AP Calculus AB Take-Home Final

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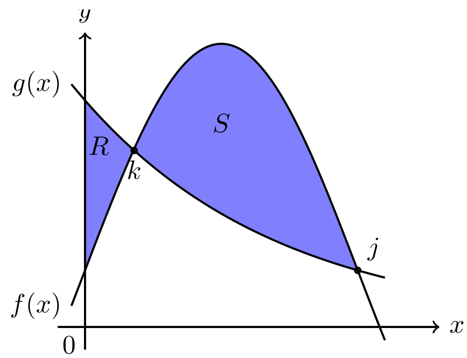
1



$$f(x) = \frac{1}{4} + \sin \pi x, \quad g(x) = 4^{-x}$$

1.1 Part A

First, we find the first intersection of $f(x)$ and $g(x)$. We'll call the x value of the intersection k. We can then use a calculator approximate the integration of the difference between $g(x)$ and $f(x)$ from 0 until k to obtain the area R.

$$R = \int_0^k [g(x) - f(x)]dx \approx 0.064$$

1.2 Part B

Next, we can find the second intersection of $f(x)$ and $g(x)$ (which we will call j. We can then integrate similarly to get the area; Note that we reverse the order of $g(x)$ and $f(x)$ because $f(x)$ has a higher value of y.

$$S = \int_k^j [f(x) - g(x)]dx \approx 0.410$$

1.3 Part C

To revolve S around the horizontal line $y = -1$, we have to adjust $f(x)$ and $g(x)$ by $+1$, and then shroud them in the circle area equation πr^2, finally integrating the resulting difference between k and j to obtain volume.

$$S_{\text{vol}} = \pi \int_k^j \left[(f(x)+1)^2 - (g(x)+1)^2\right] dx \approx 4.56$$

Sorry there's no graphic for the revolve, it's *hard.*

2

$$f(0) = 2, \;\; f'(0) = -4, \;\; f''(0) = 3$$

2.1 Part A

$$g(x) = e^{ax} + f(x)$$

First, we need to know the derivatives of $g(x)$, so we evaluate them as such:

$$\frac{d}{dx}g(x) = g'(x) = ae^{ax} + f'(x)$$

$$\frac{d}{dx}g'(x) = g''(x) = e^{ax}a^2 + f''(x)$$

Next, we can evaluate the derivatives of $g(x)$ in line with the pre-defined derivatives of $f(x)$ like so:

$$g'(0) = ae^{a(0)} + f'(0) = ae^0 - 4 = a - 4$$

$$g''(0) = e^{a(0)}a^2 + f''(0) = a^2e^0 + 3 = a^2 + 3$$

2.2 Part B

$$h(x) = \cos(kx)f(x)$$

First, we derive $h(x)$:

$$\frac{d}{dx}h(x) = h'(x) = -k\sin(kx)f(x) + \cos(kx)f'(x)$$

This gave us the slope ($h'(0)$), but we still need to find the y value of $h(0)$ and the slope at $h'(0)$

$$y = h(0) = \cos[k(0)]f(0) = \cos(0)(2) = (1)(2) = 2$$

$$m = h'(0) = -k\sin(k0)f(x) + \cos(k0)f'(0) = 0 + \cos(1)(-4) = -4$$

We can finally find the tangent equation:

$$y - 2 = -4(x - 0)$$

3

$$R(t) = 2 + 5\sin\frac{4\pi t}{25},\;\; S(t) = \frac{15t}{1+3t}$$
$$A(t) = \text{Total rate} = S(t) - R(t)$$

3.1 Part A

We can integrate the loss function to obtain the amount removed over 6 hours like so:

$$\int_0^6 R(t) \approx 6.723 \;\; yd^3$$

3.2 Part B

We can simply integrate the total rate over time and add it to the initial condition:

$$Y(t) = \int_0^t A(x)\,dx + 2500$$

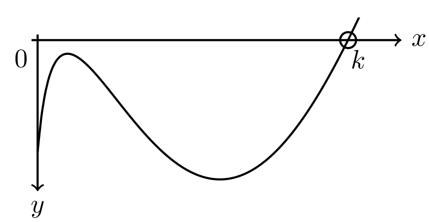
3.3 Part C

We evaluate $A(t)$ at the time point, because that's the total rate:

$$A(4.0) \approx 1.908 \;\; yd^3$$

3.4 Part D

It's intuitive to look at the graph of $A(t)$:



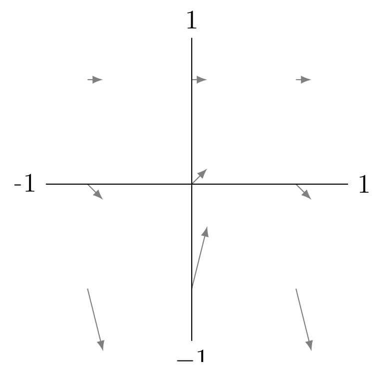
At point k, $A(t)$ intersects the x axis. Up until then, the amount of total sand has been decreasing (because $A(t)$ is below the x xis), and at k it has been decreasing the longest. We can no integrate $A(t)$ from 0 to k to obtain the total amount of sand at that point:

$$Y(k) \approx 2492.37 \;\; yd^3$$

4

4.1 Part A

$$\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$$



4.2 Part B

Despite being able to simply look at the graph, we can infer that the slope from the differential equation will be zero at $y = 1$ because the entire equation is multiplied by $(y-1)^{-2}$.

