Semantics of type theory

Recall the simply typed λ-calculus.

- ⇝ Like MLTT, except only terms (not types) can depend on contexts
- ⇝ Have unit, product, function types

Def. The syntactic category of the STLC $\mathbb{T}$ is the category $\mathcal{C}(\mathbb{T})$ whose

- objects are types
- morphisms are terms

  $$x : A \vdash f : B$$

- the identity is given by the variable rule

  $$x : A \vdash x : A$$

- composition is given by weakening and substitution

  $$x : A \vdash f : B \qquad \cfrac{\cfrac{y : B \vdash g : C}{x : A,\, y : B \vdash g : C}}{x : A \vdash g[f/y] : C}$$

https://www.cl.cam.ac.uk/teaching/1617/L108/catl-notes.pdf

- unitality / associativity follow from rules

Lem. $\mathcal{C}(\mathbb{T})$ is cartesian closed.

Pf. The rules for the product type are

FORM
$$\frac{A, B \text{ type}}{A \times B \text{ type}}$$

INTRO
$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash (a,b) : A \times B}$$

ELIM
$$\frac{\Gamma \vdash p : A \times B}{\Gamma \vdash \pi_A p : A \qquad \Gamma \vdash \pi_B p : B}$$

COMP
$$\Gamma \vdash \pi_A(a,b) \doteq a : A$$
$$\Gamma \vdash \pi_B(a,b) \doteq b : B$$

UNIQ
$$\Gamma \vdash (\pi_A p, \pi_B p) \doteq p$$

Semantically, we get an object $A \times B$ (FORM) together with morphisms $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ (ELIM). Now given any solid diagram like the following we get the dotted arrow by INTRO.

$$\begin{tikzcd} A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\ & \Gamma \arrow[ul, "a"] \arrow[u, dotted, "(a,b)"] \arrow[ur, "b"'] & \end{tikzcd}$$

This commutes since $\pi_B(a,b) \doteq b$ (similarly for a) by COMP. The dotted arrow is unique since if f were another one, $\pi_A(f) \doteq a$ and $\pi_B(f) \doteq b$, so $f \doteq (a,b)$.

The proof that the unit type presents a terminal object and the arrow type gives an internal hom are left as exercises.

Def. A *model* for the simply typed λ calculus is a cartesian closed category.

<u>Note</u>: We never explicitly interpreted contexts. This is because

$$x_0:\Gamma_0, \dots, x_n:\Gamma_n \vdash a:A \text{ is equivalent to } x:\Gamma_0\times\cdots\times\Gamma_n \vdash a:A.$$

We could have defined the syntactic category equivalently as

- objects: contexts
- morphisms: $f:\Gamma \to \Delta$ given by

  $\Gamma \vdash f_i : \Delta_i$

  for each Δ_i in $\Delta = (\Delta_0, \dots, \Delta_n)$.

<u>Note</u>: We can furthermore distinguish extra structure in $\mathcal{C}(\mathbb{T})$.

A) class of morphisms $\mathcal{D}$ of $\mathcal{C}(\mathbb{T})$ ('display maps') that are product projections of the form $\Gamma\times A \xrightarrow{\pi_\Gamma} \Gamma$. This interprets the 'judgment'

$$\gamma:\Gamma, x:A \vdash \gamma:\Gamma$$

so we often write $\Gamma.A$ for $\Gamma\times A$ to indicate that $\Gamma.A$ is the <u>extension of the context</u> Γ by A.

Note that every $X \to * \in \mathcal{D}$, every $X \cong Y \in \mathcal{D}$, and $\mathcal{D}$ is stable under pullback.

B) If we define $\mathcal{C}(\mathbb{T})$ via contexts, then there is a <u>length function</u> $\ell : \mathrm{ob}(\mathcal{C}(\mathbb{T})) \to \mathbb{N}$. Note that

1) there is a unique object of length 0 (empty context)
2) for every object Γ with $\ell(\Gamma)>0$, there is another object $ft(\Gamma)$ such that $\ell ft(\Gamma) = \ell(\Gamma)-1$ and Γ is of

the form $\#\Gamma \cdot A$

3) The pullback

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \arrow[d] \\ \Delta \arrow[r] & ft\,\Gamma \end{tikzcd}$$

always exists.

