specific class of systems:

1. Linear field theory: $E = (E, \pi, M)$ is a vector bundle.
2. Linear Lagragian dynamics: $P = Q_{\mathscr{L}}$ is a L.P.D.O.
3. M is a globally Hyperbolic space-time.
4. motion operator P is a green-hyperbolic.

Observation 35: Peierls Bracket vs Poisson Bracket

⚠[a] Paraphrasing an observation made by Sharan[14]:

The Poisson bracket determines how one quantity b(t, q, p) changes another quantity a(t, q, p) when it acts as the Hamiltonian or vice-versa. The Peierls bracket, on the other hand, determines how one quantity b(t, q, p) when added to the system Hamiltonian h with an infinitesimal coefficient Îż affects changes in another quantity a(t, q, p) and vice-versa, i.e. The Peierls bracket is related to the change in an observable when the trajectory on which it is evaluated gets shifted due to an infinitesimal change in the Lagrangian of the system by another Lagragian density.

While the Poisson bracket between two observables a and b is defined on the whole phase space and is not dependent on the existence of a Hamiltonian, the Peierls bracket refers to a specific trajectory determined by a governing Lagrangian.

[a] da aggiustare

Purpose of the Peierls' procedure is to provide a bilinear form on the space of Lagrangian densities with time-compact support. This form induces a pre-symplectic structure on suitable spaces Functionals which can be easily reduced to the subclass of suitable classical observable. The algorithm can be summarized in a few steps

1. Consider a *disturbance* χ that is a time-compact lagrangian density .
2. Construct the perturbation of a solution under the disturbance.
3. Define the effect of the disturbance on a second lagrangian functional.
4. Assemble two Effect operators to give a bracket.

2.4.1 Disturbance and Disturbed motion operator

By *"disturbance"* we mean a time-compact supported lagrangian density $\chi \in \mathsf{Lag}$[1] which act as a perturbation on the system's lagrangian:

$$\mathscr{L} \rightsquigarrow \mathscr{L}' = \mathscr{L} + \epsilon \cdot \chi$$

where ϵ is a modulation parameter. The support condition is required in order to take in account only perturbations which affect the system dynamics only for a limited time interval. The motion operator of the disturbed dynamics results:

$$P_\epsilon = \left[\partial_\mu\left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathscr{L}}{\partial \phi}\right] + \epsilon\left[\partial_\mu\left(\frac{\partial \chi}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \chi}{\partial \phi}\right] = P + \epsilon Q_\chi \tag{2.4}$$

Observation 36

P_ϵ is not necessary linear, the second Hypothesis guarantees the linearity only for P.

2.4.2 Solution of the disturbed motion

The second ingredient of the Peierls' procedure are the *perturbed a solution under the disturbance* which are solution $\phi' \in \mathtt{C}$ of P_ϵ obtainable by a linear infinitesimal perturbation of a fixed solution $\phi \in \mathtt{Sol}$. More precisely, has to be seek a configuration:

$$\phi'(x) = \phi(x) + \epsilon\eta(x) \in \mathtt{C}$$

(the linear superposition is well-defined according to hypothesis 1) such that:

$$P_\epsilon \phi'(x) = o(\epsilon)$$
$$P\phi(x) = 0$$

In other word has to be satisfied the following equation:

$$\left[P_\epsilon\right]\phi'(x) = \left[P + \epsilon Q_\chi\right](\phi(x) + \epsilon\eta(x)) = \epsilon\Big(\left[P\right]\eta(x) + \left[Q_\chi\right](\phi(x) + \epsilon\eta(x))\Big) \stackrel{!}{=} o(\epsilon)$$

The condition of linearity for operator P doesn't hold for Q_χ in general. We can work around this problem taking into account the linearization[11, pag. 31] of operator Q_χ around solution$\phi(x)$. That is the unique linear operator $big[Q_\chi^{lin}(\phi)\right]$ such that:

$$\left[Q_\chi\right](\phi(x) + \epsilon\eta(x)\big) = \left[Q_\chi\right](\phi(x)) + \epsilon\left[Q_\chi^{lin}(\phi)\right](\eta(x)) + o(eta)$$

which can be seen as the first term of a *formal* Taylor expansion of operator Q_χ around ϕ.⚠[2] This is reflected in a condition on the perturbation $\eta \in \mathtt{C}_{tc}$:

$$\left[P_\epsilon\right]\phi'(x) = \epsilon\Big(\left[P\right]\eta(x) + \left[Q_\chi \phi(x)\right]\Big) + \epsilon^2\left[Q_\chi^{lin}(\phi)\right]\eta(x) \stackrel{!}{=} o(\epsilon)$$

[1] I.e. the top form $\chi(\phi)$ is time-compact supported for all $phi \in \mathtt{C}$.

