Discussion 2
Probabilistic Machine Learning,Fall 2016

1. Decision Trees

Concepts

(a) Please provide an intuitive explanation of why the *information gain criterion* helps us choose a good split in a decision tree?

Solution It makes the leaves purer (closer to certain about the correct class) on average.

(b) Is it possible that the depth of a learned decision tree can be larger than the number of training examples used to create the tree? Explain.

Solution False. You can't have more splits than training examples.

Practice

Consider the following dataset for a problem where we try to predict the last column (can play). We're gonna be using the **ID3 algorithm** for construction of the decision tree.

(a) What is the first feature that you would split on ?

Solution First attribute we split on is HW-Due.

(b) What is the number of levels of the decision tree constructed (leaf nodes included)?

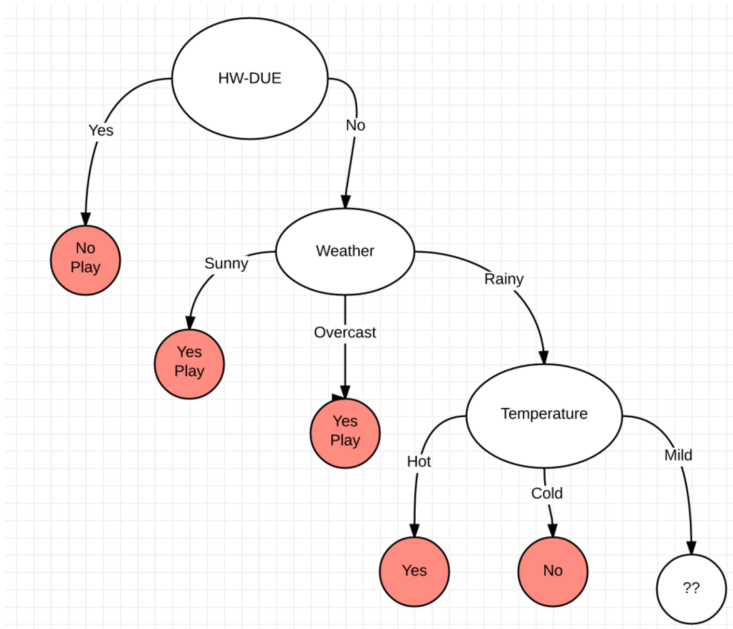
Solution The number of levels here is 4 (since we count the nodes at the bottom as well).

(c) What's the training error of this decision tree?

Solution 0.00, since we were able to classify all points to the right classes.

(d) Given the test dataset, how many testing points do we classify INCORRECTLY? Enter as an integer.

Solution Just the first testing point is predicted wrong. Thus, the answer is 1.



Training

Figure 1: Decision Trees.

2. Entropy

The entropy H of a discrete random variable X with n possible values is defined by the formula

$$H(X) = -\sum_{i=1}^{n} p(x_i)\ log\ p(x_i).$$

Similarly, the joint entropy of two random variables X with n outcomes and Y with m outcomes is

$$H(X,Y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j)\ log\ p(x_i, y_j).$$

2.1

As $p(x)$ is the probability of outcome x, $0 \leq p(x) \leq 1$ for any outcome. Use this fact to prove that $H(X) \geq 0$ for any random variable X.

Answer

$0 \leq p(x) \leq 1$ implies that $log\ p(x)$ is always negative. Since $p(x)$ is always positive, $p(x)\ log\ p(x)$ must be negative and therefore the sum $\sum_{i=1}^{n} p(x_i)\ log\ p(x_i)$ must in turn be negative. This implies that $-\sum_{i=1}^{n} p(x_i)\ log\ p(x_i)$ is positive.

2.2

Consider a discrete random variable Y with a uniform probability distribution over n outcomes, i.e. $p(y) = 1/n$. Is $H(Y)$ bounded as $n \to \infty$?

Answer

$$H(Y) = \sum_{i=1}^{n} \frac{1}{n} log\ \frac{1}{n}$$
$$= log\ \frac{1}{n}$$

$log\ \frac{1}{n}$ is unbounded as $n \to \infty$.

2.3

The conditional entropy $H(Y|X)$ is defined

$$H(Y|X) = -\sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)\ log\ p(y_j|x_i).$$

Show that $H(X,Y) = H(X) + H(Y|X)$. Here are some hints:

- $log\ \frac{a}{b} = log\ a - log\ b$
- $p(y|x) = \frac{p(x,y)}{p(x)}$
- $\sum_{j=1}^{m} p(x_i, y_j) = p(x_i)$.

Answer

$$\begin{aligned}
H(X) + H(Y|X) &= -\sum_{i=1}^{n} p(x_i)\ log\ p(x_i) - \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)\ log\ p(y_j|x_i) \\
&= -\sum_{i=1}^{n} p(x_i)\ log\ p(x_i) - \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)\ log\ \frac{p(x_i, y_j)}{p(x_i)} \\
&= -\sum_{i=1}^{n} p(x_i)\ log\ p(x_i) - \sum_{i=1}^{n}\sum_{j=1}^{m} [p(x_i, y_j)\ (log\ p(x_i, y_j) - log\ p(x_i))] \\
&= -\sum_{i=1}^{n} p(x_i)\ log\ p(x_i) - \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)\ \ log\ p(x_i, y_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)\ log\ p(x_i)) \qquad (1) \\
&= -\sum_{i=1}^{n} p(x_i)\ log\ p(x_i) - \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)\ \ log\ p(x_i, y_j) + \sum_{i=1}^{n} p(x_i)\ log\ p(x_i) \\
&= -\sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)\ log\ p(x_i, y_j) \\
&= H(X,Y)
\end{aligned}$$

Information Gain

The term "information gain" has been used in our discussion of decision trees to describe the extra information we gain about some variable X by including a new variable Y that may lend extra predictive power in a model. Formally, the information gain (also called the mutual information) is defined as:

$$I(X;Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j)\ log\ \frac{p(x_i, y_j)}{p(x_i)p(y_j)}$$

2.4

When two random variables X and Y are independent, then their joint distribution $P(X,Y)$ factorizes into $P(X,Y) = P(X)P(Y)$. What is the information gain for two independent random variables? What does this say about the ability of one variable to explain the other?

Answer

$$\begin{aligned}
I(X;Y) =& \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j) log\ \frac{p(x_i,y_j)}{p(x_i)p(y_j)} \\
=& \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j) log\ \frac{p(x_i)p(y_j)}{p(x_i)p(y_j)} \\
=& \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j) log\ 1 \\
=& 0
\end{aligned} \tag{2}$$

The information gain is zero. This means that no new information is gained when adding a conditionally independent variable.

2.5

Show that $I(X;Y) = H(X) + H(Y) - H(X,Y)$.

Answer

$$\begin{aligned}
I(X;Y) =& \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j)\ log\ \frac{p(x_i,y_j)}{p(x_i)p(y_j)} \\
=& \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j)\ log\ p(x_i,y_j) - log\ p(x_i)p(y_j) \\
=& \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j)\ log\ p(x_i,y_j) - log\ p(x_i) - log\ p(y_j) \\
=& \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j)\ log\ p(x_i,y_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j)\ log\ p(x_i) - \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j)\ log\ p(y_j) \\
=& \sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i,y_j)\ log\ p(x_i,y_j) - \sum_{i=1}^{n} p(x_i)\ log\ p(x_i) - \sum_{j=1}^{m} p(y_j)\ log\ p(y_j) \\
=& - H(X,Y) + H(X) + H(Y)
\end{aligned} \tag{3}$$

Programming machine learning algorithms

Solution Please see the folder “ExperimentalCode”.