ECE 550: Fundamentals of Computer Systems and Engineering

Combinatorial Logic

Admin

- Piazza: Updated from Friday night's rolls
  - Should have everyone now.
- Homework
  - Released now
  - Homework 1: due Sept 13
- Lab
  - 01A
  - Door code on Piazza

Last time….

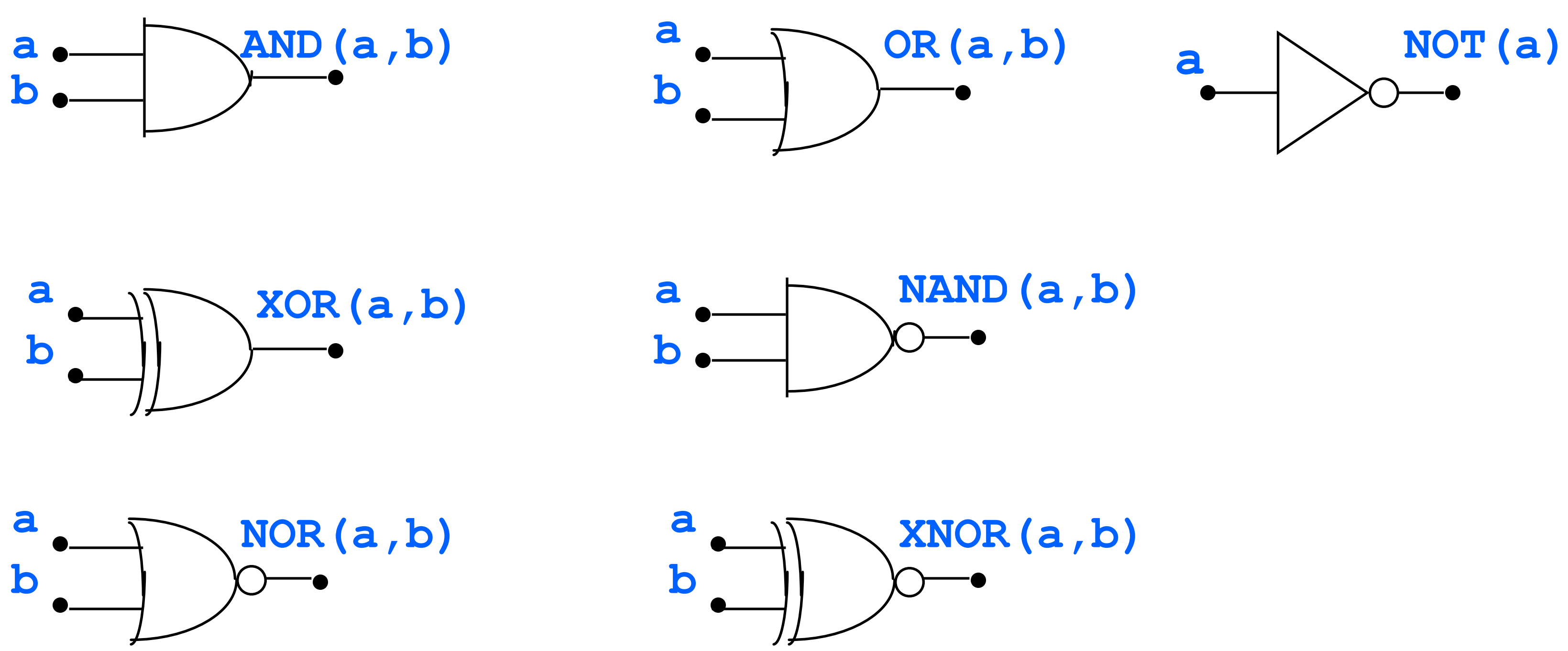
- Who can remind us what we talked about last time?

Last time….

- Who can remind us what we talked about last time?
  - Electric circuit basics
    - Vcc = 1
    - Ground = 0
  - Transistors
    - PMOS
    - NMOS
  - Gates
    - Complementary PMOS + NMOS
    - Output is logical function of inputs

Boolean Gates

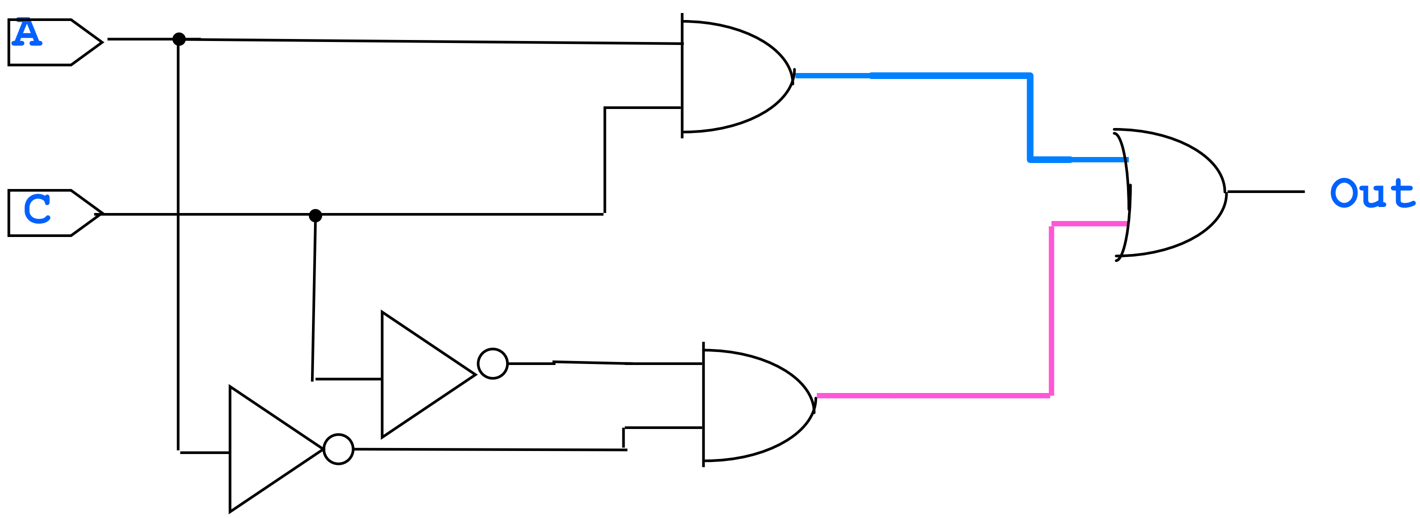
- Saw these gates
  - Mnemonic to remember them



