Chapter 1

Mathematical Preliminaries

⚠ Le interazioni matematiche sono complesse e non triviali (vedi un po' di articoli di introduzione a AQFT per ispirarti)

Tendenzialmente le teorie quantistiche di campi moderne sono di Quantizzazione.. Quindi richiedono di specificare bene la struttura del campo classico (vedi intro di Mangiaratti shardashivly)

Gli strumenti matematici per raccontare la teoria dei campi classici sono essenzialmente 3: Fibrati, S-T G-H, LDOP e GHOP.

IN questo paper non ci soffermeremo sulle strutture del framework puramente quantistico (* algebre e quant'altro). Per un primer vedere articolo di Dappiaggi o Libro Adv aqft.

Diamo per scontato un background di base in Geometria differenziale e derivate esterne (algebre di Grassman? global calculus? non so come chiamarlo!).

Potrei avere la tentazione a provare ad usare un po' di linguaggio basilare delle categorie... la mia fonte ÃÍ Joy of Cat.

⚠ Stile: Intro lapiadaria ai 3 argomenti (bundle cinematica di campo, Glob iper stage per descrivere dinamica di tipo propagativo, Operatori tipo onda). Poi smitragliata di definizioni come faceva Penati.

1.1 Fiber Bundles

⚠ intro

1.1.1 Formal Definition

Although it would be possible to present the concept *bundle* in a more general way through the language of categories, for our argument will be sufficient to consider

only the case of *smooth bundles*.

Definition 1: Fiber Bundle

A *Fiber Bundle* consists in a 4-ple (E, M, Q, π) where:

- E, M, Q : smooth manifolds called respectively *Total Space, Base Space, Typical Fiber*.

- $\pi : E \to M$ continuous smooth function (called *Bundle Projection*)

Endowed with a *Local Trivialization*:

- $\forall x \in E \; \exists$ a couple (U, χ) (called *local trivialization*)

    – U : neighborhood of x
    – $\chi : \pi^{-1}(U) \to U \times Q$: diffeomorphism [a][b]
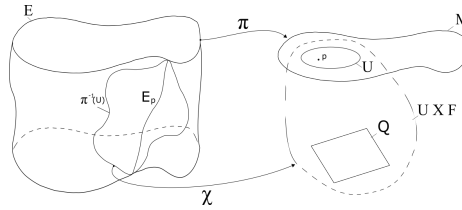
    such that: $p_1 \cdot \chi = \pi|_{\pi^{-1}(p)}$.

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\;\chi\;} & U \times Q \\
{\scriptstyle \pi} \downarrow & \swarrow {\scriptstyle p_1} & \\
U & &
\end{array}
$$

    i.e: the following graph commutes:

---
[a] surjectivity $\Rightarrow \pi^{-1}(U) \neq \emptyset$.
[b] cartesian product of topological space is a topological space with the direct product topology.

Figure 1.1: The complete fiber bundle Structure.

Notation fixing

It is customary to refer to a vector bundle specifying only its total space:

$$E = (E, \pi, M; Q)$$

In the following we adopt this convention whenever this does not lead to mis-understandings.

> **Observation 1**
>
> For all $p \in M$ we refer to the submanifold $E_p := \pi_{-1}(p) \subset E$ as *fiber over the point p*.
> Every fiber E_p is diffeomorphic to the typical fiber F through the local trivialization charts.

> **Notation fixing**
>
> We say that a smooth bundle E is *(globally) trivial* if $E \simeq M \times Q$ i.e there exists a trivialization of E which is defined everywhere. Note that definition 1 prescribes the existence of local trivializations only.

When a smooth fiber bundle $(E, \pi, M; Q)$ is considered, in addition to the typical functions of the bundle (π, χ_α) should be taken in account also the local charts $(U_{\alpha_k}, \phi_{\alpha_k})_{k=E,M,Q}$ provided by the atlases of E, M and Q:

> **Definition 2: Bundle atlas**
>
> Collection local chart which trivializes E. I.e. triples $(U_\alpha, \psi_\alpha, \chi_\alpha)$ where:
>
> - U_α open set in M such that $\bigcup_\alpha U_\alpha \supseteq M$.
>
> - χ_α is a local trivialization.
>
> - (U_α, ψ_α) local chart on E constructed from charts on the base and fiber manifold:
> $$\psi_\alpha^{(E)} = \psi_\alpha^{(M)} \times \psi_\alpha^{(Q)} \circ \chi_\alpha$$