[2] If $\mathtt{C}$ is a Frechet manifold the expansion could be made rigorous defining $\left[Q_\chi^{lin}(\phi_0)\right] = \left[\frac{\partial Q_\chi}{\partial \phi}(\phi_0)\right]$ in term of the Gateux derivative.

$$\Rightarrow P\eta = -Q_\chi\phi(x) \tag{2.5}$$

called *Jacobi Equation*. This equation is a non homogeneous P.D.E. with inhomogeneous term $(-Q_\chi\phi(x))$ fixed by the solution $\phi \in \mathtt{Sol}$ to be perturbed.

Follows from the definition of green hyperbolicity that the domain restriction of P to $\Gamma^\infty_{\rm pc}$ or $\Gamma^\infty_{\rm fc}$ admit a unique inverse G^+ and G^- respectively. Therefore, equation **??** admits a unique past compact solution η^+, called retarded perturbation of $\phi \in$ `Sol`, and a unique future compact solution η^-, called advanced perturbation:

$$\eta^\pm = G^\pm\big(-Q_\chi\phi\big) \tag{2.6}$$

Note that the time-compact support condition on χ guarantees that $Q_\chi\phi \in \mathrm{dom}(G^+)\cap \mathrm{dom}(G^-)$. Expression 2.6 reflects perfectly the orginal Peierls' notation where $\eta^\pm$ were noted as functions of the unperturbed solution: $\eta^+ \equiv \mathrm{D}_\chi\phi$ and $\eta^- \equiv \mathrm{Ꭰ}_\chi\phi$.

Observation 37

In most practical case it's possible to give a more basic characterization of $\eta^\pm$ in term of a Cauchy problem. Has to be stressed that this approach is not possible in general since Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense i.e. the well-posedness of the Cauchy problem is not guaranteed on any Cauchy surface. [4, pag 1] [3, remark 3.18][11, remark 2.1]
Consider a motion operator P which is also hyperbolic. Taking in account the time-compact support condition of χ, is possible to pick up two Cauchy surfaces $\Sigma_\pm$ (+ is after the perturbation while − stands for prior to the perturbation) such that:

$$J^\mp(\Sigma_\pm) \supset \mathrm{supp}(\chi)$$

for all time-slice foliation of the globally hyperbolic space-time.

For each of this two surfaces can be posed a Cauchy problem:

$$\begin{cases} P\eta = -Q_\chi\phi \\ (\eta, \nabla_n\eta)\big|_{\Sigma_\pm} = (0,0) \end{cases} \tag{2.7}$$

which , according to the well-posedness of the Cauchy problem, admits an unique solution. The link with the first presentation is that past/future -compact supported configuration always meet the initial data condition for some future/past Cauchy surface.

In conclusion,fixed a solution $\phi \in$ `Sol` and a perturbation χ, are uniquely determined two perturbed solution:

$$\phi^\pm_\epsilon = \phi + \epsilon\eta^\pm \tag{2.8}$$

such that:

retarded pertubation	$\eta^+ \in \Gamma^\infty_{pc}$	$(\eta^+, \nabla_n\eta^+)\big\vert_{\Sigma_-} = (0,0)$	"propagating forward"
advanced pertubation	$\eta^- \in \Gamma^\infty_{fc}$	$(\eta^-, \nabla_n\eta^-)\big\vert_{\Sigma_+} = (0,0)$	"propagating backward"

2.4.3 Effect Operator

Considering an arbitrary continuous ⚠[3] functional $B : \mathtt{Sol} \to \mathbb{R}$ (not necessarily linear) we can define the effect of a perturbation on the values of B[12, pag. 5] as a map:

$$\mathbf{E}^{\pm}_{\chi} : C^1(\mathtt{Sol}, \mathbb{R}) \to C^1(\mathtt{Sol}, \mathbb{R})$$

$$\mathbf{E}^{\pm}_{\chi} B(\phi_0) := \lim_{\epsilon \to 0} \Big(\frac{B(\phi^{\pm}_{\epsilon}) - B(\phi_0)}{\epsilon} \Big)$$

The advanced and retarded effects of χ on B are then defined by comparing the original system with a new system defined by the same kinematic configuration space $\mathtt{C}$ but with perturbed lagrangian.