4.3 Part C

$$f(1) = 0; \quad y = 0, \quad x = 1$$
$$\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$$
$$\frac{dy}{(y-1)^2} = \cos(\pi x)\, dx$$
$$(y-1)^{-2}\, dy = \cos(\pi x)\, dx$$
$$\int (y-1)^{-2}\, dy = \int \cos(\pi x)\, dx$$
$$\frac{1}{1-0} = \frac{\sin(\pi 1)}{\pi} + C_3$$
$$1 = 0 + C_3, \; C_3 = 1$$
$$\frac{1}{1-y} = \frac{\sin(\pi x)}{\pi} + 1$$
$$1 - y = \frac{\pi}{\sin(\pi x)} + 1$$
$$y = \frac{-\pi}{\sin(\pi x)}$$

5

5.1 Part A

$$\int_1^{10} (\sqrt{x-1})\,dx \approx 18$$

5.2 Part B

$$\pi\int_1^{10} (x-1)\,dx \approx 40.5$$

5.3 Part C

$$y = \sqrt{x-1}$$
$$y^2 = x-1$$
$$y^2 + 1 = x$$
$$\pi\int_0^{3} (y^2+1)^2\,dy \approx 218.65$$

6

6.1 Part A

1, 3, 5. This is because $f'(x)$ is zero (has a horizontal tangent) at these locations.

6.2 Part B

Min: $x = 4$, the graph had been negative until then, meaning that $f(x)$ decreased until this point.

Max: $x = -1$, the graph stays negative for more of the graph between $-1 \leq x \leq 5$ than it stays positive. This means that there is no higher value of $f(x)$ than the beginning.

6.3 Part C

$$g(x) = xf(x)$$

$$g'(x) = (1)f(x) + xf'(x)$$

$$m = g'(2) = (1)f(2) + xf'(2) = 6 + (2)(-1) = 6 - 2 = 4$$

$$y = f(2) = 6, \;\; x = 2$$

$$y - 6 = 4(x - 2)$$

7

7.1 Part A

$$y^2 = 2 + xy,\ \frac{dy}{dx} = \frac{y}{2y - x}$$

$$2yy' = y(1) + xy'$$

$$2yy' - xy' = y$$

$$y'(2y - x) = y$$

$$y' = \frac{y}{2y - x}$$

$$\frac{dy}{dx} = \frac{y}{2y - x}$$

7.2 Part B

$$2yy' = y(1) + xy'$$

$$y = y + \frac{x}{2}$$

$$0 = \frac{x}{2} = x$$

$$y^2 = 2 + xy$$

$$y^2 = 2 + (0)y$$

$$y^2 = 2$$

$$y = \pm\sqrt{2}$$

$$(0, -\sqrt{2}),\ (0, \sqrt{2})$$

7.3 Part C

First constraint:

$$\frac{dy}{dx} = \frac{y}{2y - x}$$

$$0 = \frac{y}{2y - x}$$

y must be zero for the first equation to be fulfilled.
Second constraint:

$$y^2 = 2 + xy$$

$$0^2 \neq 2 + x(0)$$

$$0 \neq 2 + x(0)$$

$$0 \neq 2$$

y cannot be zero for the second equation to be fulfilled.

7.4 Part D

$$y^2 = 2 + xy$$
$$3^2 = 2 + x(3)$$
$$9 = 2 + x(3)$$
$$7 = x(3)$$
$$\frac{7}{3} = x$$

$$\frac{d}{dt}[y^2 = 2 + xy] \to 2yy' = 0 + x'y + y'x$$
$$(2)(3)(6) = 3x' + 3x$$
$$(2)(3)(6) = 3x' + 7$$
$$36 = 3x' + 7$$
$$29 = 3x'$$
$$\frac{29}{3} = x'$$
$$\frac{dx}{dt} = \frac{29}{3}$$

8

$$h(x) = f(g(x)) - 6$$
$$h'(x) = f'(g(x))g'(x)$$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$
1	6	4	2	5	3	2
2	9	2	3	1	4	-8
3	10	-4	4	2	-7	21
4	-1	3	6	7		

8.1 Part A

$h(r)$ is continuous, $h(2) = 4$, and $h(3) = -7$. By IVT, $\exists\ r \in [2, 3)$ such that $h(r) = -5$.

8.2 Part B

$h'(c)$ is continuous, $h'(2) = -8$, and $h'(3) = 21$. By IVT, $\exists\ c \in [2, 3)$ such that $h'(c) = -5$.

