ECE 550:
Fundamentals of Computer Systems and Engineering

Digital Arithmetic

Admin

- Homework
  - Homework 1
- Reading:
  - Chapter 3

Last Time in ECE 550….

- Who can remind us what we talked about last time?

Last Time in ECE 550….

- Who can remind us what we talked about last time?
  - Numbers
    - One hot
    - Binary
    - Hex
  - Digital Logic
    - Sum of products
    - Encoders
    - Decoders

Implementing Addition

- First, one bit addition.
  - Three inputs: Carry In (CI), A, B
  - Two outputs Carry Out (CO), Sum (S)
- Go around room for truth table:

CI	A	B	S	CO
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

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1	1	1	1	1

Half Adder

- Ignore CI for a second (assume is 0)
  - Can simplify a lot and build “half adder”
    - Formula for S?
    - Formula for CO?

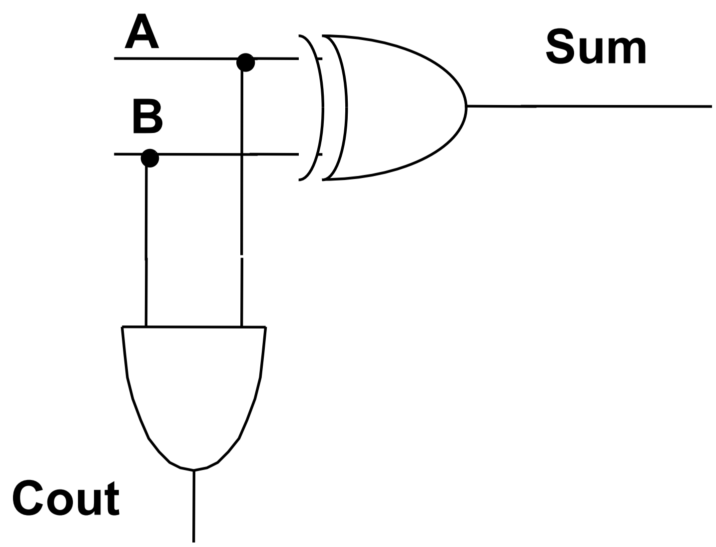
CI	A	B	S	CO
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Half Adder

- Ignore CI for a second (assume is 0)
  - Can simplify a lot and build "half adder"
    - Formula for S? A xor B
    - Formula for CO? A and B

CI	A	B	S	CO
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Half Adder



- Half adder:
- 1 XOR and 1 AND
- Can anyone guess why its called a half adder?

Implementing Addition

- Re-visit Truth table, but..
  - Use Half-Sum and Half-CO (results of Half-Adder)
- Go around room for truth table:

CI	Half-Sum	Half-CO	S	CO
0	0	0		
0	0	1		
0	1	0		
0	1	1		
1	0	0		
1	0	1		
1	1	0		
1	1	1		

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0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	0	1
1	1	1	!!!	!!!

Implementing Addition

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1	1	0	0	1
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Implementing Addition

- Formulas:
  - Sum?
  - CO?

CI	Half-Sum	Half-CO	S	CO
0	0	0	0	0
0	0	1	0	1
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
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Implementing Addition

- Formulas:
  - Sum? CI xor Half-Sum
  - CO? (CI and Half-Sum) OR Half-CO

CI	Half-Sum	Half-CO	S	CO
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0	0	1	0	1
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	0	1
1	1	1	0	1

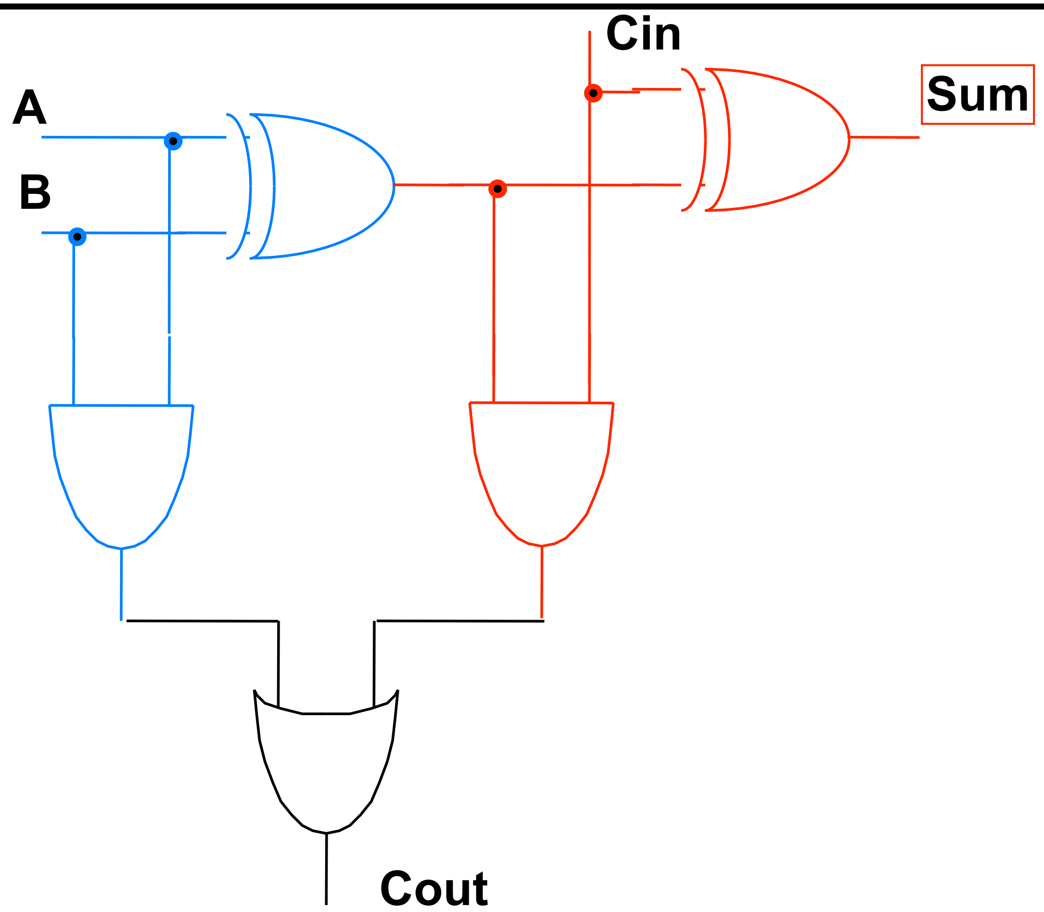
Implementing Addition

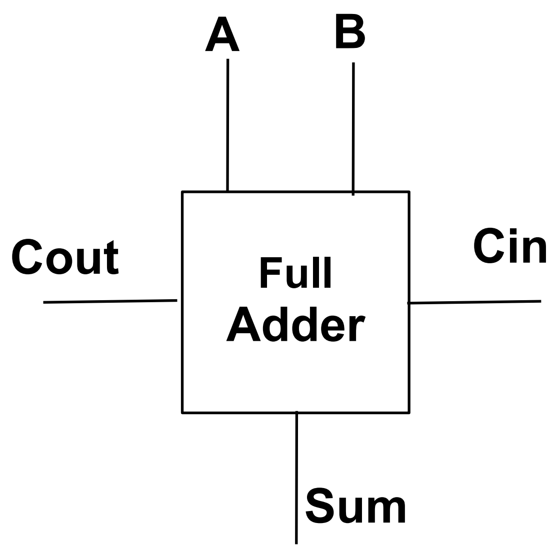
- Formulas:
  - Sum? CI xor Half-Sum
  - CO? (CI and Half-Sum) OR Half-CO

What does this look like?

