AP Calculus AB Quarter 1 Take-Home Final

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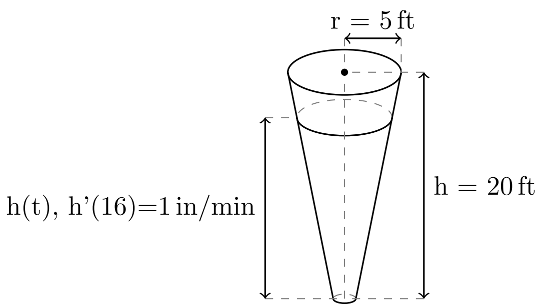
1 Fill rate of an inverted cone



We can relate the height of the cone h and the radius of the cone r by the following...:

$$\frac{r}{h} = \frac{5}{20} \tag{1}$$

$$r = \frac{5}{20}h \tag{2}$$

...So that we can find the volume of the cone based on the height function $h(t)$:

$$V = \frac{\pi r^2 h(t)}{3} = \frac{\pi(\frac{5h(t)}{20})^2 h(t)}{3} = \frac{\pi(\frac{h(t)}{4})^2 h(t)}{3} = \frac{\pi\frac{h(t)^2}{16}h(t)}{3} = \frac{\pi h(t)^3}{3*16} = \frac{\pi h(t)^3}{48} \tag{3}$$

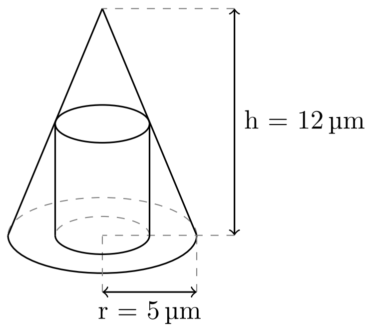
Because we want to know the rate of actual increase in volume of the cone $\frac{\Delta V}{\Delta t}$, we need to take the derivate of the above function, then plug the original values in:

$$\frac{\Delta V}{\Delta t} = \frac{3\pi h^2 h'}{48} = \frac{\pi h^2 h'}{16} = \frac{\pi(16)^2\frac{1}{12}}{16} = \frac{\pi(16)^2}{16*12} = \frac{16\pi}{12} = \frac{4\pi}{3}ft^3/min \tag{4}$$

We can then find the rate of the leak by subtracting the calulated rate from the infill rate like so:

$$8 - \frac{4\pi}{3} \approx 3.811 ft^3/min \tag{5}$$

2 Dimensions of a cylinder



Relationship between h and r:

$$\frac{h}{r} = \frac{12}{5} \tag{6}$$

$$h = 12 - \frac{12}{5}r \tag{7}$$

Note that as r increases, h must decrease so we subtract the related h from height of the cylinder

Volume of the cylinder:

$$V = \pi r^2 h \tag{8}$$

$$V = \pi r^2 \left(12 - \frac{12}{5}r\right) \tag{9}$$

$$V = 12\pi r^2 - \frac{12\pi r^3}{5} \tag{10}$$

Because this is an optimization problem, we want to find the zeroes of the derivative in order to determine the absolute maximum of the volume.

$$\frac{\Delta V}{\Delta r} = 24\pi r - \frac{36\pi r^2}{5} = 0 \tag{11}$$

$$24\pi r = \frac{36\pi r^2}{5} \tag{12}$$

$$2 = \frac{3r}{5} \tag{13}$$

$$10 = 3r \tag{14}$$

$$r = \frac{10}{3} \approx 3.3\,\mu\text{m} \tag{15}$$

We can then find h from our original relation:

$$h = 12 - \frac{12}{5}\frac{10}{3} = 4.0\,\mu\text{m} \tag{16}$$

3 Curve Sketching

First, we have to find the first and second derivatives of the function:

$$y = \frac{e^x}{x-1} \tag{17}$$

$$y' = \frac{e^x(x-1) - e^x * 1}{(x-1)^2} = \frac{e^x(x-2)}{(x-1)^2} \tag{18}$$

$$y'' = \frac{e^x(x-1) * (x-1)^2 - e^x(x-2) * 2 * (x-1)}{(x-1)^4} = \frac{e^x(x-1)^2 - e^x(x-2) * 2}{(x-1)^3} \tag{19}$$

$$= \frac{e^x\left[(x-1)^2 - (x-2) * 2\right]}{(x-1)^3} = \frac{e^x\left(x^2 - 2x + 1 - 2x + 4\right)}{(x-1)^3} = \frac{e^x\left(x^2 - 4x + 5\right)}{(x-1)^3} \tag{20}$$

