Chapter 1

Lagrangian Systems and Pierels Brackets

⚠.. introduzione dedichiamo sforzo alla definizione dei sistemi lagrangiani astratti per dare un punto di vista unificato ai sistemi a gradi libertÃă continui (continui macroscopici, fluidi, campi) e a gradi di libertÃă discreti Li vediamo come sotto classi dei sistemi lagrangiani notiamo che in entrambi i casi c'Ãĺ un possibile dato di cauchy usiamo questo ingrediente per definire l'algoritmo di peierls. ⚠

1.1   Abstract Mechanical Systems

It's possible to state a mathematical definition sufficiently broad to include all the systems in ordinary analytical mechanics regardless of the cardinality of degrees of freedom in a unified way.

Definition 1: Abstract Evolutive System

Pair (E, P) composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on manifold M called *"configuration bundle"*.

- $P : \Gamma^{\infty}(E) \to \Gamma^{\infty}(E)$
  operator called *"motion operator"*

This formultation is still very distant from the physical interpretation but has the benift to highlight the minimal mathematical objects which must be fixed in order to specify a mechanical systems.

Kinematics   The configuration bundle encompass all the kinematical structure of the system, the pivotal role is played by the smooth sections which are to be under-

stood as all the possible conformation of the system.

> **Notation fixing**
>
> $$\mathsf{C} := \Gamma^{\infty}(M, E)$$
>
> Space of kinematic configurations.

A section is not a statical configuration, equivalent to a specific point in the configuration space of ordinary classical systems, but has to be seen as a specific realization of the kinematics in the sense of a complete description of a possible motion. At this level of abstraction, since no space-time structure has been specified, terms like stasis and motion must be taken with care .The natural physical interpretation should be clearly manifested through the concrete realization of systems with discrete and continuous degree of freedom.

> **Observation 1: Mathematical structure**
>
> Mathematically speaking this set should be regarded as an infinite dimensional Manifold.
> This framework provides a geometric characterization of the notion of variations as tangent vectors on the the space of kinematic configurations .[?]

> **Observation 2: Coordinate Representation**
>
> The choice of a chart atlas $\mathscr{A}(M)$ on the base space M and $\mathscr{A}(E)$ on the total space E provides a correspondence between each configuration $\gamma \in \mathsf{C}$ and family of smooth real functions $\{f_{\alpha\beta} : A_\alpha \subset \mathbb{R}^m \to \mathbb{R}^q\}$. The process is trivial:
>
> $$\gamma \in \mathsf{C} \mapsto \{f_{A,U} = \psi_U \circ \gamma \circ \psi_A^{-1} | (A, \psi_A) \in \mathscr{A}(M), (U, \psi_U) \in \mathscr{A}(E)\}$$
>
> Since the whole section as a global object is quite difficult to handle is customary in field theory to work in the more practical local representation.

> **Observation 3: Further specification of the system's kinematics**
>
> The general formalism doesn't require any other structure to be carried forward. Additional structure on the fiber , the base or the whole bundle are to be prescribed in order to specify a precise physical model, e.g. the spin structure on E for the Dirac Field.[?]

Dynamics   The operator P is the object that contains all the information about the dynamic evolution of the system. It has the role to select the dinamically compatible configuration among all the admissible kinematic configurations of C, exactly as it happens in analytical mechanics where the dynamic equations shape the natural motions.
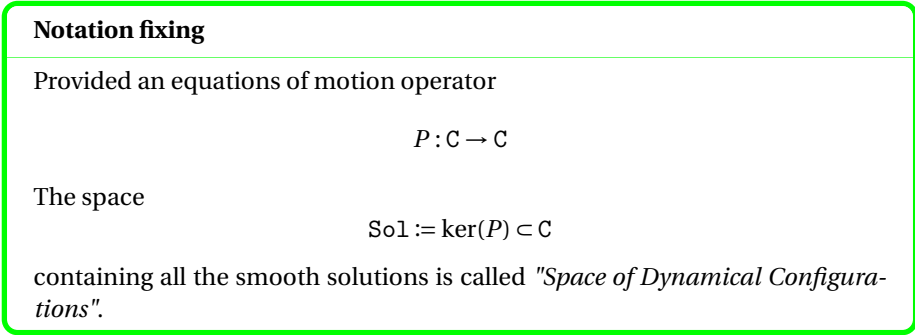
> **Notation fixing**
>
> Provided an equations of motion operator
>
> $$P : \mathtt{C} \to \mathtt{C}$$
>
> The space
>
> $$\mathtt{Sol} := \ker(P) \subset \mathtt{C}$$
>
> containing all the smooth solutions is called *"Space of Dynamical Configurations"*.

