Numerical solution to the problem of a mass (m) on a table, with friction, attached to a hanging mass (M) by a cord through a hole in the table.

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1 Definition of the problem

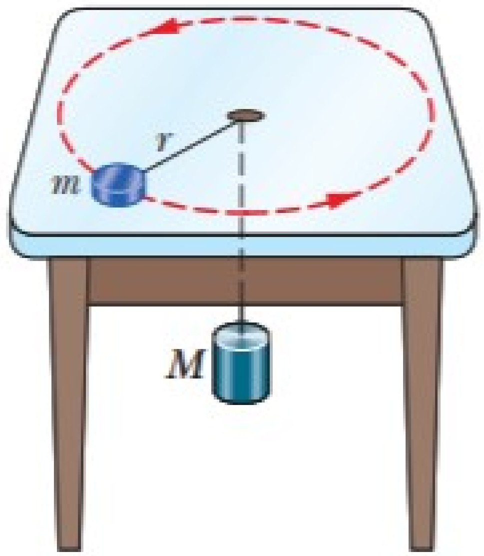

Figure 1: Representation of the physical system

The physical situation to be analysed can be visualized in Figure 2. A body of mass m is situated on the surface of a rough and boundless table, and is attached to another body of mass M by a cord. The latter is hanging by said cord through a hole in the middle of the table. The objective is then to solve for the position of both of the bodies, at each moment in time, given the initial position and velocity of said bodies. Finally, as the table's surface has been stablished to be rough, the effect of friction will be taken into account. Aside from that, we shall ignore all other effects arising from: air resistance, the non-ideallity of the cord itself (eg. torsion) and the fact that the bodies are supposed to be extended objects, also subject to external torques and capable of rotation around any axis. We shall instead solve for the position of the center of mass of bodies, whose dimensions are very short in comparison to the table or the cord itself and are subjected to no external disturbances.

2 Derivation of the equations of motion

2.1 Choosing the coordinate system

First and foremost, it is important to define our coordinate system. In this case let us situate it at the center of the table, where the through-hole lies. As we are primarily concerned with the motion of the body on the surface, and because the tension of the cord presents azimuthal invariance (since its direction is always radial and its magnitude does not depend on the value of the azimut angle), we realize that a cylindrical coordinate system is the best way to go. Having said this, the through-hole (and center of our new coordinate system) would be located then at $R = 0$ and $z = 0$, where $R = R(t)$ is the distance from a point to the center of the table, in the radial direction, and $z = z(t)$ is the positon of said point from the center of the table in the vertical direction. Aside from that, we also mention that $\phi = \phi(t)$ would be the azimut angle formed by said point with respect to the table.

Having defined our coordinate system, we proceed to identify the forces acting on all the bodies.

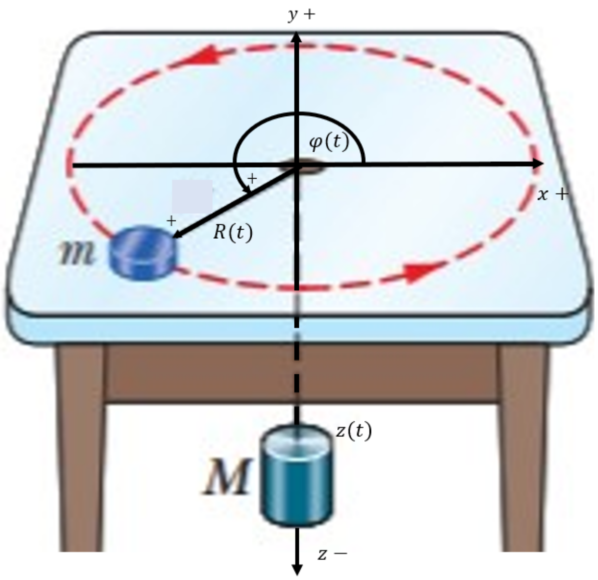

Figure 2: Coordinate system chosen for the problem

2.2 Identifying the forces

Tension forces: First, we must realize that, as said before, the direction of the tension force on the body on the surface is always radial, though directed inwards. As such, we have that the tension force on the body of mass m can be written as:

$$\begin{aligned}\vec{T}_m &= (T)(-\hat{e}_R) \\ &= -T\hat{e}_R\end{aligned} \tag{1}$$

Where, in equation (1), $\vec{T}_m$ is the tension force acting on the body of mass m, T is the magnitude of said tension force and $\hat{e}_R = \hat{e}_R(\phi)$ is the cylindrical R-unit vector. Then, analizing the z diretion yields that the tension force acting on the hanging body should, ideally, always be in the upward direction. Hence, we can write:

$$\vec{T}_M = T\hat{e}_z \tag{2}$$

Where, in equation (2), $\vec{T}_M$ is the tension force acting on the body of mass M and $\hat{e}_z$ is the cartesian z-unit vector. Let it be noted that both $\vec{T}_m$ and $\vec{T}_M$ have the same magnitude (just diferent directions), since they're both produced by the same cord.

Weigths: After that, further analyzing the z direction leads us to see that the forces on m and M, due to gravity, always act on the downward direction. Hence:

$$\begin{aligned}\vec{W}_m &= (mg)(-\hat{e}_z) \\ &= -mg\hat{e}_z\end{aligned} \tag{3}$$

$$\begin{aligned}\vec{W}_M &= (Mg)(-\hat{e}_z) \\ &= -Mg\hat{e}_z\end{aligned} \tag{4}$$

Where, in equations (3) and (4), $\vec{W}_m$ and $\vec{W}_M$ are the weights of the masses m and M, respectively, and g is the acceleration due to gravity.

Normal Force: Staying on the same page, the body on the surface experiences a normal force for the sole reason of being in contact with it. This normal force is always perpendicular to said surface, which, in this particular case, would mean the normal force would be directed completly in the upward direction. Therefore, we can write:

$$\vec{N} = N\hat{e}_z \tag{5}$$

Where, in equation (5), N denotes the magnitude of the normal force.

Friction force: Finally we also have to consider the friction force. For this problem, the model of friction chosen was that of dry kinetic friction. What this means is that the magnitude of the force of friction experienced by some object is equal to the normal force acting on said object, multiplied by some proportionality constant μ (where usually: $0 \leq \mu \leq 1$). Meanwhile, its direction is given by its tendency to opose the movement of the body upon which it acts. I.e, the direction of the friction force is always opposite to that of the velocity of the body. Having said this, if we define $\vec{r}_m = \vec{r}_m(t)$ to be the position vector of the body of mass m, then:

$$\begin{aligned} \vec{Fr} &= (\mu N)(-\frac{\dot{\vec{r}}_m}{\|\dot{\vec{r}}_m\|}) \\ &= -\frac{\mu N}{\|\dot{\vec{r}}_m\|}\dot{\vec{r}}_m \end{aligned} \tag{6}$$

To remember: The position, velocity and acceleration vectors in cylindrical coordinates, using the cylindrical basis vectors, can be written as:

$$\vec{r} = R\hat{e}_R + z\hat{e}_z \tag{7}$$

$$\dot{\vec{r}} = \dot{R}\hat{e}_R + R\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z \tag{8}$$

$$\ddot{\vec{r}} = (\ddot{R} - R\dot{\phi}^2)\hat{e}_R + (R\ddot{\phi} + 2\dot{R}\dot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z \tag{9}$$

Where $\hat{e}_\phi = \hat{e}_\phi(\phi)$ is the ϕ- unit vector. Now, after we have considered every force present in the problem, we can apply newton's laws to both bodies and find the equations of motion.

2.3 Applying Newton's laws

First, let us examine what the movement of the hanging body would look like.

2.3.1 Body of mass M:

Figure 3 shows the froces acting on the body of mass M. Then, applying Newton's second law we obtain Eq.(10).

$$M\ddot{\vec{r}}_M = \vec{W}_M + \vec{T}_M \tag{10}$$

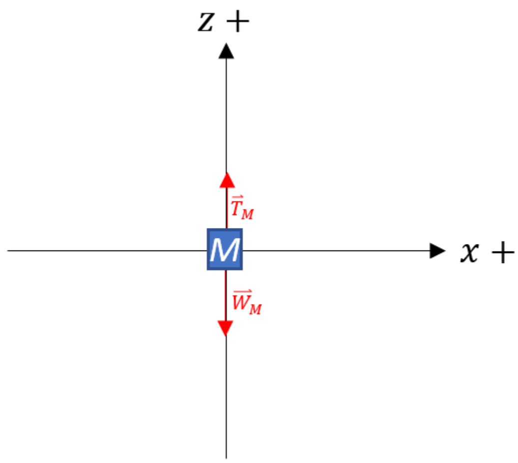

Figure 3: Force body diagram of the body of mass M

$$M\ddot{z}\hat{e}_z = -Mg\hat{e}_z + T\hat{e}_z \tag{11}$$

Where, in going from equation (10) to (11), the fact that the movement of the hanging body is confined to the z direction (and thus, for the hanging body, it follows that: $\dot{R} = \ddot{R} = 0$ and $\dot{\phi} = \ddot{\phi} = 0$) was used. Also Eqs. (2) and (4) were susbtituted in. Now, there is a way to relate the z coordinate of the hanging body to the R coordinate of the body on the surface by using the following relationship: $z = R - L$ (where L is the cord's total length and is a constant). From this expression, it is easy to see, by differentiating, that: $\ddot{z} = \ddot{R}$. Then, this last equation allows us to rewritte the vertical acceleration of the hanging body ($\ddot{z}$) completely in terms of the radial acceleration of the body on the surface ($\ddot{R}$). Applying the latter equation in Eq. (11) then yields:

$$M\ddot{R}\hat{e}_z = -Mg\hat{e}_z + T\hat{e}_z \tag{12}$$

Where $\ddot{R}$, in Eq.(12), as was previously stated, represents the radial acceleration of the body of mass m (not that of the body of mass M). Now, since the directions of all terms in Eq.(12) are equal (i.e, all their unit vectors are the same), we can drop the unit vectors since it is now, essentially, a scalar equation (or we can just say that we cancelled them, it gives the same result). Doing so and then solving for T yields:

$$M(g + \ddot{R}) = T \tag{13}$$

Eq.(13) gives us an expression which will allow us to link the equations of movement for both bodies. Now let's redirect our focus to the movement of the body on the surface.