Endowing the bundles manifolds with other additional structures, can be introduced important subclasses of smooth bundles:

> **Definition 3: Vector Bundle**
>
> Is a smooth bundle $E = (E, \pi, M; V)$ such that:
>
> - The typical fiber manifold V is a finite dimensional vector space.
>
> - All the trivialization χ_α are diffeomorphism such that:
> $$\chi_\alpha|_{\pi^{-1}(p)} \in \mathbb{GL}(n, \mathbb{R})$$

1.1.2 Cross Sections

The notion of bundle is particularly interesting from the perspective of physics because provides the rigorous description of a $Q-$valued field over the space M:

Definition 4: Smooth Section

Function $\phi : M \to E$ such that:

- ϕ smooth.

- $\phi \cdot \pi = \mathbb{1}_M$

Notation fixing

We refer to:

- *Global section* $\Leftrightarrow \mathrm{dom}(\phi) = B$

- *Local section* $\Leftrightarrow \mathrm{dom}(\phi) \subset B$ [a]

We denote the set of all the smooth sections of the bundle E as:

$$\Gamma^\infty(E)$$

[a] Usually the domain is an open set of B)

Observation 2

In general, $\Gamma^\infty(E)$ is an infinite dimensional manifolds. In case of vector bundle is also a linear space, and the section are called "vector fields".

1.1.3   Mapping between Bundles

Consider two smooth bundles $E = (E, \pi, M; Q)$ and $E' = (E', \pi', M; Q')$ on the same base space M.

Definition 5: Bundle map (*Fiber Preserving map*)

Smooth function $\phi : E \to E'$ such that:

$$\phi(E_x) = F_x \qquad \forall x \in M.$$

$$
\begin{array}{ccc}
E & \xrightarrow{\ \ \phi\ \ } & F \\
& \pi_E \searrow \quad \swarrow \pi_F & \\
& M &
\end{array}
$$

i.e.:

Observation 3

Definition 5 is a special case of *Bundle-morphism.* (see for example [?])

Consider a smooth manifold N, a (smooth) fiber bundle $E = (E, \pi, M; Q)$, and a smooth function $f : N \to M$. It's possible to induce[?] a bundle structure from the manifold M to N:

Definition 6: Pull-Back Bundle

3-ple $f^*(E) = (f^*(E) =, \pi^*, N)$ such that:

- $f^*(E) = \left\{ (b', e) \in N \times E \;\middle|\; f(b') = \pi(e) \right\}$

- $\pi^* : f^*(E) \to N$ such that $\pi * (b', e) = \mathrm{pr}_1(b', e) = b'$

$$
\begin{array}{ccc}
f^*E & & E \\
\pi' \downarrow & & \downarrow \pi \\
N & \xrightarrow{\;f\;} & M
\end{array}
$$

Proposition 1.1.1 $f^*(E) = (f^*(E) =, \pi^*, N)$ *consitute a fiber bundle of typical fiber Q.*

Proof:

To complete the fiber bundle structure is sufficient to provide a local trivialization atlas.
$\forall (U, \phi)$ local trivialization on (E, π, M) consider $\psi : f^*E \to N \times Q$ such that $\psi(b', e) = \left(b', pr_2\big(\phi(e)\big) \right)$.
Then $(f^{-1}(U), \psi)$ is a local trivialization of the pull-back bundle and the fiber of f^*E over a point $bš \in B'$ is just the fiber of E over $f(bš)$.

$\square$

It is also noteworthy that, given any two vector bundles $E = (E\pi, M, Q)$ and $E = (E'\pi, M', Q')$, we can construct naturally a third fiber bundle. Consider $\hom(E, E')$ the set of all the fiber preserving map between the two bundles:

Definition 7: Bundle of morphisms

Fiber bundle $\hom(E, E')$ over the base space M such that the fiber over a base point $p \in M$ is the infinite dimensional manifold $\hom(E_p, E'_p)$ isomorphic to $\hom(Q, Q')$.