Figure 1.1: Geometric picture of the basic mechanical system's structure.
⚠immagine

1.1.1 Lagrangian Dynamics

Lagrangian systems constitute a subclass of the abstact mechanical systems of more practical interest:

> **Definition 2: Lagrangian System**
>
> Pair $(E, \mathscr{L})$ composed of:
>
> - $E \xrightarrow{\pi} M$
>   smooth fiber bundle of typical fiber Q on the oriented manifold $(M, \mathfrak{o})$ called *"configuration bundle"*.
>
> - $\mathscr{L} : J^r E \to \wedge^m T^* M$
>   bundle-morphism from the r-th Jet Bundle to the top-dimensionial forms bundle over the base manifold M called *"Lagrangian density"* or simply *"Lagrangian"* of r-th order.

N.B. : In what follows all the systems considered will be exclusively of first order.

    In this case is the Lagrangian density the object containing all the information about the dynamic evolution of the system.
    In order to reconstruct the system's dynamic from the Lagrangian density has to be understood the mathematical nature of $\mathscr{L}$. $\mathscr{L}$ maps point q_p on the fiber $J^r_p E$ to a m-form on $T_p M$. Recalling the definition of jet bundles is clear that for each smooth section on E is associated a smooth section on the b$J^r E$:

$$\phi \in \Gamma^\infty(E) \mapsto (\phi, \partial_\mu \phi, \partial_{\mu,\nu} \phi, \dots \partial_{\vec{\alpha}} \phi)$$

where $\vec{\alpha}$ is a multi-index of length r. The correspondence is not univocal since sections equal up to the r-th order define the same jet section. The smoothness of $\mathscr{L}$

ensure that each jet bundle section is mapped to a smooth section in the top-forms bundle i.e. the most general integrable object on a orientable manifold.

It should be clear that $\mathscr{L}$ is a specific choice among the vast class of functions suitable to be a good Lagrangian density over the Configuration Bundle E:

Definition 3: Lagrangian Density on the bundle E

$$\mathsf{Lag}^r(E) := \hom\left(J^r E, \ \overset{m}{\bigwedge}(T^* M)\right) \cong \{f : \Gamma^\infty(J^r E) \to \Omega^m(M)\}$$

(where $\Omega^m(M)$ is the common name for $\Gamma^\infty\left(\bigwedge^m(T^* M)\right)$ in the context of Grassmann algebras.) The equivalence states the fact that a bundle-morphism induce a mapping between the sections.

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this choice fix the "Dynamical identity" of the considered system.

Proposition 1.1.1 $\mathsf{Lag}^r(E)$ *has an obvious vector space structure inherited by the linear structure of* $\Omega(M)$.

Thanks to the correspondence between a section $\phi \in \mathsf{C}$ and his r-th jet, it's possible to consider the Lagrangian as directly acting on the kinematic configurations. In layman terms the image $\mathscr{L}[\phi]\mathrm{d}\mu$, where $\mathrm{d}\mu$ is the measure associated to the orientation $\mathfrak{o}$, is something that can be measured over the whole base space.
This property suggests the introduction of the class of associated functionals:

Definition 4: Lagrangian functional

Is a functional on C with values on regular distribution over M associated to the generic $\mathscr{L} \in \mathsf{Lag}$.

$$\mathscr{O}_{\mathscr{L}} : \mathsf{C} \to \left(C_0^\infty(M)\right)'$$

Such that the lagrangian functional associated to $\mathscr{L}$, valued on the configuration $\phi \in \mathsf{C}$ and tested on the test-function $f \in C_0^\infty(M)$ it's given by:

$$\mathscr{O}_{\mathscr{L}}[\phi](f) = \int_M \mathscr{L}[\phi] f \mathrm{d}\mu$$

Proposition 1.1.2 *As a distribution* $\mathscr{O}_{\mathscr{L}}[\phi](f)$ *is necessarily linear in the test-functions entry but not in the configurations entry.*

Observation 4

The choice of the image of $\mathscr{O}_{\mathscr{L}}$ as a distribution it's a necessary precaution to ensure that functional is *"convergent"* whatever is the configuration on which is evaluated. In fact, despite $\mathscr{L}[\phi]$ is integrable with respect to the measure

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$\mathrm{d}\mu$, it's not necessary summable if the support of the configuration ϕ becomes arbitrarily large.