CI	Half-Sum	Half-CO	S	CO
0	0	0	0	0
0	0	1	0	1
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	0	1
1	1	1	0	1

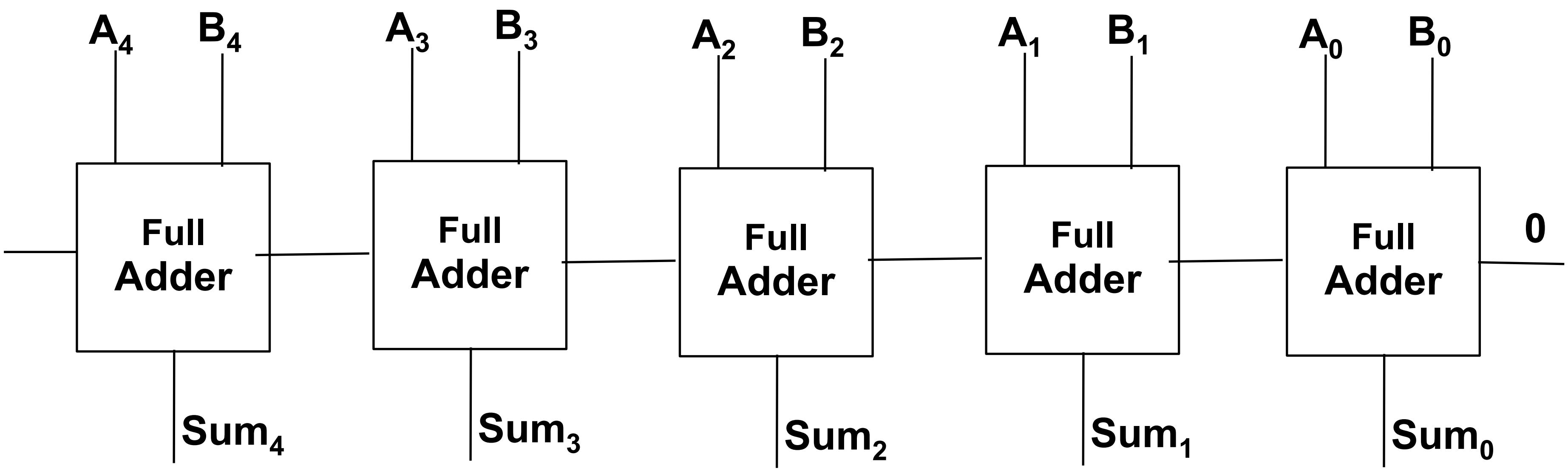
Full Adder





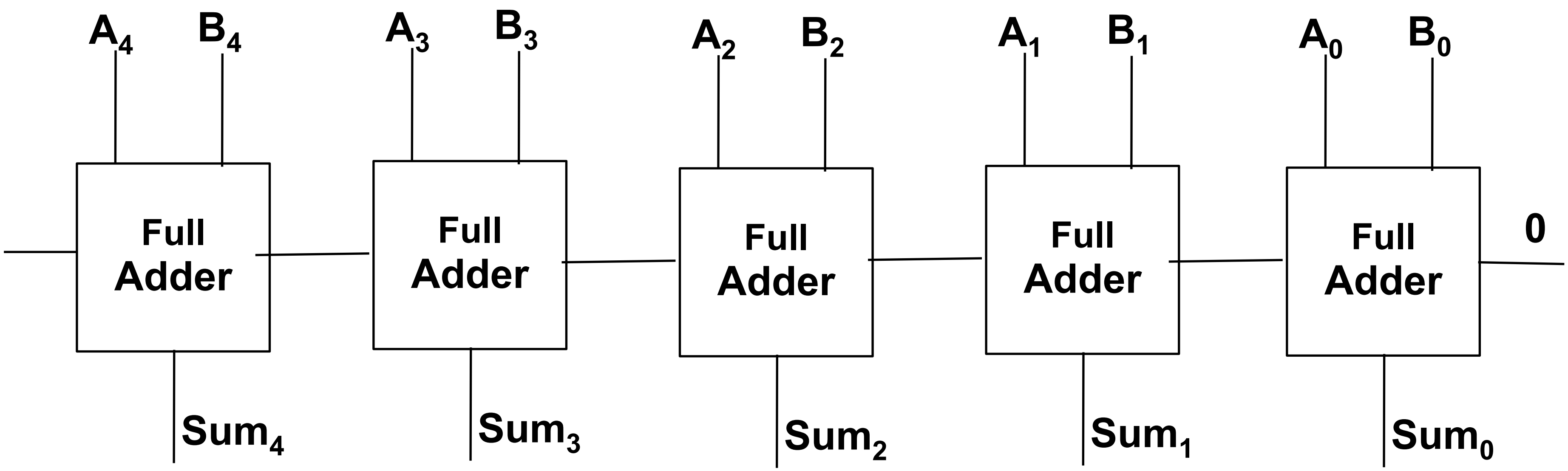
- Full Adder
- 2 Half Adders + an OR Gate

Ripple Carry



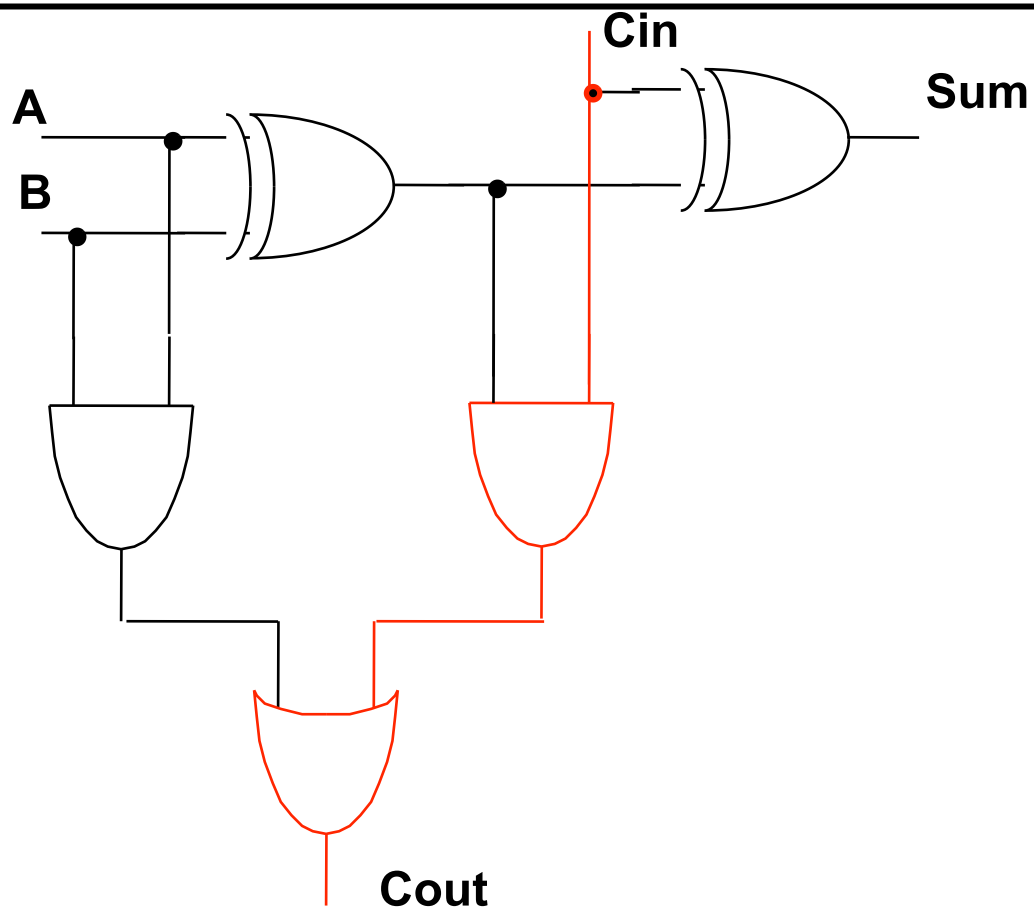
- Full Adder = Add 1 Bit
  - Can chain together to add many bits
  - Upside: Simple
  - Downside?