We can extract 2 as a critical number from $(x-2)$ in the first derivative, and 1 from $(x-1)^3$. $f(1)$ is undefined, however.
Now what we have the function, and its first and second derivatives, we can start finding out the behaviour of the function at it's critical points.

range	y'	y''	behaviour	concavity
$(-\infty, 1)$	-	-	decreasing	concave down
$(1, 2)$	-	+	decreasing	concave up
$(2, \infty)$	+	+	increasing	concave up

$$\lim_{x\to 1^-} \frac{e^x}{x-1} = \frac{e}{0^-} = -\infty \tag{21}$$

$$\lim_{x\to 1^+} \frac{e^x}{x-1} = \frac{e}{0^+} = +\infty \tag{22}$$

NOTE: SEE ATTACHED MANUAL SKETCH OF THE GRAPH

4 Definition of a derivative

$$y = \frac{5}{\sqrt{2x+1}} \tag{23}$$

$$y' = m = \lim_{h\to 0} \frac{\frac{5}{\sqrt{2(h+a)+1}} - \frac{5}{\sqrt{2a+1}}}{h} = \frac{\frac{5}{\sqrt{2(h+4)+1}} - \frac{5}{\sqrt{2(4)+1}}}{h} = \frac{\frac{5}{\sqrt{2h+9}} - \frac{5}{\sqrt{9}}}{h} \tag{24}$$

$$= \frac{\frac{5}{\sqrt{2h+9}} - \frac{5}{\sqrt{9}}}{h} * \frac{\sqrt{9}\sqrt{2h+9}}{\sqrt{9}\sqrt{2h+9}} = \frac{5(3) - 5\sqrt{2h+9}}{3h\sqrt{2h+9}} \tag{25}$$

$$= \frac{15 - 5\sqrt{2h+9}}{3h\sqrt{2h+9}} * \frac{15 + 5\sqrt{2h+9}}{15 + 5\sqrt{2h+9}} = \frac{255 - 25(9+2h)}{h(3\sqrt{9+2h})(15+5\sqrt{9+2h})} \tag{26}$$

$$= \frac{255 - 255 - 50h)}{h(3\sqrt{9+2h})(15+5\sqrt{9+2h})} = \frac{-50}{(3\sqrt{9+2h})(15+5\sqrt{9+2h})} \tag{27}$$

$$lim_{h\to 0} \frac{50}{(3\sqrt{9+2h})(15+5\sqrt{9+2h})} = \frac{-50}{(3\sqrt{9})(15+5\sqrt{9})} = \frac{-50}{9 * (15+5(3))} = \frac{-50}{270} = \frac{-5}{27} \tag{28}$$

And now that we know m we can get the line equation:

$$y - \frac{5}{3} = \frac{-5}{27}(x - 4) \tag{29}$$

5 Particle Movement

First, we write out and derive the particle motion:

$$f(t) = 40t^3 - 333t^2 + 810t + 20 \text{ where } 0 \leq t \leq 7 \tag{30}$$
$$f'(t) = 120t^2 - 666t + 810 \tag{31}$$
$$f''(t) = 120t - 666 \tag{32}$$

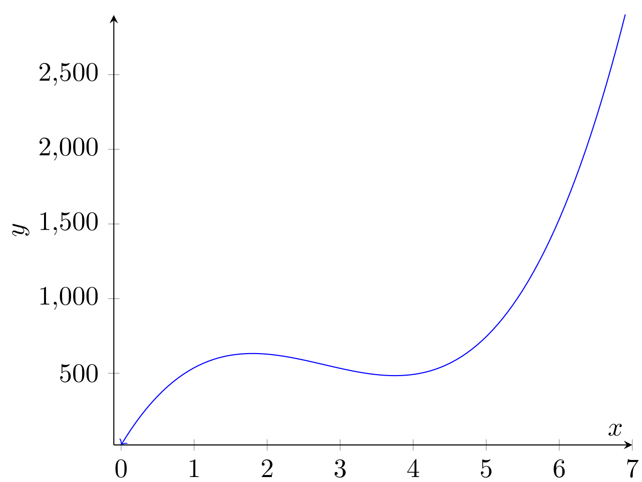
Next, we find the critical numbers by finding the zeroes of $f'(t)$ and $f''(t)$:

$$0 = f'(t) = 120t^2 - 666t + 810 \tag{33}$$
$$t = \frac{666}{2 * 120} \pm \frac{\sqrt{666^2 - 4(120)(810)}}{2 * 120} = \frac{333}{120} \pm \frac{117}{120} \approx 1.8, 3.75 \tag{34}$$

