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Résumé

Dans ce mémoire, on examine l'effet d'une variation de viscosité avec la vitesse de déformation dans une couche limite à deux dimensions. Pour des raisons mathématiques traitables, l'exemple que l'on étudie en détail est celui dans lequel le coéfficient de viscosité est une fonction linéaire de la vitesse de déformation et la vélocité du courant principal est en proportion du tiers de la puissance de distance mesurée d'un point de stagnation. On peut noter que quelques-uns des résultats qui se trouvent ici sont qualitativement les mêmes que ceux qui se trouvent quand on tient compte de la compressibilité du fluide.

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Energy Flows in a Vortex Tube

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In a recent paper [1][2]) the present writer considered the dynamics of the turbulent flow of a compressible fluid using an order-of-magnitude analysis. It is now proposed to extend this analysis to include energy fluxes in such vortex flows and, in particular, to give an explanation of the energy separation which is termed the Ranque effect.

1. The Ranque-Hilsch Vortex Tube

It will be remembered that the Ranque-Hilsch vortex tube is a device which is capable, although without moving parts, of dividing a homogeneous inflow of gas into a cooled stream and a heated stream (reference being in each case

[1]) Cavendish Laboratory.
[2]) Numbers in brackets refer to References, page 356.

to total temperature). The usual form of the device is as follows. The chief component is a tube of constant circular cross-section, its length several times its diameter. Near one end the tube wall is pierced to allow the entry of fluid (usually air), the inlet ports being so disposed that a swirling flow is set up within the tube. The fluid escapes at the ends of the vortex chamber, some through an orifice plate near the inlet, the rest through a second throttling device at the other end. Under certain operating conditions (notably, for reasonably high supply pressures and flow speeds) it is observed that the air emerging through the orifice near the inlet has a lowered total temperature and that the air leaving at the other exit has a higher temperature than the air supplied; temperature differences equivalent to several times the maximum dynamic head can be obtained.

The device and the effect it produces are associated with the names of two men: RANQUE, the French engineer who first fashioned it in 1931, and HILSCH, the German physicist who first studied its performance in detail and applied it to the liquefaction of gases in 1944. Very many other investigations of the device have been made since their early work. WESTLEY [2] has given a bibliography of work before 1952. Since then further performance data have been published by OTTEN, SPRENGER, and WESTLEY; results of internal flow measurements by HARTNETT and ECKERT, LAY, SCHELLER and BROWN; and theoretical treatments by DEISSLER and PERLMUTTER, HINZE, LAY, PENGELLEY, and ROTT ([3] to [12]). Several other studies have been made in this period, but they are less important or less easily accessible.

2. Qualitative Theories of the Ranque Effect

Early discussions of the energy separation within the vortex tube considered inviscid, non-conducting fluids: the first theory put forward by RANQUE was of this type. However, it soon became apparent that transport properties must be considered. A later theory of RANQUE, supported also by HILSCH, was that the outwards migration of energy was due to radial work transfer. They pictured a 'free' vortex being converted by viscous shearing forces into a 'forced' vortex and a consequent outwards transfer of kinetic energy. FULTON [13] carried forward these ideas of vortex conversion by noting that inwards conduction of heat would counteract the outwards work flux. He found that a net outwards flux occurred in laminar flow but estimated that it was not of sufficient magnitude to produce the observed energy separation.

It thus appeared that turbulent mixing was a more likely agent of energy transfer. This conclusion still seems valid today: it is supported by the recent calculations of DEISSLER and PERLMUTTER [9].

At first the role of turbulence was thought to be qualitatively the same as that of molecular transport, although giving rise to very much greater energy fluxes. The first discussion in which a contrary view was taken is that of KNOERNSCHILD [14]. He pointed out that a compressible fluid lump moving rapidly through a pressure gradient must experience temperature variations on expanding or contracting. Thus the temperature distribution will move towards that related to the pressure gradient by an adiabatic law. At present, KNOERNSCHILD'S concept is the most widely favoured explanation of the Ranque effect, although a work flux contribution is also thought to be important by some workers.

