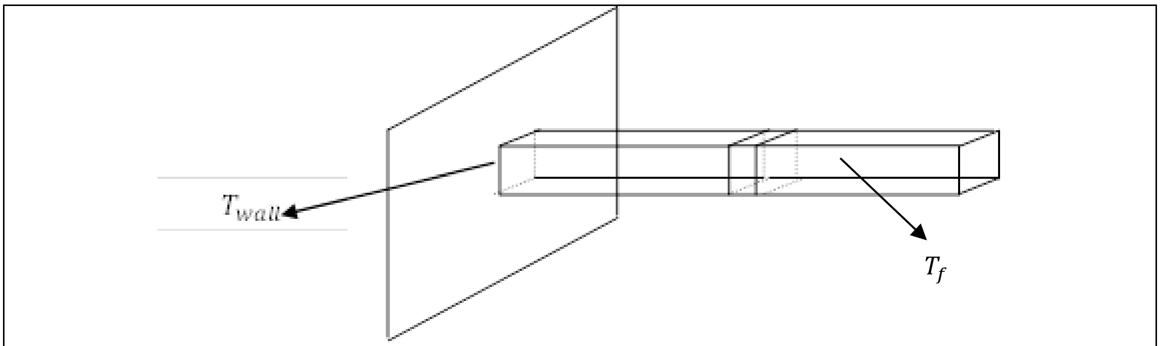
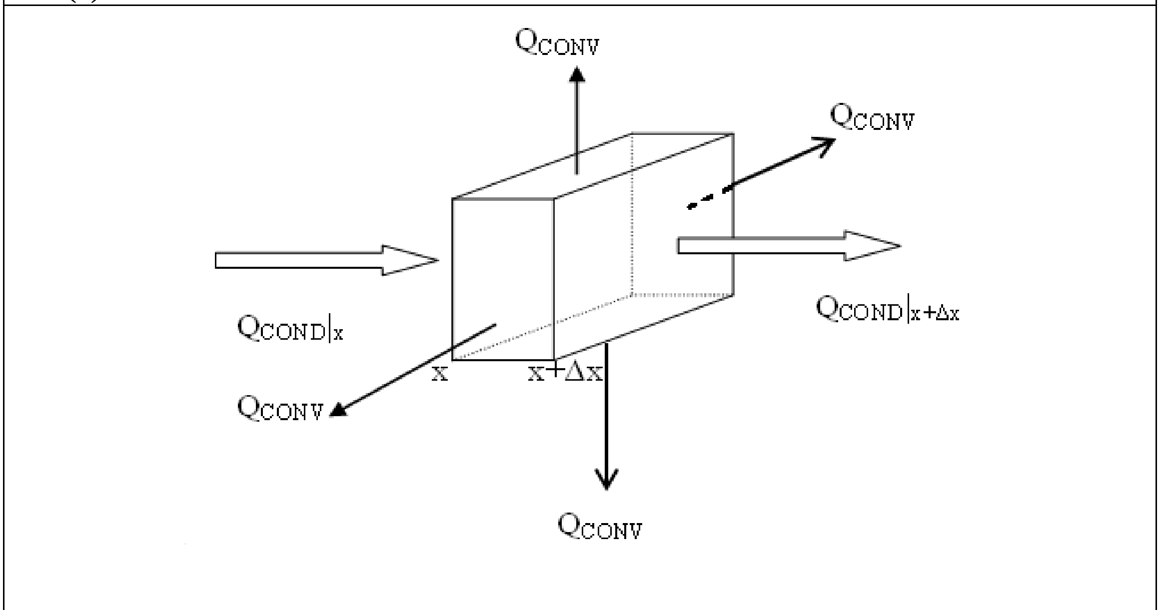
Permanent One-Dimensional Conduction

Fins with uniform cross section: As a simple illustration, consider a rectangular fin, shaped like a bar whose base is fixed to the surface of a wall at a temperature Twall [Figure 1(a)]. The fin is cooled along its surface by a fluid at temperature T_∞. The fin has a uniform cross-sectional area A, is made of a material with uniform conductivity k, and the combined heat transfer coefficient between the fin surface and the fluid is h. Admitting that the temperature in any cross section of the bar is uniform, that is, T=T(x) only, it is possible to derive an equation for the temperature distribution. In order to do that, we make a heat balance for a small fin element [Figure 1(b)]. Heat flows into the left side of the element by conduction, while heat flows out of the element in the x direction by conduction and also heat is lost by convection from the external surface of the fin element.



(a) Fin with a uniform cross section A



(b) Infinitesimal fin element, located away from the wall and fin tip.