2.3.2 Body of mass m:

Figure 4 shows the forces to which the body of mass m is subjected. Then, applying Newton's laws we obtain Eq.(14)

$$m\ddot{\vec{r}}_m = \vec{N} + \vec{W}_m + \vec{T}_m + \vec{F}r \tag{14}$$

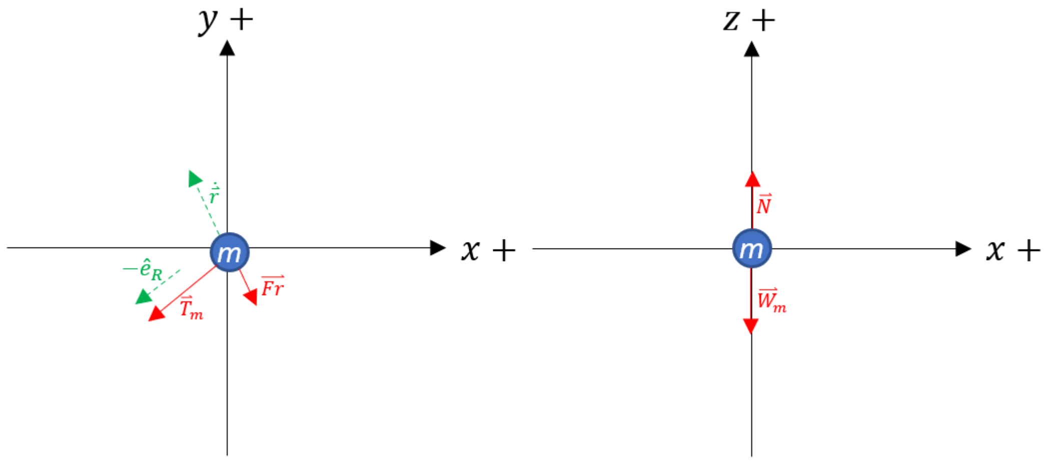

Figure 4: Force body diagram of the body of mass m

We now use Eq.(9) for the acceleration vector $\ddot{\vec{r}}_m$, while taking into account that, for the body on the surface, it holds that $\dot{z} = \ddot{z} = 0$. Apart from that, we also use Eq.(8) in Eq.(6) and we substitute the result in Eq.(14) alongside Eqs.(1), (3) and (5), yielding:

$$m[(\ddot{R} - R\dot{\phi}^2)\hat{e}_R + (R\ddot{\phi} + 2\dot{R}\dot{\phi})\hat{e}_\phi + 0\hat{e}_z] = N\hat{e}_z - mg\hat{e}_z - M(g+\ddot{R})\hat{e}_R - \frac{\mu N}{\sqrt{\dot{R}^2 + (R\dot{\phi})^2}}(\dot{R}\hat{e}_R + R\dot{\phi}\hat{e}_\phi) \tag{15}$$

Swichting now to matrix notation:

$$\begin{bmatrix} \ddot{R} - R\dot{\phi}^2 \\ R\ddot{\phi} + 2\dot{R}\dot{\phi} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{M}{m}(g+\ddot{R}) - \frac{\mu N\dot{R}}{m\sqrt{\dot{R}^2+(R\dot{\phi})^2}} \\ -\frac{\mu N R\dot{\phi}}{m\sqrt{\dot{R}^2+(R\dot{\phi})^2}} \\ \frac{N}{m} - g \end{bmatrix} \tag{16}$$

We can see from the third row of Eq.(16) that $N = mg$. Using this result in the first and second rows of Eq.(16) and rearranging terms yields:

$$\begin{bmatrix} \ddot{R}(1+\frac{M}{m}) \\ R\ddot{\phi} \\ N \end{bmatrix} = \begin{bmatrix} R\dot{\phi}^2 - \frac{M}{m}g - \frac{\mu g\dot{R}}{\sqrt{\dot{R}^2+(R\dot{\phi})^2}} \\ -2\dot{R}\dot{\phi} - \frac{\mu g R\dot{\phi}}{\sqrt{\dot{R}^2+(R\dot{\phi})^2}} \\ mg \end{bmatrix} \tag{17}$$

Finally, solving for $\ddot{R}$ and $\ddot{\phi}$ yields:

$$\begin{bmatrix} \ddot{R} \\ \ddot{\phi} \\ N \end{bmatrix} = \begin{bmatrix} (R\dot{\phi}^2 - \frac{M}{m}g - \frac{\mu g \dot{R}}{\sqrt{\dot{R}^2+(R\dot{\phi})^2}})(1+\frac{M}{m})^{-1} \\ -2\frac{\dot{R}}{R}\dot{\phi} - \frac{\mu g \dot{\phi}}{\sqrt{\dot{R}^2+(R\dot{\phi})^2}} \\ mg \end{bmatrix} \tag{18}$$

Equation (18) represents then the equations of motion for the body of mass m, where the radial and angular accelerations have been isolated.

3 Equations of motion

Rewritting and summarizing now all the equations of motion for the whole system yields:

EQUATIONS OF MOTION FOR THE BODY OF MASS m

$$\ddot{R}_m = (R_m\dot{\phi}_m^2 - \frac{M}{m}g - \frac{\mu g \dot{R}_m}{\sqrt{\dot{R}_m^2 + (R_m\dot{\phi}_m)^2}})(1+\frac{M}{m})^{-1} \tag{19}$$

$$\ddot{\phi}_m = -2\frac{\dot{R}_m}{R_m}\dot{\phi}_m - \frac{\mu g \dot{\phi}_m}{\sqrt{\dot{R}_m^2 + (R_m\dot{\phi}_m)^2}} \tag{20}$$

$$z_m = 0 \tag{21}$$

EQUATIONS OF MOTION FOR THE BODY OF MASS M

$$R_M = 0 \tag{22}$$

$$\phi_M = 0 \tag{23}$$

$$z_M = R_m - L \tag{24}$$

Note 1: *These equations hold while* $|z_M| + |R_m| \leq L$
Note 2: *The equation:* $R_m = z_M + L$ *implies that the speeds of the bodies of mass* m *and mass* M *must be equal at all times. In the real world this isn't always necessarily the case (eg. it is perfectly possible for* $\dot{z}_M(0) > 0$ *while* $\dot{R}_m(0) = 0$*), since ropes and cords can only exert force by pulling and never by pushing. Therefore, this equation (as well as the final equations of motion) has a range of validity that must be explained. Succintly, the cases where it is physically possible for* $\dot{z}_M(0) \neq \dot{R}_m(0)$ *are not accounted for here. These equations only consider, and are only valid in, the cases where* $\dot{z}_M(0) = \dot{R}_m(0)$.
Note 3: *These equations also assume that, at any given moment, the tension force will always be greater than static friction force. Hence, only the kninetic friction coefficient is used.*
Note 4: *These equations hold while* $\frac{M}{m} > \mu$. *The reason why will be expanded on in a later note.*
Note 5: *These equations hold while* $R_m > 0$ *and while* $\dot{R}_m \neq 0$ *or* $\dot{\phi}_m \neq 0$. *The reason why will be explained later on and a modification to these equations will be introduced in order to extend the range of validity of said equations.*

Where the subindices m and M have been added in order to better clarify which coordinate belongs to which body. Having completed this step and found the equations of motion we can now go on to discretize these equations and actually begin to solve the problem numerically.

