Master of Science
Sustainable Management and Technology

SUSTAINABLE AND ENTREPRENEURIAL FINANCE

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SUSTAINABLE FINANCE

Lecture 1: Fundamentals of Portfolio Management

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Objectives of the lecture

This lecture is an introduction to **Investment** (or Asset Management or Portfolio Management).

- Description of the **investment environment** (actors, markets, instruments)

- Definition of **risk and return**

- Presentation of the major steps in **asset management**:

  1- Optimal portfolio allocation
  2- Capital Asset Pricing Model (CAPM)
  3- Market efficiency

- How to measure **inputs** of the allocation process?

Objectives of the lecture

Readings

Bodie, Kane, and Marcus (2005), *Investment*, 6th edition.

⇨ **Textbook on Investment Management**

Fama, E.F. (1965), The Behavior of Stock-Market Prices, *Journal of Business*, 38(1), 34-105.

Lintner, J. (1965), "The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets", *Review of Economics and Statistics*, 47(1), 13-37.

Markowitz, H. (1952), Portfolio Selection, *Journal of Finance*, 7(1), 77-91.

Sharpe, W.F. (1964), "Capital asset prices: A Theory of Market Equilibrium under Conditions of Risk", *Journal of Finance*, 19(3), 425-442

Tobin, J. (1958), "Liquidity Preference as Behavior Towards Risk", *Review of Economic Studies*, 25.1(2), 65-86

Objectives of the lecture

➔ **Investment Environment**

- **Return and Risk**
- **Optimal Allocation**
- **Capital Asset Pricing Model**
- **Market Efficiency**
- **Predicting Risk Premia and the Covariance Matrix**

Investment Environment

Real Assets Versus Financial Assets

- **Real Assets**

- Determine the productive capacity and net income of the economy

- Examples: Land, buildings, machines, knowledge used to produce goods and services

- **Financial Assets**

- Claims on real assets, do not contribute *directly* to the productive capacity of the economy

- Examples: Fixed income or debt, Common stock or equity, Derivative securities

Real assets produce goods and services, whereas financial assets define the allocation of income or wealth among investors.

Financial assets are created and destroyed in the ordinary course of doing business.

Example: when a loan is paid off, both the creditor's claim and the debtor's obligation cease to exist. In contrast, real assets are destroyed only by accident or by time

Financial Market and the Economy

What is the role of financial markets?

1. Smoothing consumption

"Store" (e.g., using stocks or bonds) your wealth in financial assets in high earnings periods, sell these assets to provide funds for your consumption in low earnings periods (say, after retirement).

2. Allocation of risk

Virtually all real and financial assets involve some risk. If investors are uncertain about the future of GM, they can choose to buy GM's stock if they are more risk-tolerant, or they can buy GM's bonds if they are more conservative.

3. Separation of ownership and management

Let professional managers manage the firm. This generates an agency problem

- Owners can sell the firm's stocks if they do not like the incumbent management
- or "police" the managers through board of directors ("stick")
- or tie managers' income to the firm's success ("carrot")
- Other firms may acquire the firm if they observe the firm is underperforming (market discipline).

Clients of the Financial System

1 Business sector

- Firms are net borrowers
- They are concerned about how to finance their investments, through debt or equity either privately or publicly

2. Household sector

- Households are net savers
- Risk concerns: Differences in risk tolerance create demand for assets with a variety of risk-return combinations
- Tax concerns: People in different tax brackets need different financial assets with different tax characteristics

3. Government sector

- Government cannot sell equity shares
- It is restricted to borrowing to raise funds when tax revenues do not cover expenditures
- A special role of the government is in regulating the financial environment

The Financial Industry

As households are small, some financial intermediaries pool their savings and invest funds (investment companies, banks, insurance companies, pension funds)

Investment banks are hired by firms to market their securities

- Underwrite new stock and bond issues
- Sell newly issued securities to public in primary market

Investment companies pool households' savings and invest funds in exchange markets

- Advantages: Administration and record keeping, diversification and divisibility, professional management, reduced transaction costs

- Categories:
  - Exchange Traded Funds (ETF): track an index portfolio (low cost)
  - Mutual funds: more active asset management (with a benchmark)
  - Hedge funds: more sophisticated (absolute performance)
  - Real Estate Investment Trusts (REIT)

Financial Instruments

1. Money market instruments (cash for short) include short-term, marketable, liquid, low-risk debt securities

- Treasury bills

- Certificates of deposits, bankers' acceptances (banks), commercial paper (corporates)

- Eurocurrencies (deposits abroad)

- Money market mutual funds, etc.

2. Capital markets include longer-term and riskier securities.

- Bonds (Treasury notes and bonds, municipal bonds, corporate bonds, mortgages)

- Equities represent ownership shares in a corporation
  - Residual claim
  - Limited liability

- Derivatives are instruments that provide payoffs that depend on the value of other assets
  - Options
  - Futures

Stock and Bond Market Indices

Market indices are portfolios of stocks or bonds that are representative of a given market.

Example: The Dow Jones Industrial Average and the Standard & Poor's Index are representative of the U.S. market

Different ways to weigh stocks and bonds:

- **Price weighting** (DJIA): It measures the return on a portfolio that holds one share of each stock.

- **Market (or capitalization) weighting** (S&P 500, NASDAQ): It measures the return on a portfolio that is invested in proportion to the outstanding value of each stock.

- **Equal weighting** (no famous example): It measures the return on a portfolio all the constituent companies are assigned an equal value (1/N portfolio).

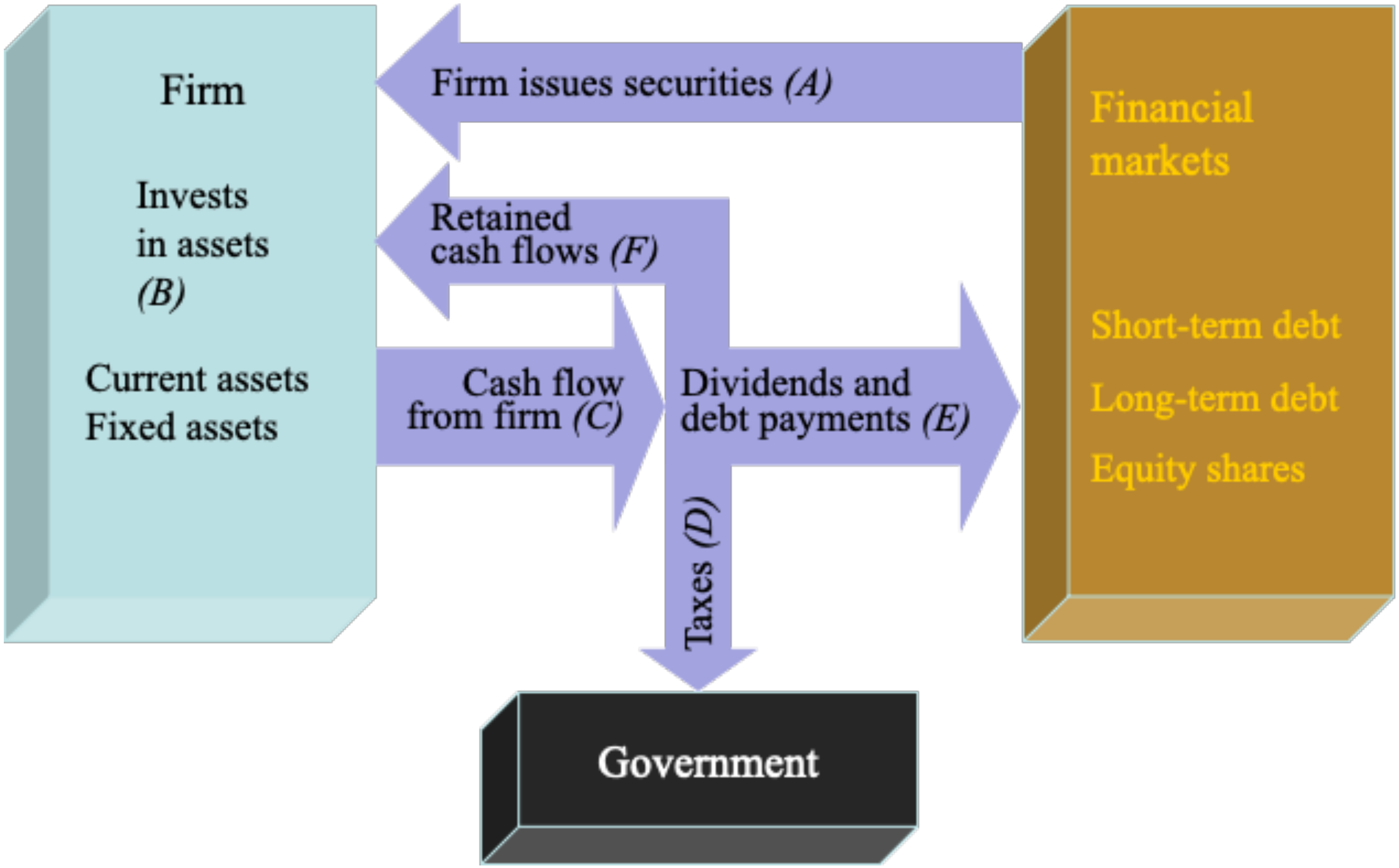
An index should be representative (DJIA 30 stocks, S&P 500, Euro Stoxx 50).