The former expression appear quite simpler in case of a linear functional:

$$\mathbf{E}^{\pm}_{\chi} B(\phi_0) = B(\eta^{\pm}) \tag{2.9}$$

2.4.4 The Bracket

Remembering that every lagrangian density define a continuous functional (Action). From that is possible to build a binary function:

$$\{\cdot, \cdot\} : \mathsf{Lag}_{\mathsf{tc}} \times \mathsf{Lag}_{\mathsf{tc}} \to \mathbb{R}$$

as follow:

$$\{\chi, \omega\}(\phi_0) := E^{+}_{\chi} F_{\omega}(\phi_0) - E^{-}_{\chi} F_{\omega}(\phi_0) \tag{2.10}$$

Proposition 2.4.1 (Bilinearity) *When restricted to Linear Lagrangian densities* $\{\cdot, \cdot\}$ *is a bilinear form*

Proof:

Linearity in the first entry follows from equation [**?**] and the linearity of the Euler-Lagrange operator $Q.$ over Lag.
Linearity in the second entry is guaranteed only for lagrangian densities ω which provide a linear Lagrangian Functional F_{ω}.

□

We don't prove the symplectic property for this general object but we will face this problem for the case of *classical observable functional*, a subclass Lagrangian functional of most practical use in algebraic quantization.

[3] The precise notion of continuity require the specification of the infinite dimensional manifold structure.

2.4.5 Extension to non-linear theories

In the previous costruction the green-hyperbolicity of motion operator P plays a primary role. Anyway the problem of searching perturbed solution of the disturbed dynamic can be stated even in presence of non-linear fields where the configuration bundle is not necessary a vector bundle and the motion operator is not linear.

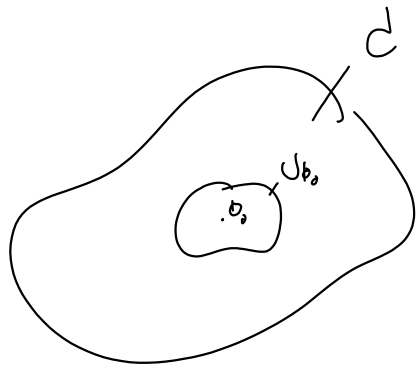


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Figure 2.2: Intrinsically, searching a variation of a solution $\gamma_0 \in \texttt{Sol}$ which solve the disturbed motion equation is equivalent to find the intersection of the perturbed solution with a local neighbourhood of Γ_0 : $U_{\gamma_0} \cap \ker(P_\epsilon)$.

The crucial point of the Peierls' procedure is to select among all the possible solution of the perturbed motion P_ϵ that configuration which can be constructed by a variation of some fixed solution of the non-perturbed dynamics $\gamma_0 \in \texttt{Sol}$. In this sense the problem results a *"linearization"* inasmuch the search of such solution is restricted to a local neighbour of the "point" $\gamma_0 \in \texttt{Sol}$.

Previously was natural restrict the choice of possible variations to the linear one only. But in the general case this preferential choice is no longer possible. A way to recover a notation similar to 2.8 is to work patchwise by chosing a coordinate representation. Fixed a solution $\gamma_0 \in \texttt{Sol}$ and a local trivializing chart (A, ϕ_A) such that $A \cap \mathrm{ran}(\gamma_0) \neq \emptyset$ we can define a local infinitesimal variation by acting on his components:

$$\gamma^i_\lambda(x) = \gamma^i_0(x) + \lambda\eta^i(x) \qquad \forall x \in \pi(A)$$

where γ^i_0 are the component of the unperturbed solution in the open set A and $\eta^i \in^q$ is a generic real q-ple (q is the dimension of the typical fiber manifold). λ is a real parameter that has to be "sufficiently small" in order to guarantee that the range of γ_λ is properly contained in A. In other words the construction of the linear variation, that for linear field theories could be done in a global way, in the general case can be recovered only locally.

Then is possible to define the effect of the a disturbance locally, searching local section $\gamma^i_\epsilon = \gamma^i_0 + \epsilon\eta^i$ solving the disturbed dynamic equation up to the first order in ϵ i.e.

$$[P_\epsilon]\gamma^i_\epsilon = o(\epsilon)$$

where $[P_\epsilon]$ has to be intended as the coordinate representation of the restriction on the local section $\Gamma^\infty(A)$. ⚠[4]

Observation 38

W.l.o.g has been taken the the same scalar ϵ to modulate both the perturbation γ_ϵ that the disturbance on the motion operator. On the contrary consider two different parameter is immaterial since only the smaller should be taken in account.

From the explicit equation of the perturbed solution:

$$\Big([P] + \epsilon[Q_\chi]\Big)(\gamma^i_0 + \epsilon\eta^i) = o(\epsilon)$$

follows an equation on the components of the local perturbation. In this case has to be dealt with the problem of non-linearity not only for Euler-Lagrange operator Q_χ but also for P. Arresting the expension to the first order in ϵ results:

$$\Big[P^{lin}_{\gamma_0}\Big]\eta^i(x) = -\Big(Q_\chi(\gamma_0)\Big)(x) \tag{2.11}$$

the *Jacobi equation* on the unperturbed solution $\gamma_0 \in \mathtt{Sol}$.

