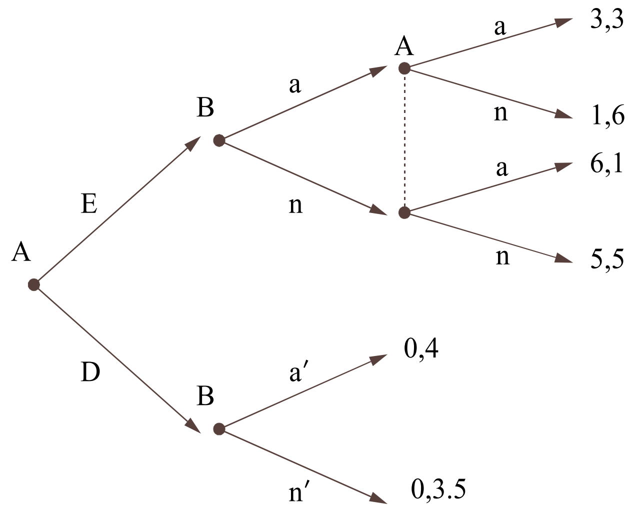
Part III
Solutions to the Exercises

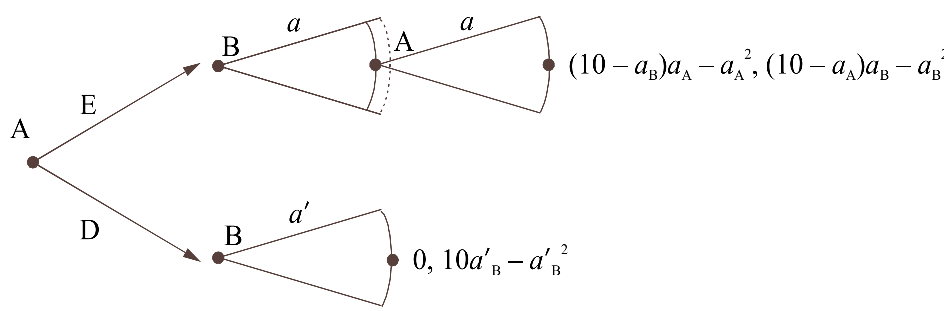
This part contains solutions to all of the exercises in the textbook. Although we worked diligently on these solutions, there are bound to be a few typos here and there. Please report any instances where you think you have found a substantial error.

2 The Extensive Form

1.

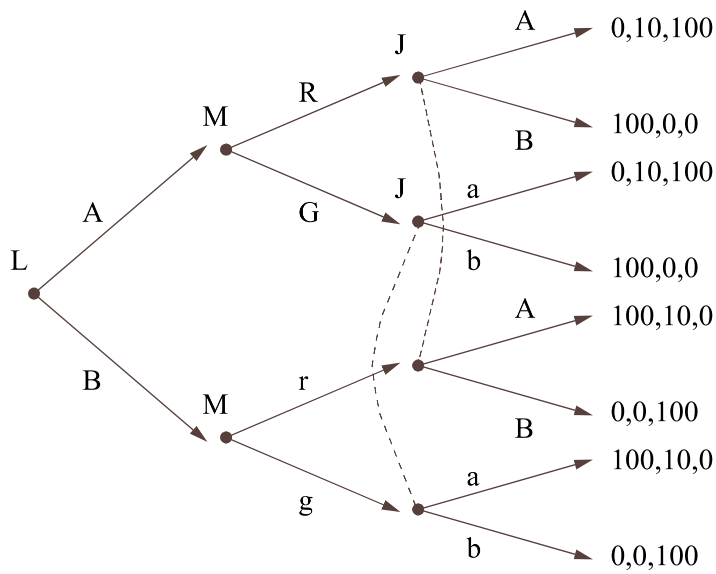


(a)



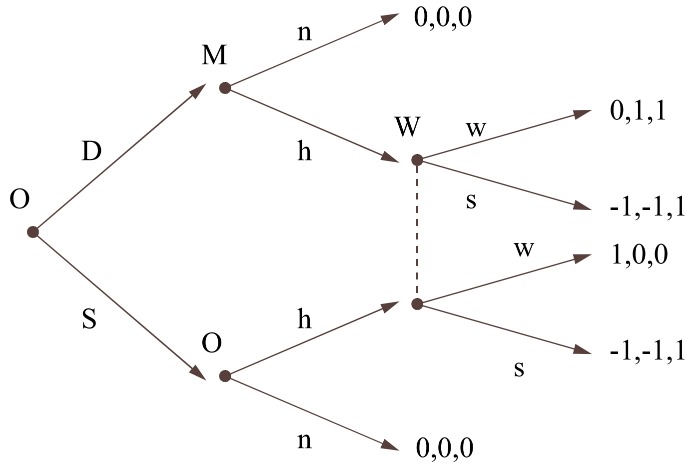
(b)

2.



3.

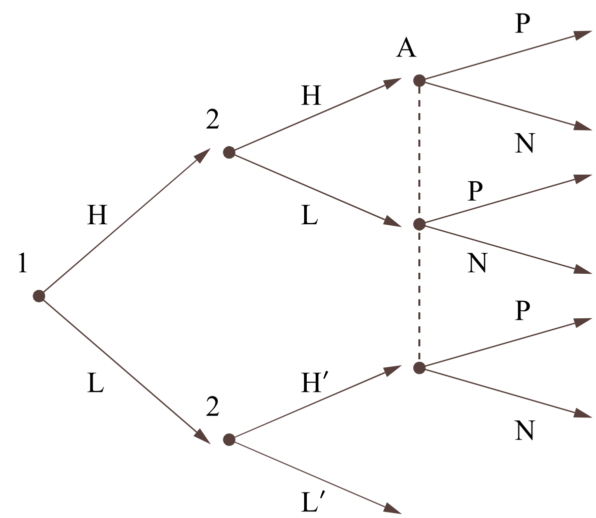
(a)



(b) Incomplete information. The worker does not know who has hired him/her.

4.

Note that we have not specified payoffs as these are left to the students.



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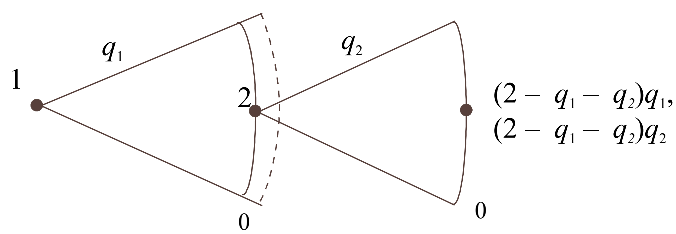

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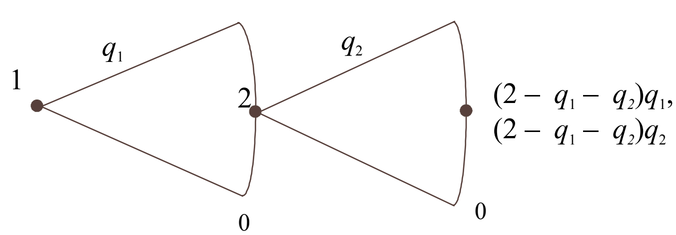

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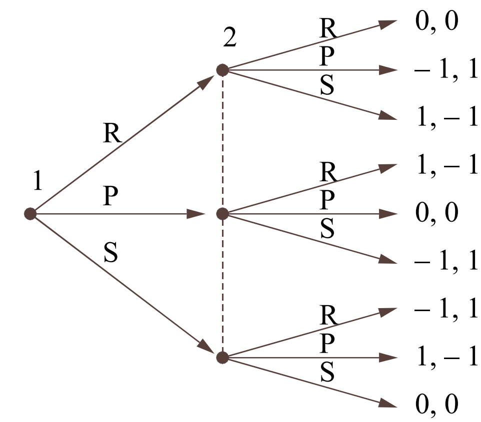
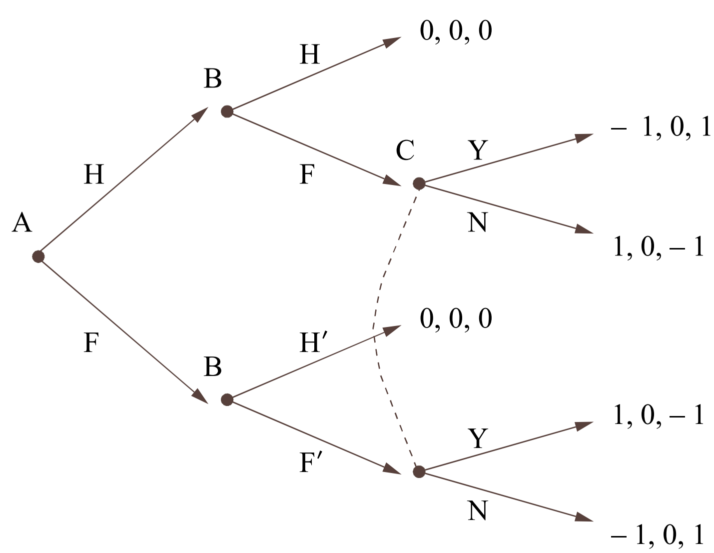


8.

It does not matter as it is a simultaneous move game.

9.

The payoffs below are in the order A, B, C.



3 Strategies

1.

Recall that a strategy for player i must describe the action to be taken at each information set where player i is on the move.

S_K = {LPR, LPE, LNR, LNE, SPR, SPE, SNR, SNE}. S_E = {P,N}.

2.

S_A = {Ea, En, Da, Dn}. S_B = {aa′, an′, na′, nn′}.

3.

S_L = {A,B}. S_M = {Rr, Rg, Gr, Gg}. S_J = {Aa, Ab, Ba, Bb }.

4.

No, "not hire" does not describe a strategy for the manager. A strategy for the manager must specify an action to be taken in every contingency. However, "not hire" does not specify any action contingent upon the worker being hired and exerting a specific level of effort.

5.

No, RR does not describe a strategy for a player. The specification of the player's action in the second play of rock/paper/scissors must be contingent on the outcome of the first play. That is, the player's strategy must specify an action to be taken (in the second play) given each possible outcome of the first play.

6.

$S_1 = \{\text{I, O}\} \times [0, \infty)$. $S_2 = \{\text{H, L}\} \times \{\text{H}', \text{L}'\}$.

4 The Normal Form

1.

A player's strategy must describe what he will do at each of his information sets.

K \ E	P	O
LPR	40,110	80,0
LPE	13,120	80,0
LNR	0,140	0,0
LNE	0,140	0,0
SPR	35,100	35,100
SPE	35,100	35,100
SNR	35,100	35,100
SNE	35,100	35,100

2.

A \ B	aa′	an′	na′	nn′
Ea	3,3	3,3	6,1	6,1
En	1,6	1,6	5,5	5,5
Da	0,4	0,3.5	0,4	0,3.5
Dn	0,4	0,3.5	0,4	0,3.5

3.

(a)

1 \ 2	CE	CF	DE	DF
A	0,0	0,0	1,1	1,1
B	2,2	3,4	2,2	3,4

(b)

1 \ 2	I	O
IU	4,0	-1,-1
ID	3,2	-1,-1
OU	1,1	1,1
OD	1,1	1,1

(c)

1 \ 2	AC	AD	BC	BD
UE	3,3	3,3	5,4	5,4
UF	3,3	3,3	5,4	5,4
DE	6,2	2,2	6,2	2,2
DF	2,6	2,2	2,6	2,2

(d)

1 \ 2	A	B
UXW	3,3	5,1
UXZ	3,3	5,1
UYW	3,3	3,6
UYZ	3,3	3,6
DXW	4,2	2,2
DXZ	9,0	2,2
DYW	4,2	2,2
DYZ	9,0	2,2

(e)

1 \ 2	U	D
A	2,1	1,2
B	6,8	4,3
C	2,1	8,7

(f)

1 \ 2	A	B
UXP	3,8	1,2
UXQ	3,8	1,2
UYP	8,1	2,1
UYQ	8,1	2,1
DXP	6,6	5,5
DXQ	6,6	0,0
DYP	6,6	5,5
DYQ	6,6	0,0

4.

The normal form specifies player, strategy spaces, and payoff functions. Here $N = \{1, 2\}$. $S_i = [0, \infty)$. The payoff to player i is give by $u_i(q_i, q_j) = (2 - q_i - q_j)q_i$.

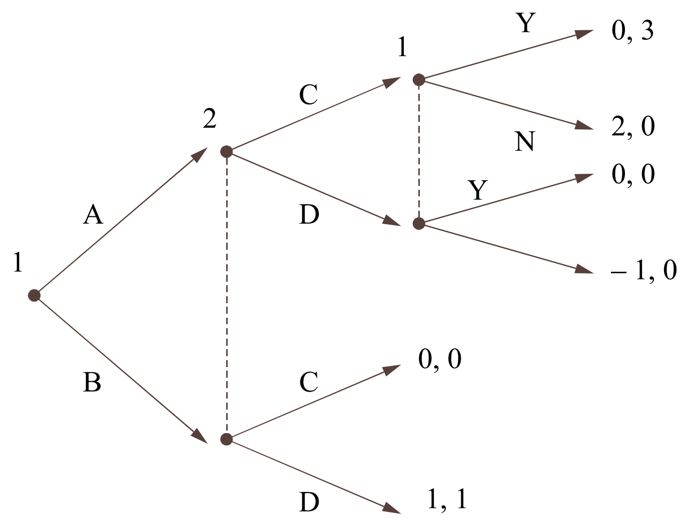
5.

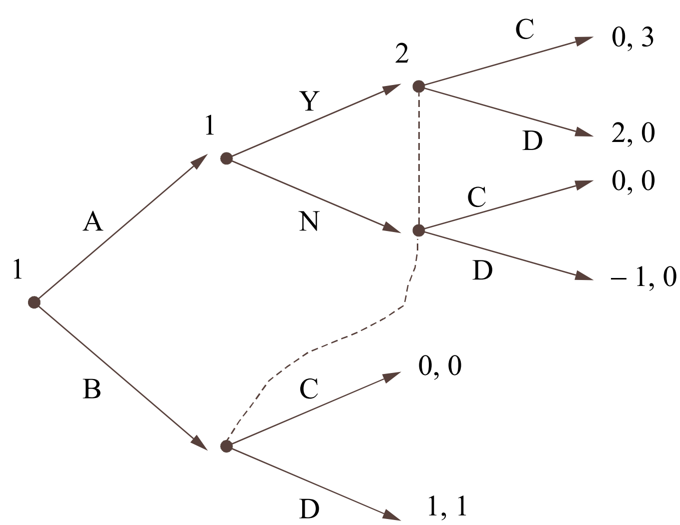
$N = \{1, 2\}$. $S_1 = [0, \infty)$. Player 2's strategy must specify a choice of quantity for each possible quantity player 1 can choose. Thus, player 2's strategy space S_2 is the set of functions from $[0, \infty)$ to $[0, \infty)$. The payoff to player i is give by $u_i(q_i, q_j) = (2 - q_i - q_j)q_i$.

6.

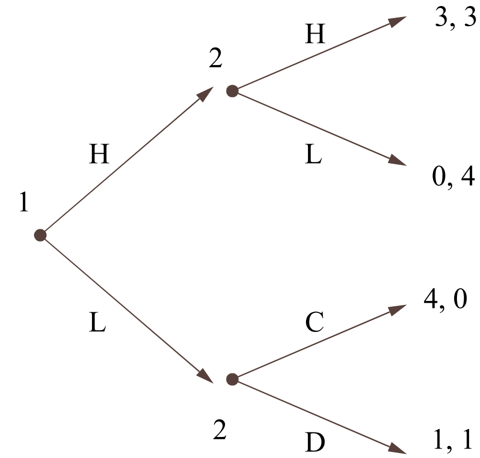
Some possible extensive forms are shown below and on the next page.

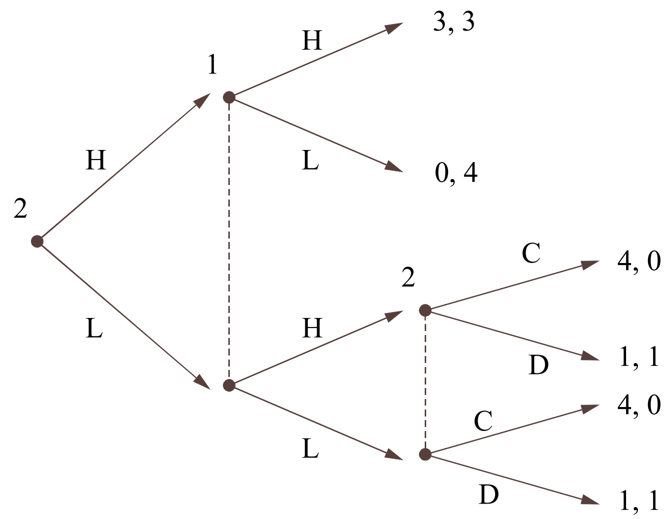
(a)





(b)





5 Beliefs, Mixed Strategies, and Expected Utility

1.

(a) u_1(U,C) = 0.

(b) u_2(M,R) = 4.

(c) u_2(D,C) = 6.

(d) For σ_1 = (1/3, 2/3, 0) $u_1(\sigma_1$,C) = 1/3(0) + 2/3(10) + 0 = 6 2/3.

(e) $u_1(\sigma_1$,R) = 5 1/4.

(f) $u_1(\sigma_1$, L) = 2.

(g) $u_2(\sigma_1$, R) = 3 1/3.

(h) $u_2(\sigma_1, \sigma_2)$ = 4 1/2.

2.

(a)

1 \ 2	X	Y
H	z, a	z, b
L	0, c	10, d

(b) Player 1's expected payoff of playing H is z. His expected payoff of playing L is 5. For $z = 5$, player 1 is indifferent between playing H or L.

(c) Player 1's expected payoff of playing L is 20/3.

3.

(a) $u_1(\sigma_1$,I) $= 1/4(2) + 1/4(2) + 1/4(4) + 1/4(3) = 11/4$.

(b) $u_2(\sigma_1$,O) $= 21/8$.

(c) $u_1(\sigma_1, \sigma_2) = 2(1/4) + 2(1/4) + 4(1/4)(1/3) + 1/4(2/3) + 3/4(1/3) + 14(2/3) = 23/12$.

(d) $u_1(\sigma_, \sigma_2) = 7/3$.

4.

Note that all of these, except "Pigs," are symmetric games.

Matching Pennies: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 0$.

Prisoners' Dilemma: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 2\ 1/2$.

Battle of the Sexes: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 3/4$.

Hawk-Dove/Chicken: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 1\ 1/2$.

Coordination: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 1/2$.

Pareto Coordination: $u_1(\sigma_1, \sigma_2) = u_2(\sigma_1, \sigma_2) = 3/4$.

Pigs: $u_1(\sigma_1, \sigma_2) = 3, u_2(\sigma_1, \sigma_2) = 1$.

5.

The expected profit of player 1 is $(100 - 28 - 20)14 - 20(14) = 448$.

6 Dominance and Best Response

1.

(a) B dominates A and L dominates R.

(b) L dominates R.

(c) 2/3 U 1/3 D dominates M. X dominates Z.

(d) none.

2.

(a) To determine the BR set we must determine which strategy of player 1 yields the highest payoff given her belief about player 2's strategy selection. Thus, we compare the payoff to each of her possible strategies.

$u_1(\text{U},\mu_2) = 1/3(10) + 0 + 1/3(3) = 13/3.$
$u_1(\text{M},\mu_2) = 1/3(2) + 1/2(10) + 1/3(6) = 6.$
$u_1(\text{D},\mu_2) = 1/3(3) + 1/3(4) + 1/3(6) = 13/3.$
$BR_1(\mu_2) = \{\text{M}\}.$

(b) $BR_2(\mu_1) = \{\text{L,R}\}.$

(c) $BR_1(\mu_2) = \{\text{U,M}\}.$

(d) $BR_2(\mu_1) = \{\text{C}\}.$

3.