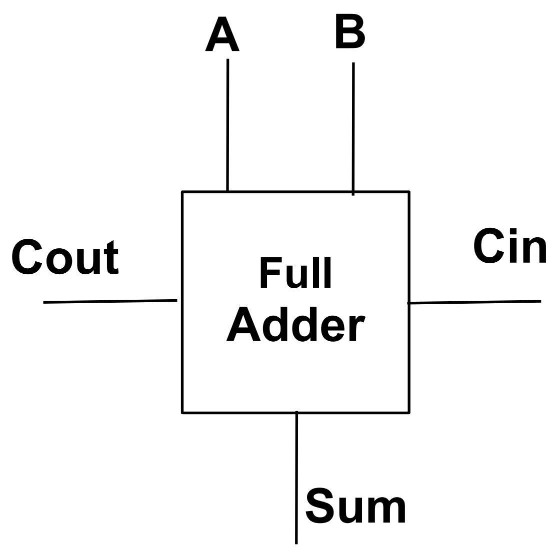
Ripple Carry



- Full Adder = Add 1 Bit
  - Can chain together to add many bits
  - Upside: Simple
  - Downside? Slow
    - Let's see why

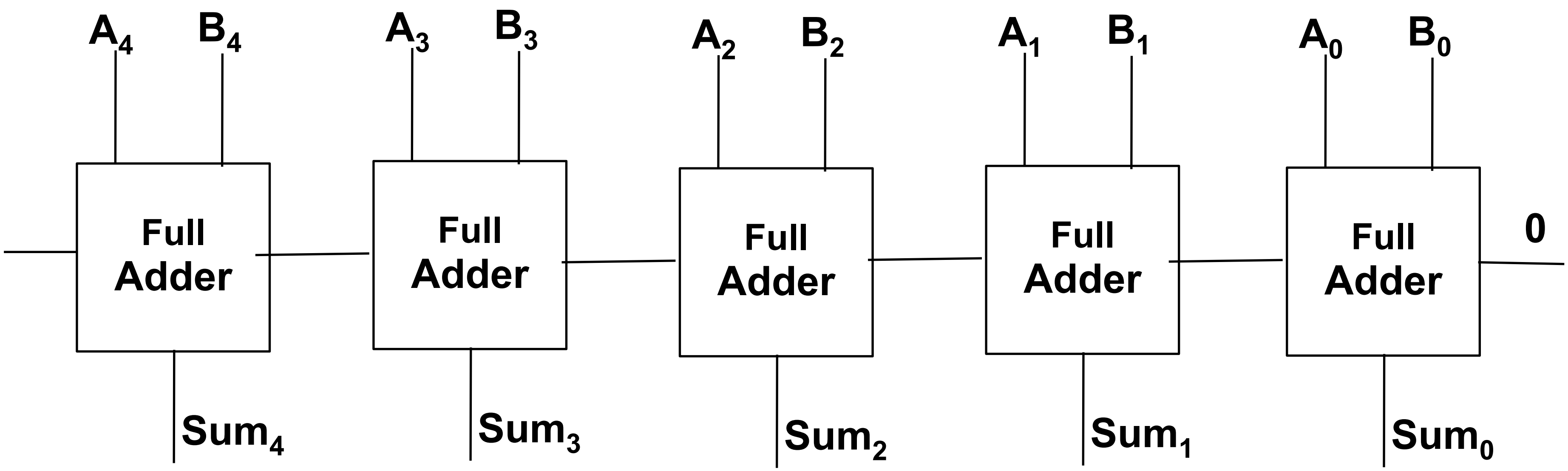
Full Adder





- Cout depends on Cin
  - 2 “gate delays” through full adder for carry

Ripple Carry



- Carries form a chain
  - Need CO of bit N is CI of bit N+1
- For few bits (e.g., 4) no big deal
  - For realistic numbers of bits (e.g., 32, 64), slow

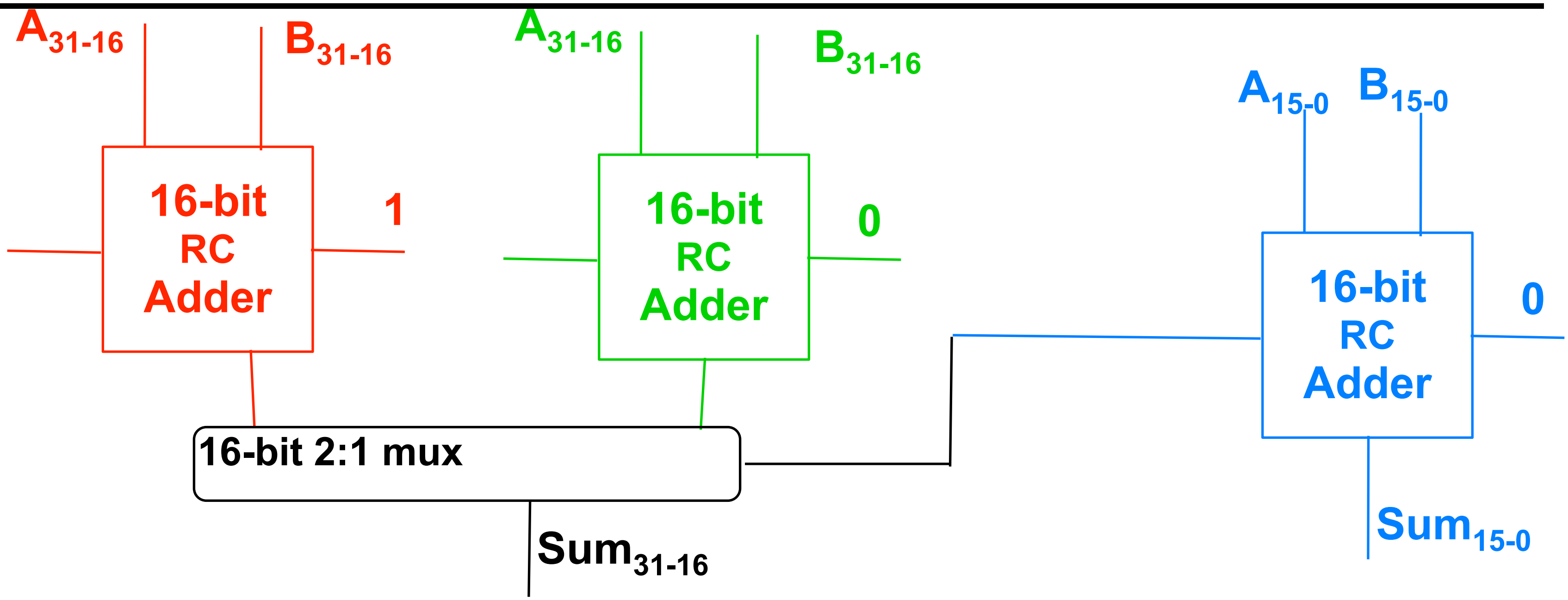
Adding

- Adding is important
  - Want to fit add in single clock cycle
    - (More on clocking soon)
    - Why? Add is ubiquitous
- Ripple Carry is slow
  - Maybe can do better?
  - But seems like Cin always depends on prev Cout
  - …and Cout always depends on Cin…

