Semantics of HoTT
Lecture Notes

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# 1 Syntactic categories

Consider a Martin-Löf type theory $\mathbb{T}$. By a Martin-Löf type theory, we mean a type theory with the structural rules of Martin-Löf type theory [Hof97]; we are agnostic about which type formers are included in $\mathbb{T}$.

**Definition 1.1.** The *syntactic category of* $\mathbb{T}$ is the category, denoted $\mathcal{C}[\mathbb{T}]$, consisting of the following.

- The objects are the contexts of $\mathbb{T}$.[1]
- The morphisms are the *context morphisms*. A *context morphism* $f : \Gamma \to \Delta$ consists of terms

$$\begin{aligned}
&\Gamma \vdash f_0 : \Delta_0 \\
&\Gamma \vdash f_1 : \Delta_1[f_0/y_0] \\
&\vdots \\
&\Gamma \vdash f_n : \Delta_n[f_0/y_0][f_1/y_1]\cdots[f_{n-1}/y_{n-1}]
\end{aligned}$$

  where $\Delta = (y_0 : \Delta_0, y_1 : \Delta_1, ..., y_n : \Delta_n)$.[2]

- Given an object/context $\Gamma$, the identity morphism $1_\Gamma : \Gamma \to \Gamma$ consists of the terms

$$\begin{aligned}
&\Gamma \vdash x_0 : \Gamma_0 \\
&\Gamma \vdash x_1 : \Gamma_1 \doteq \Gamma_1[x_0/x_0] \\
&\vdots \\
&\Gamma \vdash x_n : \Gamma_n \doteq \Gamma_n[x_0/x_0][x_1/x_1]\cdots[x_{n-1}/x_{n-1}]
\end{aligned}$$

  where $\Gamma = (x_0 : \Gamma_0, x_1 : \Gamma_1, ..., x_n : \Gamma_n)$.

[1] These are given up to judgmental equality in $\mathbb{T}$: i.e., if $\Gamma \doteq \Delta$ as contexts, then $\Gamma = \Delta$ as objects.

[2] These morphisms are given up to judgmental equality in $\mathbb{T}$: i.e., if $f_0 \doteq g_0 : \Delta_0, ..., f_n \doteq g_n : \Delta_n[\delta_0/y_0]\cdots[\delta_{n-1}/y_{n-1}]$, then $f = g$ as morphisms.

- Given morphisms $f : \Gamma \to \Delta$ and $g : \Delta \to \mathrm{E}$, the composition $g \circ f$ is given by the terms

$$\begin{aligned} &\Gamma \vdash g_0[f] : \mathrm{E}_0 \\ &\Gamma \vdash g_1[f] : \mathrm{E}_1 \\ &\vdots \\ &\Gamma \vdash g_m[f] : \mathrm{E}_m \end{aligned}$$

  where $\Delta = (y_0 : \Delta_0, ..., y_n : \Delta_n)$, $\mathrm{E} = (z_0 : \mathrm{E}_0, ..., z_m : \mathrm{E}_m)$ and where by $g_i[f]$ we mean $g_i[f_0/y_0] \cdots [f_n/y_n]$.

Now we show that left unitality, right unitality, and associativity are satisfied.

- Given $f : \Gamma \to \Delta$, we find that $f \circ 1_\Gamma$ consists of terms of the form $\Gamma \vdash f_i[x] : \Delta_i$. But $f_i[x] \doteq f_i[x_0/x_0] \cdots [x_n/x_n]$, so this is $\Gamma \vdash f_i : \Delta_i$. Thus, $f \circ 1_\Gamma = f$.

- Given $f : \Gamma \to \Delta$, we find that $1_\Gamma \circ f$ consists of terms of the form $\Gamma \vdash x_i[f] : \Gamma_i$. But $x_i[f]$ is $x_i[f_0/x_0] \cdots [f_n/x_n]$, so this is $\Gamma \vdash f_i : \Gamma_i$. Thus, $1_\Gamma \circ f = f$.