4 Discretization and finite difference scheme solution

First, let us discretize the initial conditions by finite difference. Then, using backward difference:

$$R_m^0 = R_m(0) \tag{25}$$

$$R_m^{-1} = R_m^0 - \Delta t \dot{R}_m(0) \tag{26}$$

$$\phi_m^{-1} = \phi_m^0 - \Delta t \dot{\phi}_m(0) \tag{27}$$

Note 6: *Here we didn't include an expression for $z_M(0)$ and $\dot{z}_M(0)$, since they can be rewritten in terms of R_m. More specifically: $R_m(0) = z_M(0) + L$ and $\dot{R}_m(0) = \dot{z}_M(0)$.*

Where the superindices help refer to the corresponding time step, while Δt is the actual value of said time step. Now, let us discretize de first order derivatives, again, using backward difference:

$$\begin{aligned} \dot{R}_m &\approx \frac{\Delta R_m}{\Delta t} \\ &\approx \frac{R_m^n - R_m^{n-1}}{\Delta t} = a \end{aligned} \tag{28}$$

$$\begin{aligned} \dot{\phi}_m &\approx \frac{\Delta \phi_m}{\Delta t} \\ &\approx \frac{\phi_m^n - \phi_m^{n-1}}{\Delta t} = b \end{aligned} \tag{29}$$

Then, by substituting Eqs.(28) and (29) in Eqs.(19) and (20), discretizing the radial and angular second derivatives and rearranging terms, the discrete version of Eqs.(19), (20) and (24) is obtained. These would then be, respectively:

$$R_m^{n+1} = 2R_m^n - R_m^{n-1} + \Delta t^2 (R_m^n b^2 - \frac{M}{m} g - \frac{\mu g a}{\sqrt{a^2 + (R_m^n b)^2}})(1 + \frac{M}{m})^{-1} \tag{30}$$

$$\phi_m^{n+1} = 2\phi_m^n - \phi_m^{n-1} - \Delta t^2 (\frac{2ab}{R_m^n} + \frac{\mu g b}{\sqrt{a^2 + (R_m^n b)^2}}) \tag{31}$$

$$z_M^n = R_m^n - L \tag{32}$$

Eqs.(30), (31) and (32) would then be the equations we can put into our program in order to obtain a solution to the problem and start analysing some results. Said need be, these equations may later need some adjustments in order to extent their range of validity.

5 Analysis of results

5.1 Testing criteria

Having written the python program for solving the equations of motion for both of the bodies, it is now time to analyse the results of the simulation in order to make sure they are coherent and have physical sense. In order to do achive this, we can serve us from some physical intuition:

1. In a table without friction (i.e $\mu = 0$) the total energy of the system should stay constant over time.
2. In a table with friction (i.e $\mu > 0$) the total energy of the system should decrease gradually over time.

3. In a table without friction, the conservation of energy of the system should mean that the body of mass m should never converge to the center of the table, but rather keep moving indefinetly.

4. In a table with friction, the loss of energy of the system should mean that the body of mass m should eventually converge to the center of the table and stop moving.

5. In a table without friction and with initial radial velocity equal to 0, if the initial angular velocity is equal to: $\sqrt{\frac{Mg}{mR_m(0)}}$, then the trajectory of the body of mass m should be a circle of radius $R_m(0)$.

Having defined these criteria for testing the accuracy of the simulation, we can now proceed to, actually, examine the results. However, let us admit, first, that satisfying these criteria does not gurantee that the simulation is 100% accurate. What it does do, nevertheless, is give us confindence that the simulation does make physical sense; which is a factor that, more often than not, speaks in favor of the trustworthiness of the results. Therefore, if these criteria are met (and no errors during the derivation of the equations of motion were made, and the range of validity of said equations is taken into account) we can be quite confident that the results truly ressemble reality (in so far as the simplifications of the model allow it, that is). Having said that, let us revise the each criteria down the line.

5.2 Energy equations

The total energy of the system is the sum of the kinetic energies of the masses m and M plus the potential energy of mass M. I.e:

$$\begin{aligned} KT &= KE_m + KE_M + U_M \\ &= \frac{1}{2}m\|\dot{\vec{r}}_m\|^2 + \frac{1}{2}m\|\dot{\vec{r}}_M\|^2 + Mgh \\ &= \frac{1}{2}m\|\dot{\vec{r}}_m\|^2 + \frac{1}{2}m\|\dot{\vec{r}}_M\|^2 + Mg(L + z_M) \\ &= \frac{1}{2}m(\dot{R}_m^2 + (R_m\dot{\Phi}_m)^2) + \frac{1}{2}M\dot{z}_M^2 + MgR_m \\ &= \frac{1}{2}m(\dot{R}_m^2 + (R_m\dot{\Phi}_m)^2) + \frac{1}{2}M\dot{R}_m^2 + MgR_m \\ &= \frac{1}{2}m(R_m\dot{\Phi}_m)^2 + \frac{1}{2}(M+m)\dot{R}_m^2 + MgR_m \end{aligned} \tag{33}$$

Where KT is the system's total energy, KE_m is the kinetic energy of the body of mass m, KE_M is the kinetic energy of the body of mass M, U_M is the potential energy of the body of mass M and h is the distance, still available at any given time, for the body of mass M to fall (eg. if $z = 0$ then $h = L$ and if $z = -L$ then $h = 0$). Discretized, equation (33) would be:

$$KT^n = \frac{1}{2}m(R_m^n b)^2 + \frac{1}{2}(M+m)a^2 + MgR_m^n \tag{34}$$

5.3 Verification of criteria 1 and 3

Now, with: $M = 1$ [Kg], $m = 0.5$ [Kg], $L = 6$ [m] , $R_m(0) = 2$ [m], $\phi_m(0) = 0$ [rad], $\dot{R}_m(0) = 2$ [m/s], $\dot{\phi}_m(0) = 3$ [rad/s], $\mu = 0$ and a simulation time of 8 [s], the results given by the simulation are those shown in figures 5, 6 and 7.

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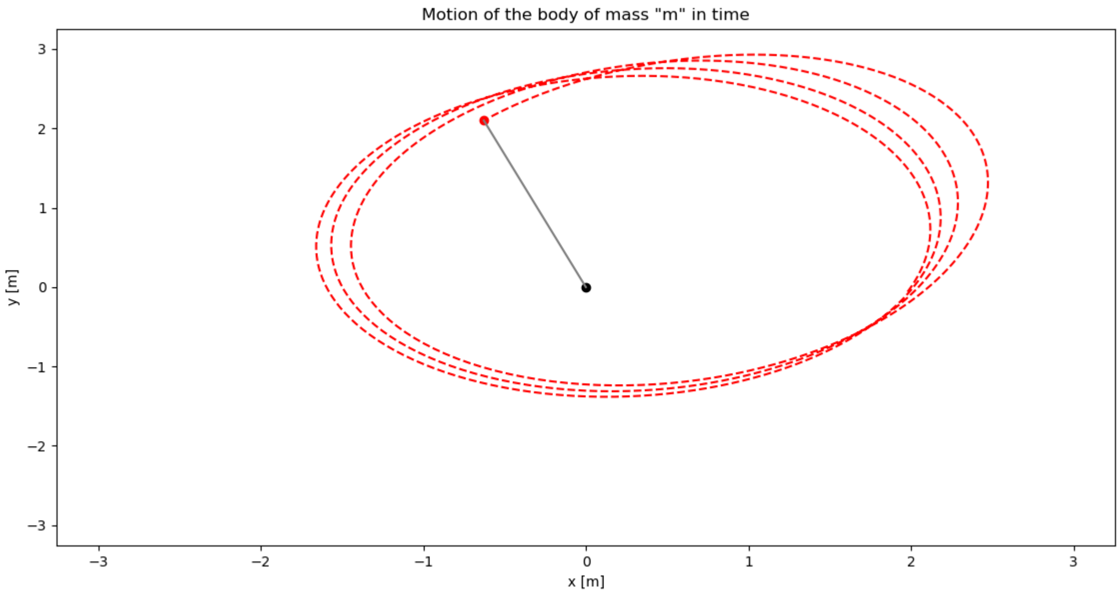

Figure 5: Trajectory of the body of mass m. $\Delta t = 10^{-3}$ [s]

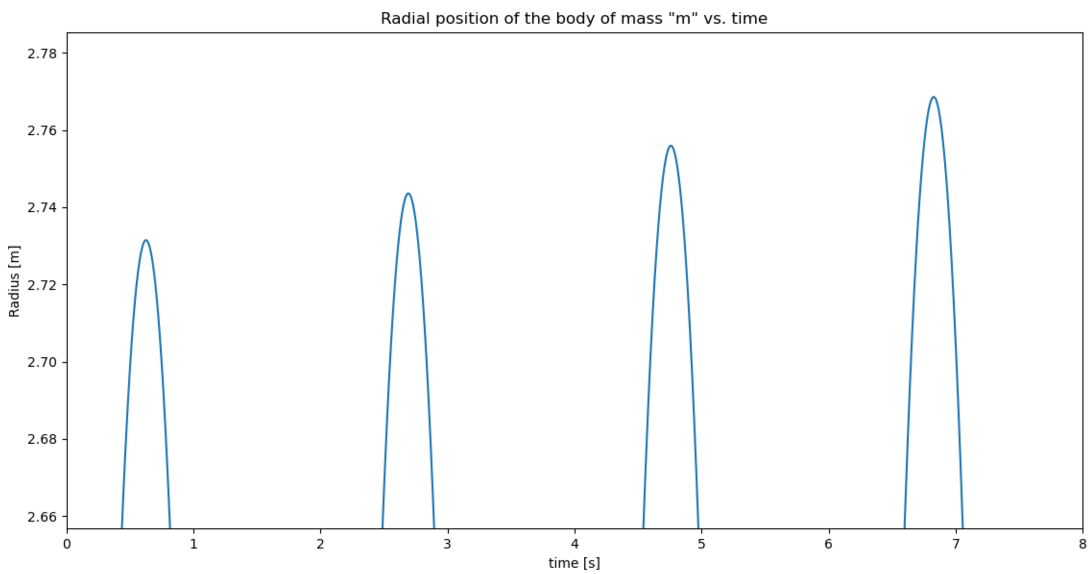

Figure 6: Radius maxima of the trajectory of the body of mass m. $\Delta t = 10^{-3}$ [s]

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As we can observe from Fig. 5 the trajectory of the body of mass m doesn't seem to decay or converge to the center of the table, which satifies criterion 3. However, it does seem, oddly, that the radius coordinate grows unbound over time, which is weird. This can be seen from the oddly-shaped elpisoides formed at the end of the trajectory. Then, looking at a graph of the radial position of the body of mass m over time (Fig. 6), and zooming in on the radius maxima, we can actually confirm our suspicion. It is easy to see that the value of said maximas increases over time. However, the most telling graph of all has to be that of Fig. 7.