Indices are not investable in general, but there are Exchange Traded Funds (ETF) that reproduce the performance of indices.

How Securities Are Issued

Primary market

New issuance of securities. The issuer receives the proceeds from the sale.



How Securities Are Issued

Primary market

Public placement: Initial public offerings (IPO) or seasoned public offerings

- Investment bank is in general in charge of the placement to the public
- Underwriting: the bank commits on proceeds to the issuing firm

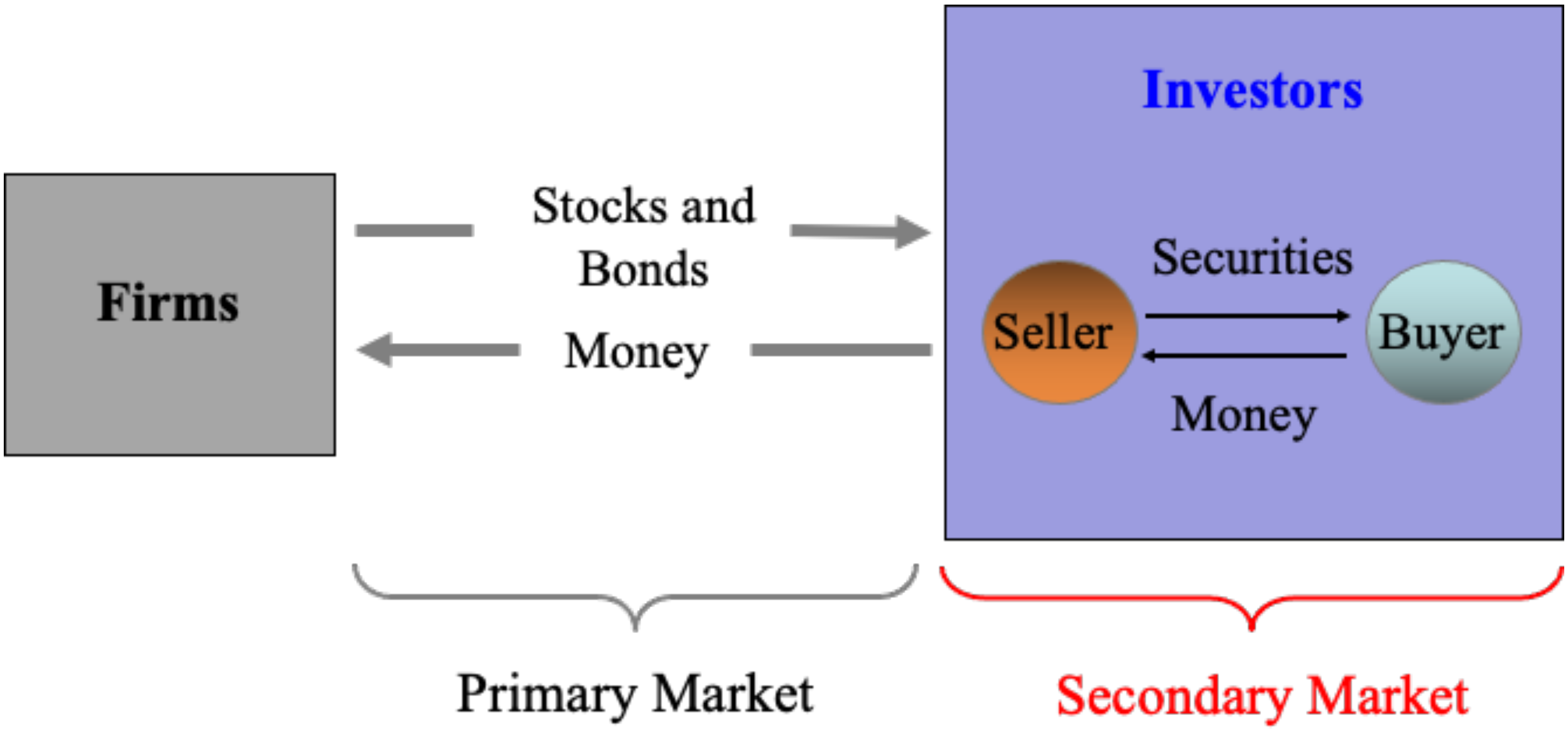
Private placement: Sale to a limited number of sophisticated investors not requiring the protection of registration

- Dominated by institutions
- Very active market for debt securities, less active for stock offerings

How Securities Are Traded

Secondary market

Existing owner sells to another party. Issuing firm does not receive proceeds.



How Securities Are Traded

Secondary market

Purchase and sale of already-issued securities take place in the secondary markets:

1. Exchanges

- There are stock exchanges in most countries
- NYSE is by far the largest stock exchange in the US (85-90% of the trading that takes place on U.S. stock exchanges)

2. Over-the-counter (OTC) market

- Many issues are traded on an OTC market and any security may be traded there
- The OTC market is not a formal exchange

- While most common stocks are traded on the exchanges, most bonds and other fixed-income securities are not. Corporate bonds are traded both on the exchanges and over the counter, but most federal and municipal bonds are traded on the OTC

3. Direct trading between two parties

How Securities Are Traded

Secondary market: Additional Aspects

Trading costs:

- Commission: fee paid to intermediary for making the transaction
- Bid-ask spread: when you trade with a dealer, the price is slightly higher when you buy than when you sell

Buying on margin:

- The investor buys an asset by borrowing part of the needed funds from a bank or broker.
- The securities bought are used as collateral.

Short sales:

- Sale of an asset or stock that the seller does not own.
- In general, the investor sells borrowed securities in anticipation of a price decline; the seller is then required to return an equal number of shares at some point in the future.