- Given $f : \Gamma \to \Delta$, $g : \Delta \to \mathrm{E}$, and $h : \mathrm{E} \to \mathrm{Z}$, we find that $h \circ (g \circ f)$ consists of terms of the form $\Gamma \vdash h_i[g[f]] : \mathrm{Z}_i$. But

$$\begin{aligned} h_i[g[f]] &\doteq h_i[g_0[f]/y_0] \cdots [g_m[f]/y_m] \\ &\doteq h_i\left[(g_0[f_0/x_0] \cdots [f_n/x_n])/y_0\right] \cdots \left[(g_n[f_0/x_0] \cdots [f_n/x_n])/y_n\right] \\ &\doteq h_i[g_0/y_0] \cdots [g_n/y_n][f_0/x_0] \cdots [f_n/x_n] \\ &\doteq h_i[g][f]. \end{aligned}$$

  Thus, $h \circ (g \circ f) = (h \circ g) \circ f$.

We think of $\mathcal{C}[\mathbb{T}]$ as the syntax of $\mathbb{T}$, arranged into a category.

**Lemma 1.2.** The empty context is the terminal object of $\mathcal{C}[\mathbb{T}]$.

*Proof.* Let $*$ denote the empty context. A morphism $\Gamma \to *$ consists of components for each component of $*$, that is, nothing. Thus, morphisms $\Gamma \to *$ are unique. □

## 2 Display map categories

**Definition 2.1.** Let $\mathcal{C}$ be a category, and consider a subclass $\mathcal{D} \subseteq \mathrm{mor}(\mathcal{C})$. $\mathcal{D}$ is a *display structure* [Tay99] if for every $d : \Gamma \to \Delta$ in $\mathcal{D}$ and every $s : \mathrm{E} \to \Delta$ in $\mathcal{C}$, there is a given pullback $s^*d \in \mathcal{D}$.

We call the elements of $\mathcal{D}$ *display maps*.

In the syntactic category $\mathcal{C}[\mathbb{T}]$, we are often interested in objects of the form $\Gamma, z : A$ for a context $\Gamma$ and a type $A$; these are often written as $\Gamma.A$. We are then often interested in morphisms of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ where each component of $\pi_\Gamma$ is given by the variable rule. We think of such a morphism as representing the type $A$ in context $\Gamma$.

**Theorem 2.2.** The class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ forms a display structure in the syntactic category $\mathcal{C}[\mathbb{T}]$.

*Proof.* Consider $\pi_\Gamma$ and $s$ as below, where $\pi_\Gamma$ is a display map and $s$ is an arbitrary map.

$$\begin{tikzcd} \Delta.A[s] \arrow[r, "s.A"] \arrow[d, "\pi_\Delta"] & \Gamma.A \arrow[d, "\pi_\Gamma"] \\ \Delta \arrow[r, "s"] & \Gamma \end{tikzcd}$$

Let $\Delta.A[s]$ denote the context $\Delta, z : A[s]$, that is more explicitly:

$$\Delta, z : A[s_0/x_0] \cdots [s_n/x_n].$$

Let $\pi_\Delta$ be the projection given by the variable rule at each component. Let $s.A$ denote the morphism consisting of $\Delta, z : A[s] \vdash s_i : \Gamma_i[s_0/x_0] \cdots [s_{i-1}/x_{i-1}]$ for each component $\Gamma_i$ of $\Gamma$ and $\Delta, z : A[s] \vdash z : A[s]$. We claim that this makes the square above into a pullback square.

Consider a context $Z$ with maps $f : Z \to \Delta$ and $g : Z \to \Gamma.A$ making the appropriate square commute. Let $h$ denote the composite $f : Z \to \Gamma$. Then all components of $g$ but the last component coincide with $h$; denote the last component of $g$ by $Z \vdash g_A : A[h]$. We can construct a map $Z \to \Delta.A[s]$ whose components are $f_i$ for each $\Delta_i$ in $\Delta$, and whose last component is $Z \vdash g_A : A[h] \doteq A[s][f]$. By construction, the two appropriate triangles commute, and any other $z : Z \to \Delta.A[s]$ making these two triangles commute will coincide with our constructed map. (The intuition being that the components of $Z \to \Delta.A[s]$ must basically coincide with the non-redundant components of $f$ and $g$.) $\square$

**Definition 2.3.** Let $\mathcal{C}$ be a category, and consider a subclass $\mathcal{D} \subseteq \mathrm{mor}(\mathcal{C})$. $\mathcal{D}$ is a *class of displays* if $\mathcal{D}$ is stable under pullback.