Hardware != Software

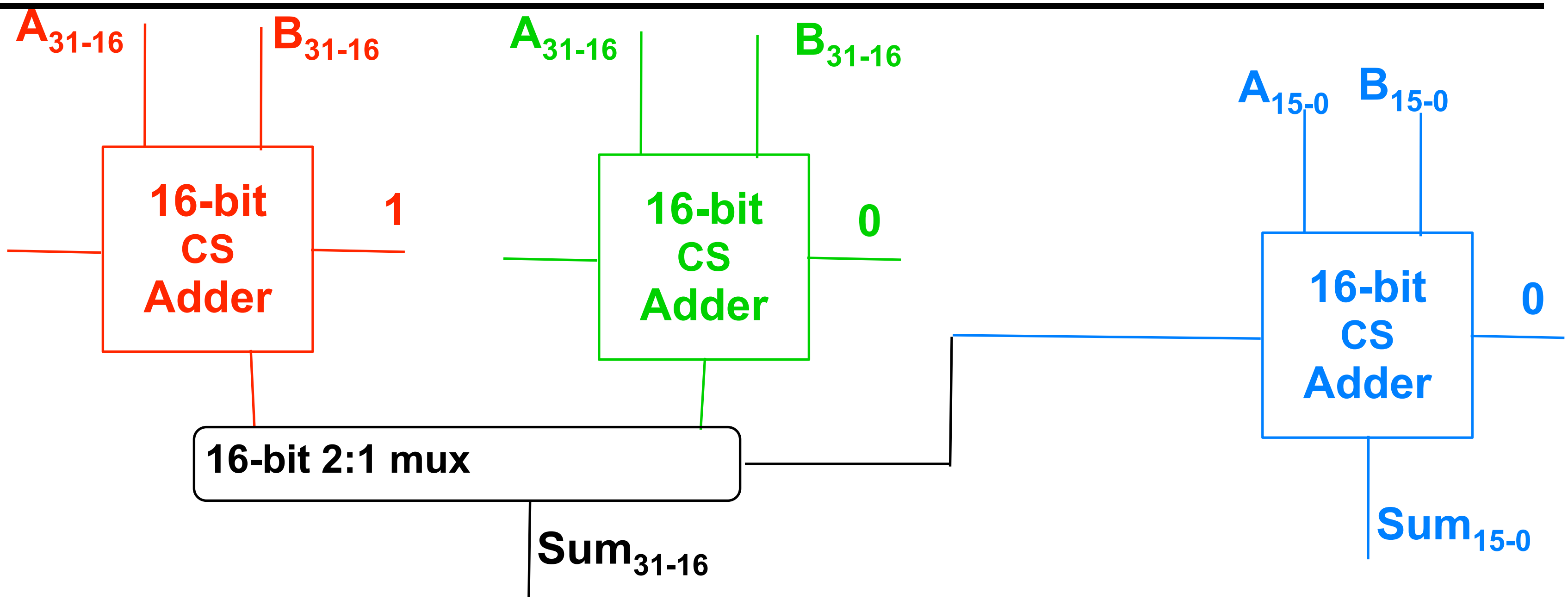
- If this were software, we'd be out of luck
  - But hardware is different
  - Parallelism: can do many things at once
  - Speculation: can guess

Carry Select



- Do three things at once (32 gates)
  - Add low 16 bits
  - Add high 16 bits assuming CI = 0
  - Add high 16 bits assuming CI =1
- Then pick correct assumption for high bits (2—3 gates)

Carry Select



- Could apply same idea again
  - Replace 16-bit RC adders with 16-bit CS adders
    - Reduce delay for 16 bit add from 32 to 18
    - Total 32 bit adder delay = 20
- So… just go nuts with this right?

Tradeoffs

- Tradeoffs in doing this
  - Power and Area (~= number of gates)
    - Roughly double every “level” of carry select we use
  - Less return on increase each time
    - Adding more mux delays
  - Wire delays increase with area
    - Not easy to count in slides
    - But will eat into real performance
- Fancier adders: recitation
  - Can do even better

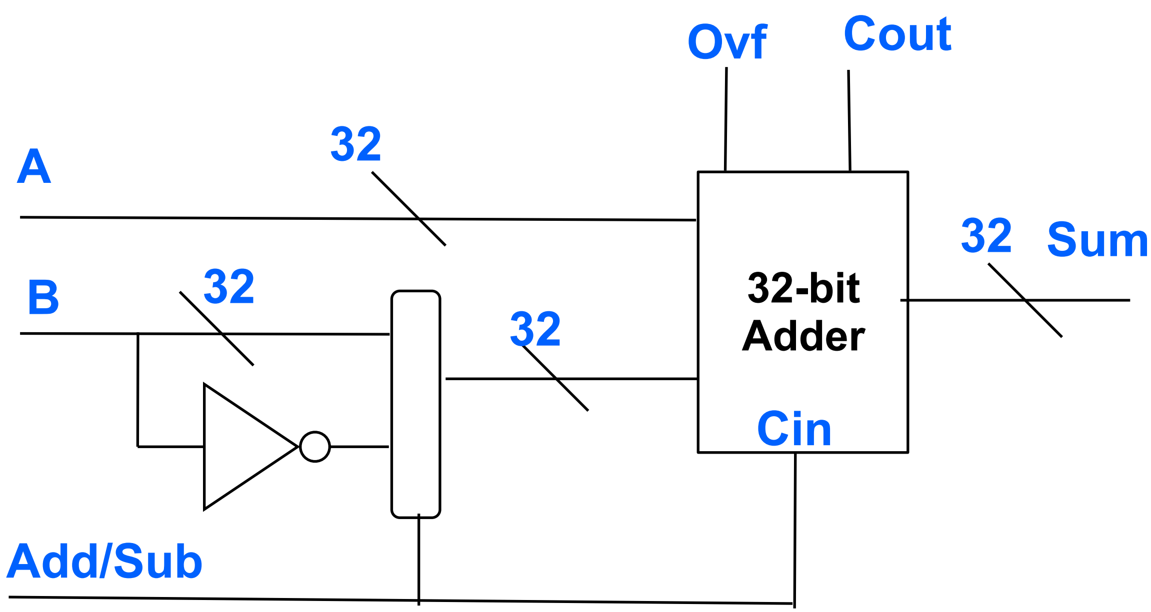
Recall: Subtraction

- 2's complement makes subtraction easy:
  - Remember: A - B = A + (-B)
  - And: -B = ~B + 1

    ↑ that means flip bits ("not")
  - So we just flip the bits and start with CI = 1
  - Fortunate for us: **makes circuits easy**