Player 1 solves $\max_{q_1}(100 - 2q_1 - 2q_2)q_1 - 20q_1$. The first order condition is $100 - 4q_1 - 2q_2 - 20 = 0$. Solving for q_1 yields $BR_1(q_2) = 20 - q_2/2$. It is easy to see that $BR_1(0) = 20$. Since $q_2 \geq 0$, it cannot be that 25 is ever a best response. Given the beliefs, player 1's best response is 15.

4.

(a) First we find the expected payoff to each strategy: $u_1(U, \mu_2) = 2/6+0+4(1/2) = 7/3$; $u_1(M, \mu_2) = 3(1/6)+1/2 = 1$; and $u_1(D, \mu_2) = 1/6+1+1 = 13/6$. As the strategy U yields a higher expected payoff to player 1, given μ_2, $BR_1(\mu_2) = \{U\}$.

(b) $BR_2(\mu_1) = \{\text{R}\}.$

(c) $BR_1(\mu_2) = \{\text{U}\}.$

(d) $BR_1(\mu_2) = \{\text{U, D}\}.$

(e) $BR_2(\mu_1) = \{\text{L, R}\}.$

5.

1 \ 2	R	P	S
R	0, 0	–1, 1	1, –1
P	1, –1	0, 0	–1, 1
S	–1, 1	1, –1	0, 0

(a) $BR_1(\mu_2) = \{\text{P}\}$.

(b) $BR_1(\mu_2) = \{\text{R, S}\}$.

(c) $BR_1(\mu_2) = \{\text{P, S}\}$.

(d) $BR_1(\mu_2) = S_1$.

7 Rationalizability and Iterated Dominance

1.

(a) $R = \{\text{U, M, D}\} \times \{\text{L, R}\}$.

(b) Here there is a dominant strategy. So we can iteratively delete dominated strategies. U dominates D. When D is ruled out, R dominates C. Thus, $R = \{\text{U, M}\} \times \{\text{L, R}\}$.

(c) $R = \{(\text{U, L})\}$.

(d) $R = \{\text{A, B}\} \times \{\text{X, Y}\}$.

(e) $R = \{\text{A, B}\} \times \{\text{X, Y}\}$.

(f) $R = \{\text{A, B}\} \times \{\text{X, Y}\}$.

(g) $R = \{(\text{D, Y})\}$.

2.

Chapter 2, problem 1(a) (the normal form is found in Chapter 4, problem 2): $R = \{(\text{Ea, aa}'), (\text{Ea, an}')\}$. Chapter 5: $R = \{\text{U, M, D}\} \times \{\text{L, C, R}\}$.

3.

No. This is because 1/2 A 1/2 B dominates C.

4.

For "give in" to be rationalizable, it must be that $x \leq 0$. The manager must believe that the probability that the employee plays "settle" is (weakly) greater than 1/2.

5.

$R = \{(w, c)\}$. The order does not matter because if a strategy is dominated (not a best response) relative to some set of strategies of the other player, then this strategy will also be dominated relative to a smaller set of strategies for the other player.

6.

$R = \{(7{:}00,\ 6{:}00,\ 6{:}00)\}$.

7.

Yes. If s_1 is rationalizable, then s_2 is a best response to a strategy of player 1 that may rationally be played. Thus, player 2 can rationalize strategy s_2.

8.

No. It may be that s_1 is rationalizable because it is a best response to some other rationalizable strategy of player 2, say $\hat{s}_2$, and just also happens to be a best response to s_2.

8 Location and Partnership

1.

Here, one can view the payoffs as being the negative of the payoffs given by the game in the chapter. For player i, all strategies in $\{2, 3, 4, \ldots, 8\}$ are dominated. Thus, $R_i = \{1, 9\}$ for $i = 1, 2$.

2.

(a) Yes, preferences are as modeled in the basic location game. When the each player's objective is to maximize his/her probability of winning, the best response set is not unique. Suppose, for example, that player 2 plays 1 then $BR_1 = \{2, 3, 4, \ldots, 8\}$.

(b) Here, we should focus on $R_i^2 = \{3, 4, 5, 6, 7\}$. It is easy to see that if the regions are divided in half between 5 and 6 that 250 is distributed to each half. So unlike in the basic location model there is not a single region that is "in the middle". Thus, $R = \{5{,}6\} \times \{5{,}6\}$. In any of these outcomes, each candidate receives the same number of votes.

(c) When $x > 75$, player i's best response to 5 is 6, and his/her best response to 6 is 6. Thus, $R = \{(6, 6)\}$.

When $x < 75$, player i's best response to 6 is 5, and his/her best response to 5 is 5. Thus, $R = \{(5, 5)\}$.

3.

(a) $S_i = [0, \infty)$. $u_1(e_1, e_2) = t[a_1e_1 + a_2e_2] - e_1^2$. $u_2(e_1, e_2) = (1-t)[a_1e_1 + a_2e_2] - e_2^2$.

(b) Player 1 solves $\max_{e_1} t[a_1e_1 + a_2e_2] - e_1^2$, which gives $e_1 = ta_1/2$. Player 2 solves $\max_{e_2}(1-t)[a_1e_1 + a_2e_2] - e_2^2$, which gives $e_2 = (1-t)a_2/2$. Note that each of these optimal levels of effort does not depend on the opponent's choice of effort. Thus, $R = \{(ta_1/2, (1-t)a_2/2)\}$.

(c) To maximize the gross profit of the firm, you should solve

$$\max_t a_1(ta_1/2) + a_2((1-t)a_2/2) = \max_t a_2^2/2 + t[a_1^2/2 - a_2^2/2].$$

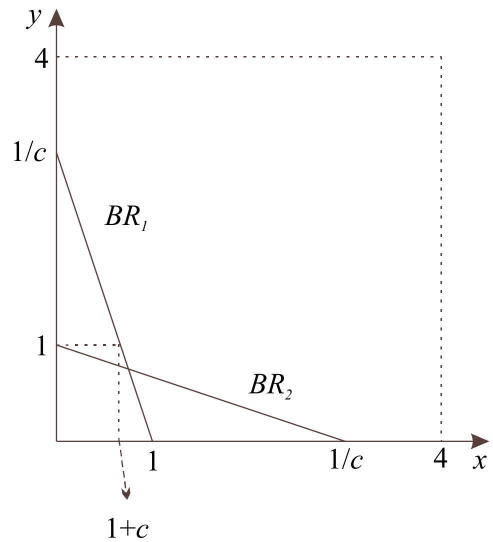
Note that the objective function is linear in t. Thus, maximization occurs at a "corner," where either $t = 0$ or $t = 1$. If $a_1 > a_2$ then it is best to set $t = 1$; otherwise, it is best to set $t = 0$. One can also consider maximizing the firm's return minus the partners' effort costs. Then the problem is

$$\max_t a_1(ta_1/2) + a_2((1-t)a_2/2) - (ta_1/2)^2 - ((1-t)a_2/2)^2$$

and the solution is to set $t = a_1^2/(a_1^2 + a_2^2)$.

4.

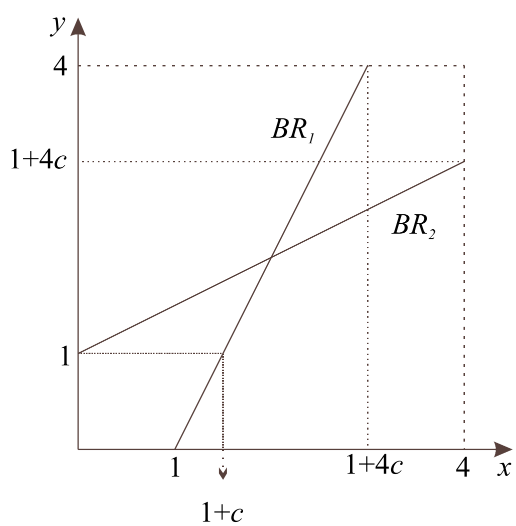
Recall from the text that $BR_1(\overline{y}) = 1 + c\overline{y}$, and $BR_2(\overline{x}) = 1 + c\overline{x}$. Assume $-1 < c < 0$. This yields the following graph of best response functions.



As neither player will ever optimally exert effort that is greater than 1, $R_i^1 = [0, 1]$. Realizing that player j's rational behavior implies this, $R_i^2 = [1 + c, 1]$. Continuing yields $R_i^3 = [1 + c, 1 + c + c^2]$. Repeating yields $R_i = \{\frac{1+c}{1-c^2}\} = \frac{1}{1-c}$.

5.

Recall from the text that $BR_1(\overline{y}) = 1 + c\overline{y}$, and $BR_2(\overline{x}) = 1 + c\overline{x}$. Assume $1/4 < c \leq 3/4$. This yields the following graph of best response functions.



Because player i will never optimally exert effort that is either less than 1 or greater than $1+4c$, we have $R_i^1 = [1, 1+4c]$. Because the players know this about each other, we have $R_i^2 = [1+c, 1+c(1+4c)]$. Repeating yields $R_i = \{\frac{1+c}{1-c^2}\} = \frac{1}{1-c}$.

Next suppose that $c > 3/4$. In this case, the functions $x = 1 + c\overline{y}$ and $y = 1 + c\overline{x}$ suggest that players would want to select strategies that exceed 4 in response to some beliefs. However, remember that the players' strategies are constrained to be less than or equal to 4. Thus, the best response functions are actually

$$BR_1(\overline{y}) = \begin{cases} 1 + c\overline{y} & \text{if } 1 + c\overline{y} \leq 4 \\ 4 & \text{if } 1 + c\overline{y} > 4 \end{cases}$$

and

$$BR_2(\overline{x}) = \begin{cases} 1 + c\overline{x} & \text{if } 1 + c\overline{x} \leq 4 \\ 4 & \text{if } 1 + c\overline{x} > 4 \end{cases}.$$

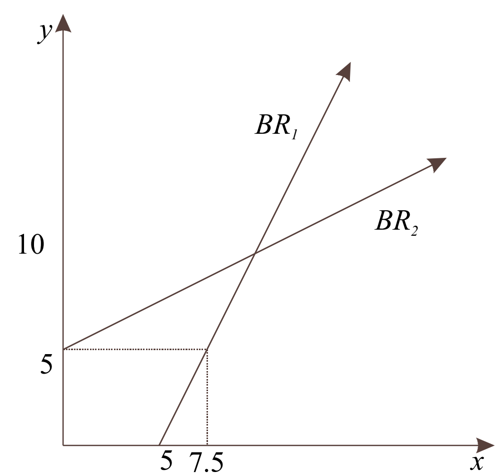
In this case, the best response functions cross at $(4,4)$, and this is the only rationalizable strategy profile.

6.

(a) $u_1(p_1, p_2) = [10 - p_1 + p_2]p_1$. $u_2(p_1, p_2) = [10 - p_2 + p_1]p_2$.

(b) $u_i(p_1, p_2) = 10p_i - p_i^2 + p_j p_i$. As above, we want to solve for p_i that maximizes i's payoff given $\overline{p}_j$. Solving for the first order condition yields $p_i(\overline{p}_j) = 5 + 1/2\overline{p}_j$.

(c) Here there is no bound to the price a player can select. Thus, we do not obtain a unique rationalizable strategy profile. The best response functions are represented below.



Similar to the above, we have $R_i^1 = [5, \infty)$ and $R_i^2 = [15/2, \infty)$. Repeating the analysis yields $R_i = [10, \infty)$ for $i = 1, 2$.

7.

We label the regions as shown below.

1	2	3
4	5	6
7	8	9

We first find the best response sets. Noticing the symmetry makes this easier. $BR_i(1) = \{2, 4, 5\}$; $BR_i(2) = \{5\}$; $BR_i(3) = \{2, 5, 6\}$; $BR_i(4) = \{5\}$; $BR_i(5) = \{5\}$; $BR_i(6) = \{5\}$; $BR_i(7) = \{4, 5, 8\}$; $BR_i(8) = \{5\}$; and $BR_i(9) = \{5, 6, 8\}$. It is easy to see that $\{1, 3, 7, 9\}$ are never best responses. Thus, $R_i^1 = \{2, 4, 5, 6, 8\}$. Since player i knows that player j is rational, he/she knows that j will never play $\{1, 3, 7, 9\}$. This implies $R_i^2 = R_i = \{5\}$.

8.

(a) No.

(b) $\sigma_i = (0, p, 0, 0, 1 - p, 0)$ dominates locating in region 1, for all $p \in (1/2, 1)$.

9 Congruous Strategies and Nash Equilibrium

1.

(a) The Nash equilibria are (B, CF) and (B, DF).

(b) The Nash equilibria are (IU, I), (OU, O) and (OD, O).

(c) The Nash equilibria are (UE, BD), (UF, BD), (DE, AC), and (DE, BC).

(d) There is no Nash equilibrium.

2.

(a) The set of Nash equilibria is $\{(\text{B, L})\} = R$.

(b) The set of Nash equilibria is $\{(\text{U, L}),(\text{M, C})\}$. $R = \{\text{U, M, D}\} \times \{\text{L, C}\}$.

(c) The set of Nash equilibria is $\{(\text{U, X})\} = R$.

(d) The set of Nash equilibria is $\{(\text{U, L}), (\text{D, R})\}$. $R = \{\text{U, D}\} \times \{\text{L, R}\}$.

3.

Figure 7.1: The Nash equilibrium is (B,Z).

Figure 7.3: The Nash equilibrium is (M,R).

Figure 7.4: The Nash equilibria are (stag,stag) and (hare,hare).

Exercise 1: (a) No Nash equilibrium. (b) The Nash equilibria are (U,R) and (M,L). (c) The Nash equilibrium is (U,L). (d) The Nash equilibria are (A,X) and (B,Y). (e) The Nash equilibria are (A,X) and (B,Y). (f) The Nash equilibria are (A,X) and (B,Y). (g) The Nash equilibrium is (D,Y).

Chapter 4, Exercise 2: The Nash equilibria are (Ea,aa′) and (Ea,an′).

Chapter 5, Exercise 1: The Nash equilibrium is (D,R).

Exercise 3: No Nash equilibrium.

4.

Only at (1/2, 1/2) would no player wish to unilaterally deviate. Thus, the Nash equilibrium is (1/2, 1/2).

5.

Player 1 solves $\max_{s_1} 3s_1 - 2s_1s_2 - 2s_1^2$. Taking s_2 as given and differentiating with respect to s_1 yields the first order condition $3 - 2s_2 - 4s_1 = 0$. Rearranging, we obtain player 1's best response function: $s_1(\overline{s}_2) = 3/4 - \overline{s}_2/2$. player 2 solves $\max_{s_2} s_2 + 2s_1s_2 - 2s_2^2$. This yields the best response function $s_2(\overline{s}_1) = 1/4 + \overline{s}_1/2$. The Nash equilibrium is found by finding the strategy profile that satisfies both of these equations. Substituting player 2's best response function into player 1's, we have $s_1 = 3/4 - 1/2[1/4 + s_1/2]$. This implies that the Nash equilibrium is (1/2, 1/2).

6.

(a) The congruous sets are S, {(z, m)}, and {w, y} × {k, l}.

(b) They will agree to {w, y} × {k, l}.

(c) No, there are four possible strategy profiles.

7.

Consider the game represented below. $X = S$ is best response complete. However, R $\notin BR_2(\mu_{-i})$ for any μ_{-i}.

1 \ 2	L	R
A	5, 5	5, 0
B	0, 1	0, 0

Consider the game represented below. $X = \{(\text{U, R})\}$ is weakly congruous. However, $BR_1(\text{R}) = \{\text{U, D }\}$.

1 \ 2	L	R
U	6, 2	3, 4
D	1, 5	3, 4

8.

The best response function for player i is given by $p_i(\overline{p}_j) = 5 + (1/2)\overline{p}_j$. Solving the system of equations for the two players yields $p_i^* = 5 + (1/2)[5 +$

$(1/2)p_i^*]$. Solving results in $p_i^* = 10$. The Nash equilibrium is given by the intersection of the best response functions (hence each player is best responding to the other). Here, the rationalizable set does not shrink on the upper end because no strategies higher than 10 can be ruled out.

9.

(a) The Nash equilibria are (2, 1), (5/2, 5/2), and (3, 3).

(b) $R = [2, 3] \times [1, 4]$.

10.

Consider the following game, in which (H, X) is an efficient strategy profile that is also a non-strict Nash equilibrium.

1 \ 2	X	Y
H	2, 2	1, 2
L	0, 0	0, 0

11.

(a) Play will converge to (D, D), because D is dominant for each player.

(b) Suppose that the first play is (opera, movie). Recall that BR_i(movie) = {movie}, and BR_i(opera) = {opera}. Thus, in round two, play will be (movie, opera). Then in round three, play will be (opera, movie). This cycle will continue with no equilibrium being reached.

(c) In the case of strict Nash equilibrium, it will be played all of the time. The non-strict Nash equilibrium will not be played all of the time. It must be that one or both players will play a strategy other than his part of such a Nash equilibrium with positive probability.

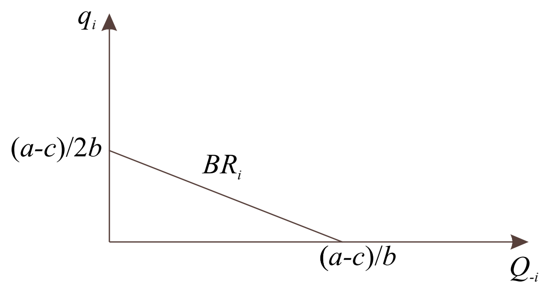
(d) Strategies that are never best responses will eventually be eliminated by this rule of thumb. Thus, in the long run s_i will not be played.

10 Oligopoly, Tariffs, and Crime and Punishment

1.

(a) $S_i = [0, \infty)$. $u_i(q_i, Q_{-i}) = [a - bQ_{-i} - bq_i]q_i - cq_i$, where $Q_{-i} \equiv \sum_{j \neq i} q_j$.

(b) Firm i solves $\max_{q_i} [a - b\overline{Q}_{-i} - bq_i]q_i - cq_i$. This yields the first order condition $a - b\overline{Q}_{-i} - c = 2bq_i$. Player i's best response function is $q_i(\overline{Q}_{-i}) = (a-c)/2b - \overline{Q}_{-i}/2$. This is represented in the graph below.



(c) By symmetry, total equilibrium output is $Q^* = nq^*$, where q^* is the equilibrium output of an individual firm. Thus, $Q^*_{-i} = (n-1)q^*$. So $q^* = [a - c - b(n-1)q^*]/2b$. Thus, $q^* = [a-c]/b(n+1)$ and $Q^* = n[a-c]/b(n+1)$. We also have

$$\begin{aligned} p^* &= a - bn[a-c]/b(n+1) = n[a-c]/(n+1) \\ &= [an + a - an + nc]/(n+1) = [a + cn]/(n+1]. \end{aligned}$$

and

$$\begin{aligned} u^* &= p^*q^* - cq^* \\ &= ([a+cn]/(n+1])[n[a-c]/b(n+1)] - cn[a-c]/b(n+1)\ . \\ &= (a-c)2/b(n+1)2 \end{aligned}$$

(d) In the duopoly case $q_i(\overline{q}_j) = (a-c)/2b - \overline{q}_j/2$. The Nash equilibrium is found by solving the system of two equations given by the best response functions of the two players (alternatively, one can just set $n = 2$ in the above result). Thus, $q^* = (a-c)/3b$. By examining the best response function, we can identify the sequence R_i^k and inspection reveals that $R_i = \{(a-c)/3b\}$ for $i = 1, 2$.

2.

(a) $S_i = [0, \infty]$, $u_i(p_i, p_{-i}) = \begin{cases} \frac{1}{m}(a - p_i)[p_i - c] & \text{if } p_i = \underline{p} \\ 0 & \text{if } p_i > \underline{p}, \end{cases}$ where m denotes the number of players $k \in \{1, 2, \ldots, n\}$ such that $p_k = \underline{p}$.

(b) The Nash equilibrium is: $p_i = c$ for all i. For $n > 2$, there are other Nash equilibria in which one or more players selects a price greater than c (but at least two players select c).

(c) The notion of best response is not well defined. Let $\underline{p}_{-i}$ denote the minimum p_j selected by any player $j \neq i$. If $c < \underline{p}_{-i}$, player i's best response is to select $p_i < \underline{p}_{-i}$, but as close to $\underline{p}_{-i}$ as possible. However there is no such number.

