Interpretazione Geometrica delle parentesi di Peierls nella quantizzazione algebrica del campo Geodetico.

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Abstract

La prima parte della tesi ÃÍ stata rivolta allo studio del framework matematico necessario per dare una formulazione rigorosa dei sistemi classici continui, punto di partenza di ogni schema di quantizzazione algebrica. Nello specifico viene fatta una digressione sui Fibrati Topologici e viene sfruttata la definizione di fibrato liscio per presentare l'approccio geometrico alla meccanica classica sia per sistemi a gradi di libertÃă finiti che continui.

Nella seconda parte viene presentato l'algoritmo di Peierls che rappresenta una âĂIJricettaâĂİ efficace per attribuire una struttura pre-simplettica allo spazio delle configurazioni dinamiche di un sistema qualunque. Dalla ricerca bibliografica ÃÍ evidente come questo strumento a partire dal suo esordio (nel 1952) fino ad oggi non abbia mai ricevuto particolare attenzione. Questo sembra dovuto soprattutto alla mancanza di una convincente interpretazione geometrica.
Per fare un passo verso la comprensione di questo oggetto viene studiato lâĂŹestremamente noto problema della geodetica vedendolo come un sistema campo. Emerge sin da subito come il calcolo delle parentesi Peierls per questo sistema sia legato intrinsecamente al problema del calcolo dei campi di Jacobi lungo una geodetica.

Nella terza parte vengono descritte due realizzazione dello schema di quantizzazione algebrico per i campi bosonici. La prima sfrutta le parentesi di Peierls mentre la seconda interviene sui dati iniziali della dinamica di campo.
Il campo di Jacobi si presta ad essere quantizzato secondo entrambe le prescrizioni. Confrontando le 2 forme simplettiche cosÃň ottenute si cerca di fornire nuovi tasselli per attribuire un'interpretazione geometrica al metodo originale di Peierls.

List of Symbols

Contents

Chapter 1

Mathematical Preliminaries

⚠ Le interazioni matematiche sono complesse e non triviali (vedi un po' di articoli di introduzione a AQFT per ispirarti)

Tendenzialmente le teorie quantistiche di campi moderne sono di Quantizzazione.. Quindi richiedono di specificare bene la struttura del campo classico (vedi intro di Mangiaratti shardashivly)

Gli strumenti matematici per raccontare la teoria dei campi classici sono essenzialmente 3: Fibrati, S-T G-H, LDOP e GHOP.

IN questo paper non ci soffermeremo sulle strutture del framework puramente quantistico (* algebre e quant'altro). Per un primer vedere articolo di Dappiaggi o Libro Adv aqft.

Diamo per scontato un background di base in Geometria differenziale e derivate esterne (algebre di Grassman? global calculus? non so come chiamarlo!).

Potrei avere la tentazione a provare ad usare un po' di linguaggio basilare delle categorie... la mia fonte ÃÍ Joy of Cat.

⚠ Stile: Intro lapiadaria ai 3 argomenti (bundle cinematica di campo, Glob iper stage per descrivere dinamica di tipo propagativo, Operatori tipo onda). Poi smitragliata di definizioni come faceva Penati.

1.1 Fiber Bundles

⚠intro

1.1.1 Formal Definition

Although it would be possible to present the concept *bundle* in a more general way through the language of categories, for our argument will be sufficient to consider only the case of *smooth bundles*.

Definition 1: Fiber Bundle

A *Fiber Bundle* consists in a 4-ple (E, M, Q, π) where:

- E, M, Q : smooth manifolds called respectively *Total Space, Base Space, Typical Fiber*.
- $\pi : E \to M$ continuous smooth function (called *Bundle Projection*)

Endowed with a *Local Trivialization*:

- $\forall x \in E\ \exists$ a couple (U, χ) (called *local trivialization*)
  - U : neighborhood of x
  - $\chi : \pi^{-1}(U) \to U \times Q$: diffeomorphism [a] [b]

  such that: $p_1 \cdot \chi = \pi|_{\pi^{-1}(p)}$.

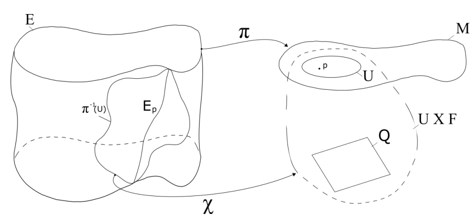
  i.e: the following graph commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\chi"] \arrow[d, "\pi"'] & U \times Q \arrow[dl, "p_1"] \\ U & \end{tikzcd}$$

[a] surjectivity $\Rightarrow \pi^{-1}(U) \neq \emptyset$.

[b] cartesian product of topological space is a topological space with the direct product topology.

Figure 1.1: The complete fiber bundle Structure.



Notation fixing

It is customary to refer to a vector bundle specifying only its total space:

$$E = (E, \pi, M; Q)$$

In the following we adopt this convention whenever this does not lead to misunderstandings.

Observation 1

For all $p \in M$ we refer to the submanifold $E_p := \pi_{-1}(p) \subset E$ as *fiber over the point p.*
Every fiber E_p is diffeomorphic to the typical fiber F through the local trivialization charts.

Notation fixing

We say that a smooth bundle E is *(globally) trivial* if $E \simeq M \times Q$ i.e there exists a trivialization of E which is defined everywhere. Note that definition 1 prescribes the existence of local trivializations only.

When a smooth fiber bundle $(E, \pi, M; Q)$ is considered, in addition to the typical functions of the bundle (π, χ_α) should be taken in account also the local charts $(U_{\alpha_k}, \phi_{\alpha_k})_{k=E,M,Q}$ provided by the atlases of E, M and Q:

Definition 2: Bundle atlas

Collection local chart which trivializes E. I.e. triples $(U_\alpha, \psi_\alpha, \chi_\alpha)$ where:

- U_α open set in M such that $\bigcup_\alpha U_\alpha \supseteq M$.
- χ_α is a local trivialization.
- (U_α, ψ_α) local chart on E constructed from charts on the base and fiber manifold:
$$\psi_\alpha^{(E)} = \psi_\alpha^{(M)} \times \psi_\alpha^{(Q)} \circ \chi_\alpha$$

Endowing the bundles manifolds with other additional structures, can be introduced important subclasses of smooth bundles:

Definition 3: Vector Bundle

Is a smooth bundle $E = (E, \pi, M; V)$ such that:

- The typical fiber manifold V is a finite dimensional vector space.
- All the trivialization χ_α are diffeomorphism such that:

$$\chi_\alpha|_{\pi^{-1}(p)} \in \mathbb{GL}(n, \mathbb{R})$$

1.1.2 Cross Sections

The notion of bundle is particularly interesting from the perspective of physics because provides the rigorous description of a $Q-$valued field over the space M:

Definition 4: Smooth Section

Function $\phi : M \to E$ such that:

- ϕ smooth.
- $\phi \cdot \pi = \mathbb{1}_M$

Notation fixing

We refer to:

- *Global section* $\Leftrightarrow$ $\mathrm{dom}(\phi) = B$
- *Local section* $\Leftrightarrow$ $\mathrm{dom}(\phi) \subset B$ [a]

We denote the set of all the smooth sections of the bundle E as:

$$\Gamma^{\infty}(E)$$

[a] Usually the domain is an open set of B)

Observation 2

In general, $\Gamma^{\infty}(E)$ is an infinite dimensional manifolds. In case of vector bundle is also a linear space, and the section are called "vector fields".

1.1.3 Mapping between Bundles

Consider two smooth bundles $E = (E, \pi, M; Q)$ and $E' = (E', \pi', M; Q')$ on the same base space M.

Definition 5: Bundle map (*Fiber Preserving map*)

Smooth function $\phi : E \to E'$ such that:

$$\phi(E_x) = F_x \qquad \forall x \in M.$$

i.e.:

$$\begin{tikzcd}
E \arrow[rr, "\phi"] \arrow[dr, "\pi_E"'] & & F \arrow[dl, "\pi_F"] \\
& M &
\end{tikzcd}$$

Observation 3

Definition 5 is a special case of *Bundle-morphism.* (see for example [8])

Consider a smooth manifold N, a (smooth) fiber bundle $E = (E, \pi, M; Q)$, and a smooth function $f : N \to M$. It's possible to induce[**?**] a bundle structure from the manifold M to N:

Definition 6: Pull-Back Bundle

3-ple $f^*(E) = (f^*(E) =, \pi^*, N)$ such that:

- $f^*(E) = \{(b', e) \in N \times E \quad | \, f(b') = \pi(e)\}$
- $\pi^* : f^*(E) \to N$ such that $\pi * (b', e) = \mathrm{pr}_1(b', e) = b'$

$$\begin{tikzcd}
f^*E \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\
N \arrow[r, "f"] & M
\end{tikzcd}$$

Proposition 1.1.1 *$f^*(E) = (f^*(E) =, \pi^*, N)$ consitute a fiber bundle of typical fiber Q.*

Proof:

To complete the fiber bundle structure is sufficient to provide a local trivialization atlas.
$\forall (U, \phi)$ local trivialization on (E, π, M) consider $\psi : f^*E \to N \times Q$ such that $\psi(b', e) = \Big(b', pr_2\big(\phi(e)\big)\Big)$.
Then $(f^{-1}(U), \psi)$ is a local trivialization of the pull-back bundle and the fiber of f^*E over a point bâĂš $\in B'$ is just the fiber of E over $f(b$âĂš$)$.

$\square$

It is also noteworthy that, given any two vector bundles $E = (E\pi, M, Q)$ and $E = (E'\pi, M', Q')$, we can construct naturally a third fiber bundle. Consider $\hom(E, E')$ the set of all the fiber preserving map between the two bundles:

Definition 7: Bundle of morphisms

Fiber bundle $\hom(E, E')$ over the base space M such that the fiber over a base point $p \in M$ is the infinite dimensional manifold $\hom(E_p, E'_p)$ isomorphic to $\hom(Q, Q')$.

Notation fixing

We shall write $End(E)$ for $\hom(E,E)$ and call it bundle of endomorphism, whose typical fiber is $\mathrm{End}(Q)$.

Remark:

If F, F' are vector bundle then the fiber of $\hom(F,F')$ over a base point $p \in M$ is $\hom(F_p, F'_p)$, which is a vector space isomorphic to the vector space $\hom(V,V')$ of linear applications from V to V'

1.1.4 Tangent Bundles

The *tangent* bundle is a natural structure defined on any smooth manifold, represent the canonical example of non-trivial vector bundle.