The ideas just discussed form the main stream of thought concerning the Ranque effect. But several suggestions have been made which do not fit into this evolutionary pattern. The most challenging is that put forward by SPRENGER [4], but seemingly due originally to ACKERET, that organized unsteadiness (as distinct from turbulence) might produce the energy separation. This hypothesis is based on analogy with resonance tube phenomena. The present writer has already put forward his reasons for feeling that this mechanism offers a less satisfactory explanation of the Ranque effect [15].

Summarizing: it seems probable that the energy separation of the vortex tube is the result of turbulent activity within the vortex chamber. Our task is to discover what are the mechanisms of turbulence by which the energy fluxes are set up.

3. The Energy Equation

If we neglect molecular transport properties and body forces, the energy equation can be written

$$\frac{\partial}{\partial t}(\varrho\, h_0) + \nabla\cdot(\varrho\, h_0\, \boldsymbol{u}) = \frac{\partial p}{\partial t},$$

where $h_0 = h + \boldsymbol{u}^2/2$ is the total enthalpy of the fluid, the sum of the enthalpy and the kinetic energy per unit mass. Taking mean values with respect to time and requiring that the mean motion be steady, we have

$$\nabla\cdot(\overline{\varrho\, h_0\, \boldsymbol{u}}) = 0$$

which gives

$$\frac{1}{r}\,\frac{\partial}{\partial r}(r\,\overline{\varrho\, u\, h_0}) + \frac{\partial}{\partial z}(\overline{\varrho\, w\, h_0}) = 0$$

for axisymmetric mean flow. We have

$$\overline{\varrho\, u\, h_0} = \overline{(R+\varrho')\,(U+u')\,(H_0+h_0')}$$

and

$$H_0 = H + \frac{1}{2}\,[U^2 + V^2 + W^2 + \overline{u'^2} + \overline{v'^2} + \overline{w'^2}]\,,$$

$$h_0' = h' + u'\,U + v'\,V + w'\,W\,.$$

4. Order-of-Magnitude Analysis

We shall specialize the energy equation given above by introducing a system of approximation suitable for vortex tube flows. The scheme to be used is that set forth in Section 3 of the previous paper on the dynamics of vortex flows:

$$V \sim O(1)\,, \quad U \sim O(\varepsilon)\,; \quad \frac{\partial}{\partial r} \sim O(1)\,, \quad \frac{\partial}{\partial z} \sim O(\delta)\,.$$

As before, we specify the turbulence by

$$\overline{u'^2},\ \overline{v'^2},\ \overline{w'^2} \sim O(\beta)\,, \quad \varrho' \sim O(\eta)\,,$$

adding here $h' \sim O(\alpha)$.

We choose units such that r, $V \sim O(1)$, knowing from the previous analysis that $R \sim O(1)$ for consistency. We take also $\Delta H_0 \sim O(1)$, knowing that energy separations are typically a few times the dynamic head. This assumption will be seen to be consistent with the others a posteriori. It will be assumed, of course, that $\alpha, \beta, \delta, \varepsilon, \eta$ are all small compared to unity so that terms of higher orders in them can be neglected.

Introducing these approximations into the appropriate form of the continuity equation, we find that

$$W \sim O\left(\frac{\varepsilon}{\delta}\right),$$

at most, so that

$$\underset{O(\varepsilon)}{\frac{1}{r}\,\frac{\partial}{\partial r}(r\,R\,U)} + \underset{O(\varepsilon)}{\frac{\partial}{\partial z}(R\,W)} + \underset{O(\beta^{1/2}\,\eta)}{\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\overline{\varrho'\,\mu'}\right)} = O(\beta^{1/2}\,\delta\,\eta)$$

is the approximate continuity relation for this case.