Remark:

If F, F' are vector bundle then the fiber of $\hom(F, F')$ over a base point $p \in M$ is $\hom(F_p, F'_p)$, which is a vector space isomorphic to the vector space $\hom(V, V')$ of linear applications from V to V'

1.1.4 Tangent Bundles

The *tangent* bundle is a natural structure defined on any smooth manifold, represent the canonical example of non-trivial vector bundle.

Definition 8: Tangent Bundle

The smooth vector bundle $TM = (TM, \tau, M; \mathbb{R}^m)$ such that:

- The total space is the union of all tangent spaces to

$$M : TM := \bigsqcup_{p \in M} T_p M \equiv \bigcup_{x \in M} x \times T_x M$$

- The bundle projection maps each tangent vector $v \in T_p M$ to the correspondent base point p;

$$\tau : (p, v_p) \longmapsto p$$

Observation 4

In a similar fashion it's possible to construct a vector bundle relatively to any tensor product of the tangent spaces. i.e.:

- *Cotangent Bundle* $T^* M$ is build by disjoint union of the dual tangent space:

$$T_p^* M \ \forall p \in M$$

- *Tensor Bundle* $T^{(k,l)} M$ is build by disjoint union of the tensor product of tangent space with itself:

$$T_p^{(k,l)} M = \underbrace{T_p^* M \otimes \cdots \otimes T_p^* M}_{\text{k-times}} \otimes \underbrace{T_p M \otimes \cdots \otimes T_p M}_{\text{l-times}}$$

- *k-forms Bundle* $\bigwedge^m (T^* M)$ is build by disjoint union of the antisimmetrized tensor product of the dual tangent space with itself.

Tautological one-form and simplectic form.

> **Notation fixing**
>
> In the context of Classical mechanics is customary to refer to the cotangent bundle T^*Q over the smooth manifold Q - called *Configuration Space* - as *Phase Space.*

Since TQ and T^*Q are diffeomorphic , it might seem that there is no particular reason in treating this two spaces separately, but it is not so. There are certain geometrical objects that live naturally on T^*Q , not on TQ.
Of greatest interest in mathematical-physics are the PoincarÃÍ forms[**?**].
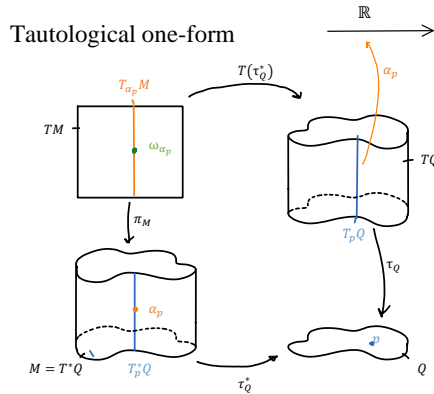 Consider a smooth manifold Q and call $\mathcal{M} = T^*Q$ the corresponding cotangent bundle.

> **Definition 9: Tautological (Poincare) 1-form**
>
> Is the 1-form over $\mathcal{M}$:
> $$\theta_0 \in \Gamma^\infty(T^*\mathcal{M})$$
> such that the action on a generic point $\omega_{\alpha_p} \in T_{\alpha_p} M$ (in the fiber of α_p, which in turn is a one-form on the fiber of $p \in Q$) is given by:
> $$\theta_0(\alpha_p) : T_{\alpha_p}\mathcal{M} \to \mathbb{R} \qquad : \omega_{\alpha_p} \mapsto \alpha_q \circ T\tau_Q^*(\omega_{\alpha_p})$$

Tautological one-form

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Figure 1.2: The definition of tautological 1-form is achieved exploiting the concept of *Tangent map* and remembering that $\alpha_p : T_p\mathcal{M} \to \mathcal{M}$ is a linear functional.

Observation 5

In canonical coordinate the tautological one-form assumes the famous expression:

$$\theta_0 = \sum_{i=1}^n p_i dq^i$$

(note that dq^i is a 1-form on $T^* M$ calculated with respect to the coordinate on the bundle. Has not to be confused with the 1-natural form $dq^i \in T_p^* M$.)

The claim is proved by the following definition:

Definition 10: Canonical (Poincare) symplectic form

Symplectic form:

$$\omega_0 := -d\theta_0$$

In canonical coordinates assumes the famous expression:

$$\omega_0 := \sum_{i=1}^n dq^i \wedge dp_i$$

1.1.5 Jet Bundles

The jet bundle is a certain construction that makes a new smooth fiber bundle out of a given smooth fiber bundle. The first step is to identify the typical fiber for this construction.