Definition 8: Tangent Bundle

The smooth vector bundle $TM = (TM, \tau, M; \mathbb{R}^m)$ such that:

- The total space is the union of all tangent spaces to

$$M : TM := \sqcup_{p \in M} T_p M \equiv \bigcup_{x \in M} x \times T_x M$$

- The bundle projection maps each tangent vector $v \in T_p M$ to the correspondent base point p;

$$\tau : (p, v_p) \mapsto p$$

Observation 4

In a similar fashion it's possible to construct a vector bundle relatively to any tensor product of the tangent spaces. i.e.:

- *Cotangent Bundle* $T^* M$ is build by disjoint union of the dual tangent space:

$$T_p^* M \; \forall p \in M$$

- *Tensor Bundle* $T^{(k,l)} M$ is build by disjoint union of the tensor product of tangent space with itself:

$$T_p^{(k,l)} M = \underbrace{T_p^* M \otimes \cdots \otimes T_p^* M}_{\text{k-times}} \otimes \underbrace{T_p M \otimes \cdots \otimes T_p M}_{\text{l-times}}$$

- *k-forms Bundle* $\bigwedge^m(T^*M)$ is build by disjoint union of the antisimmetrized tensor product of the dual tangent space with itself.

Tautological one-form and simplectic form.

Notation fixing

In the context of Classical mechanics is customary to refer to the cotangent bundle T^*Q over the smooth manifold Q - called *Configuration Space* - as *Phase Space*.

Since TQ and T^*Q are diffeomorphic , it might seem that there is no particular reason in treating this two spaces separately, but it is not so. There are certain geometrical objects that live naturally on T^*Q , not on TQ.
Of greatest interest in mathematical-physics are the PoincarÃÍ forms[**?**].

Consider a smooth manifold Q and call $\mathscr{M} = T^*Q$ the corresponding cotangent bundle.

Definition 9: Tautological (PoincarÃÍ) 1-form

Is the 1-form over $\mathscr{M}$:

$$\theta_0 \in \Gamma^\infty(T^*\mathscr{M})$$

such that the action on a generic point $\omega_{\alpha_p} \in T_{\alpha_p}M$ (in the fiber of α_p, which in turn is a one-form on the fiber of $p \in Q$) is given by:

$$\theta_0(\alpha_p) : T_{\alpha_p}\mathscr{M} \to \mathbb{R} \qquad : \omega_{\alpha_p} \mapsto \alpha_q \circ T\tau^*_Q(\omega_{\alpha_p})$$

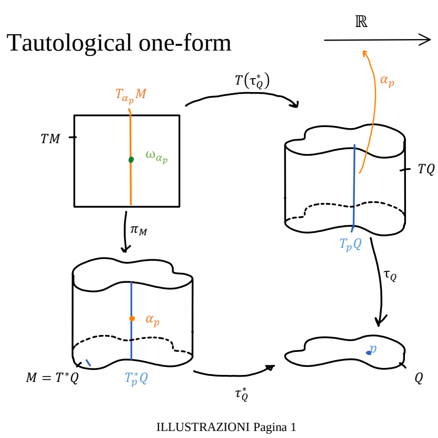


Figure 1.2: The definition of tautological 1-form is achieved exploiting the concept of *Tangent map* and remembering that $\alpha_p : T_p\mathscr{M} \to \mathscr{M}$ is a linear functional.

Notation fixing

Canonical coordinates are defined as a special set of coordinates on the cotangent bundle of a manifold. They are usually written as a set of (q^i, p_j) where q_i are denoting the coordinates on the underlying manifold and the p_j are denoting the conjugate momentum, which are decomposition of 1-forms in $T_p^* M$ on the dual natural basis dq^J in the cotangent bundle at point q in the manifold.

Observation 5

In canonical coordinate the tautological one-form assumes the famous expression:

$$\theta_0 = \sum_{i=1}^{n} p_i dq^i$$

(note that dq^i is a 1-form on $T^* M$ calculated with respect to the coordinate on the bundle. Has not to be confused with the 1-natural form $dq^i \in T_p^* M$.)

The claim is proved by the following definition:

Definition 10: Canonical (PoincarÃĺ) symplectic form

Symplectic form:

$$\omega_0 := -d\theta_0$$

In canonical coordinates assumes the famous expression:

$$\omega_0 := \sum_{i=1}^{n} dq^i \wedge dp_i$$

1.1.5 Jet Bundles

Ob(A) hom(A, B) *Mor coddomran*

1.2 Globally Hyperbolic Space-times

⚠ Mettere solo le definizioni che uso prese dagli articoli di review delle Fonti

Appunti che mi ero preso scrivendo il secono capitolo:

This condition is strictly connected to the dynamic behaviour of the system.

⚠ Def di dominio di dipendendenza footnote di definizione di spazio tempo def cauchy surface Remark causal future past def globally hyperbolic Teo-

⚠ rema sulle caratterizzazioni

Notation fixing

We denote the set of all the cauchy surfaces as $\mathscr{P}_C(M)$.

Glon iperbolic determina la fogliazione dello spazio tempo per superfici di cauchy La superficie di cauchy ÃÍ questa:

Definition 11: Cauchy surface

questo da la possibilitÃă della buona posizione dei problemi di cauchy.. fisicamente ÃÍ la condizione minima per definire i dati iniziali dell'evoluzione dinamica. definisco data...

⚠ Rapporto con la condizione sugli operatori...

No! La definizione di green hyperbolicity non garantisce invece l'esistenza e unicitÃă del problema di cauchy associata

e non solo, anche l'esistenza degli operatori di green associati che sono ingrediente fondamentale della costruzione di peierls

M ÃÍ glob iper e P ÃÍ green iper per tener conto del comporatamento propagativo definire sup cauchy definire s-t iperbolico (solo la caratterizzazione di ammetre una sup di cauchy) definire op green iperbolico su spazio tempo iperbolico (cioÃÍ ha delle green ope) Propr di buona definizione esistenza e unicita della soluzione

Di particolare ricorrenza fisica sono gli operatori normally iperbolic espressione in coordinate esempio K-g!

⚠ Far notare che minkowski e tanti esempi importanti sono GH

Observation 6

(che serve dopo) lo spazio R ÃÍ banalmente iperbolico in quanto tutti i punti posso essere visti come superfici di cauchy.

(sono ripetizioni inutili per la tesi, sono informazioni che si ritrovano ovunque... sono informazioni adatta al knowledge base)

Recurring definitions in general Relativity (excluding the general smooth manifold prolegomena).

Definition 12: Space-Time

A quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ such that:

- (M, g) is a time-orientable n-dimensional manifold ($n > 2$)
- $\mathfrak{o}$ is a choice of orientation
- $\mathfrak{t}$ is a choice of time-orientation

Definition 13: Lorentzian Manifold

A pair (M, g) such that:

- M is a n-dimensional ($n \geq 2$), Hausdorff, second countable, connected, orientable smooth manifold.
- g is a Lorentzian metric.

Definition 14: Metric

A function on the bundle product of TM with itself:

$$g : TM \times_M TM \to \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is a non-degenerate bilinear form.

Notation fixing

- *Riemman* if the sign of g is positive definite, *Pseudo-Riemman* otherwise.
- *Lorentzian* if the signature is $(+,-,\dots,-)$ or equivalently $(-,+,\dots,+)$.

Observation 7: Causal Structure

If a smooth manifold is endowed with a Lorentzian manifold of signature $(+,-,\dots,-)$ then the tangent vectors at each point in the manifold can be classed into three different types.

Notation fixing

$\forall p \in M, \quad \forall X \in T_pM$, the vector is:

- *time-like* if $g(X,X) > 0$.
- *light-like* if $g(X,X) = 0$.
- *space-like* if $g(X,X) < 0$.

Observation 8: Local Time Orientability

$\forall p \in M$ the timelike tangent vectors in p can be divided into two equivalence classes taking

$$X \sim Y \text{ iff } g(X,Y) > 0 \qquad \forall X, Y \in T_p^{\text{time-like}}M:$$

We can (arbitrarily) call one of these equivalence classes "future-directed" and call the other "past-directed". Physically this designation of the two classes of future- and past-directed timelike vectors corresponds to a choice of an arrow of time at the point. The future- and past-directed designations can be extended to null vectors at a point by continuity.

Definition 15: Time-orientation

A global tangent vector field $\mathfrak{t} \in \Gamma^\infty(TM)$ over the Lorenzian manifold M such that:

- $\operatorname{supp}(\mathfrak{t}) = M$
- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Observation 9

The fixing of a time-orientation is equivalent to a consistent smooth choice of a local time-direction.

Definition 16: Time-Orientable Lorentzian Manifold

A Lorentzian Manifold (M,g) such that exist at least one time-orientation $\mathfrak{t} \in \Gamma^\infty(TM)$.

Notation fixing

Consider a piece-wise smooth curve $\gamma : \mathbb{R} \supset I \to M$ is called:

- *time-like* (resp. light-like, space-like) iff $\dot{\gamma}(p)$ is time-like (resp. light-like, space-like) $\forall p \in M$.
- *causal* iff $\dot{\gamma}(p)$ is nowhere spacelike.
- *future directed* (resp. past directed) iff is causal and $\dot{\gamma}(p)$ is future (resp. past) directed $\forall p \in M$.

Definition 17: Chronological $\substack{\textbf{future}\\\textbf{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{I}^{\pm}_{M}(p) := \left\{q \in M \middle| \, \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ time-like } \substack{\text{future}\\\text{past}} - \text{directed } : \gamma(0) = p,\, \gamma(1) = q\right\}$$

Definition 18: Causal $\substack{\textbf{future}\\\textbf{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{J}^{\pm}_{M}(p) := \left\{q \in M \middle| \, \exists \gamma \in C^{\infty}\big((0,1), M\big) \text{ causal } \substack{\text{future}\\\text{past}} - \text{directed } : \gamma(0) = p,\, \gamma(1) = q\right\}$$

Notation fixing

Former concept can be naturally extended to subset $A \subset M$:

- $\mathbf{I}^{\pm}_{M}(A) = \bigcup_{p \in A} \mathbf{I}^{\pm}_{M}(p)$
- $\mathbf{J}^{\pm}_{M}(A) = \bigcup_{p \in A} \mathbf{J}^{\pm}_{M}(p)$

Definition 19: Achronal Set

Subset $\Sigma \subset M$ such that every inextensible timelike curve intersect Σ at most once.

Definition 20: $\substack{\textbf{future}\\\textbf{past}}$ Domain of dependence of an Achronal set

The two subset related to the generic achornal set $\Sigma \subset M$:

$$\mathbf{D}^{\pm}_{M}(\Sigma) := \left\{q \in M \middle| \, \forall \gamma \ \substack{\text{past}\\\text{future}} \text{inextensible causal curve passing through } q : \gamma(I) \cap \Sigma \neq \emptyset\right\}$$

Notation fixing

$\mathbf{D}_{M}(\Sigma) :- \mathbf{D}^{+}_{M}(\Sigma) \cup \mathbf{D}^{-}_{M}(\Sigma)$ is called *total domain of dependence*.

Definition 21: Cauchy Surface

Is a subset $\Sigma \subset M$ such that:

- closed
- achronal
- $\mathbf{D}_M(\Sigma) \equiv M$

1.3 Green Hyperbolic Operators

⚠ Mettere solo le definizioni che uso prese dagli articoli di review delle Fonti

⚠ Pensavo di utilizzare la definizione di Green hyperbolic data da Bar che si avvale del concetto di formally dual (che non richiede la presenza del pairing) invece di quella usata in Advances AQFT che richiede solo che ammetta almeno un $G^{\pm}$ per poi dimostrare tramite teorema che se ÃÍ anche autoaggiunto vale l'unicitÃă. Si tratta solo di una piccola sfumatura.. Deve essere chiarito che in tutto ciÃš che faccio interessano che

$$\forall P \exists 1! G^{\pm}$$

. Che poi questa condizione derivi da GH secondo bar o Gh secondo dap+selfadj ÃÍ una di quelle questioni propriamente matematiche che poco interessa ai fisici della commissione.

