Review.

The formation & introduction rules for Id-types say that

for every display map $p : \begin{array}{c}\Gamma.A\\ \downarrow\\ \Gamma\end{array}$ there is

- a display map $\begin{array}{c}\mathrm{Id}(A)\\ \downarrow\\ \Gamma.A.A\end{array}$

- a morphism $r_A : \Gamma.A \to \mathrm{Id}(A)$ making the following commute

$$\begin{tikzcd} & \mathrm{Id}(A) \arrow[d, two heads] \\ \Gamma.A \arrow[ur, "r_A"] \arrow[r, "\Delta"'] & \Gamma.A.A \end{tikzcd}$$

Thus, in $\mathscr{C}/\Gamma$, (call $\Gamma.A \to \Gamma$ by $A$) have

$$\begin{tikzcd} A \arrow[r, "r_A"] \arrow[dr, dotted] & \mathrm{Id}(A) \arrow[r, "\varepsilon_A"] \arrow[d, dotted] & A \times A \arrow[dl, dotted] \\ & \Gamma & \end{tikzcd}$$

In a wfs, so model category, we can always factor

$$A \xrightarrow{\lambda_\Delta \,\in\, \mathcal{L}} M\Delta \xrightarrow{\rho_\Delta \,\in\, \mathcal{R}} A \times A$$

*Ex.* In Gpd, there is a wfs with

$$\mathcal{L} = \text{equivalences} \cap \text{injective on objects}$$
$$\mathcal{R} = \text{isofibrations}$$

We can factor a diagonal as

$$\mathcal{C} \xrightarrow[\in \mathcal{L}]{i} \mathcal{C}^{\cong} \xrightarrow[\in \mathcal{R}]{\mathrm{dom}\times\mathrm{cod}} \mathcal{C}\times\mathcal{C}$$

$\left\{\begin{array}{l}\text{objects: isos of } \mathcal{C} \\ \text{morphisms: commuting squares}\end{array}\right.$

*Ex.* In Top, there is a wfs with

$$\mathcal{L} = \{\text{maps w/ homotopy extension + homotopies}\}$$
$$\mathcal{R} = \{\text{maps w/ homotopy lifting}\}$$

ho ext
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X^I \arrow[d] \\ C \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

ho lift
$$\begin{tikzcd} X \arrow[r] \arrow[d] & E \arrow[d] \\ X\times I \arrow[r] & B \end{tikzcd}$$
(transport)

We can almost factor a diagonal as

$$X \to X^I \to X\times X.$$

Instead

"Moore paths": paths of any finite length

$$X \xrightarrow{\in \mathcal{L}} \Gamma X \xrightarrow{\in \mathcal{R}} X \times X$$

Ex. In Kan, there is a wfs where we factor the diagonal as

$$X \longrightarrow X^{\Delta 1} \longrightarrow X \times X.$$

Exercise. Given a wfs $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$, show that there is a wfs $(\mathcal{L}/X, \mathcal{R}/X)$ on any slice $\mathcal{C}/X$.

Obs. Given a wfs, by factoring the diagonal, we obtain a model of the formation + introduction rules of the identity type.

Elimination and computation rules tell us that

$$\{r_A \mid \Gamma.A \to \Gamma \in \mathcal{D}\} \boxslash \mathcal{D} \quad .$$

Thm. A model for these rules that is stable under pb/substitution corresponds to

- a wfs
- equipped with chosen factorizations of each diagonal
  $$X \xrightarrow{r} \mathrm{Id}X \xrightarrow{\in \mathcal{D}} X \times X$$
- such that the factorization of $X \xrightarrow{f} Y$ is given by the mapping path space
  $$X \longrightarrow X \times_{\bullet} \mathrm{Id}Y \xrightarrow{\in \mathcal{D}} Y$$

(& in a wfs the factorization always determines the classes $\mathcal{L}, \mathcal{R}$).

The mapping path space

<u>Obs</u>. $X \overset{\times}{\bullet} \mathrm{Id} Y \to Y$ is the universal way to endow $f$ with transport:

$$\begin{tikzcd} * \arrow[r] \arrow[d] & X \overset{\times}{\bullet} \mathrm{Id} Y \arrow[d] \\ I \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

<u>Obs</u>. $X \overset{\times}{\bullet} \mathrm{Id} Y$ in the wfs for groupoids produces

$F \downarrow Y$

the universal isofibration generated by $F$.

