Introduction

Quantum Field Theory (QFT) is the synthesis of ~~Quantum Theory~~ and Special Relativity and ~~is~~ the general framework for the description of the physics of relativistic quantum systems. Its most direct applications, quantum electrodynamics ~~and~~ standard model of particles, are both been experimentally verified to an outstanding degree of precision and ~~allowed~~ us to have an almost fully satisfactory and unified description of the electro-weak forces.

In any case it is by no mean a definitive theory. It is clear that the intrinsic quantum description of elementary particles clash with the the structure of a deterministic theory, as ~~it is the~~ general relativity. It is almost unanimously accepted that a quantum theory of gravity is needed in order to reconcile general relativity with the principles of quantum mechanics. Yet, despite countless efforts, a quantum theory of the gravitational interaction remains an open problem

While quantum field theory has been a well established topic for the past 50 years, the quest to finding of a *theory of everything* has often lead the community to neglect the role of two important aspect concerning the QFT.

The first is the existence of an intermediate regime, ~~namely the~~ *QFT in curved background*, which is expected to provide an accurate description of quantum phenomena in ~~a~~ regions where the effects of curved spacetime may be significant, but effects of quantum gravity itself may be neglected. Many successful application of this idea can be found in context of the theory of cosmological inflation or black holes thermodynamic.

The second is the construction of a mathematically rigorous description of quantum field, in particular of ~~its~~ non-perturbative aspects, based on a sound and shared set of first principles. In other words an *axiomatic foundation of QFT*

At the moment, *algebraic quantum field theory* (AQFT) is ~~proven to be the most~~ promising way to complete the picture of the quantum theory of fields regarding ~~these~~ two aspects. Its aim is to reach a general and mathematical rigorous description of the foundations of quantum fields on a sufficiently large class of curved, but fixed, backgrounds.

The algebraic approach, as the majority of contemporary quantum field theory, is developed ~~as~~ quantization of classical fields. ~~From that should be clear that~~ a mathematically rigorous *classical field theory* is thus a necessary step towards the understanding of the foundations of the theory. ~~Has~~ to be noted that classical fields such the force fields of analytical mechanics or the ~~material~~ tensor in fluid dynamics are not of much interest ~~in what concerns to QFT~~. What is essential to determine is a proper definition of the (pre-quantum) classical analogue of the ~~"fundamentals~~" fields. Particularly, it is crucial the identification of the field-theoretic equivalent of the geometric structures underlying the canonical formalism of classical mechanics, namely the *phase space* and ~~*classical observables algebra*~~. From an abstract point of view the first is a symplectic manifold, namely a smooth manifold endowed with a non-degenerate 2-form, while the second is a Poisson algebra constituted by functionals on the phase space.

It is important to emphasize that the algebraic quantization is not a unique and well-defined algorithm that reads in a system of classical mechanics and returns a corresponding ~~system of quantum mechanics~~, ~~but rather should~~ be seen as a *quantization scheme* which can be realized by several specific procedures. All of them are based on a set set of first principles, essentially proposed by Dimock [**?**] as an extension of ~~axioms~~ of Haag and Kastler ~~stated~~ on Minkowski spacetime, ~~which prescribe~~ the mathematical structure of the quantum observables algebra. What in which such realizations of the algebraic scheme essentially differ, ~~it~~ is in the different identification of the suitable symplectic manifold to be associated with the pre-quantum version of the field under examination.

The most common way to building these structures requires an explicit choice of a time function even when no ~~one such~~ choice is natural. This leads to a realization of the algebraic quantization known as *quantization via initial data*.

Of much greater interest is the , ~~completely~~ covariant, construction based on the so-called *covariant phase space* ~~and~~ *Peierls brackets*. The first is defined as the *space of ~~dynamic~~ configurations*, *i.e.* the infinite-dimensional space of solutions ~~of~~ equations of motion, while the second is a particular choice of Poisson brackets ~~attributable to a field system.~~

The construction of such Poisson bracket is achieved following what we call the *Peierls' algorithm*, a procedure originally proposed by Rudolf E. Peierls in a seminal paper dated 1952 [**?**]. This is an effective, but rather convoluted, "recipe" to prescribe a pre-symplectic structure on the space of ~~dynamic~~ configurations~~,~~ Browsing through the literature, it is clear that the Peierls' construction never received particular attention since its ~~debut~~. This can be ascribed mainly to the lack of a convincing geometric interpretation which had the effect of limiting its reception often relegating its role to that of a mere "mathematical trick".

