ECE 587 / STA 563: Lecture 2 – Measures of Information

Information Theory
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Outline of lecture:

2.1 Quantifying Information

- How much "information" do the answers to the following questions provide?

  (i) Will it rain today in Durham? (two possible answers)

  (ii) Will it rain today in Death Valley? (two possible answers)

  (iii) What is today's winning lottery number? (for the Mega Millions Jackpot, there are 258,890,850 possible answers)

- The amount of "information" is linked to the number of possible answers. In 1928, Ralph Hartley gave the following definition:

$$\text{Hartley Information} = \log \# \text{ answers}$$

- Hartley's measure of information is additive. The number of possible answers for two questions corresponds to the *product* of the number of answers for each question. Taking the logarithm turns the product into a sum.

  - What is today's winning lottery number?

$$\log_2(258,890,850) \approx 28(\text{bits})$$

  - What are the winning lottery numbers for today and tomorrow?

$$\log_2(258,890,850 \times 258,890,850) = \log_2(258,890,850) + \log_2(258,890,850) \approx 56(\text{bits})$$

- But Hartley's information does not distinguish between likely and unlikely answers (e.g. rain in Durham vs. rain in Death Valley).
- In 1948, Shannon introduced measures of information which depend on the *probabilities* of the answers.

2.2 Entropy and Mutual Information

2.2.1 Entropy

- Let X be discrete random variable with pmf $p(x)$ and finite support $\mathcal{X}$.
- The entropy of X is
$$H(X) = \sum_{x \in \mathcal{X}} p(x) \log\left(\frac{1}{p(x)}\right)$$
- Note that this is the expect value of the random variable $g(X)$ where $g(x) = \log(1/p(x))$, i.e.
$$H(X) = \mathbf{E}\left[\log\left(\frac{1}{p(X)}\right)\right]$$
- **Binary Entropy:** If X is a Bernoulli(p) random variable (i.e. $\mathbf{P}[X = 1]p$ and $\mathbf{P}[X = 0] = 1 - p$), then its entropy is given by the binary entropy function
$$H_b(p) = -p \log p - (1 - p) \log(1 - p)$$

  - $H_b(p)$ is concave, the maximum at $H_b(1/2) = \log(2)$ and has minimum is at $H_b(0) = H_b(1) = 0$.
- **Example:** Two Questions
  - Will it rain Today in Durham?
$$H_b\left(\frac{104}{365}\right) \approx 0.862 \quad \text{bits}$$
  - Will it rain Today in Death Valley?
$$H_b\left(\frac{1}{365}\right) \approx 0.027 \quad \text{bits}$$
- **Fundamental Inequality** For any bases $b > 0$ and $x > 0$,
$$\left(1 - \frac{1}{x}\right) \log_b(e) \leq \log_b(x) \leq (x - 1) \log_b(e)$$
with equalities on both sides if, and only if, $x = 1$.

  Proof:

- For the natural log, this simplifies to

$$\left(1 - \frac{1}{x}\right) \le \ln(x) \le (x-1)$$

- To prove the upper bound, note that equality is attained at $x = 1$. For $x > 1$,

$$(x-1) - \ln(x) = \int_1^x \underbrace{\left(1 - \frac{1}{x}\right)}_{\text{strictly positive}} dx > 0$$

and for $x < 1$,

$$(x-1) - \ln(x) = \int_x^1 \underbrace{\left(\frac{1}{x} - 1\right)}_{\text{strictly positive}} dx > 0$$

- To prove the lower bound, let $y = 1/x$ so that

$$\ln(y) \le y - 1 \Rightarrow \frac{1}{y-1} \le \ln\left(\frac{1}{y}\right) \Rightarrow \frac{x}{1-x} \le \ln(x)$$

- **Theorem:** $0 \le H(X) \le \log |\mathcal{X}|$
  - Left side follows from $0 \le p(x) \le 1$.
  - For right side, observe that

$$\begin{aligned}
\sum_x p(x) \log\left(\frac{1}{p(x)}\right) &= \sum_x p(x) \log\left(\frac{|\mathcal{X}|}{p(x)|\mathcal{X}|}\right) \\
&= \log(|\mathcal{X}|) + \sum_x p(x) \log\left(\frac{1}{p(x)|\mathcal{X}|}\right) \\
&\le \log(|\mathcal{X}|) + \sum_x p(x) \log(e)\left(\frac{1}{p(x)|\mathcal{X}|} - 1\right) && \text{Fundamental Inq.} \\
&= \log(|\mathcal{X}|) + \log(e) - \log(e) \\
&= \log(|\mathcal{X}|)
\end{aligned}$$