Observation 39

We've moved from an operator P definited on $\mathtt{C}$ to an operator $P^{lin}_{\gamma_0}$ defined on the space of variation. From a global point of view this variation can be seen as the tangent vector i.e. s $\eta \in T_{\gamma_0}\mathtt{C}$. In the case of the linear system this passage was unnecessary, the Jacobi equation was directly defined on $\mathtt{C}$ since, for linear system, any section could be seen as a generator of an infinitesimal variation. This behaviour mimics perfectly what happens in ordinary classical mechanics where the configuration space of a linear system is a vector space i.e a "flat" manifold[a] which is isomorphic to his tangent space in every point.

[a] In sense that admits a global coordinate chart.

Provided that the linearized motion operator (which is now properly a linear partial differential operator) is Green-Hyperbolic, the Peierels construction can continue as before. Has to be noted that now the advanced/retarded perturbation are formally identical to the former:

$$\eta^{\pm\, i} = G^\pm\big(-Q_\chi\gamma^j_0\big)$$

[4] non mi Ãĺ evidente se la rappresentazione in coordinate di un operatore agente sulle sezioni si realizza in modo ovvio, ma non vedo nemmeno ostruzioni! Di sicuro l'operazione Ãĺ ben definita per gli L.P.D.O visto che la definizione prevede proprio che su ogni carta locale trivilizzante l'operatore sia lineare alle derivate parziali

with the important difference that $G^{\pm}$ are now the Green operators of the linearized motion operator and depend on the fixed solution γ_0^j.

In conclusion the perturbed solution:

$$\gamma_\chi^{\pm\, i} = \gamma_0^i \pm G^{\pm}\big(- Q_\chi \gamma_0^j \big)$$

has to be intended as the "glueing" of all the local chart representations covering the chosen solution.

2.4.6 Finite Dimensional case

⚠

.. c’ÃÍ su i miei appunti ma potrebbe essere superfluo.. No, mi conviene metterlo qui piuttosto che nel capitolo su Jacobi!

⚠

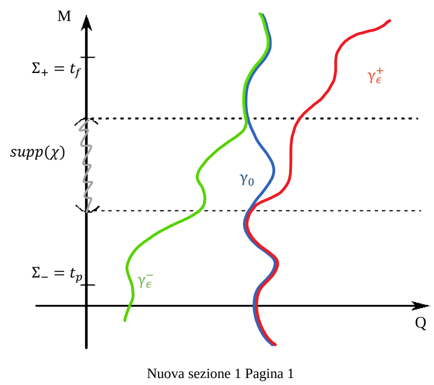

Figure 2.3: Intrinsically, searching a variation of a solution $\gamma_0 \in \mathtt{Sol}$ which solve the disturbed motion equation is equivalent to find the intersection of the perturbed solution with a local neighbourhood of Γ_0 : $U_{\gamma_0} \cap \ker(P_\epsilon)$.

2.5 Dubbi

- quando parlo della cinematica mi piacerebbe dare indicazioni sulla struttura matematica dello spazio delle configurazioni cinematiche:

  1. costituisce una frechet manifold (gli unici risultati che ho trovato sono quelli di Palais di "non linear global analysis"
  2. le curve parametrizzate sono le variazioni
  3. classi di equivalenza definiscono delle variazioni infinitesime che costituiscono lo spazio tangente allo spazio delle configurazioni cinematiche
  4. questo spazio tangente ÃÍ isomorfo allo spazio delle sezioni del pullback rispetto alla sezione $\phi \in C$ del verical bundle (vedere forger romero)
  5. il problema dell'atlante e della rappresentazione delle sezioni in carta locale (da scegliere sia sul total space E che sul base space M)

- fare riferimento al teorema di Ostrowsky per giustificare il fatto che consideriamo solo il primo ordine. le langrangiana con termini cinetici esotici sono instabili (nel senzo che non ammetto come soluzioni sezioni globali ma solo locali).

- Devo mettere la trafila di definizioni? direi di no, qualcosa va detto qualcosa va messo in footnote

  (sono ripetizioni inutili per la tesi, sono informazioni che si ritrovano ovunque... sono informazioni adatta al knowledge base)

  Recurring definitions in general Relativity (excluding the general smooth manifold prolegomena).

  Definition 42: Space-Time

  A quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ such that:

  - (M, g) is a time-orientable n-dimensional manifold $(n > 2)$
  - $\mathfrak{o}$ is a choice of orientation
  - $\mathfrak{t}$ is a choice of time-orientation

  Definition 43: Lorentzian Manifold

  A pair (M, g) such that:

  - M is a n-dimensional $(n \geq 2)$, Hausdorff, second countable, connected, orientable smooth manifold.
  - g is a Lorentzian metric.

