ON THE L_∞-ALGEBRA ASSOCIATED TO HIGHER COURANT ALG.OIDS

$$\left(\begin{array}{l}\text{TO ANY}\\ \text{SMOOTH MFD } M\\ r\in\mathbb{N}\quad r\geq 1\end{array}\right) \xrightarrow[\text{[Zam10]}]{\text{WE CAN ASSOCIATE}} \left(\begin{array}{l}\text{A } L_\infty\text{-ALGEBRA}\\ \text{CONCENTRATED IN DEGREES}\\ (-r+1,\dots,0)\end{array}\right)$$

IDEA

1) CONSIDER THE GRADED MANIFOLD

$$Q = T^*[r]T[1]M$$

2) OBSERVE THAT $\mathcal{C} = \mathcal{C}(Q)$ FORMS A DGLA

3) APPLY "GETZLER CONSTRUCTION" [Get10]
(YIELDS A L_∞-ALGEBRA OUT OF ANY DGLA)

WHY?

a) THE ABOVE L_∞-ALGEBRA SUITABLY EXTENDS THE STANDARD COURANT BRACKETS ON $E^{r-1} = TM \oplus \Lambda^{r-1}T^*M$

(HIGHER COURANT ALG.OID OR VINOGRADOV ALG.OID)

b) CONSIDER $\omega \in \Omega^{r+1}(M)$ CLOSED

$\rightsquigarrow$ TWIST THE COURANT ALGEBROID

$\rightsquigarrow$ TWIST THE L_∞-ALGEBRA : $L_\infty(E,\omega)$

c) CONSIDER $\omega \in \Omega^{r+1}(M)$ r-PLECTIC

THM [MZ22] THERE'S AN EMBEDDING OF L_∞-ALGEBRAS

$$L_\infty(M,\omega) \hookrightarrow L_\infty(E,\omega)$$

ROGER'S MULTISYMPLECTIC OBSERVABLES

HONEST SCOPE OF THE TALK : DISCUSS THE GRADED GEOMETRY UNDERLYING THE DGLA (STEP 2)

DISCLAIMER : I'M A NOVICE IN GRADED GEOMETRY!
IN THE SPIRIT OF THE "WORKING GROUP" WE SKIP DETAILS ON SIGNS'

TOC: 1) PREAMBLE ON GRADED MULTIDERIVATIONS

2) REMINDER ON GRADED MFDS

3) GEO/ALG. STRUCTURE OF $\mathcal{C}(T^*[r]T[1]M)$

MAINLY TAKEN FROM [Cat06, §2]

⊛ MY NOTATIONS IN GRADED LINEAR ALGEBRA

- GRADED V. SPACES ARE FAMILIES

$$V = \{V_i\}_{i\in\mathbb{Z}} \qquad \text{with} \qquad V_i \in \mathrm{Vect}$$

- i.e. $V \in \mathrm{GVect}$ IS A FUNCTOR $\mathbb{Z} \longrightarrow \mathrm{Vect}$

- DENOTE $V^{\oplus} = \bigoplus_{i\in\mathbb{Z}} V_i$ THE CORRESPONDING ORDINARY V. SPACE

- GIVEN V, W GVECTOR SPACES

  - $\mathrm{Hom}(V,W) = \left(\begin{array}{l}\text{GRADED V. SPACE OF GRADED HOMOMORPHISMS} \\ \phi : V \to W \ \text{ s.t. } \ \phi_i(V_i) \subseteq W_i \ \forall i\end{array}\right)$

  - $\underline{\mathrm{Hom}}_k(V,W) = \mathrm{Hom}(V, W[k])^{\oplus}$ $\left(\begin{array}{l}\text{ORDINARY V.SPACE OF GRADED MAPS IN DEGREE } k \\ \phi : V \to W \ \text{ s.t. } \\ \phi_j(V_j) \subseteq W_{k+j}\end{array}\right)$

  - $\underline{\mathrm{Hom}}_k(V,W) =$ GRADED V. SPACE OF GRADED MAPS.

(1) GRADED MULTIDERIVATIONS

- CONSIDER $\mathcal{A} = (A, \cdot, 1)$ UNITAL GRADED COMMUTATIVE ASSOCIATIVE GRADED ALGEBRA

DEF: DERIVATIONS

$$\mathrm{Der}(A) = \left\{ D \in \underline{\mathrm{Hom}}(A, A) \;\middle|\; D(ab) = D(a)\cdot b + (-)^{|D||a|} a D(b) \right\}$$