**Lemma 2.4.** Any class of displays is closed under isomorphism.

**Corollary 2.5** (to Theorem 2.2)**.** Let $\mathcal{D}$ denote the closure under isomorphism of the class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ in $\mathcal{C}[\mathbb{T}]$. Then $\mathcal{D}$ is a class of displays.

Now suppose that we close the class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ under composition. This is then the class of maps of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ where $\Gamma$ and $\Delta$ are arbitrary contexts.

**Lemma 2.6.** Now let $\mathcal{D}$ denote the class of maps of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ in $\mathcal{C}[\mathbb{T}]$. Then

1. $\mathcal{D}$ is closed under composition,

2. $\mathcal{D}$ contains all the maps to the terminal object,

3. every identity is in $\mathcal{D}$

*Proof.* Consider two composable maps in $\mathcal{D}$. Then they must be of the form $\pi_{\Gamma,\Delta} : \Gamma, \Delta, \mathrm{E} \to \Gamma, \Delta$ and $\pi_\Gamma : \Gamma, \Delta \to \Gamma$. Then their composition can be written as $\pi_\Gamma : \Gamma, \Delta, \mathrm{E} \to \Gamma$. Then $\mathcal{D}$ is closed under composition.

Since any context $\Gamma$ can be written as $*, \Gamma$ or $\Gamma, *$, the unique map $\pi_* : \Gamma \to *$ and the identity $\pi_\Gamma : \Gamma \to \Gamma$ are in $\mathcal{D}$. □

**Definition 2.7.** A clan [Joy17] is a category $\mathcal{C}$ with a terminal object $*$ and a distinguished class $\mathcal{D}$ of maps such that

1. $\mathcal{D}$ is closed under isomorphisms,

2. $\mathcal{D}$ contains all isomorphisms,

3. $\mathcal{D}$ is closed under composition,

4. $\mathcal{D}$ is stable under pullbacks, and

5. $\mathcal{D}$ contains all maps to the terminal object.

Note that the first requirement follows from the others.

**Theorem 2.8.** Let $\mathcal{D}$ denote the closure under isomorphism of morphisms of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ in $\mathcal{C}[\mathbb{T}]$. This is a clan.

*Proof.* The first requirement holds by construction.

By Lemma 2.6, $\mathcal{D}$ contains all identities. Since it is then closed under isomorphism, it contains all isomorphism.

The closure under isomorphisms of a class that is closed under composition is still closed under composition, so $\mathcal{D}$ is closed under composition by Lemma 2.6.

Consider any $\pi_\Gamma : \Gamma, \Delta \to \Gamma$. We can write this as a composition of the form

$$\Gamma.\Delta_0 ... \Delta_n \xrightarrow{\pi_{\Gamma.\Delta_0 ... \Delta_{n-1}}} \Gamma.\Delta_0 ... \Delta_{n-1} \xrightarrow{\pi_{\Gamma.\Delta_0 ... \Delta_{n-2}}} \dots \xrightarrow{\pi_\Gamma} \Gamma$$

To take a pullback of $\pi_\Gamma : \Gamma, \Delta \to \Gamma$, we can take pullbacks of each of the component maps (which are in $\mathcal{D}$ by Theorem 2.2) and compose. Since $\mathcal{D}$ is closed under composition, the pullback of $\pi_\Gamma$ is in $\mathcal{D}$.

$\mathcal{D}$ contains all maps to the terminal object by Lemma 2.6. □

The presence of $\Sigma$-types and a unit type allows us to conflate contexts and types.

**Theorem 2.9.** If $\mathbb{T}$ has $\Sigma$-types (with both computation/$\beta$ and uniqueness/$\eta$ rules [nLaa]) and a unit type, then the closure under isomorphism of the class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ is a clan (and indeed, is the same class as in Theorem 2.8).