Definition 44: Metric

A function on the bundle product of TM with itself:

$$g : TM \times_M TM \to \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a non-degenerate bilinear form.

Notation fixing

- *Riemman* if the sign of g is positive definite, *Pseudo-Riemman* otherwise.
- *Lorentzian* if the signature is $(+,-,\dots,-)$ or equivalently $(-,+,\dots,+)$.

Observation 40: Causal Structure

If a smooth manifold is endowed with a Lorentzian manifold of signature $(+,-,\dots,-)$ then the tangent vectors at each point in the manifold can be classed into three different types.

Notation fixing

$\forall p \in M, \quad \forall X \in T_pM$, the vector is:

- *time-like* if $g(X,X) > 0$.
- *light-like* if $g(X,X) = 0$.
- *space-like* if $g(X,X) < 0$.

Observation 41: Local Time Orientability

$\forall p \in M$ the timelike tangent vectors in p can be divided into two equivalence classes taking

$$X \sim Y \text{ iff } g(X,Y) > 0 \qquad \forall X, Y \in T_p^{\text{time-like}} M :$$

We can (arbitrarily) call one of these equivalence classes "future-directed" and call the other "past-directed". Physically this designation of the two classes of future- and past-directed timelike vectors corresponds to a choice of an arrow of time at the point. The future- and past-directed designations can be extended to null vectors at a point by continuity.

Definition 45: Time-orientation

A global tangent vector field $\mathfrak{t} \in \Gamma^{\infty}(TM)$ over the Lorenzian manifold M such that:

- $\text{supp}(\mathfrak{t}) = M$
- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Observation 42

The fixing of a time-orientation is equivalent to a consistent smooth choice of a local time-direction.

Definition 46: Time-Orientable Lorentzian Manifold

A Lorentzian Manifold (M, g) such that exist at least one time-orientation $\mathfrak{t} \in \Gamma^{\infty}(TM)$.

Notation fixing

Consider a piece-wise smooth curve $\gamma : \mathbb{R} \supset I \to M$ is called:

- *time-like* (resp. light-like, space-like) iff $\dot{\gamma}(p)$ is time-like (resp. light-like, space-like) $\forall p \in M$.
- *causal* iff $\dot{\gamma}(p)$ is nowhere spacelike.
- *future directed* (resp. past directed) iff is causal and $\dot{\gamma}(p)$ is future (resp. past) directed $\forall p \in M$.

Definition 47: Chronological $\substack{\textbf{future}\\\textbf{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{I}^{\pm}_M(p) := \left\{ q \in M \middle| \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ time-like } \substack{\text{future}\\\text{past}}\text{−directed} \,:\, \gamma(0) = p,\, \gamma(1) = q \right\}$$

Definition 48: Causal $\substack{\textbf{future}\\\textbf{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{J}^{\pm}_M(p) := \left\{ q \in M \middle| \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ causal } \substack{\text{future}\\\text{past}}\text{−directed} \,:\, \gamma(0) = p,\, \gamma(1) = q \right\}$$

Notation fixing

Former concept can be naturally extended to subset $A \subset M$:

- $\mathbf{I}^{\pm}_M(A) = \bigcup_{p\in A} \mathbf{I}^{\pm}_M(p)$
- $\mathbf{J}^{\pm}_M(A) = \bigcup_{p\in A} \mathbf{J}^{\pm}_M(p)$

Definition 49: Achronal Set

Subset $\Sigma \subset M$ such that every inextensible timelike curve intersect Σ at most once.

Definition 50: $\substack{\text{future}\\\text{past}}$ Domain of dependence of an Achronal set

The two subset related to the generic achornal set $\Sigma \subset M$:

$$\mathbf{D}^{\pm}_M(\Sigma) := \left\{q \in M \middle| \ \forall \gamma \ {}^{\text{past}}_{\text{future}}\text{inextensible causal curve passing through } q: \ \gamma(I) \cap \Sigma \neq \emptyset\right\}$$

Notation fixing

$\mathbf{D}_M(\Sigma) :- \mathbf{D}^{+}_M(\Sigma) \cup \mathbf{D}^{-}_M(\Sigma)$ is called *total domain of dependence*.