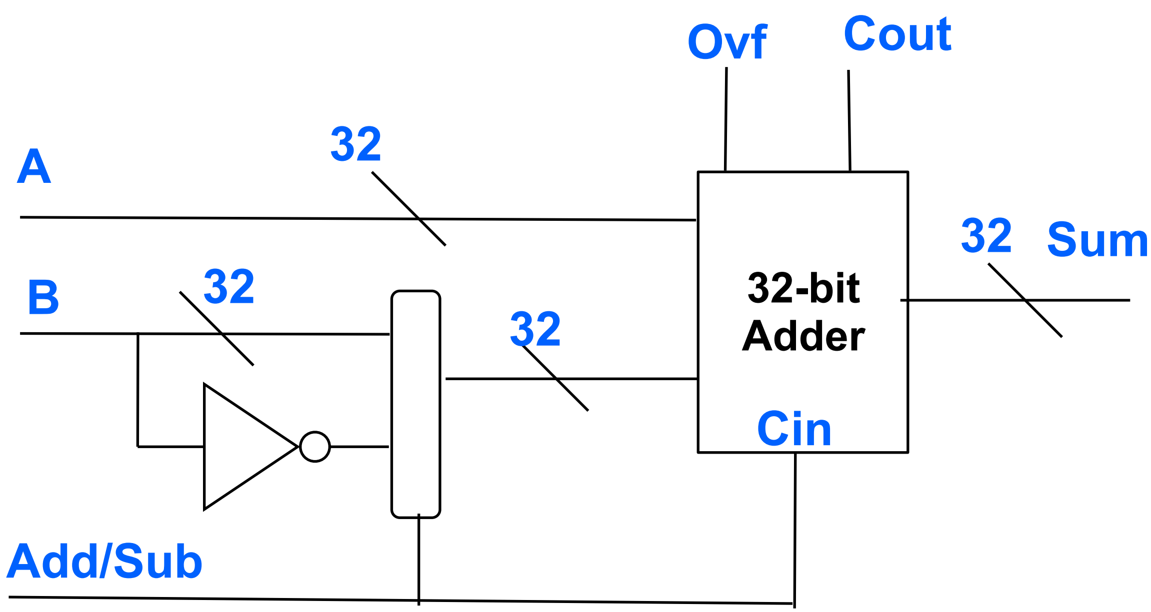
```
                            1
    0110101     ->    0110101
  - 1010010         + 0101101
```

32-bit Adder/subtractor



- Inputs: A, B, Add/Sub (0=Add,1 = Sub)
- Outputs: Sum, Cout, Ovf (Overflow)

32-bit Adder/subtractor



- By the way:
  - That thing has about 3,000 transistors
  - Aren't you glad we have abstraction?

Arithmetic Logic Unit (ALU)

- ALUs do a variety of math/logic
  - Add
  - Subtract
  - Bit-wise operations: And, Or, Xor, Not
  - Shift (left or right)
- Take two inputs (A,B) + operation (add,shift..)
  - Do a variety in parallel, then mux based on op

Bit-wise operations: SHIFT

- Left shift (<<)
  - Moves left, bringing in 0s at right, excess bits "fall off"
  - 10010001 << 2 = 01000100
  - x << k corresponds to $x * 2^k$
- Logical (or unsigned) right shift (>>)
  - Moves bits right, bringing in 0s at left, excess bits "fall off"
  - 10010001 >> 3 = 00010010
  - x >>k corresponds to $x / 2^k$ for unsigned x
- Arithmetic (or signed) right shift (>>)
  - Moves bits right, brining in (sign bit) at left
  - 10010001 >> 3= 11110010
  - x >>k corresponds to $x / 2^k$ for signed x

Shift: Implementation…?