⚠ Devo richiedere che il green operator sia unico? sia negli schemi di quantizzazione che nella definizione di peierls faccio largo uso dell'unicitÃă. Per provare questa unicitÃă si passa per la definizione di una forma bilineare che permette di parlare di aggiunto formale e quindi avvalersi del teorema.

Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. [**?**] [2]

(sono ripetizioni inutili per la tesi, sono informazioni che si ritrovano ovunque... sono informazioni adatta al knowledge base) Basic Definition in L.P.D.O. on smooth vector sections.
Consider $F = F(M,\pi,V), F' = F'(M,\pi',V')$ two linear vector bundle over M with different typical fiber

Definition 22: Linear Partial Differential operator **(of order at most $s \in \mathbb{N}_0$)**

Linear map $L : \Gamma(F) \to \Gamma(F')$ such that:
$\forall p \in M$ exists:

- (U, ϕ) local chart on M.
- (U, χ) local trivialization of F
- (U, χ') local trivialization of F'

for which:

$$L(\sigma\big|_U) = \sum_{|\alpha| \le s} A_\alpha \partial^\alpha \sigma \qquad \forall \sigma \in \Gamma(M)$$

Remark:

(multi-index notation)
A multi-index is a natural valued finite dimensional vector $\alpha = (\alpha_0, \dots, \alpha_n - 1) \in \mathbb{N}_0^n$ with $n < \infty$.
On $\mathbb{R}^n$ a general differential operator can be identified by a multi-index:

$$\partial^\alpha = \prod_{\mu=0}^{n-1} \partial_\mu^{\alpha_\mu}$$

(Until the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|\alpha| := \sum_{\mu=0}^{n-1} \alpha_\mu$$

?????????????????????

Hp:

Proposition 1.3.1 (Existence and uniqueness for the Cauchy Problem)

- $\mathbf{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ *a globally hyperbolic space-time.*
- $\Sigma \subset M$ *a spacelike cauchy surface with future-pointing unit normal vector field* $\vec{n}$.
-

Th:

Observation 10

"Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense

and that they cannot be characterized in general by well-posedness of a Cauchy problem. " [?] [2]
However the existence and uniqueness can be proved for the large class of the *Normally-Hyperbolic Operators*.

Chapter 2

Lagrangian Systems and Peierls Brackets

In this chapter we will put to good use all the effort invested in the study of mathematical methods for classical field to provide a good notion of Abstract Mechanical system. *Abstract* in the sense that we rely on a refined definition, sufficiently broad to encompass mechanical systems with degrees of freedom both discrete and continuous.
Taking advantage of this language will be possible to formalize precisely each step of the original Peierls' procedure[**?**] and thus establish the class of applicability of this algorithm.

2.1 Abstract Mechanical Systems

It's possible to state a mathematical definition sufficiently broad to encode all the classical mechanical systems regardless of the cardinality of degrees of freedom in a unified way.

Definition 23: Abstract Evolutive System

Pair (E, P) composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on manifold M, called *"configuration bundle"*.
- $P : \Gamma^{\infty}(E) \to \Gamma^{\infty}(E)$
  operator called *"motion operator"*

This formulation is still very distant from the physical interpretation but has the benefit to highlight the minimal mathematical objects which must be fixed in order to specify a mechanical systems.

Kinematics The configuration bundle encompass all the kinematical structure of the system. A pivotal role is played by the smooth sections which are to be understood as all the possible conformation of the system.

Notation fixing

$$\mathtt{C} := \Gamma^{\infty}(M, E)$$

Space of kinematic configurations.

A section is not a statical configuration, equivalent to a specific point in the configuration space of ordinary classical systems, but has to be seen as a specific realization of the kinematics in the sense of a complete description of a possible motion. At this level of abstraction, since no space-time structure has been specified, terms like stasis and motion must be taken with care .The natural physical interpretation should be clearly manifested through the concrete realization of systems with discrete and continuous degree of freedom.

Observation 11: Mathematical structure

Mathematically speaking this set should be regarded as an infinite dimensional Manifold.
This framework provides a geometric characterization of the notion of variations as tangent vectors on the the space of kinematic configurations .[6]

Observation 12: Coordinate Representation

The choice of a chart atlas $\mathscr{A}(M)$ on the base space M and $\mathscr{A}(E)$ on the total space E provides a correspondence between each configuration $\gamma \in \mathtt{C}$ and family of smooth real functions $\{f_{\alpha\beta} : A_\alpha \subset \mathbb{R}^m \to \mathbb{R}^q\}$. The process is trivial:

$$\gamma \in \mathtt{C} \mapsto \{f_{A,U} = \psi_U \circ \gamma \circ \psi_A^{-1} | (A, \psi_A) \in \mathscr{A}(M), (U, \psi_U) \in \mathscr{A}(E)\}$$

Since the whole section is quite difficult to handle as a global object, is customary in field theory to work in the more practical local representation.

Observation 13: Further specification of the system's kinematics

The general formalism doesn't require any other structure to be carried forward. Additional structure on the fiber , the base or the whole bundle are to be prescribed in order to specify a precise physical model, e.g. the spin structure on E for the Dirac Field.[4]

Dynamics The operator P is the object that contains all the information about the dynamic evolution of the system. It has the role to select the dinamically compatible configuration among all the admissible kinematic configurations of $\mathtt{C}$, exactly as it happens in analytical mechanics where the dynamic equations shape the natural motions.

Notation fixing

Provided an equations of motion operator

$$P : \mathtt{C} \to \mathtt{C}$$

The space

$$\mathtt{Sol} := \ker(P) \subset \mathtt{C}$$

containing all the smooth solutions is called *"Space of Dynamical Configurations"*.

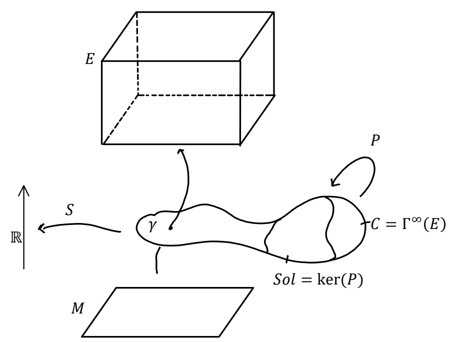


Figure 2.1: Mathematimatical framework of the mechanics of abstract systems.

2.1.1 Lagrangian Dynamics

Lagrangian systems constitute a subclass of the abstract mechanical systems of great interest:

Definition 24: Lagrangian System

Pair $(E, \mathscr{L})$ composed of:

- $E \xrightarrow{\pi} M$
  smooth fiber bundle of typical fiber Q on the oriented manifold $(M, \mathfrak{o})$ called *"configuration bundle"*.

- $\mathscr{L} : J^r E \to \wedge^m T^* M$
  bundle-morphism from the r-th Jet Bundle to the top-dimensionial forms bundle over the base manifold M called *"Lagrangian density"* or simply *"Lagrangian"* of r-th order.

N.B. : In what follows all the systems considered will be exclusively of first order.

In this case, is the Lagrangian density the object containing all the information about the dynamic evolution of the system.

In order to reconstruct the system's dynamic from the Lagrangian density has to be understood the mathematical nature of $\mathscr{L}$. $\mathscr{L}$ maps point q_p on the fiber $J^r_p E$ to a m-form on $T_p M$. Recalling the definition of jet bundles is clear that for each smooth section on E is associated a smooth section on $J^r E$:

$$\phi \in \Gamma^\infty(E) \mapsto (\phi, \partial_\mu \phi, \partial_{\mu,\nu}\phi, \dots \partial_{\vec{\alpha}}\phi)$$

where $\vec{\alpha}$ is a multi-index of length r. The correspondence is not univocal since sections equal up to the r-th order define the same jet section. The smoothness of $\mathscr{L}$ ensure that each jet bundle section is mapped to a smooth section in the top-forms bundle i.e. the most general integrable object on a orientable manifold.

It should be clear that $\mathscr{L}$ is a specific choice among the vast class of functions suitable to be a good Lagrangian density over the Configuration Bundle E:

Definition 25: Lagrangian Density on the bundle E

$$\mathsf{Lag}^r(E) := \hom\Big(J^r E, \quad \bigwedge^m (T^* M)\Big) \cong \{f : \Gamma^\infty(J^r E) \to \Omega^m(M)\}$$

(where $\Omega^m(M)$ is the common name for $\Gamma^\infty\big(\bigwedge^m (T^* M)\big)$ in the context of Grassmann algebras.) The equivalence states the fact that a bundle-morphism induce a mapping between the sections.

this choice fix the "Dynamical identity" of the considered system.

Proposition 2.1.1 $\mathsf{Lag}^r(E)$ *has an obvious vector space structure inherited by the linear structure of* $\Omega(M)$.

Thanks to the correspondence between a section $\phi \in \mathtt{C}$ and his r-th jet, it's possible to consider the Lagrangian as directly acting on the kinematic configurations. In layman terms the image $\mathscr{L}[\phi]\mathrm{d}\mu$, where $\mathrm{d}\mu$ is the measure associated to the orientation $\mathfrak{o}$, is something that can be measured over the whole base space.
This property suggests the introduction of the class of associated functionals:

Definition 26: Lagrangian functional

Is a functional on $\mathtt{C}$ with values on regular distribution over M associated to the generic $\mathscr{L} \in \mathsf{Lag}$.

$$\mathscr{O}_{\mathscr{L}} : \mathtt{C} \to \left(C_0^\infty(M)\right)'$$

Such that the lagrangian functional associated to $\mathscr{L}$, valued on the configuration $\phi \in \mathtt{C}$ and tested on the test-function $f \in C_0^\infty(M)$ it's given by:

$$\mathscr{O}_{\mathscr{L}}[\phi](f) = \int_M \mathscr{L}[\phi] f \, \mathrm{d}\mu$$

Proposition 2.1.2 *As a distribution $\mathscr{O}_{\mathscr{L}}[\phi](f)$ is necessarily linear in the test-functions entry but not in the configurations entry.*

Observation 14

The choice of the image of $\mathscr{O}_{\mathscr{L}}$ as a distribution it's a necessary precaution to ensure the *convergence* of the functional, whatever is the configuration on which is evaluated. In fact, despite $\mathscr{L}[\phi]$ is integrable with respect to the measure $\mathrm{d}\mu$, it's not necessary summable if the support of the configuration ϕ becomes arbitrarily large.
This is a simple consequence of the well known sequence of inclusions:

$$\mathscr{L}[\phi] \in C_0^\infty(M) \subset L^1_{\mathrm{loc}}(M,\mu) \supsetneq L^1(M,\mu)$$

of the functional analysis . Indeed, the functional

$$\mathscr{O}_{\mathscr{L}}[\phi] = \int_{\mathrm{supp}(\phi)} \mathscr{L}[\phi] \mathrm{d}\mu$$

is well defined for all $\mathscr{L} \in \mathsf{Lag}^r(E)$ only over the compactly supported sections. To take account of the global sections it's sufficient to dampen the integral smearing the integrand with an arbitrary test-function.