Boolean Functions, Gates and Circuits

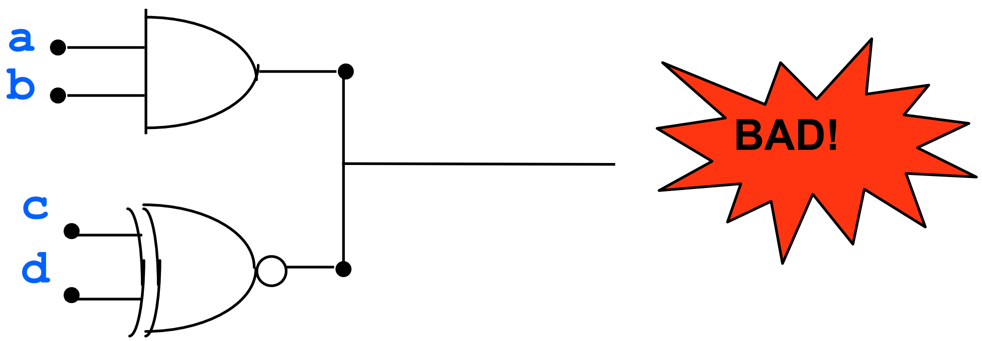
- Circuits are made from a network of gates.

`(!A & !C)|(A & C)`



A few more words about gates

- Gates have inputs and outputs
  - If you try to hook up two outputs, get short circuit
    - (Think of the transistors each gate represents)



  - If you don't hook up an input, it behaves kind of randomly

(also not good, but not set-your-chip-on-fire bad)

Let's Make a Useful Circuit

- Pick between 2 inputs (called 2-to-1 MUX)
  - Short for multiplexor
- What might we do first?

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      - S=0, pick A
      - S=1, pick B

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
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      - S=1, pick B
- Next: **sum-of-products**
  - Always works to find formula

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Sum of Products

- Find the rows where the output is 1

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	**1**	**1**	**1**
1	**0**	**0**	**1**
1	0	1	0
1	**1**	**0**	**1**
1	**1**	**1**	**1**

Sum of Products

- Find the rows where the output is 1
- Write a formula that exactly specifies each row

	A	B	S	Output
	0	0	0	0
	0	0	1	0
	0	1	0	0
!A & B & S	**0**	**1**	**1**	**1**
A & !B & !S	**1**	**0**	**0**	**1**
	1	0	1	0
A & B & !S	**1**	**1**	**0**	**1**
A & B & S	**1**	**1**	**1**	**1**

Sum of Products

- Find the rows where the output is 1
- Write a formula that exactly specifies each row
- OR these all together
  - Possible ways to get 1.

	A	B	S	Output
	0	0	0	0
	0	0	1	0
	0	1	0	0
!A & B & S →	**0**	**1**	**1**	**1**
A & !B & !S →	**1**	**0**	**0**	**1**
	1	0	1	0
A & B & !S →	**1**	**1**	**0**	**1**
A & B & S →	**1**	**1**	**1**	**1**

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```
(!A & B & S) |
(A & !B & !S) |
(A & B & !S ) |
(A & B & S)
```

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Let’s Make a Useful Circuit

- Pick between 2 inputs (called 2-to-1 MUX)
  - Short for multiplexor
- What might we do first?
  - Make a truth table?
    - S is selector:
      - S=0, pick A
      - S=1, pick B
- Next: sum-of-products
- Simplify!

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Simplifying The SOP Formula

- Simplifying this formula:

```
(!A & B & S) |
(A & !B & !S) |
(A & B & !S ) |
(A & B & S)
```

B doesn't matter

Simplifying The SOP Formula

- Simplifying this formula:

```
(!A & B & S) |
(A & !S) |
(A & B & S)
```

A doesn't matter

Simplifying The SOP Formula

- Simplifying this formula:

```
(A & !S) |
(B & S)
```

Let’s Make a Useful Circuit

- Pick between 2 inputs (called 2-to-1 MUX)
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- What might we do first?
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- Simplify

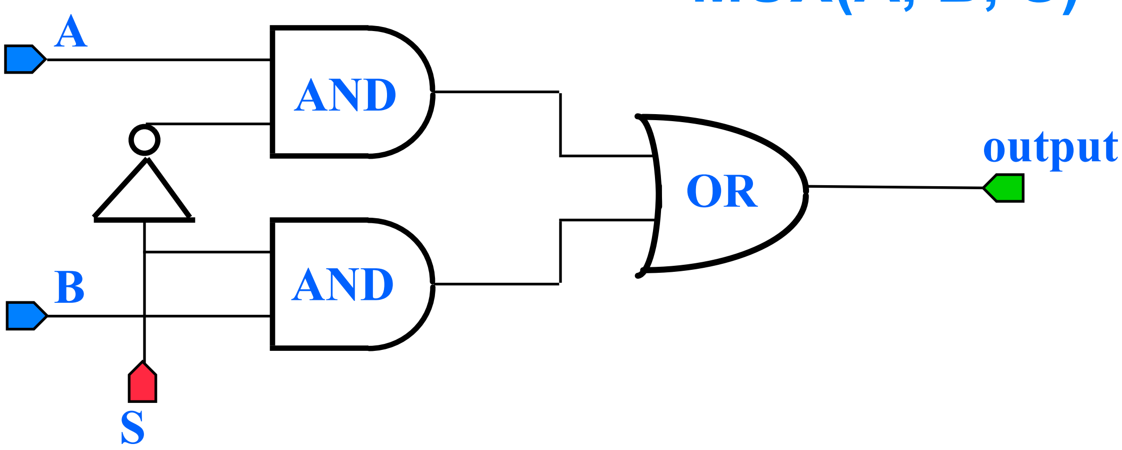
```
(A & !S) |
(B &  S)
```

A	B	S	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

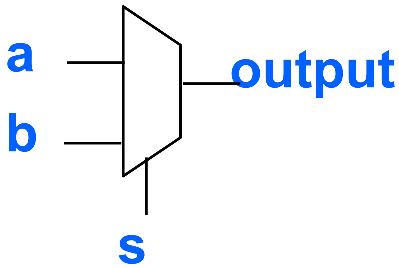
Circuit Example: 2x1 MUX

Draw it in gates:

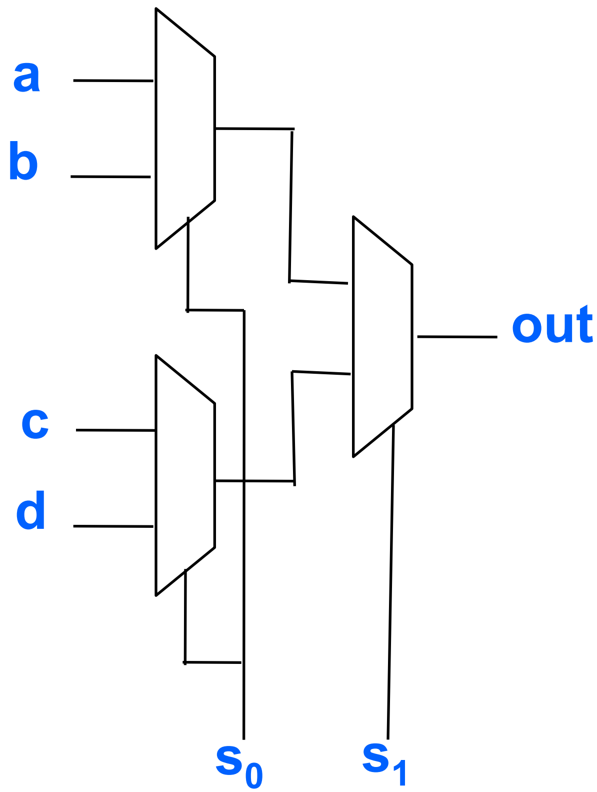
MUX(A, B, S) = (A & !S) | (B & S)

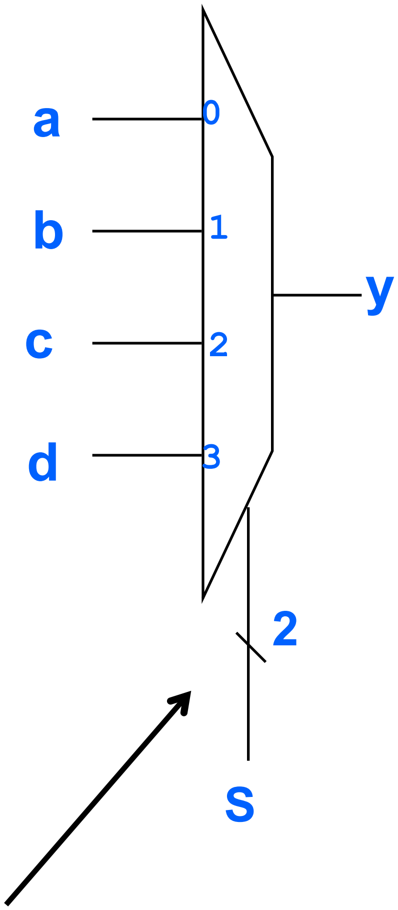


So common, we give it
its own symbol:



Example 4x1 MUX





The / 2 on the wire means "2 bits"

Boolean Function Simplification

- Boolean expressions can be simplified by using the following rules (bitwise logical):
  - `A & A = A` `A | A = A`
  - `A &  0 = 0` `A | 0 = A`
  - `A & 1 = A` `A | 1 = 1`
  - `A & !A = 0` `A | !A = 1`
  - `!!A  = A`
  - & and | are both commutative and associative
  - & and | can be distributed: `A & (B | C) = (A & B) | (A & C)`
  - & and | can be subsumed: `A | (A & B) = A`
  - Can typically just let synthesis tools do this dirty work, but good to know

DeMorgan’s Laws

- Two (less obvious) Laws of Boolean Algebra:
  - Let us push negations inside, flipping & and |

```
!(A & B) = (!A) | (!B)
```

```
!(A | B) = (!A) & (!B)
```

Alluded to these last time

Very good rules to know in general

Next piece of logic: Adder

- Computers do one thing: math
  - And they do it well/fast
  - Fundamental rule of computation: “Everything is a number”
    - Computers can only work with numbers
    - Represent things as numbers

Next: logic to work with numbers

- Computers do one thing: math
  - And they do it well/fast
  - Fundamental rule of computation: “Everything is a number”
    - Computers can only work with numbers
    - Represent things as numbers
  - Specifically: good at binary math
    - Base 2 number system: matches circuit voltages
      - 1 (Vcc)
      - 0 (Ground)
    - Use fixed sized numbers
      - How many **bits**
  - Quick primer on binary numbers/math
    - Then how to make circuits for it

Numbers for computers

- We usually use base 10:
  - $12345 = 1 * 10^4 + 2 * 10^3 + 3 * 10^2 + 4 * 10^1 * 5 * 10^0$
  - Recall from third grade: 1's place, 10's place, 100's place…
    - Yes, we are going to re-cover 3rd grade math, but in binary
  - What is the biggest digit that can go in any place?
- Base 2:
  - 1's place, 2's place, 4's place, 8's place, ….
  - What is the biggest digit that can go in any place?

Binary continued:

- Binary Number Example: 101101
  - Take a second and figure out what number this is

Binary continued:

- Binary Number Example: 101101
  - Take a second and figure out what number this is

1 in 32's place = 32

0 in 16's place

1 in 8's place = 8

1 in 4's place = 4

0 in 2's place

1 in 1's place = 1

—-

45

Converting Numbers

- Converting Decimal to Binary

  Suppose I want to convert 4872 to binary

  Think for a second about how to do this

Converting Numbers

- Converting Decimal to Binary

  Suppose I want to convert 4872 to binary

  Think for a second about how to do this

$$\begin{array}{r} 4872 \\ \underline{-\ 4096} \\ 776 \end{array}$$

4096	1
2048	
1024	
512	
256	
128	
64	
32	
16	
8	
4	
2	
1	

Converting Numbers

- Converting Decimal to Binary

  Suppose I want to convert 4872 to binary

  Think for a second about how to do this