<u>Def</u>. Given a Martin-Löf type theory $\mathbb{T}$, its syntactic category $\mathcal{C}(\mathbb{T})$ is given by:

- objects: contexts
- morphisms: context morphisms:

  an $f : \Gamma \to \Delta$ consists of

  $\Gamma \vdash f_0 : \Delta_0$

  $\Gamma \vdash \Delta_1 [f_0/x_0]$

  $\vdots$

  $\Gamma \vdash \Delta_n [f_0/x_0][f_1/x_1] \cdots [f_{n-1}/x_{n-1}]$

<u>Note</u>. We could have equivalently defined $\mathcal{C}(\mathbb{T})$ to have objects types and morphisms terms, using Σ-types.

Now we can similarly identify a class $\mathcal{D}$ of <u>display maps</u> in $\mathcal{C}(\mathbb{T})$: these consist of maps of the form $(\Gamma, A) \xrightarrow{\pi} \Gamma$ where π is repeated applications of the variable rule. We write $\Gamma . A$ for (Γ, A). They contain all $X \to *$.

__Lem__. For every diagram of the form below, there is a pullback.

$$\begin{tikzcd} & \Gamma.A \arrow[d, "\pi \in \mathcal{D}"] \\ \Delta \arrow[r, "f"', "\underset{\text{mor}(\mathcal{C}(\mathcal{T}))}{\cap}"] & \Gamma \end{tikzcd}$$

__Pf__. If Γ is empty, then (Δ, Γ, A) is a pullback. It makes the square commute, and maps $Z \to \Delta$, $Z \to \Gamma.A$ uniquely factor through (Δ, Γ, A).

Otherwise, take $(\Delta, A[f])$ where if

$$x_0 : \Gamma_0, \dots, x_n : \Gamma_n \vdash A \text{ type}$$

then $A[f]$ is $A[f_0/x_0][f_1/x_1]\cdots[f_n/x_n]$.

This makes the square commute. If there are maps $Z \xrightarrow{\delta} \Delta$, $Z \xrightarrow{\alpha} \Gamma.A$, then there is a unique map $Z \to (\Delta, A[f])$ given by

$$Z \vdash \delta_j : \Delta_j[\delta_0/y_0]\cdots[\delta_{j-1}/y_{j-1}]$$

$$Z \vdash \quad (A[f_0(y_0,\dots,y_m)/x_0]\cdots[f_n(y_0,\dots,y_m)/x_n])[\delta_0/y_0]\cdots[\delta_m/y_m]$$

$$\shortparallel$$

$$A[f_0(\delta_0,\dots,\delta_m)/x_0]\cdots[f_n(\delta_0,\dots,\delta_m)/x_n]$$

$$\therefore$$

$$\alpha[f_0(\delta_0,\dots,\delta_m)/\alpha_0]\cdots[f_n(\delta_0,\dots,\delta_m)/\alpha_n]$$

where $\quad Z \vdash \alpha : A[\alpha_0/x_0][\alpha_1/x_1]\cdots[\alpha_n/x_n]$. $\qquad\square$

__NB__: Substitution is represented semantically by pullback.

__Def__. A __display map category__ is a category $\mathcal{C}$ with a distinguished class of morphisms $\mathcal{D}$ called __display maps__ such that

1) every $X \to * \in \mathcal{D}$
2) every iso $\in \mathcal{D}$
3) $\mathcal{D}$ is closed under pullback

This is the <u>weakest</u> notion of model.

<u>Thm</u>. The category of groupoids is a display map category where the display maps are isofibrations.

<u>Thm</u>. The category of Kan complexes is a display map category where the display maps are Kan fibrations.

<u>Def</u>. A <u>C system</u> is a category $\mathcal{C}$ with

1) $\ell : \mathrm{ob}\,\mathcal{C} \to \mathbb{N}$
2) a terminal object $*$ such that $\ell^{-1}(0) = \{*\}$
3) $ft : \mathrm{ob}\,\mathcal{C} \setminus * \to \mathrm{ob}\,\mathcal{C}$ such that $\ell ft(\Gamma) = \ell(\Gamma) - 1$
4) for Γ with $\ell(\Gamma) > 0$, $\pi_\Gamma : \Gamma \to ft\,\Gamma$
5) pullbacks of diagrams of the form

$$\begin{tikzcd} \pi^*\Delta \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \arrow[d, "\pi"] \\ \Delta \arrow[r] & ft\,\Gamma \end{tikzcd}$$

where $ft(\pi^*\Delta) = \Delta$.

More notions of model

$$\begin{tikzcd} \text{Display map categories} \arrow[r, hook] & \text{Comprehension categories} & \text{Categories with attributes} \arrow[l, hook'] \arrow[r, phantom, "\overset{\text{Natural models is}}{\simeq}"] & \text{Categories with families} & \text{C-system} \arrow[l] \end{tikzcd}$$