Objectives of the lecture

- Investment Environment

➔ **Return and Risk**

- Optimal Allocation

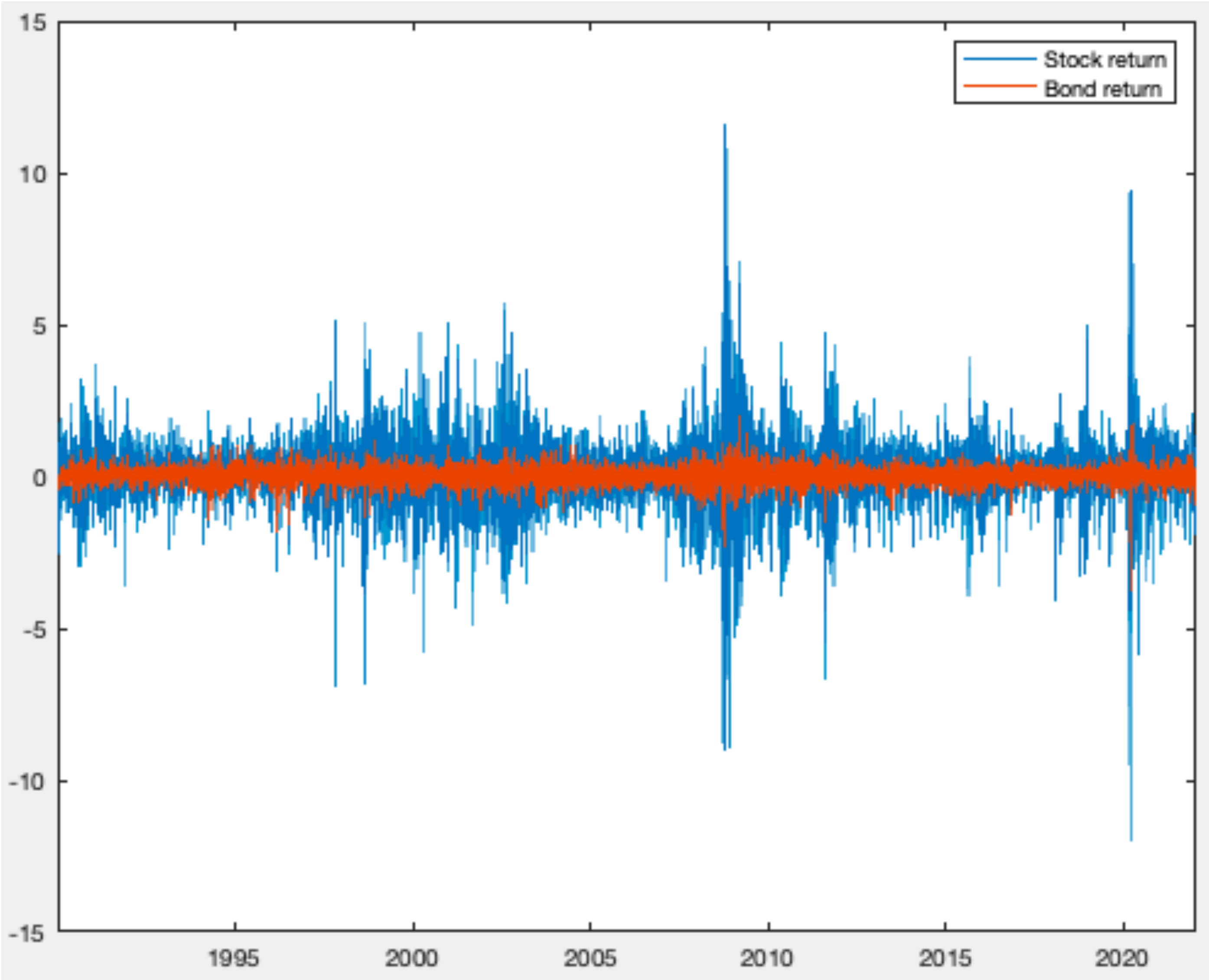
- Capital Asset Pricing Model

- Market Efficiency

- Predicting Risk Premia and the Covariance Matrix

Stock and Bond Returns

Daily Evolution of S&P 500 and BofA US Corporate bond index (total return)



Expected Return and Variance

Definition of return:

- Simple holding period return in period 1: $R_1 = \frac{P_1 - P_0 + D_1}{P_0}$

- Log holding period return in period 1: $r_1 = \log(1 + R_1) = \log(P_1 + D_1) - \log(P_0)$

where P_0 is the beginning price, P_1 is the ending price, D_1 is the dividend of the stock (or the coupon of the bond)

Consider the following world with two risky securities. There is a 1/3 chance of each state of the economy and the only securities are a stock and a bond.

Scenario	Probability	Holding period return	
		Stock	Bond
Recession	33.3%	-21.0%	17%
Normal	33.3%	1.3%	7%
Boom	33.3%	22.4%	-3%

Expected Return and Variance

Scenario	Stock		Bond	
	Return	Squared deviation	Return	Squared deviation
Recession	-7%	3.24%	17%	1.00%
Normal	12%	0.01%	7%	0.00%
Boom	28%	2.89%	-3%	1.00%
Expected return	11.00%		7.00%	
Variance	2.05%		0.67%	
Standard deviation	14.3%		8.2%	

- $E[R_S] = \frac{1}{3}(-7\%) + \frac{1}{3}(12\%) + \frac{1}{3}(28\%) = 11\%$

$[(-7\%) - (11\%)]^2 = 3.24\%$

- $E[R_B] = \frac{1}{3}(17\%) + \frac{1}{3}(7\%) + \frac{1}{3}(-3\%) = 7\%$
- $\sigma_S^2 = \frac{1}{3}(3.24\%) + \frac{1}{3}(0.01\%) + \frac{1}{3}(2.89\%) = 2.05\%$
- $\sigma_S = \sqrt{2.05\%} = 14.3\%$

Portfolio Construction

Three levels of decision:

1. **Capital allocation decision:** choice of the proportion of the overall portfolio to invest in safe but low-return money market securities versus risky but higher return securities

2. **Asset allocation decision:** distribution of risky investments across broad asset classes – stocks, bonds, real estate, foreign assets, and so on

3. **Security selection decision:** choice of which particular securities to hold within each asset class

Most institutional investors follow a **top-down approach**:

- Capital allocation and asset allocation decisions are made at a high organizational level
- Choice of the specific securities to hold within each asset class is delegated to particular portfolio managers

Most individuals adopt a **bottom-up approach**:

- Investment based on the price-attractiveness, which may result in unintended weights in the portfolio

Return and Risk of a Portfolio

Stocks have a higher expected return and higher risk than bonds. What is the risk-return tradeoff of a portfolio that is 50% invested in stocks and 50% invested in bonds?

Scenario	Rate of return		Portfolio	
	Stock	Bond	Return	Squared deviation
Recession	-7%	17%	5.0%	0.160%
Normal	12%	7%	9.5%	0.003%
Boom	28%	-3%	12.5%	0.123%
Expected return	11.0%	7.0%	9.0%	
Variance	2.05%	0.67%	0.095%	
Standard deviation	14.3%	8.2%	3.08%	

- $R_P = \alpha_S R_S + \alpha_B R_B = 50\% \times (-7\%) + 50\% \times (17\%) = 5\%$

- $E[R_P] = \alpha_S E[R_S] + \alpha_B E[R_B] = 50\% \times (11\%) + 50\% \times (7\%) = 9\%$

- $\sigma_P^2 = \frac{1}{3}(0.160\%) + \frac{1}{3}(0.003\%) + \frac{1}{3}(0.123\%) = 0.095\%$

Return and Risk of a Portfolio

Scenario	Rate of return		Portfolio	
	Stock	Bond	Return	Squared deviation
Recession	-7%	17%	5.0%	0.160%
Normal	12%	7%	9.5%	0.003%
Boom	28%	-3%	12.5%	0.123%
Expected return	11.0%	7.0%	9.0%	
Variance	2.05%	0.67%	0.095%	
Standard deviation	14.3%	8.2%	3.08%	

The portfolio variance can also be computed as follows:

$$\sigma_P^2 = (\alpha_B \sigma_B)^2 + (\alpha_S \sigma_S)^2 + 2(\alpha_B \sigma_B)(\alpha_S \sigma_S)\rho_{BS} = 0.512\% + 0.166\% - 0.585\% = 0.095\%$$

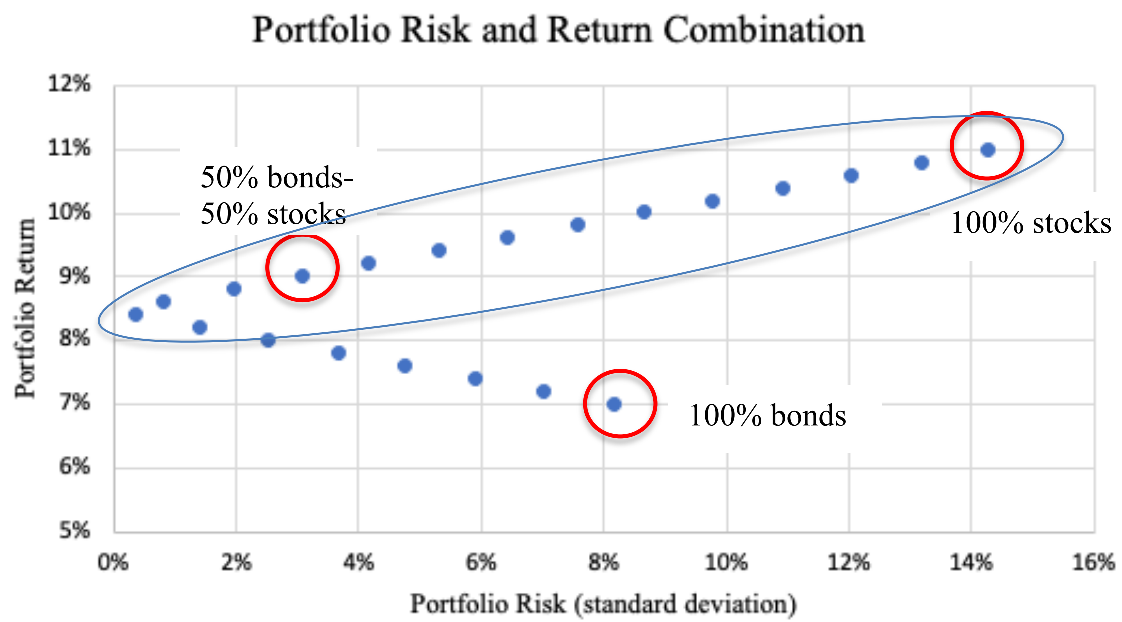
where $\rho_{BS} = \frac{Cov[R_B, R_S]}{\sigma_B \sigma_S} = -0.9988$ is the correlation coefficient between the stock and bond returns

The diversification offers a decrease in risk: An equally weighted portfolio (50% in stocks and 50% in bonds) has less risk than stocks or bonds held in isolation