```
  4872
- 4096
------
   776
-  512
------
   264
```

4096	1
2048	0
1024	0
512	1
256	
128	
64	
32	
16	
8	
4	
2	
1	

Converting Numbers

- Converting Decimal to Binary

    Suppose I want to convert 4872 to binary

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```
  4872
- 4096
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   776
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------
   264
-  256
------
     8
```

4096	1
2048	0
1024	0
512	1
256	1
128	
64	
32	
16	
8	
4	
2	
1	

Converting Numbers

- Converting Decimal to Binary

  Suppose I want to convert 4872 to binary

  Think for a second about how to do this

$$
\begin{array}{r}
4872 \\
\underline{-\ 4096} \\
776 \\
\underline{-\ \ 512} \\
264 \\
\underline{-\ \ 256} \\
8 \\
\underline{-\ \ \ \ 8} \\
0
\end{array}
$$

4096	1
2048	0
1024	0
512	1
256	1
128	0
64	0
32	0
16	0
8	1
4	
2	
1	

Converting Numbers

- Converting Decimal to Binary

  Suppose I want to convert 4872 to binary

  Think for a second about how to do this

```
   4872
 - 4096
 ------
    776
 -  512
 ------
    264
 -  256
 ------
      8
 -    8
 ------
      0
```

4096	1
2048	0
1024	0
512	1
256	1
128	0
64	0
32	0
16	0
8	1
4	0
2	0
1	0

Hexadecimal: Convenient shorthand for

- Binary is unwieldy to write
  - 425,000 decimal = 1100111110000101000 binary
  - Generally about 3x as many binary digits as decimal
  - Converting (by hand) takes some work and thought
- Hexadecimal (aka "hex")—base 16—is convenient:
  - Easy mapping to/from binary
  - Same or fewer digits than decimal
  - 425,000 decimal = 0x67C28
  - Generally write "0x" on front to make clear "this is hex"
  - Digits from 0 to 15, so use A—F for 10—15.

Binary ⇔ Hex

- Binary ⇔ Hex conversion is straightforward
  - Every 4 binary bits = 1 hex digit
  - If # of bits not a multiple of 4, add implicit 0s on left as needed

0000 ⇔ 0	1000 ⇔ 8
0001 ⇔ 1	1001 ⇔ 9
0010 ⇔ 2	1010 ⇔ A
0011 ⇔ 3	1011 ⇔ B
0100 ⇔ 4	1100 ⇔ C
0101 ⇔ 5	1101 ⇔ D
0110 ⇔ 6	1110 ⇔ E
0111 ⇔ 7	1111 ⇔ F

Binary ⇔ Hex

- **1100111110000101000**
- **6 7 C 2 8**

Binary Math : Addition

- Suppose we want to add two numbers:

```
      00011101
+     00101011
```

- How do we do this?

Binary Math : Addition

- Suppose we want to add two numbers:

```
    00011101          695
+   00101011        + 232
```

- How do we do this?
  - Let's revisit decimal addition
  - Think about the process as we do it

Binary Math : Addition

- Suppose we want to add two numbers:

```
    00011101          695
+   00101011        + 232
                        7
```

- First add one's digit 5+2 = 7

Binary Math : Addition

- Suppose we want to add two numbers:

```
       00011101            1
   +   00101011           695
                        + 232
                           27
```

- First add one's digit 5+2 = 7
- Next add ten's digit 9+3 = 12 (2 carry a 1)

Binary Math : Addition

- Suppose we want to add two numbers:

```
                          1
     00011101            695
+    00101011          + 232
                         927
```

- First add one's digit 5+2 = 7
- Next add ten's digit 9+3 = 12 (2 carry a 1)
- Last add hundred's digit 1+6+2 = 9

Binary Math : Addition

- Suppose we want to add two numbers:

```
      00011101
  +   00101011
```

- Back to the binary:
- First add 1's digit 1+1 = ...?

Binary Math : Addition

- Suppose we want to add two numbers:

```
          1
    00011101
 +  00101011
    --------
           0
```

- Back to the binary:
- First add 1's digit 1+1 = 2 (0 carry a 1)

Binary Math : Addition

- Suppose we want to add two numbers:

```
           11
     00011101
 +   00101011
 -------------
           00
```

- Back to the binary:
- First add 1's digit 1+1 = 2 (0 carry a 1)
- Then 2's digit: 1+0+1 =2 (0 carry a 1)
- You all finish it out….