Similarly, the approximate energy equation is found to be

$$\underset{O(\varepsilon)}{\frac{1}{r}\,\frac{\partial}{\partial r}(r\,R\,U\,H_0)} + \underset{O(\varepsilon)}{\frac{\partial}{\partial z}(R\,W\,H_0)} + \underset{O(\beta^{1/2}\eta)}{\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,H_0\,\overline{\varrho'\,u'}\right)}$$

$$+ \underset{O(\alpha\,\beta^{1/2})}{\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,R\,\overline{u'\,h'}\right)} + \underset{O(\beta)}{\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,R\,V\,\overline{u'\,v'}\right)} + \underset{O(\beta\,\varepsilon/\delta)}{\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,R\,W\,\overline{u'\,w'}\right)}$$

$$= O(\beta\,\delta,\ \beta\,\varepsilon,\ \alpha\,\beta^{1/2}\,\delta)\,.$$

Only the leading error terms have been indicated.

This result relates the several important energy fluxes into an elemental volume. It can be cast into another form without altering the orders of the errors if we use the continuity equation given above. Thus

$$R \frac{DH_0}{Dt} = -\frac{1}{r} \frac{\partial}{\partial r}(r R \overline{u' h'}) - \overline{\varrho' u'} \frac{\partial H_0}{\partial r}$$
$$- \frac{1}{r} \frac{\partial}{\partial r}(r V \overline{u' v'}) - \frac{1}{r} \frac{\partial}{\partial r}(r W R \overline{u' w'})$$

where D/Dt indicates the rate-of-change following the mean motion. Here we have a relation among the radial energy fluxes and the energy input to a lump of fluid moving through the system.

5. The Important Energy Fluxes

The simplified forms of the energy equation suggest that the significant energy movements within the vortex are due to four energy fluxes:

a) The heat flux $R \overline{u' h'}$ is produced by turbulent mixing through the radial temperature and pressure gradients. This is the contribution whose full significance was pointed out by KNOERNSCHILD.

b) The flux of total energy $\overline{\varrho' u'} H_0$ is associated with the 'Archimedean' correlation which was discussed briefly in connection with vortex dynamics.

c) The work fluxes $R V \overline{u' v'}$ and $R W \overline{u' w'}$ associated with the two important Reynolds stresses, those acting on cylindrical surfaces.

A rough estimate of the relative magnitudes of these contributions can be made using data from extensive tests on vortex tubes. These data were used in a similar manner in the previous analysis of flow dynamics [1]. The vortex tube within which the flow was investigated was about 7·5 cm in diameter and about 120 cm long. Total and static pressure and temperature traverses were made at several stations down the tube with various blockages at the ends. In [1] it is found that the results suggest that we may represent a typical internal flow by taking

$$\delta \sim \frac{1}{50}, \quad \varepsilon \sim \frac{1}{200}, \quad \beta \sim \frac{1}{100}, \quad \eta \sim \frac{1}{100}.$$

Thus we estimate the terms of the energy equation obtained above as

$$\underset{O(1/200)}{\frac{1}{r} \frac{\partial}{\partial r}(r R U H_0)} + \underset{O(1/200)}{\frac{\partial}{\partial z}(R W H_0)} + \underset{O(1/1000)}{\frac{1}{r} \frac{\partial}{\partial r}(r H_0 \overline{\varrho' u'})}$$
$$+ \underset{O(\alpha/10)}{\frac{1}{r} \frac{\partial}{\partial r}(r R \overline{u' h'})} + \underset{O(1/100)}{\frac{1}{r} \frac{\partial}{\partial r}(r R V \overline{u' v'})} + \underset{O(1/400)}{\frac{1}{r} \frac{\partial}{\partial r}(r R W \overline{u' w'})}$$
$$= O\left(\frac{1}{5000}, \frac{\alpha}{500}\right).$$

We note that the Archimedean effect seems to be somewhat less important that the other contributions. The work fluxes appear to be of the same order as the heating/cooling effects (the first two terms). There is no reason to suppose that the energy flux due to the Knoernschild effect is more important than that produced by the radial work fluxes. The quantity α will have to be $\sim 1/20$ if the thermal convection term is to equal the work flux in importance.

We next consider the physical processes corresponding to these important terms of the energy equation. In this way it will be possible to obtain a better idea of the relative importance of the several contributions to the radial energy flow. We adopt now the so-called Lagrangian point of view and attempt to build up idealized models of the internal motions of the turbulent fluid in order to see how the energy flows are influenced by mean-value gradients in the vortex.