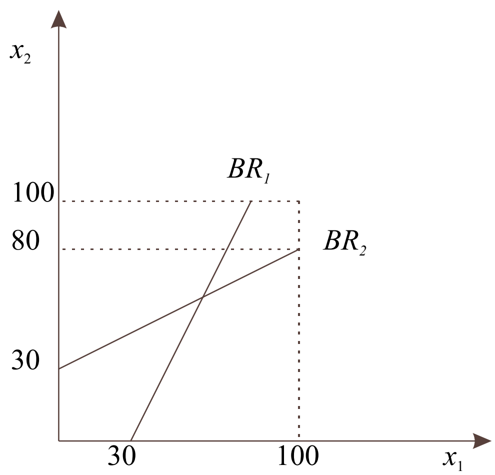
3.

(a) $BR_i(\overline{x}_j) = 30 + \overline{x}_j/2$.

(b) The Nash equilibrium is (60, 60).

(c) $u_i(60, 60) = 200$. $u_i(0, 0) = 2000$.

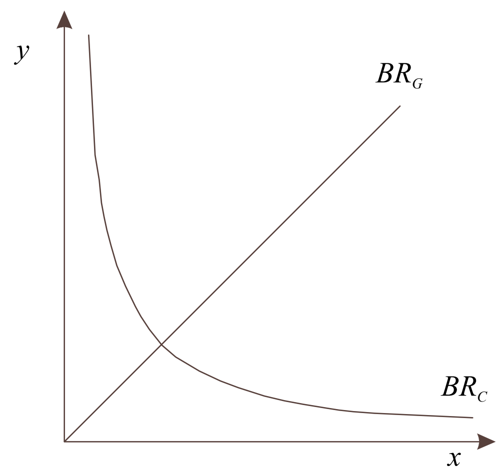
(d) The best response functions are represented below.



It is easy to see that player i will never set $x_i < 30$ or $x_i > 80$. Thus, $R_i^1 = [30, 80]$, $R_i^2 = [45, 70]$, and so on. Thus, $R_i = \{60\}$ for $i = 1, 2$.

4.

(a) G solves $\max_x -y^2x^{-1} - xc^4$. This yields the first order condition $\frac{y^2}{x^2} - c^4 = 0$. Rearranging, we find G's best response function to be $x(\overline{y}) = \overline{y}/c^2$. C solves $\max_y y^{1/2}(1 + xy)^{-1}$. This yields the first order condition $\frac{1}{2y^{1/2}(1+xy)} - \frac{y^{1/2}x}{(1+xy)^2} = 0$. Rearranging, we find C's best response function to be $y(\overline{x}) = 1/\overline{x}$. These are represented at the top of the next page.



(b) We find x and y such that $x = y/c^2$ and $y = 1/x$. The Nash equilibrium is $x = 1/c$ and $y = c$.

(c) As the cost of enforcement c increases, enforcement x decreases and criminal activity y increases.

5.

In equilibrium $b_1 = b_2 = 15,000$. Clearly, neither player wishes to bid higher than 15,000 as she will receive a negative payoff. Further, neither does better by unilaterally deviating to a bid that is less than 15,000 because this leads to a payoff of zero.

6.

(a) The normal form is given by $N = \{P, D\}$, $e_i \in [0, \infty)$, $u_P(e_P, e_D) = 8e_P/(e_P + e_D) - e_P$, and $u_D(e_P, e_D) = 8e_D/(e_P + e_D) - e_D$.

(b) The prosecutor solves $\max_{e_P} 8e_P/(e_P + e_D) - e_P$. The first order condition is $8/(e_P + e_D) - 8e_P/(e_P + e_D)^2 = 1$. This implies $8(e_P + e_D) - 8e_P = (e_P + e_D)^2$, or $8e_D = (e_P + e_D)^2$. Taking the square root of both sides yields $2\sqrt{2e_D} = e_P + e_D$. Rearranging, we find $e_P^*(e_D) = 2\sqrt{2e_D} - e_D$. By symmetry, $e_D^*(e_P) = 2\sqrt{2e_P} - e_P$.

(c) By symmetry, it must be that $e_P^* = 2\sqrt{2e_p^*} - e_P^*$. Thus, $e_P^* = e_D^* = 2$. The probability that the defendant wins in equilibrium is $1/2$.

(d) This is not efficient.

7.

In equilibrium, 6 firms locate downtown and 4 locate in the suburbs. Each firm earns a profit of 44.

11 Mixed-Strategy Nash Equilibrium

1.

(a) (N, L) and (L, N).

(b) Firm Y chooses q so that Firm X is indifferent between L and N. This yields $-5q + (x-15)(1-q) = 10 - 10q$. Rearranging yields $q = \frac{25-x}{20-x}$. Firm X chooses p so that firm Y is indifferent between L and N. This yields $-5p + 15 - 15p = 10 - 10p$. Rearranging yields $p = 1/2$.

(c) The probability of (L, N) $= p(1-q) = (1/2)[\frac{20-x-25+x}{20-x}] = (1/2)[\frac{5}{x-20}]$.

(d) As x increases, the probability of (L, N) decreases. However, as x becomes larger, (L, N) is a "better" outcome.

2.

(a) $\sigma_1 = (1/5, 4/5)$ $\sigma_2 = (3/4, 1/4)$.

(b) It is easy to see that M dominates L, and that $(2/3, 1/3, 0)$ dominates D. Thus, player 1 will never play D, and player 2 will never play L. We need to find probabilities over U and C such that player 2 is indifferent between M and R. This requires $5p + 5 - 5p = 3p + 8 - 8p$ or $p = 3/5$. Thus, $\sigma_1 = (3/5, 2/5, 0)$. We must also find probabilities over M and R such that player 1 is indifferent between U and C. This requires $3q + 5 - 5q = 6q + 4 - 4q$ or $q = 1/4$. Thus, $\sigma_2 = (0, 1/4, 3/4)$.

3.

When $x < 1$, the Nash equilibria are (U, L) and $((0, 1/2, 1/2), (0, 1/2, 1/2))$. When $x > 1$, Nash equilibrium is (U, L). Further, for $0 < x < 1$, there is an equilibrium of $((1-x, x/2, x/2), (1-x, x/2, x/2))$.

4.

(a) $\sigma_i = (1/2, 1/2)$.

(b) (D, D)

(c) There are no pure strategy Nash equilibria. $\sigma_1 = (1/2, 1/2)$ and $\sigma_2 = (1/2, 1/2)$.

(d) (A, A), (B, B), and $\sigma_1 = (1/5, 4/5)$, $\sigma_2 = (1/2, 1/2)$.

(e) (A, A), (B, B), and $\sigma_1 = (2/3, 1/3)$, $\sigma_2 = (3/5, 2/5)$.

(f) Note that M dominates L@. So player 2 chooses probabilities over M and R such that player 1 is indifferent between at least two strategies. Let q denote the probability with which M is played. Notice that the q

which makes player 1 indifferent between any two strategies makes him indifferent between all three strategies. To see this note that $q = 1/2$ solves $4 - 4q = 4q = 3q + 1 - q$. Thus, $\sigma_2 = (0, 1/2, 1/2)$. It remains to find probabilities such that player 2 is indifferent between playing M and R. Here p denotes the probability with which U is played and r denotes the probability with which C is played. Indifference between M and R requires $2p + 4r + 3(1 - p - r) = 3p + 4(1 - p - r)$. This implies $r = 1/5$. Thus, $\sigma_1 = (x, 1/5, y)$, where $x, y \geq 0$ and $x + y = 4/5$.

5.

First game: The normal form is represented below.

1 \ 2	X	Y
A	8,8	0,0
B	2,2	6,6
C	5,5	5,5

Player 2 mixes over X and Y so that player 1 is indifferent between those strategies on which player 1 puts positive probability. Let q be the probability that player 2 selects X. The comparison of $8q$ to $2q + 6 - 6q$ to 5 shows that we cannot find a mixed strategy in which player 1 places positive probability on all of his strategies. So we can consider each of the cases where player 1 is indifferent between two of his strategies. Clearly, at $q = 5/8$ player 1 is indifferent between A and C. Indifference between A and B requires $8q = 6 - 4q$ or $q = 1/2$. However, note that $BR_1(1/2, 1/2)$ $= \{\text{C}\}$ and, thus, there is no equilibrium in which player 1 mixes between A and B. Finally, indifference between B and C requires $6 - 4q = 5$ or $q = 1/4$. Further, note that $BR_1(1/4, 3/4) = \{\text{B, C}\}$.

Turning to player 2's incentives, there is clearly no equilibrium in which player 1 mixes between A and C; this is because player 2 would strictly prefer X, and then player 1 would not be indifferent between A and C. Likewise, there is no equilibrium in which player 1 mixes between B and C; in this case, player 2 would strictly prefer Y, and then player 1 would not be indifferent between B and C. There are, however, mixed strategy equilibria in which player 1 selects C with probability 1 (that is, plays a pure strategy) and player 2 mixes between X and Y. This is an equilibrium for every $q \in [1/4, 5/8]$.

Second game: The normal form of this game is represented below.

1 \ 2	I	O
IU	4,-1	-1,0
ID	3,2	-1,0
OU	1,1	1,1
OD	1,1	1,1

Clearly, there is no equilibrium in which player 1 selects ID with positive probability. There is also no equilibrium in which player 1 selects IU with positive probability, for, if this were the case, then player 2 strictly prefers O and, in response, player 1 should not pick IU. Note that player 1 prefers OU or OD if player 2 selects O with a probability of at least $3/5$. Further, when player 1 mixes between OU and OD, player 2 is indifferent between his two strategies. Thus, the set of mixed strategy equilibria is described by $\sigma_1 = (0, 0, p, 1-p)$ and $\sigma_2 = (q, 1-q)$, where $p \in [0, 1]$ and $q \leq 2/5$.

6.

(a) The symmetric mixed strategy Nash equilibrium requires that each player call with the same probability, and that each player be indifferent between calling and not calling. This implies that $(1 - p^{n-1})v = v - c$ or $p = (c/v)^{\frac{1}{1-n}}$.

(b) The probability that at least one player calls in equilibrium is $1 - p^n = 1 - (c/v)^{\frac{n}{n-1}}$. Note that this *decreases* as the number of bystanders n goes up.

7.

(a) If the game has a pure-strategy Nash equilibrium, we are done.

(b) Assume the game has no pure-strategy Nash equilibrium, and proceed as follows. That (U,L) is not a Nash equilibrium implies $e > a$ and/or $d > b$. That (U,R) is not a Nash equilibrium implies $g > c$ and/or $b > d$. That (D,R) is not a Nash equilibrium implies $c > g$ and/or $f > h$. That (D,L) is not a Nash equilibrium implies $a > e$ and/or $h > f$. It is easy to see that if there is no pure strategy Nash equilibrium, then only one of each of these pairs of conditions can hold. This implies that each pure strategy of each player is a best response to some other pure strategy of the other. Further, it must be that there is a mixture for each player i such that the other player j is indifferent between his two strategies.

Consider player 1. It must be that either $e > a$ and $g > c$ or $a > e$ and $c > g$. It is easy to show that there exists a $q \in [0, 1]$ such that $aq + c(1-q) = eq + g(1-q)$. Rearranging yields $(a-e) = (g-c)(1-q)/q$. It is the case that $(a-e)$ and $(g-c)$ have the same sign. The analogous argument can be made with respect to player 2.

8.

No, it does not have any pure strategy equilibria. The mixed equilibrium is $((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))$.

9.

(a) When $\mu_2 > 2/3$, 001 should choose route a. When $\mu_2 < 2/3$, 001 should choose route d. When $\mu_2 = 2/3$, 001 should choose either route a, route c, or route d.

(b) It is advised that 001 never take route b. Route b is dominated by a mixture of routes a and c. One such mixture is 2/3 probability on a and 1/3 probability on c. It is easy to see that $12(2/3)+10(1/3) = 34/11 > 11$, and $4(1/3) > 1$.

(c) As 002's payoff is the same, regardless of his strategy, when 001 chooses c, we should expect that the equilibrium with one player mixing and the other playing a pure strategy will involve 001 choosing c. Clearly 002 is indifferent between x and y when 001 is playing c. Further, 002 can mix so that c is a best response for 001. A mixture of 2/3 and 1/3 implies that 001 receives a payoff of 8 from all of his undominated strategies. This equilibrium is $s_1 =$c and $\sigma_2 = (2/3, 1/3)$.

Since b is dominated, we now consider a mixture by 001 over a and d. In finding the equilibrium above, we noticed that 002's mixing with probability $(2/3, 1/3)$ makes 001 indifferent between a, c, and d. Thus, we need only to find a mixture over a and d that makes 002 indifferent between x and y. Let p denote the probability with which 001 plays a, and $1-p$ denote the probability with which he plays d. Indifference on the part of 002 is reflected by $3 - 3p = 6p$. This implies $p = 1/3$, which means that 002 receives a payoff of 2 whether he chooses x or y. This equilibrium is $\sigma = ((1/3, 0, 0, 2/3), (2/3, 1/3)$.

In considering whether there are any more equilibria, it is useful to notice that in both of the above equilibria that 002's payoff from choosing x is the same as that from y. Thus we should expect that, so long as the ratio of a to d is kept the same, 001 could also play c with positive probability. Let p denote the probability with which 001 plays a, and let q denote the probability with which he plays c. Since he never plays b, the probability with which d is played is $1-p-q$. Making 002 indifferent between playing x and y requires that $2q + 3(1 - p - q) = 6p + 2q$. This implies that any p and q such that $1 = 3p + q$ will work. One such case is $(1/9, 6/9, 2/9)$, implying an equilibrium of $((1/9, 6/9, 2/9), (2/3, 1/3))$

12 Strictly Competitive Games and Security Strategies

1.

(a) No. Note that u_1(A, Z) = u_1(C, Z), but u_2(A, Z) > u_2(C, Z).

(b) Yes.

(c) Yes.

(d) No. Note that u_1(D, X) > u_1(D, Y), but u_2(D, X) > u_2(D, Y).

2.

(a) 1: C, 2: Z

(b) 1: C, 2: Z

(c) 1: A, 2: X

(d) 1: D, 2: Y

3.

Note that security strategies have been defined in terms of pure strategies. Suppose a security strategy is dominated by a mixed strategy. Consider the game below.

1 \ 2	X	Y
A	4,1	0,1
B	0,-1	4,1
C	1,0	2,-1

C is player 1's security strategy, but C is dominated by a mixture of A and B.

4.

In the game below, B is player 1's security strategy yet B is not rationalizable.

1 \ 2	X	Y
A	3,5	-1,1
B	2,6	1,2

5.

Let i be one of the players and let j be the other player. Because s is a Nash equilibrium, we have $u_i(s) \geq u_i(t_i, s_j)$. Because t is a Nash equilibrium, we have $u_j(t) \geq u_j(t_i, s_j)$; strict competition further implies that $u_i(t) \leq u_i(t_i, s_j)$. Putting these two facts together, we obtain $u_i(s) \geq u_i(t_i, s_j) \geq u_i(t)$. Switching the roles of s and t, the same argument yields $u_i(t) \geq u_i(s_i, t_j) \geq u_i(s)$. Thus, we know that $u_i(s) = u_i(s_i, t_j) = u_i(t_i, s_j) = u_i(t)$ for $i = 1, 2$, so the equilibria are equivalent. To see that the equilibria are also interchangeable, note that, because s_i is a best response to s_j and $u_i(s) = u_i(t_i, s_j)$, we know that t_i is also a best response to s_j. For the same reason, s_i is a best response to t_j.

6.

Examples include chess, checkers, tic-tac-toe, and Othello.

13 Contract, Law, and Enforcement in Static Settings

1.

(a) A contract specifying (I, I) can be enforced under expectations damages because neither player has the incentive to deviate from (I, I).

1 \ 2	I	N
I	4, 4	4, 1
N	-6, 4	0, 0

(b) Yes.

1 \ 2	I	N
I	4, 4	5, 0
N	0, -2	0, 0

(c) No, player 2 still has the incentive to deviate.

1 \ 2	I	N
I	4, 4	0, 5
N	-2, 0	0, 0

(d)

1 \ 2	I	N
I	4, 4	$-c$, $5 - c$
N	$-2 - c$, $-c$	0, 0

(e) $c > 1$.

(f) Consider (I,N). Player 1 sues if $-c > -4$ or $c < 4$. Consider (N,I). Player 2 sues if $-c > -4$ or $c < 4$. Thus, suit occurs if $c < 4$.

(g) $c > 1/2$.

2.

(a) 10

(b) 0

3.

(a) Now the payoff to i when no one calls is negative. Let d denote the fine for not calling. Consider the case where the fine is incurred regardless of whether anyone else calls. This yields the new indifference relationship of $(1-p^{n-1})v-d = v-c$. This implies that, if $c > d$, then $p = [(c-d)/v]^{\frac{1}{n-1}}$. If $c < d$ then $p = 0$ in equilibrium.

Now consider the case where the fine is incurred only when no one calls. The indifference relationship here implies $(1-p^{n-1})v - dp^{n-1} = v - c$. This implies $p = [c/(d+v)]^{\frac{1}{n-1}}$.

(b) (1) Given that if i doesn't call then he pays the fine with certainty, the fine can be relatively low. (2) Here, if i doesn't call then he pays the fine with a low probability. Thus, the fine should be relatively large.

(c) Either type of fine can be used to induce any particular p value, except for $p = 0$ which results only if the type (1) fine is imposed. The required type (2) fine may be much higher than the required type (1) would be. The type (2) fine may be easier to enforce, because in this case one only needs to verify whether the pedestrian was treated promptly and who the bystanders were. The efficient outcome is for exactly one person to call. There are pure strategy equilibria that achieve this outcome, but it never happens in the symmetric mixed strategy equilibrium.

4.

Verifiability is more important. It must be possible to convey information to the court in order to have a transfer imposed.

5.

Expectations damages gives the non-breaching player the payoff that he expected to receive under the contract. Restitution damages takes from the breacher the amount of his gain from breaching. Expectations damages is more likely to achieve efficiency. This is because it gives a player the incentive to breach when it is efficient to do so.

6.

(a)

1 \ 2	I	N
I	6, 8	6, 3
N	– 1, 8	0,0

(b) No, to prevent player 1 from deviating requires a transfer of at least 1 from player 1 to player 2, but this gives player 2 even more incentive to deviate.

7.

(a) For technology A, the self-enforced component is to play (I, I). The externally-enforced component is a transfer of at least 1 from player 2 to player 1 when (I, N) occurs, a transfer of at least 2 from player 1 to player 2 when (N, I) occurs, and none otherwise. For technology B, the self-enforced component is to play (I, I). The externally-enforced component is a transfer of at least 4 from player 1 to player 2 when (N, I) occurs, and none otherwise.

(b) Now for technology A, the self-enforced component is to play (N, N). There is no externally-enforced component. For B the self-enforced component is to transfer 4 from player 1 to player 2 when someone plays N, and no transfer when both play I.

(c) Expectations damages gives the non-breaching player the amount that he expected to receive under the contract. The payoffs under this remedy are depicted for each case as shown here:

1 \ 2	I	N
I	3, 8	3, 0
N	– 4, 8	0, 0

A

1 \ 2	I	N
I	6, 7	6, – 4
N	– 2, 7	0, 0

B

Reliance damages seek to put the non-breaching party back to where he would have been had he not relied on the contract. The payoffs under reliance damages are depicted below.

1 \ 2	I	N
I	3, 8	0, 3
N	4, 0	0, 0

A

1 \ 2	I	N
I	6, 7	0, 5
N	5, 0	0, 0

B

Restitution damages take the gain that the breaching party receives due to breaching. The payoffs under restitution damages are depicted below.

1 \ 2	I	N
I	3, 8	3, 0
N	0, 5	0, 0

A

1 \ 2	I	N
I	6, 7	5, 0
N	0, 5	0, 0

B

8.

(a) $S_1 = [0, \infty), S_2 = [0, \infty)$. If $y > x$, then the payoffs are $(Y, -X)$. If $x \geq y$, the payoffs are $(y, X - y)$.

