Semantics of HoTT
Lecture Notes

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1 Syntactic categories

Consider a Martin-Löf type theory $\mathbb{T}$. By a Martin-Löf type theory, we mean a type theory with the structural rules of Martin-Löf type theory [Hof97]; we are agnostic about which type formers are included in $\mathbb{T}$.

Definition 1.1. The *syntactic category of* $\mathbb{T}$ is the category, denoted $\mathcal{C}[\mathbb{T}]$, consisting of the following.

- The objects are the contexts of $\mathbb{T}$.[1]

- The morphisms are the *context morphisms*. A *context morphism* $f : \Gamma \to \Delta$ consists of terms

$$\Gamma \vdash f_0 : \Delta_0$$
$$\Gamma \vdash f_1 : \Delta_1[f_0/y_0]$$
$$\vdots$$
$$\Gamma \vdash f_n : \Delta_n[f_0/y_0][f_1/y_1]\cdots[f_{n-1}/y_{n-1}]$$

  where $\Delta = (y_0 : \Delta_0, y_1 : \Delta_1, ..., y_n : \Delta_n)$.[2]

- Given an object/context Γ, the identity morphism $1_\Gamma : \Gamma \to \Gamma$ consists of the terms

$$\Gamma \vdash x_0 : \Gamma_0$$
$$\Gamma \vdash x_1 : \Gamma_1 \doteq \Gamma_1[x_0/x_0]$$
$$\vdots$$
$$\Gamma \vdash x_n : \Gamma_n \doteq \Gamma_n[x_0/x_0][x_1/x_1]\cdots[x_{n-1}/x_{n-1}]$$

  where $\Gamma = (x_0 : \Gamma_0, x_1 : \Gamma_1, ..., x_n : \Gamma_n)$.

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[1] These are given up to judgmental equality in $\mathbb{T}$: i.e., if $\Gamma \doteq \Delta$ as contexts, then $\Gamma = \Delta$ as objects.

[2] These morphisms are given up to judgmental equality in $\mathbb{T}$: i.e., if $f_0 \doteq g_0 : \Delta_0, ..., f_n \doteq g_n : \Delta_n[\delta_0/y_0]\cdots[\delta_{n-1}/y_{n-1}]$, then $f = g$ as morphisms.

- Given morphisms $f : \Gamma \to \Delta$ and $g : \Delta \to \mathrm{E}$, the composition $g \circ f$ is given by the terms

$$\Gamma \vdash g_0[f] : \mathrm{E}_0$$
$$\Gamma \vdash g_1[f] : \mathrm{E}_1$$
$$\vdots$$
$$\Gamma \vdash g_m[f] : \mathrm{E}_m$$

  where $\Delta = (y_0 : \Delta_0, ..., y_n : \Delta_n)$, $\mathrm{E} = (z_0 : \mathrm{E}_0, ..., z_m : \mathrm{E}_m)$ and where by $g_i[f]$ we mean $g_i[f_0/y_0] \cdots [f_n/y_n]$.

Now we show that left unitality, right unitality, and associativity are satisfied.

- Given $f : \Gamma \to \Delta$, we find that $f \circ 1_\Gamma$ consists of terms of the form $\Gamma \vdash f_i[x] : \Delta_i$. But $f_i[x] \doteq f_i[x_0/x_0] \cdots [x_n/x_n]$, so this is $\Gamma \vdash f_i : \Delta_i$. Thus, $f \circ 1_\Gamma = f$.

- Given $f : \Gamma \to \Delta$, we find that $1_\Gamma \circ f$ consists of terms of the form $\Gamma \vdash x_i[f] : \Gamma_i$. But $x_i[f]$ is $x_i[f_0/x_0] \cdots [f_n/x_n]$, so this is $\Gamma \vdash f_i : \Gamma_i$. Thus, $1_\Gamma \circ f = f$.