Definition 51: Cauchy Surface

Is a subset $\Sigma \subset M$ such that:

- closed
- achronal
- $\mathbf{D}_M(\Sigma) \equiv M$

Basic Definition in L.P.D.O. on smooth vector sections.
Consider $F = F(M,\pi,V), F' = F'(M,\pi',V')$ two linear vector bundle over M with different typical fiber

Definition 52: Linear Partial Differential operator **(of order at most $s \in \mathbb{N}_0$)**

Linear map $L : \Gamma(F) \to \Gamma(F')$ such that:
$\forall p \in M$ exists:

- (U,ϕ) local chart on M.
- (U,χ) local trivialization of F
- (U,χ') local trivialization of F'

for which:

$$L\left(\sigma\big|_U\right) = \sum_{|\alpha|\leq s} A_\alpha \partial^\alpha \sigma \qquad \forall \sigma \in \Gamma(M)$$

Remark:

(multi-index notation)
A multi-index is a natural valued finite dimensional vector $\alpha = (\alpha_0, \dots, \alpha_{n}-1) \in \mathbb{N}_0^n$ with $n < \infty$.
On $\mathbb{R}^n$ a general differential operator can be identified by a multi-index:

$$\partial^\alpha = \prod_{\mu=0}^{n-1} \partial_\mu^{\alpha_\mu}$$

(Until the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|\alpha| := \sum_{\mu=0}^{n-1} \alpha_\mu$$

?????????????????????

Hp:

Proposition 2.5.1 (Existence and uniqueness for the Cauchy Problem) –

$\mathbf{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ *a globally hyperbolic space-time.*

- $\Sigma \subset M$ *a spacelike cauchy surface with future-pointing unit normal vector field* $\vec{n}$.
-

Th:

Observation 43

"Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. " [**?**] [3]
However the existence and uniqueness can be proved for the large class of the *Normally-Hyperbolic Operators.*

Chapter 3

Algebraic Quantization

The point we want to get, that we will face in the next chapter, is the algebraic quantization of geodesic system. For this purpose it is necessary to devote a chapter to the description of algebraic quantization scheme. We will show two realizations of the scheme applicable to a class of systems sufficiently broad to encompass the system that we want to examine.

3.1 Overview on the Algebraic Quantization Scheme.

Contemporary quantum field theory is mainly developed as quantization of classical fields. Classical field theory thus is a necessary step towards quantum field theory.⚠[1] The *"Quantization process"* has to be considered as an algorithm , in the sense of self-conteining succession of instruction, that has to be performed in order to establish a correspondence between a classical field theory and its quantum counterpart. ⚠[2]

On this basis the axiomatic theory of quantum fields takes the role of "validity check". It provide a set of conditions that must be met in order to establish whether the result can be consider a proper quantum field theory. Basically there are no physical/philosophical principles which justifies "a priori" the relation between mathematical objects (e.g the classical state versus quantum states) individually. The scheme can only be ratified " a posteriori" as whole verifying the agreement with the experimental observations.

However this is by no means different from what is discussed in ordinary quantum mechanics where there are essentially two plane: the basic formalism of quantum mechanics, which is substantially axiomatic and permits to define an abstract quantum mechanical system, and the quantization process that determine how to construct the quantum analogous of a classical system realizing the basic axioms.

[1] Cito testualmente Mangiarotti, shardanashivly

[2] forse l'nlab esprime la cosa meglio di me `http://ncatlab.org/nlab/show/quantization`. Sono d'accordo con il loro approccio ma non voglio usare la loro formulazione perchÃĺ in fondo ci sono arrivato anche da solo :P

We refer to the algebraic quantization as a *scheme of quantization* because it's not a single specific procedure but rather a class of algorithms. These algorithms are the same concerning the quantization step per se (costruction of the *-algebra of classical observable) but they differ in the choice of the classical objects (essentially the classical observables and the bilinear form) to be subjected to the procedure.

Basically an algebraic quantization is achieved in three steps:

1. **Classical Step**
   Identify all the mathematical structures necessary to define the field, i.e. the pair (E, P).
   In general every quantization process exploit some conditions on the quantum field structure that has to be met.

2. **Pre-Quantum Step**
   ⚠Are implemented some additional mathematical over-structure on the classic framework. The aim is to establish the specific objects which will be submitted to the quantization process in the next step. Generally these object don't have any a classical meaning, their only purpose is to represent the classical analogous of the crucial structures of the quantum framework. From that we say *Pre-Quantum*, their introduction doesn't have a proper *a priori*explanation but has to be treated as an anstatz and justified *a posteriori* within the quantum treatment.

   Essentially has to be chosen a suitable space of *Classical observable* and this space has to be rigged with a well-behaved bilinear form.

   The ordinary quantum mechanics equivalent step is the choice of a particular Poisson bracket on $C^\infty(T^*Q)$, which tipically implement the *canonical commutation relations* $\{q, p\} = i\hbar$, among all the possible Poisson structure. Note that this is a "pre-quantum" step because in classical Hamiltonian mechanics is considered only the Poisson structure carried from the natural symplectic form [2].