The aim of this thesis is to review the original Peierls' procedure in every single ~~step~~ adapting it to a more rigorous and modern mathematical formalism. To take a step closer to the comprehension of this object we study the well-known geodesic problem regarding it as a special case of a field-like system. Essentially this example is noteworthy from two aspect:

1. ~~Is~~ a system with discrete degrees of freedom. Its "field ~~conformations~~" are parametrized curves on a Riemannian manifold and in this sense ~~represent~~ an example complementary to the basic real scalar field.

2. This system is dynamically ruled by the well-known *geodesic equation*. A typical realization of the algebraic scheme requires to ~~pass~~ through the linearization of these, highly non-linear, equations of motion which ~~takes~~ the name of *Jacobi equations*. The solutions of these linearized equations, named *Jacobi*

> *fields*, are extensively studied from the point of view of differential geometry (where they are introduced as a tangent field over a geodesic variation) ~~but rarely are approached~~ as a field-like dynamical system.

Since the Jacobi fields ~~lends itself to~~ be quantized both according to the *Peierls procedure* ~~than~~ according the *initial data procedure*, we hope that the comparison between the two symplectic spaces thus obtained, allow us to give a new geometric insight on the original Peierls' method.

We briefly summarize the content of the thesis.

The first chapter is devoted to reviewing the mathematical framework underlying ~~to~~ the rigorous formulation of continuous classical system, starting point of every algebraic quantization ~~realizations~~. We begin by introducing the notion of *smooth bundles*, these are the suitable objects to encode the kinematical structure of a generic field system. Subsequently we define the notion of globally hyperbolic spacetime as the natural arena for the mathematical theory of hyperbolic (systems of) partial differential equations, ~~in~~ which the Cauchy problem is well posed. ~~Finally~~ we outline the theory of Green hyperbolic operators, the class of linear differential operators on a vector bundle to which the ~~Peierls brackets construction~~ as well as the corresponding quantization procedure applies.

Chapter 2 is dedicated to introducing the procedure ~~of construction of~~ the Peierls brackets. In the first part we make use of the mathematical language developed above in order to formalize the correct abstract mechanical system for which ~~it~~ the Peierls procedure is well defined. Furthermore we will show how the most familiar mechanical systems, namely the point particle and the ~~fundamental~~ fields over a spacetime, can be treated in a unified way as special cases of the aforementioned abstract system.

In the end we propose an extended version of the original Peierls' algorithm obtained combining the construction proposed in his paper[**?**] with some recent references, mainly [**?**][**?**][**?**][**?**][**?**]. Instead of ~~limit~~ ourselves ~~to the case of scalar field only,~~ we extend the step-by-step procedure proposed by Peierls to a large class of abstact mechanical systems, not necessarily linear.

In order to pursue the study of the system under investigation it is necessary to introduce the ~~scheme of algebraic~~ quantization. To this end, the third chapter is focused ~~to~~ presenting two realizations of the algebraic quantization scheme. The two are distinguished by the different construction of the pre-quantum symplectic space associated to the classical system.

The first one is based on the restriction of the Peierls brackets to the class of *classical observables*, the resulting symplectic form is sometimes prescribed axiomatically [**?**][**?**][**?**].

The second exploits the hyperbolicity ~~of equations~~ of motion of the system and it is known as *quantization* via the *initial data*[**?**].

In the last chapter we will apply all the formalism developed so far to the case of the geodesic field. We construct the Peierls bracket for such system and, as a concrete example, we carry out all calculations for the specific case of a FRW spacetime ~~spatially flat~~. Then we will construct two equivalent pre-quantum symplectic space related to the geodesic system following the step-by-step algorithm presented in chapter 3 . At last we propose a geometric ~~picture~~ of the whole Peierls construction.