6. The Knoernschild Effect

KNOERNSCHILD observed [14] that it is not correct to think of the turbulent mixing of a compressible fluid across a pressure gradient in terms of the simple model used for an incompressible fluid – a fluid lump shifted unchanged to a new position and there mixing with the ambient fluid. A lump of compressible fluid will not be transported through a pressure gradient with all its properties unchanged and without interactions with the fluid through which it moves. A work interaction with the surroundings will occur during such a movement, the particle being compressed on moving radially outwards to a region of higher pressure or expanding on moving inwards. Thus the work interaction with the matrix of fluid through which mixing takes place results in the cooling of the inwards-moving particles and heating of those moving outwards.

It has generally been assumed that the convecting particles do not interact with their surroundings except by expanding and contracting. That is, it is held that they are very nearly adiabatic while in motion. DEISSLER and PERLMUTTER [9] mention work supporting this idea. In any case, the utility of the theory does not depend on strictly adiabatic lump movements. The most recent derivation of the expression for $\overline{u' h'}$ in terms of mean-flow gradients, that of HINZE [10], takes account of non-adiabatic lump motions[3]).

Assuming the mixing fluid to be a perfect gas, HINZE gets a result equivalent to

$$\overline{u' h'} = -\varepsilon_H \left(\frac{\partial H}{\partial r} - \frac{\gamma - 1}{\gamma} \frac{H}{P} \frac{\partial P}{\partial r}\right).$$

[3]) It should be noted that we are able in this way to take account of the effect of molecular transport properties on the small-scale motions of the turbulence, even though viscosity and thermal conductivity were neglected in the mean-value equations with which the analysis was begun.

For this case we have simply $H = c_p T$, where T is the absolute temperature. Here ε_H is a positive coefficient expressing the activity of the turbulence – an eddy-diffusion coefficient. This result shows clearly the tendency of the radial distribution to move towards that defined by an adiabatic law applied to the pressure dustribution.

Using the radial equilibrium condition:

$$\frac{\partial P}{\partial r} = R \frac{V^2}{r} + O(\beta)$$

an alternative form can be given.

$$\overline{u' h'} = -\varepsilon_H \left(\frac{\partial H}{\partial r} - \frac{V^2}{r} \right).$$

The simple dependence of the radial flux on the local centripetal acceleration is thus made clear. This result is valid only for a perfect gas, even though the nature of the fluid does not enter explicitly.

When the radial temperature gradient is less steep than that defined by the adiabatic law (as was the case in the vortex tube flows investigated) the heat flux due to this effect is everywhere outwards. The influence of this term may be expected to be strongest in the core of the vortex where centripetal accelerations are highest.

7. Archimedean Effects

So far the influence of the heat flux, $R\,\overline{u' h'}$, on the overall energy transfer has appeared straightforward. But YUDINE, in the discussion of HINZE's paper on energy transfer in the atmosphere [10], pointed out that his treatment is not quite complete. In addition to the vertical transfer due to non-selective mixing by turbulent stirring (that considered by HINZE), there exists in the atmosphere an upwards heat flux set up by Archimedean forces. A counterpart of this phenomenon must be expected in vortex flows. A buoyancy effect ($\overline{\varrho' u'}$) did emerge in the simplified energy equation for vortex flow. But this term does not wholly account for the effects of Archimedean forces. This can be seen as follows. It is apparent that a particle lighter than its surroundings will tend inwards under the action of buoyancy forces. But a deficiency in density can be thought of as resulting either from a lower than ambient pressure or from a higher than ambient temperature. The first of these would seem to contribute primarily to the energy flux associated with the Archimedean correlation function, $\overline{\varrho' u'}$: its discussion will be deferred. The second possibility, low density corresponding to a temperature above ambient, is of immediate interest, for it contributes directly to the correlation function $\overline{u' h'}$ (as well as to $\overline{\varrho' u'}$).