Return and Risk of a Portfolio

We can consider other portfolio weights besides 50% in stocks and 50% in bonds

% in stocks	Portfolio Risk	Portfolio Return
0%	8.2%	7.0%
5%	7.0%	7.2%
10%	5.9%	7.4%
15%	4.8%	7.6%
20%	3.7%	7.8%
25%	2.6%	8.0%
30%	1.4%	8.2%
35%	0.4%	8.4%
40%	0.9%	8.6%
45%	2.0%	8.8%
50%	3.1%	9.0%
55%	4.2%	9.2%
60%	5.3%	9.4%
65%	6.4%	9.6%
70%	7.6%	9.8%
75%	8.7%	10.0%
80%	9.8%	10.2%
85%	10.9%	10.4%
90%	12.1%	10.6%
95%	13.2%	10.8%
100%	14.3%	11.0%



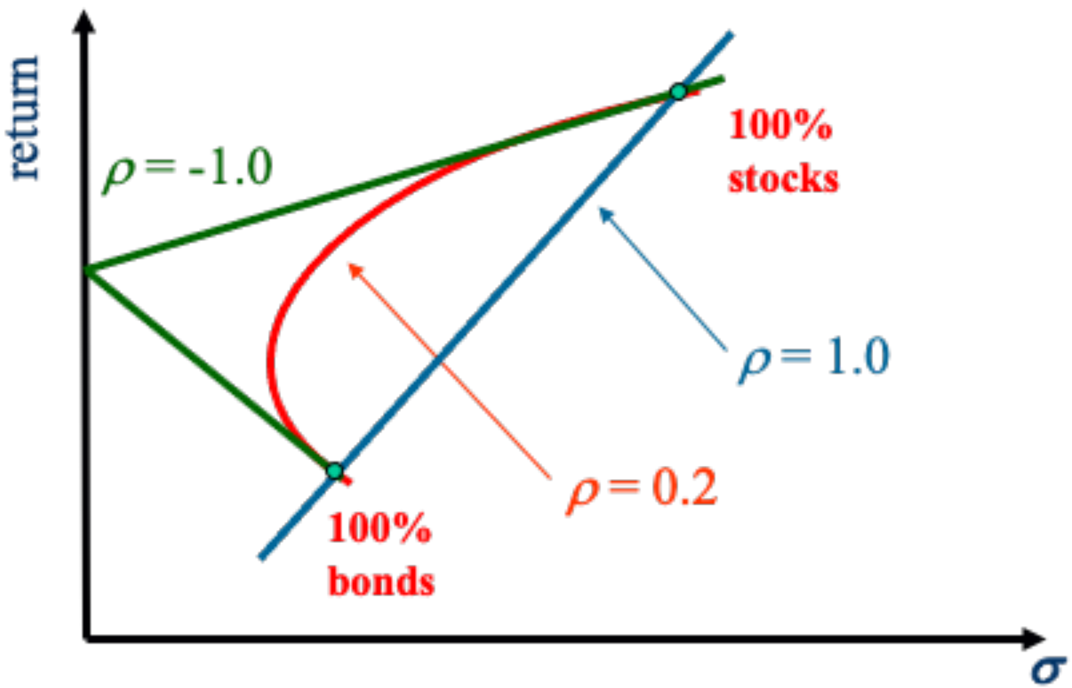
Portfolios at the top of the figure have higher returns for the same level of risk or less. These are the **efficient frontier**.

A risk-averse investor will select a portfolio on the left of the efficient frontier. A risk-lover investor will select of portfolio on the right of the frontier.

Effect of Correlation between Securities

The relationship between portfolio risk and return depends on correlation between stocks and bonds ($-1.0 < \rho < +1.0$):

- If $\rho = +1.0$, no risk reduction is possible
- If $\rho = -1.0$, complete risk reduction is possible



Portfolio of one Risky Asset and the Risk-Free Asset

Suppose that there is an asset *f* that is risk free. It satisfies $E\left[R_f\right] = R_f$ and $V\left[R_f\right] = 0$

The investor allocates α to the risky asset and $(1 - \alpha)$ to the risk-free asset. The combined portfolio is defined as:

- $R_C = \alpha R_P + (1 - \alpha) R_f$
- $E[R_C] = \alpha E[R_P] + (1 - \alpha) R_f = R_f + \alpha(E[R_P] - R_f)$
- $\sigma_C = \alpha\, \sigma_P$

If the investor does not want to take any risk ($\alpha = 0$), the portfolio return is R_f

If the investor takes risk ($\alpha > 0$), the portfolio should earn a higher return as a compensation for the risk taken. The "risk premium" is $(E[R_P] - R_f)$

Portfolio of one Risky Asset and the Risk-Free Asset

These equations give:

$$E[R_C] = R_f + \alpha(E[R_P] - R_f) = R_f + \frac{\sigma_C}{\sigma_P}(E[R_P] - R_f)$$

The expected return of the combined portfolio as a function of its standard deviation is a straight line, called the **capital allocation line**.

This relation also defines what we call the **Sharpe ratio** (reward-to-variability ratio):

$$SR = \frac{E[R_C] - R_f}{\sigma_C} = \frac{E[R_P] - R_f}{\sigma_P}$$

Example: $R_f = 5\%$, $E[R_P] = 11\%$, $\sigma_P = 14.3\%$, then $SR = 0.42$.

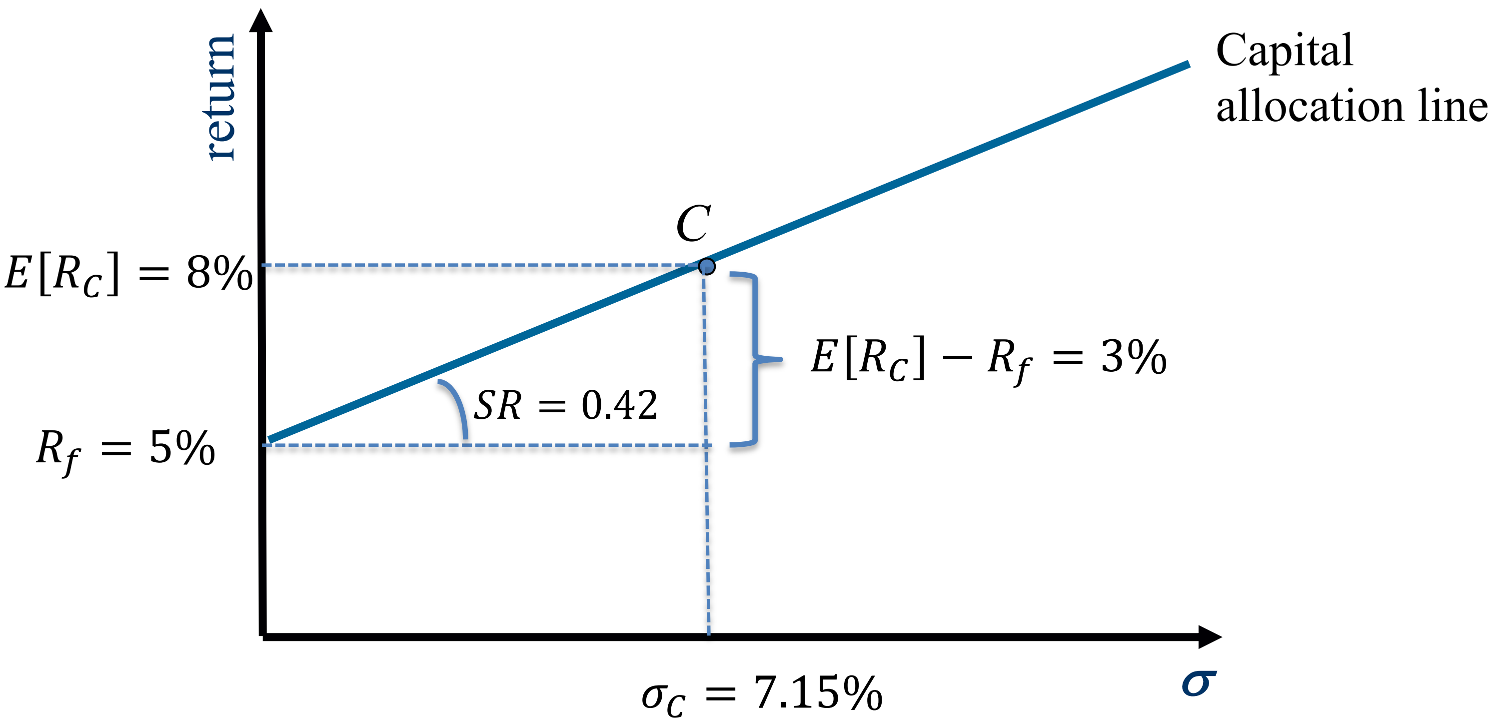
Assume $\alpha = 0.5$, then $E[R_C] = 8\%$ and $\sigma_C = 7.15\%$

Portfolio of one Risky Asset and the Risk-Free Asset

Capital allocation line

Assuming a negative weight for the risk-free asset is equivalent to borrowing at the risk-free rate. This moves the portfolio to the right of the *C* portfolio.