Binary Math : Addition

- Suppose we want to add two numbers:

```
      111111
     00011101    = 29
 +   00101011    = 43
 ------------
     01001000    = 72
```

- Can check our work in decimal

Negative Numbers

- May want negative numbers too!
- Many ways to represent negative numbers:
  - Sign/magnitude
  - Biased
  - 1's complement
  - 2's complement

2's Complement Integers

- To negate, flip bits, add 1:
  - 1's complement + 1
- Pros:
  - Easy to compute with
  - One representation of 0
- Cons:
  - More complex negation
  - Extra negative number (-8)
- Ubiquitous choice

0000	0
0001	1
0010	2
0011	3
0100	4
0101	5
0110	6
0111	7
1000	-8
1001	-7
1010	-6
1011	-5
1100	-4
1101	-3
1110	-2
1111	-1

Binary Math : Addition

- Revisit binary math for a minute:

```
      01011101
  +   01101011
```

Binary Math : Addition

- What about this one:

```
      1111111
      01011101   =   93
  +   01101011   =  107
      --------
      11001000   =  -56
```

- But… that can't be right?
  - What do you expect for the answer?
  - What is it in 8-bit signed 2's complement?
-

Integer Overflow

- Answer should be 200
  - Not representable in 8-bit signed representation
  - No right answer
- Called Integer Overflow
  - Signed addition: CI != CO of last bit
  - Unsigned addition: CO != 0 of last bit
- Can detect in hardware
  - Signed: XOR CI and CO of last bit
  - Unsigned: CO of last bit
  - What processor does: depends

Subtraction

- 2’s complement makes subtraction easy:
  - Remember: A - B = A + (-B)
  - And: -B = ~B + 1

    ↑ that means flip bits (“not”)
  - So we just flip the bits and start with CI = 1
  - Fortunate for us: makes circuits easy (next time)

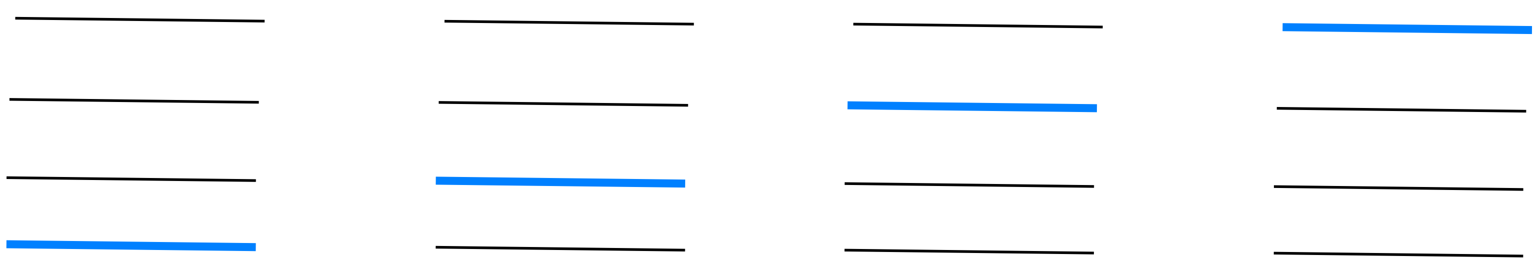
```
                          1
   0110101    ->    0110101
 - 1010010        + 0101101
 ---------        ---------
```

Signed and Unsigned Ints

- Most programming languages support two int types
  - Signed: negative and positive
  - Unsigned: positive only, but can hold larger positive numbers
- Addition and subtraction:
  - Same, except overflow detection
  - x86: one add instruction, sets two different flags for overflows
- Inequalities
  - Different operations for signed/unsigned
  - Can someone give an example? (Let’s say 4-bit numbers)

One hot representation

- Binary representation convenient for math
- Another representation:
  - One hot: one wire per number
  - At any time, one wire = 1, others = 0



  - Wow, that is one hot representation!
  - Very convenient in many cases (e.g., homework 1)

Converting to/from one hot

- Converting from 2^N bits one hot to N bits binary=**encoder**
  - E.g., "an 8-to-3 encoder"
- Converting from N bits binary to 2^N bits one hot=**decoder**
  - E.g., "a 4-to-16 decoder"
    (which may be quite useful on hwk1)

Lets build a 4-to-2 encoder

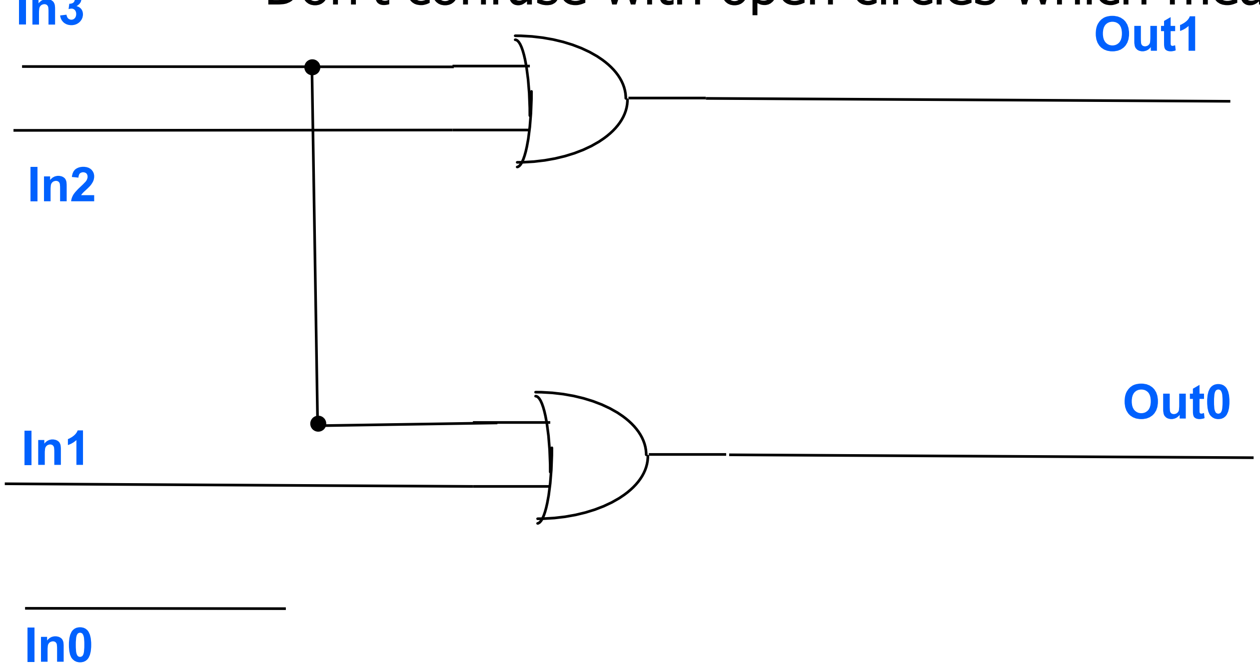
- Start with a truth table
  - Input constrained to 1-hot: don't care about invalid inputs
    - Can do anything we want