8.3 Part C

$$\int_a^b f(x) = F(b) - F(a)$$
$$w(x) = \int_1^{g(x)} f(t)\, dt = F(1) - F(g(x))$$
$$w'(x) = f(g(x))g'(x) - f(1)$$
$$w'(3) = f(g(3))g'(3) - f(1) = f(4)(2) - 6 = -2 - 6 = -8$$

8.4 Part D

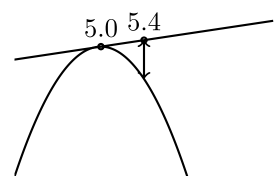
$$y = g^{-1}(x),\ \ x = 2$$
$$y = g^{-1}(2),\ \ g(1) = 2,\ \ g^{-1}(2) = 1$$
$$m = g^{-1\prime}(2),\ \ g'(3) = 2,\ \ g^{-1\prime}(2) = 3$$
$$y - 1 = 3(x - 2)$$

9

9.1 Part A

$$r - 30 = 2.0(t-5)$$

$$r - 30 = 2.0(5.4-5), \;\; r = 30.8ft$$



The approximation is above the actual value, because the line approximation will overshoot the decreasing (concave down) function.

9.2 Part B

$$r(5) = 30, \;\; r'(5) = 2.0$$

$$\frac{dV}{dt} = \frac{d}{dt}\frac{4}{3}\pi r^3 = 4\pi r^2 r'$$

$$4\pi(30.0)^2(2.0) = 7200\pi \, ft^3/min$$

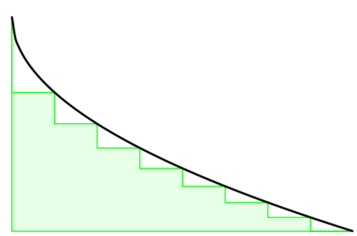
9.3 Part C

$$\int_0^{12} r'(t)\,dt \approx \sum_{i=1}^{6} r'(t)_i\,\Delta t_i \approx$$

$$(0.5)(1) + (0.6)(4) + (1.2)(2) + (2.0)(3) + (4.0)(2) = 19.3\,ft$$

$\underline{\int_0^{12} r'(t)\,dt}$ represents the number of feet the balloon has expanded (in radius) past it's original condition, and it is **not** the absolute radius of the balloon.

9.4 Part D



The RHS will under approximate the graph because it's slope is always negative, meaning any nonzero area under the curve from the right will undershoot it travelling left.

10

$$V = \pi r^2 h,\ r = 100cm,\ h = 0.5cm,\ r' = 2.5cm/min,\ \frac{dV}{dt} = 2000cm^3/min$$

10.1 Part A

$$h = \frac{V}{\pi r^2}$$

$$\frac{dh}{dt} = \frac{(V')(\pi r^2) - (V)(2r'r)}{(\pi r^2)^2} = 0.056cm/min$$

10.2 Part B

Total rate $= f(x) = 2000 - 400\sqrt{t}$
The total rate decreases over time, so when the rate runs below zero that is the peak of the function.

$$0 = 2000 - 400\sqrt{t}$$

$$2000 = 400\sqrt{t}$$

$$5 = \sqrt{t}$$

$$t = 25min$$

10.3 Part B

$$\int_0^{25} (2000 - 400\sqrt{t})\, dt + 60,000$$

11

11.1 Part A

$$y = f(e^2) = \frac{\ln e^2}{e^2} = \frac{2}{e^2}$$

$$m = f'(e^2) = \frac{1 - \ln e^2}{(e^2)^2} = \frac{-1}{e^4}$$

$$y - \frac{2}{e^2} = \frac{-1}{e^4}(x - e^2)$$

11.2 Part B

$$0 = f'(x) = \frac{1 - \ln x}{x^2}, \;\; ln(k) = 1, \;\; k = e, \;\; f'(e) = 0$$

$$f''(x) = \frac{(\frac{1}{x})(x^2) - (2x)(1 - \ln x)}{x^4}$$

$$f''(e) = \frac{(\frac{1}{e})(e^2) - (2e)(1 - \ln e)}{e^4} = \frac{e - 2e + 2e}{e^4} = \frac{e}{e^4} = e^{-3} > 0$$

$f''(e)$ is positive, so $f(e)$ is just about to increase at this critical point, meaning that this is a local minimum.

11.3 Part C

$$0 = f''(x) = \frac{x - (2x)(1 - \ln x)}{x^4}$$

$$0 = x - (2x)(1 - \ln x), \;\; x = 0$$

11.4 Part D

$$\text{if } \lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}, \text{ then } \frac{f'(x)}{g'(x)} = \frac{f(x)}{g(x)}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

12

12.1 Part A

$$\frac{150-126}{7-4} = 8ppl/hr$$

12.2 Part B

$$\frac{120+156(2)+176(2)+126}{6} = 151.6ppl$$

12.3 Part C

3 times, once each time the slope changes direction.

12.4 Part D

$$\int_0^3 550te^{-t/2} \approx 973ppl$$