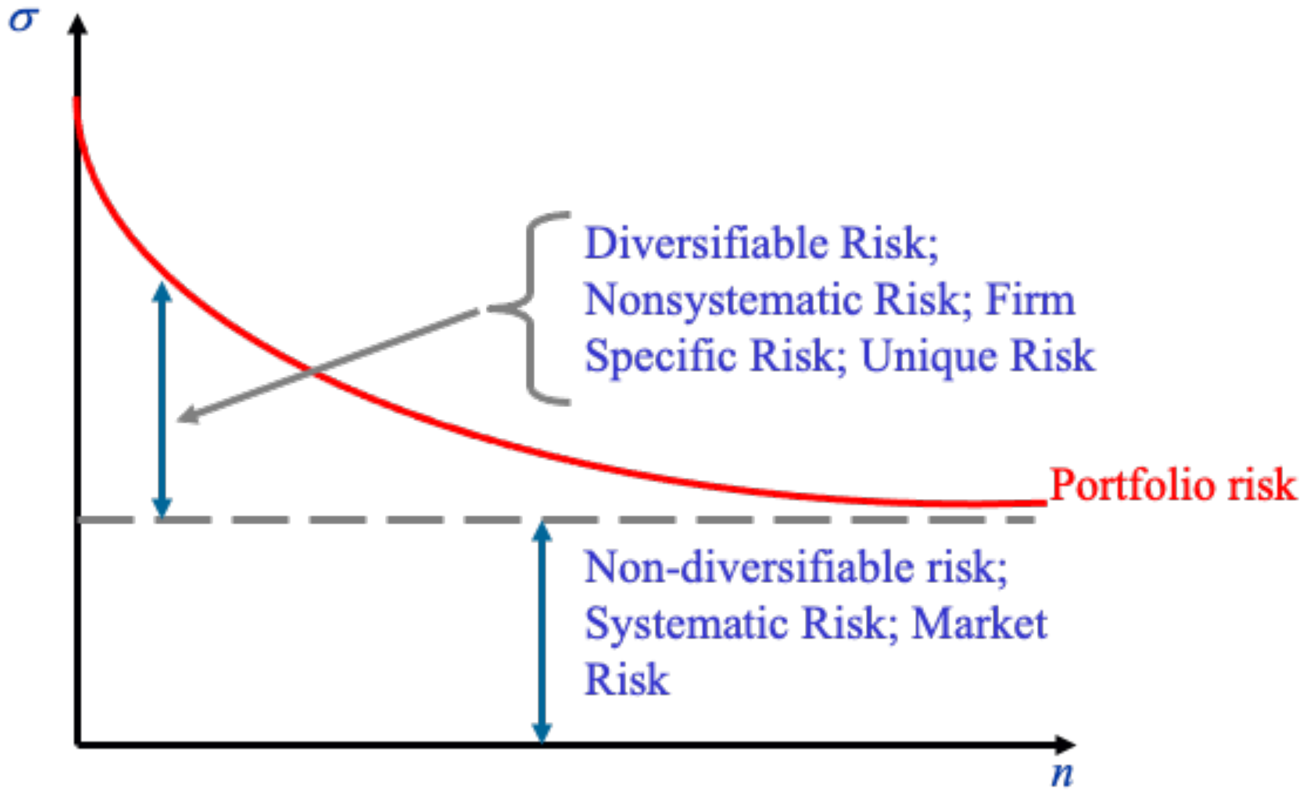
Higher levels of risk aversion result in lower proportions of the portfolio of risky assets.



Effect of the Number of Securities

In a large portfolio, the variance terms are diversified away but covariance terms are not.

Thus, diversification can eliminate some, but not all the risk of individual securities.



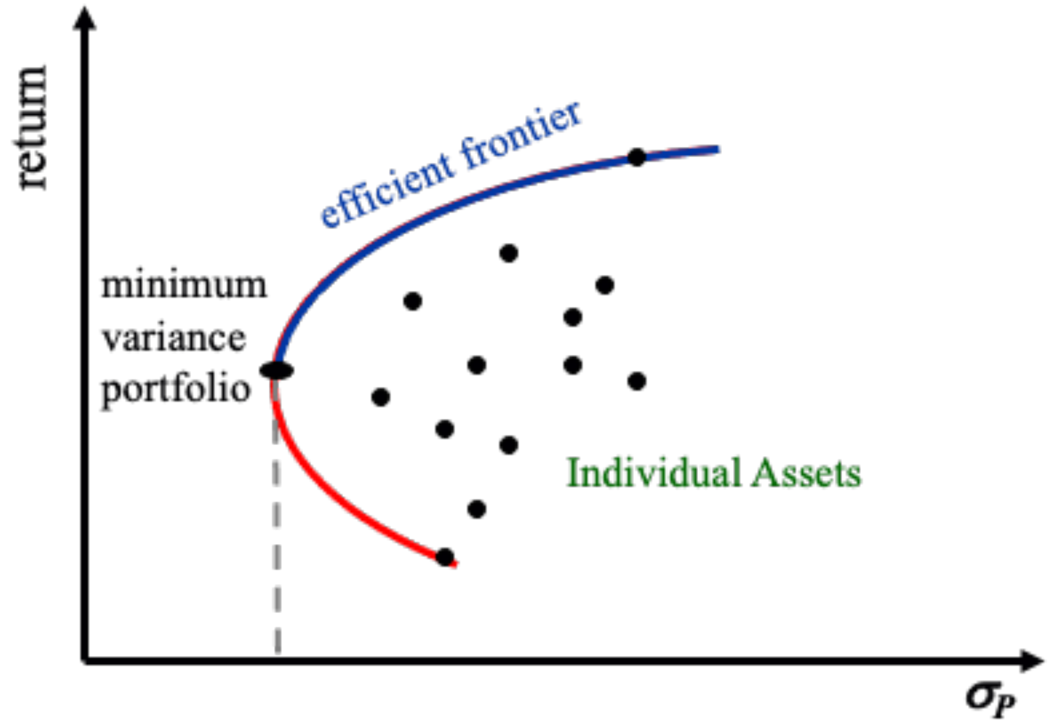
Remark: Assume $\alpha_i = 1/N$, $\sigma_i^2 = \bar{\sigma}^2$, σ_{ij}=$\bar{\rho}\bar{\sigma}^2$, for all i and j, and $N \to \infty$. Then:

$$V[R_P] = \sum_{i=1}^{N} \alpha_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j \neq i}^{N} \alpha_i \alpha_j \sigma_{ij} = N\left(\frac{1}{N}\right)^2 \bar{\sigma}^2 + N(N-1)\left(\frac{1}{N}\right)^2 \bar{\rho}\bar{\sigma}^2 \to \bar{\rho}\bar{\sigma}^2$$

Efficient Frontier with Many Risky Securities

Consider a world with many risky securities; we can still identify the **opportunity set** of risk-return combinations of various portfolios.

Given the opportunity set, we identify the **minimum variance portfolio**. The section of the opportunity set above the minimum variance portfolio is the **efficient frontier**.



Objectives of the lecture

- Investment Environment
- Return and Risk

➔ **Optimal Allocation**

- Capital Asset Pricing Model
- Market Efficiency
- Predicting Risk Premia and the Covariance Matrix

Major Steps in Asset Management

1- **Optimal portfolio allocation** (Markowitz, 1952): The optimization problem reduces to a simple mean-variance criterion.

2- **Capital Asset Pricing Model (CAPM)** (Sharpe, 1964, Lintner, 1965): At equilibrium, there is a unique factor explaining cross-section differences between firms' expected return, i.e., their **beta**

3- **Market efficiency** (Fama, 1965): If markets are efficient, abnormal returns should be unpredictable.

We now discuss these 3 steps.

Notations with Many Securities

Assume that we have *N* risky securities.

Securities returns are denoted by: $R = (R_1, \dots, R_N)'$.

Expected returns are denoted by: $E[R] = \mu = (\mu_1, \dots, \mu_N)'$.

The covariance matrix is denoted by: $\Sigma = E[(R-\mu)(R-\mu)'] = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{1N} & \cdots & \sigma_N^2 \end{pmatrix}$

Portfolio weights are denoted by: $\alpha = (\alpha_1, \dots, \alpha_N)'$, with $\alpha' e = \sum_{i=1}^N \alpha_i = 1$

Portfolio return is denoted by: $R_P = \alpha' R = \sum_{i=1}^N \alpha_i R_i$

Portfolio expected return is denoted by: $E[R_P] = \mu_P = \alpha' \mu = \sum_{i=1}^N \alpha_i \mu_i$

Portfolio variance is denoted by: $V[R_P] = \sigma_P^2 = \alpha' \Sigma \, \alpha = \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \sigma_{ij}$

Mean-Variance case (Markowitz, 1952)

On the efficient frontier, there are many portfolios. Are we really indifferent between all these portfolios? It depends on our **risk aversion** and therefore on how we trade-off risk and return.