- The entropy of an n-dimensional random vector $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ with pmf $p(\boldsymbol{x})$ is defined as

$$H(\boldsymbol{X}) = H(X_1, X_2, \cdots, X_n) = \sum_{\boldsymbol{x} \in \mathcal{X}} p(\boldsymbol{x}) \log\left(\frac{1}{p(\boldsymbol{x})}\right)$$

- The **joint entropy** of random variables X and Y is simply the entropy of the vector (X, Y)

$$H(X, Y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log\left(\frac{1}{p(x, y)}\right)$$

- **Conditional Entropy**: The entropy of a random variable Y conditioned on the event $\{X = x\}$ is a function of the conditional distribution $p_{Y|X}(\cdot|x)$ and is given by:

$$H(Y|X = x) = \sum_{y \in \mathcal{Y}} p(y|x) \log\left(\frac{1}{p(y|x)}\right)$$

The conditional entropy of Y given X is a function of the joint distribution $p(x, y)$:

$$H(Y|X) = \sum_{x \in \mathcal{X}} p(x) H(Y|X = x) = \sum_{x,y} p(x, y) \log\left(\frac{1}{p(y|x)}\right)$$

- **Warning:** Note that $H(Y|X)$ is *not* a random variable! This is differs from the usual convention for conditioning where, for example, $\mathbf{E}[Y|X]$ is a random variable.

- **Chain Rule:** The joint entropy of X and Y can be decomposed as

$$H(X, Y) = H(X) + H(Y|X)$$

and more generally,

$$H(X_1, X_2, \cdots, X_n) = \sum_{i=1}^{n} H(X_i|X_{i-1}, \cdots, X_1)$$

Proof of chain rule:

$$\begin{aligned}
H(X;Y) &= \sum_{x,y} p(x, y) \log\left(\frac{1}{p(x, y)}\right) \\
&= \sum_{x,y} p(x, y) \log\left(\frac{1}{p(x)} \frac{1}{p(y|x)}\right) \\
&= \sum_{x,y} p(x, y) \left[\log\left(\frac{1}{p(x)}\right) + \log\left(\frac{1}{p(y|x)}\right)\right] \\
&= H(X) + H(Y|X)
\end{aligned}$$

2.2.2 Mutual Information

- Measure of the amount of information that one RV contains about another RV

$$I(X;Y) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log\left(\frac{p(x, y)}{p(x)p(y)}\right)$$

It can also be expressed as an expectation:

$$I(X;Y) = \mathbf{E}[i(X, Y)] \qquad \text{where} \qquad i(x, y) = \log\left(\frac{p(x, y)}{p(x)p(y)}\right)$$

- Mutual information between X and Y can be expressed as the amount by which knowledge of X reduces the entropy of Y:

$$I(X;Y) = H(Y) - H(Y|X)$$

and by symmetry

$$I(X;Y) = H(X) - H(X|Y)$$

Proof:

$$\begin{aligned}\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log\left(\frac{p(x,y)}{p(x)p(y)}\right) &= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\left[\log\left(\frac{1}{p(y)}\right) - \log\left(\frac{1}{p(y|x)}\right)\right] \\ &= \underbrace{\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log\left(\frac{1}{p(y)}\right)}_{H(Y)} - \underbrace{\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log\left(\frac{1}{p(y|x)}\right)}_{H(Y|X)}\end{aligned}$$

- The mutual information between X and itself is equal to entropy:

$$I(X;X) = H(X) - H(X|X) = H(X)$$

Thus entropy is sometimes known as "self-information"

- Venn diagram

- The conditional mutual information between X and Y given Z is

$$I(X;Y|Z) = \sum_{x,y,z} p(x,y,z)\log\left(\frac{p(x,y|z)}{p(x|z)p(y|z)}\right)$$

or equivalently

$$I(X;Y|Z) = \mathbf{E}[i(X,Y|Z)], \qquad i(x,y|z) = \log\left(\frac{p(x,y|z)}{p(x|z)p(y|z)}\right)$$