3. **Quantization**
   Finally are introduced the rules which realize the correspondence between the chosen classical objects and their quantum analogues.⚠[3] The algebraic approach characterizes the quantization of any field theory as a two-step procedure. In the first, one assigns to a physical system a suitable âĹŮ-algebra A of observables, the central structure of the algebraic theory which encodes all structural relations between observables. The second step consists of selecting a so-called *Hadamard state* which allows us to recover the interpretation of the elements of A as linear operators on a suitable Hilbert space.

⚠[4] As said previously, the realization of the Algebraic scheme are many: Fedosov's procedure, by Deformation, Peierls' procedure, by Initial Data etc . In the next section we review the last two.

[3] Sto Cito direttamente [6].

[4] Frase che non mi piace ma voglio far presente che le realizzazioni dello schema algebrico sono molteplici!

3.2 Quantization with Peierls Bracket.

⚠Temp ⚠da contestualizzare (e spostare)

Observation 44

In the algebraic quantization scheme the choice of the bundle bilinear form take a pivotal role since it is the basis of the so-called *pairing*. In effect this is the only discretionary parameter of the whole procedure. The prescription on the symmetry properties determine the Bosonic/Fermionic character of the quantized theory:

Pairing	Observables linear form	Quantum Theory
symmetric	anti-symmetric	Bosonic
anti-symmetric	symmetric	Fermionic

Observation 45

What we are going to show is a quantization procedure strictly defined for a specific class of classical theories:

1. Linear Fields.
2. Lagrangian Dynamics.
3. On Globally-Hyperbolic Space-time.
4. with Green-hyperbolic motion Operator.

Fall into this category prominent examples like Klein-Gordon and Proca Field Theory.[5] Has to be noted that the Lagrangian condition is ancillary. This has the purpose to justify the shape of the symplectic form on the classical observables space as consequent from the Peierls bracket. It's customary to overlook to the origin of this object and jump directly to the expression **??** in term of the Green's operator that no longer present any direct link to the Lagrangian and therefore can be extended to any green-hyperbolic theory.

Briefly the procedure can be resumed in few steps:

1. Classical Step
   Has to be stated the mathematical structure of the classical theory under examination.
   (a) Kinematics: is encoded in the configuration bundle of the theory.
      i. Specify the base manifold M.
         Has to be a Globally-Hyperbolic Space-time.
      ii. Specify the Fiber and the total Space E auxiliary structure, e.g: spin-structure or trasformation laws under diffeomorphism on the base space.
         E has to be at least a vector bundle.

(b) Dinamics: has to be specified the local coordinate expression of the motion operator $P : \Gamma^{\infty}(E) = \mathtt{C} \to \mathtt{C}$.

   i. Is P Green-hyperbolic?

   ii. Is P derived from a lagrangian: $P = Q_{\mathscr{L}}$?

2. Pre-Quantum Step

   (a) Pairing: construct a basic bilinear form on the space of kinematic configurations.

      i. Choose $< \cdot, \cdot >$ a bilinear form on the bundle E.
      Generally this object is suggested by the m

      ii.

   (b) Classical Observables

      i.

      ii.

   (c) Symplectic structure

      i.

      ii.

3. Quantization Step

   (a) Quantum Observables Algebra
   A concrete realization is achieved in three step.

      i.

      ii.

   (b) Hadamard State

      i.

      ii.

3.2.1 Classical Step

Applicability of the procedure.

3.2.2 PreQuantum Step.

3.2.3 Second Quantization Step.

3.3 Quantization by Initial Data.

Chapter 4

Geodesic Fields

Usually, in the context of differential geometry, a *geodesic curve* is characterized as a self-parallel curve in order to generalize the *straight lines*. Considering a differential manifold M endowed with an affine connection ∇ we define:

Definition 53: Geodesic

A curve ⚠[a] $\gamma : [a, b] \to M$ such that:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0 \tag{4.1}$$

where $\dot{\gamma}^{\mu} := \frac{d\gamma^{\mu}}{dt}$ is the tangent vector to the curve.

[a] Devo dire smooth o piecewise?

Notation fixing

In local chart the previous equation assume the popular expression:

$$\ddot{\gamma}^i + \Gamma^i_{\ jk}\dot{\gamma}^j\dot{\gamma}^k = 0 \tag{4.2}$$

Where $\Gamma^i_{\ jk}$ is the coordinate representation of the Christoffel symbols of the connection.

In presence of a pseudo-Riemannian metric is possible to present the geodesic in a metric sense i.e. as the curve which extremizes the *Energy Functional*[1]:

Definition 54: Energy functional

[1] Remember that for arc-length parametrized curves the Energy functional coincide with the length functional.[10, Lemma 1.4.2]

$$E(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|^2 dt \tag{4.3}$$

Considering only the proper variation (that keep the end-point fixed), the extremum condition corresponds to equation where ∇ is the unique Levi-Civita connection (torsion-free and metric-compatible).