Notation fixing

The functional $\mathscr{O}_{\mathscr{L}}$, when calculated for the specific Lagrangian of the system, takes the name of *Action* or *Total Lagrangian*.

The introduction of the Lagrangian density is meaningless without the prescription of a dynamical principle which allows to determine univocally a motion operator P on the kinematics configurations space $\mathtt{C}$. This fundamental principle is the *least action principle*. A proper justification of this claim should require the presentation of the differential calculus on the infinite dimensional manifolds $\mathtt{C}$.
Jumping straight to the conclusion we can state this correspondence as a law which assign for all Lagrangian densities an operator on the kinematic configurations space. In the case of first order Lagrangian we define:

Definition 27: Euler-Lagrange operator

It's the differential operator

$$Q_\chi : \mathtt{C} \to \mathtt{C}$$

relative to the Lagrangian density $\chi \in \mathsf{Lag}^1(E)$, such that:

$$Q_\chi(\gamma) = \left(\partial_\mu \left(\frac{\partial \chi}{\partial(\partial_\mu \phi)} \Big|_\gamma \right) - \frac{\partial \chi}{\partial \phi} \Big|_\gamma \right) \qquad \forall \gamma \in \mathtt{C} \tag{2.1}$$

($\frac{\partial \chi}{\partial(\partial_\mu \phi)}$ has the be intended as the lagrangian density constructed differentiating $\chi(\phi, \partial_\mu)$ as an ordinary function, treating its functional entries as an usual scalar variable.)

Observation 15

The whole theory of both Lagrangian densities class and Euler-Lagrange equation could be stated in a more syntetic way in terms of the Grassmann-graded variational bicomplex.[7][12]

2.2 Concrete Realization

In the previous section we claim that the abstract definition of Lagrangian systems is broad enough to encompass all the classical lagrangian systems with both discrete degrees of freedom, like particles, and continuous degree of freedom, like fluids or fields. Let' show two of the most significant examples.

2.2.1 Classical Field Theory

Basically a (classical) *Fields System* is nothing more than an abstract Field System (E, P) where the base space M is a suitable spacetime manifold[3]. At this stage the question about the Lagrangian nature of the dynamics is purely ancillary.
The idea of taking bundles on a space-time manifold is physically intuitive, kinematically speaking a fields configuration is simply the association of some element of the fiber Q for each point of the space-time M.

However, there are a few more requirements that are often prescribed in commonly studied field theories.

Linear Sistem Condition

- The configuration bundle $E \xrightarrow{\pi} M$ is a vector bundle.
- The motion operator P is a *L.P.D.O.*.

Even if it might make sense to speak of nonlinear fields in some more general context, this condition it's a necessary element in case some form of the *superposition principles* as to be taken in account. Obviously this hypothesis is not sufficient to

formulate the principle in the strong classical way, i.e.:"the response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually" mostly because only free systems can be considered at this stage and any statement about stimulus can make sense.
However the first condition assure that $\mathtt{C}$ is a vector space, in other words every linear combination of kinematic configuration it's still a kinematic configuration. This condition, in conjunction with the linearity of motion operator P, guarantees that also $\mathtt{Sol} = \ker(P)$ is a linear subspace.

Propagative Dynamic Condition

- The base manifold M is a Globally Hyperbolic Spacetime.
- The motion operator P is PDE-hyperbolic.

The first condition ensures the existence of Cauchy surfaces and therefore permits to state *Cauchy Problems* assigning an initial data on such regions. Furthermore, PDE-hyperbolicity of the motion operator P guarantees that for every Cauchy surface $\Sigma \subset M$ the corresponding initial data problem is well posed, that is:

$$\begin{cases} Pu = 0 \\ u = u_0 \\ \nabla_{\vec{n}} u = u_1 \end{cases} \tag{2.2}$$

admit a unique solution $u \in \Gamma(E)$ for all $(u_0, u_1) \in \Gamma(\Sigma) \times \Gamma(\Sigma)$.
This suggests the following definition:

Notation fixing

The set of all the smooth initial data which can be given on the Cauchy Surface Σ is: $\mathsf{Data}(\Sigma) := \left\{ (f_0, f_1) \middle| f_i \in \Gamma^\infty(\Sigma) \right\} \equiv \Gamma^\infty(\Sigma) \times \Gamma^\infty(\Sigma)$

Observation 16

$\mathsf{Data}(\Sigma)$ inherit the linear structure of its component $\Gamma^\infty(\Sigma)$.

In this term the well-posedness of the Cauchy problem can be stated as follow:

Proposition 2.2.1 *The map* $\mathbf{s} : \mathsf{Data}(\Sigma) \to \mathtt{Sol}$ *which assign to* $(u_0, u_1) \in \mathsf{Data}(\Sigma)$ *the unique solution of the cauchy problem 2.2 is linear and bijective.*

Since any solution, when restricted to a generic Cauchy surface Σ', determines another pair of initial data, i.e.:

$$\phi \equiv \mathbf{s}(\phi|_{\Sigma'}, \nabla_{\vec{n}'} \phi|_{\Sigma'}) \quad \forall \phi \in \mathtt{Sol}$$

we can define the set of initial data regardless of the particular Cauchy surface:

Definition 28: Set of smooth initial Data

$$\mathsf{Data} := \frac{\bigsqcup_{\Sigma \in \mathscr{P}_C(M)} \mathsf{Data}(\Sigma)}{\sim}$$

where $\sim$ is such that:

$$(f_0, f_1)|_\Sigma \sim (g_0, g_1)|_{\Sigma'} \Leftrightarrow \mathbf{s}(f_0, f_1) = \mathbf{s}(g_0, g_1)$$

Initial data, associated with different surface, are similar if they lead to the same solution.

Proposition 2.2.2 Data *is still a vector space.*

Proof:

It's sufficient to prove that:

$$[\phi_a + \phi_b] = [\phi_a] + [\phi_b]$$

where $[\phi] = \left\{(\phi|_\Sigma, \nabla_{\vec{n}}\phi|_\Sigma)\middle|\Sigma \in \mathscr{P}_C\right\}$. In fact:

$$\begin{aligned}\mathbf{s}_{\Sigma'}\big([(a', b')] + [(c', d')]\big) = \mathbf{s}_\Sigma\big([(a, b)] + [(c, d)]\big) = \mathbf{s}_\Sigma\big([(a, b)]\big) + \mathbf{s}_\Sigma\big([(c, d)]\big) = \\ = \mathbf{s}_{\Sigma'}\big([(a', b')]\big) + \mathbf{s}_{\Sigma'}\big([(c', d')]\big) = \mathbf{s}_{\Sigma'}\big([(a', b')] + [(c', d')]\big)\end{aligned}$$

$\square$

Corollary 2.2.1 *The function* $: \mathsf{Data}(\Sigma) \to \mathtt{Sol}$ *which map every equivalence class to the associated solution is linear and bijective.*

Observation 17

The Propagative dynamic condition is the main ingredient in the *initial data quantization* procedure[14].

Green-Hyperbolic Dynamic Condition

- The dynamic is generated by a Lagrangian, i.e. $P = Q_{\mathscr{L}}$.
- The motion operator P is a Green Hyperbolic L.P.D.O.

It's customary in Algebraic quantum field theory to identify quantizable systems through this condition. In fact this condition is a necessary hypothesis to carry on the Peierls construction and the brackets underlie to the definition of the classical symplectic structure.
We remark that Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense, therefore the last two condition are not equivalent.

Example: 1

It's straightforward to frame the well-known *Real Scalar Field on curved background* inside this abstract picture:

Configuration bundle	real scalar field: $F \equiv \mathbb{R}$ curved background: M is globally hyperbolic
Kinematic Configurations	$\mathtt{C} = C^{\infty}(M, \mathbb{R})$
Motion Operator	$P = \Box_M + m^2 + \xi R$ normally hyperbolic operator

Where P is the Klein-Gordon Operator: ξ and m^2 are two real numbers, R stands for the scalar curvature build out of g and $\Box_M := g^a b \nabla_a \nabla_b$ is the d'Alambert operator associated to the Levi-Civita connection.

Note that this system satisfies all the above three condition.

2.2.2 Finite Degree systems as a Field

With a little more effort is possible to see every ordinary mechanic system -with discrete degrees of freedom- as a special case of classical field.

Consider a lagrangian system (Q, L) with configuration space Q and Lagrangian function $L : TQ \to \mathbb{R}$.
Remembering the intuitive meaning of Q as set of all the statical conformation of the system, it's natural to read as kinematic configuration all and only the trajectories compatible with the constraints. In other words the space of kinematic configurations is the set of all the parametrized smooth curves on Q:

$$\mathtt{C} = C^{\infty}(\mathbb{R}, Q)$$

Therefore, the configuration bundle of this system is the trivial smooth bundle:

$$E = (\mathbb{R} \times Q, \pi, \mathbb{R})$$

of typical fiber Q, since the corresponding space of sections coincides to the required configuration space:

$$\Gamma^{\infty}(E) \equiv C^{\infty}(\mathbb{R}, Q)$$

Also the "field theoretic" picture of the dynamic is not difficult to achieve. It's enough to recall how the Euler-Lagrange of ordinary mechanical system can be derived from the *least action principle*:

> **Least Action Principle:** The natural motions of the mechanical system (Q, L) are the stationary points of the action functional:
>
> $$S[\gamma] = \int_{\mathbb{R}} L(t, \gamma^i, \dot{\gamma}^i) dt$$
>
> constructed from the Lagrangian of the system.

Remembering that we have interpreted the space $\mathbb{R}$ of the parameter t as the base manifold of the configuration bundle E, it's immediate to see S as the *Total Lagrangian* related to the lagrangian density:

$$\mathscr{L}[\gamma] := L \circ (\gamma)^{\uparrow} dt$$

obtained evaluating the ordinary Lagrangian on the lifted curve $\gamma^{\uparrow}$ (lift of $\gamma \in \mathtt{C}$ from Q to TQ).

Summing up, the mathematical structure of such mechanical system is enconded as follows:

Configuration bundle	Trivial bundle $E = Q \times \mathbb{R}$ Base manifold: $M = \mathbb{R}$
Kinematic Configurations	$\mathtt{C} = C^{\infty}(\mathbb{R}, Q)$
Lagrangian Density	$\mathscr{L}[\gamma] := \left(L \circ \gamma^{\uparrow}\right) dt = L(t, \gamma^i, \dot{\gamma}^i) dt$
Motion Operator	$P = Q_{\mathscr{L}} \equiv \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^i} L\right) - \frac{\partial}{\partial x^i} L$ normally hyperbolic operator (⚠?)