- Chain Rule for mutual information:

$$I(X;Y_1,Y_2) = I(X;Y_1) + I(X;Y_2|Y_1)$$

and more generally

$$I(X;Y_1,Y_2,\cdots,Y_n) = \sum_{i=1}^{n} I(X;Y_i|Y_1,Y_2,\cdots,Y_{k-1})$$

2.2.3 Example: Testing for a disease

There is a 1% chance I have a certain disease. There exists a test for this disease which is 90% accurate (i.e. $\mathbf{P}[\text{test is pos}|\text{I have disease}] = \mathbf{P}[\text{test is neg}|\text{I don't have disease}] = 0.9$). Let

$$X = \begin{cases} 1, & \text{I have disease} \\ 0, & \text{I don't have disease} \end{cases} \quad \text{and} \quad Y_i = \begin{cases} 1, & i\text{th test is positive} \\ 0, & i\text{th test is negative} \end{cases}$$

Assume the the test outcomes $\boldsymbol{Y} = (Y_1, Y_2)$ are conditionally independent given X.

- The probability mass functions can be computed as

$p(x, \boldsymbol{y})$	$\boldsymbol{y} = (0,0)$	$\boldsymbol{y} = (0,1)$	$\boldsymbol{y} = (1,0)$	$\boldsymbol{y} = (1,1)$
$x = 0$	0.8019	0.0891	0.0891	0.0099
$x = 1$	0.0001	0.0009	0.0009	0.0081

  and

	p(x)
$x = 0$	0.99
$x = 1$	0.01

	$p(\boldsymbol{y})$
$\boldsymbol{y} = (0,0)$	0.8020
$\boldsymbol{y} = (0,1)$	0.0900
$\boldsymbol{y} = (1,0)$	0.0900
$\boldsymbol{y} = (1,1)$	0.0180

	$p(y_1)$
$y_1 = 0$	0.8920
$y_1 = 1$	0.1080

- The individual entropies are

$$H(X) = H_b(0.01) \approx 0.0808$$

$$H(Y_1) = H(Y_2) = H_b(0.1080) \approx 0.4939$$

- The conditional entropy of X given Y_1 is computed as follows:

$$H(X|Y_1 = 1) = H_b(0.9167) \approx 0.4137$$

$$H(X|Y_1 = 0) = H_b(0.0011) \approx 0.0126$$

  and so

$$H(X|Y) = \mathbf{P}[Y_1 = 1]H(X|Y_1 = 1) + \mathbf{P}[Y = 0]H(H|Y_1 = 0) \approx 0.0559$$

- Furthermore

$$H(X|Y_1, Y_2) = H(X, Y_1, Y_2) - H(Y_1, Y_2) \approx 0.0339$$

$$H(Y_1|Y_2) = H(Y_1, Y_2) - H(Y_2) \approx 0.4930$$

- The mutual information is

$$I(X; Y_1) = H(X) - H(X|Y_1) \approx 0.0249$$

$$I(X; Y_1, Y_2) = H(X) - H(X|Y_1, Y_2) \approx 0.0469$$

- The conditional mutual information is

$$I(X; Y_2|Y_1) = H(X|Y_1) - H(X|Y_1, Y_2) \approx 0.0220$$

2.2.4 Relative Entropy

- The relative entropy between a distributions p and q is defined by

$$D(p||q) = \sum_{x \in \mathcal{X}} p(x) \log\left(\frac{p(x)}{q(x)}\right)$$

  This is also known as the Kullback-Leibler divergence. It can be expressed as the expectation of the expectation of the log likelihood ratio

$$D(p||q) = \mathbf{E}[\Lambda(X)], \qquad X \sim p, \qquad \Lambda(x) = \log\left(\frac{p(x)}{q(x)}\right)$$

- Note that if there exists x such that $p(x) > 0$ and $q(x) = 0$, then $D(p||q) = \infty$.
- **Warning:** $D(p||q)$ is not a metric since it is not symmetric and it does not satisfy the triangle inequality.
- Mutual information between X and Y is equal to the relative entropy between $p_{X,Y}(x, y)$ and $p_X(x)p_Y(y)$,

$$I(X; Y) = D(p_{X,Y}(x, y)||p_X(x)p_Y(y))$$

- **Theorem:** Relative entropy is nonnegative, i.e $D(p||q) \geq 0$. It is equal to zero if and only if $p = q$. The proof is given in CT problem 2.26.