- Suppose an 8-bit number

  $b_7b_6b_5b_4b_3b_2b_1b_0$

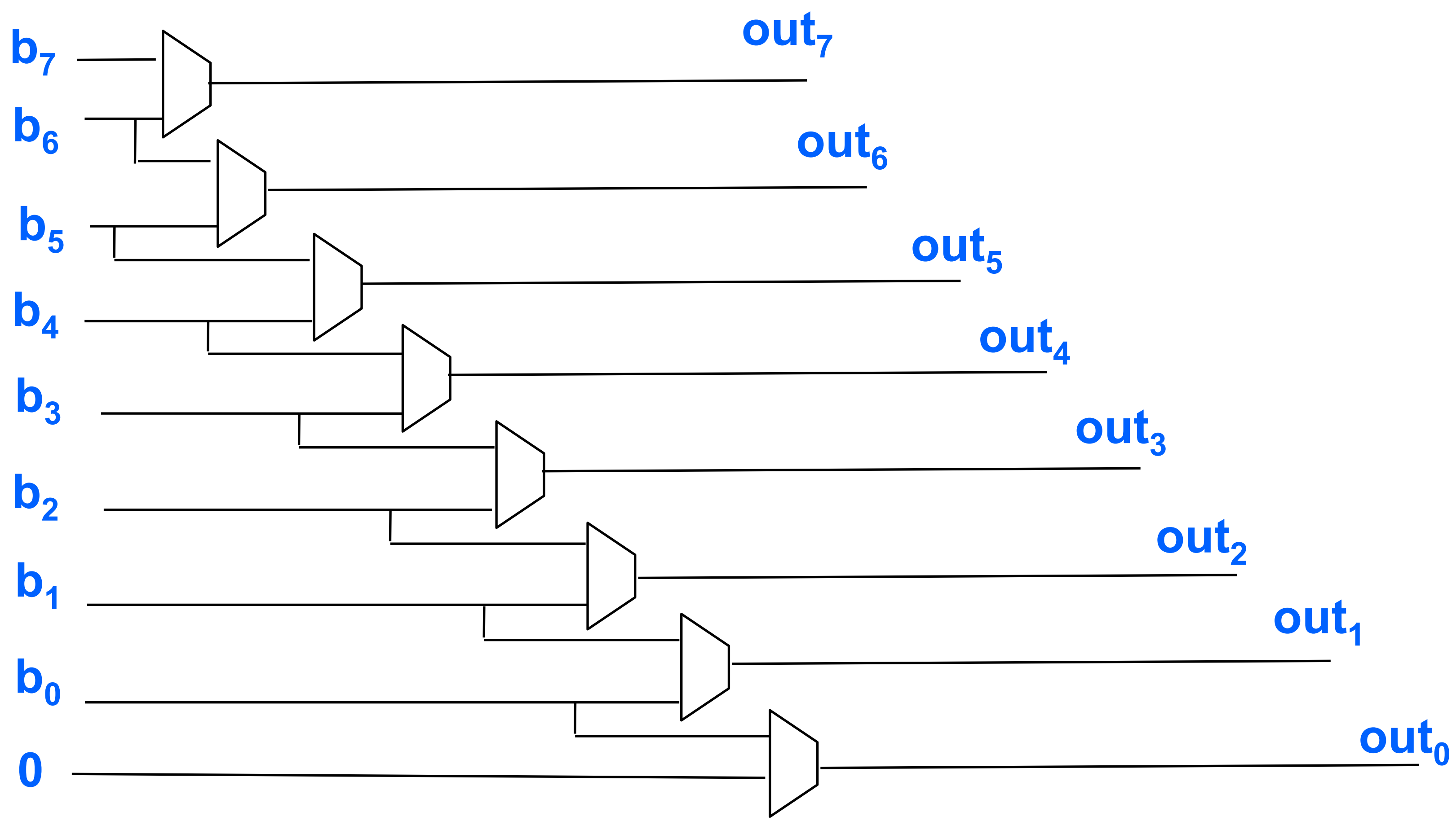
Shifted left by a 3 bit number

  $s_2s_1s_0$

- Option 1: Truth Table?
  - 2048 rows? Not appealing

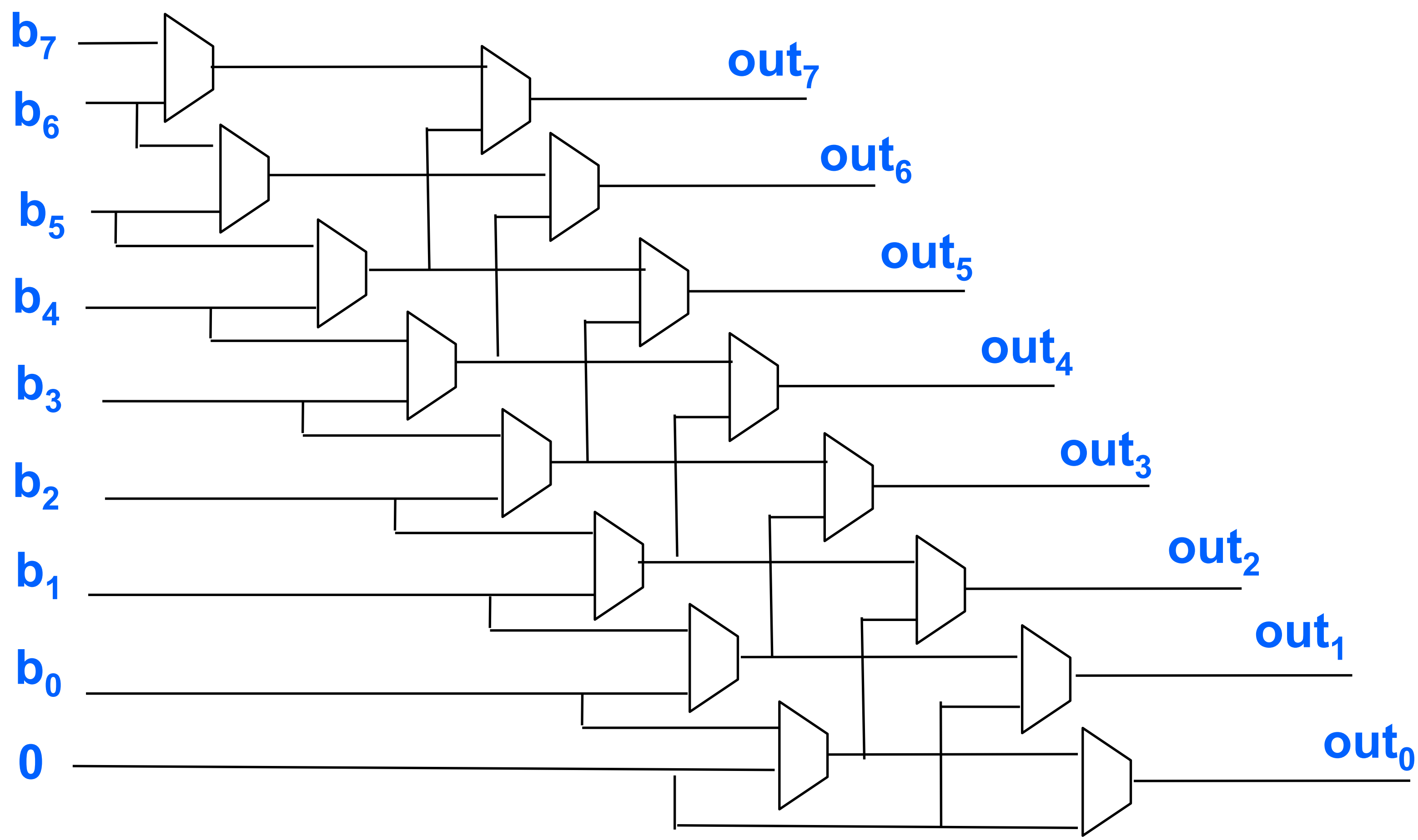
Lets simplify

- Simpler problem: 8-bit number shifted by 1 bit number (shift amount selects each mux)



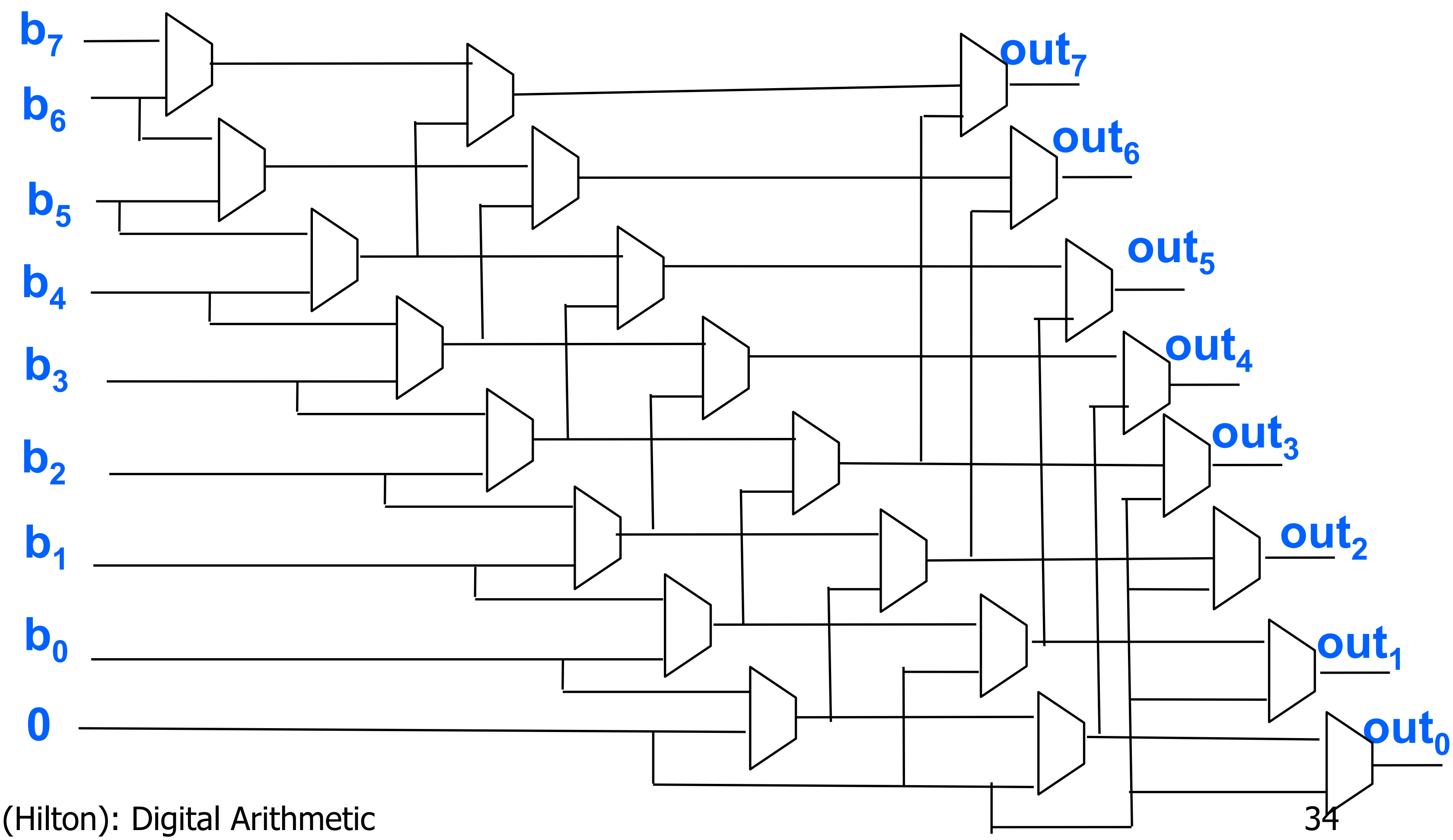
Lets simplify

- Simpler problem: 8-bit number shifted by 2 bit number (new muxes selected by 2nd bit)