Suppose M is an m-dimensional manifold and that (E, π, M) is a fiber bundle.Consider the set of all the local sections whose domain contains p:

$$\Gamma^\infty(p) := \left\{ \sigma \in \Gamma^\infty(E) \quad \middle| \, p \in dom(\sigma) \right\}$$

We define an equivalence relation between such section *up to r-th order*:

Definition 11: r-jet equivalence

Two such section $\sigma, \eta \in \Gamma^\infty(p)$ have the <u>same *r-jet* at p</u> $(\sigma \sim \eta)$ iff:

$$\left.\frac{\partial^{|I|}\sigma^\alpha}{\partial x^I}\right|_p = \left.\frac{\partial^{|I|}\eta^\alpha}{\partial x^I}\right|_p \quad \forall I \in \mathbb{N}_0^m \,|\, 0 \le |I| \le r.$$

where I is a *Multi-index*.

Remark:

(Multi-index notation)
A multi-index is a natural valued finite dimensional vector $I = (i_1, i_2, \ldots, i_m) \in \mathbb{N}_0^m$ with $m < \infty$.
On $\mathbb{R}^n$ a general differential operator can be identified by a multi-index:

$$\frac{\partial^{|I|}}{\partial x^I} := \prod_{i=1}^m \left(\frac{\partial}{\partial x^i}\right)^{I(i)}$$

(Until the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|I| := \sum_{i=1}^m I(i)$$

We define the *r-th Jet in p* as the equivalence class under this relation.

Definition 12: Space of r-th Jet in p

$$J_p^r(E) := \frac{\Gamma^\infty(p)}{\sim}$$

where $\sim$ is the r-Jet equivalence.

Notation fixing

A r-jet with representative σ is denoted as $j_p^r\sigma$.
The integer r is also called the order of the jet, p is its source and $\sigma(p)$ is its target.

Glueing all the jet fiber $J_p^r(E)$ together for all the base point $p \in M$, as done for the tangent bundle, we obtain the desired bundle:

Definition 13: r-th Jet Bundle of E

The smooth bundle $(J^r(E), \pi_r, M)$ where:

- $J^r(E) := \bigsqcup_{p \in M} J_p^r(E) \equiv \left\{j_p^r\sigma \quad | \, p \in M, \, \sigma \in \Gamma^\infty(p)\right\}$

- $\pi_r : J^r(E) \to M$ such that $j_p^r \sigma \mapsto p$

1.2 Globally Hyperbolic Spacetimes

⚠ Mettere solo le definizioni che uso prese dagli articoli di review delle Fonti

Appunti che mi ero preso scrivendo il secono capitolo:
This condition is strictly connected to the dynamic behaviour of the system.

1.2.1 A reprise in Spacetimes Geometry

Recurring definitions in general Relativity (excluding the general smooth manifold prolegomena).

Definition 14: Space-Time

A quadruple $(M, g, \mathfrak{o}, \mathfrak{t})$ such that:

- (M, g) is a time-orientable n-dimensional manifold ($n > 2$)

- $\mathfrak{o}$ is a choice of orientation

- $\mathfrak{t}$ is a choice of time-orientation

Definition 15: Lorentzian Manifold

A pair (M, g) such that:

- M is a n-dimensional ($n \geq 2$), Hausdorff, second countable, connected, orientable smooth manifold.

- g is a Lorentzian metric.

Definition 16: Metric

A function on the bundle product of TM with itself:

$$g : TM \times_M TM \to \mathbb{R}$$

such that the restriction on each fiber

$$g_p : T_p M \times T_p M \to \mathbb{R}$$

is a non-degenerate bilinear form.

Notation fixing

- *Riemman* if the sign of g is positive definite, *Pseudo-Riemman* otherwise.

- *Lorentzian* if the signature is $(+, -, \ldots, -)$ or equivalently $(-, +, \ldots, +)$.

Observation 6: Causal Structure

If a smooth manifold is endowed with a Lorentzian manifold of signature $(+, -, \ldots, -)$ then the tangent vectors at each point in the manifold can be classed into three different types.