This is a simple consequence of the well known sequence of inclusions:

$$\mathscr{L}[\phi] \in C_0^\infty(M) \subset L_{\text{loc}}^1(M,\mu) \supsetneqq L^1(M,\mu)$$

of the functional analysis . Indeed, the functional

$$\mathscr{O}_{\mathscr{L}}[\phi] = \int_{\text{supp}(\phi)} \mathscr{L}[\phi]\mathrm{d}\mu$$

is well defined for all $\mathscr{L} \in \mathsf{Lag}^r(E)$ only over the compactly supported sections. To take account of the global sections it's sufficient to dampen the integral multiplying the integrand with an arbitrary test-function.

Notation fixing

When calculated for the specific density of the Lagrangian system $\mathscr{O}_{\mathscr{L}}$ takes the name of *Action* or *Total Lagrangian*.

The introduction of the Lagrangian density is meaningless without the prescription of a dynamical principle which allows to determine univocally a differential operator P on the kinematics configurations space C. This fundamental principle is the *least action principle*. A proper justification of this claim should require the presentation of the differential calculus on the infinite dimensional manifolds C. Jumping straight to the conclusion we can state this correspondence as a principle in term of a function which assign for all lagrangian densities an operator on the kinematic configurations space. In the case of first order lagrangian we define

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Definition 5: Euler-Lagrange operator

It's the differential operator
$$Q_\chi : \mathsf{C} \to \mathsf{C}$$
relative to the lagrangian density $\chi \in \mathsf{Lag}^1(E)$, such that:

$$Q_\chi(\gamma) = \left(\partial_\mu \left(\frac{\partial \chi}{\partial(\partial_\mu\phi)} \Big|_\gamma \right) - \frac{\partial\chi}{\partial\phi} \Big|_\gamma \right) \qquad \forall \gamma \in \mathsf{C} \tag{1.1}$$

(where $\left(\frac{\partial\chi}{\partial(\partial_\mu\phi)} \right.$ is the be intended as the lagrangian density constructed differentiating $\chi(\phi,\partial_\mu)$ as an ordinary function treating its functional entries as an usual scalar variable.)

Observation 5

The whole theory of both Lagrangian densities class and Euler-Lagrange equa-

tion could be stated in a more syntetic way in terms of the Grassmann-graded variational bicomplex.[?][?]

1.2 Concrete Realization

In the previous section we claim that the abstract definition of Lagrangian systems is broad enough to encompass all the classical lagrangian systems with both discrete degrees of freedom, like particles, and continuous degree of freedom, like fluids or fields. Let' show two of the most significant examples.

1.2.1 Classical Linear Field over a Space-Time

⚠ cosÃň com'ÃÍ la definizione ÃÍ troppo forte!

The field systems are a subset of the lagrangian systems:

Definition 6: Linear Fields on curved Background

It's a Lagragian system $(E, \mathscr{L})$ such that:

- the configuration bundle $E \xrightarrow{\pi} M$ is a <u>vector bundle</u>.

- the base manifold M is a <u>Globally Hyperbolic Spacetime</u>.

- the Euler-Lagrange operator $P = Q_{\mathscr{L}}$ is a <u>Green Hyperbolic operator</u>.

- For each Cauchy surface $\Sigma \subset M$ can be defined a well-posed Cauchy problem for the motion equation of P.[a]

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[a]Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. [?] [?]

The idea of taking bundles on a space-time manifold it's physically intuitive, kinematically speaking a fields configuration it's not more than an than an association of some element of the fiber Q for each point of the space-time M. But the other three condition are worth a deeper insight:

Vector Bundle Condition   Even if it might make sense to speak of nonlinear fields in some more general context, this condition it's a necessary element in case some form of the *superposition principles* as to be taken in account. Obviously this hypothesis is not sufficient to formulate the principle in the strong classical way, i.e.:"the response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually" mostly because only free systems can be considered at this stage and any statement about stimulus can make sense.