In0	In1	In2	In3	Out1	Out0
1	0	0	0	0	0
0	1	0	0	0	1
0	0	1	0	1	0
0	0	0	1	1	1

- Simplest formulas:
  - Out0 = In1 or In3 [alternatively: Out0 = In0 nor In2]
  - Out1 = In2 or In3 [alternatively: Out1 = In0 nor In1]

4-to-2 encoder

- Our 4-to-2 encoder
  - Note: the dots here show connections
    - Don’t confuse with open circles which mean NOT



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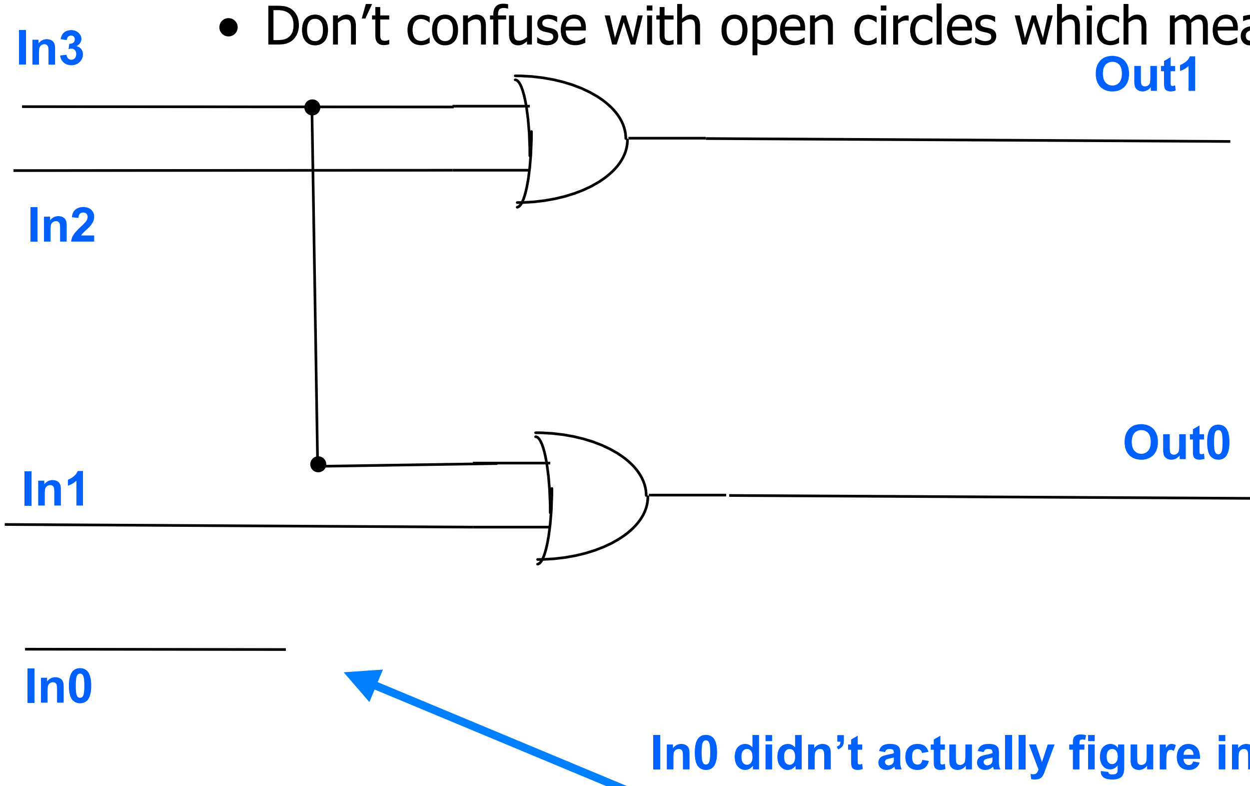


In0 didn’t actually figure into the logic. That’s ok
Synthesis will eliminate it if its not used elsewhere…
And whatever logic creates it, etc..

Lets build a 2-to-4 decoder

- Start with a truth table
  - Now input unconstrained

In1	In0	Out0	Out1	Out2	Out3
0	0	1	0	0	0
0	1	0	1	0	0
1	0	0	0	1	0
1	1	0	0	0	1

Now SOP more useful, do for each of the 4 outputs:

Lets build a 2-to-4 decoder

- Start with a truth table
  - Now input unconstrained

In1	In0	Out0	Out1	Out2	Out3
0	0	1	0	0	0
0	1	0	1	0	0
1	0	0	0	1	0
1	1	0	0	0	1

Now SOP more useful, do for each of the 4 outputs:

Out0 = (Not In1) and (Not In0)