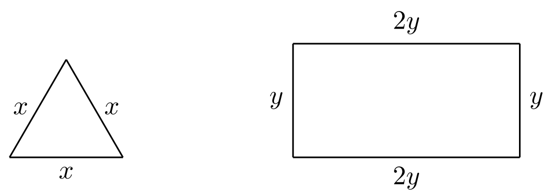
range	y'	y''	direction	action
(0, 1.8)	+	-	forward	slowing down
(1.8, 3.75)	-	-	backward	slowing down
(3.75, 7)	+	+	forward	speeding up

We can now find the total distance travelled by computing:

$$|k(0) - k(1.8)| + |k(1.8) - k(3.75)| + |k(3.75) - k(7)| = 3369.595m \tag{35}$$



6 Wire Length



The wire length relation is so:

$$\begin{align} 15 &= 6y + 3x \tag{36} \\ y &= \frac{15 - 3x}{6} \tag{37} \end{align}$$

The combined area of the shapes is:

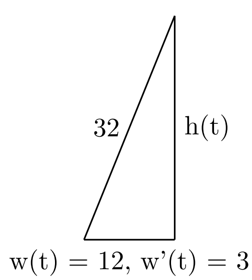
$$\frac{x^2\sqrt{3}}{4} + y * 2y = \frac{x^2\sqrt{3}}{4} + 2y^2 = \frac{x^2\sqrt{3}}{4} + 2\left(\frac{15 - 3x}{6}\right)^2 \tag{38}$$

To maximize the area of both shapes, we need to take the derivative of the area function, like so:

$$\begin{align} A' = \frac{4 * 2x\sqrt{3} - 0 * x^2\sqrt{3}}{4^2} + 2 * 2 * \left(\frac{15 - 3x}{6}\right) * \frac{-3x}{6} &= \frac{8x\sqrt{3}}{16} - 12x\left(\frac{15 - 3x}{6}\right) \tag{39} \\ = \frac{8x\sqrt{3}}{16} - \frac{36x^2 - 180x}{6} = \frac{1}{2}\sqrt{3}x + 6x^2 - 30x = 6x^2 + x\left(\frac{1}{2}\sqrt{3} - 30\right) &= 0 \tag{40} \\ x = \frac{-b}{2a} = \frac{-\frac{1}{2}\sqrt{3} + 30}{2 * 6} &\approx 2.4278 \tag{41} \end{align}$$

Which means we cut at about 2.4278 meters after the beginning of the wire.

7 Ladder and related rates



The height and width of the triangle can be related by pythagorean theorem:

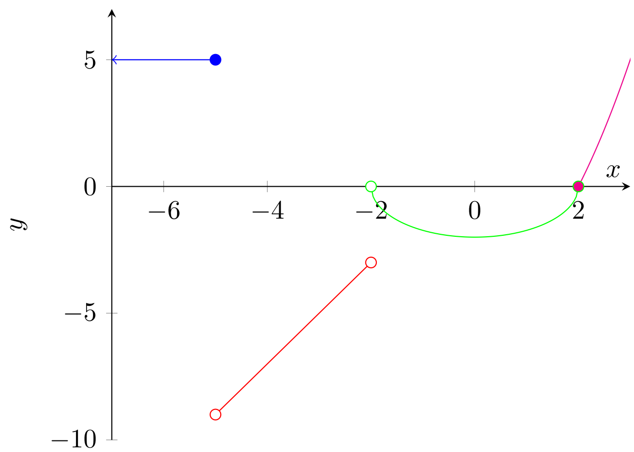
$$\begin{align} 32 &= \sqrt{h(t)^2 + w(t)^2} \tag{42} \\ 32^2 &= h(t)^2 + w(t)^2 \tag{43} \\ \sqrt{32^2 - w(t)^2} &= h(t) \tag{44} \end{align}$$

Now, height can be defined as a function of time:

$$\frac{\Delta h}{\Delta t} h(t) = \frac{w'(t)}{\sqrt{32^2 - w(t)^2}} = \frac{3}{\sqrt{32^2 - (12)^2}} \approx .101 ft/sec \tag{45}$$

8 Graphing a function

$$f(x) = \begin{cases} 5 & x \leq -5 \\ 2x+1 & -4 < x < -2 \\ -\sqrt{4-x^2} & -2 < x \leq 2 \\ x^2 - 4 & 2 < x \end{cases} \tag{46}$$



The function is non-differentiable at $x = -2$ and continuous at $(-\infty, -2) \cup (-2, \infty)$