We denote by λ the risk aversion parameter

This allows us to define the mean-variance criterion (Markowitz, 1952):

The optimal portfolio weight α^* is the solution of the problem:

$$\begin{cases} \max_{\alpha} \; \mu_P - \frac{\lambda}{2}\sigma_P^2 = \alpha'\mu - \frac{\lambda}{2}\alpha'\Sigma\,\alpha \\ \text{s.t.} \quad \alpha' e = 1 \qquad (\gamma\text{: Lagrange multiplier}) \end{cases}$$

The Lagrangian is: $\mathcal{L} = \alpha'\mu - \frac{\lambda}{2}\alpha'\Sigma\,\alpha - \gamma(\alpha' \mathrm{e} - 1)$

The derivatives are:

$$\frac{\partial L}{\partial \alpha} = \mu - \lambda\Sigma\alpha - \gamma e = 0 \qquad \text{and} \qquad \frac{\partial L}{\partial \gamma} = \alpha' e - 1 = 0$$

Mean-Variance case (Markowitz)

Proposition: When there are no restrictions on portfolio weights, the weights of the **optimal portfolio** are:

$$\alpha^* = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e} + \frac{1}{\lambda}\Sigma^{-1}\left(\mu - \frac{e'\Sigma^{-1}\mu}{e'\Sigma^{-1}e}e\right) = \alpha_{\text{gmv}} + \frac{1}{\lambda}\Sigma^{-1}\left(\mu - \frac{e'\Sigma^{-1}\mu}{e'\Sigma^{-1}e}e\right)$$

where α_{gmv} is independent from expected returns μ. Rearranging terms, we obtain:

$$\alpha^* = \left(1 - \frac{e'\Sigma^{-1}\mu}{\lambda}\right)\alpha^*_{\text{min}} + \left(\frac{e'\Sigma^{-1}\mu}{\lambda}\right)\alpha^*_{\text{spec}}$$

We obtain the well-known **Mutual Fund Separation Theorem**. Investors invest in:

- **the minimum-variance portfolio:** $\alpha^*_{\text{min}} = \frac{\Sigma^{-1}e}{e'\Sigma^{-1}e}$ with weight $\left(1 - \frac{e'\Sigma^{-1}\mu}{\lambda}\right)$

- **a speculative portfolio**: $\alpha^*_{\text{spec}} = \frac{\Sigma^{-1}\mu}{e'\Sigma^{-1}\mu}$ with weight $\left(\frac{e'\Sigma^{-1}\mu}{\lambda}\right)$

Mean-Variance case (Markowitz)

The solution without a risk-free asset – Alternative derivations

An investor minimizing the portfolio variance subject to a return constraint or maximizing the portfolio return subject to a volatility constraint will find the same solution (with no particular value for the risk aversion parameter):

$$\begin{cases} \min\limits_{\alpha} \ \sigma_P^2 = \alpha' \Sigma \, \alpha \\ \text{s.t.} \ \ \mu_P \geq \tilde{\mu}_P \ \text{ and } \ \alpha' e = 1 \end{cases} \quad \text{or} \quad \begin{cases} \max\limits_{\alpha} \ \mu_P = \alpha' \mu \\ \text{s.t.} \ \ \sigma_P \leq \tilde{\sigma}_P \ \text{ and } \ \alpha' e = 1 \end{cases}$$

Proposition: When there are no restrictions on the portfolio weights, the weights of the **minimum variance portfolio (MVP)** for a required return $\tilde{\mu}_P$ are:

$$\alpha^* = \Lambda_1 + \Lambda_2 \tilde{\mu}_P$$

where $\Lambda_1 = \frac{\Sigma^{-1}}{D}[Be - A\mu]$ and $\Lambda_2 = \frac{\Sigma^{-1}}{D}[C\mu - Ae]$

with $A = e'\Sigma^{-1}\mu$, $B = \mu'\Sigma^{-1}\mu$, $C = e'\Sigma^{-1}e$ and $D = BC - A^2$.

Remark: The collection of MVPs for various $\tilde{\mu}_P$ gives the **mean-variance efficient frontier**

Mean-Variance case (Markowitz)

The efficient frontier

The efficient frontier is given by the relation:

$$\tilde{\mu}_P = \frac{A}{C} + \sqrt{\frac{D}{C}\left(\tilde{\sigma}_P^2 - \frac{1}{C}\right)}$$

Remarks:

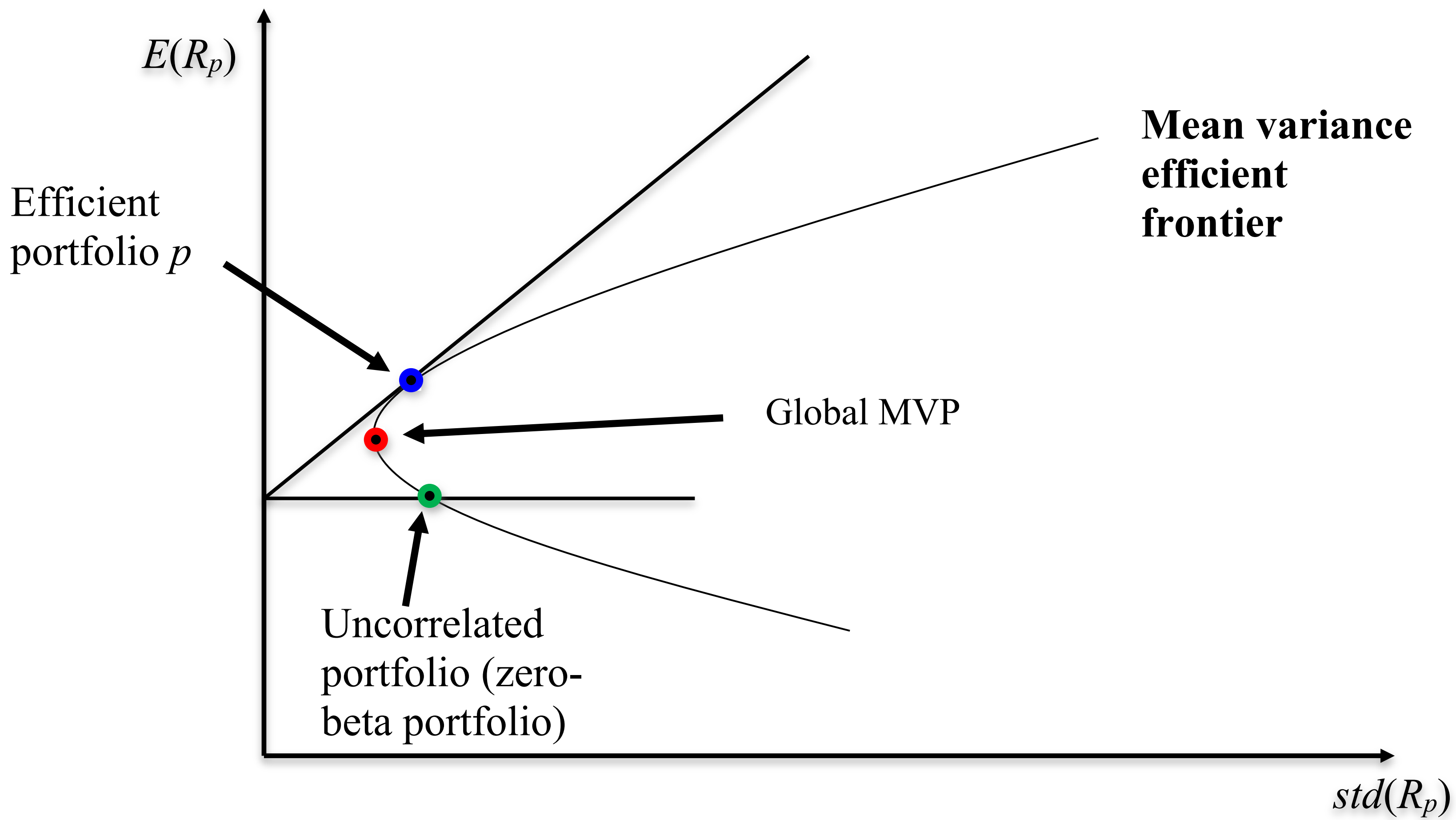
- Any portfolio of MVPs is also an MVP.
- The covariance between two efficient MVPs is always positive.
- The **Global Minimum Variance Portfolio** (Global MVP) is given by

$$\alpha_g = \frac{\Sigma^{-1}e}{C} \qquad \text{with} \qquad \mu_g = \frac{A}{C} \qquad \text{and} \qquad \sigma_g^2 = \frac{1}{C}$$

- The covariance of any asset or portfolio return R_P with the Global MVP is $Cov[R_g, R_P] = \frac{1}{C}$.