The general effect of this contribution to the heat flux in a vortex is not hard to see. Warmer lumps will tend inwards so that an additional heat flux is set up, one which exists even if the mean-value temperature distribution is adiabatically related to the pressure distribution and which is inwards in all circumstances. The estimation of this buoyancy contribution in comparison with that suggested by KNOERNSCHILD and studied by HINZE and others is not easy. We shall not study this difficult problem in detail here: PRIESTLEY and SWINBANK [16] have discussed the analogous atmospheric case. But it is impossible to leave this matter without reaching some decision, however arbitrary. We shall then assume that this Archimedean heat transfer is masked by the Knoernschild effect so that the heat flux associated with the correlation function $\overline{u' h'}$ is outwards almost everywhere in the vortex. Support for this assumption is given by the following considerations. First, the Archimedean contribution to the heat flux can be of decisive importance only when the temperature distribution is near that prescribed by the adiabatic law applied to the pressure distribution. And, in the vortex tube flows studied, this was seldom the case. Second, it seems evident on considering the temperature distributions found in the Ranque-Hilsch tube that the outwards fluxes near the core do, in fact, overpower this additional, opposed heat flow. Lastly, the effect is of secondary importance in the analogous atmospheric situation.

We have tried to separate out three influences of compressibility on energy transfer in a vortex flow – the Knoernschild and two Archimedean effects. The separation is a somewhat arbitrary one: each of the effects must in reality interact with the others. This approach does display some of the possible phenomena even if it does not clearly define the relative importance of the contributions.

We now turn to the buoyancy effect which did emerge explicitly from the energy equation. As this contribution has not been dealt with previously, it will be given fairly detailed consideration here. Later an attempt will be made to relate the Archimedean correlation function, $\overline{\varrho' u'}$, to the mean-flow gradients. For simplicity, we restrict consideration at that point to density perturbations induced by non-uniform pressures in the turbulent field. The methods used can be applied to other models.

As was pointed out in the discussion of flow dynamics, the correlation function $\overline{\varrho' u'}$ may be expected to be positive in a vortex in a compressible fluid: lighter lumps of fluid will tend inwards under the action of buoyancy forces. Then the associated energy flux, $\overline{\varrho' u'}\, H_0$, will be everywhere radially outwards. Surprisingly, the overall effect of this outwards flux, as indicated by the term $-\overline{\varrho' u'}\, \partial H_0/\partial r$, seems to be to cool the fluid throughout the tube (since, except very near the tube wall, $\partial H_0/dr > 0$ in almost all of the situations investigated.)

This conundrum can be resolved as follows: the overall effect of the energy flux is to redistribute energy in the various *regions* of the tube; but, when continuity requirements are taken into account, this results in a reduction in the energy per unit mass of the whole of the *moving fluid*. Consider a simple example: purely radial flow. Continuity is

$$\frac{\partial}{\partial r}(r \overline{\varrho u}) = 0;$$

whence, for zero net source flow, we get

$$R\,U = -\overline{\varrho' u'}\,.$$

We shall take $\overline{\varrho' u'}$ to be positive. Then the requirement of continuity is an inwards mean velocity, $U = -\overline{\varrho' u'}/R$. The energy equation derived from continuity is

$$R\,U\,\frac{\partial H_0}{\partial r} = R\,\frac{\partial H_0}{Dt} = -\overline{\varrho' u'}\,\frac{\partial H_0}{\partial r}\,.$$

Thus we see than the rate-of-change following the mean motion is, in fact, the rate-of-change as the fluid moves inwards in the direction opposite to the energy flux.

We shall next attempt to relate the correlation function $\overline{\varrho' u'}$ to the mean flow as has already been done for the function $\overline{u' h'}$. Let us clarify the situation by relating the turbulent velocity fluctuation at a point to the perturbation velocity of a lump at that point. We write $u' = u_1' + u_2'$ where u' is the deviation from the time-mean value at a point; u_1' is the velocity of the coincident particle due to the Archimedean buoyancy force per unit volume, $\varrho'\,V^2/r$; and u_2' is the part of the local motion induced by the surrounding turbulence. The term u_1' is random with respect to the fixed point but not so far as the passing eddy is concerned. Then $\overline{\varrho' u'} = \overline{\varrho' u_1'}$.