- Given $f : \Gamma \to \Delta$, $g : \Delta \to \mathrm{E}$, and $h : \mathrm{E} \to \mathrm{Z}$, we find that $h \circ (g \circ f)$ consists of terms of the form $\Gamma \vdash h_i[g[f]] : \mathrm{Z}_i$. But

$$
\begin{aligned}
h_i[g[f]] &\doteq h_i[g_0[f]/y_0] \cdots [g_m[f]/y_m] \\
&\doteq h_i\left[(g_0[f_0/x_0] \cdots [f_n/x_n])/y_0\right] \cdots \left[(g_n[f_0/x_0] \cdots [f_n/x_n])/y_n\right] \\
&\doteq h_i[g_0/y_0] \cdots [g_n/y_n][f_0/x_0] \cdots [f_n/x_n] \\
&\doteq h_i[g][f].
\end{aligned}
$$

  Thus, $h \circ (g \circ f) = (h \circ g) \circ f$.

We think of $\mathcal{C}[\mathbb{T}]$ as the syntax of $\mathbb{T}$, arranged into a category.

Lemma 1.1. *The empty context is the terminal object of* $\mathcal{C}[\mathbb{T}]$.

Proof. Let $*$ denote the empty context. A morphism $\Gamma \to *$ consists of components for each component of $*$, that is, nothing. Thus, morphisms $\Gamma \to *$ are unique. $\square$

2 Display map categories

Definition 2.1. Let $\mathcal{C}$ be a category, and consider a subclass $\mathcal{D} \subseteq \mathrm{mor}(\mathcal{C})$. $\mathcal{D}$ is a *display structure* [Tay99] if for every $d : \Gamma \to \Delta$ in $\mathcal{D}$ and every $s : \mathrm{E} \to \Delta$ in $\mathcal{C}$, there is a given pullback $s^*d \in \mathcal{D}$.

  We call the elements of $\mathcal{D}$ *display maps*.

In the syntactic category $\mathcal{C}[\mathbb{T}]$, we are often interested in objects of the form $\Gamma, z : A$ for a context Γ and a type A; these are often written as $\Gamma.A$. We are then often interested in morphisms of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ where each component of π_Γ is given by the variable rule. We think of such a morphism as representing the type A in context Γ.

Theorem 2.1. *The class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ forms a display structure in the syntactic category $\mathcal{C}[\mathbb{T}]$.*

Proof. Consider π_Γ and s as below, where π_Γ is a display map and s is an arbitrary map.

$$\begin{array}{ccc} \Delta.A[s] & \xrightarrow{\ s.A\ } & \Gamma.A \\ {\scriptstyle \pi_\Delta}\downarrow & & \downarrow{\scriptstyle \pi_\Gamma} \\ \Delta & \xrightarrow{\ s\ } & \Gamma \end{array}$$

Let $\Delta.A[s]$ denote the context $\Delta, z : A[s]$, that is more explicitly:

$$\Delta, z : A[s_0/x_0]\cdots[s_n/x_n].$$

Let π_Δ be the projection given by the variable rule at each component. Let $s.A$ denote the morphism consisting of $\Delta, z : A[s] \vdash s_i : \Gamma_i[s_0/x_0]\cdots[s_{i-1}/x_{i-1}]$ for each component Γ_i of Γ and $\Delta, z : A[s] \vdash z : A[s]$. We claim that this makes the square above into a pullback square.

Consider a context Z with maps $f : Z \to \Delta$ and $g : Z \to \Gamma.A$ making the appropriate square commute. Let h denote the composite $f : Z \to \Gamma$. Then all components of g but the last component coincide with h; denote the last component of g by $Z \vdash g_A : A[h]$. We can construct a map $Z \to \Delta.A[s]$ whose components are f_i for each Δ_i in Δ, and whose last component is $Z \vdash g_A : A[h] \doteq A[s][f]$. By construction, the two appropriate triangles commute, and any other $z : Z \to \Delta.A[s]$ making these two triangles commute will coincide with our constructed map. (The intuition being that the components of $Z \to \Delta.A[s]$ must basically coincide with the non-redundant components of f and g.) $\qquad\square$

Definition 2.2. Let $\mathcal{C}$ be a category, and consider a subclass $\mathcal{D} \subseteq \mathrm{mor}(\mathcal{C})$. $\mathcal{D}$ is a *class of displays* if $\mathcal{D}$ is stable under pullback.