*Proof.* It is clear the the class of maps considered here is contained in the class of Theorem 2.8.

Thus, we show that any map of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ is isomorphism to one of the form $\pi_\Gamma : \Gamma, A \to \Gamma$. We let $A$ be the following iterated $\Sigma$-type in context $\Gamma$.

$$\sum_{x_0:\Delta_0} \sum_{x_1:\Delta_1} \cdots \sum_{x_{n-1}:\Delta_{n-1}} \Delta_n$$

Then we claim that $\Gamma, \Delta \cong \Gamma.A$ and this commutes with the projections to $\Gamma$.

Let the morphism $f : \Gamma, \Delta \cong \Gamma.A$ have components given by the variable rule for each component $\Gamma_i$ in $\Gamma$. For the component corresponding to $A$, we take

$$\Gamma, x_0 : \Delta_0, ..., x_n : \Delta_n \vdash \langle x_0, ..., x_n \rangle : \sum_{x_0:\Delta_0} \sum_{x_1:\Delta_1} \cdots \sum_{x_{n-1}:\Delta_{n-1}} \Delta_n.$$

For the morphism $g : \Gamma.A \to \Gamma, \Delta$, we again let the components corresponding to each $\Gamma_i$ be given by the variable rule. For the component at a $\Delta_i$, we take

$$\Gamma, y : \sum_{x_0:\Delta_0} \sum_{x_1:\Delta_1} \cdots \sum_{x_{n-1}:\Delta_{n-1}} \Delta_n \vdash \pi_i y : \Delta_i[\pi_0 y/x_0] \cdots [\pi_{i-1} y/x_{i-1}].$$

These morphisms clearly commute with the projections to $\Gamma$, since every component of all the morphisms in question at a $\Gamma_i$ is given by the variable rule.

The fact that $f$ and $g$ are inverse to each other amount to the computation and uniqueness rules for $\Sigma$-types. $\square$

# 3 Categories with families

**Definition 3.1.** A *category with families* consists of the following.

- A category $\mathcal{C}$.
- A presheaf $\mathcal{T} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$.
- A copresheaf $\mathcal{S} : \int \mathcal{T} \to \mathcal{S}et$ where $\int$ denotes the Grothendieck construction. In other words, for every $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}(\Gamma)$, there is a set $\mathcal{S}(\Gamma, A)$; for every $s : \Delta \to \Gamma$, there is a function $\mathcal{S}(f, A) : \mathcal{S}(\Gamma, A) \to \mathcal{S}(\Delta, \mathcal{T}(f)A)$; and this is functorial.
- For each object $\Gamma$ of $\mathcal{C}$ and for each $A \in \mathcal{T}(\Gamma)$, there is an object $\pi_\Gamma : \Gamma.A \to \Gamma$ of $\mathcal{C}/\Gamma$ with the following universal property.

$$\hom_{\mathcal{C}/\Gamma}(s, \pi_\Gamma) \cong \mathcal{S}(\mathcal{T}(s)A).$$

**Theorem 3.2.** The syntactic category $\mathcal{C}[\mathbb{T}]$ has the structure of a category with families.

*Proof.* The underlying category is $\mathcal{C}[\mathbb{T}]$.

For the presheaf $\mathcal{T} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$, we set $\mathcal{T}(\Gamma)$ to be the types of $\mathbb{T}$ in context $\Gamma$. Given $s : \Gamma \to \Delta$, we set $\mathcal{T}(\Delta) \to \mathcal{T}(\Gamma)$ to be substitution by $s$, which we have previously denoted $-[s]$.

For the copresheaf $\mathcal{S} : \int \mathcal{T} \to \mathcal{S}et$, we set $\mathcal{S}(\Gamma, A)$ to be the terms of $A$ in context $\Gamma$. Given $s : \Delta \to \Gamma$, the function $\mathcal{S}(f, A) : \mathcal{S}(\Gamma, A) \to \mathcal{S}(\Delta, \mathcal{T}(f)A)$ is also given by substitution by $s$.