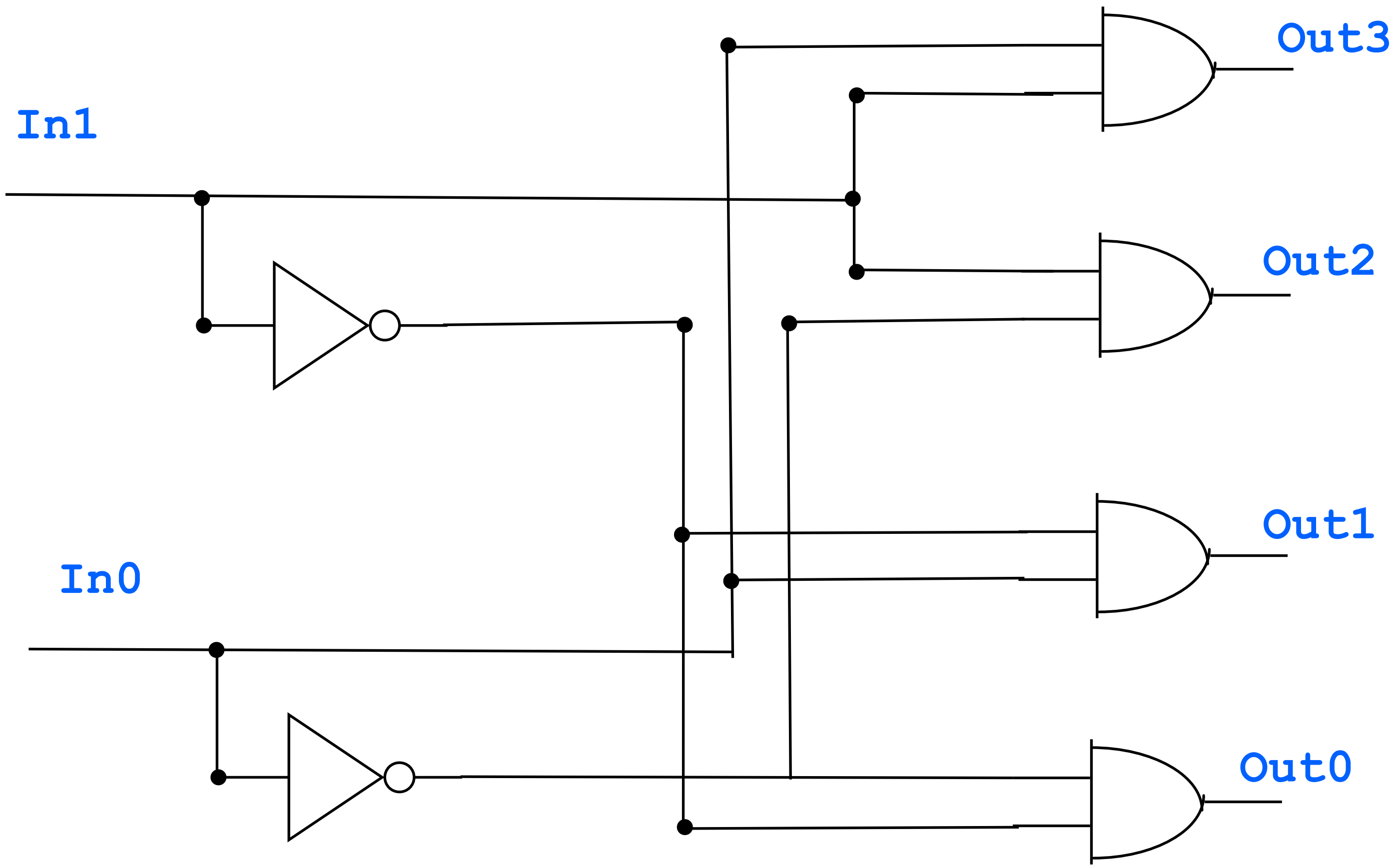
Out1 = (Not In1) and In0

Out2 = In1 and (Not In0)