Lem:
- $\mathrm{Der}(A)$ IS AN A-MODULE
- $\mathrm{Der}(A)$ IS A GLA w/

$$[D_1, D_2] = D_1 \circ D_2 + (-)^{|D_1||D_2|} D_2 \circ D_1$$

DEF. MULTIDERIVATIONS IN DEGREE ℓ AND ARITY k

$$\mathrm{MultiDer}^k_\ell(A) = \left\{ P \in \underline{\mathrm{Hom}}_\ell(\wedge^k A, A) \;\middle|\; P(ab, \dots) = \pm P(a, \dots)\cdot b + (-)^{|P||a|} a P(b, \dots) \right\}$$

$(-)^{|b|\cdot|\cdots|}$

GRADED SKEW MULTILINEAR MAPS

DERIVATIONS IN EACH SEPARATE ENTRY

- ALLTOGETHER THEY GIVE A BIGRADED VECTOR SPACE

NOTATION: $\mathfrak{X}^k_\ell(A) = \mathrm{MultiDer}^k_\ell(A) \subseteq \underline{\mathrm{Hom}}_\ell(\wedge^k A, A)$

DEF: WEDGE OF MULTIDERIVATIONS

$$\wedge : \mathfrak{X}^p_k(A) \otimes \mathfrak{X}^q_\ell(A) \longrightarrow \mathfrak{X}^{p+q}_{k+\ell}(A)$$

$$P, Q \longmapsto (P \wedge Q)(f_1, \dots, f_{p+q}) = \sum_{\sigma \in S_{p,q}} \chi(\sigma) (-)^{|Q|\cdot(|f_{\sigma_1}| + \dots + |f_{\sigma_p}|)} P(f_{\sigma_1} \dots f_{\sigma_p}) \cdot Q(f_{\sigma_{p+1}} \dots f_{\sigma_{p+q}})$$

UNSHUFFLES PERMUTATION

DEF: SCHOUTEN BRACKET

$$[\cdot,\cdot]_S : \mathfrak{X}^p_k(A) \otimes \mathfrak{X}^q_\ell(A) \longrightarrow \mathfrak{X}^{p+q-1}_{k+\ell}(A$$

where

$$[P,Q]_S(a_1 \cdots a_{p+q-1}) = \sum_{\sigma\in S_{q,p-1}} \chi(\sigma)\, P\big(Q(a_{\sigma_1}\cdots a_{\sigma_q}), a_{\sigma_{q+1}} \cdots a_{\sigma_{q+p-1}}\big)$$

$$- (-)^{\|P\|\cdot\|Q\|} \sum_{\sigma\in S_{p,q-1}} \chi(\sigma)\, Q\big(P(a_{\sigma_1}\cdots a_{\sigma_p}), a_{\sigma_{p+1}} \cdots a_{\sigma_{p+q-1}}\big)$$

with $\|P\| = p+k-1$

NOTATION $\mathcal{X}^\bullet(A) = \mathrm{Tot}(\mathfrak{X}^\bullet_\bullet(A))$

Lem: • $\mathcal{X}^\bullet(A)$ IS AN A-MODULE

• $(\underbrace{\mathcal{X}^\bullet(A)}, \underbrace{\wedge}, \underbrace{[\cdot,\cdot]_S})$ IS A 1-POISSON ALGEBRA

GRADED V.SPACE

GRADED COMMU ASSOCIATIVE PRODUCT

DEGREE −1 LIE BRACKET SATISFYING n-SHIFTED (n−1) LEIBNIZ

$$[a, b\wedge c] = [a,b]\wedge c + (-)^{|b|(|a|+n)}\, b\cdot[a,c]$$

SEE • [CFL06] § 1.2. : DEF OF n-POISSON ALGEBRA

• [LGPV 06] § 3.3.2. : PROOF OF 1-POISSON (UNGRADED CASE $k=\ell=0$)

UPSHOT : $\mathcal{X}(A)$ IS NATURALLY 1-POISSON

EXAMPLE: let M be a SMOOTH MFD

$$\mathfrak{X}(M) := \Gamma(TM) \cong \mathrm{Der}(C^\infty(M))$$

$$\mathcal{X}^k(M) := \Gamma(\wedge^k TM) \cong \mathrm{Mult.Der}(C^\infty(M))$$

IS

$$\bigwedge^k_{C^\infty(M)} \Gamma(TM) \cong \bigwedge^k_{C^\infty(M)} \mathrm{Der}(C^\infty(M)) = S^k\left(\mathrm{Der}(C^\infty(M))[1]\right)$$

GEOMETRIC CHARACTERIZATION OF MULTIVECTOR FIELDS

(here $[\cdot,\cdot]_S$ is CALLED SCHOUTEN-NIJENHUIS BRACKET)