Figure 1 - Fin with a uniform cross section A heated by a wall at temperature T_{wall}

A one-dimensional model for the fin is desired, so the necessary hypotheses are listed below:

1. The ambient temperature does not vary, since the amount of energy that goes into the environment is negligible in relation to large size of the environment;
2. The values of k and h are independent of temperature;
3. Edge effects are negligible;
4. The length/perimeter ratio >1, this fact indicates the validity of the consideration of one-dimensional flow;
5. Radiative heat transfer is negligible, as it involves temperatures not exceeding 300ºC;
6. There is no generation of heat, which could exist if a chemical reaction happened in the rod. There is no other form of energy present, such as electrical energy, transforming into heat;
7. There is no accumulation of energy. Since the focus of this study is on the steady-state process and not on time variation. It is desired to know how the process works after the initial transient phase, which will be the most significant portion of the operating time.

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Using steady-state development, with the fact that the accumulation term is equal to zero, the following equation is obtained after energy balance:

[1] $$0 = k.A_T.\frac{d^2T}{dx^2} - h.P.(T - T_{ar})$$

Dividing by $k.A_T$:

[2] $$\frac{d^2T}{dx^2} - \frac{hP}{kA_T}.(T - T_{ar}) = 0$$

Naming $m^2 = \dfrac{hP}{kA_T}$:

[3] $$\frac{d^2T}{dx^2} - m^2.(T - T_{ar}) = 0$$

Numerical Resolution [Steady Regime with convective boundary]

Using co-located mesh with 5 equally spaced volumes as shown in the figure below:

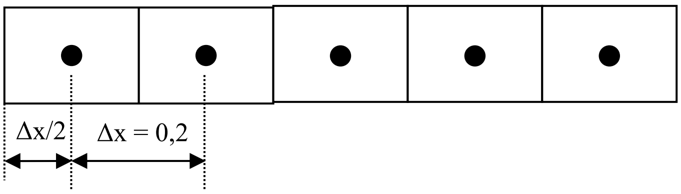

Figure 1.2 – Mesh for the one-dimensional problem considered.

Internal points

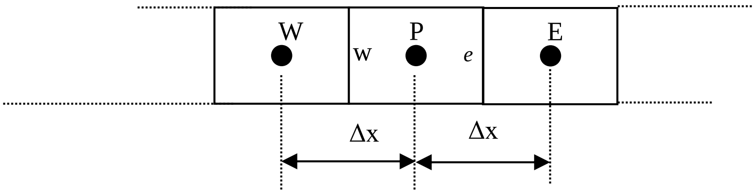

Figure 1.3 – Scheme for internal points.

Energy Balance for volume P, integrating equation [3]

[4] $$\int_w^e \left(.\frac{d^2T}{dx^2} \right) dx + \int_w^e \left(-\mathrm{m}^2.(T - T_{ar}) \right) dx = 0$$

We obtain:

[5] $$\left[\left(\frac{dT}{dx} \right)_e - \left(\frac{dT}{dx} \right)_w \right] - (\mathrm{m}^2).(T_p - T_{ar})\Delta x = 0$$

Using the Central Difference Scheme [CDS]:

[6] $$\left[\left(\frac{T_E - T_P}{\Delta x} \right) - \left(\frac{T_P - T_W}{\Delta x} \right) \right] - (\mathrm{m}^2).(T_P - T_{ar})\Delta x = 0$$

Grouping the terms:

[7] $$\left(\frac{1}{\Delta x} + \frac{1}{\Delta x} + \mathrm{m}^2 \Delta x \right) T_P = \frac{1}{\Delta x} T_E + \frac{1}{\Delta x} T_W + \mathrm{m}^2 T_{ar} \Delta x$$

In terms of coefficients:

[8] $A_P T_P = A_E T_E + A_W T_W + B$

where:

[9] $A_P = \dfrac{2}{\Delta x} + \mathrm{m}^2 \Delta x$

[10] $A_E = \dfrac{1}{\Delta x}$

[11] $A_W = \dfrac{1}{\Delta x}$

[12] $B = \mathrm{m}^2 T_{ar} \Delta x$

West boundary (W) (first control volume)

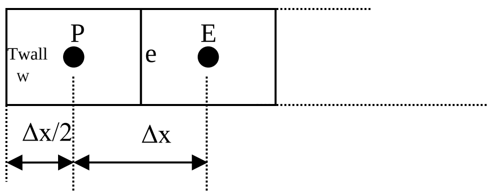

Figure 1.4 – Scheme for the west boundary

Energy Balance for the volume of the western boundary, integrating Equation [2]

[13] $$\left[\left(\frac{dT}{dx}\right)_e - \left(\frac{dT}{dx}\right)_w\right] - \left(\mathrm{m}^2\right)\left(T_p - T_{ar}\right)\Delta x = 0$$

Using the central differences interpolation scheme [CDS], we have:

[14] $$\left[\left(\frac{T_E - T_P}{\Delta x}\right) - \left(\frac{\left(T_P - T_{wall}\right)}{\Delta x / 2}\right)\right] - \left(\mathrm{m}^2\right)\left(T_P - T_{ar}\right)\Delta x = 0$$

rearranging:

[15] $$\left(\frac{1}{\Delta x} + \frac{2}{\Delta x} + \mathrm{m}^2 \Delta x\right) T_P = \frac{1}{\Delta x} T_E + \left[\mathrm{m}^2 T_{ar} \Delta x + \frac{2}{\Delta x} T_{wall}\right]$$

In terms of coefficients:

[16] $A_P T_P = A_E T_E + B$

where:

[17] $A_P = \dfrac{3}{\Delta x} + \mathrm{m}^2 \Delta x$

[18] $A_E = \dfrac{1}{\Delta x}$

[19] $B = \mathrm{m}^2 T_{ar} \Delta x + \dfrac{2}{\Delta x} T_{wall}$

East boundary (E) (last control volume)

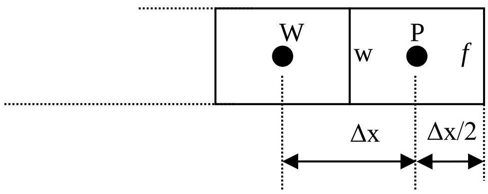

Figure 1.5 – Scheme for the East boundary

Energy Balance for the volume of the eastern boundary, integrating Equation [2]

[20] $\left[\left(\dfrac{dT}{dx}\right)_e - \left(\dfrac{dT}{dx}\right)_w\right] - (\mathrm{m}^2)(T_p - T_{ar})\Delta x = 0$

Applying the boundary condition for the convective boundary:

[21] $-k \left.\dfrac{dT}{dx}\right|_e = h(T_f - T_{ar})$

Using Central Difference Scheme (CDS):

[22] $k\left(\dfrac{2(T_f - T_P)}{\Delta x}\right) = h(T_{ar} - T_f)$

isolating T_f :

[23] $$T_f = \frac{\frac{2k}{\Delta x}T_p + hT_{ar}}{\frac{2k}{\Delta x} + h}$$

Substituting in Equation [21]:

[24] $$\left.\frac{dT}{dx}\right|_e = \frac{h}{k}\left(T_{ar} - \frac{\frac{2k}{\Delta x}T_p + hT_{ar}}{\frac{2k}{\Delta x} + h}\right)$$

Simplifying:

[25] $$\left.\frac{dT}{dx}\right|_e = \frac{h}{k}\left(\frac{1}{1+\frac{h\Delta x}{2k}}\right)\left(T_{ar} - T_p\right)$$

Substituting in Equation [19]:

[26] $$\frac{h}{k}\left(\frac{1}{1+\frac{h\Delta x}{2k}}\right)\left(T_{ar} - T_p\right) - \left(\frac{T_P - T_W}{\Delta x}\right) - \left(\mathrm{m}^2\right).\left(T_P - T_{ar}\right)\Delta x = 0$$

Grouping the terms:

[27] $$\left(\frac{h}{k}\left(\frac{1}{1+\frac{h\Delta x}{2k}}\right) + \frac{1}{\Delta x} + \mathrm{m}^2\Delta x\right)T_P = \frac{1}{\Delta x}T_W + \left[\mathrm{m}^2 T_{ar}\Delta x + \frac{h}{k}\left(\frac{1}{1+\frac{h\Delta x}{2k}}\right)T_{ar}\right]$$

In term of coefficients:

[28] $$A_P T_P = A_W T_W + B$$

where:

[29] $$A_P = \frac{h}{k}\left(\frac{1}{1+\frac{h\Delta x}{2k}}\right) + \frac{1}{\Delta x} + \mathrm{m}^2 \Delta x$$

[30] $$A_W = \frac{1}{\Delta x}$$

[31] $$B = \mathrm{m}^2 T_{ar} \Delta x + \frac{h}{k}\left(\frac{1}{1+\frac{h\Delta x}{2k}}\right) T_{ar}$$

As there are few points, the corresponding linear system of the problem under analysis can be presented:

	1	2	3	4	5		
1	[17]	- [18]				T_1	[19]
2	- [11]	[9]	- [10]			T_2	[12]
3		- [11]	[9]	- [10]		T_3	[12]
4			- [11]	[9]	- [10]	T_4	[12]
5				- [30]	[29]	T_5	[31]

Data:

$Tw = 20^o C$	$k = 100 \frac{W}{m.K}$	$L = 1m$
$T_{ar} = 20^o C$	$h = 10 \frac{W}{m^2 K}$	$P = 0,4m$
$A_T = 0,01\ m^2$		

It is asked:

1. Obtain the solution for the rod problem using either matlab, python, Excell, Excell VBA or any other software you are used to;
2. Solve for at least three different mesh sizes until results are mesh-independent;
3. Obtain the analytical solution;
4. Compare the analytical and numerical solutions. Compare the results especially at the center of the control volumes, where the results are calculated for the Finite Volume Method;
5. The report should have at maximum 5 pages, including figures. Do upload in Moodle both the report and the code you used, as a separate file. The report should be typed (nothing should be hand-written). Take especial care at the final

form of the report, since this will also be taken into consideration when correcting the reports. If you use Latex, send the tex file along with the images used.