Out3 = In1 and In0

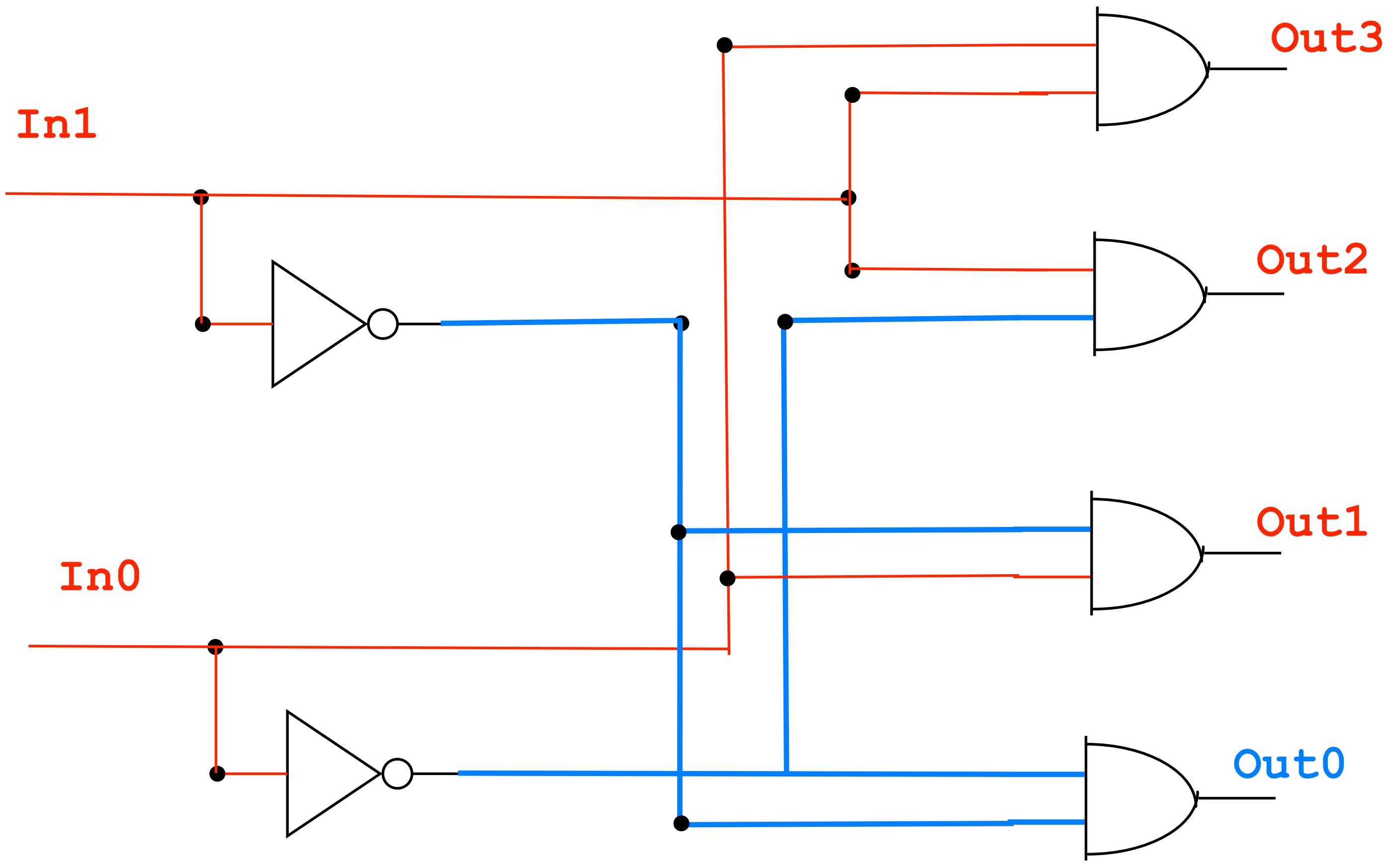
2-to-4 decoder

- 2-to-4 decoder



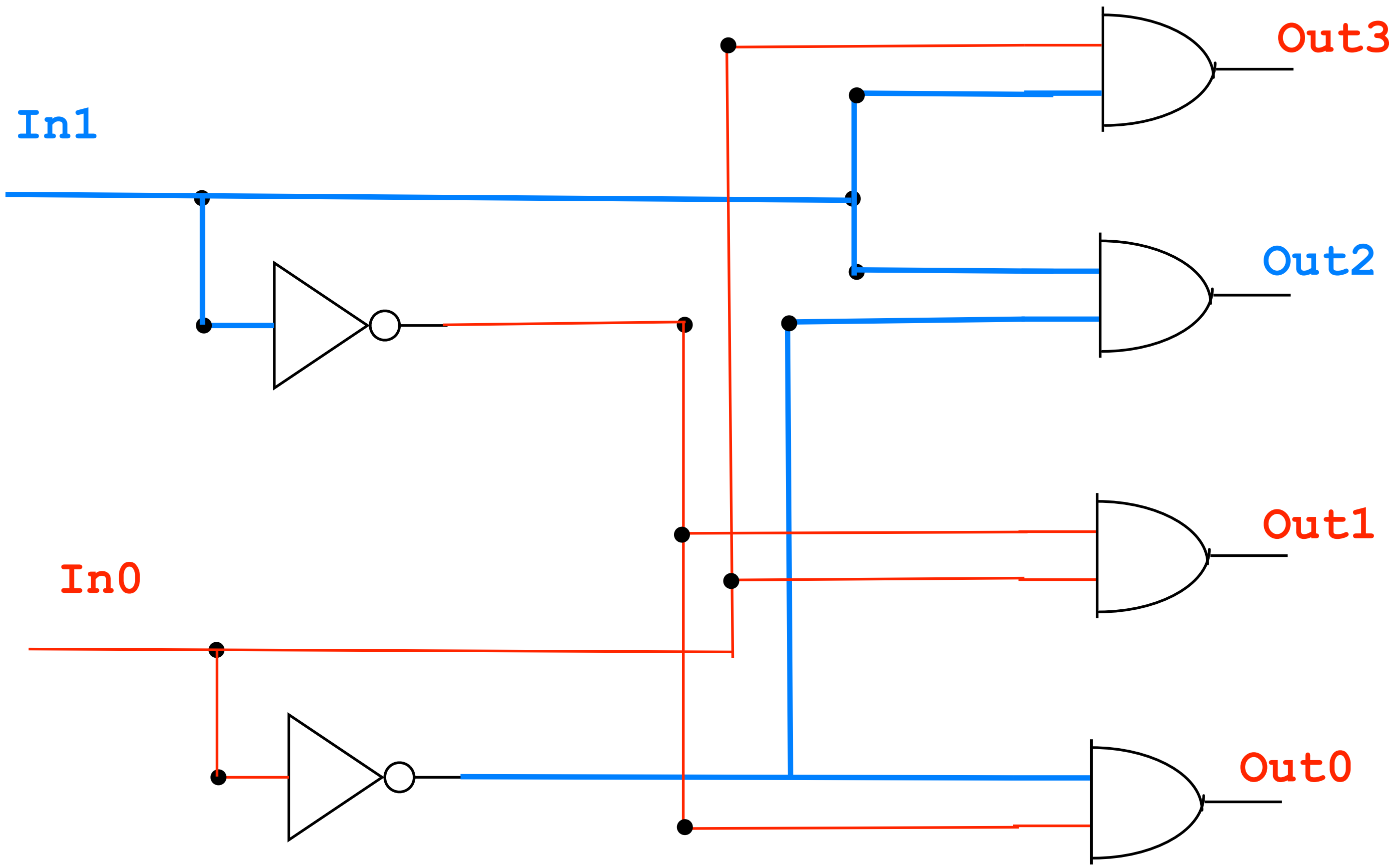
2-to-4 decoder

- 2-to-4 decoder



2-to-4 decoder

- 2-to-4 decoder



Delays

- Mentioned before: switching not instant
  - Not going to try to calculate delays by hand (tools can do)
  - But good to know where delay comes from, to tweak/improve
- Gates:
  - Switching the transistors in gates takes time
  - More gates (in series) = more delay
- Fan-out: how many gates the output drives
  - Related to capacitance
  - High fan-out = slow
  - Sometimes better to replicate logic to reduce its fan-out
- Wire delay:
  - Signals take time to travel down wires

Wrap Up

- Combinatorial Logic
  - Putting gates together
  - Sum-of-products
  - Simplification
  - Muxes, Encoders, Decoders
- Number Representations
  - One Hot
  - 2's complement