However It assure that C is a vector space and , in conjunction with the linearity of motion operator P, Sol = $\ker(P)$ is a linear subspace. In other words every linear combination of kinematic configuration it's still a kinematic configuration.

Global hyperbolicity condition. This condition is strictly connected to the dynamic behaviour of the system.

⚠ Def di dominio di dipendendenza footnote di definizione di spazio tempo def cauchy surface Remark causal future past def globally hyperbolic Teorema sulle caratterizzazioni

> **Notation fixing**
>
> We denote the set of all the cauchy surfaces as $\mathscr{P}_C(M)$.

Glon iperbolic determina la fogliazione dello spazio tempo per superfici di cauchy La superficie di cauchy Ãĺ questa:

> **Definition 7: Cauchy surface**
>

questo da la possibilitÃă della buona posizione dei problemi di cauchy.. fisicamente Ãĺ la condizione minima per definire i dati iniziali dell'evoluzione dinamica. definisco data...

No! La definizione di green hyperbolicity garantisce invece l'esistenza e unicitÃă del problema di cauchy associata

e non solo, anche l'esistenza degli operatori di green associati che sono ingrediente fondamentale della costruzione di peierels

M Ãĺ glob iper e P Ãĺ green iper per tener conto del comporatamento propagativo definire sup cauchy definire s-t iperbolico (solo la caratterizzazione di ammetre una sup di cauchy) definire op green iperbolico su spazio tempo iperbolico (cioÃĺ ha delle green ope) Propr di buona definizione esistenza e unicita della soluzione

Di particolare ricorrenza fisica sono gli operatori normally iperbolic espressione in coordinate esempio K-g! ⚠

⚠

Secondo bar e ginoux per parlare di campo classico non serve specificare nient'altro... la condizione di $\exists 1!$ operatore di green di P insieme a quella di Essere un sistema lagrangiano Ãĺ un requisito minimo per definire senza ambiguitÃă le parentesi di peierls.

la condizione di green-hyperbolicity (che garantisce di $\exists 1!$ $E^{\mp}$ ma non che $\exists 1!$ soluzione del PC) corredata della scelta di un pairing permette di quantizzare secondo lo schema algebrico

La condizione di well-posedness del problema di cauchy da la possibilitÃă di quantizzare secondo lo schema dei dati iniziali

in tutti questi casi la candizione di Globally -hyperbolic per lo spazio tempo sottostante Ãĺ necessaria ⚠

Green-Hyperbolicity condition. The third condition ensures the existence of the Green's Operator as follows directly from definition.

⚠ Devo richiedere che il green operator sia unico? sia negli schemi di quantizzazione che nella definizione di peierels faccio largo uso dell'unicitÃă. Per provare

questa unicitĂă si passa per la definizione di una forma bilineare che permette di parlare di aggiunto formale e quindi avvalersi del teorema.

Cauchy condition. While the existence of a Cauchy surface allows to assign the data of initial value problems, the forth condition ensure the well -posedness of the problem for on every Cauchy surface Σ. I.e:

$$\begin{cases} Pu = 0 \\ u = u_0 \\ \nabla_{\vec{n}} u = u_1 \end{cases} \tag{1.2}$$

admit a unique solution $u \in \Gamma(E)$ for all $(u_0, u_1) \in \Gamma(\Sigma) \times \Gamma(\Sigma)$.
This suggests the following definition:

Notation fixing

The set of all the smooth initial data which can be given on the Cauchy Surface Σ is: $\mathsf{Data}(\Sigma) := \left\{ (f_0, f_1) \big| f_i \in \Gamma^\infty(\Sigma) \right\} \equiv \Gamma^\infty(\Sigma) \times \Gamma^\infty(\Sigma)$

Observation 6

$\mathsf{Data}(\Sigma)$ inherit the linear structure of its component $\Gamma^\infty(\Sigma)$.