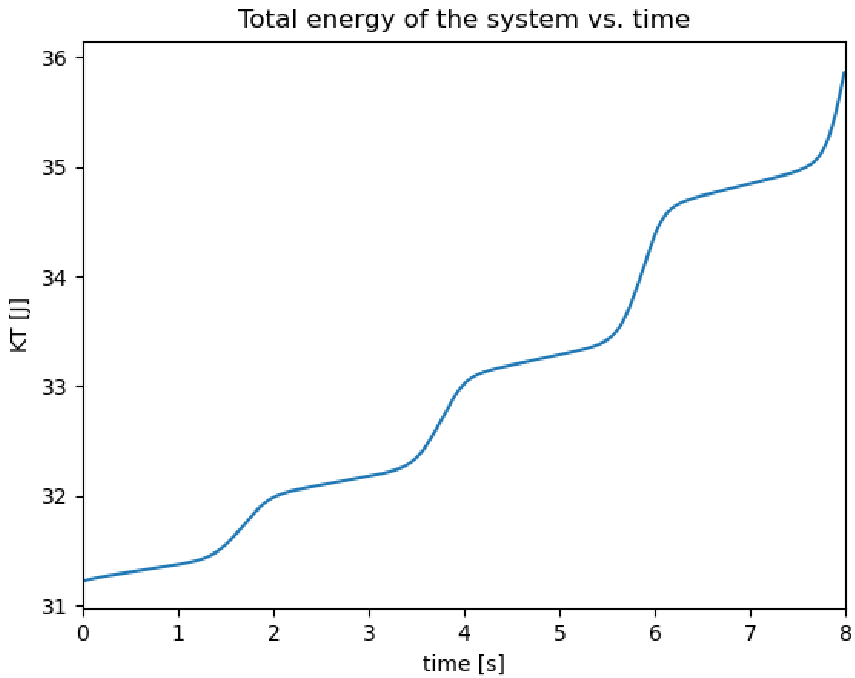

Figure 7: Total energy of the system vs. time. $\Delta t = 10^{-3}$ [s]

Here, it is clear that energy is not being conserved in the system, but rather increases, almost in a linear fashion, with time. This, of course, doesn't make any physical sense and is in clear contradiction to criterion 1. Therefore, we must conclude that this simulation is untrustworthy and wrong. However, not all is lost, this is in fact a problem that can be readily "fixed", by just decreasing the value of the Δt . Doing so yields Figs. (8), (9) and (10).

Fig. 8 shows a trajectory that makes much more sense. It also never converges nor decays, but it displays a trajectory which always formes an elipse of the same dimensions, which just precesses. Now, focusing again on the graph of the radial position of the body of mass m over time, but now with a smaller Δt (Fig. 9), we see that the upward trend of the value of radius maxima, seen previously, is no longer discernible. Finally, looking again at the total energy of the system (Fig. 10) it is easy to see that this time the energy is well conserved. The graph of KT vs time is a flat line, which indicates a constant value. With all this in mind, we can now more confidently affirm that these last results are trustworthy and actually ressemble reality, since they now check criteria 1 and 3. However, one last thing to note: The numerical scheme here used, it would seem, is still unstable. I.e, the total energy of the system still increases over time, even if it is ever so slightly. However, the rate of this growth decreases greatly and rapidly for smaller and smaller values of Δt. Hence, for reasonable simulation times and a small enough Δt, we can say that, for practical purposes, it is exact. And anytime more accurate results are needed, it suffices with further decreasing the value of Δt.

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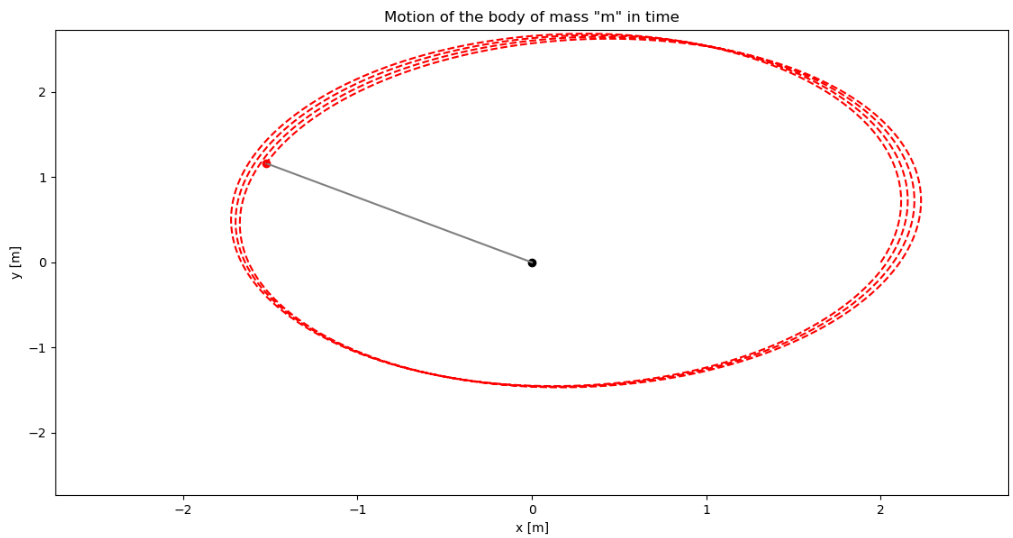

Figure 8: Trajectory of the mass m. $\Delta t = 10^{-5}$ [s]

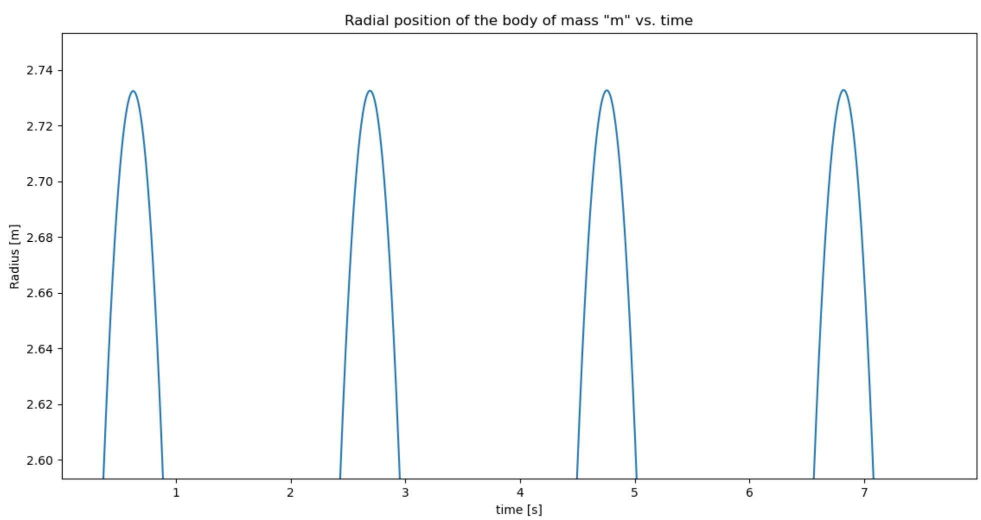

Figure 9: Radius maxima of the trajectory of the body of mass m. $\Delta t = 10^{-5}$ [s]

5.4 Verification of criterion 5

Now, with: $M = 1$ [Kg], $m = 0.5$ [Kg], $L = 6$ [m] , $R(0) = 2$ [m], $\phi(0) = 0$ [rad], $\dot{R}(0) = 0$ [m/s], $\dot{\phi}(0) = 3.1304$ [rad/s], $\mu = 0$ and a simulation time of 8 [s], the results given by the simulation are those seen in figures 11 and 12. As we can detail from said figures, criterion 5 is fully satisfied. The trajectory of the body of mass m is a circle, with its center coinciding the the table's center and with a radius of constant length equal to $R(0)$. It also can be to seen, easily, that energy is conserved, since the trajectory desn't decay or grow without bounds. However, whoever may prefer, can print the energy graph in the simulation too.