(b) There are multiple equilibria in which the players report $x = y = \alpha$, where $\alpha \in [Y, X]$.

(c) There is an equilibrium in which player 1 reports $y = Y$ and player 2 reports $x = X$. There are also multiple equilibria in which player 1 reports $y \geq Y$ and player 2 reports $x \leq Y$.

(d) It is efficient, because the plant is shut down if and only if it is efficient to do so.

9.

Examples include the employment contracts of salespeople, attorneys, and professors.

14 Details of the Extensive Form

1.

From the definition, dashed lines in the extensive form imply imperfect information. This can be applied all of the extensive forms.

2.

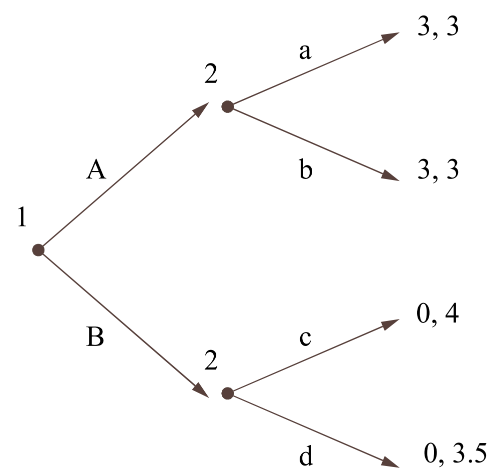
No general rule. Consider, for example, the prisoners' dilemma. Clearly, the extensive form of this game will contain dashed lines. Consider Exercise 3 (a) of Chapter 4. The normal form of this does not exhibit imperfect information.

3.

Suppose not. Then it must be that some pure strategy profile induces at least two paths through the tree. Since a strategy profile specifies an action to be taken in every contingency (at every node), having two paths induced by the same pure strategy profile would require that Tree Rule 3 not hold.

4.

In the following extensive form game the strategy profiles (A, ac) and (A, ad) induce the same path.



15 Backward Induction and Subgame Perfection

1.

(a) (I, C, X)

(b) (AF, C)

(c) (BHJKN, CE)

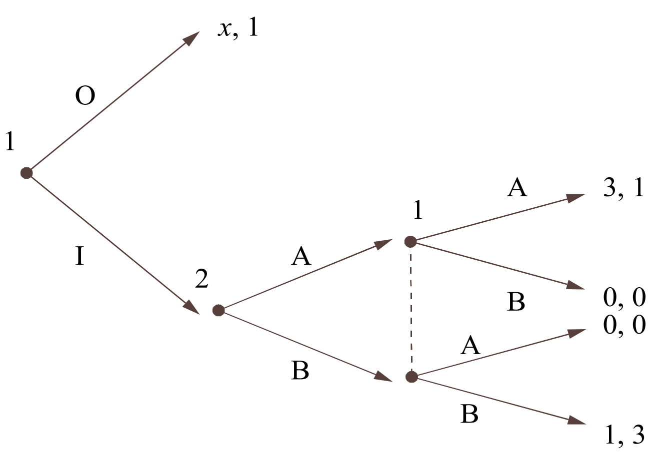
2.

(a) The subgame perfect equilibria are (WY, AC) and (ZX, BC). The Nash equilibria are (WY, AC), (ZX, BC), (WY, AC), (ZY, BC), and (WX, BD).

(b) The subgame perfect equilibria are (UE, BD) and (DE, BC). The Nash equilibria are (UE, BD), (DE, BC), (UF, BD), and (DE, AC).

3.

(a)



1 \ 2	A	B
OA	x, 1	x, 1
OB	x, 1	x, 1
IA	3, 1	0, 0
IB	0, 0	1, 3

(b) If $x > 3$, the equilibria are (OA,A), (OB,A), (OA,B), (OB,B). If $X = 3$, add (IA, A) to this list. If $1 < x < 3$, the equilibria are (IA,A), (OA,B), (OB,B). If $x = 1$, add (IB, B) to this list. If $x < 1$, the equilibria are (IA,A), (IB,B).

(c) If $x > 3$ any mixture with positive probabilities over OA and OB for player 1, and over A and B for player 2.

If $1 < x < 3$, then IB is dominated. Any mixture (with positive probabilities) over OA and OB will make player 2 indifferent. Player 2 plays A with probability $x/3$, and plays B with probability $1 - x/3$.

For $3/4 \leq x \leq 1$, let $1-p-q$ denote the probability with which player 1 plays OA or OB. Let p denote the probability with which player 1 plays IA, and q denotes the probability with which she plays IB. Then any p and q (where both are positive and sum to not more than 1) such that $p = 3q$ will make player 2 indifferent between A and B. Player 2 plays A with probability 1/4.

For $x < 3/4$ OA and OB are dominated. In equilibrium, player 1 chooses IA with probability 3/4 and IB with probability 1/4. In equilibrium, player 2 chooses A with probability 1/4, and B with probability 3/4.

(d)

1 \ 2	A	B
A	1, 3	0, 0
B	0, 0	3, 1

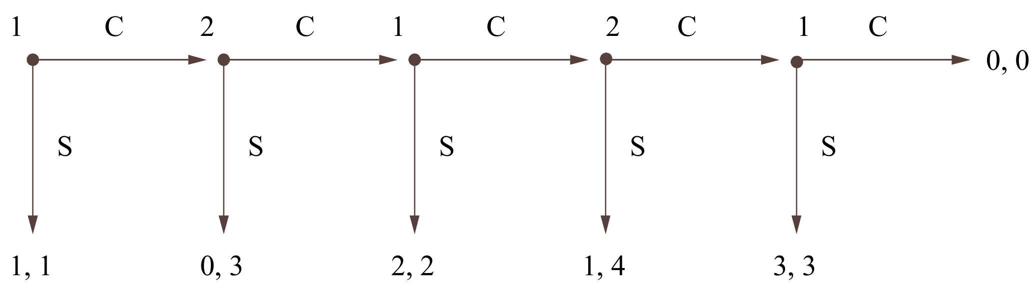
The pure strategy equilibria are (A, A) and (B, B). There is also a mixed equilibrium $(3/4, 1/4; 1/4, 3/4)$.

(e) The Nash equilibria that are not subgame perfect include (OB, A), (OA, B), and the above mixed equilibria in which, once the proper subgame is reached, player 1 does not play A with probability 3/4 and/or player 2 does not play A with probability 1/4.

(f) The subgame perfect mixed equilibria are those in which, once the proper subgame is reached, player 1 does plays A with probability 3/4 and player 2 does plays A with probability 1/4.

4.

(a)

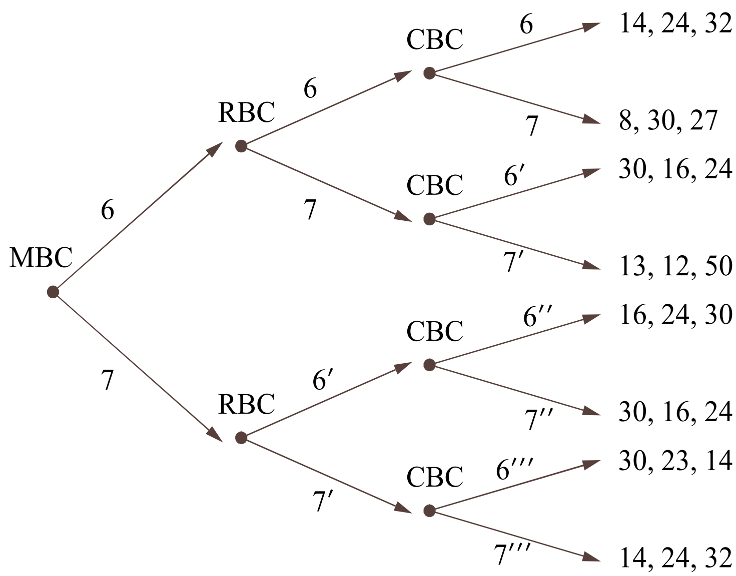


(b) Working backward, it is easy to see that in round 5 player 1 will choose S. Thus, in round 4 player 2 will choose S. Continuing in this fashion, we find that, in equilibrium, each player will choose S any time he is on the move.

(c) For any finite k, the backward induction outcome is that player 1 chooses S in the first round and each player receives one dollar.

5.

Payoffs in the extensive form representation are in the order RBC, CBC, and MBC.



In the subgame perfect equilibrium, MBC chooses 7, RBC chooses $76'$, and CBC chooses $76'6''7'''$. The outcome differs from the simultaneous move case because of the sequential play.

6.

Player 2 accepts any $x \geq 0$ and player 1 offers $x = 0$.

7.

(a) $S_i = \{\text{A, B}\} \times (0, \infty) \times (0, \infty)$. Each player selects A or B, picks a positive number when (A, B) is chosen, and picks a positive number when (B, A) is chosen.

(b) It is easy to see that $0 < (x_1 + x_2)/(1 + x_1 + x_2) < 1$, and that $(x_1 + x_2)/(1 + x_1 + x_2)$ approaches 1 as $(x_1 + x_2) \to \infty$. Thus, each has a higher payoff when both choose A. Further, B will never be selected in equilibrium. The Nash equilibria of this game are given by $(\text{A}x_1, \text{A}x_2)$, where x_1 and x_2 are any positive numbers.

(c) There is no subgame perfect equilibrium because the subgames following (A, B) and (B, A) have no Nash equilibria.

16 Topics in Industrial Organization

1.

From the text, $z_1(a) = a^2/9 - 2a^3/81$. If the firms were to write a contract that specified a, they would choose a to maximize their joint profit (with m set to divide the profit between them). This advertising level solves $\max_a 2a^2/9 - 2a^3/81$, which is $a^* = 6$.

2.

The subgame perfect equilibrium is $a = 0$ and $p_1 = p_2 = 0$.

3.

Because this is a simultaneous move game, we are just looking for the Nash equilibrium of the following normal form.

1 \ 2	H	L	N
H	– 85, – 85	– 15, – 10	27 ½ , 0
L	– 10, – 15	20, 20	30, 0
N	0, 27 ½	0, 30	0, 0

The equilibrium is (L, L). Thus, in the subgame perfect equilibrium both players invest 50,000 in the low production plant.

4.

(a) $u_2(q_1, q_2^*(q_1)) = (1000 - 3q_1 - 3q_2)q_2 - 100q_2 - F$. Maximizing by choosing q_2 yields the first order condition $1000 - 3q1 - 6q2 - 100 = 0$. Thus, $q_2^*(q_1) = 150 - (1/2)q_1$.

(b) $u_1(q_1, q_2^*(q_1)) = (1000 - 3q_1 - 3[150 - 1/2q_1])q_1 - 100q_1 - F$. Maximizing by choosing q_1 yields the first order condition $550 - 3q_1 - 100 = 0$. Thus, $q_1^* = 150$. $q_2^* = 150 - (1/2)(150) = 75$. Solving for equilibrium price yields $p^* = 100 - 3(150 + 75) = 325$. $u_1^* = 325(150) - 100(150) = 33,750 - F$. $u_2^* = 325(75) - 100(75) - F = 16875 - F$.

(c) Find q_1 such that $u_2(q_1, q_2^*(q_1)) = 0$. We have

$$
\begin{aligned}
&(1000 - 3q_1 - 3[150 - (1/2)q_1])[150 - (1/2)q_1] - 100[150 - (1/2)q_1] - F \\
&= (900 - 3q_1)[150 - (1/2)q_1] - 3[150 - (1/2)q_1]2 - F \\
&= 6[150 - (1/2)q_1]^2 - 3[150 - (1/2)q_1]^2 - F \\
&= 3[150 - (1/2)q_1]^2 - F.
\end{aligned}
$$

Setting profit equal to zero implies $F = 3[150 - (1/2)q_1]^2$ or $(F/3)^{1/2} = 150 - (1/2)q_1$. Thus, $\overline{q}_1 = 300 - 2(F/3)^{1/2}$. Note that

$$\begin{aligned}\overline{u}_1 &= (1000 - 3[300 - 2(F/3)^{1/2}])[300 - (F/3)^{1/2}] \\ &\quad -100[300 - 2(F/3)^{1/2}] - F \\ &= 900[300 - 2(F/3)^{1/2}] - 3[300 - 2(F/3)^{1/2}]2 - F.\end{aligned}$$

d) (i) $F = 18,723$ implies $\overline{q}_1 = 142 < q_1^*$. So firm 1 will produce q_1^* and $u_1 = 48,777$. (ii) $F = 8112$: In this case, $\overline{q}_1 = 300 - 2(8112/3)^{1/2} = 196$ and $\overline{pi}_1 = 900(196) - 3(196)2 - 8,112 = 53,040$. $u_1^* = 33,750 - 8,112 = 25,630$. Thus firm 1 will produce $q_1 = 196$, resulting in $u_1 = 53,040$. (iii) $F = 1728$: Here, $\overline{q}_1 = 300 - 2(1,728/3)^{1/2} = 252$ and $\overline{u}_1 = 900(252) - 3(252)2 - 1,728 = 34,560$. $u_1* = 33,750 - 1,728 = 32,022$. Thus, firm 1 will produce $q_1 = 252$, resulting in $u_1 = 34,560$. (iv) $F = 108$: In this case, $\overline{q}_1 = 300 - 2(108/3)^{1/2} = 288$ and $\overline{u}_1 = 900(288) - 3(288)2 - 108 = 10,260$. $u_1^* = 33,750 - 108 = 33,642$. Thus, firm 1 will produce $q_1 = 150$, resulting in $u_1 = 33,642$.

5.

(a) If Hal does not purchase the monitor in period 1, then $p_2 = 200$ is not optimal because $p_2 = 500$ yields a profit of 500, while $p_2 = 200$ yields a profit of 400. The optimal pricing scheme is $p_1 = 1,400$ and $p_2 = 200$. Tony would gain from being able to commit to not sell monitors in period 2. This would allow him to make a profit of $1,700$ instead of $1,600$.

(b) The optimal prices are $p_1 = 1,400$ and $p_2 = 200$. Hal buys in period 1 and Laurie buys is period 2. Here, Tony would not benefit from being able to commit not to sell monitors in period 2.

6.

For scheme A to be optimal, it must be that twice Laurie's (the low type) value in period 1 is at least as great as Hal's (high type) period 1 value plus his period 2 value. An example of this is below.

	Period 1	Period 2
Benefit to Hal	1200	300
Benefit to Laurie	1000	200

For scheme B to be optimal, it must be that Laurie's (low type) value in period 2 is at least as large as both Hal's (high type) period 1 value and Laurie's period 1 value. An example of this is below.

	Period 1	Period 2
Benefit to Hal	150	300
Benefit to Laurie	100	200

7.

(a) The subgame perfect equilibrium is for player 1 to locate in region 5, and for player 2 to use the strategy 234555678 (where, for example, 2 denotes that player 2 locates in region 2 when player 1 has located in region 1).

(b) A subgame perfect equilibrium is for player 1 to locate in region 4, for player 2 to use the strategy 662763844 (where the first 6 denotes player 2's location given that player 1 located in region 1), and for player 3 to use the following strategy.

− 3	3
1,1	2
1,2	3
1,3	4
1,4	5
1,5	6
1,6	7
1,7	2
1,8	2
1,9	2

− 3	3
2,1	3
2,2	3
2,3	4
2,4	5
2,5	6
2,6	7
2,7	3
2,8	3
2,9	3

− 3	3
3,1	4
3,2	4
3,3	4
3,4	5
3,5	6
3,6	7
3,7	4
3,8	4
3,9	4

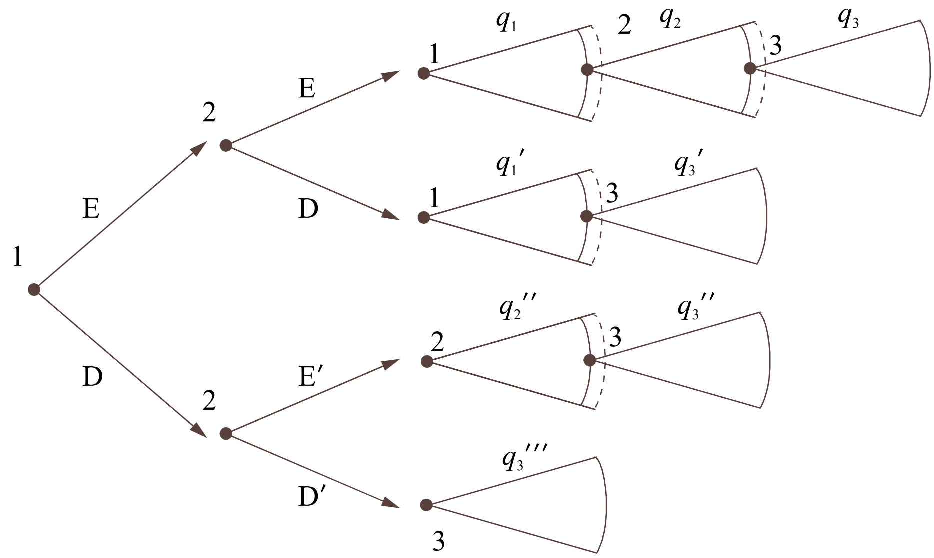
− 3	3
4,1	5
4,2	5
4,3	5
4,4	5
4,5	6
4,6	3
4,7	3
4,8	3
4,9	3

− 3	3
5,1	6
5,2	6
5,3	6
5,4	6
5,5	4
5,6	4
5,7	4
5,8	4
5,9	4

Note that this chart specifies player 3's location without regard to the issue of specifically which of the other two players (-3 players) locates in each position.

8.

(a) Without payoffs, the extensive form is as follows.



In the subgame perfect equilibrium, player 1 selects E, player 2 chooses DE′, and the quantities are given by $q_1 = q_2 = q_3 = 3$, $q_1' = q_3' = 4$, $q_2'' = q_3'' = 4$, and $q_3''' = 6$.

(b) Player 1 enters.

9.

(a) Given x, the retailer solves $\max_q 200q - q^2/100 - xq$. The first order condition implies $q^*(x) = 10,000 - 50x$. Knowing this, the manufacturer solves $\max_x (10,000 - 50x)x - (10,000 - 50x)10$. The first order condition implies $x^* = 105$. Thus, in equilibrium $q = 4,750$. This implies $p = 152.50$.

(b) Now the manufacturer solves $\max_q [200 - q/100]q - 10q$. The first order condition implies $\hat{q} = 9,500$. This implies $p = 105$.

(c) The joint profit in part (a) is $(152.50 - 10)4,750 = 676,875$. The manufacturer's profit in part (b) is $95(9,500) = 902,500$. The difference is because in part (b) the manufacturer sets quantity to maximize profits given its marginal cost of 10. However, in part (a) the retailer sets quantity to maximize profits given it marginal cost of x.

10.

(a) The government solves $\max_{\dot{p}} 30 + \dot{p} - \dot{W} - \dot{p}/2 - 30$ or $\max_{\dot{p}} \dot{p}/2 - \dot{W}$. This implies that they want to set $\dot{p}$ as high as possible, regardless of the level of $\dot{W}$. So $\dot{p}^* = 10$.

Knowing how the government will behave, the ASE solves $\max_{\dot{W}} -(\dot{W} - 10)^2$. The first order condition implies $\dot{W}^* = \dot{p}^* = 10$. So in equilibrium $y = 30$.

(b) If the government could commit ahead of time, it would solve $\max_{\dot{W}} -\dot{W}/2$. This implies that it would commit to $\dot{p} = 0$ and the ASE would set $\dot{W} = 0$. In (a) $u = 0$ and $v = -5$. Now, when commitment is possible, $u = 0$ and $v = 0$.

(c) One way is to have a separate central bank that does not have a politically elected head that states its goals.

17 Parlor Games

1.

(a) Use backward induction to solve this. To win the game, a player must not be forced to enter the top-left cell Z; thus, a player would lose if he must move with the rock in either cell 1 or cell 2 as shown in the following diagram.

Z	1
2	

A player who is able to move the rock into cell 1 or cell 2 thus wins the game. This implies that a player can guarantee victory if he is on the move when the rock is in one of cells 3, 4, 5, 6, or 7, as shown in the diagram below.