Notation fixing

$\forall p \in M, \quad \forall X \in T_p M$, the vector is:

- *time-like* if $g(X, X) > 0$.

- *light-like* if $g(X, X) = 0$.

- *space-like* if $g(X, X) < 0$.

Observation 7: Local Time Orientability

$\forall p \in M$ the timelike tangent vectors in p can be divided into two equivalence classes taking

$$X \sim Y \text{ iff } g(X, Y) > 0 \qquad \forall X, Y \in T_p^{\text{time-like}} M :$$

We can (arbitrarily) call one of these equivalence classes "future-directed" and call the other "past-directed". Physically this designation of the two classes of future- and past-directed timelike vectors corresponds to a choice of an arrow of time at the point. The future- and past-directed designations can be extended to null vectors at a point by continuity.

Definition 17: Time-orientation

A global tangent vector field $\mathfrak{t} \in \Gamma^\infty(TM)$ over the Lorenzian manifold M such that:

- $\text{supp}(\mathfrak{t}) = M$

- $\mathfrak{t}(p)$ is time-like $\forall p \in M$.

Observation 8

The fixing of a time-orientation is equivalent to a consistent smooth choice of a local time-direction.

Definition 18: Time-Orientable Lorentzian Manifold

A Lorentzian Manifold (M, g) such that exist at least one time-orientation $t \in \Gamma^\infty(TM)$.

Notation fixing

Consider a piece-wise smooth curve $\gamma : \mathbb{R} \supset I \to M$ is called:

- *time-like* (resp. light-like, space-like) iff $\dot{\gamma}(p)$ is time-like (resp. light-like, space-like) $\forall p \in M$.

- *causal* iff $\dot{\gamma}(p)$ is nowhere spacelike.

- *future directed* (resp. past directed) iff is causal and $\dot{\gamma}(p)$ is future (resp. past) directed $\forall p \in M$.

Definition 19: Chronological $^{\text{future}}_{\text{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{I}_M^\pm(p) := \left\{ q \in M \,\middle|\, \exists \gamma \in C^\infty\big((0,1), M\big) \text{ time-like } ^{\text{future}}_{\text{past}} - \text{directed} : \gamma(0) = p, \gamma(1) = q \right\}$$

Definition 20: Causal $^{\text{future}}_{\text{past}}$ of a point

Are two subset related to the generic point $p \in M$:

$$\mathbf{J}_M^\pm(p) := \left\{ q \in M \,\middle|\, \exists \gamma \in C^\infty\big((0,1), M\big) \text{ causal } ^{\text{future}}_{\text{past}} - \text{directed} : \gamma(0) = p, \gamma(1) = q \right\}$$

Notation fixing

Former concept can be naturally extended to subset $A \subset M$:

- $\mathbf{I}_M^\pm(A) = \bigcup_{p \in A} \mathbf{I}_M^\pm(p)$

- $\mathbf{J}_M^\pm(A) = \bigcup_{p \in A} \mathbf{J}_M^\pm(p)$

Definition 21: Achronal Set

Subset $\Sigma \subset M$ such that every inextensible timelike curve intersect Σ at most once.

Definition 22: $^{\text{future}}_{\text{past}}$ Domain of dependence of an Achronal set

The two subset related to the generic achornal set $\Sigma \subset M$:

$$\mathbf{D}^{\pm}_M(\Sigma) := \left\{ q \in M \,\middle|\, \forall \gamma\, ^{\text{past}}_{\text{future}} \text{inextensible causal curve passing through } q : \gamma(I) \cap \Sigma \neq \emptyset \right\}$$

Notation fixing

$\mathbf{D}_M(\Sigma) :- \mathbf{D}^+_M(\Sigma) \cup \mathbf{D}^-_M(\Sigma)$ is called *total domain of dependence.*

Definition 23: Cauchy Surface

Is a subset $\Sigma \subset M$ such that:

- closed

- achronal

- $\mathbf{D}_M(\Sigma) \equiv M$

1.2.2 Causal Structure

1.2.3 Globally Hyperbolic Spacetimes

⚠ Def di dominio di dipendendenza footnote di definizione di spazio tempo (o subsection fatta apposta?) def cauchy surface Remark causal future past def globally hyperbolic Teorema sulle caratterizzazioni

Notation fixing

We denote the set of all the cauchy surfaces as $\mathscr{P}_C(M)$.