Our next task is to relate the velocity u_1' to the buoyancy force producing it. It seems plausible to compare the inwards-moving lighter lumps to microscopic particles whose net drift through a gas is superposed upon Brownian motion. In the present case, the turbulence of the fluid supplies the random disturbance to the drift due to the Archimedean force. In his theory of Brownian motion EINSTEIN [17] has assumed that the drift speed is linearly related to the force on a particle, in fact, that the drift motion obeys STOKES' law. The results of his theory have been verified experimentally. Similarly in electricity: a linear resistance law – OHM's law – is found to apply when the electrons' drift speed is small compared to the speed characterizing the random thermal motion.

By analogy with STOKES' law we relate a typical density fluctuation and a typical drift speed by

$$\varrho'\,\frac{V^2}{r} \propto R\,\varepsilon_M\,u_1'$$

where $R\,\varepsilon_M$ replaces the molecular viscosity, ε_M being the eddy viscosity. The constant of proportionality is dependent on the scale of the turbulence.

Introducing once again (as in [1]) the hypothesis that density fluctuations at a point are the result of the convection past that point of eddies in which density gradients are maintained by pressure gradients holding the eddies together, we obtain

$$p' \propto R\,\overline{u'^2}; \qquad \frac{\varrho'}{p'} \simeq K;$$

where K is the compressibility of the fluid. Then

$$\overline{\varrho'\,u'} = \varepsilon_A\,R\,K^2\,\frac{V^2}{r}$$

where ε_A is a positive coefficient dependent on the kinematic aspects of the turbulence.

Alternatively, introducing the radial equilibrium condition, we get

$$\overline{\varrho'\,u'} = \varepsilon_A\,K^2\,\frac{\partial P}{\partial r}$$

for the Archimedean correlation function.

8. Work Flux due to Tangential Stresses

To investigate the work fluxes we need not consider the details of the turbulent motion. The previous investigation of the dynamics of vortex tube flows provides the means for the study of the work transfers. It was found that tangential momentum conservation could be represented with reasonable accuracy by

$$R\,U\left(\frac{\partial V}{\partial r} + \frac{V}{r}\right) + R\,W\,\frac{\partial V}{\partial z} = -\frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2\,R\,\overline{u'\,v'})\,.$$

In its present form this result does not easily yield information about the stress $R\,\overline{u'\,v'}$ which gives rise to the work flux of interest here. An obvious simplification is the restriction to swirl distributions independent of axial position. Fortunately, this idealization has close counterparts in the vortex flows studied experimentally; and in particular it applies to the configurations for which the Ranque effect was especially large. The conclusions reached for this class will thus have some bearing on the problem of present interest. Subject to this restriction we have

$$(r\,R\,U)\,\frac{\partial}{\partial r}(r\,V) = -\frac{\partial}{\partial r}(r^2\,R\,\overline{u'\,v'})\,.$$

We shall study this relationship by considering simple examples, beginning with the case of constant radial flow: $r\,R\,U = c$. For this case the torque is found to be

$$r^2\,R\,\overline{u'\,v'} = -c\,r\,V$$

on requiring that both velocity and mixing stress be zero on the axis. The work flux is

$$r\,V\,R\,\overline{u'\,v'} = -c\,V^2\,,$$

inwards for $c > 0$, outwards for $c < 0$; that is, opposed in direction to the radial mass flow. The source term is

$$-\frac{1}{r}\,\frac{\partial}{\partial r}(r\,V\,R\,\overline{u'\,v'}) = \frac{c}{r}\,\frac{\partial\,V^2}{\partial r}\,.$$

Local heating occurs for inflow combined with swirl decreasing outwards and for outflow with swirl increasing outwards. The other combinations result in cooling. Practically, this means that radial inflow will result in the cooling of fluid in the core and the heating of that further out.