  Proof:

$$\begin{aligned} -D(p||q) &= \sum_x p(x) \log \frac{q(x)}{p(x)} \\ &\leq \sum_x p(x) \log(e) \left(\frac{q(x)}{p(x)} - 1\right) && \text{Fundamental Inq.} \\ &= \log(e) \sum_x q(x) - \log(e) \sum_x p(x) \\ &= 0 \end{aligned}$$

- Important consequences of the non-negativity of relative entropy:
  - Mutual information is nonnegative, $I(X; Y) \geq 0$, with equality if an only if X and Y are independent.
  - This means that $H(X) - H(X|Y) \geq 0$, and thus **conditioning cannot increase entropy**,

$$H(X|Y) \leq H(X)$$

  - **Warning**: Although conditioning cannot increase entropy (in expectation), it is possible that the entropy of X conditioned on an specific event, say $\{Y = y\}$, is greater than $H(X)$, i.e. $H(X|Y = y) > H(X)$.

2.3 Convexity & Concavity

- A function $f(x)$ is convex over an interval $(a, b) \subseteq \mathbb{R}$ if for every $x_1, x_2 \in (a, b)$ and $0 \le \lambda \le 1$,

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2).$$

  The function is strictly convex if equality holds only if $\lambda = 0$ or $\lambda = 1$.

- Illustration of convexity. Let $x^* = \lambda x_1 + (1-\lambda)x_2$

- **Theorem:** $H(X)$ is a concave function $p(x)$, i.e.

$$H(\underbrace{\lambda p_1 + (1-\lambda)p_2}_{p^*}) \ge \lambda H(p_1) + (1-\lambda) H(p_2)$$

  - This can be proved using the fundamental inequality (try it yourself)
  - Here is an alternative proof which uses the fact that conditioning cannot increase entropy. Let Z be Bernoulli(λ)and let

$$X \sim \begin{cases} p_1, & Z = 1 \\ p_2, & Z = 0 \end{cases}$$

    Then,

$$H(X) = H(\lambda p_1 + (1-\lambda)p_2)$$

    Since conditioning cannot increase entropy,

$$H(X) \ge H(X|Z) = \lambda H(X|Z=1) + (1-\lambda) H(X|Z=0).$$

    Combining the displays completes the proof.

- **Jensen's Inequality** If $f(\cdot)$ is a convex function over an interval $\mathcal{I}$ and X is a random variable with support $\mathcal{X} \subset \mathcal{I}$ then

$$\mathbf{E}[f(X)] \ge f(\mathbf{E}[X])$$

  Moreover, if $f(\cdot)$ is strictly convex, equality occurs if and only if $X = \mathbf{E}[X]$ is a constant.

- The proof of Jensen's Inequality is given in [C&T]

- **Example** For any set $\{x_i\}_{i=1}^n$, the geometric mean is no greater than the arithmetic mean:

$$\left(\prod_{i=1}^n x_i\right)^{1/n} \le \frac{1}{n}\sum_i^n x_i$$

  Proof: Let Z be uniformly distributed on $\{x_i\}$ so that $\mathbf{P}[Z = x_i] = 1/n$. By Jensen's inequality,

$$\log\left(\prod_{i=1}^n x_i\right)^{1/n} = \frac{1}{n}\sum_{i=1}^n \log x_i = \mathbf{E}[\log(Z)] \le \log(\mathbf{E}[Z]) = \log\left(\frac{1}{n}\sum_i^n x_i\right)$$

2.4 Data Processing Inequality

- **Markov Chain:** Random variables X, Y, Z form a Markov chain, denoted

$$X \to Y \to Z$$

  if X and Z are independent conditioned on Y.

$$p(x, z|y) = p(x|y)p(z|y)$$

  - alternatively

$$\begin{aligned} p(x, y, z) &= p(x)p(y, z|x) && \text{always true} \\ &= p(x)p(y|x)p(z|x, y) && \text{always true} \\ &= p(x)p(y|x)p(z|y) && \text{if Markov chain} \end{aligned}$$

  - Note $X \to Y \to Z$ implies $Z \to Y \to X$
  - If $Z = f(Y)$ then $X \to Y \to Z$.