Glon iperbolic determina la fogliazione dello spazio tempo per superfici di cauchy La superficie di cauchy ÃÍ questa:

Definition 24: Cauchy surface

questo da la possibilitÃă della buona posizione dei problemi di cauchy.. fisicamente
ÃÍ la condizione minima per definire i dati iniziali dell'evoluzione dinamica. definisco
data...

Rapporto con la condizione sugli operatori...

No! La definizione di green hyperbolicity non garantisce invece l'esistenza
e unicitÃă del problema di cauchy associata

e non solo, anche l'esistenza degli operatori di green associati che sono
ingrediente fondamentale della costruzione di peierls

M ÃÍ glob iper e P ÃÍ green iper per tener conto del comporatamento
propagativo definire sup cauchy definire s-t iperbolico (solo la caratteriz-
zazione di ammetre una sup di cauchy) definire op green iperbolico su spazio
tempo iperbolico (cioÃÍ ha delle green ope) Propr di buona definizione es-
istenza e unicita della soluzione

Di particolare ricorrenza fisica sono gli operatori normally iperbolic espres-
sione in coordinate esempio K-g!

Far notare che minkowski e tanti esempi importanti sono GH

Observation 9

(che serve dopo) lo spazio R ÃÍ banalmente iperbolico in quanto tutti i punti
posso essere visti come superfici di cauchy.

1.3   Green Hyperbolic Operators

Mettere solo le definizioni che uso prese dagli articoli di review delle Fonti

Pensavo di utilizzare la definizione di Green hyperbolic data da Bar che si
avvale del concetto di formally dual (che non richiede la presenza del pair-
ing) invece di quella usata in Advances AQFT che richiede solo che ammetta
almeno un $G^{\pm}$ per poi dimostrare tramite teorema che se ÃÍ anche autoag-
giunto vale l'unicitÃă. Si tratta solo di una piccola sfumatura.. Deve essere
chiarito che in tutto ciÃš che faccio interessano che

$$\forall P \, \exists 1! \, G^{\pm}$$

. Che poi questa condizione derivi da GH secondo bar o Gh secondo dap+selfadj
ÃÍ una di quelle questioni propriamente matematiche che poco interessa ai
fisici della commissione.

Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. [?] [?]

Basic Definition in L.P.D.O. on smooth vector sections.

Consider $F = F(M, \pi, V), F' = F'(M, \pi', V')$ two linear vector bundle over M with different typical fiber

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Definition 25: Linear Partial Differential operator (**of order at most** $s \in \mathbb{N}_0$)

Linear map $L : \Gamma(F) \to \Gamma(F')$ such that:
$\forall p \in M$ exists:

- (U, ϕ) local chart on M.

- (U, χ) local trivialization of F

- (U, χ') local trivialization of F'

for which:
$$L\big(\sigma\big|_U\big) = \sum_{|\alpha| \le s} A_\alpha \partial^\alpha \sigma \qquad \forall \sigma \in \Gamma(M)$$

Remark:

(multi-index notation)
A multi-index is a natural valued finite dimensional vector $\alpha = (\alpha_0, \ldots, \alpha_n - 1) \in \mathbb{N}_0^n$ with $n < \infty$.
On $\mathbb{R}^n$ a general differential operator can be identified by a multi-index:

$$\partial^\alpha = \prod_{\mu=0}^{n-1} \partial_\mu^{\alpha_\mu}$$

(Until the Schwartz theorem holds, the order of derivation is irrelevant.)
The order of the multi-index is defined as:

$$|\alpha| := \sum_{\mu=0}^{n-1} \alpha_\mu$$

?????????????????????

Proposition 1.3.1 (Existence and uniqueness for the Cauchy Problem)
> $\mathbf{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ *a globally hyperbolic space-time.*

- $\Sigma \subset M$ *a spacelike cauchy surface with future-pointing unit normal vector field $\vec{n}$.*

-

-

Observation 10

"Green-hyperbolic operators are not necessarily hyperbolic in any PDE-sense and that they cannot be characterized in general by well-posedness of a Cauchy problem. " [?] [?]
However the existence and uniqueness can be proved for the large class of the *Normally-Hyperbolic Operators.*