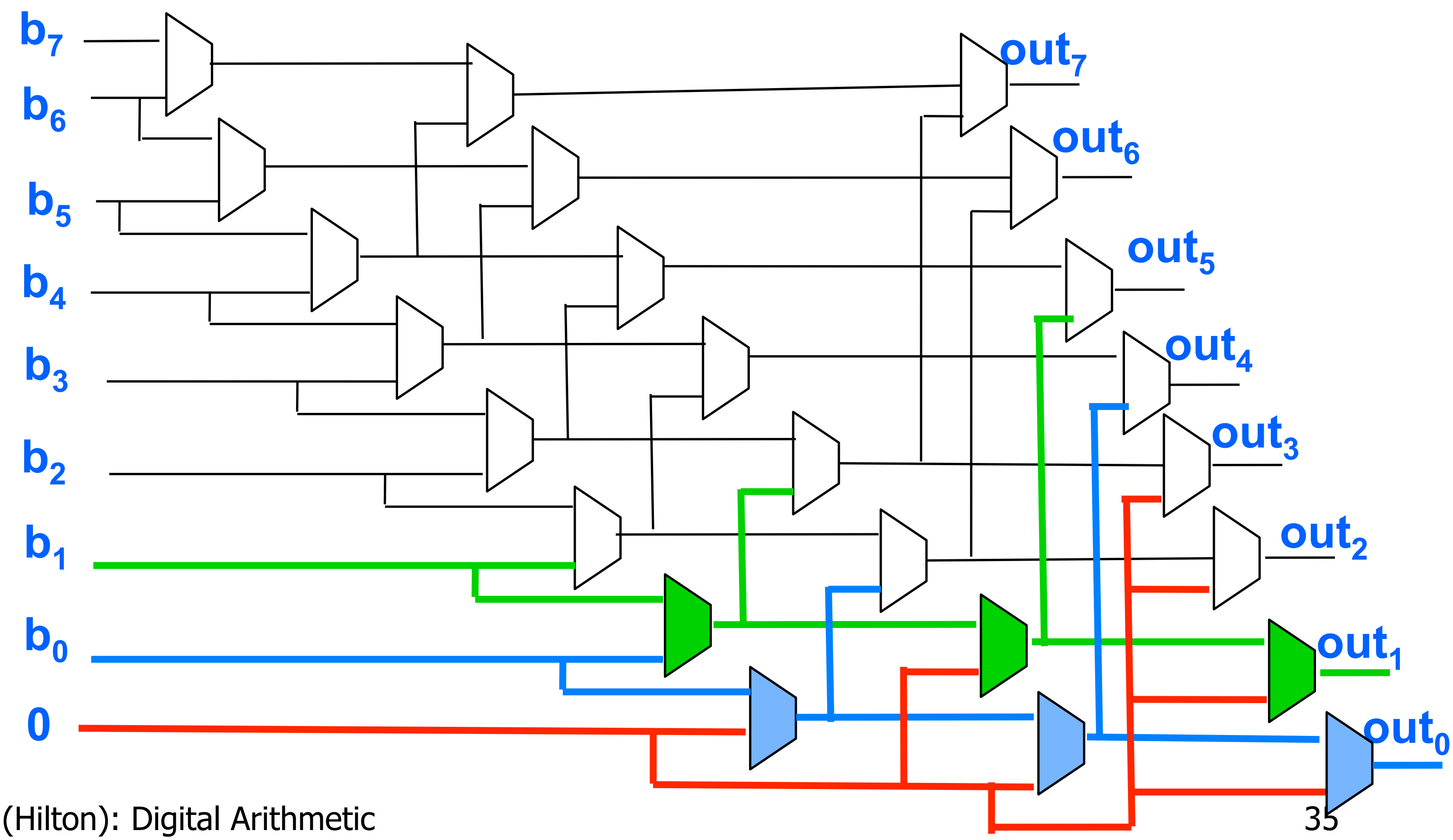
Now shifted by 3-bit number

- Full problem: 8-bit number shifted by 3 bit number (new muxes selected by 3rd bit)