Considering that the space M is trivially globally hyperbolic, since every point $t \in \mathbb{R}$ is a genuine "Cauchy Surface", it's evident how this system can be seen as a special case of field theory: a "field of curves".

Observation 18

Unless the configuration space is a vector space, the corresponding field theory cannot be linear.

2.3 Hamiltonian Mechanics of Finite Degree systems

The picture of the systems with discrete degrees of freedom as a field brings out only a part of the typical mathematical ingredients proper of the *Geometric* approach to the classical mechanics

For example we have totally overlook to mention the hamiltonian formalism in the case of abstract fields. Even though it should be possible, in the spirit of what has been done for Lagrangian picture, to extend the canonical treatment to include systems with continuous degrees of freedom (see for example [**?**]),we will not expand this topic since the protagonist of this paper is essentially a single-particle system.

However in the next chapter we will need to realize a field-theoretic versions of some Canonical objects.
To fulfill these quantization procedures we will need to draw inspiration from their correspondent classical versions. Let's briefly review them in the finite dimensional case,

Phase Space We recalled in chapter 1 the definition of *Phase Space* in ordinary classical mechanics as the cotangent bundle T^*Q of the classical configuration space Q. We showed that every classical phase space is symplectic trough the natural

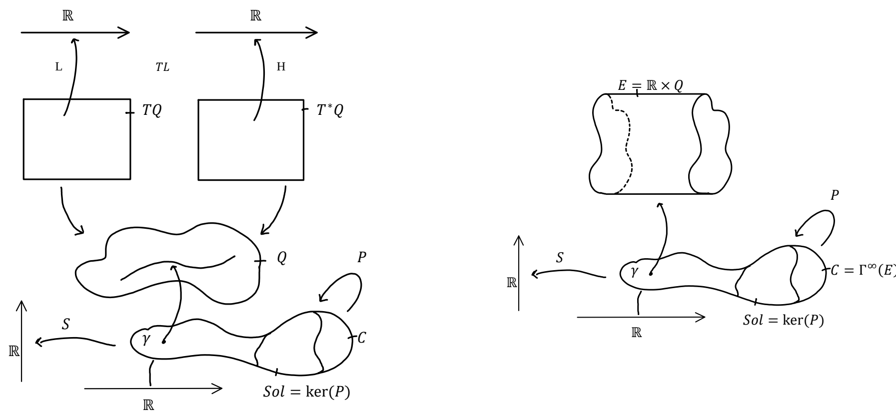


Figure 2.2: An "impressionistic comparison" between the mathematical framework of geometrical mechanics and the field-theoretic picture

PoincarÃÍ form. However, every quantization procedure require a modification of this standard symplectic form in order to implement the canonical commutation rules.

This leads us to make use of the abstract formulation of Hamiltonian systems[1]:

Definition 29: (Ordinary) Hamiltonian System

Triple $(\mathscr{M}, \omega, H)$ composed of:

- $(\mathscr{M}, \omega)$
  finite dimensional symplectic manifold called *"Phase space"*.
- $H : \mathscr{M} \to \mathbb{R}$
  smooth function called *"Hamiltonian"*

Observation 19

In classical mechanics Hamiltonian systems could be seen as a subset of Lagrangian systems.
The key is the definition of the Legendre Map $TL : TQ \to T^*Q$, in the case that the Lagrangian L is *hyperegular* - i.e. TL is a diffeomorphism - is possible to push-forward L to give a proper Hamiltonian on $\mathscr{M} = T^*Q$. (see for example [1])

Remember that there's an important theorem attributed to Darboux that state that, at least locally, every symplectic form can be represented in the canonical form:

Theorem 2.3.1 (Darboux)

Hp:

- *$\mathscr{M}$ is a 2n-dimensional smooth manifold.*
- *ω is a symplectic form on $\mathscr{M}$ i.e. a non degenerate closed ($d\omega = 0$) two-form.*

Th:

$\forall m \in \mathscr{M}, \exists$ a local chart (U, φ) (where $\varphi(u) = \left(x^1(u), \ldots, x^n(u); y^1(u), \ldots, y^n(u)\right)$) such that:

- $\varphi(m) = 0$
- $\omega\big|_U = \sum_{i=1}^n dx^i \wedge dy^i$

Proof:

See Theorem 3.2.2 [1].

□

Classical Observables Observables in classical mechanics are represented by real valued smooth function on $\mathscr{M}$:

Notation fixing

The *Classical Observables* space is denoted as:

$$\mathscr{E} \equiv C^\infty(\mathscr{M}, \mathbb{R})$$

Observation 20

Trivially, the space $C^\infty(\mathscr{M}, \mathbb{R})$ of smooth real valued function on $\mathscr{M}$, inherits the structure of commutive algebra over $\mathbb{R}$ from its codomain $\mathbb{R}$.

The symplectic structure on $\mathscr{M}$ give rise to a second algebraic structure on the vector space of observables. At first it is necessary to introduce the Hamiltonian fields:

Definition 30: Hamiltonian field with energy function $H \in \mathscr{E}$

Is the vector field $\mathbf{X}_H$ determined by the condition:

$$\omega(\mathbf{X}_H, \cdot) \equiv dH(\cdot)$$

Nondegeneracy of ω guarantees that $\mathbf{X}_H$ exists for all classical observables $H \in \mathscr{E}$. From that follows the definition of the bracket:

Definition 31: Poisson Bracket

Is the bilinear function $\{\cdot,\cdot\} : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ such that:

$$\{f,g\} := \omega\big(\mathbf{X}_f, \mathbf{X}_g\big) = df(\mathbf{X}_g) \tag{2.3}$$

In the canonical coordinates, provided by Darboux's theorem, Poisson bracket assumes there typical expression:

Proposition 2.3.1 (Symplectic char representation) *In canonical coordinates* $(q^1,\dots,q^{;} p_1,\dots,p_n)$ *we have:*

$$\{f,g\} = \sum_{i=1}^{n}\Big(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}\Big) \qquad \forall f,g \in \mathscr{E} \tag{2.4}$$

Proof:

See Corollary 3.3.14 [1].

□

Solution Space In this framework the Hamilton's equations of motions can be stated in term of the Hamiltonian Fields:

> **Hamilton dynamics principle** The dynamically possible motions of the Hamiltonian system $(\mathscr{M}, H)$ corresponds to the *integral curves*[1] of the hamiltonian vector field $X_H \in \Gamma(T\mathscr{M})$

Follows immediately that the specification of a point is appropriate initial data for determining a solution of Hamilton's equations of motion, i.e. each point $y \in \mathscr{M}$ give rise to a unique solution of the dynamical evolution. Therefore:

$$\mathscr{M} \cong \mathsf{Data} \cong \mathtt{Sol}$$

2.3.1 Linear dynamical systems

Most of the physical systems that are encountered in the theory of fields are linear. Of course is possible to come across linear dynamical systems also in ordinary mechanics.

Remark:

Linear Hamiltonian System

- $\mathscr{M}$ has a natural structure of vector space.
- H is a quadratic function on $\mathscr{M}$.[a]

[a]Equations of motion are then linear on the affine canonical coordinates. Dynamics is thus

simply a collection of coupled harmonic oscillators.

In this case the the difference between the underlying geometric entities tend to fade out as a consequence of the flatness of the configuration space.

The key consequence of the vector structure of $\mathscr{M}$ is that it allows us to identify the tangent space at any point $y \in \mathscr{M}$ with the Phase space $\mathscr{M}$ itself. Under this identification, the symplectic form became a bilinear function $\Omega : \mathscr{M} \times \mathscr{M} \to \mathbb{R}$ on $\mathscr{M}$, i.e. It can be seen as acting directly on the points of the phase space rather than on tangent vectors. In this way, the phase space of a linear dynamical system, may be viewed simply as a *symplectic vector space* $(\mathscr{M}, \Omega)$.

Due to the identification of the phase space and the solution space, follow that the symplectic structure Ω is directly transferred from $\mathscr{M}$ to `Sol`. This symplectic vector space structure $(\texttt{Sol}, \Omega)$ of the manifold of solutions for a linear dynamical system is the fundamental classical structure that underlies the construction of the *Inital Data* quantization procedure.

In light of the linear structure the *Linear Observables* take a primary role:

$$\mathscr{E}_{\text{lin}} = \mathscr{M}^*$$

this set is a vector space and every choice of linear canonical coordinate $\psi^\alpha = (q^a, p^b)$ on $\mathscr{M}$ constitutes a basis on $\mathscr{E}_{\text{lin}}$:

$$T\mathscr{M} \simeq \mathscr{M} \;\Rightarrow\; d\psi^\alpha \equiv \psi^\alpha$$

$\mathscr{M}_{\text{lin}}$ forms a Poisson subalgebra. Moreover, the presence of the bilinear form provide the usual identification $\mathscr{M} \simeq \mathscr{M}^*$ and therefore the simplectic form on $\mathscr{M}$ can directly reproduced on $\mathscr{M}_{\text{lin}}$:

$$\{\Omega(y_1, \cdot), \Omega(y_2, \cdot)\} = -\Omega(y_1, y_2) \qquad \forall\, y_i \in \mathscr{M} \Rightarrow \Omega(y_1, \cdot) \in \mathscr{E}_{\text{lin}}$$

Take Away Message

In summary the essential aspects that characterize the geometry of linear systems are the following:

- The symplectic form of $\mathscr{M}$ is directly defined on the points of the Phase space.
- Since the points of the phase space can be put in correspondence with the solutions (can be considered as the *inital data*) the symplectic form is directly transported to `Sol`.
- The same simplectic structure can be reproduced on the space $\mathscr{E}_{\text{lin}}$ and coincides with the reduction of the Poisson bracket to this subspace.

2.4 Peierls Brackets

Purpose of the Peierls' procedure is to provide a bilinear form on the space of Lagrangian densities with time-compact support. This form induces a pre-symplectic structure on suitable subspaces of functionals to which can be recognized the role of *classical observables* of the theory.

Observation 21: Relation between Peierls Bracket and Poisson Bracket

Intuitively we can say that the Peierls Brackets implement a sorts of "comparison relation" between two observables similar to the case of Poisson Bracket in ordinary hamiltonian mechanics.

As we will see there are important differences between the two definitions:

- The Poisson bracket determines how one "quantity" b changes another "quantity" a when it acts as the generator (tipically the Hamiltonian) of the dynamical evolution or vice-versa. [13]
  The Peierls bracket, on the other hand, determines how one "quantity" b when added to the system dynamics (usually the Lagrangian or the total action) with an infinitesimal coefficient λ affects changes in another "quantity" a and vice-versa. In other words the Peierls bracket is related to the change in an observable when the trajectory on which it is evaluated gets shifted due to an infinitesimal change in the Lagrangian of the system by another Lagrangian density.

- While the Poisson bracket between two observables a and b is defined on the whole phase space and is not dependent on the existence of a Hamiltonian, the Peierls bracket refers to a specific trajectory determined by a governing Lagrangian.

A rigorous treatment of the notions of *observable* and *Phase Space* should require some further specification depending on which is the considered bracket. However ,we can read the "observable" as an object with a double nature. Essentialy it can act both as the generator of the dynamics than as a quantity which can be evaluated on the system configurations.