We have objects $\pi_\Gamma : \Gamma.A \to \Gamma$ of $\mathcal{C}/\Gamma$. For the universal property, consider an arbitrary $s : \Delta \to \Gamma$. Then an $f \in \hom_{\mathcal{C}/\Gamma}(s, \pi_\Gamma)$ consists of (many components which must coincide with $s$ and) and one component

$$\Delta \vdash f : A[s],$$

which is exactly an element of $\mathcal{S}(\mathcal{T}(s)A)$. □

**Theorem 3.3.** Consider a category $\mathcal{C}$ and a display structure $\mathcal{D}$. Assume that for every object $\Gamma$ of $\mathcal{C}$, the collection of display maps with codomain $\Gamma$ form a set[3] and that pullback is functorial.

Then there is a category with families on $\mathcal{C}$.

*Proof.* We construct the category with families as follows.

- The underlying category is $\mathcal{C}$.
- The presheaf $\mathcal{T}$ is given by sending an object $\Gamma$ of $\mathcal{C}$ to the set of display maps with domain $\Gamma$. The contravariant functorial action is given by pullback: i.e. given $T \in \mathcal{T}(\Gamma)$ and $f : \Delta \to \Gamma$, set $\mathcal{T}(f)T := f^*T$.
- The copresheaf $\mathcal{S}$ is given by setting $\mathcal{S}(\Gamma, T)$ to be the set of sections of $T$. The functorial action is again given by pullback.
- For each object $\Gamma$ of $\mathcal{C}$ and $T \in \mathcal{T}(\Gamma)$, we take $\Gamma.T$ to be the domain of the display map $T$, and take $\pi_\Gamma$ to be $T$ itself. Now, under the assignments that we have made, the isomorphism we need to establish says that $\hom_{\mathcal{C}/\Gamma}(s, T)$ is in natural bijection with sections of $s^*T$. But this is given by the universal property of $s^*T$, and thus holds (and holds naturally). □

**Theorem 3.4.** Given a category with families $(\mathcal{C}, \mathcal{T}, \mathcal{S})$, there is a display structure $\mathcal{D}$ on $\mathcal{C}$.

*Proof.* We take $\mathcal{D}$ to be all morphisms of the form $\pi_\Gamma : \Gamma.A \to \Gamma$.

Let $s : \Delta \to \Gamma$. We want to show that $\pi_\Delta : \Delta.(\mathcal{T}(s)A) \to \Delta$ is a pullback of $\pi_\Gamma$ along $s$. Then we will say that $\pi_\Delta$ is the chosen pullback of $\pi_\Gamma$ along $s$.

First, let $\iota_x$ denote the composition of the following bijections (where the first and third are part of the definition of category with families, and the middle is functoriality of $\mathcal{T}$).

$$\hom_{\mathcal{C}/\Delta}(x, \pi_\Delta) \cong \mathcal{S}(\mathcal{T}(x)\mathcal{T}(s)A) \cong \mathcal{S}(\mathcal{T}(sx)A) \cong \hom_{\mathcal{C}/\Gamma}(sx, \pi_\Gamma)$$

[3]In general, they may form a proper class. If this hypothesis is not satisfied, you can prove a version of this theorem by introducing a notion of 'smallness' for display maps.

We need to complete the pullback square. Let $\tilde{s}$ denote $\iota_{\pi_\Delta}(1_{\pi_\Delta}) \in \hom_{\mathcal{C}/\Gamma}(s\pi_\Delta, \pi_\Gamma)$. Now we claim that the square below is a pullback.

To this end, consider a $x : Z \to \Delta$ and $y : Z \to \Gamma.A$.

$$\begin{tikzcd}
Z \arrow[rrr, bend left, "y"] \arrow[dr, bend right, "x"'] & \Delta.(\mathcal{T}(s)A) \arrow[rr, "\tilde{s}"] \arrow[d, "\pi_\Delta"] & & \Gamma.A \arrow[d, "\pi_\Gamma"] \\
& \Delta \arrow[rr, "s"] & & \Gamma
\end{tikzcd}$$

Then we have a morphism $y : sx \to \pi_\Gamma$ in the slice $\mathcal{C}/\Gamma$, and so $\iota_x^{-1}y$ is a morphism $x \to \pi_\Delta$ in $\mathcal{C}/\Delta$. That is, $\iota_x^{-1}y$ is a morphism $Z \to \Delta.(\mathcal{T}(s)A)$ making the bottom-left triangle commute. Naturality of $\iota_x$ ensures that the upper triangle commutes.