In this term the well-posedness of the cauchy problem can be stated as follow:

Proposition 1.2.1 *The map* $\mathbf{s} : \mathsf{Data}(\Sigma) \to \mathsf{Sol}$ *which assign to* $(u_0, u_1) \in \mathsf{Data}(\Sigma)$ *the unique solution of the cauchy problem 1.2 is linear and bijective.*

Since any solution, when restricted to a generic Cauchy surface Σ', determines another pair of initial data, i.e.:

$$\phi \equiv \mathbf{s}(\phi|_{\Sigma'}, \nabla_{\vec{n'}} \phi|_{\Sigma'}) \quad \forall \phi \in \mathsf{Sol}$$

we can define the set of initial data regardless of the particular Cauchy surface:

Definition 8: Set of smooth initial Data

$$\mathsf{Data} := \frac{\bigsqcup_{\Sigma \in \mathscr{P}_C(M)} \mathsf{Data}(\Sigma)}{\sim}$$

where $\sim$ is such that:

$$(f_0, f_1)|_\Sigma \sim (g_0, g_1)|_{\Sigma'} \Leftrightarrow \mathbf{s}(f_0, f_1) = \mathbf{s}(g_0, g_1)$$

Initial data, associated with different surface, are similar if they lead to the same solution.

Proposition 1.2.2 Data *is still a vector space.*

Proof:

⚠ It's sufficient to prove that:

$$\left[\phi_a + \phi_b\right] = \left[\phi_a\right] + \left[\phi_b\right]$$

where $[\phi] = \left\{(\phi|_\Sigma, \nabla_{\vec{n}}\phi|_\Sigma)\,\middle|\,\Sigma \in \mathscr{P}_C\right\}$. In fact:

$$\mathbf{s}_{\Sigma'}\big([(a',b')] + [(c',d')]\big) = \mathbf{s}_\Sigma\big([(a,b)] + [(c,d)]\big) = \mathbf{s}_\Sigma\big([(a,b)]\big) + \mathbf{s}_\Sigma\big([(c,d)]\big) =$$
$$= \mathbf{s}_{\Sigma'}\big([(a',b')]\big) + \mathbf{s}_{\Sigma'}\big([(c',d')]\big) = \mathbf{s}_{\Sigma'}\big([(a',b')] + [(c',d')]\big)$$

$\square$

Corollary 1.2.1 *The function* $: \mathrm{Data}(\Sigma) \to \mathrm{Sol}$ *which map every equivalence class to the associated solution is linear and bijective.*

1.2.2   Finite Degree systems

1.3   Geometric mechanics of Finite Degree systems

1.3.1   Linear dynamical systems

1.4 Peierls Brackets

The Peierls's construction algorithm is well defined for a specific class of systems:

1. Linear field theory $E = (E, \pi, M)$

2. Lagragian dynamics $P = Q_{\mathscr{L}}$.

3. M is a globally Hyperbolic space-time.

4. motion operator P is a green-hyperbolic L.P.D.O.

Observation 7: Peierls Bracket vs Poisson Bracket

⚠ [a] Paraphrasing an observation made by Sharan[**?**]:

The Poisson bracket determines how one quantity b(t, q, p) changes another quantity a(t, q, p) when it acts as the Hamiltonian or vice-versa. The Peierls bracket, on the other hand, determines how one quantity b(t, q, p) when added to the system Hamiltonian h with an infinitesimal coefficient Îż affects changes in another quantity a(t, q, p) and vice-versa, i.e. The Peierls bracket is related to the change in an observable when the trajectory on which it is evaluated gets shifted due to an infinitesimal change in the Lagrangian of the system by another Lagragian density.

While the Poisson bracket between two observables a and b is defined on the whole phase space and is not dependent on the existence of a Hamiltonian, the Peierls bracket refers to a specific trajectory determined by a governing Lagrangian.

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[a]da aggiustare

Purpose of the Peierls' procedure is to provide a bilinear form on the space of Lagrangian densities Lag. This form induces a pre-symplectic structure on the space of Lagrangian Functionals which can be easily reduced to the subclass of suitable classical observable. The algorithm can be summarized in a few steps

1. Consider a *disturbance χ* that is a time-compact lagrangian density .

2. Construct the perturbation of a solution under the disturbance.

3. Define the effect of the disturbance on a second lagrangian functional.

4. Assemble two Effect operators to give a bracket.

Disturbed motion operator   Consider a perturbation by $χ \in$ Lag of the system's lagrangian:

$$\mathscr{L} \rightsquigarrow \mathscr{L}' = \mathscr{L} + \epsilon \cdot χ$$

where ϵ is a modulation parameter. The motion operator of the disturbed dynamics results:

$$P_\epsilon = \left[\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathscr{L}}{\partial \phi} \right] + \epsilon \left[\partial_\mu \left(\frac{\partial χ}{\partial(\partial_\mu \phi)} \right) - \frac{\partial χ}{\partial \phi} \right] = P + \epsilon Q_χ \tag{1.3}$$

⚠ sono autorizzato ad affermare che P_ϵ ÃÍ lineare?