In this section we present more extensively the original Peierls' construction. Please note that we are not trying to provide the state of the art on the Peierls bracket (see for example [10] for the treatment in presence of gauge freedom) but only to expand and modernize the first approach given by Peierls. Instead of considering only scalar theory we extend the algorithm to a broader class of systems.

2.4.1 Peierls' construction.

The Peierls's construction algorithm is well defined for a specific class of systems:

1. Linear field theory: $E = (E, \pi, M)$ is a vector bundle.
2. Linear Lagragian dynamics: $P = Q_{\mathscr{L}}$ is a L.P.D.O.
3. M is a globally Hyperbolic space-time.
4. Motion operator P is a green-hyperbolic.

The procedure can be summarized in a few steps:

1. Consider a *disturbance* χ that is a time-compact Lagrangian density .
2. Construct the *perturbation of a solution under the disturbance.*
3. Define the *effect of the disturbance* on a second Lagrangian functional.
4. Assemble the mutual effects of two different Lagrangian densities to give a *bracket.*

Let's review each step in depth.

Disturbance and Disturbed motion operator

By *"disturbance"* we mean a time-compact supported lagrangian density $\chi \in \mathtt{Lag}$[1] which act as a perturbation on the system's lagrangian:

$$\mathscr{L} \rightsquigarrow \mathscr{L}' = \mathscr{L} + \epsilon \cdot \chi$$

where ϵ is a modulation parameter. The support condition is required in order to take in account only perturbations which affect the dynamic for a definite time interval. The motion operator of the disturbed dynamics results:

$$P_\epsilon = \left[\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathscr{L}}{\partial \phi}\right] + \epsilon \left[\partial_\mu \left(\frac{\partial \chi}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \chi}{\partial \phi}\right] = P + \epsilon Q_\chi \tag{2.5}$$

Observation 22

P_ϵ is not necessary linear, the second Hypothesis guarantees the linearity only for P.

Solution of the disturbed motion

The second ingredient of the Peierls' procedure is the calculus of the *perturbed solutions* under the considered *disturbance.*
These are the solutions $\phi' \in \mathtt{C}$ of P_ϵ obtainable by a infinitesimal linear perturbation of a fixed solution $\phi \in \mathtt{Sol}$. The good definition of linear superposition is guaranteed by the hypothesis 1). More precisely, has to be seek a configuration:

$$\phi'(x) = \phi(x) + \epsilon\eta(x) \in \mathtt{C}$$

such that:

$$P_\epsilon \phi'(x) = o(\epsilon)$$
$$P\phi(x) = 0$$

[1] I.e. the top form $\chi(\phi)$ is time-compact supported for all $\phi \in \mathtt{C}$.

In other word has to be satisfied the following equation:

$$[P_\epsilon]\phi'(x) = [P+\epsilon Q_\chi](\phi(x)+\epsilon\eta(x)) = \epsilon\Big([P]\eta(x) + [Q_\chi](\phi(x)+\epsilon\eta(x))\Big) \stackrel{!}{=} o(\epsilon)$$

The condition of linearity for operator P doesn't hold for Q_χ in general. We can work around this problem considering the linearization[10, pag. 31] of operator Q_χ around the unperturbed solution $\phi(x)$. The linearization of Q_χ is the unique linear operator $[Q_\chi^{lin}(\phi)]$ such that:

$$[Q_\chi](\phi(x)+\epsilon\eta(x)) = [Q_\chi](\phi(x)) + \epsilon[Q_\chi^{lin}(\phi)](\eta(x)) + o(eta)$$

which can be seen as the first term of a *formal* Taylor expansion of operator Q_χ around ϕ[2] This is reflected in a condition on the perturbation $\eta \in \mathtt{C}_{tc}$:

$$[P_\epsilon]\phi'(x) = \epsilon\Big([P]\eta(x) + [Q_\chi\phi(x)]\Big) + \epsilon^2[Q_\chi^{lin}(\phi)]\eta(x) \stackrel{!}{=} o(\epsilon)$$
$$\Rightarrow P\eta = -Q_\chi\phi(x) \tag{2.6}$$

called *Jacobi Equation.* This equation is a non homogeneous P.D.E. with inhomogeneous term $(-Q_\chi\phi(x))$ fixed by the solution $\phi \in \mathtt{Sol}$ to be perturbed.

Follows from the definition of green hyperbolicity that the domain restrictions of P to $\Gamma^\infty_{\mathrm{pc}}$ or $\Gamma^\infty_{\mathrm{fc}}$ admit a unique inverse G^+ and G^- respectively. Therefore, equation 2.6 admits a unique past compact solution η^+, called retarded perturbation of $\phi \in \mathtt{Sol}$, and a unique future compact solution η^-, called advanced perturbation:

$$\eta^\pm = G^\pm(-Q_\chi\phi) \tag{2.7}$$

Note that the time-compact support condition on χ guarantees that $Q_\chi\phi \in dom(G^+) \cap dom(G^-)$. Expression 2.7 reflects perfectly the orginal Peierls' notation where $\eta^\pm$ were noted as functions of the unperturbed solution: $\eta^+ \equiv \mathrm{D}_\chi\phi$ and $\eta^- \equiv \mathrm{Ɑ}_\chi\phi$.

Observation 23

In most practical case it's possible to give a more "down to earth" characterization of $\eta^\pm$ in term of a Cauchy problem. Has to be stressed that this approach is not possible in general since Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense i.e. the well-posedness of the Cauchy problem is not guaranteed on any Cauchy surface. [3, pag 1] [2, remark 3.18][10, remark 2.1]
Consider a motion operator P which is also hyperbolic. Taking in account the time-compact support condition of χ, is possible to pick up two Cauchy surfaces $\Sigma_\pm$ (+ is after the perturbation while − stands for prior to the perturbation) such that:

$$J^\mp(\Sigma_\pm) \supset \mathrm{supp}(\chi)$$

for all time-slice foliation of the globally hyperbolic space-time.

[2] If $\mathtt{C}$ is a Frechet manifold the expansion could be stated rigorous defining $[Q_\chi^{lin}(\phi_0).] = [\frac{\partial Q_\chi}{\partial\phi}(\phi_0)]$ in term of the Gateaux derivative.

For each of this two surfaces can be posed a Cauchy problem:

$$\begin{cases} P\eta = -Q_\chi \phi \\ (\eta, \nabla_n \eta)\big|_{\Sigma_\pm} = (0,0) \end{cases} \tag{2.8}$$

which , according to the well-posedness of the Cauchy problem, admits an unique solution. The link with the previous presentation is that past/future -compact supported configurations trivially meet the initial data condition for some future/past Cauchy surface.

In conclusion,fixed a solution $\phi \in \mathtt{Sol}$ and a perturbation χ, are uniquely determined two perturbed solution:

$$\phi^\pm_\epsilon = \phi + \epsilon\eta^\pm \tag{2.9}$$

such that:

retarded pertubation	$\eta^+ \in \Gamma^\infty_{pc}$	$(\eta^+, \nabla_n \eta^+)\big|_{\Sigma_-} = (0,0)$	"propagating forward"
advanced pertubation	$\eta^- \in \Gamma^\infty_{fc}$	$(\eta^-, \nabla_n \eta^-)\big|_{\Sigma_+} = (0,0)$	"propagating backward"

Effect Operator

Considering an arbitrary continuous[3] functional $B : \mathtt{Sol} \to \mathbb{R}$ (not necessarily linear) we can define the effect of a perturbation on the values of B[11, pag. 5] as a map:

$$\mathbf{E}^\pm_\chi : C^1(\mathtt{Sol}, \mathbb{R}) \to C^1(\mathtt{Sol}, \mathbb{R})$$

$$\mathbf{E}^\pm_\chi B(\phi_0) := \lim_{\epsilon \to 0} \left(\frac{B(\phi^\pm_\epsilon) - B(\phi_0)}{\epsilon} \right) \tag{2.10}$$

The advanced and retarded effects of χ on B are then defined by comparing the original system with a new system defined by the same kinematic configuration space $\mathtt{C}$ but with perturbed lagrangian.

Observation 24

Expression 2.10 is clearly a special case of Gateaux derivative.[**?**]

The former expression appear quite simpler in case of a linear functional:

$$\mathbf{E}^\pm_\chi B(\phi_0) = B(\eta^\pm) \tag{2.11}$$

The Bracket

Remembering that every lagrangian density define a continuous functional (Action). From that is possible to build a binary function:

$$\{\cdot,\cdot\} : \mathsf{Lag}_{\mathrm{tc}} \times \mathsf{Lag}_{\mathrm{tc}} \to \mathbb{R}$$

[3] The precise notion of continuity require the specification of a (infinite dimensional) manifold structure on $\mathtt{C}$.

as follow:

$$\{\chi,\omega\}(\phi_0) \coloneqq E^+_\chi F_\omega(\phi_0) - E^-_\chi F_\omega(\phi_0) \tag{2.12}$$

Proposition 2.4.1 (Bilinearity) *When restricted to Linear Lagrangian densities* $\{\cdot,\cdot\}$ *is a bilinear form*

Proof:

Linearity in the first entry follows from equation [**?**] and the linearity of the Euler-Lagrange operator $Q.$ over Lag.
Linearity in the second entry is guaranteed only for lagrangian densities ω which provide a linear Lagrangian Functional F_ω.

□

We don't care to probe the cases for which the simplectic property is met on this general ground. In the next chapter we will face the problem to determine symmetry and non-degeneracy properties for the case of *classical observable functional*, a subclass of Lagrangian functionals of most practical use in the quantization schemes.

2.4.2 Extension to non-linear theories

In the previous construction the green-hyperbolicity of motion operator P plays a primary role. Anyway the problem of searching perturbed solution of the disturbed dynamic can be stated even in presence of non-linear fields where the configuration bundle is not necessary a vector bundle or the motion operator is not linear.

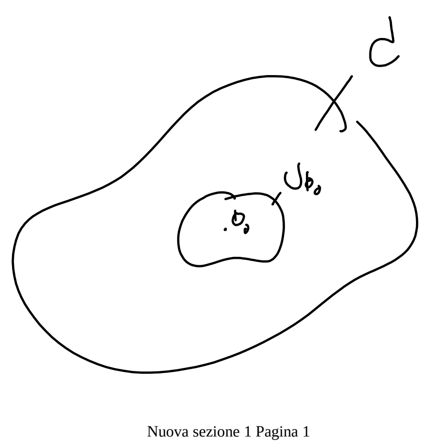


Figure 2.3: Intrinsically, searching a variation of a solution $\gamma_0 \in \mathtt{Sol}$ which solve the disturbed motion equation is equivalent to find the intersection of the perturbed solution with a local neighbourhood of Γ_0 : $U_{\gamma_0} \cap \ker(P_\epsilon)$.

The crucial point of the Peierls' procedure is to select among all the possible solution of the perturbed motion P_ϵ that configuration which can be constructed by a variation of some fixed solution of the non-perturbed dynamics $\gamma_0 \in \mathtt{Sol}$. In this sense the problem results a *"linearization"* inasmuch the search of such solution is restricted to a local neighbourhood of the "point" $\gamma_0 \in \mathtt{Sol}$.