Mean-Variance case (Markowitz)

Mean-variance efficient frontier



Mean-Variance case (Tobin, 1958)

The solution with a risk-free asset

There is a risk-free asset, which the investor can borrow or lend with unlimited amount. The solution is the portfolio weight α^* that maximizes:

$$\mu_P - \frac{\lambda}{2}\sigma_P^2 = [\alpha'\mu + (1 - \alpha'e)R_f] - \frac{\lambda}{2}\alpha'\Sigma\,\alpha$$

Proposition: If there is a risk-free asset, the weights of the **optimal portfolio** are:

$\alpha^* = \frac{1}{\lambda}\Sigma^{-1}(\mu - R_f e)$ in risky assets

$1 - e'\alpha^*$ in the risk-free asset

The sum of the α^* does not necessarily sum to 1, because the investor can hold some amount of risk-free asset.

Remark: If there is one risky asset only (μ_M, σ_M^2), then $\alpha^* = \frac{\mu_M - R_f}{\lambda\,\sigma_M^2}$

Mean-Variance case (Tobin)

The solution with a risk-free asset – Alternative derivation

An investor minimizing the portfolio variance subject to a return constraint or maximizing the portfolio return subject to a volatility constraint will find the same solution:

$$\begin{cases} \min\limits_{\alpha} \sigma_P^2 = \alpha' \Sigma \, \alpha \\ \text{s.t.} \quad \mu_P = \alpha' \mu + (1 - e'\alpha) R_f \geq \tilde{\mu}_P \end{cases} \quad \text{or} \quad \begin{cases} \max\limits_{\alpha} \mu_P = \alpha' \mu + (1 - e'\alpha) R_f \\ \text{s.t.} \quad \sigma_P = \sqrt{\alpha' \Sigma \, \alpha} \leq \tilde{\sigma}_P \end{cases}$$

Proposition: If there is a risk-free asset, the weights of the risky assets in the **optimal portfolio** for a required return $\tilde{\mu}_P$ are:

$$\alpha^* = \frac{\tilde{\mu}_P - R_f}{\mu^{e\prime} \Sigma^{-1} \mu^e} \Sigma^{-1} \mu^e = \gamma \Sigma^{-1} \mu^e$$

where $\mu^e = \mu - R_f e$ is the vector of excess returns and γ is the Lagrange multiplier.

Mean-Variance case (Tobin)

The solution with a risk-free asset – Alternative derivation

Tobin's Two-fund Separation Theorem: Every mean-variance efficient portfolio is a combination of the risk-free asset and the **tangency portfolio** with weight:

$$\alpha_T = \frac{\alpha^*}{e'\alpha^*} = \frac{\Sigma^{-1}\mu^e}{e'\Sigma^{-1}\mu^e}$$

The tangency portfolio is also characterized by:

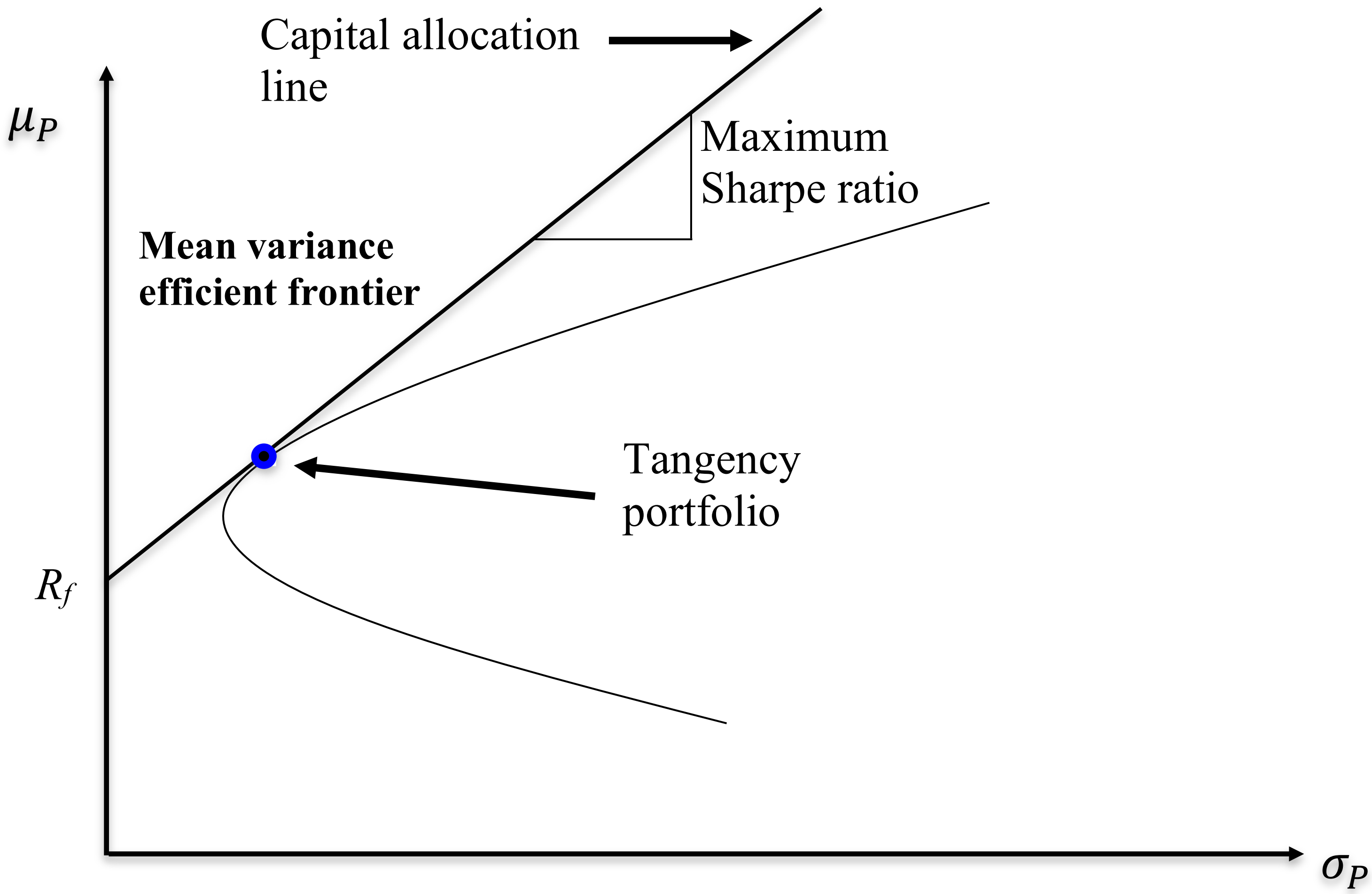
$$\mu_T - R_f = \alpha_T'\mu^e = \frac{\mu^e\Sigma^{-1}\mu^e}{e'\Sigma^{-1}\mu^e} \qquad \text{and} \qquad \sigma_T^2 = \alpha_T'\Sigma^{-1}\alpha_T = \frac{\mu^e\Sigma^{-1}\mu^e}{(e'\Sigma^{-1}\mu^e)^2}$$

so that its Sharpe ratio is

$$\frac{\mu_T - R_f}{\sigma_T} = \sqrt{(\mu - R_f e)\Sigma^{-1}(\mu - R_f e)}$$

Mean-Variance case (Tobin)

Mean-variance efficient frontier



Remark: The tangency portfolio is the risky portfolio with the maximum Sharpe ratio. It is often referred to as the **market portfolio** (true at equilibrium).

Objectives of the lecture

- Investment Environment
- Return and Risk
- Optimal Allocation

➔ **Capital Asset Pricing Model**

- Market Efficiency
- Predicting Risk Premia and the Covariance Matrix

Market Equilibrium and CAPM

Assumptions

A1. Investors are **price-takers** (believe security prices are unaffected by their own trades)

A2. All investors plan for one **identical holding period**. This behavior is myopic in that it ignores everything that might happen after the end of the single-period horizon

A3. Investments are limited to a **universe of publicly traded financial assets**, such as stocks and bonds, and to risk-free borrowing or lending arrangement. Investors may borrow or lend any amount at the risk-free rate

A4. Investors pay **no taxes** on returns and **no transaction costs** on trades in securities

A5. All investors are **rational mean-variance optimizers**, meaning that they all use the Markowitz/Tobin portfolio selection model.