OSS: FOR A GENERAL CASE WE CAN MORE WEAKLY CONCLUDE THAT

$$S^k_A \mathrm{Der}(A)[1] = \wedge^k_A \mathrm{Der}(A) \longleftrightarrow \mathrm{Mult.Der}(A)$$

$$D_1 \wedge \dots \wedge D_k \longmapsto \left(f_1 \cdots f_k \longmapsto \pm \sum_{\sigma \in S_n} D_1(f_{\sigma_1}) \cdots D_k(f_{\sigma_k}) \right)$$

SIGN COMING FROM CONVENTION
$D_1 \otimes D_2 (f_1, f_2) = (-)^{|D_2||f_1|} D_1(f_1) \otimes D_2(f_2)$

REM: IN SEVERAL GG. REFERENCES ([CaH06], [CFL06, §2.1]) ONE DIRECTLY STARTS FROM THE SUBCASE SAYING THAT

$$\mathcal{X}^k(A) \equiv S^k(\mathrm{Der}(A)[-1])$$

FURTHER DEFINING "n-SHIFTED j-DERIVATION" AS ELEMENT IN

$$\mathcal{X}^k(A,n) \equiv S^k(\mathrm{Der}(A)[-n])$$

COROLL: $\mathcal{X}^\bullet(A,n)$ IS A n-POISSON ALGEBRA

LEM [CFL06, §2.1] GIVEN $\mathfrak{g}$ a GLA
$\Rightarrow S(\mathfrak{g}[n])$ IS A n-POISSON ALGEBRA

Proof: • ON $S(g[m])$ THERE IS AN ASSOCIATIVE ALGEBRA STRUCTURE INDUCED BY THE SYMMETRIC TENSOR PRODUCT

$$S(g[m]) \times S(g[m]) \xrightarrow{\;\cdot\;} S(g[m])$$

• THE $[m]$-SUSPENSION OF $[\cdot,\cdot]_g$ GIVES THE POISSON BRACKET IN LOWER DEGREES

$$\{\cdot,\cdot\} : \underset{g[m]}{\underset{\parallel}{S^1(g[m])}} \times \underset{g[m]}{\underset{\parallel}{S^1(g[m])}} \longrightarrow \underset{g[m]}{\underset{\parallel}{S^1(g[m])}}$$

$$a_{[m]} \quad b_{[m]} \longmapsto [a,b]_{g\,[m]}$$

• OTHER CASES ARE DEFINED INDUCTIVELY ENFORCING THE LEIBNIZ RULE

$$\{a, b\cdot c\} = \{a,b\}\cdot c + (-)^{|b|(|a|+m)}\, b\cdot\{a,c\}$$

i.e. $\{\cdot,\cdot\} : S^k g[m] \times S^e g[m] \longrightarrow S^{k+e-1} g[m]$

$$a_1, \dots a_k,\ b_1,\dots, b_e \longmapsto \sum_{i,j} \pm \{a_i, b_j\}\cdot a_1 \cdots \hat{a}_i \cdots a_k \cdot b_1 \cdots \hat{b}_j \cdots b_e$$

$\square$

Rem: TECHNICAL DETAIL! LET V a GVS

$S(V)$ = FREE GRADED COMM. ASSOCIAT. ALG. OVER V

$$= \bigoplus_{m\geq 0} S^m V = \bigoplus_{m\geq 0} \ker\left(P_a = \frac{1}{m!}\sum_{\sigma\in S_m} \sigma : V^{\otimes m} \to V^{\otimes m}\right)$$

DIRECT SUM: ELEMENTS ARE FINITE STRINGS OF $x_1 \odot \cdots \odot x_m$

$\not\parallel$

$$\hat{S}(V) = \prod_{m\geq 0} S^m V$$

DIRECT PRODUCT ELEMENTS ARE POTENTIALLY INFINITE STRINGS OF ELEMENTS

I.e. DIRECT LIMIT

e.g.: $S(V^*)$ = POLYNOMIAL FUNCTIONS ON V TO $\mathbb{R}$

$\cap$ FORMAL COMPLETION

$\hat{S}(V^*)$ = SMOOTH (ANALYTIC) FUNCTIONS ON V TO $\mathbb{R}$

⚠ (I WILL GLOSS OVER THIS DETAIL!) ⚠

② GRADED MANIFOLDS

Def: GRADED MANIFOLD

a PAIR $\mathcal{M} = (M, A_M)$

M — SMOOTH MFD

A_M — UNITAL ASSOCIATIVE GRADED COMMUTATIVE GRADED ALGEBRA

SUCH THAT $A_M \cong \Gamma(S(E^*))$

FOR SOME GRADED VEC. BUN. $E \to M$

NOT: "ALGEBRA OF FUNCTIONS OF M" $\mathcal{C}(\mathcal{M}) = A_M$

Def: GRADED VEC. BUNDLE OVER M $(E_\bullet \to M) = E_\bullet$

IS A COLLECTION OF V.BUNS. $\{E_k \to M\}_{k \in \mathbb{Z}}$

$$\begin{array}{ccccccc} \dots & E_{-1} & \dots & E_0 & & E_1 & \dots \\ & & \searrow & \downarrow & \swarrow & & \\ & & & M & & & \end{array}$$