Previously the choice to consider only the linear variation was quite natural but in the general case this preferential restriction is no longer possible. Anyway, it's possible to recover a notation similar to 2.9 by working patchwise, under the choice of a particular coordinate representation.
Fixed a solution $\gamma_0 \in \mathtt{Sol}$ and a local trivializing chart (A, ϕ_A) such that $A \cap ran(\gamma_0) \neq \emptyset$ we can define a local infinitesimal variation by acting on his components:

$$\gamma^i_\lambda(x) = \gamma^i_0(x) + \lambda\eta^i(x) \qquad \forall x \in \pi(A)$$

where γ^i_0 are the component of the unperturbed solution in the open set A and $\eta^i \in^q$ is a generic real q-ple (q is the dimension of the typical fiber manifold). λ is a real parameter that has to be "sufficiently small" in order to guarantee that the range of γ_λ is properly contained in A.
In other words the construction of the linear variation, that for linear field theories could be done in a global way, in the general case can be recovered only locally varying the components.

Therefore is possible to define the effect of a disturbance locally, searching a local section $\gamma^i_\epsilon = \gamma^i_0 + \epsilon\eta^i$ which solves the disturbed dynamic equation up to the first order in ϵ i.e.

$$\left[P_\epsilon\right]\gamma^i_\epsilon = o(\epsilon)$$

where $\left[P_\epsilon\right]$ has to be intended as the coordinate representation of the operator with restricted domain to the local sections $\Gamma^\infty(A)$.

Observation 25

W.l.o.g has been taken the the same scalar ϵ to modulate both the perturbation γ_ϵ that the disturbance on the motion operator.

Consider two different parameter is totally immaterial since, in that case,only the smaller one should be taken in account.

From the explicit equation of the perturbed solution:

$$\Big(\left[P\right] + \epsilon\left[Q_\chi\right]\Big)\left(\gamma^i_0 + \epsilon\eta^i\right) = o(\epsilon)$$

follows an equation on the components of the local perturbation. In this case has to be dealt with the problem of non-linearity not only for Euler-Lagrange operator Q_χ but also for P. Arresting the expension to the first order in ϵ results:

$$\left[P^{lin}_{\gamma_0}\right]\eta^i(x) = -\Big(Q_\chi(\gamma_0)\Big)(x) \tag{2.13}$$

the *Jacobi equation* on the unperturbed solution $\gamma_0 \in \mathtt{Sol}$.

Observation 26

We've moved from an operator P definited on $\mathtt{C}$ to an operator $P^{lin}_{\gamma_0}$ defined on the space of variation. From a global point of view this variation can be seen as the tangent vector i.e. s $\eta \in T_{\gamma_0}\mathtt{C}$. In the case of the linear system this passage was unnecessary, the Jacobi equation was directly defined on $\mathtt{C}$ since, for linear system, any section could be seen as a generator of an infinitesimal variation. This behaviour mimics perfectly what happens in ordinary classical mechanics where the configuration space of a linear system is a vector space i.e a "flat" manifold[a] which is isomorphic to his tangent space in every point.

[a] In sense that admits a global coordinate chart.

Provided that the linearized motion operator (which is now properly a linear partial differential operator) is Green-Hyperbolic, the Peierels construction can continue as before. Has to be noted that now the advanced/retarded perturbation are formally identical to the former:

$$\eta^{\pm\, i} = G^{\pm}\big(-Q_\chi \gamma_0^j\big)$$

with the important difference that $G^{\pm}$ are now the Green operators of the linearized motion operator and depends strictly on the fixed solution γ_0^j.

In conclusion the perturbed solution:

$$\gamma_\chi^{\pm\, i} = \gamma_0^i \pm G^{\pm}\big(-Q_\chi \gamma_0^j\big)$$

has to be intended as the "glueing" of all the local chart representations covering the chosen solution.

Example: Finite Dimensional Case

As an example of such process we can consider a *field of curves* i.e. an ordinary classical mechanical system in the field theoretic picture. We have shown in section 2.2.2 that such systems are generally non linear: the configuration bundle is not a vector one and then linearity of P cannot be defined .

However the base manifold is very simple. Indeed $M = \mathbb{R}$ can be seen as a trivial globally-hyperbolic space-time where every real scalar $t \in M$ is a Cauchy surface. This allows us to specify the above equations in a more intuitive way:

- γ_0^j is a simple local chart representation of the curve.
- for a suitable small ϵ γ_0^j and $\gamma_\chi^{\pm\, i}$ can be pictured in the same local chart.
- the variation $\eta^{\pm}$ is then the field over the unperturbed curve whose components compute the separation between γ_0 an $\gamma_\chi^{\pm}$

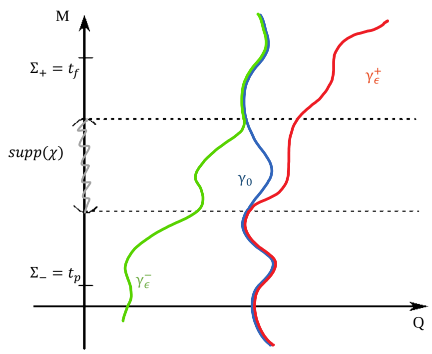


Figure 2.4: Picture of the perturbed solution in case of a finite dimensional system.

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Chapter 3

Algebraic Quantization

The point we want to get, that we will face in the next chapter, is the algebraic quantization of geodesic system. For this purpose it is necessary to devote a chapter to the description of algebraic quantization scheme. We will show two realizations of the scheme applicable to a class of systems sufficiently broad to encompass the system that we want to examine.

3.1 Overview on the Algebraic Quantization Scheme.

Contemporary quantum field theory is mainly developed as quantization of classical fields. Classical field theory thus is a necessary step towards quantum field theory.⚠[1] The *"Quantization process"* has to be considered as an algorithm , in the sense of self-conteining succession of instruction, that has to be performed in order to establish a correspondence between a classical field theory and its quantum counterpart. ⚠[2]

On this basis the axiomatic theory of quantum fields takes the role of "validity check". It provide a set of conditions that must be met in order to establish whether the result can be consider a proper quantum field theory. Basically there are no physical/philosophical principles which justifies "a priori" the relation between mathematical objects (e.g the classical state versus quantum states) individually. The scheme can only be ratified " a posteriori" as whole verifying the agreement with the experimental observations.

However this is by no means different from what is discussed in ordinary quantum mechanics where there are essentially two plane: the basic formalism of quantum mechanics, which is substantially axiomatic and permits to define an abstract quantum mechanical system, and the quantization process that determine how to construct the quantum analogous of a classical system realizing the basic axioms.

[1] Cito testualmente Mangiarotti, shardanashivly

[2] forse l'nlab esprime la cosa meglio di me `http://ncatlab.org/nlab/show/quantization`. Sono d'accordo con il loro approccio ma non voglio usare la loro formulazione perchÃĺ in fondo ci sono arrivato anche da solo :P

We refer to the algebraic quantization as a *scheme of quantization* because it's not a single specific procedure but rather a class of algorithms. These algorithms are the same concerning the quantization step per se (costruction of the *-algebra of classical observable) but they differ in the choice of the classical objects (essentially the classical observables and the bilinear form) to be subjected to the procedure.

Basically an algebraic quantization is achieved in three steps:

1. **Classical Step**
   Identify all the mathematical structures necessary to define the field, i.e. the pair (E,P).
   In general every quantization process exploit some conditions on the quantum field structure that has to be met.

2. **Pre-Quantum Step**
   ⚠Are implemented some additional mathematical over-structure on the classic framework. The aim is to establish the specific objects which will be submitted to the quantization process in the next step. Generally these object don't have any a classical meaning, their only purpose is to represent the classical analogous of the crucial structures of the quantum framework. From that we say *Pre-Quantum*, their introduction doesn't have a proper *a priori*explanation but has to be treated as an anstatz and justified *a posteriori* within the quantum treatment.

   Essentially has to be chosen a suitable space of *Classical observable* and this space has to be rigged with a well-behaved bilinear form.

   The ordinary quantum mechanics equivalent step is the choice of a particular Poisson bracket on $C^\infty(T^*Q)$, which tipically implement the *canonical commutation relations* $\{q,p\} = i\hbar$, among all the possible Poisson structure. Note that this is a "pre-quantum" step because in classical Hamiltonian mechanics is considered only the Poisson structure carried from the natural symplectic form [1].

3. **Quantization**
   Finally are introduced the rules which realize the correspondence between the chosen classical objects and their quantum analogues.⚠[3] The algebraic approach characterizes the quantization of any field theory as a two-step procedure. In the first, one assigns to a physical system a suitable âĹŮ-algebra A of observables, the central structure of the algebraic theory which encodes all structural relations between observables. The second step consists of selecting a so-called *Hadamard state* which allows us to recover the interpretation of the elements of A as linear operators on a suitable Hilbert space.

⚠[4] As said previously, the realization of the Algebraic scheme are many: Fedosov's procedure, by Deformation, Peierls' procedure, by Initial Data etc . In the next section we review the last two.

[3] Sto Cito direttamente [5].

[4] Frase che non mi piace ma voglio far presente che le realizzazioni dello schema algebrico sono molteplici!

3.2 Quantization with Peierls Bracket.

⚠Temp ⚠da contestualizzare (e spostare)

Observation 27

In the algebraic quantization scheme the choice of the bundle bilinear form take a pivotal role since it is the basis of the so-called *pairing*. In effect this is the only discretionary parameter of the whole procedure. The prescription on the symmetry properties determine the Bosonic/Fermionic character of the quantized theory:

Pairing	Observables linear form	Quantum Theory
symmetric	anti-symmetric	Bosonic
anti-symmetric	symmetric	Fermionic

Observation 28

What we are going to show is a quantization procedure strictly defined for a specific class of classical theories:

1. Linear Fields.
2. Lagrangian Dynamics.
3. On Globally-Hyperbolic Space-time.
4. with Green-hyperbolic motion Operator.

Fall into this category prominent examples like Klein-Gordon and Proca Field Theory.[4] Has to be noted that the Lagrangian condition is ancillary. This has the purpose to justify the shape of the symplectic form on the classical observables space as consequent from the Peierls bracket. It's customary to overlook to the origin of this object and jump directly to the expression **??** in term of the Green's operator that no longer present any direct link to the Lagrangian and therefore can be extended to any green-hyperbolic theory.

Briefly the procedure can be resumed in few steps:

1. Classical Step
   Has to be stated the mathematical structure of the classical theory under examination.
   (a) Kinematics: is encoded in the configuration bundle of the theory.
      i. Specify the base manifold M.
         Has to be a Globally-Hyperbolic Space-time.
      ii. Specify the Fiber and the total Space E auxiliary structure, e.g: spin-structure or trasformation laws under diffeomorphism on the base space.
         E has to be at least a vector bundle.

