

We are given two rays, defined by a total of four vectors:

$$\vec{A}(t) = \vec{P} + t\vec{R}$$

$$\vec{B}(k) = \vec{G} + k\vec{D}$$

Where \vec{R} and \vec{D} are the directions of the rays and \vec{P} and \vec{G} are their origins respectively.

We want to find the closest approach between these vectors, so first we'll define that "close" means the smallest euclidean distance between the two vectors:

$$f(t, k) = \sqrt{\left(\vec{A}(t) - \vec{B}(k)\right)^2}$$

Because $\text{sqr}(x)$ is nondecreasing, we can leave it off to simplify our calculations:

$$L(t, k) = \left(\vec{A}(t) - \vec{B}(k)\right)^2$$

In summary, we want to find:

$$\arg \min_{t, k} L(t, k)$$

For simplification, we will group our origins P and G into one factor:

$$\vec{J} = \vec{P} - \vec{G}$$

$$L(t, k) = (\vec{A}(t) - \vec{B}(k))^2 = (\vec{P} + t\vec{R} - \vec{G} - k\vec{D})^2 = (t\vec{R} - k\vec{D} + \vec{J})^2$$

Next we'll foil this dot product:

$$= t^2 R^2 - 2tk\vec{R} \cdot \vec{D} + 2t\vec{R} \cdot \vec{J} + k^2 D^2 - 2k\vec{D} \cdot \vec{J} + J^2$$

Because the closest approach is unique, any variation to t or k at this point will cause an increase in distance. Therefore the gradient at $L(t, k)$ will be zero. So we will begin by taking the partial derivatives of it:

$$\frac{d}{dt} L(t, k) = 2tR^2 - 2k\vec{R} \cdot \vec{D} + 2\vec{R} \cdot \vec{J}$$

$$\frac{d}{dk} L(t, k) = 2kD^2 - 2t\vec{R} \cdot \vec{D} - 2\vec{D} \cdot \vec{J}$$

Now we set the gradient to zero, from which we obtain the following set of equations:

$$0 = 2tR^2 - 2k\vec{R} \cdot \vec{D} + 2\vec{R} \cdot \vec{J}$$

$$0 = 2kD^2 - 2t\vec{R} \cdot \vec{D} - 2\vec{D} \cdot \vec{J}$$

Now, we solve for t and k :

$$2tR^2 = 2k\vec{R} \cdot \vec{D} - 2\vec{R} \cdot \vec{J}$$

$$t = \frac{k\vec{R} \cdot \vec{D} - \vec{R} \cdot \vec{J}}{R^2}$$

$$\begin{aligned}
0 &= 2kD^2 - 2(\vec{R} \cdot \vec{D}) \frac{k\vec{R} \cdot \vec{D} - \vec{R} \cdot \vec{J}}{R^2} - 2\vec{D} \cdot \vec{J} \\
0 &= 2kD^2 - 2(\vec{R} \cdot \vec{D}) \frac{k\vec{R} \cdot \vec{D}}{R^2} + (\vec{R} \cdot \vec{D}) \frac{\vec{R} \cdot \vec{J}}{R^2} - 2\vec{D} \cdot \vec{J} \\
0 &= k \left[2D^2 - 2(\vec{R} \cdot \vec{D}) \frac{\vec{R} \cdot \vec{D}}{R^2} \right] + 2(\vec{R} \cdot \vec{D}) \frac{\vec{R} \cdot \vec{J}}{R^2} - 2\vec{D} \cdot \vec{J} \\
k \left[2D^2 - 2(\vec{R} \cdot \vec{D}) \frac{\vec{R} \cdot \vec{D}}{R^2} \right] &= -2(\vec{R} \cdot \vec{D}) \frac{\vec{R} \cdot \vec{J}}{R^2} + 2\vec{D} \cdot \vec{J} \\
k \left[2D^2 - 2 \frac{(\vec{R} \cdot \vec{D})(\vec{R} \cdot \vec{D})}{R^2} \right] &= - \frac{2(\vec{R} \cdot \vec{D})(\vec{R} \cdot \vec{J})}{R^2} + 2\vec{D} \cdot \vec{J} \\
k &= \frac{\frac{2(\vec{R} \cdot \vec{D})(\vec{R} \cdot \vec{J})}{R^2} + 2\vec{D} \cdot \vec{J}}{2D^2 - 2 \frac{(\vec{R} \cdot \vec{D})(\vec{R} \cdot \vec{D})}{R^2}} \\
k &= \frac{\vec{D} \cdot \vec{J} - \frac{(\vec{R} \cdot \vec{D})(\vec{R} \cdot \vec{J})}{R^2}}{D^2 - \frac{(\vec{R} \cdot \vec{D})^2}{R^2}}
\end{aligned}$$

Our desired distance is simply $f(t, k)$!