Z	1	3
2	4	6
5	7	

We next see that a player who must move from cell 8, cell 9 or cell 10 (shown below) will lose.

Z	1	3	9
2	4	6	
5	7	8	
10			

Continuing the procedure reveals that, starting from a cell marked with an X in the following picture, the next player to move will win.

Z	1	X		X		X
2	X	X	X	X	X	X
X	X		X		X	
	X	X	X	X	X	X
X	X		X		X	Y

Since the dimensions of the matrix are 5×7, player 2 has a strategy that guarantees victory.

(b) In general, player 2 has a winning strategy when $m, n > 1$ and both are odd, or when m or n equals 1 and the other is even. Otherwise, player 1 has a winning strategy.

2.

To win, a player must leave her opponent with 1 red ball and 0 blue balls. This implies that the winning player must be left with either 1 red ball and 1 blue ball, or 2 red balls. Note that this is an even number of total balls. If $m + n$ is an even number, player 1 can force player 2 to take the last red ball. If $m + n$ is an odd number, player 2 can force player 1 to take the last red ball.

3.

This can be solved by backward induction. Let (x, y) denote the state where the red basket contains x balls and the blue basket contains y balls. To win this game, a player must leave her opponent with either (0,1) or (1,0). Thus, in order to win, a player must not leave her opponent with either any of the following $(0, z), (1, z), (z, 1)$, or $(z, 0)$, $z > 1$. So, to win, a player should leave her opponent with $(2, 2)$. Thus, a player must not leave her opponent with either $(w, 2)$ or $(2, w)$, where $w > 2$. Continuing with this logic and assuming $m, n > 0$, we see that player 2 has a winning strategy when $m = n$ and player 1 has a winning strategy when $m \neq n$.

4.

Player 1 has a strategy that guarantees victory. This is easily proved using a contradiction argument. Suppose player 1 does not have a strategy guaranteeing victory. Then player 2 must have such a strategy. This means that, for every opening move by player 1, player 2 can guarantee victory from this point. Let X be the set of matrix configurations that player 1 can create in his first move, which player 2 would then face. A configuration refers to the set of cells that are filled in.

We have that, starting from each of the configurations in X, the next player to move can guarantee victory for himself. Note, however, that if player 1 selects cell (m, n) in his first move, then, whatever player 2's following choice is, the configuration of the matrix induced by player 2's selection will be in X (it is a configuration that player 1 could have created in his first move). Thus, whatever player 2 selects in response to his choice of cell (m, n), player 1 can guarantee a victory following player 2's move. This means that player 1 has a strategy that guarantees him a win, which contradicts what we assumed at the beginning. Thus, player 1 actually does have a strategy that guarantees him victory, regardless of what player 2 does.

This game is interesting because player 1's winning strategy in arbitrary $m \times n$ Chomp games is not known. A winning strategy is known for the special case in which $m = n$. This strategy selects cell $(2, 2)$ in the first round.

5.

(a) In order to win, in the matrix below, a player must avoid entering a cell marked with an X. As player 1 begins in cell Y, he must enter a cell marked with an X. Thus, player 2 has a strategy that ensures a win.

Z	X		X		X	
X	X	X	X	X	X	X
	X		X		X	
X	X	X	X	X	X	X
	X		X		X	Y

(b) There are many subgame perfect equilibria in this game, because players are indifferent between moves at numerous cells. There is a subgame perfect equilibrium in which player 1 wins, another in which player 2 wins, and still another in which player 3 wins.

6.

(a) Yes.

(b) No.

(c) Player 1 can guarantee a payoff of 1 by choosing cell (2,1). Player 2 will then rationally choose cell (1,2) and force player 3 to move into cell (1,1).

18 Bargaining Problems

1.

(a) $v^* = 50,000$; $u_J^* = u_R^* = 25,000$; $t = 15,000$.

(b) Solving $\max_x 60,000 - x^2 + 800x$ yields $x^* = 400$. This implies $v^* = 220,000$, $u_J^* = u_R^* = 110,000$, $v_J = -100,000$, and $v_R = 320,000$. Thus, $t = 210,000$.

(c) From above, $x^* = 400$ and $v^* = 220,000$. $u_J^* = 40,000 + (220,000 - 40,000 - 20,000)/4 = 80,000$ and $u_R^* = 20,000 + (3/4)(220,000 - 60,000) = 140,000$. This implies $t = 180,000$.

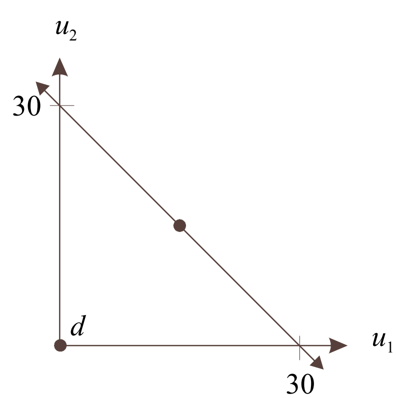
2.

(a) The surplus with John working as a programmer is $90,000 - w$. The surplus with him working as a manager is $x - 40,000 - w > 110,000 - w$. Thus, the maximal joint value is attained by John working as a manager. John's overall payoff is $w + \pi_J[x - 40,000]$ which is equal to $(1 - \pi_J)w + \pi_J[x - 40,000]$. The firm's payoff is $\pi_F[x - 40,000 - w]$. Knowing that John's payoff must equal $t - 40,000$, we find that $t = [1 - \pi_J][w - 40,000] + \pi_j x$.

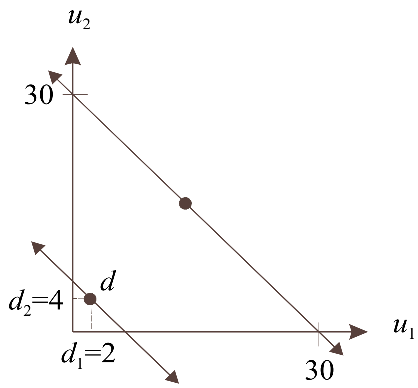
(b) John should undertake the activity that has the most impact on t, and hence his overall payoff, per time/cost. A one-unit increase in x will raise t by π_J. A one unit increase in w raises t by $1 - \pi_J$. Assuming that x and w can be increased at the same cost, John should increase x if $\pi_j > 1/2$; otherwise, he should increase w.

3.

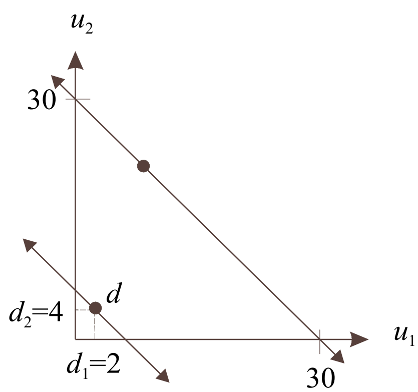
(a) $x = 15, t = 0$, and $u_1 = u_2 = 15$.



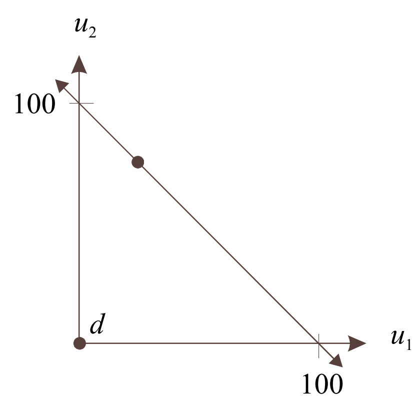
(b) $x = 15, t = -1, u_1 = 14$, and $u_2 = 16$.



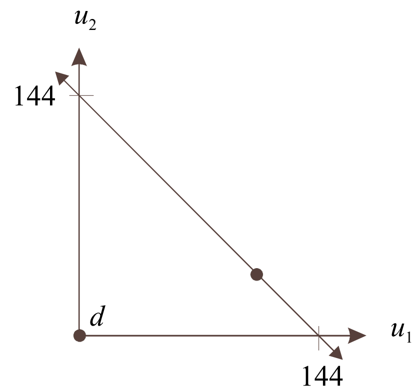
(c) $x = 15, t = -7, u_1 = 8$, and $u_2 = 22$.



(d) $x = 10, t = -175, u_1 = 25$, and $u_2 = 75$.



(e) $x = 12, t = 144\pi_1 - 336, u_1 = 144\pi_1$, and $u_2 = 144\pi_2$.



4.

The other party's disagreement point influences how much of v^* you get because it influences the size of the surplus.

5.

Assuming that it costs the same to raise either, and that your bargaining weight is less than one, you should raise your disagreement payoff by ten units. This is because you receive all of the gain in your disagreement payoff. This is not efficient.

6.

Possible examples would include salary negotiations, merger negotiations, and negotiating the purchase of an automobile.

19 Analysis of Simple Bargaining Games

1.

(a) The superintendent offers $x = 0$, and the president accepts any x.

(b) The president accepts x if $x \geq \min\{z, |y|\}$.

(c) The superintendent offers $x = \min\{z, |y|\}$, and the president accepts.

(d) The president should promise $z = |y|$.

2.

(a) Here you should make the first offer, because the current owner is very impatient and will be quite willing to accept a low offer in the first period. More precisely, since $\delta < 1/2$, the responder in the first period prefers accepting less than one-half of the surplus to rejecting and getting all of the surplus in the second period. Thus, the offerer in the first period will get more than half of the surplus.

(b) In this case, you should make the second offer, because you are patient and would be willing to wait until the last period rather than accepting a small amount at the beginning of the game. More precisely, in the least, you can wait until the last period, at which point you can get the entire surplus (the owner will accept anything then). Discounting to the first period, this will give you more than one-half of the surplus available in the first period.

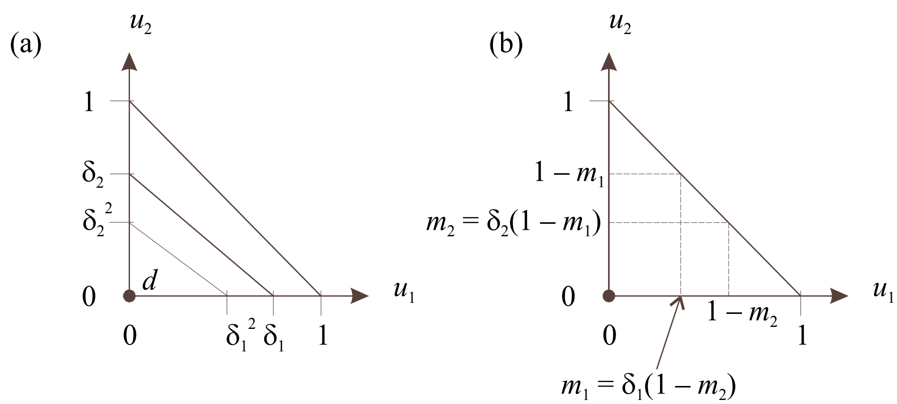
3.

In the case of $T = 1$, player 1 offers $m = 1$ and player 2 accepts. If $T = 2$, player 1 offers $1 - \delta$ in the first period and player 2 accepts, yielding the payoff vector $(1-\delta, \delta)$. For $T = 3$, the payoff vector is $(1-\delta(1-\delta), \delta(1-\delta))$. The payoff is $(1-\delta^2(1-\delta), \delta^2(1-\delta))$ in the case of $T = 4$. For $T = 5$, the payoff is $(1-\delta-\delta^2(1-\delta+\delta^2), \delta-\delta^2(1-\delta+\delta^2))$. As T approaches infinity, the payoff vector converges to $([1-\delta]/[1-\delta^2], [\delta-\delta^2]/[1-\delta^2]$, which is the subgame perfect equilibrium payoff vector of the infinite-period game.

4.

Note that $BR_i(m_j) = 1 - m_j$. The set of Nash equilibria is given by $\{m_1, m_2 \in [0,1] \mid m_1 + m_2 = 1\}$. One can interpret the equilibrium demands (the m_i's) as the bargaining weights.

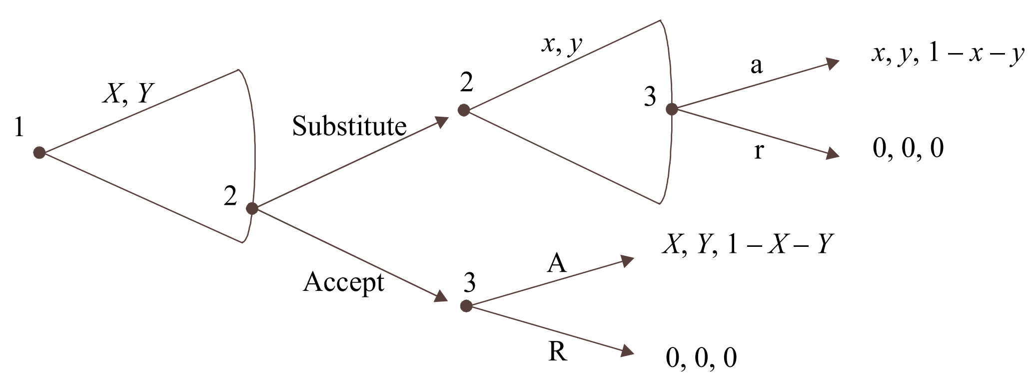
5.



6.

For simplicity, assume that the offer is always given in terms of the amount player 1 is to receive. Suppose that the offer in period 1 is x, the offer in period 2 it is y, and the offer in period 3 is z. If period 3 is reached, player 2 will offer $z = 0$ and player 1 will accept. Thus, in period 2, player 2 will accept any offer that gives her at least δ. Knowing this, in period 2 (if it is reached) player 1 will offer y such that player 2 is indifferent between accepting and rejecting to receive 1 in the next period. This implies $y = 1 - \delta$. Thus, in period 1, player 2 will accept any offer that gives her at least $\delta(1 - \delta)$. In the first period, player 1 will offer x so that player 2 is indifferent between accepting and rejecting to receive $1 - \delta$ in the second period. Thus, player 1 offers $x = 1 - \delta + \delta^2$ and it is accepted.

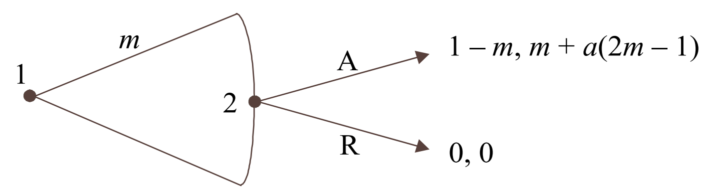
7.



Player 3 accepts any offer such that his share is at least zero. Player 2 substitutes an offer of $x = 0, y = 1$ for any offer made by player 1. Player 1 makes any offer of X and Y. Also, it may be that player 2 accepts $X = 0, Y = 1$.

8.

(a)



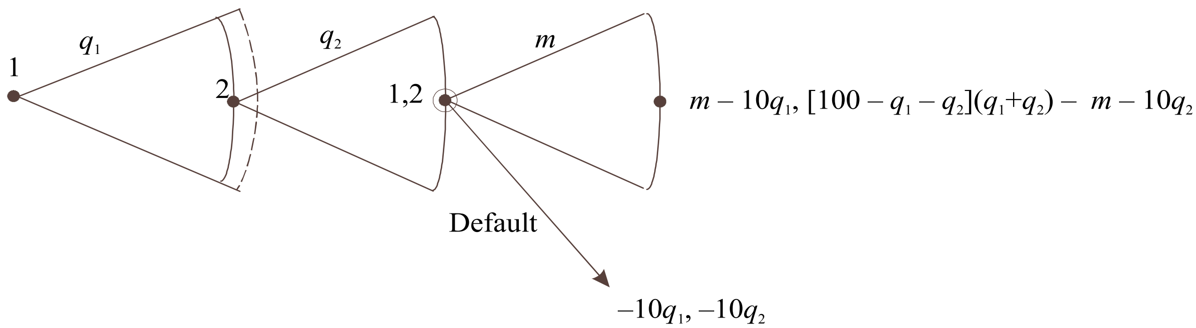
(b) Player 2 accepts any m such that $m + a(2m - 1) \geq 0$. This implies accepting any $m \geq a/(1 + 2a)$. Thus, player 1 offers $a/(1 + 2a)$.

(c) As a becomes large the equilibrium split is 50:50. This is because, when a is large, player 2 cares very much about how close his share is to player 1's share and will reject any offer in which a is not close to $1 - a$.

20 Games with Joint Decisions; Negotiation Equilibrium

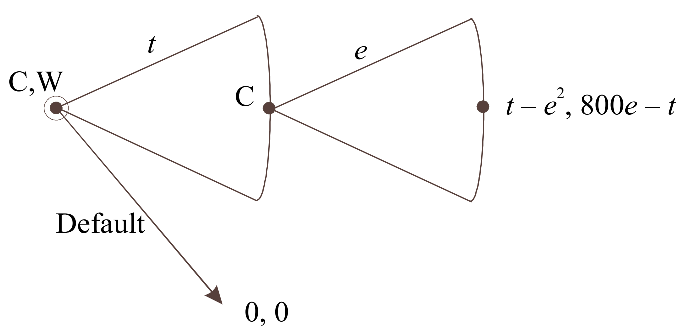
1.

The game is represented as below. Note that $m \in [0, (100 - q_1 - q_2)(q_1 + q_2)]$.



2.

(a)



Carina expends no effort ($e^* = 0$) and Wendy sets $t = 0$.

(b) Carina solves $\max_e 800xe - e^2$. This yields the first order condition of $800x = 2e$. This implies $e^* = 400x$. Wendy solves $\max_x 800[400x] - 800x[400x]$. This yields $x^* = 1/2$.

(c) Given x and t, Carina solves $\max_e 800xe + t - e^2$. This implies $e^* = 400x$. To find the maximum joint surplus, hold t fixed and solve $\max_x 800[400x] - [400x]^2$. This yields $x^* = 1$. The joint surplus is $320,000 - 160,000 = 160,000$. Because of the players' equal bargaining weights, the transfer is $t^* = 80,000$.

3.

(a) Since the cost is sunk, the surplus is $[100 - q_1 - q_2](q_1 + q_2)$. Thus, $u_i = -10q_i + \pi_i[100 - q_1 - q_2](q_1 + q_2)$.

(b) $u_1 = (1/2)[100 - q_1 - q_2](q_1 + q_2) - 10q_1$ and $u_2 = (1/2)[100 - q_1 - q_2](q_1 + q_2) - 10q_2$.

(c) Firm 1 solves $\max_{q_1} (1/2)[100 - q_1 - q_2](q_1 + q_2) - 10q_1$. The first order condition implies $q_1^*(q_2) = 40 - q_2$. By symmetry $q_2^*(q_1) = 40 - q_1$. In equilibrium, $q_1 + q_2 = 40$. Since there are many combinations of q_1 and q_2 that satisfy this equation, there are multiple equilibria. Each firm wants to maximize its share of the surplus less cost. The gain from having the maximum surplus outweighs the additional cost. Note that the total quantity (40) is less than both the standard Cournot output and the monopoly output. Since it is less than the monopoly output, it is not efficient from the firms' point of view.

(d) Now each firm solves $\max_{q_i} \pi_i[100 - q_i - q_j](q_i + q_j) - 10q_i$. This implies best response functions given by $q_i^*(q_j) = 50 - 5/\pi_i - q_j$ that *cannot* be simultaneously satisfied with positive quantities. This is because the player with the smaller π_i would wish to produce a negative amount. In the equilibrium, the player with the larger bargaining weight π produces $50 - 5/\pi$ units and the other firm produces zero.

(e) The player with the smaller bargaining weight does not receive enough gain in his share of the surplus to justify production.

4.

(a) Each gets 100,000. Thus, $x = y = 100,000$.

(b) Working backward, the surplus when Frank and Cathy bargain is $300,000 - x$. Frank's disagreement payoff is $-t$, whereas Cathy's is 0. Thus, Frank's payoff following the negotiation is $150,000 - x/2 - t$. Cathy's payoff is $150,000 - x/2$.