- **Theorem:** (Data Processing Inequality) If $X \to Y \to Z$, then

$$I(X; Y) \geq I(X; Z)$$

- In particular, for any function $g(\cdot)$, we have $X \to Y \to g(Y)$ and so

$$I(X; Y) \geq I(X; g(Y)).$$

  No clever manipulation of Y can increase the mutual information!

- **proof:** By chain rule, we can expand mutual information two different ways:

$$\begin{aligned} I(X; Y, Z) &= I(X; Z) + I(X; Y|Z) \\ &= I(X; Y) + I(X; Z|Y) \end{aligned}$$

  Since X and Z are conditionally independent given Y, we have $I(X; Z|Y) = 0$. Since $I(X; Y|Z) \geq 0$, we have

$$I(X; Y) \geq I(X; Z)$$

2.5 Fano's Inequality

- Suppose we want to estimate a random variable X from an observation Y.
- The probability of error for an estimator $\hat{X} = \phi(Y)$ is

$$P_e = \mathbf{P}\left[\hat{X} \neq X\right]$$

- **Theorem:** (Fano's Inequality) For any estimator $\hat{X}$ such that $X \to Y \to \hat{X}$,

$$H_b(P_e) + P_e \log(|\mathcal{X}|) \geq H(X|Y)$$

  and thus

$$P_e \geq \frac{H(X|Y) - 1}{\log(|\mathcal{X}|)}$$

- **Remark:** Fano's Inequality provides a lower bound on P_e for any possible function of Y!
- **Proof:**
  - Let E be a random variable that indicates whether an error has occured:

$$E = \begin{cases} 1, & \hat{X} = X \\ 0, & \hat{X} \neq X \end{cases}$$

  - By the chain rule, the entropy of (E, X) given $\hat{X}$ can be expanded two different ways

$$\begin{aligned} H(E, X|\hat{X}) &= H(X|\hat{X}) + \underbrace{H(E|X, \hat{X})}_{=0} \\ &= \underbrace{H(E|\hat{X})}_{\leq H_b(P_e)} + \underbrace{H(X|E, \hat{X})}_{\leq P_e \log|\mathcal{X}|} \end{aligned}$$

  - $H(E|\hat{X}) \leq H(E) = H_b(P_e)$ since conditioning cannot increase entropy
  - $H(E|X, \hat{X}) = 0$ since E is a deterministic function of X and $\hat{X}$.
  - By the data processing inequality,

$$H(X|\hat{X}) \geq H(X|Y)$$

  - Furthermore,

$$H(X|E, \hat{X}) = \mathbf{P}[E = 1] \underbrace{H(X|\hat{X}, E = 1)}_{=0} + \mathbf{P}[E = 0] \underbrace{H(X|\hat{X}, E = 0)}_{\leq \log|\mathcal{X}|}$$

  - Putting everything together proves the desire result.

2.6 Summary of Basic Inequalities

- **Jensen's inequality:**
  - If f is a convex function then

$$\mathbf{E}[f(X)] \geq f(\mathbf{E}[X])$$

  - if f is a concave function then

$$\mathbf{E}[f(X)] \leq f(\mathbf{E}[X])$$

- **Data Processing Inequality:** If $X \to Y \to Z$ form a Markov chain, then

$$I(X; Y) \geq I(X; Z)$$

- **Fano's Inequality:** If $X \to Y \to \hat{X}$ forms a Markov chain, then

$$\mathbf{P}\Big[X \neq \hat{X}\Big] \geq \frac{H(X|Y) - 1}{\log(|\mathcal{X}|)}$$

2.7 Axiomatic Derivation of Mutual Information [Optional]

This section is based on lecture notes from Toby Berger.

- Let X, Y denote discrete random variables with respective alphabets $\mathcal{X}$ and $\mathcal{Y}$. (Assume $|\mathcal{X}| < \infty$ and $|\mathcal{Y}| < \infty$.)
- Let $i(x, y)$ be the amount of information about event $\{X = x\}$ conveyed by learning $\{Y = y\}$
- Let $i(x, y|z)$ be the amount of information about event $\{X = x\}$ conveyed by learning $\{Y = y\}$ conditioned on the event $\{Z = z\}$
- Consider the four postulates:

  (A) **Bayesianness:** $i(x, y)$ depends only on $p(x, y)$, i.e.

  $$i(x,y) = f(\alpha,\beta)\Big|_{\substack{\alpha=p(x)\\ \beta=p(x|y)}}$$

  for some function $f : [0, 1]^2 \to \mathbb{R}$.