<u>Thm</u>. A wfs satisfies the above properties (ignoring constructive issues) if $\mathcal{L}$ is stable under pb along $\mathcal{R}$ (Frobenius condition) and every $X \to * \in \mathcal{R}$.

<u>Model category jargon</u>.

Cisinski model str + right proper $\Rightarrow$ Frobenius

every object is fibrant

<u>Kan complexes</u>

Let $\Delta$ be the category of finite totally ordered sets and order-preserving functions.

$$0 \;\underset{\delta_1}{\overset{\delta_0}{\rightleftarrows}}\; 1 \;\substack{\to\\ \leftarrow \\ \to \\ \leftarrow \\ \to}\; 2 \cdots$$

**Def.** The *category of simplicial sets* is the presheaf topos $\hat{\Delta}$.

Given a simplicial set $S$, we think of it geometrically.

We think of $S_0$ as being the points of $S$.

We think of $S_1$ as being the 'line segments' of $S$.

$S_2$ 'triangles'

There is a functor

$$sSet \underset{S}{\overset{|-|}{\rightleftarrows}} Top$$

called 'geometric realization' making this precise.

'up to homotopy'

**Thm.** $|-|$ gives a Quillen equivalence $sSet \simeq Top$.

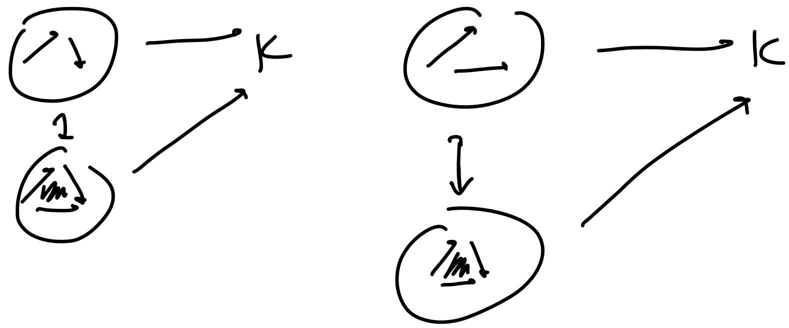
This equivalence 'throws away' all objects in $sSet$ except those for which $X \to * \in \mathcal{D}$; the *Kan complexes*.

**Def.** A *Kan complex* is a simplicial set $K$ such that

\begin{tikzcd}
\Lambda^i_j \arrow[r] \arrow[d, "\delta_i"'] & K \\
\Delta^i \arrow[ur, dotted] &
\end{tikzcd}

A *Kan fibration* is a map s.t.

\begin{tikzcd}
\Lambda^i_j \arrow[r] \arrow[d, "\delta_i"'] & K \\
\Delta^i \arrow[ur, dotted] &
\end{tikzcd}

Ex. 

So the 'line segments of K' behave like paths in a topological space: they are invertible and composable.

Thm. There is a wfs on the category of Kan complexes s.t.

$\mathcal{L}$ = monomorphisms $\cap$ ho eq

$\mathcal{R}$ = Kan complexes.

This gives a model of Id types.

It also has $\Sigma, \Pi$ types (as a topos, locally cartesian closed).

## Univalence in Kan complexes

UA was added to TT from the model in Kan complexes.

Have a universe $\tilde{U} \xrightarrow{\pi_U} U$ st. (ignoring size issues) every Kan fibration is a pb

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d] \\ B \arrow[r] & U \end{tikzcd}$$

That is, we have a CwF from this universe.

But, not only are Kan fibrations recoverable as pb of $\pi_U$, but equivalences are too.

$$\begin{tikzcd} E \arrow[r, "\simeq"] \arrow[d, "p"'] & E' \arrow[r] \arrow[dl, "p'"] & \tilde{U} \arrow[d, "\pi"] \\ B \arrow[rr, shift left] \arrow[rr, shift right, "s"'] & & U \end{tikzcd}$$

Taking $B := *$, have equivalences $E \simeq E'$ correspond to paths in $U$.