Def: DUAL G.V.BUN $(E^*)_k = (E_{-k})^*$

SHIFT GVBUN $(E[m])_k = E_{k+m}$

Rem: $(E[m])^* = E^*[-m]$

NOT: "ALGEBRA OF FUNCTIONS OF $E_\bullet$": $\mathcal{C}(E_\bullet) = \Gamma(S(E^*))$

MOTIVATING EXAMPLE: $E_\bullet = V \to *$ $\Rightarrow$ $\mathcal{C}(E_\bullet)$ = POLYNOMIAL FUNCTION OVER V

OR SMOOTH IF I UNDERSTAND S AS ITS COMPLETION

- GIVEN $(M, A_M = \mathcal{C}(M))$ A G.MFD WE CAN DEFINE MULTI-VECTOR FIELDS IN AN ALGEBRAIC FASHION (AS MULTIDERIVATIONS)

Def: SPACE OF V. FIELDS OVER M

$$\mathfrak{X}(M) := \mathrm{Der}(C^\infty(M))$$

← INTERPRETED AS THE $C^\infty(M)$-MODULE

Def: SPACE OF MULTI V.FIELDS OVER M

$$\mathcal{X}(M) = \mathrm{MultiDer}(C^\infty(M)) \cong S_{C^\infty(M)}\left(\mathrm{Der}(C^\infty(M))[-1]\right)$$

$C^\infty(M)$-MODULE AND ASSO. GRADED COMM ALGEBRA W/ $\wedge$

Def: SPACE OF n-SHIFTED MULTIVECTOR FIELDS

$$\mathcal{X}(M,n) := S_{C^\infty(M)}\left(\mathrm{Der}(C^\infty(M))[-n]\right)$$

NATURALLY n-POISSON WITH SCHOUTEN CONSTRUCTION

* LET'S CONSIDER SPECIAL EXAMPLES RELEVANT TO THE LATTER

Ex.1: $E = T[1]M \ (= TM[1])$ ← IS THE TANGENT BUNDLE OF M SEEN AS CONCENTRATED IN DEG = -1

by DEFINITION:

$$\begin{aligned}\mathcal{C}(T[1]M) &:= \Gamma\left(S\left(T[1]M\right)^*\right)\\ &= \Gamma\left(S(T^*M[1])\right)\\ &= \Gamma(\wedge T^*M) \cong \Omega(M)\end{aligned}$$

Ex.2 $E = T^*[1]M \ (= T^*M[1])$

by DEFINITION

$$\begin{aligned}\mathcal{C}(T^*[1]M) &:= \Gamma\left(S\left(T^*[1]M\right)^*\right)\\ &= \Gamma\left(S(TM[-1])\right) = \Gamma(\wedge TM) = \mathrm{MultiDer}(C^\infty(M))\\ &\cong \wedge_{C^\infty}\Gamma(TM) = \wedge_{C^\infty}\mathrm{Der}(C^\infty(M)) = \mathcal{X}(M)\end{aligned}$$

(IN BOTH CASES $\mathcal{C}(E)$ IS CONCENTRATED IN DEGREES $(0,\dots,\dim(M))$)

Ex. 3: $E = T^*[r]M$

$$\mathcal{C}(T^*[r]M) = \Gamma(S(T^*M[r])^*$$
$$= S_{C^\infty(M)}\Gamma(TM)[-r]$$
$$= S_{C^\infty(M)}\mathrm{Der}(C^\infty(M))[-r] = \mathfrak{X}(M,r)$$

HENCE: NATURALLY r-POISSON!

(*) WE ARE READY TO DEAL WITH OUR MANIFOLD

$Q = T^*[r]T[1]M$

$$\mathcal{C} = \Gamma(S((T^*[r]T[1]M)^*))$$
$$= \Gamma(S(T(T[1]M)[-r]))$$
$$= \Gamma(S(\mathrm{Der}(\mathcal{C}(T[1]M))[-r])) = \underbrace{\mathfrak{X}(T[1]M, r)}$$
$$= \Gamma(S(\mathrm{Der}(\Omega(M))[-r]))$$

NATURALLY POISSON!

STILL A LITTLE BIT OBSCURE!
TO UNDERSTAND IT BETTER ONE SHOULD WORK IN G.V.BUN. LOCAL COORDINATES AND LOOK FOR AN EXPLICIT EXPRESSION OF THE CANONICAL POISSON BRACKET