  (B) **Smoothness:** partial derivatives of $f(\cdot, \cdot)$ exist.

  $$f_1(\alpha,\beta) = \frac{\partial f(\alpha,\beta)}{\partial \alpha}, \quad f_2(\alpha,\beta) = \frac{\partial f(\alpha,\beta)}{\partial \beta}$$

  (C) **successive revelation:** Let $y = (w, z)$. Then

  $$i(x,y) = i(x,w) + i(x,z|w)$$

  where $i(x, w) = f(p(x), p(x|w))$ and $i(x, z|w) = f(p(x|w), p(x|z, w))$ and so the function $f(\cdot, \cdot)$ must obey

  $$f(\alpha,\gamma) = f(\alpha,\beta) + f(\beta,\gamma), \quad 0 \le \alpha,\beta,\gamma \le 1$$

  (D) **Additivity:** If (X, Y) and (U, V) are independent, i.e. $p(x, y, u, v) = p(x, y)p(u, v)$, then

  $$i((x,u),(y,v)) = i(x,y) + i(u,v)$$

  where $i(x, u) = f(p(x, u), p(x, u|y, v)) = f(p(x)p(u), p(x|y)p(u|v))$ and so the function $f(\cdot, \cdot)$ must obey

  $$f(\alpha\gamma,\beta\delta) = f(\alpha,\beta) + f(\gamma,\delta) \quad 0 \le \alpha,\beta,\gamma,\delta \le 1$$

- **Theorem:** The function

  $$i(x,y) = \log\left(\frac{p(x,y)}{p(x)p(y)}\right)$$

  is the is the only function which satisfies our four postulates above.

2.7.1 Proof of uniqueness of $i(x, y)$

- Because of B, we can apply $\frac{\partial}{\partial \beta}$ to left and right sides of C

$$0 = f_2(\alpha, \beta) + f_1(\beta, \gamma)$$

so

$$f_2(\alpha, \beta) = -f_1(\beta, \gamma)$$

Thus $f_2(\alpha, \beta)$ must be a function only of β, say $g'(\beta)$.

Integrating w.r.t. β gives

$$\int f_2(\alpha, \beta) d\beta = f(\alpha, \beta) + c(\alpha)$$

i.e.

$$\int g'(\beta) d\beta = g(\beta) = f(\alpha, \beta) + c(\alpha)$$

and so

$$f(\alpha, \beta) = g(\beta) - c(\alpha)$$

- Put this back into C

$$\begin{aligned} f(\alpha, \gamma) = g(\gamma) - c(\alpha) &= g(\beta) - c(\alpha) + g(\gamma) - c(\beta) \\ &\Rightarrow c(\beta) = g(\beta) \\ &\Rightarrow f(\alpha, \beta) = g(\beta) - g(\alpha) \end{aligned}$$

- Next, write D in terms of $g(\cdot)$

$$g(\beta\delta) - g(\alpha\gamma) = g(\beta) - g(\alpha) + g(\delta) - g(\gamma)$$

Take derivative w.r.t δ of both sides to get

$$\beta g'(\beta\delta) = g'(\delta)$$

Set $\delta = 1/2$ (could be $\delta = 1$ but scared to try)

$$\beta g'(\beta/2) = g'(1/2) = K, \quad \text{a constant}$$

and so

$$g'(\beta/2) = K/\beta$$

Take the integral of both sides with respect to β to get

$$g(\beta/2) = K \ln(\beta) + C$$

So

$$g(x) = K \ln(2x) + C$$

or

$$g(x) = K \ln(x) + \tilde{C}$$

Thus

$$f(\alpha, \beta) = g(\beta) - g(\alpha) = K \ln(\beta) - K \ln(\alpha) = K \ln(\beta/\alpha)$$

- By A,
$$i(x, y) = K \ln\left(\frac{p(x|y)}{p(x)}\right)$$
Choosing K is equivalent to choosing the log base:
  - $K = 1$ corresponds to measuring information in nats
  - $K = \log_2(e)$ corresponds to measuring information in bits