In general relativity the problem of the geodesic equation linearization, named *Jacobi equations* takes a central role. [2] ⚠(nel file di ripasso di geometria riemmaniana ho scritto gran parte delle definizione conviene vedere cosa mi serve effettivamente... Di certo mi avvalgo della seguente equazione

Notation fixing

In local charts the Jacobi fields along the geodesic γ solve a linear O.D.E.:

$$\left(X''\right)^{\mu} + R^{\mu}_{i\alpha j} T^i X^{\alpha} T^j = 0 \tag{4.4}$$

where:

- $\left(X'\right)^{\mu} := \left(\nabla_{\dot{\gamma}(t)} X\right)^{\mu}$ is the covariant derivative along the curve γ.
- $T \equiv \dot{\gamma}(t)$ stands for the tangent vector to the curve γ.

The rest of this chapter will be dedicated to presenting the physical approach to the Geodesic.

4.1 Geodesic Problem as a Mechanical Systems

The basic idea is very simple, portray the geodesic curve as the natural motion of a free particle constrained on the Pseudo-Riemannian manifold Q.

⚠

obvious enough this problem can be seen as a generalization of the calculation of the motions of free falling parcticles In terms of general relativity this problem can be instantly recognized as the derivation of the free-falling particles motion.

However, there is no lack of alternative viewpoints . The framework of the classical Geometric Mechanics teach us to picture the "static" configurations of a constrained, complex, classical system as a point on the *Configuration space* manifold. According to that, the geodesic motion can be seen as a realization of a particular dynamics on each mechanical system endowed with a pseudo-Riemannian configuration space[3].

⚠

[2] Usually in this context takes the name of *Geodesic deviation* problem[**?**, pag. 46].

[3] Such systems can be depicted as "geodesic" even in presence of a position-dependant potential.[2, Cap 3.7]

Theorem 4.1.1 (Geodesic Motion) *The geodesics on the Pseudo-Riemmanian manifold* (Q, g) *are the natural motions of the ordinary Lagrangian system* (Q, L) *where:*

$$L(V_q) := \frac{1}{2} g_q(V, V) \tag{4.5}$$

Proof:

The Euler-Lagrange equation of L coincides with the geodesic equation 4. ⚠.. ÃÍ sul quaderno non so se metterla

□

Observation 46

The geodesic system is not simply Lagrangian but also Hamiltonian. This property follows from the hyperregularity[2] of L.

⚠Anyway we will neglect this fact inasmuch in what follows only the Lagrangian character assumes a role.

As shown in chapter 2, every system with discrete degrees of freedom can be seen as the trivial field system. From that follows the alternative characterization of geodesic as a lagrangian field:

Corollary 4.1.1 (Geodesic field) *The geodesics on the Pseudo-Riemmanian manifold* (Q, g) *can be seen as the* Dynamical Configurations *of the lagrangian field system* $(E, \mathscr{L})$ *where:*

- $E = (Q \times \mathbb{R}, \pi, \mathbb{R})$ *trivial smooth bundle on the real line.*
- $\mathscr{L}[\gamma] = \frac{1}{2} g(\dot{\gamma}, \dot{\gamma})(t)\, dt$

Proof:

Is simple application of the correspondence seen in chapter 2.3.

□

From this perspective is clear that the Energy Functional can be seen as the action in the geodesic field dynamics and equation 4 is nothing more than the motion equation under the *least action principle*.

Figure 4.1: Impressionistic view of the geometric mechanics structure.

4.2 Peierls Bracket of the Geodesic field

⚠
Da dire: espressione in coordinate della lagrangiana, ÃÍ altmente non lineare perchÃÍ implicitamente ÃÍ $g_{\mu\nu}(\gamma^i(t))$ non polinomiale in γ^i ed esplicitamente ÃÍ quadratica, Mostrare esplicitamente che l'equazione di jacobi per il sistema ÃÍ effittivamente l'equjazione di jacobi (questo ÃÍ triviale se vedi come definisce il campo di jacobi jurgen

⚠

4.2.1 Example: Geodesic field on FRW space-time.

4.3 Algebraic quantization of the Geodesic Field

⚠
va ripetuto che la geodetica ÃÍ non lineare quindi ciÃš che effettivamente si quantizza ÃÍ jacobi lungo una prefissata geodetica. questo ÃÍ un campo lineare.

⚠

4.3.1 Peierls Approach

4.3.2 Inital data Approach

4.4 Interpretations??????

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