Lemma 2.1. *Any class of displays is closed under isomorphism.*

Corollary 2.1 (to Theorem 2.1)**.** *Let $\mathcal{D}$ denote the closure under isomorphism of the class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ in $\mathcal{C}[\mathbb{T}]$. Then $\mathcal{D}$ is a class of displays.*

Now suppose that we close the class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ under composition. This is then the class of maps of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ where Γ and Δ are arbitrary contexts.

Lemma 2.2. *Now let $\mathcal{D}$ denote the class of maps of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ in $\mathcal{C}[\mathbb{T}]$. Then*

1. $\mathcal{D}$ *is closed under composition,*

2. $\mathcal{D}$ *contains all the maps to the terminal object,*

3. *every identity is in* $\mathcal{D}$

Proof. Consider two composable maps in $\mathcal{D}$. Then they must be of the form $\pi_{\Gamma.\Delta} : \Gamma.\Delta, E \to \Gamma.\Delta$ and $\pi_\Gamma : \Gamma.\Delta \to \Gamma$. Then their composition can be written as $\pi_\Gamma : \Gamma.\Delta, E \to \Gamma$. Then $\mathcal{D}$ is closed under composition.

Since any context Γ can be written as $*.\Gamma$ or $\Gamma.*$, the unique map $\pi_* : \Gamma \to *$ and the identity $\pi_\Gamma : \Gamma \to \Gamma$ are in $\mathcal{D}$. $\qquad\square$

Definition 2.3. A clan [Joy17] is a category $\mathcal{C}$ with a terminal object $*$ and a distinguished class $\mathcal{D}$ of maps such that

1. $\mathcal{D}$ is closed under isomorphisms,

2. $\mathcal{D}$ contains all isomorphisms,

3. $\mathcal{D}$ is closed under composition,

4. $\mathcal{D}$ is stable under pullbacks, and

5. $\mathcal{D}$ contains all maps to the terminal object.

Note that the first requirement follows from the others.

Theorem 2.2. *Let* $\mathcal{D}$ *denote the closure under isomorphism of morphisms of the form* $\pi_\Gamma : \Gamma.\Delta \to \Gamma$ *in* $\mathcal{C}[\mathbb{T}]$. *This is a clan.*

Proof. The first requirement holds by construction.

By Lemma 2.2, $\mathcal{D}$ contains all identities. Since it is then closed under isomorphism, it contains all isomorphism.

The closure under isomorphisms of a class that is closed under composition is still closed under composition, so $\mathcal{D}$ is closed under composition by Lemma 2.2.

Consider any $\pi_\Gamma : \Gamma.\Delta \to \Gamma$. We can write this as a composition of the form

$$\Gamma.\Delta_0...\Delta_n \xrightarrow{\pi_{\Gamma.\Delta_0...\Delta_{n-1}}} \Gamma.\Delta_0...\Delta_{n-1} \xrightarrow{\pi_{\Gamma.\Delta_0...\Delta_{n-2}}} ... \xrightarrow{\pi_\Gamma} \Gamma$$

To take a pullback of $\pi_\Gamma : \Gamma.\Delta \to \Gamma$, we can take pullbacks of each of the component maps (which are in $\mathcal{D}$ by Theorem 2.1) and compose. Since $\mathcal{D}$ is closed under composition, the pullback of π_Γ is in $\mathcal{D}$.

$\mathcal{D}$ contains all maps to the terminal object by Lemma 2.2. $\qquad\square$

The presence of Σ-types and a unit type allows us to conflate contexts and types.

Theorem 2.3. *If* $\mathbb{T}$ *has* Σ*-types (with both computation/*β *and uniqueness/*η *rules [nLa]) and a unit type, then the closure under isomorphism of the class of maps of the form* $\pi_\Gamma : \Gamma.A \to \Gamma$ *is a clan (and indeed, is the same class as in Theorem 2.2).*

Proof. It is clear the the class of maps considered here is contained in the class of Theorem 2.2.

Thus, we show that any map of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ is isomorphism to one of the form $\pi_\Gamma : \Gamma, A \to \Gamma$. We let A be the following iterated Σ-type in context Γ.

$$\sum_{x_0:\Delta_0} \sum_{x_1:\Delta_1} \cdots \sum_{x_{n-1}:\Delta_{n-1}} \Delta_n$$

Then we claim that $\Gamma, \Delta \cong \Gamma.A$ and this commutes with the projections to Γ.