Solution of the disturbed motion as a perturb By we mean a solution $\phi' \in \mathtt{C}$ of P_ϵ obtainable by a linear perturbation of a fixed solution $\phi \in \mathtt{Sol}$.

1.4.1 Extension to non-linear theories

(Nel caso di teorie non lineari, come la geodetica, posso tentare di applicare questo criterio alle linearizzazioni!)

1.4.2 Finite Dimensional case

1.5 Dubbi

- quando parlo della cinematica mi piacerebbe dare indicazioni sulla struttura matematica dello spazio delle configurazioni cinematiche:

  1. costituisce una frechet manifold (gli unici risultati che ho trovato sono quelli di Palais di "non linear global analysis"

  2. le curve parametrizzate sono le variazioni

  3. classi di equivalenza definiscono delle variazioni infinitesime che costituiscono lo spazio tangente allo spazio delle configurazioni cinematiche

  4. questo spazio tangente ÃÍ isomorfo allo spazio delle sezioni del pullback rispetto alla sezione $\phi \in C$ del verical bundle (vedere forger romero)

  5. il problema dell'atlante e della rappresentazione delle sezioni in carta locale (da scegliere sia sul total space E che sul base space M)

- fare riferimento al teorema di Ostrowsky per giustificare il fatto che consideriamo solo il primo ordine. le langrangiana con termini cinetici esotici sono instabili (nel senzo che non ammetto come soluzioni sezioni globali ma solo locali).

- Devo mettere la trafila di definizioni? direi di no, qualcosa va detto qualcosa va messo in footnote

  (sono ripetizioni inutili per la tesi, sono informazioni che si ritrovano ovunque... sono informazioni adatta al knowledge base)

  Recurring definitions in general Relativity (excluding the general smooth manifold prolegomena).

Definition 9: Space-Time

A quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ such that:

- (M, g) is a time-orientable n-dimensional manifold ($n > 2$)
- $\mathfrak{o}$ is a choice of orientation
- $\mathfrak{t}$ is a choice of time-orientation

Definition 10: Lorentzian Manifold

A pair (M, g) such that:

- M is a n-dimensional ($n \geq 2$), Hausdorff, second countable, connected, orientable smooth manifold.
- g is a Lorentzian metric.

Definition 11: Metric

A function on the bundle product of TM with itself:

$$g : TM \times_M TM \to \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a non-degenerate bilinear form.

Notation fixing

- *Riemman* if the sign of g is positive definite, *Pseudo-Riemman* otherwise.
- *Lorentzian* if the signature is $(+, -, \ldots, -)$ or equivalently $(-, +, \ldots, +)$.

Observation 8: Causal Structure

If a smooth manifold is endowed with a Lorentzian manifold of signature $(+, -, \ldots, -)$ then the tangent vectors at each point in the manifold can be classed into three different types.

Notation fixing

$\forall p \in M, \quad \forall X \in T_pM$, the vector is:

- *time-like* if $g(X, X) > 0$.
- *light-like* if $g(X, X) = 0$.
- *space-like* if $g(X, X) < 0$.

Observation 9: Local Time Orientability

$\forall p \in M$ the timelike tangent vectors in p can be divided into two equivalence classes taking

$$X \sim Y \text{ iff } g(X, Y) > 0 \qquad \forall X, Y \in T_p^{\text{time-like}}M :$$

We can (arbitrarily) call one of these equivalence classes "future-directed" and call the other "past-directed". Physically this designation of the two classes of future- and past-directed timelike vectors corresponds to a choice of an arrow of time at the point. The future- and past-directed designations can be extended to null vectors at a point by continuity.

Definition 12: Time-orientation

A global tangent vector field $\mathfrak{t} \in \Gamma^\infty(TM)$ over the Lorenzian manifold M such that:

- $\text{supp}(\mathfrak{t}) = M$
- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Observation 10

The fixing of a time-orientation is equivalent to a consistent smooth choice of a local time-direction.