That is, for any $z : x \to \pi_\Delta$, naturality produces the following commutative diagram where $z^*$ denotes precomposition.

$$\begin{tikzcd}
\hom_{\mathcal{C}/\Delta}(\pi_\Delta, \pi_\Delta) \arrow[r, "\iota_{\pi_\Delta}"] \arrow[d, "z^*"] & \hom_{\mathcal{C}/\Gamma}(s\pi_\Delta, \pi_\Gamma) \arrow[d, "z^*"] \\
\hom_{\mathcal{C}/\Delta}(x, \pi_\Delta) \arrow[r, "\iota_x"] & \hom_{\mathcal{C}/\Gamma}(sx, \pi_\Gamma)
\end{tikzcd}$$

It tells us that $z^*\iota_{\pi_\Delta}1_{\pi_\Delta} = \iota_x z^* 1_{\pi_\Delta}$. But $z^*\iota_{\pi_\Delta}1_{\pi_\Delta} = z^*\tilde{s} = \tilde{s}z$ and $\iota_x z^* 1_{\pi_\Delta} = \iota_x z$. Thus, $\tilde{s}z = \iota_x z$. When $z = \iota_x^{-1}y$, we find that $\tilde{s}(\iota_x^{-1}y) = y$ and the upper triangle commutes.

To show that this is unique, consider another $z : Z \to \Delta.(\mathcal{T}(s)A)$ making the diagram commute. If $\iota_x z = y$, then $z = \iota_x^{-1}y$ since $\iota_x$ is a bijection. But by our above calculation, $\iota_x z = \tilde{s}z$, and thus $\iota_x z = y$. $\square$

**Exercise 3.5** (Open ended)**.** What is the relationship between the two above constructions?

# 4 Semantic universes

**Definition 4.1.** Consider a category $\mathcal{C}$. Say that a morphism $\pi_U : \tilde{U} \to U$ is a *universe* if for any $A : \Gamma \to U$ in $\mathcal{C}$, there exists a chosen pullback, which will be denoted as in the following.

$$\begin{tikzcd}
\Gamma.A \arrow[r] \arrow[d, "\pi_\Gamma"] & \tilde{U} \arrow[d, "\pi_U"] \\
\Gamma \arrow[r, "A"] & U
\end{tikzcd}$$

**Theorem 4.2.** Consider a category $\mathcal{C}$ with a universe $\pi_U : \tilde{U} \to U$. Let $\mathcal{D}$ denote the class of all pullbacks of $\pi_U$. Then $\mathcal{D}$ is a display structure.

*Proof.* We need to show that there exist chosen pullbacks of any $\pi_G amma$ : $\Gamma.A \to \Gamma$ along any $f : \Delta \to \Gamma$. We let the chosen pullback be $\Gamma_D elta$ : $\Delta.(Af) \to \Delta$, i.e., the chosen pullback of $\pi_U$ along $Af$. By the pullback-pasting law this is a pullback. $\square$

**Theorem 4.3.** Consider a category $\mathcal{C}$ with a universe $\pi_U : \tilde{U} \to U$. Then $\mathcal{C}$ has the structure of a category with families.

*Proof.* We need to show that there exist chosen pullbacks of any $\pi_G amma$ : $\Gamma.A \to \Gamma$ along any $f : \Delta \to \Gamma$. We let the chosen pullback be $\Gamma_D elta$ : $\Delta.(Af) \to \Delta$, i.e., the chosen pullback of $\pi_U$ along $Af$. By the pullback-pasting law this is a pullback. We construct the category with families as follows.

