We are given two rays, defined by a total of four vectors:

$$\vec{A}(t) = \vec{P} + t\vec{R}$$

$$\vec{B}(k) = \vec{G} + k\vec{D}$$

Where $\vec{R}$ and $\vec{D}$ are the directions of the rays and $\vec{P}$ and $\vec{G}$ are their origins respectively.

We want to find the closest approach between these vectors, so first we'll define that "close" means the smallest euclidean distance between the two vectors:

$$f(t,k) = \sqrt{\left(\vec{A}(t) - \vec{B}(k)\right)^2}$$

Because $sqrt(x)$ is nondecreasing, we can leave it off to simplify our calculations:

$$L(t,k) = \left(\vec{A}(t) - \vec{B}(k)\right)^2$$

In summary, we want to find:

$$\arg\min_{t,k} L(t,k)$$

For simplification, we will group our origins P and G into one factor:

$$\vec{J} = \vec{P} - \vec{G}$$

$$L(t,k) = (\vec{A}(t) - \vec{B}(k))^2 = (\vec{P} + t\vec{R} - \vec{G} - k\vec{D})^2 = (t\vec{R} - k\vec{D} + J)^2$$

Next we'll foil this dot product:

$$= t^2R^2 - 2tk\vec{R}\cdot\vec{D} + 2t\vec{R}\cdot\vec{J} + k^2D^2 - 2k\vec{D}\cdot\vec{J} + J^2$$

Because the closest approach is unique, any variation to t or k at this point will cause an increase in distance. Therefore the gradient at $L(t,k)$ will be zero. So we will begin by taking the partial derivatives of it:

$$\frac{d}{dt}L(t,k) = 2tR^2 - 2k\vec{R}\cdot\vec{D} + 2\vec{R}\cdot\vec{J}$$

$$\frac{d}{dk}L(t,k) = 2kD^2 - 2t\vec{R}\cdot\vec{D} - 2\vec{D}\cdot\vec{J}$$

Now we set the gradient to zero, from which we obtain the following set of equations:

$$0 = 2tR^2 - 2k\vec{R}\cdot\vec{D} + 2\vec{R}\cdot\vec{J}$$

$$0 = 2kD^2 - 2t\vec{R}\cdot\vec{D} - 2\vec{D}\cdot\vec{J}$$

Now, we solve for t and k:

$$2tR^2 = 2k\vec{R}\cdot\vec{D} - 2\vec{R}\cdot\vec{J}$$

$$t = \frac{k\vec{R}\cdot\vec{D} - \vec{R}\cdot\vec{J}}{R^2}$$

$$0 = 2kD^2 - 2(\vec{R}\cdot\vec{D})\frac{k\vec{R}\cdot\vec{D} - \vec{R}\cdot\vec{J}}{R^2} - 2\vec{D}\cdot\vec{J}$$

$$0 = 2kD^2 - 2(\vec{R}\cdot\vec{D})\frac{k\vec{R}\cdot\vec{D}}{R^2} + (\vec{R}\cdot\vec{D})\frac{\vec{R}\cdot\vec{J}}{R^2} - 2\vec{D}\cdot\vec{J}$$

$$0 = k\left[2D^2 - 2(\vec{R}\cdot\vec{D})\frac{\vec{R}\cdot\vec{D}}{R^2}\right] + 2(\vec{R}\cdot\vec{D})\frac{\vec{R}\cdot\vec{J}}{R^2} - 2\vec{D}\cdot\vec{J}$$

$$k\left[2D^2 - 2(\vec{R}\cdot\vec{D})\frac{\vec{R}\cdot\vec{D}}{R^2}\right] = -2(\vec{R}\cdot\vec{D})\frac{\vec{R}\cdot\vec{J}}{R^2} + 2\vec{D}\cdot\vec{J}$$

$$k\left[2D^2 - 2\frac{(\vec{R}\cdot\vec{D})(\vec{R}\cdot\vec{D})}{R^2}\right] = -\frac{2(\vec{R}\cdot\vec{D})(\vec{R}\cdot\vec{J})}{R^2} + 2\vec{D}\cdot\vec{J}$$

$$k = \frac{-\frac{2(\vec{R}\cdot\vec{D})(\vec{R}\cdot\vec{J})}{R^2} + 2\vec{D}\cdot\vec{J}}{2D^2 - 2\frac{(\vec{R}\cdot\vec{D})(\vec{R}\cdot\vec{D})}{R^2}}$$

$$k = \frac{\vec{D}\cdot\vec{J} - \frac{(\vec{R}\cdot\vec{D})(\vec{R}\cdot\vec{J})}{R^2}}{D^2 - \frac{(\vec{R}\cdot\vec{D})^2}{R^2}}$$

Our desired distance is simply $f(t,k)$!