Knowing this, when Frank and Gwen bargain, the surplus is $150,000 + x/2$. Note that $x \leq 300,000$ is required. Thus, the optimal choice of x is 300,000, which yields a surplus of 300,000. They split this evenly, which implies $t = -150,000$. This implies that Frank and Cathy agree to a contract specifying $y = 0$.

(c) Frank and Gwen each receive 150,000, and Cathy receives 0. This is because Frank receives a larger payoff by reducing the surplus of the relationship with Cathy to zero. The 150,000 that he receives from Gwen is sunk. This is efficient.

5.

(a) The players need enforcement when (H, L) is played. In this case, player 2 would not select "enforce." For player 1 to have the incentive to choose "enforce," it must be that $t \geq c$. Player 2 prefers not to deviate from (H, H) only if $t \geq 4$. We also need $t - c \leq 2$, or otherwise player 1 would prefer to deviate from (H, H) and then select "enforce." Combining these inequalities, we have $c \in [t-2, t]$ and $t \geq 4$. A value of t that satisfies these inequalities exists if and only if $c \geq 2$. Combining this with the legal constraint that $t \leq 10$, we find that (H, H) can be enforced (using an appropriately chosen t) if and only if $c \in [2, 10]$.

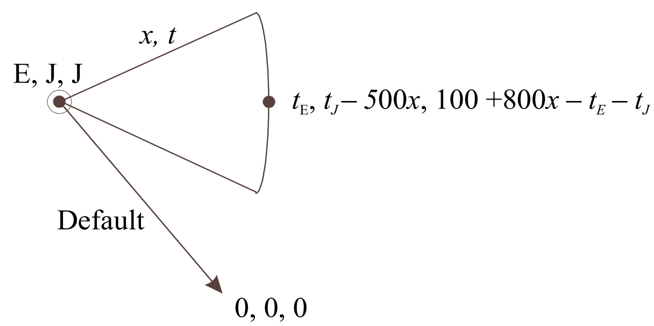
(b) We need t large to deter player 2, and $t - c$ small to deter player 1. It is not possible to do both if c is close to 0. In other words, the legal fee deters frivolous suits from player 1, while not getting in the way of justice in the event that player 2 deviates.

(c) In this case, the players would always avoid court fees by negotiating a settlement. This prevents the support of (H, H).

21 Investment, Hold Up, and Ownership

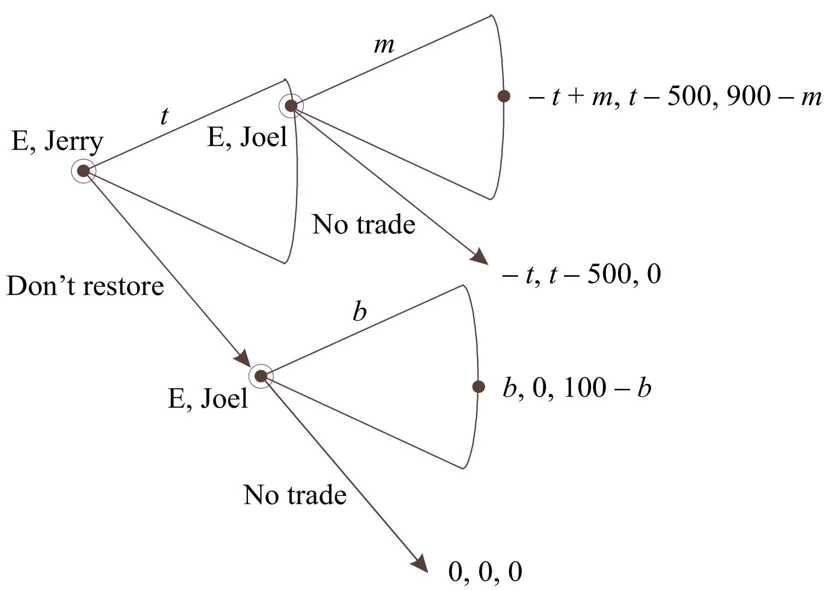
1.

(a) Let $x = 1$ denote restoration and $x = 0$ denote no restoration. Let t_E denote the transfer from Joel to Estelle, and let t_J denote the transfer from Joel to Jerry. The order of the payoffs is Estelle, Jerry, Joel. Here is the extensive form with joint decisions:



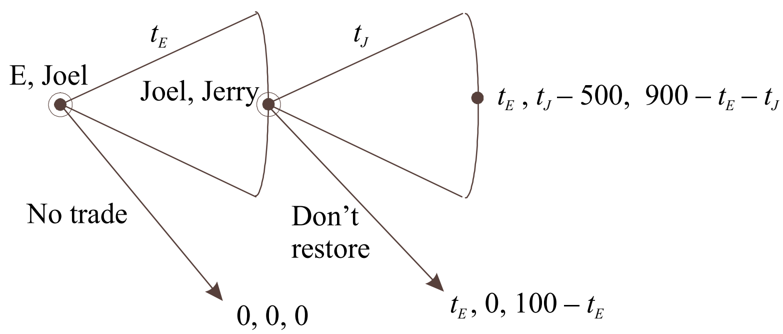
The surplus is $900-500 = 400$. The standard bargaining solution requires that each player i receive $d_i + \pi_i[v^* - d_i - d_l - d_k]$, where l and k denote the other players. Thus, Joel buys the desk, Joel pays Estelle 400/3, Joel pays Jerry 1900/3, and Jerry restores the desk. This is efficient.

(b) Let t denote the transfer from Estelle to Jerry. Let m denote the transfer from Joel to Estelle when the desk has been restored. Let b denote the transfer from Joel to Estelle when the desk has not been restored.



In equilibrium, the desk is not restored and Joel buys the desk for 50. This is not efficient.

(c) Let t_E denote the transfer from Joel to Estelle, and let t_J denote the transfer from Joel to Jerry.



In equilibrium, Joel buys the desk for 125, and pays Jerry 650 to restore it. This is efficient. However, Jerry's payoff is greater here than in part (a) because Jerry can hold up Joel during their negotiation, which occurs after Joel has acquired the desk from Estelle.

(d) Estelle (and Jerry) do not value the restored desk. Thus, Estelle can be held up if she has the desk restored and then tries to sell it to Joel.

2.

If the worker's bargaining weight is less than 1, then he gets more of an increase in his payoff from increasing his outside option by a unit than from increasing his productivity with the single employer. Thus, he does better to increase his general human capital.

3.

(a) The worker chooses A if $w \geq bx$ and R otherwise. Thus, the firm offers $w = bx$. To maximize his payoff at the initial node, the worker selects $x = b/2$.

(b) Here, the firm will accept if $ax \geq w$, so the worker offers $w = ax$. This gives the worker the incentive to choose $x^* = a/2$ at the initial node.

(c) In this case, the wage will be $w = bx + (1/2)(a + b)x$ (from the standard bargaining solution). At the initial node, the worker selects x to solve $\max_x (1/2)(a + b)x - x^2$. The optimal choice is $x^* = (1/4)(a + b)$.

(d) More bargaining power to the worker implies a larger investment x. An increase in b raises the equilibrium investment, which increases the joint value. The maximum value of the relationship is achieved in the setting of part (b), where the worker obtains the full value of his investment and therefore is not held up.

4.

(a) The efficient investment level is the solution to $\max_x x - x^2$m which is $x^* = 1/2$.

(b) Player 1 selects $x = 1/2$. Following this investment, the players demand $m_2(1/2) = 0$ and $m_1(1/2) = x$. In the event that player 1 deviates by choosing some $x \neq 1/2$, then the players are prescribed to make the demands $m_2(x) = x$ and $m_1(x) = 0$.

(c) One way to interpret this equilibrium is that player 1's bargaining weight is 1 if he invests 1/2, but it drops to zero if he makes any other investment. Thus, player 1 obtains the full value of his investment when he selects 1/2, but he obtains none of the benefit of another investment level.

5.

(a) The union makes a take-it-or-leave-it offer of $w = (R - M)/n$, which is accepted. This implies that the railroad will not be built, since the entrepreneur can foresee that it will lose F.

(b) The surplus is $R - M$. The entrepreneur gets $\pi_E[R - M]$ and the union gets $n\overline{w} + \pi_U[R - M]$. The railroad is built if $\pi_E[R - M] > F$.

(c) The entrepreneur's investment is sunk when negotiation occurs, so he does not generally get all of the returns from his investment. When he has all of the bargaining power, he does extract the full return. To avoid the hold-up problem, the entrepreneur may try to negotiate a contract with the union before making his investment.

6.

(a) The efficient outcome is high investment and acceptance.

(b) If $p_0 \geq p_1 - 5$ then the buyer always accepts. The seller will not choose H.

(c) In the case that L occurs, the buyer will not accept if $p_1 \geq 5 + p_0$. In the case that H occurs, the buyer will accept if $p_1 \leq 20 + p_0$. Thus, it must be that $20 + p_0 \geq p_1 \geq 10 + p_0$. Because the seller invests high if $p_1 \geq 10 + p_0$, there are values of p_0 and p_1 that induce the efficient outcome.

(d) The surplus is 10. Each gets 5. Thus, $p_1 = 15$ and $p_0 \in [-5, 5]$. The seller chooses H. The buyer chooses A if H, and R if L.

7.

Stock options in a start-up company, stock options for employees, and options to buy in procurement settings are examples.

8.

If it is not possible to verify whether you have abused the computer or not, then it is better for you to own it. This gives you the incentive to treat it with care, because you will be responsible for necessary repairs.

22 Repeated Games and Reputation

1.

(U, L) can be supported as follows. If player 2 defects ((U,M) is played) in the first period, then the players coordinate on (C, R) in the second period. If player 2 defects ((C, L) is played) in the first period, then the players play (D, M) in the second period. Otherwise, the players play (D, R) in the second period.

2.

(a) To support cooperation, δ must be such that $2/(1-\delta) \geq 4+\delta/(1-\delta)$. Solving for δ, we see that cooperation requires $\delta \geq 2/3$.

(b) To support cooperation by player 1, it must be that $\delta \geq 1/2$. To support cooperation by player 2, it must be that $\delta \geq 3/5$. Thus, we need $\delta \geq 3/5$.

(c) Cooperation by player 1 requires $\delta \geq 4/5$. Player 2 has no incentive to deviate in the short run. Thus, it must be that $\delta \geq 4/5$.

3.

(a) To find player i's best response function, solve $\max_{x_i} x_j^2 + x_j - x_j x_i$. It is easy to see that player i's best response is always $x_i = 0$. This is not efficient. To see this, consider $x_1 > 0$ and $x_2 = 0$.

(b) Since $x = x_1 = x_2$, $u_i = x^2 + x - x^2 = x$. The optimal deviation by player i is to set $x_i = 0$. This yields a payoff of x^2+x. Using the stage Nash profile for punishment, supporting cooperation requires $x/(1-\delta) \geq x^2+x$, which simplifies to $\delta x/(1-\delta) \geq x^2$. Rearranging, we obtain $\delta \geq x/(1+x)$.

(c) The level of x that can be achieved is increasing in the players' patience.

4.

In period 2, subgame perfection requires play of the only Nash equilibrium of the stage game. As there is only one Nash equilibrium of the stage game, selection of the Nash equilibrium to be played in period 2 cannot influence incentives in period 1. Thus, the only subgame perfect equilibrium is play of the Nash equilibrium of the stage game in both periods. For any finite T, the logic from the two period case applies, and the answer does not change.

5.

Alternating between (C, C) and (C, D) requires that neither player has the incentive to deviate. Clearly, however, player 1 can guarantee himself at least 2 per period, yet he would get less than this starting in period 2 if the players alternated as described. Thus, alternating between (C,C) and (C,D) cannot be supported.

On the other hand, alternating between (C,C) and (C,D) can be supported. Note first that, using the stage Nash punishment, player 2 has no incentive to deviate in odd or even periods. Player 1 has no incentive to deviate in even periods, when (D, D) is supposed to be played. Furthermore, player 1 prefers not to deviate in an even period if

$$7 + \frac{2\delta}{1-\delta} \leq 3 + 2\delta + 3\delta^2 + 2\delta^3 + 3\delta^4 + \ldots,$$

which simplifies to

$$7 + \frac{2\delta}{1-\delta} \leq \frac{3+2\delta)}{1-\delta^2}.$$

Solving for δ yields $\delta \geq \sqrt{\frac{4}{5}}$.

6.

A long horizon ahead.

7.

(a) The (pure strategy) Nash equilibria are (U, L, B) and (D, R, B).

(b) Any combination of the Nash equilibria of the stage game are subgame perfect equilibria. These yield the payoffs (8, 8, 2), (8, 4, 10), and (8, 6, 6). There are two other subgame perfect equilibria. In the first, the players select (U, R, A) in the first round, and then if no one deviated, they play (D, R, B) in the second period; otherwise, they play (U, L, B) in the second period. This yields payoff (9, 7, 10). In the other equilibrium, the players select (U, R, B) in the first round and, if player 2 does not cheat, (U, L, B) in the second period; if player 2 cheats, they play (D, R, B) in the second period. This yields the payoff (8, 6, 9).

8.

(a) Player 2^t plays a best response to player 1's action in the stage game.

(b) Consider the following example. There is a subgame perfect equilibrium, using stage Nash punishment, in which, in equilibrium, player 1 plays T and player 2^t plays D.

1 \ 2	E	D
T	3, –1	6, 0
A	5, 5	7, 0

(c) Consider, for example, the prisoners' dilemma. If only one player is a long-run player, then the only subgame perfect equilibrium repeated game will involves each player defecting in each period. However, from the text we know that cooperation can be supported when both are long-run players.

9.

(a) As $x < 10$, there is no gain from continuing. Thus, neither player wishes to deviate.

(b) If a player selects S, then the game stops and this player obtains 0. Since the players randomize in each period, their continuation values from the start of a given period are both 0. If the player chooses C in a period, he thus gets an expected payoff of $10\alpha - (1 - \alpha)$. Setting this equal to 0 (which must be the case in order for the players to be indifferent between S and C) yields $\alpha = 1/11$.

(c) In this case, the continuation value from the beginning of each period is αx. When a player selects S, he expects to get αz; when he chooses C, he expects $10\alpha + (1 - \alpha)(-1 + \delta\alpha x)$. The equality that defines α is thus $\alpha z = 10\alpha + (1 - \alpha)(-1 + \delta\alpha x)$.

23 Collusion, Trade Agreements, and Goodwill

1.

(a) Consider all players selecting $p_i = p = 60$, until and unless someone defects. If someone defects, then everyone chooses $p_i = p = 10$ thereafter.

(b) The quantity of each firm when they collude is $q^c = (110 - 60)/n = 50/n$. The profit of each firm under collusion is $(50/n)60 - 10(50/n) = 2500/n$. The profit under the Nash equilibrium of the stage game is 0. If player i defects, she does so by setting $p_i = 60 - \varepsilon$, where ε is arbitrarily small. Thus, the stage game payoff of defecting can be made arbitrarily close to $2,500$.

To support collusion, it must be that $[2500/n][1/(1-\delta)] \geq 2500+0$, which simplifies to $\delta \geq 1 - 1/n$.

(c) Collusion is "easier" with fewer firms.

2.

(a) The best response function of player i is given by $BR_i(x_j) = 30+x_j/2$. Solving for equilibrium, we find that $x_i = 30+\frac{1}{2}[30+\frac{x_i}{2}]$ which implies that $x_1^* = x_2^* = 60$. The payoff to each player is equal to $2,000-30(60) = 1,100$.

(b) Under zero tariffs, the payoff to each country is 2,000. A deviation by player i yields a payoff of $2,000 + 60(30) - 30(30) = 2,900$. Thus, player i's gain from deviating is 900. Sustaining zero tariffs requires that

$$900 + \frac{1100\delta}{1-\delta} \leq \frac{2000\delta}{1-\delta}.$$

Solving for δ, we get $\delta \geq 1/2$.

(c) The payoff to each player of cooperating by setting tariffs equal to k is $2000 + 60k + k^2 - k^2 - 90k = 2000 - 30k$. The payoff to a player from unilaterally deviating is equal to

$$\begin{aligned}&2,000 + 60\left[30 + \tfrac{k}{2}\right] + \left[30 + \tfrac{k}{2}\right]k - \left[30 + \tfrac{k}{2}\right]^2 - 90k\\ &= 2,000 + \left[30 + \tfrac{k}{2}\right]^2 - 90k.\end{aligned}$$

Thus, the gain to player i of unilaterally deviating is

$$\left[30 + \frac{k}{2}\right]^2 - 60k.$$

In order to support tariff setting of k, it must be that

$$\left[30+\frac{k}{2}\right]^2-60k+\frac{1100\delta}{1-\delta}\leq\frac{[2000-30k]\delta}{1-\delta}.$$

Solving yields the condition

$$\frac{[30+\frac{k}{2}]^2-60k}{900-90k-[30+\frac{k}{2}]^2}\leq\delta.$$

3.

The Nash equilibria are (A, Z) and (B, Y). Obviously, there is an equilibrium in which (A, Z) is played in both periods and player 2^1 sells the right to player 2^2 for 8α. There is also a "goodwill" equilibrium that is like the one constructed in the text, although here it may seem undesirable from player 2^1's point of view. Players coordinate on (A, X) in the first period and (A, Z) in the second period, unless player 2^1 deviated from X in the first period, in which case (B,Y) is played in the second period. Player 2^1 sells the right to player 2^2 for 8α if he did not deviate in the first period, whereas he sells the right for 4α if he deviated. This is an equilibrium (player 2^1 prefers not to deviate) if $\alpha > 3/4$.

4.

(a) Each player 2^t cares only about his own payoff in period t, so he will play D. This implies that player 1 will play D in each period.

(b) Suppose players select (C, C) unless someone defects, in which case (D, D) is played thereafter. For this to be rational for player 1, we need $2/(1-\delta)\geq 3+\delta/(1-\delta)$ or $\delta\geq 1/2$. For player 2^t, this requires that $2+\delta p^G\geq 3+\delta p^B$, where p^G is the price he gets with a good reputation and p^B is the price he gets with a bad reputation. (Trade occurs at the beginning of the next period, so the price is discounted). Cooperation can be supported if $\delta(p^G-p^B)\geq 1$.

Let α be the bargaining weight of each player2^t in his negotiation to sell the right to player 2^{t+1}. We can see that the surplus in the negotiation between players 2^t and 2^{t+1} is $2+\delta p^G$, because this is what player 2^{t+1} expects to obtain from the start of period $t+1$ if he follows the prescribed strategy of cooperating when the reputation is good. This surplus is divided according to the fixed bargaining weights, implying that player 2^t obtains $p^G=\alpha[2+\delta p^G]$. Solving for p^G yields $p^G=2\alpha/(1-\delta\alpha)$. Similar calculations show that $p^B=\alpha/(1-\delta\alpha)$. Substituting this into the condition $\delta(p^G-p^B)\geq 1$ and simplifying yields $\delta\alpha\geq 1/2$. In words, the discount factor and the owner's bargaining weight must be sufficiently large in order for cooperation to be sustained over time.

5.

(a) The Nash equilibria are (x, x) and (y, y).

(b) They would agree to (y, y).

(c) In the first period, they play (z, z). If no one defected in the first period, then they are supposed to play (y, y) in the second period. If someone defected in the first period, then they play (x, x) in the second period. It is easy to verify that this strategy is a subgame perfect equilibrium.

(d) Probably not. They would renegotiate to play (x, x).

6.

(a) The Nash equilibria are (x, x), (x, z), (z, x), and (y, y).

(b) They would agree to play (y,y).

(c) In the first round, they play (z, z). If no one defected in the first period, then they are supposed to play (y, y) in the second period. If player 1 defected in the first period, then they coordinate on (z, x) in the second period. If player 2 defected in the first period, then they coordinate on (x, z) in the second period. It is easy to verify that this strategy is a subgame perfect equilibrium.