Let the morphism $f : \Gamma, \Delta \cong \Gamma.A$ have components given by the variable rule for each component Γ_i in Γ. For the component corresponding to A, we take

$$\Gamma, x_0 : \Delta_0, ..., x_n : \Delta_n \vdash \langle x_0, ..., x_n \rangle : \sum_{x_0:\Delta_0} \sum_{x_1:\Delta_1} \cdots \sum_{x_{n-1}:\Delta_{n-1}} \Delta_n.$$

For the morphism $g : \Gamma.A \to \Gamma, \Delta$, we again let the components corresponding to each Γ_i be given by the variable rule. For the component at a Δ_i, we take

$$\Gamma, y : \sum_{x_0:\Delta_0} \sum_{x_1:\Delta_1} \cdots \sum_{x_{n-1}:\Delta_{n-1}} \Delta_n \vdash \pi_i y : \Delta_i[\pi_0 y/x_0] \cdots [\pi_{i-1} y/x_{i-1}].$$

These morphisms clearly commute with the projections to Γ, since every component of all the morphisms in question at a Γ_i is given by the variable rule.

The fact that f and g are inverse to each other amount to the computation and uniqueness rules for Σ-types. $\square$

3 Categories with families

Definition 3.1. A *category with families* consists of the following.

- A category $\mathcal{C}$.

- A presheaf $\mathcal{T} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$.

- A copresheaf $\mathcal{S} : \int \mathcal{T} \to \mathcal{S}et$ where $\int$ denotes the Grothendieck construction. In other words, for every $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}(\Gamma)$, there is a set $\mathcal{S}(\Gamma, A)$; for every $s : \Delta \to \Gamma$, there is a function $\mathcal{S}(f, A) : \mathcal{S}(\Gamma, A) \to \mathcal{S}(\Delta, \mathcal{T}(f)A)$; and this is functorial.

- For each object Γ of $\mathcal{C}$ and for each $A \in \mathcal{T}(\Gamma)$, there is an object $\pi_\Gamma : \Gamma.A \to \Gamma$ of $\mathcal{C}/\Gamma$ with the following universal property.

$$\hom_{\mathcal{C}/\Gamma}(s, \pi_\Gamma) \cong \mathcal{S}(\mathcal{T}(s)A).$$

Theorem 3.1. *The syntactic category $\mathcal{C}[\mathbb{T}]$ has the structure of a category with families.*

Proof. The underlying category is $\mathcal{C}[\mathbb{T}]$.

For the presheaf $\mathcal{T} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$, we set $\mathcal{T}(\Gamma)$ to be the types of $\mathbb{T}$ in context Γ. Given $s : \Gamma \to \Delta$, we set $\mathcal{T}(\Delta) \to \mathcal{T}(\Gamma)$ to be substitution by s, which we have previously denoted $-[s]$.

For the copresheaf $\mathcal{S} : \int \mathcal{T} \to \mathcal{S}et$, we set $\mathcal{S}(\Gamma, A)$ to be the terms of A in context Γ. Given $s : \Delta \to \Gamma$, the function $\mathcal{S}(f, A) : \mathcal{S}(\Gamma, A) \to \mathcal{S}(\Delta, \mathcal{T}(f)A)$ is also given by substitution by s.

We have objects $\pi_\Gamma : \Gamma.A \to \Gamma$ of $\mathcal{C}/\Gamma$. For the universal property, consider an arbitrary $s : \Delta \to \Gamma$. Then an $f \in \hom_{\mathcal{C}/\Gamma}(s, \pi_\Gamma)$ consists of (many components which must coincide with s and) and one component

$$\Delta \vdash f : A[s],$$

which is exactly an element of $\mathcal{S}(\mathcal{T}(s)A)$. $\square$

Exercise 3.1. *Construct a category with families from an arbitrary display structure, and show that when you apply this to the display structure of Theorem 2.1, you obtain the same category with families as above.*

Exercise 3.2. *Show that from any category with families you obtain a display structure. What is the relationship between this construction and the above construction?*

- categories with families

- correspondence with display map categories

- category with families from a universe

- isofibrations and the grothendieck construction

- fibrations and transport

- therefore identity?

- Sigma types?

- Pi types?

References

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