Having considered the effect of swirl on the work flux for a particular radial flow, we now take the other point of view. We take $V = k\,r^{-1/2}$ and consider a variety of radial flows. Swirl distributions approximating to this form are commonly found in actual vortex flows; this was the case in the series of tests referred to here, and the agreement was particularly good for the configurations which exhibit the Ranque effect strongly. In any case, the qualitative results obtained for this particular swirl distribution are more generally applicable; a specialized form is chosen for simplicity. We table the results for some simple examples:

	Radial Flow	*Work Flux*	*Source Term*
	$r\,R\,U$	$r\,V\,R\,\overline{u'\,v'}$	$-\frac{1}{r}\,\frac{\partial}{\partial r}(r\,V\,R\,\overline{u'\,v'})$
(1)	$c\,r^2$	$-\frac{c\,k^2}{5}\,r$	$\frac{c\,k^2}{5\,r}$
(2)	$c\,r$	$-\frac{c\,k^2}{3}$	0
(3)	c	$-\frac{c\,k^2}{r}$	$-\frac{c\,k^2}{r^3}$
(4)	$c\,(a - r)\,,\quad r \leqq a$	$-c\,k^2\left(\frac{a}{r} - \frac{1}{3}\right)$	$-\frac{a\,c\,k^2}{r^3}$
(5)	$c\,r^2\,(a - r)\,,\quad r \leqq a$	$-c\,k^2\left(\frac{a}{5} - \frac{r}{7}\right)r$	$-c\,k^2\left(\frac{2}{7} - \frac{a}{5\,r}\right)$

Case (1) models flow in the core; cases (2) and (3), that in the body of the flow; case (4), that near the tube wall. Case (5) covers the whole radius from axis to wall. Again we have chosen solutions such that the torque approaches zero in the core.

These examples follow the pattern that emerged on considering constant radial mass flow. Similar results are found for the more general swirl distribution $V = k\,r^n$ where $n > -1$. Stability considerations suggest that cases for which $n < -1$ are not of physical interest. They were not found in experimental situations in the present study.

We conclude that the radial work flux is usually in the direction opposed to the radial mass flux. But in the Ranque-Hilsch vortex tube the gas is injected at the periphery and escapes through exits nearer the axis. Then radial flow must be predominantly inwards. We can expect that the core will commonly be cooled and the peripheral fluid heated through the working of the tangential stresses.

9. Work Flux due to Axial Stresses

The effect of the other important Reynolds' stress is not simple. The approximate equation governing axial momentum has been shown [1] to be

$$\frac{1}{r}\,\frac{\partial}{\partial r}(r\,R\,\overline{u'\,w'}) = -\frac{\partial P}{\partial z}\,.$$

Requiring that the stress be bounded at the axis, we find the work flux to be

$$r\,W\,R\,\overline{u'\,w'} = -W\int\limits_0^r r\,\frac{\partial P}{\partial z}\,dr\,.$$

In order to demonstrate the properties that this function can be expected to possess typically, we consider a simple example:

$$W = -\frac{\partial P}{\partial z} = r - 1 \quad \text{for} \quad r \leqq 3\,,$$

incorporating the property that W and $\partial P/\partial z$ are usually of opposite sign in the vortex flows studied. The work flux is

$$\begin{aligned} r\,R\,W\,\overline{u'\,w'} &= \frac{1}{3}\,r^2\,(r-1)\left(r-\frac{3}{2}\right) \\ &> 0 \quad \text{for} \quad r < 1\,,\quad r > \frac{3}{2}\,, \\ &< 0 \quad \text{for} \quad \frac{3}{2} > r > 1\,. \end{aligned}$$

This example illustrates the trends of more complex cases. Usually W and $-\int\limits_0^r r\,(\partial P/\partial z)\,dr$ have the same sign in an outer annulus, indicating an outwards flux there. Most of the fluid in this outer annulus will be cooled; only that very near the wall will be heated. A complex flow pattern will be found in the core; it will not likely play an important part in overall energy separation.