(d) The answer depends on whether one believes that the players' bargaining powers would be affected by the history of play. If deviation by a player causes his bargaining weight to suddenly drop to, say, 0, then the equilibrium described in part (c) seems consistent with the opportunity to renegotiate before the second period stage game. Another way of interpreting the equilibrium is that the prescribed play for period 2 is the *disagreement point* for renegotiation, in which case there is no surplus of renegotiation. However, perhaps a more reasonable theory of renegotiation would posit that each player's bargaining weight is independent of the history (it is related to institutional features) and that each player could insist on some neutral stage Nash equilibrium, such as (x, x) or (y, y). In this case, as long as bargaining weights are positive, it would not be possible to sustain (x, z) or (z, x) in period 2. As a result, the equilibrium of part (c) would not withstand renegotiation.

7.

(a) If a young player does not expect to get anything when he is old, then he optimizes myopically when young and therefore gives nothing to the older generation.

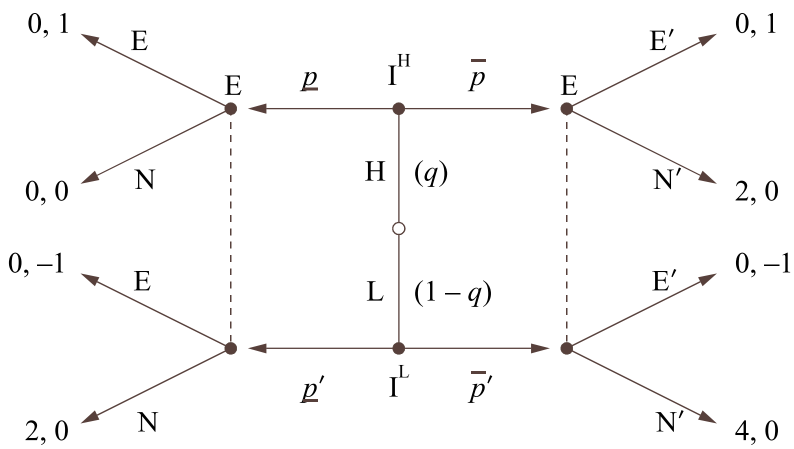
(b) If player $t-1$ has given $x_{t-1} = 1$ to player $t-2$, then player t gives $x_t = 1$ to player $t-1$. Otherwise, player t gives nothing to player $t-1$ ($x_t = 0$). Clearly, each young player thus has the incentive to give 1 to the old generation.

(c) Each player obtains 1 in the equilibrium from part (a), 2 in the equilibrium from part (b). Thus, a reputation-based intergenerational-transfer equilibrium is best.

24 Random Events and Incomplete Information

1.

(a)



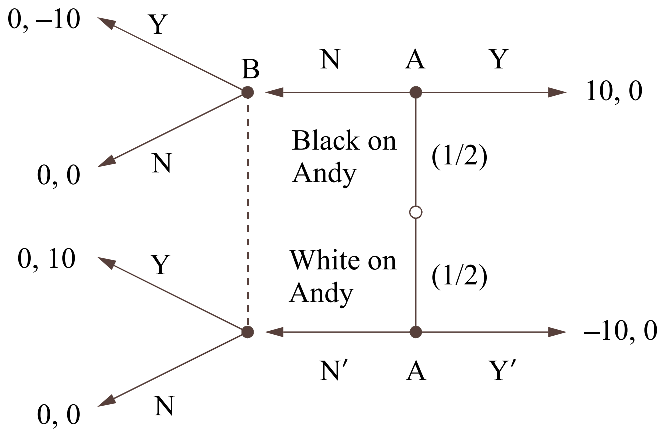
(b) This extensive form game has *no* proper subgames, so subgame perfection is the same as Nash equilibrium.

(c)

I \ E	EE′	EN′	NE′	NN′
$\overline{p}\overline{p}'$	$0, 2q-1$	$4-2q, 0$	$0, 2q-1$	$4-2q, 0$
$\overline{p}\underline{p}'$	$0, 2q-1$	$2q, q-1$	$2-2q, q$	$2, 0$
$\underline{p}\overline{p}'$	$0, 2q-1$	$4-4q, q$	$0, q-1$	$4-4q, 0$
$\underline{p}\underline{p}'$	$0, 2q-1$	$0, 2q-1$	$2-2q, 0$	$2-2q, 0$

2.

(a)



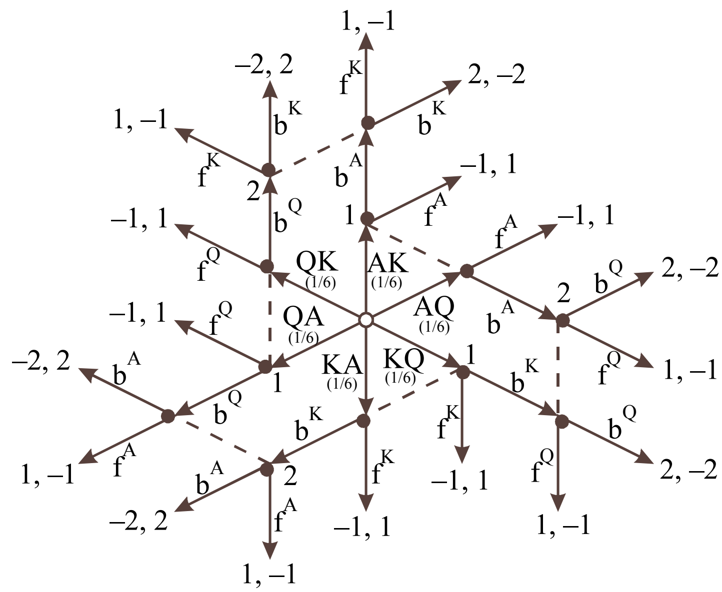
(b)

A \ B	Y	N
YY′	0, 0	0, 0
YN′	5, 5	5, 0
NY′	–5, –5	–5, 0
NN′	0, 0	0, 0

3.

1 \ 2	U	D
LL′	2, 0	2, 0
LR′	1, 0	3, 1
RL′	1, 2	3, 0
RR′	0, 2	4, 1

4.


1, –1
–2, 2
2, –2
1, –1
–1, 1
–1, 1
–1, 1
2, –2
–1, 1
–2, 2
1, –1
1, –1
–1, 1
2, –2
–2, 2
–1, 1
1, –1
1, –1
QK
(1/6)
AK
(1/6)
QA
(1/6)
AQ
(1/6)
KA
(1/6)
KQ
(1/6)

25 Risk and Incentives in Contracting

1.

Examples include stock brokers, commodities traders, and salespeople.

2.

This requires that $v(20) > (1/4)v(100) + (3/4)v(0)$. One function that meets these requirements is $u(x) = \sqrt{x}$.

3.

That lottery A is preferred to lottery B implies $(1/8)v(100) + (7/8)v(0) > (1/2)v(20) + (1/2)v(0)$. Subtracting $(1/2)v(0)$ from each side and then multiplying by 2 yields $(1/4)v(100) + (3/4)v(0) > v(20)$, which contradicts the preference given in Exercise 2.

4.

The probability of a successful project is p. This implies an incentive compatibility constraint of

$$p(w + b - 1)^{\alpha} + (1 - p)(w - 1)^{\alpha} \geq w^{\alpha}$$

and a participation constraint of

$$p(w + b - 1)^{\alpha} + (1 - p)(w - 1)^{\alpha} \geq 1.$$

Thus, we need

$$p(w + b - 1)^{\alpha} + (1 - p)(w - 1)^{\alpha} = 1 = w^{\alpha}.$$

This implies that $b = p^{-\alpha}$.

5.

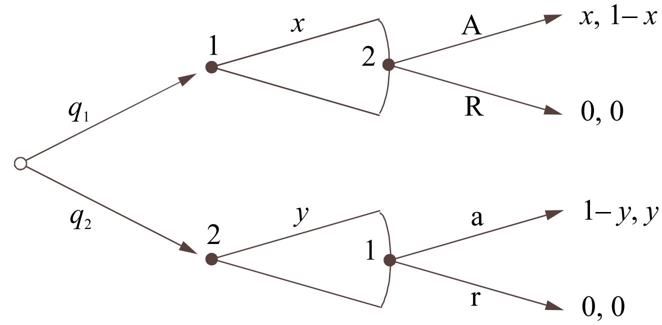
(a) The wage offer must be at least $100 - y$, so the firm's payoff is $180 - (100 - y) = 80 + y$.

(b) In this case, the worker accepts the job if and only if $w + 100q \geq 100$, which means the wage must be at least $100(1 - q)$. The firm obtains $200 - 100(1 - q) = 100(1 + q)$.

(c) When $q = 1/2$, it is optimal to offer the risky job at a wage of 50 if $y \leq 70$, whereas the safe job at a wage of $100 - y$ is optimal otherwise.

6.

(a) Below is a representation of the extensive form for $T = 1$.



(b) Regardless of T, whenever player 1 gets to make the offer, he offers $q_2\delta$ to player 2 (and demands $1 - q_2\delta$ for himself). When player 1 offers $q_2\delta$ or more, then player 2 accepts. When player 2 gets to offer, she offers $q_1\delta$ to player 1. When player 2 offers $q_1\delta$ or more, player 1 accepts.

(c) The expected equilibrium payoff for player i is q_i. Thus, the probability with which player i gets to make an offer can be viewed as his bargaining weight.

(d) The more risk averse a player is, the lower is the offer that he is willing to accept. Thus, an increase in a player's risk aversion should lower the player's equilibrium payoff.

26 Bayesian Nash Equilibrium and Rationalizability

1.

(a) The Bayesian normal form is:

1 \ 2	V	W
X	3, 0	2, 1
B	3, 0	2, 1
C	5, 1	3, 0

(Z, Y) is the only rationalizable strategy profile.

(b) The Bayesian normal form is:

1 \ 2	V	W
X^AX^B	3, 0	2, 1
X^AY^B	6, 0	4, 1
X^AZ^B	5.5, .5	3.5, .5
Y^AX^B	0, 0	0, 1
Y^AY^B	3, 0	2, 1
Y^AZ^B	2.5, .5	1.5, .5
Z^AX^B	2.5, .5	1.5, .5
Z^AY^B	5.5, .5	3.5, .5
Z^AZ^B	5, 1	3, 0

X^AY^B is a dominant strategy for player 1. Thus, the rationalizable set is (X^AY^B, W).

(c) False.

2.

Player 1's payoff is given by

$$u_1 = (x_1 + x_{2L} + x_1 x_{2L}) + (x_1 + x_{2H} + x_1 x_{2H}) - x_1^2.$$

The low type of player 2 gets the payoff

$$u_{2L} = 2(x_1 + x_{2L} + x_1 x_{2L}) - 2x_{2L}^2,$$

whereas the high type of player 2 obtains

$$u_{2H} = 2(x_1 + x_{2H} + x_1 x_{2H}) - 3x_{2H}^2.$$

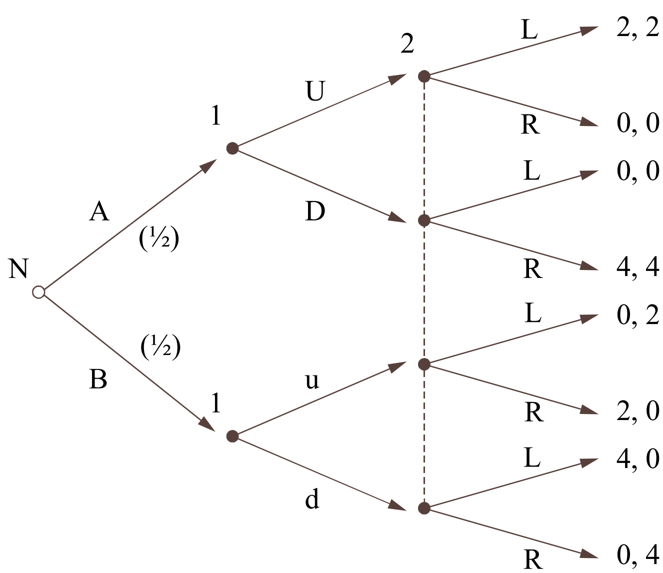
Player 1 solves

$$\max_{x_1} (x_1 + x_{2L} + x_1 x_{2L}) + (x_1 + x_{2H} + x_1 x_{2H}) - x_1^2.$$

The first-order condition is $1 + x_{2L} - x_1 + 1 + x_{2H} - x_1 = 0$. This implies that $x_1^*(x_{2L}, x_{2H}) = 1 + (x_{2L} + x_{2H})/2$. Similarly, the first-order condition of the low type of player 2 yields $x_{2L}^*(x_1) = (1 + x_1)/2$. The first order condition of the high type of player 2 implies $x_{2H}^*(x_1) = (1+x_1)/3$. Solving this system of equations, we find that the equilibrium is given by $x_1^* = \frac{17}{7}$, $x_{2L}^* = \frac{12}{7}$, and $x_{2H}^* = \frac{8}{7}$.

3.

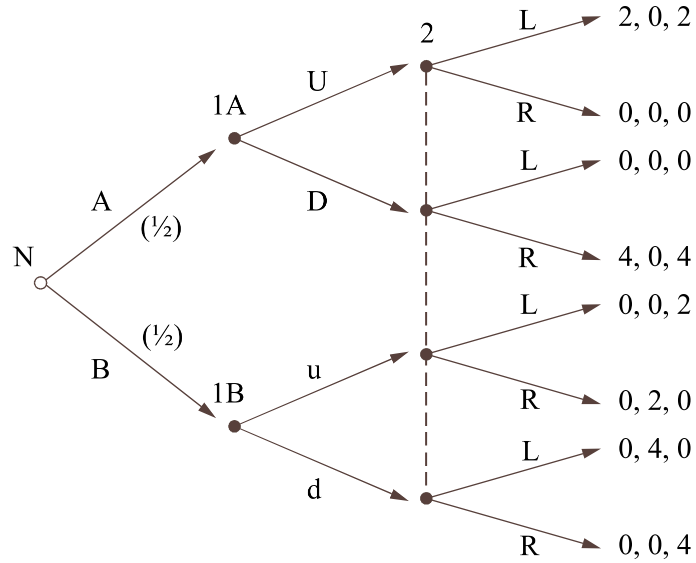
(a) The extensive form and normal form representations are:



1 \ 2	L	R
Uu	1, 2	1, 0
Ud	3, 1	0, 2
Du	0, 1	3, 2
Dd	2, 0	2, 4

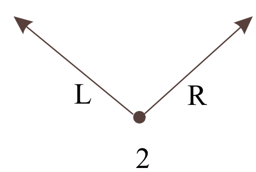
The set of Bayesian Nash equilibria is equal to the set of rationalizable strategies, which is $\{(\text{Du, R})\}$.

(b) The extensive form and normal form representations in this case are:



1A \ 1B	u	d
U	1, 0, 2	1, 2, 1
D	0, 0, 1	0, 2, 4

1A \ 1B	u	d
U	0, 1, 0	0, 0, 2
D	2, 1, 2	2, 0, 4



The equilibrium is (D, u, R). The set of rationalizable strategies is S.

(c) Regarding rationalizability, the difference between the settings of parts (a) and (b) is that in part (b) the beliefs of players 1A and 1B do not have to coincide. In equilibrium, the beliefs of player 1A and 1B must be the same.

4.

Recall that player 1's best response function is given by $BR_1(q_2^L, q_2^H) = 1/2 - q_2^L/4 - q_2^H/4$. The low type of player 2 has a best response function of $BR_2^L(q_1) = 1/2 - q_1/2$. The high type of player 2 has a best response function of $BR_2^H(q_1) = 3/8 - q_1/2$. If $q_1 = 0$, then player 2's optimal quantities are $q_2^L = 1/2$ and $q_2^H = 3/8$. Note that player 2 would never produce more than these amounts. To the quantities $q_2^L = 1/2$ and $q_2^H = 3/8$, player 1's best response is $q_1 = 5/16$. Thus, player 1 will never produce more than $q_1 = 5/16$. We conclude that each type of player 2 will never produce more than her best response to $5/16$. Thus, q_2^L will never exceed $11/32$, and q_2^H will never exceed $7/32$. Repeating this logic, we find that the rationalizable set is the single strategy profile that simultaneously satisfies the best response functions, which is the Bayesian Nash equilibrium.

5.

$R = \{(\text{YN}', \text{Y})\}$.

6.

(LL′, U).

7.

(a)

1 \ 2	X	Y
AA′	0, 1	1, 0
AB′	1/3, 2/3	2/3, 1/3
BA′	2/3, 1/3	5/3, 2/3
BB′	1, 0	4/3, 1

(b) (BA′, Y)

8.

If $x_i \leq \alpha$, then player i folds. Thus, when $x_i = \alpha$, it must be that $-1 = \text{Prob}(x_j \leq \alpha) - 2\text{Prob}(x_j > \alpha)$. This implies $\alpha = 1/3$.

27 Trade with Incomplete Information

1.

There is always an equilibrium in this game. Note that, regardless of p, there is an equilibrium in which neither the lemon nor the peach is traded (Jerry does not trade and Freddie trades neither car). When either $1000 < p \leq 2000$ or $p > 1000 + 2000q$, the only equilibrium involves no trade whatsoever.

2.

(a) Clearly, if $p < 200$ then John would never trade, so neither player will trade in equilibrium. Consider two cases for p between 200 and 1000.

First, suppose $600 \leq p \leq 1,000$. In this case, Jessica will not trade if her signal is $x_2 = 200$, because she then knows that 600 is the most the stock could be worth. John therefore knows that Jessica would only be willing to trade if her signal is $1,000$. However, if John's signal is $1,000$ and he offers to trade, then the trade could occur only when $v = 1000$, in which case he would have been better off not trading. Realizing this, Jessica deduces that John would only be willing to trade if $x_1 = 200$, but then she never has an interest in trading. Thus, the only equilibrium has both players choosing "not," regardless of their types.

Similar reasoning establishes that trade never occurs in the case of $p < 600$ either. Thus, trade never occurs in equilibrium. Interestingly, we reached this conclusion by tracing the implications of common knowledge of rationality (rationalizability), so the result does not rely on equilibrium.

(b) It is not possible for trade to occur in equilibrium with positive probability. This may seem strange compared to what we observe about real stock markets, where trade is usually vigorous. In the real world, players may lack common knowledge of the fundamentals or each other's rationality, trade may occur due to liquidity needs, and there may be differences in owners' abilities to run firms.

(c) Intuitively, the equilibrium strategies can be represented by numbers $\underline{x}_1$ and $\underline{x}_2$, where John trades if and only if $x_1 \leq \underline{x}_1$ and Jessica trades if and only if $x_2 \geq \underline{x}_2$. For John, trade yields an expected payoff of

$$\int_{100}^{\underline{x}_2} (1/2)(x_1 + x_2)F_2(x_2)dx_2 + \int_{\underline{x}_2}^{1000} pF_2(x_2)dx_2 - 1.$$

Not trade yields

$$\int_{100}^{1000} (1/2)(x_1 + x_2)F_2(x_2)dx_2.$$

Simplifying, we see that John's trade payoff is greater than is his no-trade payoff when

$$\int_{\underline{x}_2}^{1000} [p - (1/2)(x_1 + x_2)]F_2(x_2)dx_2 \geq 1. \ (*)$$

For Jessica, trade implies an expected payoff of

$$\int_{100}^{\underline{x}_1} [(1/2)(x_1 + x_2) - p]F_1(x_1)dx_1.$$

No trade gives her a payoff of zero. Simplifying, she prefers trade when

$$\int_{100}^{\underline{x}_1} [(1/2)(x_1 + x_2) - p]F_1(x_1)dx_1 \geq 1. \ (**)$$

By the definitions of $\underline{x}_1$ and $\underline{x}_2$, (*) holds for all $x_1 \leq \underline{x}_1$ and (**) holds for all $x_2 \geq \underline{x}_2$. Integrating (*) over $x_1 < \underline{x}_1$ yields

$$\int_{100}^{\underline{x}_1} \int_{\underline{x}_2}^{1000} [p - (1/2)(x_1 + x_2)]F_2(x_2)F_1(x_1)dx_2dx_1 \geq \int_{100}^{\underline{x}_1} F_1(x_1)dx_1.$$

Integrating (**) over $x_2 > \underline{x}_2$ yields

$$\int_{100}^{\underline{x}_1} \int_{\underline{x}_2}^{1000} [(1/2)(x_1 + x_2) - p]F_2(x_2)F_1(x_1)dx_2dx_1 \geq \int_{\underline{x}_2}^{1000} F_2(x_2)dx_2.$$

These inequalities cannot be satisfied simultaneously, unless trade never occurs in equilibrium—so that $\underline{x}_1$ is less than 100 and $\underline{x}_2$ exceeds $1,000$, implying that all of the integrals in these expressions equal zero.