Definition 13: Time-Orientable Lorentzian Manifold

A Lorentzian Manifold (M, g) such that exist at least one time-orientation $\mathfrak{t} \in \Gamma^\infty(TM)$.

Notation fixing

Consider a piece-wise smooth curve $\gamma : \mathbb{R} \supset I \to M$ is called:

- *time-like* (resp. light-like, space-like) iff $\dot\gamma(p)$ is time-like (resp. light-like, space-like) $\forall p \in M$.
- *causal* iff $\dot\gamma(p)$ is nowhere spacelike.
- *future directed* (resp. past directed) iff is causal and $\dot\gamma(p)$ is future (resp. past) directed $\forall p \in M$.

Definition 14: Chronological $^{\text{future}}_{\text{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{I}_M^\pm(p) := \left\{ q \in M \,\big|\, \exists \gamma \in C^\infty\big((0,1), M\big) \text{ time-like } {}^{\text{future}}_{\text{past}}\text{-directed} : \gamma(0) = p,\, \gamma(1) = q \right\}$$

Definition 15: Causal $^{\text{future}}_{\text{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{J}_M^\pm(p) := \left\{ q \in M \,\big|\, \exists \gamma \in C^\infty\big((0,1), M\big) \text{ causal } {}^{\text{future}}_{\text{past}}\text{-directed} : \gamma(0) = p,\, \gamma(1) = q \right\}$$

Notation fixing

Former concept can be naturally extended to subset $A \subset M$:

  – $\mathbf{I}_M^{\pm}(A) = \bigcup_{p \in A} \mathbf{I}_M^{\pm}(p)$
  – $\mathbf{J}_M^{\pm}(A) = \bigcup_{p \in A} \mathbf{J}_M^{\pm}(p)$

Definition 16: Achronal Set

Subset $\Sigma \subset M$ such that every inextensible timelike curve intersect Σ at most once.

Definition 17: $^{\text{future}}_{\text{past}}$ Domain of dependence of an Achronal set

The two subset related to the generic achornal set $\Sigma \subset M$:

$$\mathbf{D}_M^{\pm}(\Sigma) := \left\{ q \in M \,\middle|\, \forall \gamma \, ^{\text{past}}_{\text{future}} \text{inextensible causal curve passing through } q : \gamma(I) \cap \Sigma \neq \emptyset \right.$$

Notation fixing

$\mathbf{D}_M(\Sigma) :- \mathbf{D}_M^{+}(\Sigma) \cup \mathbf{D}_M^{-}(\Sigma)$ is called *total domain of dependence.*

Definition 18: Cauchy Surface

Is a subset $\Sigma \subset M$ such that:

  – closed
  – achronal
  – $\mathbf{D}_M(\Sigma) \equiv M$

Basic Definition in L.P.D.O. on smooth vector sections.

Consider $F = F(M, \pi, V), F' = F'(M, \pi', V')$ two linear vector bundle over M with different typical fiber

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Definition 19: Linear Partial Differential operator (of order at most $s \in \mathbb{N}_0$)

Linear map $L : \Gamma(F) \to \Gamma(F')$ such that:
$\forall p \in M$ exists:

- (U, ϕ) local chart on M.
- (U, χ) local trivialization of F
- (U, χ') local trivialization of F'

for which:
$$L\big(\sigma\big|_U\big) = \sum_{|\alpha| \le s} A_\alpha \partial^\alpha \sigma \qquad \forall \sigma \in \Gamma(M)$$

Remark:

(multi-index notation)
A multi-index is a natural valued finite dimensional vector $\alpha = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^n$ with $n < \infty$.
On $\mathbb{R}^n$ a general differential operator can be identified by a multi-index:

$$\partial^\alpha = \prod_{\mu=0}^{n-1} \partial_\mu^{\alpha_\mu}$$

(Until the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|\alpha| := \sum_{\mu=0}^{n-1} \alpha_\mu$$

?????????????????????

Hp:

Proposition 1.5.1 (Existence and uniqueness for the Cauchy Problem) —
   $\mathbf{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ *a globally hyperbolic space-time.*

- $\Sigma \subset M$ *a spacelike cauchy surface with future-pointing unit normal vector field $\vec{n}$.*

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Th:

Observation 11

"Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. " [?] [?]
However the existence and uniqueness can be proved for the large class of the *Normally-Hyperbolic Operators.*