10. The Efficient Mechanisms of Energy Redistribution

From the results of the four previous sections we conclude that the overall effect of each of the four important energy fluxes will be to transfer energy from the fluid in the core of the vortex to the fluid of the periphery. This agrees qualitatively with the behaviour of the Ranque-Hilsch vortex tube from which cooled fluid is withdrawn from the core at the inlet end of the tube. It was shown in Section 5 that the energy transfer mechanisms of turbulence are of the same order of magnitude as the observed energy redistributions.

We have already seen that it is to the thermal flux $R\,\overline{u'\,h'}$ and the work fluxes that we must look for important energy movements. The consideration of the details of the physical processes giving rise to these suggests the following conclusions regarding their relative importance. The efficient mechanism of energy redistribution in the core is thermal transfer by turbulent mixing across the large radial pressure gradients found there. The further separation in the outer fluid is primarily due to work fluxes associated with turbulent mixing stresses.

The analysis presented here gives a theoretical framework for the most plausible of the qualitative ideas on the Ranque effect that have been developed over the past thirty years. It is realized, of course, that this treatment contains such radical simplifying assumptions that it may accurately be described as crude. However, most of the assumptions have at least empirical support; many were, indeed, suggested by consideration of experimental results. It is felt that the conclusions reached should be applicable to the particular vortex tube configuration investigated by the present writer and probably for most of the configurations considered in previous studies of the Ranque effect.

Symbols

r, z	radial and axial co-ordinates;
u, v, w	radial, tangential and axial velocity components (U, u', etc. are time-mean values and perturbations to them);
ϱ, p, h, h_0	fluid density, pressure, enthalpy, and total enthalpy (R, ϱ', etc. are time-mean values and perturbations to them);
$\alpha, \beta, \delta, \varepsilon, \eta$	quantities small with respect to unity.

Abstract

The energy separation within a vortex tube filled with turbulent compressible fluid is investigated with an order-of-magnitude analysis of the energy equation. The physical processes corresponding to the important terms are: a heat flux due to turbulent mixing of the compressible fluid through radial pressure and temperature gradients, a flux of total energy produced by Archimedean forces, and work fluxes associated with the two most important Reynolds' stresses. All these fluxes will commonly be outwards and will tend to cool the vortex core. Experimental results are used to estimate the relative magnitudes of the contributions. The Archimedean effect seems to be the least important.

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Zusammenfassung

Die Energieverteilung in einem mit turbulenter kompressibler Flüssigkeit gefüllten Wirbelrohr wird durch Berechnung der Gliedergrösse der Energiegleichung untersucht. Die physikalischen Vorgänge, die den wichtigen Gliedern entsprechen, sind folgende: ein Wärmefluss, veranlasst durch turbulente Mischung der kompressiblen Flüssigkeit infolge der radialen Druck- und Temperaturgradienten; ein Fluss des Gesamtwärmeinhaltes, verursacht durch die Schwimmkraft; und die Arbeitsflüsse, die von den beiden wichtigsten Reynoldsschen scheinbaren Spannungen abhängen. Alle diese Flüsse sind gewöhnlich nach aussen gerichtet und dienen zum Kühlen des Wirbelkerns. Die experimentellen Ergebnisse werden zu einer Abschätzung der Grösse dieser Beiträge gebraucht. Der Schwimmkrafteffekt scheint am wenigsten wichtig.

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Note on the Large Deflection of a Circular Plate under a Concentrated Load

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Introduction

For the large deflection of a plate we usually get non-linear equations which cannot be exactly solved. BERGER [1][2]) has shown that if, in deriving the differential equations from strain energy, the strain energy due to second strain invariant in the middle plane of the plate is neglected, a simple fourth order differential equation coupled with a non-linear second order equation is obtained. He has solved these differential equations for the problem of circular plates under various boundary conditions subjected to *normal uniform load* throughout the plate. Though his approximation lacks sufficient technical interpretation for its justification, he has shown that his results for deflection, displacement, and stresses tally with the known results for all practical purposes. In this note following BERGER'S method, an attempt has been made to obtain the deflection for a circular plate under a *concentrated load* at the centre.

1. In the case of circular symmetry if h is the thickness of the plate, w the displacement perpendicular to its middle plane, u the radial displacement in the

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[2]) Numbers in brackets refer to References, page 362.