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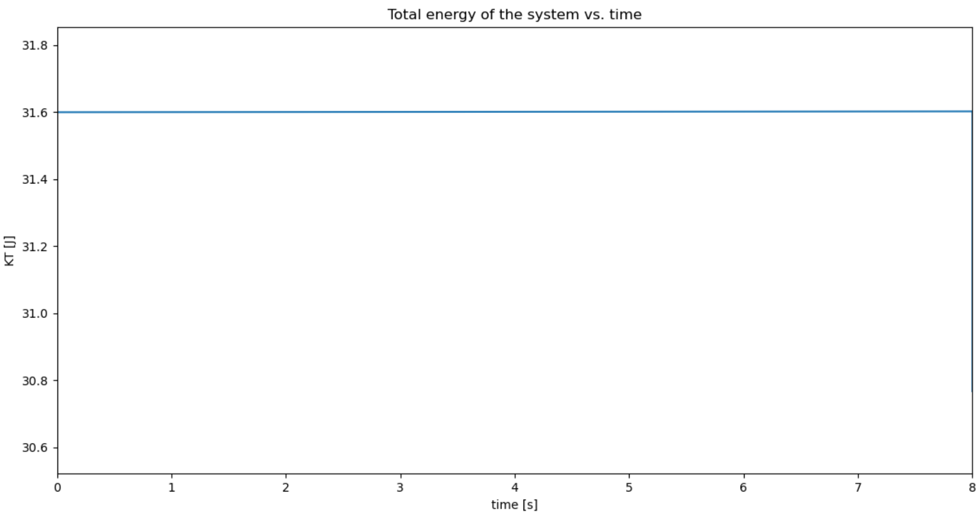

Figure 10: Total energy of the system vs. time. $\Delta t = 10^{-5}$ [s]

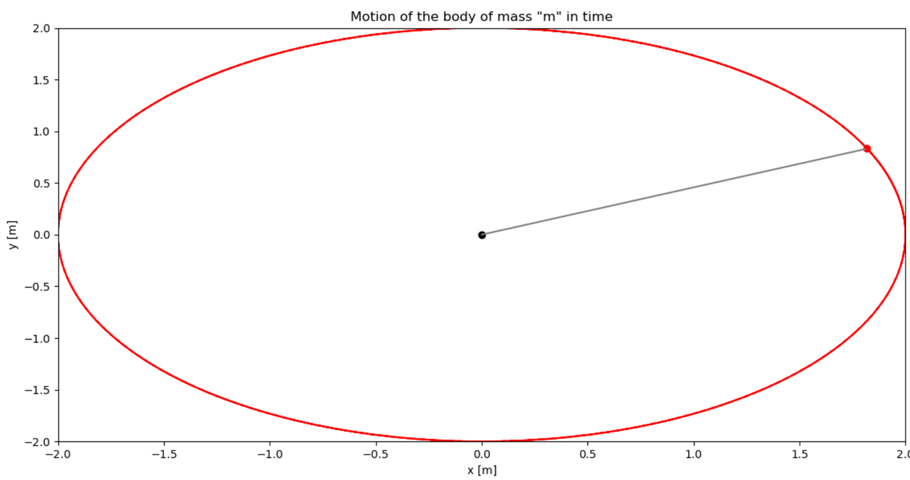

Figure 11: Trajectory of mass m vs. time. $\Delta t = 10^{-5}$ [s]

5.5 Corrections

Before we verify the last two criteria in our list, some corrections to our equations are necessary. Here it will be seen that some modifications to the original equations of motion will have to be made, in order to further increase the range of validity of the equations. In order to do this, let us expand then on a couple of points.

5.5.1 Problematic Terms:

First, let us note that in Eqs.(20) and (21) there are three terms that could pose a problem in the simulation. Namely:

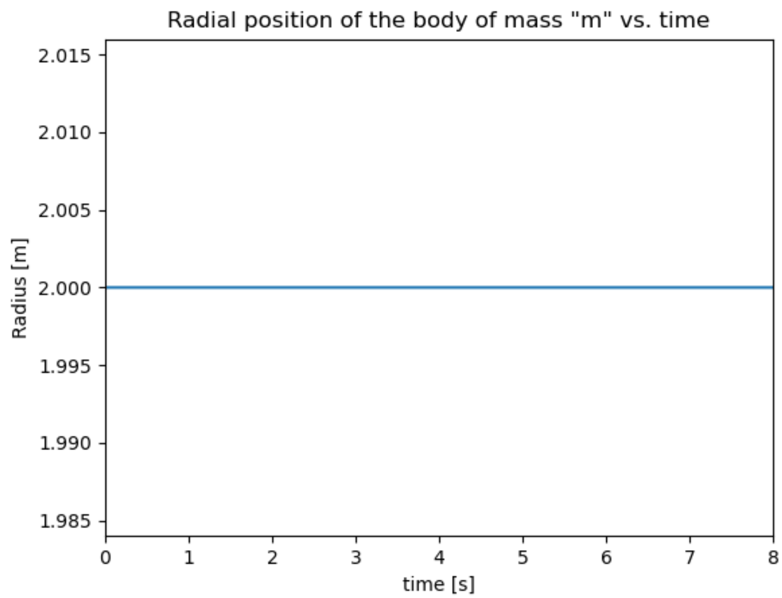

Figure 12: Total energy of the system vs. time. $\Delta t = 10^{-5}$ [s]

$$\frac{\mu g \dot{R}_m}{\sqrt{\dot{R}_m^2 + (R_m \dot{\phi}_m)^2}} \tag{35}$$

$$\frac{\mu g \dot{\phi}_m}{\sqrt{\dot{R}_m^2 + (R_m \dot{\phi}_m)^2}} \tag{36}$$

$$-2\frac{\dot{R}_m}{R_m}\dot{\phi}_m \tag{37}$$

Looking at expressions (35) and (36), it is easy to see that they become undeterminate when $\dot{R}_m = 0$ and $\dot{\phi}_m = 0$ or when $\dot{R}_m = 0$ and $R_m = 0$. The latter case is rare and of less consequence and, therefore, we shall prioritize fixing the undetermination for the first case. For this, we can serve us, again, from some physical intuition. when $\dot{R}_m = 0$ and $\dot{\phi}_m = 0$, that means that the body is not moving at all. Now, since kinetic friction is a force that opposes movement, it follows that, when the body is completely still, in that moment, the force should be 0 (remember that we don't consider the force of static friction since we assume that it is instantly overcome by the force of tension). Therefore, expressions (35) and (36) should be 0 too, when $\dot{R}_m = 0$ and $\dot{\phi}_m = 0$.

Now, looking over to expression (37), we identify the real troublemaker. This term usually tends to blow up and produce huge angular velocity spikes during the simulation (although very briefly). In real life, R_m would never truly be 0 (just approach it), thus avoiding this problem. This is because the body of mass m, as well as the hole, should, in reality, be extended objects/features, with bigger dimensions. Here, however, we are considering, essentially, point-like bodies. Hence the problem. One possible solution to this conundrum, would be to also make this term 0, when R_m is smaller than a certain value. Implementing this proposal, then, into the code, yields that by disregarding expression (37) as R_m becomes smaller than the threshold value, does, in fact, reduce the magnitude of the brief angular velocity spikes. However, depending on the value of said threshold, the value for the total energy of the system could be negatively affected (instead of seeing improvements, which is mostly what we are after). Therefore, it is recommended that, this value should be chosen carefully, by means of trial and error, by determining which one gives the most coherent results. One thing to note, though, is that since the spikes in angular velocity are so brief (specially for small values of Δt), the value of the angular position of the body of mass m is not really impacted by them, so this is a thing that shouldn't be of concern. And that even big irregularities on the values the total energy of the system, tend to improve exponentially and smooth out greatly by just introducing smaller values of Δt.

5.5.2 Crossing over the origin:

Finally, let us look again at Eq.(19). There's actually a simple case where the equations of motion for the body of mass m can be solved analythicaly and without much trouble. I.e, when: $\dot{\phi}_M(0) = 0$. Looking at equations (19) and (20), with this initial condition, both of them become decoupled and equation (20) simplifies to $\ddot{\phi}_m = 0$, which, in this case gurantees that $\phi_m(t) = \phi_m(0) = \text{constant}$, for all time t. On the other hand, equation (19) simplifies to equation (38).

$$\begin{aligned}\ddot{R}_m &= -\frac{\frac{M}{m} + \frac{\dot{R}_m}{|\dot{R}_m|}\mu}{\frac{M}{m}+1}g \\ &= -\frac{\frac{M}{m} + sgn(\dot{R}_m)\mu}{\frac{M}{m}+1}g \\ &= -\frac{\frac{M}{m}+\mu}{\frac{M}{m}+1}g, \dot{R}_m > 0 \\ &= -\frac{\frac{M}{m}-\mu}{\frac{M}{m}+1}g, \dot{R}_m < 0 \\ &= -\frac{\frac{M}{m}}{\frac{M}{m}+1}g, \dot{R}_m = 0\end{aligned} \tag{38}$$