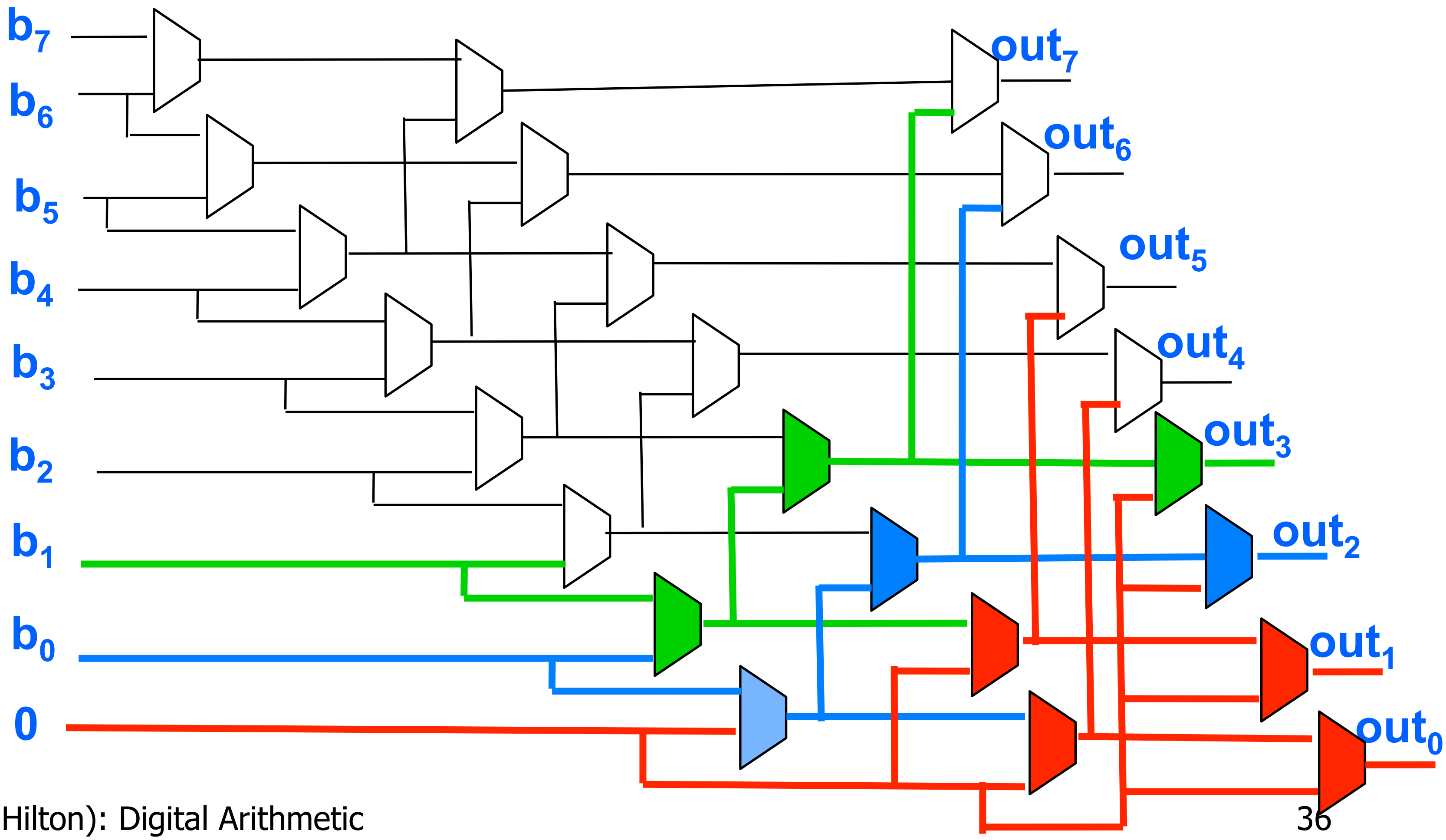
Now shifted by 3-bit number

- Shifter in action: shift by 000



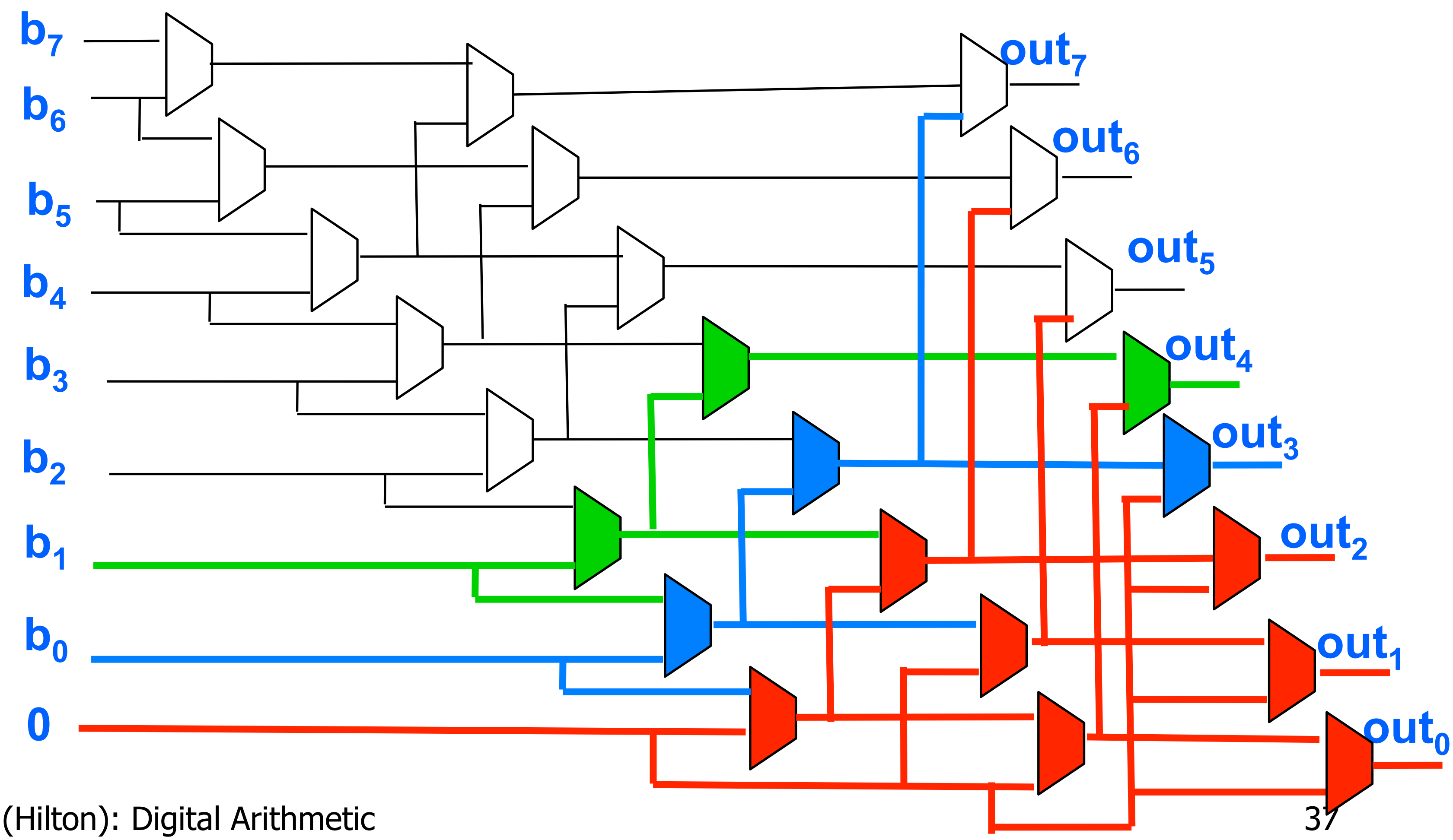
Now shifted by 3-bit number

- Shifter in action: shift by 010



Now shifted by 3-bit number

- Shifter in action: shift by 011



What About Non-integer Numbers?

- There are infinitely many real numbers between two integers
- Many important numbers are real
  - Pi = 3.145…
  - ½ = 0.5
- How could we represent these sorts of numbers?
  - Fixed Point
  - Rational
  - Floating Point (IEEE Single Precision)

Floating Point

- Think about scientific notation for a second:
- For example:

  $6.02 * 10^{23}$

- Real number, but comprised of ints:
  - 6 generally only 1 digit here
  - 2 any number here
  - 10 always 10 (base we work in)
  - 23 can be positive or negative
- Can we do something like this in binary?

Floating Point

- How about:
- $+/- X.YYYYYY * 2^{+/-N}$
- Big numbers: large positive N
- Small numbers (<1): negative N
- Numbers near 0: small N
- This is "floating point" : most common way

IEEE single precision floating point

- Specific format called IEEE single precision:
- $+/-\ \ 1.YYYYY * 2^{(N-127)}$
- "float" in Java, C, C++,...

- Assume X is always 1 (save a bit)
- 1 sign bit (+ = 0, 1 = -)
- 8 bit biased exponent (do N-127)
- Implicit 1 before binary point
- 23-bit mantissa (YYYYY)

Binary fractions

- 1.YYYY   has a binary point
  - Like a decimal point but in binary
  - After a decimal point, you have
    - tenths
    - hundredths
    - Thousandths
    - ….
- So after a binary point you have…

Binary fractions

- 1.YYYY   has a binary point
  - Like a decimal point but in binary
  - After a decimal point, you have
    - Tenths
    - Hundredths
    - Thousandths
    - ….
- So after a binary point you have…
  - Halves
  - Quarters
  - Eights
  - ….

Floating point example

- Binary fraction example:
  - 101.101 = 4 + 1 + ½ + $^1/_8$ = 5.625
- For floating point, needs normalization:
  - 1.01101 * 2^2
- Sign is +, which = 0
- Exponent = 127 + 2 = 129 = 1000 0001
- Mantissa = 1.011 0100 0000 0000 0000 0000

```
31 30        23 22                          0
0 |1000 0001|011 0100 0000 0000 0000 0000
```

Floating Point Representation

Example:

What floating-point number is:

0xC1580000?

Answer

What floating-point number is

0xC1580000?

1100 0001 0101 1000 0000 0000 0000 0000

Sign = **1** which is negative

Exponent = **(128+2)-127 = 3**

Mantissa = **1.1011**

$-1.1011 \times 2^3 = -1101.1 =$ **-13.5**

Trick question

- How do you represent 0.0?
  - Why is this a trick question?

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  - But need 1.XXXXX representation?

Trick question

- How do you represent 0.0?
  - Why is this a trick question?
  - 0.0 = 000000000
  - But need 1.XXXXX representation?
- Exponent of 0 is denormalized
  - Implicit 0. instead of 1. in mantissa
  - Allows 0000….0000 to be 0
  - Helps with very small numbers near 0
- Results in +/- 0 in FP (but they are “equal”)

Other weird FP numbers

- Exponent = 1111 1111 also not standard
  - All 0 mantissa: +/- ∞

    $1/0 = +\infty$

    $-1/0 = -\infty$
  - Non zero mantissa: Not a Number (NaN)

    sqrt(-42) = NaN

Floating Point Representation

- Double Precision Floating point:

  64-bit representation:
  - 1-bit sign
  - 11-bit (biased) exponent
  - 52-bit fraction (with implicit 1).

- “double” in Java, C, C++, …

S	Exp	Mantissa
1	11-bit	52 - bit

Danger: floats cannot hold all ints!

- Many programmers think:
  - Floats can represent all ints
  - NOT true

31	30 Exp 23	22 Mantissa 0
1	8-bit	23- bit

- First summer internship I had:
  - Need some floats and some ints: just use floats!
  - Bug in their code!
  - Other developers shocked as I demonstrated problem…
- Doubles can represent all 32-bit ints
- (but not all 64-bit ints)

S	Exp	Mantissa
1	11-bit	52 - bit

Wrap Up

- Implementation of Math
  - Addition/Subtraction
  - Shifting
- Floating Point Numbers
  - IEEE representation
  - Denormalized Numbers
- Next Time:
  - Storage
  - Clocking