   (b) Dinamics: has to be specified the local coordinate expression of the motion operator $P : \Gamma^{\infty}(E) = \mathbb{C} \to \mathbb{C}$.
      i. Is P Green-hyperbolic?
      ii. Is P derived from a lagrangian: $P = Q_{\mathscr{L}}$?

2. Pre-Quantum Step
   (a) Pairing: construct a basic bilinear form on the space of kinematic configurations.
      i. Choose $< \cdot, \cdot >$ a bilinear form on the bundle E.
      Generally this object is suggested by the m
      ii.
   (b) Classical Observables
      i.
      ii.
   (c) Symplectic structure
      i.
      ii.

3. Quantization Step
   (a) Quantum Observables Algebra
   A concrete realization is achieved in three step.
      i.
      ii.
   (b) Hadamard State
      i.
      ii.

Da ricopiare!

3.2.1 Classical Step

Applicability of the procedure.

Da ricopiare!

3.2.2 PreQuantum Step.

Da ricopiare!

3.2.3 Second Quantization Step.

⚠ Da ricopiare!

3.3 Quantization by Initial Data.

⚠ Da ricopiare!

3.3.1 PreQuantum Step.

⚠ Da ricopiare!

3.4 Link between the two realizations

⚠ Intro da Ricopiare

3.4.1 Equivalence of the Classical Observables

⚠ Intro da Ricopiare

3.4.2 Equivalence of the Brackets

⚠ Non completata! vedi email del 9 luglio.

Chapter 4

Geodesic Fields

In the context of differential geometry, *geodesic curves* are a generalization of *straight lines* in the sense of self-parallel curves. Considering a differential manifold M endowed with an affine connection ∇ we define:

Definition 32: Geodesic

A curve ⚠[a] $\gamma : [a, b] \to M$ such that:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0 \tag{4.1}$$

where $\dot{\gamma}^\mu := \frac{d\gamma^\mu}{dt}$ is the tangent vector to the curve.

[a] Devo dire smooth o piecewise?

Notation fixing

In local chart the previous equation assume the popular expression:

$$\ddot{\gamma}^i + \Gamma^i{}_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0 \tag{4.2}$$

Where $\Gamma^i{}_{jk}$ is the coordinate representation of the Christoffel symbols of the connection.

In presence of a pseudo-Riemannian metric is possible to present the geodesic in a metric sense i.e. as the curve which extremizes the *Energy Functional*[1]:

Definition 33: Energy functional

[1] Remember that for arc-length parametrized curves the Energy functional coincide with the length functional.[9, Lemma 1.4.2]

$$E(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\|^2 dt \tag{4.3}$$

Considering only the proper variation (that keep the end-point fixed), the extremum condition corresponds to equation 4.2 where ∇ is the unique Levi-Civita connection (torsion-free and metric-compatible).

In general relativity the problem of the geodesic equation linearization, named *Jacobi equations* takes a central role. [2]

⚠ (nel file di ripasso di geometria riemmaniana ho scritto gran parte delle definizione conviene vedere cosa mi serve effettivamente... Di certo mi avvalgo della seguente equazione

Notation fixing

In local charts the Jacobi fields along the geodesic γ solve a linear O.D.E.:

$$\left(X''\right)^\mu + R^\mu_{i\alpha j} T^i X^\alpha T^j = 0 \tag{4.4}$$

where:

- $\left(X'\right)^\mu := \left(\nabla_{\dot\gamma(t)} X\right)^\mu$ is the covariant derivative along the curve γ.
- $T \equiv \dot\gamma(t)$ stands for the tangent vector to the curve γ.

The rest of this chapter will be dedicated to presenting the physical approach to the Geodesic.

4.1 Geodesic Problem as a Mechanical Systems

The basic idea is very simple, portray the geodesic curve as the natural motion of a free particle constrained on the Pseudo-Riemannian manifold Q.

⚠ obvious enough this problem can be seen as a generalization of the calculation of the motions of free falling parcticles In terms of general relativity this problem can be instantly recognized as the derivation of the free-falling particles motion.

However, there is no lack of alternative viewpoints . The framework of the classical Geometric Mechanics teach us to picture the "static" configurations of a constrained, complex, classical system as a point on the *Configuration space* manifold. According to that, the geodesic motion can be

[2] Usually in this context takes the name of *Geodesic deviation* problem[**?**, pag. 46].

seen as a realization of a particular dynamics on each mechanical system endowed with a pseudo-Riemannian configuration space[a].

[a] Such systems can be depicted as "geodesic" even in presence of a position-dependant potential.[1, Cap 3.7]

Theorem 4.1.1 (Geodesic Motion) *The geodesics on the Pseudo-Riemmanian manifold* (Q, g) *are the natural motions of the ordinary Lagrangian system* (Q, L) *where:*

$$L(V_q) := \frac{1}{2} g_q(V, V) \tag{4.5}$$

Proof:

The Euler-Lagrange equation of L coincides with the geodesic equation 4.2. ⚠.. ÃÍ sul quaderno non so se metterla

□

Observation 29

The geodesic system is not simply Lagrangian but also Hamiltonian. This property follows from the hyperregularity[1] of L.

Observation 29

Anyway we will neglect this fact inasmuch in what follows only the Lagrangian character assumes a role.

As shown in chapter 2, every system with discrete degrees of freedom can be seen as the trivial field system. From that follows the alternative characterization of geodesic as a lagrangian field:

Corollary 4.1.1 (Geodesic field) *The geodesics on the Pseudo-Riemmanian manifold* (Q, g) *can be seen as the* Dynamical Configurations *of the lagrangian field system* $(E, \mathscr{L})$ *where:*

- $E = (Q \times \mathbb{R}, \pi, \mathbb{R})$ *trivial smooth bundle on the real line.*
- $\mathscr{L}[\gamma] = \frac{1}{2} g(\dot{\gamma}, \dot{\gamma})(t) dt$

Proof:

Is simple application of the correspondence seen in chapter 2.2.2.

□

From this perspective is clear that the Energy Functional can be seen as the action in the geodesic field dynamics and equation 4.2 is nothing more than the motion equation according to the *least action principle*.

Figure 4.1: Impressionistic view of the geometric mechanics structure.

4.2 Peierls Bracket of the Geodesic field

The local coordinate expression of the lagrangian density of the geodesic field results:

$$\mathscr{L}\left(t,\gamma^i(t),\dot{\gamma}^i(t)\right) := \frac{1}{2} g_{\mu,\nu}\left(\gamma^i(t)\right)\dot{\gamma}^\mu\dot{\gamma}^\nu \tag{4.6}$$

this is highly non-linear. Explicitly is quadratic in the velocity components $\dot{\gamma}^i$ and implicitly, through $g_{\mu\nu}(\gamma^i(t))$, is non-polynomial in coordinate γ^i.

As show in section **??**, for this type of systems, the calculation of Peierls bracket can be realized only locally around a predetermined solution. Let's repeat the Peierls' procedure for the system under investigation.

Introduzione da rividere, dimostro che l'operatore linearizzato senza termine inomogeneo corrisponde all'equazione di Jacobi vera e propria mentre con termine inomogeneo dato dalle E-P del disturbo da l'equazione che definisce la perturbazione ritardata e anticipata.

As a consequence of our introduction on the geodesic as a field, we can state the unperturbed dynamic as a L.P.D.O :

$$Q_{\mathscr{L}}\left(q^\mu\right) = \left[\ddot{q}^\mu + \Gamma^\mu_{\ ij}\dot{q}^i\dot{q}^j\right] \tag{4.7}$$

where $\dot{q}^\mu = \frac{d}{dt}q^\mu(t) = \dot{q}^i\partial_i q^\mu$.

A linear variation of $q_0^\mu + \epsilon\eta^\mu$ constructed from the coordinate representation q_0^μ of the geodesic $\gamma_0 \in \mathtt{Sol}$, solves the original motion equations when

$$Q_{\mathscr{L}}\left(q_0^\mu + \epsilon\eta^\mu\right) = \frac{d^2}{dt^2}\left(q_0^\mu + \epsilon\eta^\mu\right) + \left[\Gamma^\mu_{\ ij}\left(\vec{q}_0 + \epsilon\vec{\eta}\right)\right]\left(\dot{q}_0{}^i + \epsilon\dot{\eta}^i\right)\left(\dot{q}_0{}^j + \epsilon\dot{\eta}^j\right) = 0 \stackrel{!}{=} o(\epsilon) \tag{4.8}$$

If we consider only the first order in the parameter ϵ we can expand the expression of the Christoffel symbols:

$$\left[\Gamma^\mu_{\ ij}\left(\vec{q}_0 + \epsilon\vec{\eta}\right)\right] = \left[\Gamma^\mu_{\ ij}(\vec{q}_0) + \epsilon\eta^\alpha\left(\partial_\alpha\Gamma^\mu_{\ ij}\right)\Big|_{\vec{q}_0} + o(\epsilon)\right]$$

Collecting all the terms in equation 4.8 up the first order in ϵ follows a condition on the perturbation:

$$0 = \ddot{\eta}^{\mu} + \eta^{\alpha} \left(\partial_{\alpha} \Gamma^{\mu}_{ij}\right)\Big|_{\vec{q}_0} \dot{q_0}^{i} \dot{q_0}^{j} + \Gamma^{\mu}_{ij} \left(\dot{\eta}^{i} \dot{q_0}^{j} + \dot{q_0}^{i} \dot{\eta}^{j}\right) == \left\{ g^{\mu}_{\alpha} \frac{d^2}{dt^2} + \Gamma^{\mu}_{i\alpha}(\vec{q}_0) \left[2 \dot{q_0}^{i} \frac{d}{dt}\right] + \left[\partial_{\alpha}\right] \right\} \eta^{\alpha}$$

⚠ Da dire: espressione in coordinate della lagrangiana, ÃĺI altmente non lineare perchÃĺ implicitamente ÃĺI $g_{\mu\nu}(\gamma^{i}(t))$ non polinomiale in γ^{i} ed esplicitamente ÃĺI quadratica, Mostrare esplicitamente che l'equazione di jacobi per il sistema ÃĺI effittivamente l'equjazione di jacobi (questo ÃĺI triviale se vedi come definisce il campo di jacobi jurgen

4.2.1 Example: Geodesic field on FRW space-time.

4.3 Algebraic quantization of the Geodesic Field

⚠ va ripetuto che la geodetica ÃĺI non lineare quindi ciÃš che effettivamente si quantizza ÃĺI jacobi lungo una prefissata geodetica. questo ÃĺI un campo lineare.

⚠ INtro: paragrafo sul quaderno: "qual'ÃĺI l'interesse che spinge a quantizzare questo sistema "Campo di Jacobi"?

⚠ Disclaimer: Non approfondisco piÃź di tanto gli step di quantizzazione vera e propria. il ruolo di peierls ÃĺI nel prequantistico, definisce il bracket che poi va implementato sull'algebra. Una volta decisa la parentesi la macchinetta procede in automatico.

4.3.1 Peierls Approach

⚠ Paragrafo sul Quaderno: ...

4.3.2 Inital data Approach

⚠ Ancora su fogli di Brutta!

4.4 Interpretations??????

⚠ Speriamo bene.. :S

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