3.

To show that bidding v_i is weakly preferred to bidding any $x < v_i$, consider three cases, with respect to x, v_i, and the other player's bid b_j. In the first case, $x < b_j < v_i$. Here, bidding x causes player i to lose, but bidding v_i allows player i to win and receive a payoff of $v_i - b_j$. Next consider the case in which $x < v_i < b_j$. In this case, it does not matter whether player i bids x or v_i; he loses either way, and receives a payoff of 0. Finally, consider the case where $b_j < x < v_i$. Here, bidding either x or v_i ensures that player i wins and receives the payoff $v_i - b_j$.

4.

(a) Colin wins and pays 82.

(b) Colin wins and pays 82 (or 82 plus a very small number).

(c) The seller should set the reserve price at 92. Colin wins and pays 92.

5.

As discussed in the text, without a reserve price, the expected revenue of the auction is 1000/3. With a reserve price r, player i will bid at least r if $v_i > r$. The probability that $v_i < r$ is $r/1000$. Thus, the probability that both players have a valuation that is less than r is $(r/1000)^2$. Consider, for example, setting a reserve price of 500. The probability that at least one of the players' valuations is above 500 is $1 - (1/2)^2 = 3/4$. Thus, the expected revenue of setting $r = 500$ is at least $500(3/4) = 385$, which exceeds 1000/3.

6.

Assume that the equilibrium strategies take the form $b_i = av_i$. Then, given that the other players are using this bidding strategy (for some constant a), player i's expected payoff of bidding x is $(v_i - x)[x/1000a]^{n-1}$. The first-order condition for player i's best response is $(n-1)(v_i - x)x^{n-2} - x^{n-1} = 0$. Solving for x yields $x = v_i(n-1)/n$, which means $a = (n-1)/n$. Note that, as $n \to \infty$, a approaches 1.

7.

Let $v_i = 20$. Suppose player i believes that the other players' bids are 10 and 25. If player i bids 20 then she loses and obtains a payoff of 0. However, if player i bids 25 then she wins and obtains a payoff of $20 - 10 = 10$. Thus, bidding 25 is a best response, but bidding 20 is not.

8.

Your optimal bidding strategy is $b = v/3$, you should bid $b(3/5) = 1/5$.

28 Perfect Bayesian Equilibrium

1.

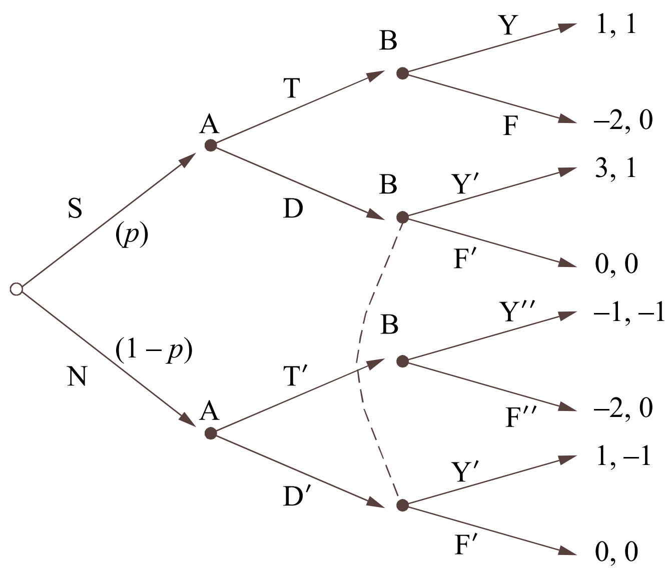
Let $w = \text{Prob}(H \mid \overline{p})$ and let $r = \text{Prob}(H \mid \underline{p})$.

(a) The separating equilibrium is ($\overline{p}\underline{p}'$, NE′) with beliefs $w = 1$ and $r = 0$.

(b) For $q \leq 1/2$, there is a pooling equilibrium with strategy profile ($\overline{p}\overline{p}'$, NN′) and beliefs $w = q$ and any $r \leq 1/2$. There are also similar pooling equilibria in which the entrant chooses E and has any belief $r \geq 1/2$. For $q > 1/2$, there is a pooling equilibrium in which the strategy profile is ($\underline{p}\underline{p}'$, EE′) and the beliefs are $w = q$ and any $r \leq 1/2$. There are also pooling equilibria in which the incumbent plays $\underline{p}\underline{p}'$.

2.

(a) The extensive form is below. Amy's payoffs are given first.



(b) Yes. Let q denote Brenda's posterior probability that the shoes are on sale, given that the non-singleton information set is reached. The equilibrium is (TD′,YF′F″) with $q = 0$.

(c) Yes, if $p \geq 1/2$. Again, let q denote Brenda's posterior probability that the shoes are on sale, given that the non-singleton information set is reached. In the equilibrium, Amy's plays strategy DD′, Brenda's posterior belief is $q = p$, and Brenda chooses YY′F″. There is no pooling equilibrium if $p \leq 1/2$.

3.

(a) Yes, it is (RL′, U) with $q = 1$.

(b) Yes, it is (LL′,D) with $q \leq 1/3$.

4.

Yes. Player 1's actions may signal something of interest to the other players. This sort of signaling can arise in equilibrium as long as, given the rational response of the other players, player 1 is indifferent or prefers to signal.

5.

(a) The perfect Bayesian equilibrium is given by $\mathrm{E}^0\mathrm{N}^1$, $\overline{y} = 1$, $\underline{y} = 0$, $q = 1$, and $y = 1$.

(b) The innocent type provides evidence, whereas the guilty type does not.

(c) In the perfect Bayesian equilibrium, each type $x \in \{0, 1, \ldots, K-1\}$ provides evidence and the judge believes that he faces type K when no evidence is provided.

6.

(a) $c \geq 2$. The separating perfect Bayesian equilibrium is given by OB′, FS′, $r = 0$, and $q = 1$.

(b) $c \leq 2$. The following is such a pooling equilibrium: OO′, SF′, $r = 0$, and $q = 1/2$.

7.

(a) If the worker is type L, then the firm offers $z = 0$ and $w = 35$. If the worker is type H, then the firm offers $z = 1$ and $w = 40$.

(b) Note that the H type would obtain $75 + 35 = 110$ by accepting the safe job. Thus, if the firm wants to give the H type the incentive to accept the risky job, then the firm must set w^1 so that $100(3/5) + w^1 \geq 110$, which means $w^1 \geq 50$. The firm's optimal choice is $w^1 = 50$, which yields a higher payoff than would be the case if the firm gave to the H type the incentive to select the safe job.

(c) The answer depends on the probabilities of the H and L types. If the firm follows the strategy of part (b), then it expects $150p + 145(1 - p) = 145 + 5p$. If the firm only offers a contract with the safe job and wants to employ both types, then it is best to set the wage at 35, which yields a payoff of 145. Clearly, this is worse than the strategy of part (b). Finally, the firm might consider offering only a contract for the risky job, with the intention of only attracting the H type. In this case, the optimal wage is 40 and the firm gets an expected payoff of $160p$. This "H-only" strategy is best if $p \geq 145/155$; otherwise, the part (b) strategy is better.

8.

In the perfect Bayesian equilibrium, player 1 bids with both the Ace and the King, player 2 bids with the Ace and folds with the Queen. When player 1 is dealt the Queen, he bids with probability 1/3. When player 2 is dealt the King and player 1 bids, player 2 folds with probability 1/3.

29 Job-Market Signaling and Reputation

1.

Education would not be a useful signal in this setting. If high types and low types have the same cost of education, then they would have the same incentive to become educated.

2.

Consider separating equilibria. It is easy to see that NE′ cannot be an equilibrium, by the same logic conveyed in the text. Consider the worker's strategy of EN′. Consistent beliefs are $p = 0$ and $q = 1$, so the firm plays MG′. Neither the high nor low type has the incentive to deviate.

Next consider pooling equilibria. It is easy to see that EE′ cannot be a pooling equilibrium, because the low type is not behaving rationally in this case. There is a pooling equilibrium in which NN′ is played, $p = 1/2$, the firm selects M′, q is unrestricted, and the firm's choice between M and C is whatever is optimal with respect to q.

3.

(a) There is no separating equilibrium. The low type always wants to mimic the high type.

(b) Yes, there is such an equilibrium provided that p is such that the worker accepts. This requires $2p-(1-p) \geq 0$, which simplifies to $p \geq 1/3$. The equilibrium is given by ($\mathrm{O^H O^L}$, A) with belief $q = p$.

(c) Yes, there is such an equilibrium regardless of p. The equilibrium is given by ($\mathrm{N^H N^L}$, R) with belief $q \leq 1/3$.

4.

Clearly, the PBE strategy profile is a Bayesian Nash equilibrium. In fact, there is no other Bayesian Nash equilibrium, because the presence of the C type in this game (and rationality of this type) implies that player 2's information set is reached with positive probability. This relation does not hold in general, of course, because of the prospect of unreached information sets.

5.

As before, player 1 always plays S, I′, and B′. Also, player 2 randomizes so that player 1 is indifferent between I and N, which implies that $s = 1/4$. Player 1 randomizes so that player 2 is indifferent between I and N. This requires $2q - 2(1 - q) = 0$, which simplifies to $q = 1/2$. However, $q = p/(p + r - pr)$. Substituting and solving for r, we get $r = p/(1 - p)$. Thus, in equilibrium, player 1 selects action I with probability $r = p/(1 - p)$, and player 2 has belief $q = 1/2$ and plays I with probability 1/4.

If $p > 1/2$, then player 2 always plays I when her information set is reached. This is because $2p - 2(1 - p) = 4p - 2 > 0$. Thus, equilibrium requires that player 1's strategy is II′SB′, that player 2 has belief $q = p$, and that player 2 selects I.

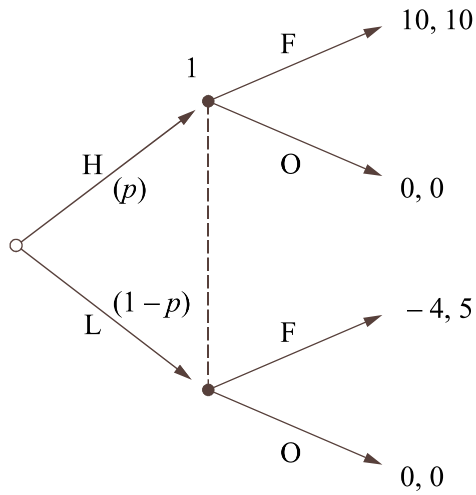
6.

(a) Working backward, it is easy to see that player 2's optimal decision in the second period is to offer a price of 0, because player 1 will be indifferent between accepting and rejecting. In this case, player 2's payoff would be δv. In the first period, player 2 will accept any price that is at or below $v(1 - \delta)$, so player 1 should offer either a price of $2(1 - \delta)$ or $1 - \delta$. If player 1 offers $2(1 - \delta)$ then only the high type will accept and player 1 expects a payoff of $r2(1 - \delta)$. If player 1 offers a price of $1 - \delta$, then he gets $1 - \delta$ with certainty. Thus, player 1 should offer a price of $2(1 - \delta)$ when $r2(1 - \delta) \geq 1 - \delta$, which simplifies to $r \geq 1/2$.

(b) In this setting, player 2 will accept any price that does not exceed $v(1 - \delta)$. If player 1 offers a price p, then it will be accepted by all types of player 2 with $v \geq p/(1 - \delta)$. The probability that $v > a$ is $1 - a$. Thus, player 1's expected payoff from offering p is $p[1 - p/(1 - \delta)]$. To maximize this expected payoff, player 1 selects $p^* = (1 - \delta)/2$.

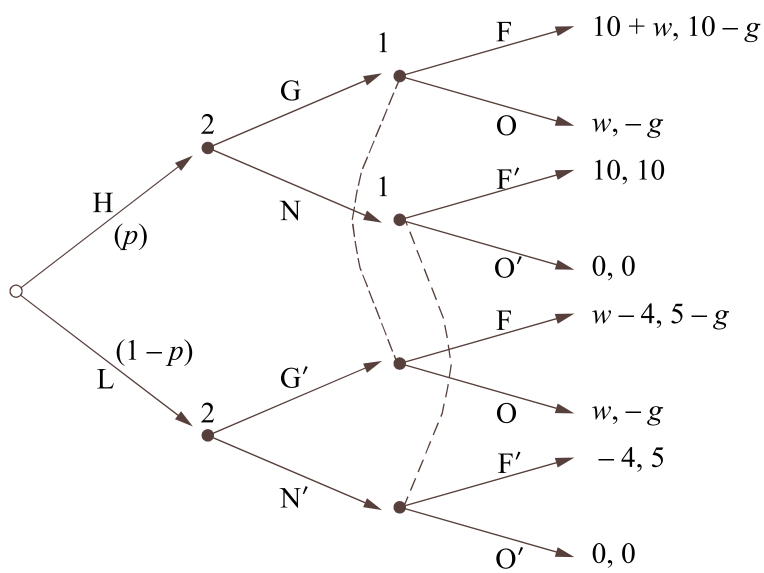
7.

(a) The extensive form is:



In the Bayesian Nash equilibrium, player 1 forms a firm (F) if $10p - 4(1 - p) \geq 0$, which simplifies to $p \geq 2/7$. Player 1 does not form a firm (O) if $p < 2/7$.

(b) The extensive form is:



(c) Clearly, player 1 wants to choose F with the H type and O with the L type. Thus, there is a separating equilibrium if and only if the types of player 2 have the incentive to separate. This is the case if $10 - g \geq 0$ and $0 \geq 5 - g$, which simplifies to $g \in [5, 10]$.

(d) If $p \geq 2/7$ then there is a pooling equilibrium in which NN′ and F′ are played, player 1's belief conditional on no gift is p, player 1's belief conditional on a gift is arbitrary, and player 1's choice between F and O is optimal given this belief. If, in addition to $p \geq 2/7$, it is the case that $g \in [5, 10]$, then there is also a pooling equilibrium featuring GG′ and FO′. If $p \leq 2/7$ then there is a pooling equilibrium in which NN′ and OO′ are played (and player 1 puts a probability on H that is less than 2/7 conditional on receiving a gift).

8.

(a) A player is indifferent between O and F when he believes that the other player will choose O for sure. Thus, (O, O; O, O) is a Bayesian Nash equilibrium.

(b) If both types of the other player select Y, the H type prefers Y if $10p - 4(1-p) \geq 0$, which simplifies to $p \geq 2/7$. The L type weakly prefers Y, regardless of p. Thus, such an equilibrium exists if $p \geq 2/7$.

(c) If the other player behaves as specified, then the H type expects $-g + p(w + 10) + (1 - p)0$ from giving a gift. He expects pw from not giving a gift. Thus, he has the incentive to give a gift if $10p \geq g$. The L type expects $-g + p9w + 5) + (1 - p)0$ if he gives a gift, whereas he expects pw if he does not give a gift. The L type prefers not to give if $g \geq 5p$. The equilibrium, therefore, exists if $g \in [5p, 10p]$.

9.

(a) The manager's optimal contract solves $\max_{\hat{e},\hat{x}} \hat{e} - \hat{x}$ subject to $\hat{x} - \alpha\hat{e}^2 \geq 0$ (which is necessary for the worker to accept). Clearly, the manager will pick $\hat{x}$ and $\hat{e}$ so that the constraint binds. Using the constraint to substitute for $\hat{x}$ yields the unconstrained problem $\max_{\hat{e}} \hat{e} - \alpha\hat{e}^2$. Solving the first-order condition, we get $\hat{e} = 1/(2\alpha)$ and $\hat{x} = 1/(4\alpha)$.

(b) Using the solution of part (a), we obtain $\underline{e} = 4$, $\underline{x} = 2$, $\overline{e} = 4/3$, and $\overline{x} = 2/3$.

(c) The worker will choose the contract that maximizes $\hat{x} - \alpha\hat{e}^2$. The high type of worker would get a payoff of -4 if he chooses contract $(\underline{e}, \underline{x})$, whereas he would obtain 0 by choosing contract $(\overline{e}, \overline{x})$. Thus, he would choose the contract that is meant for him. On the other hand, the low type prefers to select contract $(\overline{e}, \overline{x})$, which gives him a payoff of 4/9, rather than getting 0 under the contract designed for him.

(d) The incentive compatibility conditions for the low and high types, respectively, are

$$x_L - \frac{1}{8}e_L^2 \geq x_H - \frac{1}{8}e_H^2$$

and

$$x_H - \frac{3}{8}e_H^2 \geq x_L - \frac{3}{8}e_L^2.$$

The participation constraints are

$$x_L - \frac{1}{8}e_L^2 \geq 0$$

and

$$x_H - \frac{3}{8}e_H^2 \geq 0.$$

(e) Following the hint, we can substitute for x_L and x_H using the equations

$$x_L = x_H - \frac{1}{8}e_H^2 + \frac{1}{8}e_L^2$$

and

$$x_H = \frac{3}{8}e_H^2.$$

Note that combining these gives $x_L = \frac{1}{4}e_H^2 + \frac{1}{8}e_L^2$. Substituting for x_L and x_H yields the following unconstrained maximization problem:

$$\max_{e_L, e_H} \frac{1}{2}\left[e_H - \frac{3}{8}e_H^2\right] + \frac{1}{2}\left[e_L - \frac{1}{4}e_H^2 - \frac{1}{8}eL^2\right].$$

Calculating the first-order conditions, we obtain $e_L^* = 4$, $x_L^* = 54/25$, $e_H^* = 4/5$, and $x_H^* = 6/25$.

(f) The high type exerts less effort than is efficient, because this helps the manager extract more surplus from the low type.

30 Appendix B

1.

(a) Suppose not. Then it must be that $B(R^{k-1}) = \emptyset$, which implies that $B_i(R^{k-1}) = \emptyset$ for some i. However, we know that the best response set is nonempty (assuming the game is finite), which contradicts what we assumed at the start.

(b) The operators B and UD are *monotone*, meaning that $X \subset Y$ implies $B(X) \subset B(Y)$ and $UD(X) \subset UD(Y)$. This follows from the definitions of B_i and UD_i. Note, for instance, that any belief for player i that puts positive probability only on strategies in X_{-i} can also be considered in the context of the larger Y_{-i}. Furthermore, if a strategy of player i is dominated with respect to strategies Y_{-i}, then it also must be dominated with respect to the smaller set X_{-i}. Using the monotone property, we see that $UD(S) = R^1 \subset S = R^0$ implies $R^2 = UD(R^1) \subset UD(R^0) = R^1$. By induction, $R^k \subset R^{k-1}$ implies $R^{k+1} = UD(R^k) \subset R^k = UD(R^{k-1})$.

(c) Suppose not. Then there are an infinite number of rounds in which at least one strategy is removed for at least one player. However, from (b), we know strategies that are removed are never "put back," which means an infinite number of strategies are eventually deleted. This contradicts that S is finite.

2.

This is discussed in the lecture material for Chapter 7 (see Part II of this manual).

3.

(a) For any p such that $0 \leq p \leq 1$, it cannot be that $6p > 5$ and $6(1-p) > 5$.

(b) Let p denote the probability that player 1 plays U and let q denote the probability that player 2 plays M. Suppose that $C \in BR$. Then it must be that the following inequalities hold: $5pq \geq 6pq$, $-100(1-p)q \geq 0$, $-100p(1-q) \geq 0$, and $5(1-p)(1-q) \geq 6(1-p)(1-q)$. This requires that $(1-p)q = p(1-q)$, which contradicts the assumption of uncorrelated beliefs.

(c) Consider the belief μ_{-1} that (U, M) is played with probability 1/2 and that (D, N) is played with probability 1/2. We have that $u_1(\text{C}, \mu_{-1}) = 5$ and $u_1(\text{B}, \mu_{-1}) = u_1(\text{A}, \mu_{-1}) = 3$.