- The underlying category is $\mathcal{C}$.
- We set the presheaf $\mathcal{T} := \hom(-, U)$.
- For each object $\Gamma$ of $\mathcal{C}$ and $A \in \mathcal{T}(\Gamma)$, we take $\pi_\Gamma : \Gamma.A \to \Gamma$ to be the specified pullback of $\pi_U$ along $A$.
- The copresheaf $\mathcal{S}$ is given by setting $\mathcal{S}(\Gamma, T)$ to be the set of sections of $\pi_\Gamma$. The functorial action is given by pullback.
- Now, under the assignments that we have made, the isomorphism we need to establish says that $\hom_{\mathcal{C}/\Gamma}(s, \pi_\Gamma)$ is in natural bijection with sections of $s^*(\pi_\Gamma)$. But this is given by the universal property of $s^*(\pi_\Gamma)$ (and is a nice diagram chase to sketch out). $\square$

**Exercise 4.4** (Open ended)**.** Here, we have constructed display structures and categories with families out of categories with universes. Above, we constructed display structures from categories with families and vice versa. What is the relationship between all these constructions? Notice that there is something of a mismatch when these constructions are applied to the following example.

**Example 4.5.** Consider the (1-)category $\mathcal{G}$ of groupoids; this is closed under pullback. Let $U$ denote a groupoid whose objects are small groupoids and whose isomorphisms are functors which are isomorphisms. Let $\pi_U : \tilde{U} \to U$ denote the Grothendieck construction of the identity $U \to U$. Then this is a universe in $\mathcal{G}$, and thus $\mathcal{G}$ has the structure of a display structure (where display maps are pullbacks of $\pi_U$, equivalently 'small' isofibrations) and a category with families (where types are functors into $U$).

See [nLac] and [nLab] for details about isofibrations and the Grothendieck construction.

**Exercise 4.6.** Show that every pullback of $\pi_U$ is an isofibration. Show that every isofibration with small fibers is a pullback of $\pi_U$. You should use the Grothendieck construction.

# 5 Type formers in display maps

In this section, fix a category $\mathcal{C}$ with display maps $\mathcal{D}$. In this section we explain how to interpret various type formers in a class of display maps. The rules that comprise a type former stipulate some structure involving contexts, types, and terms. Thus, we will interpret these rules as stipulating some structure in such a category with display maps involving objects (contexts), display maps (types), and morphisms (terms).

## 5.1 Products

**Theorem 5.1.** Consider a context $\Gamma$ of $\mathbb{T}$ and a simple type types $B$ in $\mathbb{T}$ (i.e., type in the empty context). Then in the syntactic category $\Gamma.A$ has the universal property of the product of $\Gamma$ and $A$.

*Proof.* First, we have maps $\pi_\Gamma : \Gamma.A \to A$ and $\pi_A : \Gamma.A \to A$ using the variable rule. Given any maps $\gamma : Z \to \Gamma$ and $a : Z \to A$ we can construct a map $\langle \gamma, a \rangle : Z \to A.B$, which is unique making the appropriate diagram commute. $\square$

**Exercise 5.2.** Compare this with the substitution $B[\pi_*]$ where $\pi_* : A \to *$.

Thus, we interpret simple types $A$ as display maps with codomain $*$ and context extension $\Gamma.A$ of a context $\Gamma$ by $A$ as the product.

## 5.2 The unit type

Consider the rules for the unit type. Interpreted semantically, they say that there exists a unit type, i.e. a display map $\mathtt{unit} \to *$ (by formation), together with a section $\mathtt{tt} : * \to \mathtt{unit}$ (introduction). The elimination rule tells us that given any object $\Gamma$ and display map $\pi_{\Gamma.\mathtt{unit}} : \Gamma.\mathtt{unit}.D \to \Gamma.\mathtt{unit}$ together with a morphism $d : \Gamma \to \Gamma.\mathtt{unit}.D$ making the solid diagram below commute, there is a section $\overline{d}$ of $\pi_{\Gamma.\mathtt{unit}}$. The computation rule tells us that this section makes the diagram commute.

$$\begin{tikzcd}
 & \Gamma.\mathtt{unit}.D \arrow[d, "\pi_{\Gamma.\mathtt{unit}}"] \\
\Gamma \arrow[ur, "d"] \arrow[r, "\Gamma.\mathtt{tt}"] & \Gamma.\mathtt{unit} \arrow[u, dotted, bend left, "\overline{d}"]
\end{tikzcd}$$