Note 7: *Where, in Eq.(38), the fact that $\vec{Fr} = 0$ when $\dot{R}_m = 0$ was, again, used.*
Note 8: *Eq.(38) explains why $\frac{M}{m} > \mu$ has to hold in order for the equations of motion to be valid. If $\frac{M}{m} < \mu$, that would imply that the force of kinetic friction is greater than that of tension, which would mean that the body of mass m would have to remain at rest, which goes against the definition of kinetic fricion itself.*

Where $sgn()$ is the sign function. Then, if $R_m(0) > 0$ and $\dot{R}_m(0) < 0$ the solution to Eq. (38) is:

$$R_m(t) = -\frac{g(\frac{M}{m}-\mu)}{2(\frac{M}{m}+1)}t^2 + \dot{R}_m(0)t + R_m(0), 0 \le t \tag{39}$$

While, if $R_m(0) > 0$ and $\dot{R}_m(0) > 0$ the solution to Eq. (38) is:

$$\begin{aligned}R_m(t) &= -\frac{g(\frac{M}{m}+\mu)}{2(\frac{M}{m}+1)}t^2 + \dot{R}_m(0)t + R_m(0), 0 \le t < t_c \\ &= -\frac{g(\frac{M}{m}-\mu)}{2(\frac{M}{m}+1)}t^2 + c_1 t + c_2, t_c < t\end{aligned} \tag{40}$$

Where in Eq.(40): $t_c = \frac{\dot{R}_m(0)(\frac{M}{m}+1)}{g(\frac{M}{m}+\mu)}$, $c_1 = \dot{R}_m(0)\frac{\frac{M}{m}-u}{\frac{M}{m}+u}$ and $c_2 = -\frac{\mu g}{\frac{M}{m}+1}t_c^2 + (\dot{R}_m(0) - c_1)t_c + R_m(0)$.

Now, in principle, in Eq. (39), R_m should always be greater than 0. However, by examining further, we note that, whatever the initial conditions may be, if we graph out Eq.(39) or Eq.(40), the function will eventually cross the R_m-axis and we will obtain negative values for R_m. Now, in cylindrical coordinates, a value of less than 0 for the R-coordinate doesn't really make much sense, so we may conclude that, beyond this point, Eqs.(39) and (40) lose all practical meaning. However, this is not the case. Using our intuition of the physical situation, we know that if the body of mass m was approaching the center of the table (because of the pull of the cord) with some radial velocity (and no other velocity), the moment it reached said center, it wouldn't just stop, but rather pass it and continue straight ahead to the other side. This is more or less what Eqs.(39) and (40), tell us when R_m becomes negative. The minus sign just lets us know that the $\phi-$coordinate suffered an instant shift of π radians. I.e, that the body crossed over to the other side of the table. Now, let us note, however, one thing about the situation previously described. Even though we say that the $\phi-$coordinate *suffered an instant shift of* π *radians*, this is something that is not and should not be reflected in the angular velocity of the body. The body doesn't adquire any angular velocity beacuse of this shift. It is just a matter of convention. However, even though *we* know that it is just a convention (that denotes that a crossing over the origin has ocurred and whose main advantage is to be in agreement with the traditional sign convention of the cylindrical coordinate sytem), in-code, it turns out to be quite troublesome. Therefore, we shall stick with the negative radius convention (at a minimum, when converting from cylindrical to cartesian coordinates, both conventions function equally well), and we shall account for this, in other ways.

With all this in mind, we are now ready to modify the equations of motion, so they can handle cossings over the origin. Now, since we have chosen the negative radius convention, to account for this, in the equations, we will have to change the sign of the tension force, after the crossing over the origin has ocurred. If we didn't do this, the body would just accelerate to infinity. Then, modifying Eq.(38), to include this change, yields:

$$\begin{aligned}\ddot{R}_m &= -\frac{\frac{R_m}{|R_m|}\frac{M}{m} + \frac{\dot{R}_m}{|\dot{R}_m|}\mu}{\frac{M}{m}+1}g \\ &= -\frac{sgn(R_m)\frac{M}{m} + sgn(\dot{R}_m)\mu}{\frac{M}{m}+1}g\end{aligned} \tag{41}$$

Whose numerical solution (found using the ode45 solver of MATLAB) with initial conditions: $R_m(0) = 2$ and $\dot{R}_m$, can be seen in figure 13. Now, an analythical solution to Eq.(41) shouldn't be too hard to find, since the procedure is, in principle, identical to the one used for solving Eq.(38). However, it would be long, tedius and filled with steps. Therefore, using a proven numerical solver from a third party is a better way to go in this case. Then, extrapolating the modifications made in this equation to Eq.(33) and from Eq.(19) through Eq. (24), yields:

EQUATIONS OF MOTION FOR THE BODY OF MASS m (Modified)

$$\ddot{R}_m = (|R_m|\dot{\phi}_m^2 - sgn(R_m)\frac{M}{m}g - \frac{\mu g \dot{R}_m}{\sqrt{\dot{R}_m^2 + (R_m\dot{\phi}_m)^2}})(1+\frac{M}{m})^{-1} \tag{42}$$

$$\ddot{\phi}_m = -2\frac{sgn(R_m)\dot{R}_m}{|R_m|}\dot{\phi}_m - \frac{\mu g \dot{\phi}_m}{\sqrt{\dot{R}_m^2 + (R_m\dot{\phi}_m)^2}} \tag{43}$$

$$z_m = 0 \tag{44}$$

EQUATIONS OF MOTION FOR THE BODY OF MASS M (Modified)

$$R_M = 0 \tag{45}$$

$$\phi_M = 0 \tag{46}$$

$$z_M = |R_m| - L \tag{47}$$

ENERGY EQUATION OF THE SYSTEM (Modified)

$$KT = \frac{1}{2}m(R_m\dot{\Phi}_m)^2 + \frac{1}{2}(M+m)\dot{R}_m^2 + Mg|R_m| \tag{48}$$

Let us observe now, various things. First, as it is the case with, R_m in Eq.(41), (42) and (43), $\dot{R}_m$ also doesn't follow the normal sign convention used in cylindrical coordinates. This unorthodox sign convention, though very useful for calculations and computations, is not so great for physical interpretation, since the physical properties of the system do follow the cylindrical coordinate convention. Hence, we would like a formula for converting between conventions, in order to facilitate understanding and analysis. Fortunatly, this is not difficult:

The radial position of the body of mass m using the normal convention for cylindrical coordinates would be, simply:

$$|R_m| \tag{49}$$

Likewise, the radial velocity of the body of mass m using the normal convention would be:

$$\frac{d|R_m|}{dt} = sgn(R_m)\dot{R}_m, R_m \neq 0 \tag{50}$$

And the radial acceleration using said convention would be given by:

$$\frac{d^2|R_m|}{dt^2} = \frac{d[sgn(R_m)\dot{R}_m]}{dt} = \dot{R}_m^2\frac{d[sgn(R_m)]}{dR_m} + sgn(R_m)\ddot{R}_m = sgn(R_m)\ddot{R}_m, R_m \neq 0 \wedge \dot{R}_m \neq 0 \tag{51}$$

Meanwhile, the angular position of the body of mass m, using the normal convention, is obtained by just adding (to its current value) π radians everytime the body of mass m crosses the center of the table. Finally, two more things. First, Eqs.(43) through (48) do follow the cylindrical coordinate convention, so they do represent system variables. Second, to mention that notes 1 through 4 also apply to these equations.

5.5.3 Final equations of motion

Then using everything learned from this subsection we are now ready to give the final equations of motion and of the system:

EQUATIONS OF MOTION FOR THE BODY OF MASS m (Final)

$$\ddot{R}_m = (|R_m|\dot{\phi}_m^2 - sgn(R_m)\frac{M}{m}g - \frac{\mu g\dot{R}_m}{\sqrt{\dot{R}_m^2 + (R_m\dot{\phi}_m)^2}})(1+\frac{M}{m})^{-1}, \dot{R}_m \neq 0 \vee \dot{\phi}_m \neq 0 \tag{52}$$

$$\ddot{R}_m = (|R_m|\dot{\phi}_m^2 - sgn(R_m)\frac{M}{m}g)(1+\frac{M}{m})^{-1}, \dot{R}_m = 0 \wedge \dot{\phi}_m = 0 \tag{53}$$

$$\ddot{\phi}_m = -2\frac{sgn(R_m)\dot{R}_m}{|R_m|}\dot{\phi}_m - \frac{\mu g\dot{\phi}_m}{\sqrt{\dot{R}_m^2 + (R_m\dot{\phi}_m)^2}}, R_m > rr \wedge (\dot{R}_m \neq 0 \vee \dot{\phi}_m \neq 0) \tag{54}$$

$$\ddot{\phi}_m = -2\frac{sgn(R_m)\dot{R}_m}{|R_m|}\dot{\phi}_m, R_m > rr \wedge \dot{R}_m = 0 \wedge \dot{\phi}_m = 0 \tag{55}$$

$$\ddot{\phi}_m = -\frac{\mu g\dot{\phi}_m}{\sqrt{\dot{R}_m^2 + (R_m\dot{\phi}_m)^2}}, R_m \leq rr \wedge (\dot{R}_m \neq 0 \vee \dot{\phi}_m \neq 0) \tag{56}$$

$$\ddot{\phi}_m = 0, R_m \leq rr \wedge \dot{R}_m = 0 \wedge \dot{\phi}_m = 0 \tag{57}$$

$$z_m = 0 \tag{58}$$

EQUATIONS OF MOTION FOR THE BODY OF MASS M (Final)

$$R_M = 0 \tag{59}$$

$$\phi_M = 0 \tag{60}$$

$$z_M = |R_m| - L \tag{61}$$

ENERGY EQUATION OF THE SYSTEM (Final)

$$KT = \frac{1}{2}m(R_m\dot{\Phi}_m)^2 + \frac{1}{2}(M+m)\dot{R}_m^2 + Mg|R_m| \tag{62}$$

Where rr is the threshold radius that we had defined earlier in this subsection. Note that from equation (61), it follows that $\dot{z} = \frac{d|R_m|}{dt}$ and that $\ddot{z} = \frac{d^2|R_m|}{dt^2}$.