**Theorem 5.3.** Any terminal object of $\mathcal{C}$ gives an interpretation of the unit type.

*Proof.* If we take $\mathtt{unit}$ to be $*$ and $\mathtt{tt}$ to be $1_*$, then $\Gamma.\mathtt{unit} \cong \Gamma$. Thus, we take $\overline{d} := d(\Gamma.\mathtt{tt})^{-1}$. $\square$

# 6 Dependent sum types

Consider the rules in the empty context for the $\Sigma$ type. Interpreted semantically, they say that for every display map $\pi_\Gamma : \Gamma.A \to \Gamma$, there is a simple type $\Sigma(\Gamma.A) \to *$. This has the property that for any display map $\pi_{\Sigma(\Gamma.A)} : \Sigma(\Gamma.A).B \to \Sigma(\Gamma.A)$ and any morphism ...

# 7 Homotopic models of type theory

**Definition 7.1.** Consider a category $\mathcal{C}$ with two morphisms $c : A \to B$ and $f : X \to Y$. Say that $f$ has the *right lifting property with respect to* $c$ (resp. that $c$ has the *left lifting property with respect to* $f$) if for every $x : A \to X$ and $y : B \to Y$ making the solid square below commute, there is a morphism $\ell : B \to X$ making the whole diagram commute.

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[d, "c"'] & X \arrow[d, "f"] \\ B \arrow[r, "y"] \arrow[ur, dotted, "\ell"] & Y \end{tikzcd}$$

We say that a square such as the above solid square is a *lifting problem*, and such an $\ell$ is a *solution*. We write $c \boxslash f$.

Given a classes $\mathcal{A}$ and $\mathcal{D}$ of morphisms of $\mathcal{C}$, we write $\mathcal{A} \boxslash \mathcal{C}$ and say that $\mathcal{A}$ has the *left lifting property with respect to* $\mathcal{C}$ or that $\mathcal{C}$ has the *right lifting property with respect to* $\mathcal{A}$ if $c \boxslash f$ for every $c \in \mathcal{A}$ and $f \in \mathcal{D}$.

Given a class $\mathcal{A}$, we write $\mathcal{A}^{\boxslash}$ for the class of morphisms $f$ such that $c \boxslash f$ for every $c \in \mathcal{A}$. That is, $\mathcal{A}^{\boxslash}$ is the largest class such that $\mathcal{A} \boxslash \mathcal{A}^{\boxslash}$. Dually, given a class $\mathcal{D}$, we write ${}^{\boxslash}\mathcal{D}$ for class of morphisms $c$ such that $c \boxslash f$ for every $f \in \mathcal{D}$.

**Theorem 7.2.** Consider a category $\mathcal{C}$ with all finite limits and a class $\mathcal{A}$ of morphisms of $\mathcal{C}$. Then $\mathcal{A}^{\boxslash}$ has the following properties.

1. $\mathcal{D}$ is closed under isomorphisms,
2. $\mathcal{D}$ contains all isomorphisms,
3. $\mathcal{D}$ is closed under composition,
4. $\mathcal{D}$ is stable under pullbacks.

*Proof.* This is basically the HW. □

**Example 7.3.** A model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{M}$ consists of three classes of morphisms, $\mathcal{C}$, $\mathcal{W}$, and $\mathcal{F}$ such that $\mathcal{F} = (\mathcal{C} \cap \mathcal{W})^{\boxslash}$ (among other requirements). We call $\mathcal{F}$ the *fibrations* and $(\mathcal{C} \cap \mathcal{W})$ the *acyclic (or trivial) cofibrations*. Thus the class of fibrations in any model category satisfies the hypotheses of the above theorem.

We say that an object $X$ of $\mathcal{M}$ is *fibrant* if the unique morphism from $X$ to the terminal object is a fibration. If every object of $\mathcal{M}$ is fibrant, then $(\mathcal{M}, \mathcal{F})$ is a clan.

**Definition 7.4.** Given a category $\mathcal{C}$ with a class of display maps $\mathcal{D}$, say that an object $X$ is *fibrant* if the unique map from $X$ to the terminal object is in $\mathcal{D}$.

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