5.5.4 Discretization of the final equations of motion:

And then, the discrete version of the equations of montion would be:

EQUATIONS OF MOTION FOR THE BODY OF MASS m (Final and discrete)

$$R_m^{n+1} = 2R_m^n - R_m^{n-1} + \Delta t^2 (R_m^n b^2 - np.sign(R_m^n)\frac{M}{m}g - \frac{\mu g a}{\sqrt{a^2 + (R_m^n b)^2}})(1 + \frac{M}{m})^{-1}, a \neq 0 \vee b \neq 0 \tag{63}$$

$$R_m^{n+1} = 2R_m^n - R_m^{n-1} + \Delta t^2 (R_m^n b^2 - np.sign(R_m^n)\frac{M}{m}g)(1 + \frac{M}{m})^{-1}, a = 0 \wedge b = 0 \tag{64}$$

$$\phi_m^{n+1} = 2\phi_m^n - \phi_m^{n-1} - \Delta t^2 (\frac{2ab}{|R_m^n|}np.sign(R_m^n) + \frac{\mu g b}{\sqrt{a^2 + (R_m^n b)^2}}), R_m^n > rr \wedge (a \neq 0 \vee b \neq 0) \tag{65}$$

$$\phi_m^{n+1} = 2\phi_m^n - \phi_m^{n-1} - \Delta t^2 \frac{2ab}{|R_m^n|}np.sign(R_m^n), R_m^n > rr \wedge a = 0 \wedge b = 0 \tag{66}$$

$$\phi_m^{n+1} = 2\phi_m^n - \phi_m^{n-1} - \frac{\mu g b}{\sqrt{a^2 + (R_m^n b)^2}}, R_m^n \leq rr \wedge (a \neq 0 \vee b \neq 0) \tag{67}$$

$$\phi_m^{n+1} = 2\phi_m^n - \phi_m^{n-1}, R_m^n \leq rr \wedge a = 0 \wedge b = 0 \tag{68}$$

$$z_m^n = 0 \tag{69}$$

EQUATIONS OF MOTION FOR THE BODY OF MASS M (Final and discrete)

$$R_M^n = 0 \tag{70}$$

$$\phi_M^n = 0 \tag{71}$$

$$z_M^n = |R_m^n| - L \tag{72}$$

ENERGY EQUATION OF THE SYSTEM (Final and discrete)

$$KT^n = \frac{1}{2}m(R_m^n b)^2 + \frac{1}{2}(M+m)a^2 + Mg|R_m^n| \tag{73}$$

Where $np.sign()$ is the numpy library sign function and where, in order to convert to the normal angular position (i.e, the one that agrees with the cylindrical coordinate sign convention), we could use the following equation: Angular Position$^n = \phi_m^n + 2\pi q$. Where q is a variable that starts at 0 and increases by one, each time R_m goes from positive to negative or viceversa. Finally, since the initial conditions (i.e: $R_m(0)/R_m^0$, $\dot{R}_m(0)/R_m^{-1}$, $\phi_m(0)/\phi_m^{-1}$ and $\dot{\phi}_m(0)/\phi_m^{-1}$) are expected to follow the normal convention for cylindrical coordinates, Eqs.(25), (26) and (27), shouldn't need, at least in theory, any changes. Now, having all this in mind, let us now, once and for all, verify criteria 2 and 4.

5.6 Verification of criteria 2 and 4

With: $M = 1$ [Kg], $m = 0.5$ [Kg], $L = 6$ [m] , $R(0) = 2$ [m], $\phi(0) = 0$ [rad], $\dot{R}(0) = 1$ [m/s], $\dot{\phi}(0) = 0$ [rad/s], $\mu = 0$ and a simulation time of 8 [s], we obtain figures 13 and 14. In this particular case, equations (52) and (53) reduce to equation (41). Hence, this is the perfect oportunity to compare the solution given by our code to the solution given by a third party numerical solver that has already been proven to be trustworthy. With all this into account, the results are clear. Both, figure 13, which is the solution given by MATLAB's ode45 numerical solver, and figure 14, which is the solution given by our code, are identical. This is something great, since it practically validates the numerical scheme used here (at least, for this special case). Let us now analyse the results for another, more complex, case.

With: $M = 1$ [Kg], $m = 0.5$ [Kg], $L = 6$ [m] , $R(0) = 2$ [m], $\phi(0) = 0$ [rad], $\dot{R}(0) = 2$ [m/s], $\dot{\phi}(0) = 3$ [rad/s], $\mu = 0.3$ and a simulation time of 8 [s], we obtain figures (15) through (20). The difference is that in figs. (15), (16) and (17) a Δt of 10 μs was used, while in figs. (18), (19) and (20) a Δt of 1 μs was used. We note, that though the former and latter figures are very similar, some important differences are notable. The most remarkable one can be seen from comparing figure (17) and figure (20). In the former, we can see a spike in total potential energy of the system around the 4.3 seconds mark. This spike is an error, since it does not make any physical sense. Fortunately, reducing the value of the Δt, greatly improved the results from the simulation (there are other initial conditions which may actually cause a bigger discontinuity than this one, but, in every case, reducing the value of Δt improves the results). Once done this, we can actually go on to see that the simulation does satisfy criteria 2 and 4 with ease. The system's energy decreases gradually overtime and the body of mass "m" tends to converge to the center of the table beacause of this loss in energy. Hence, having satisfied now all the criteria, we can now, more confidently say, that our simulation is trustworthy.

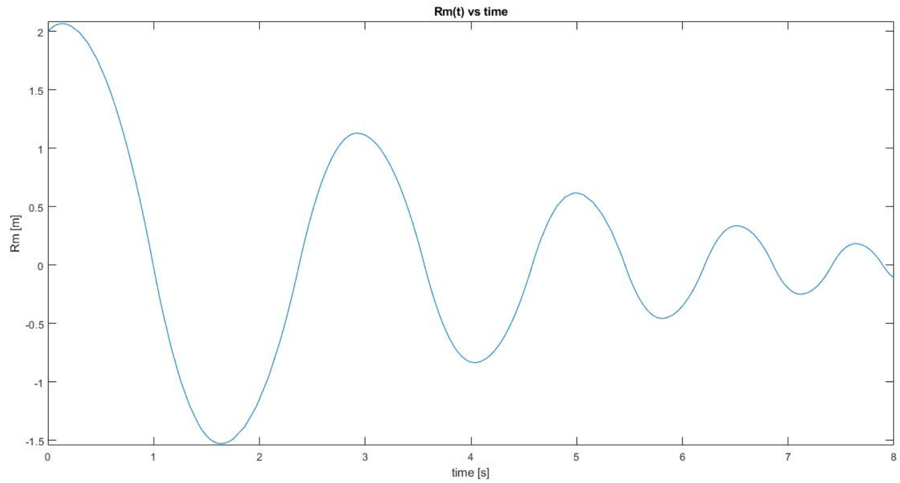

Figure 13: Numerical solution of Eq.(41) using MATLAB's ode45 numerical solver.

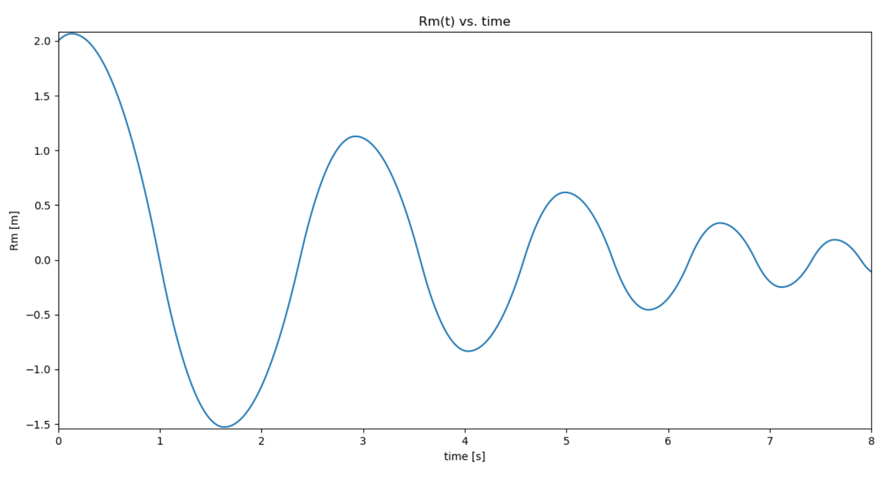

Figure 14: Numerical solution of Eq.(41) using our python code. $\Delta t = 10^{-5}$

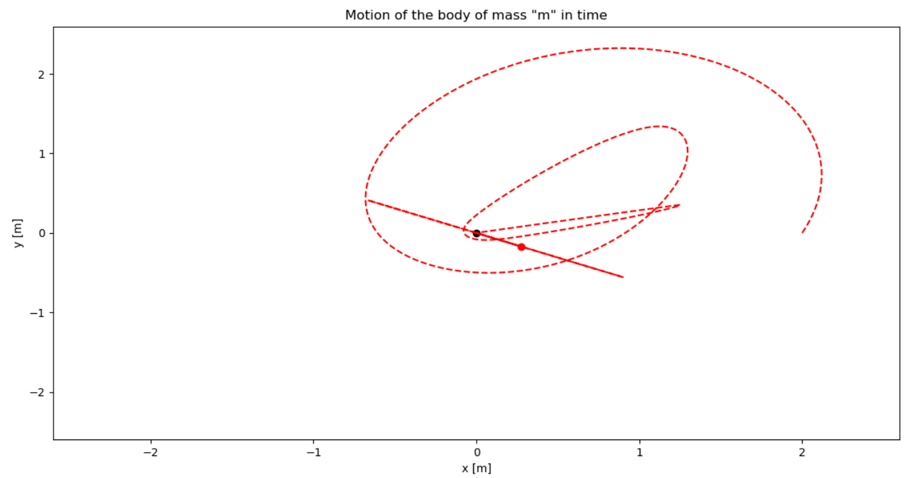

Figure 15: Trajectory of the body of mass m over time. $\Delta t = 10^{-5}$.

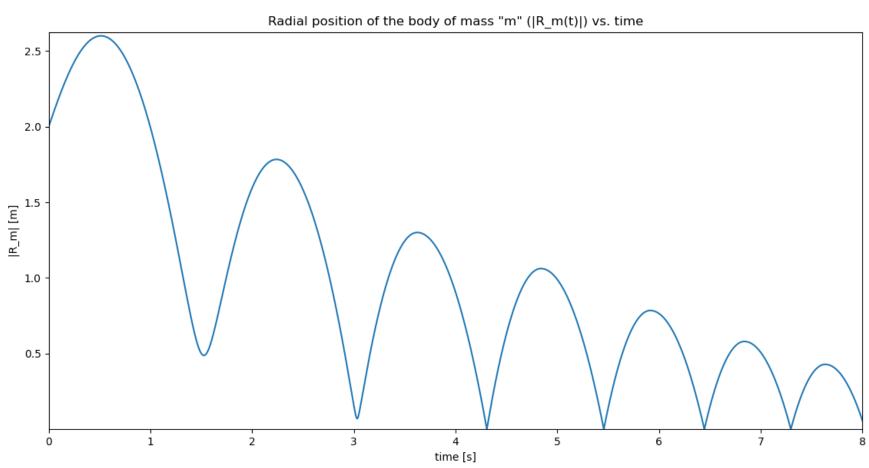

Figure 16: Radial position of the body of mass m vs. time. $\Delta t = 10^{-5}$.

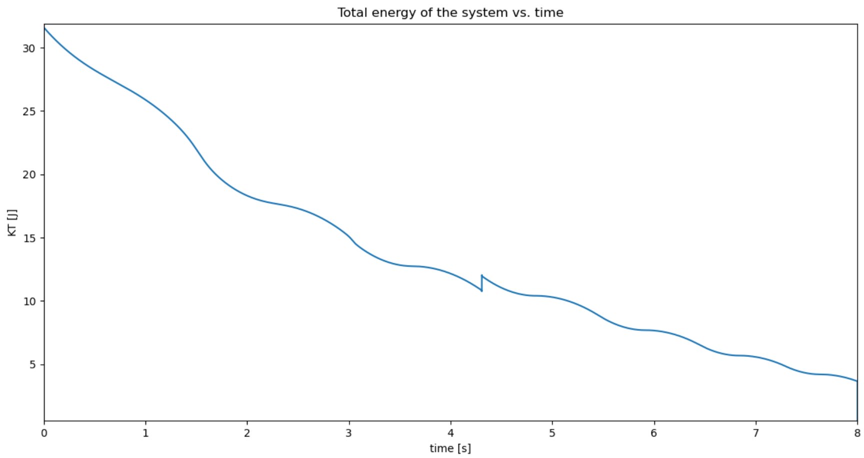

Figure 17: System's total energy vs. time. $\Delta t = 10^{-5}$.

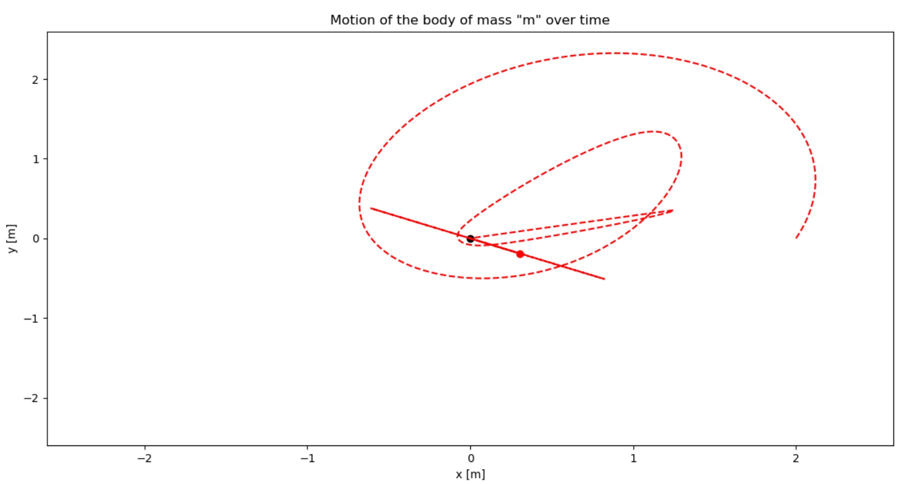

Figure 18: Trajectory of the body of mass m over time. $\Delta t = 10^{-6}$.

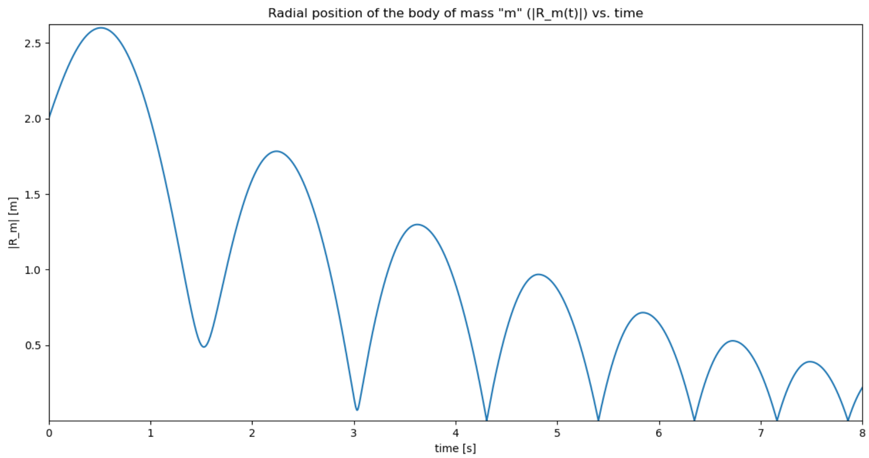

Figure 19: Radial position of the body of mass m vs. time. $\Delta t = 10^{-6}$.

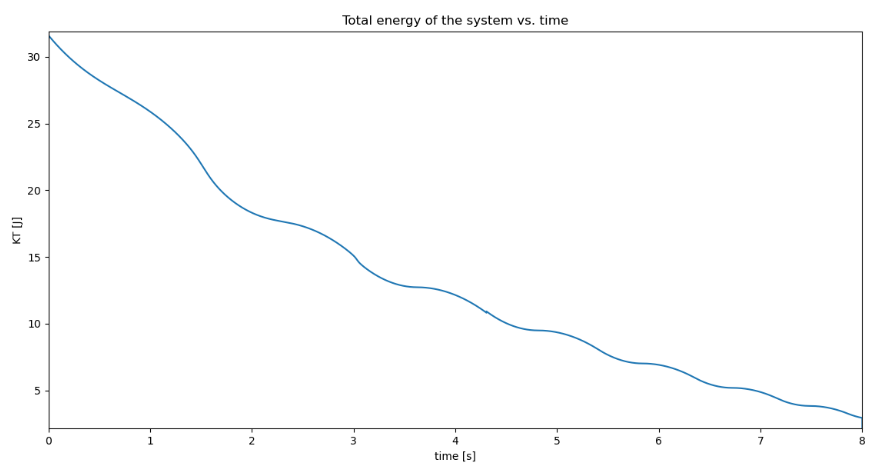

Figure 20: System's total energy vs. time. $\Delta t = 10^{-6}$.