A6. **Homogeneous expectations**: All investors use the same expected returns and covariance matrix of security returns to generate the efficient frontier and the unique optimal risky portfolio. They may have different aversion to risk.

Market Equilibrium and CAPM

Results

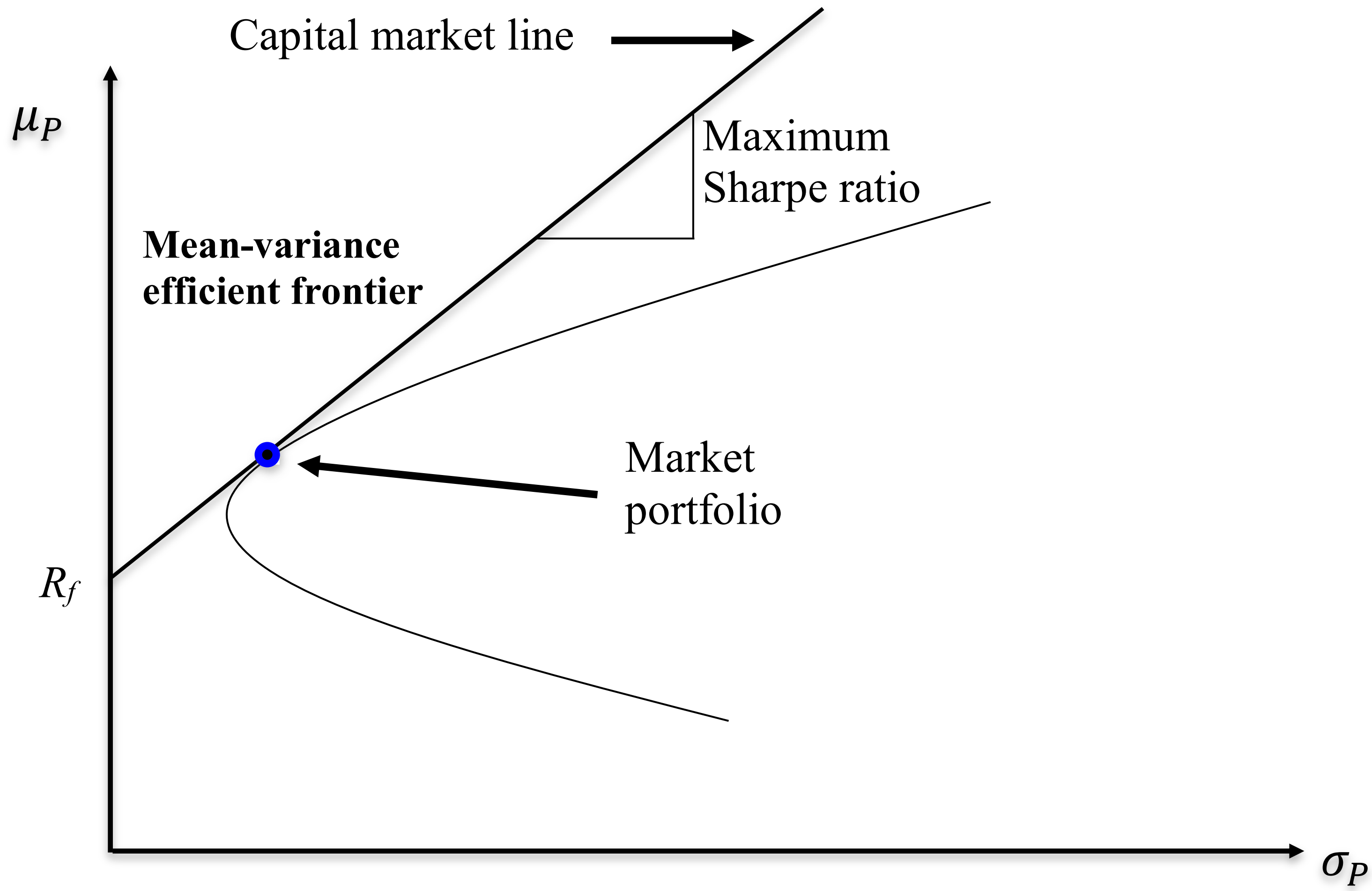
R1. All investors will choose to hold the same **market portfolio** (M), which is a market-value-weighted portfolio of all existing securities

R2. Not only will the market portfolio be on the efficient frontier, but it also will be the tangency portfolio to the optimal **capital allocation line** derived by each investor. As a result, the **capital market line (CML),** the line from the risk-free rate through the market portfolio, M, is also the best attainable capital allocation line.

All investors hold M as their optimal risky portfolio, differing only in the amount invested in it versus in the risk-free asset.

R3. The **risk premium on the market portfolio** will be proportional to its risk and the degree of risk aversion of the representative investor: $E[R_M] - R_f = \lambda\, \sigma_M^2 \quad (\alpha^* = 1)$

Market Equilibrium and CAPM



Market Equilibrium and CAPM

Results (cont'd)

R4. The **risk premium on individual assets** is proportional the risk premium on the market portfolio, *M*, and the beta coefficient of the security relative to the market portfolio. Beta measures the extent to which returns on the stock and the market move together. Beta is defined as $\beta_i = \frac{cov[R_i,R_M]}{V[R_M]} = \frac{e_i'\Sigma\alpha}{\alpha'\Sigma\alpha}$

The risk premium on individual securities is (price times quantity of risk)

$$\underbrace{E[R_i] - R_f}_{\substack{\text{Expected return} \\ \text{of asset } i}} = \underbrace{\frac{E\left[R_M - R_f\right]}{\sigma_M}}_{\text{Price of risk}} \times \underbrace{\frac{cov[R_i, R_M]}{\sigma_M}}_{\substack{\text{Systematic risk} \\ \text{of asset } i}}$$

or

$$E[R_i] - R_f = \frac{cov[R_i, R_M]}{V[R_M]} E\left[R_M - R_f\right] = \beta_i E\left[R_M - R_f\right]$$

CAPM: The risk premium of the asset *i* is equal to its beta times the excess return of the market portfolio. Assets with higher systematic risk (higher beta) are priced to offer higher expected return.

CAPM and Multi-factor Models

CAPM was first published by Sharpe (1964) and Lintner (1965)

Limitations to CAPM

- Market Portfolio is not directly observable
- Research shows that other factors affect returns

Fama French Three-Factor Model:

- Market beta
- Size
- Book value relative to market value

Fund managers are often evaluated based on running the adjusted CAPM regression:

$$R_{i,t} - R_f = \alpha_i + \beta_i (R_{b,t} - R_f) + \varepsilon_{i,t}$$

where $R_{i,t}$ is the fund return and $R_{b,t}$ is the return of the benchmark adopted by the fund

Passive management = beta **versus** **Active management = alpha**

Objectives of the lecture

- Investment Environment
- Return and Risk
- Optimal Allocation
- Capital Asset Pricing Model

➔ **Market Efficiency**

- Predicting Risk Premia and the Covariance Matrix

Efficient Market Hypothesis (EMH)

In an **efficient** capital market, stock prices fully reflect available information (Fama, 1965)

Efficient Market Hypothesis (EMH) has implications for investors and firms:

– Since information is reflected in security prices quickly, knowing information *when it is released* does an investor no good.
– Firms should expect to receive the fair value for securities that they sell. Firms cannot profit from fooling investors in an efficient market.

Three versions of the EMH:

- **Weak Form**
  - Security prices reflect all information found in past prices and volume
  - Technical analysis (using prices and volumes) fails
- **Semi-Strong Form**
  - Security prices reflect all publicly available information
  - Fundamental analysis (using economic and accounting information) fails
- **Strong Form**
  - Security prices reflect all information (public and private)
  - Even insider trading fails

Implications of Efficient Market Hypothesis

Time-series implications

- A **stock price** is a **random walk** with a positive trend (compensation for risk taking)

- **Expected returns** (risk premia) are given by an equilibrium model (e.g., CAPM)

- As the flow of information is random, **abnormal returns** are **unpredictable** conditional on a given information set (depending on the version of the EMH)

- There is no free lunch: in efficient markets, underpriced or overpriced securities do not exist, so it is not possible to design a strategy that will generate an abnormal return on a regular basis.

- Timing the market does not work.

Implications of Efficient Market Hypothesis

Investment implications

- **Fundamental analysis** does not work because the quality of firms is already public. The only form that can work is security analysis: find firms that are better than everyone else's estimated

- Under EMH, active management is wasted effort and costly

- The only worthy strategy is **passive management**:

  - Investors should invest in a buy-and-hold strategy
  - Investors should invest in a highly diversified portfolio (market portfolio)
  - Investment firms should create an index fund and the fund manager should only tailor the portfolio to the needs of the investors

Efficient Market Hypothesis (EMH)

Evidence

1. The weak form of EMH is not rejected by the data

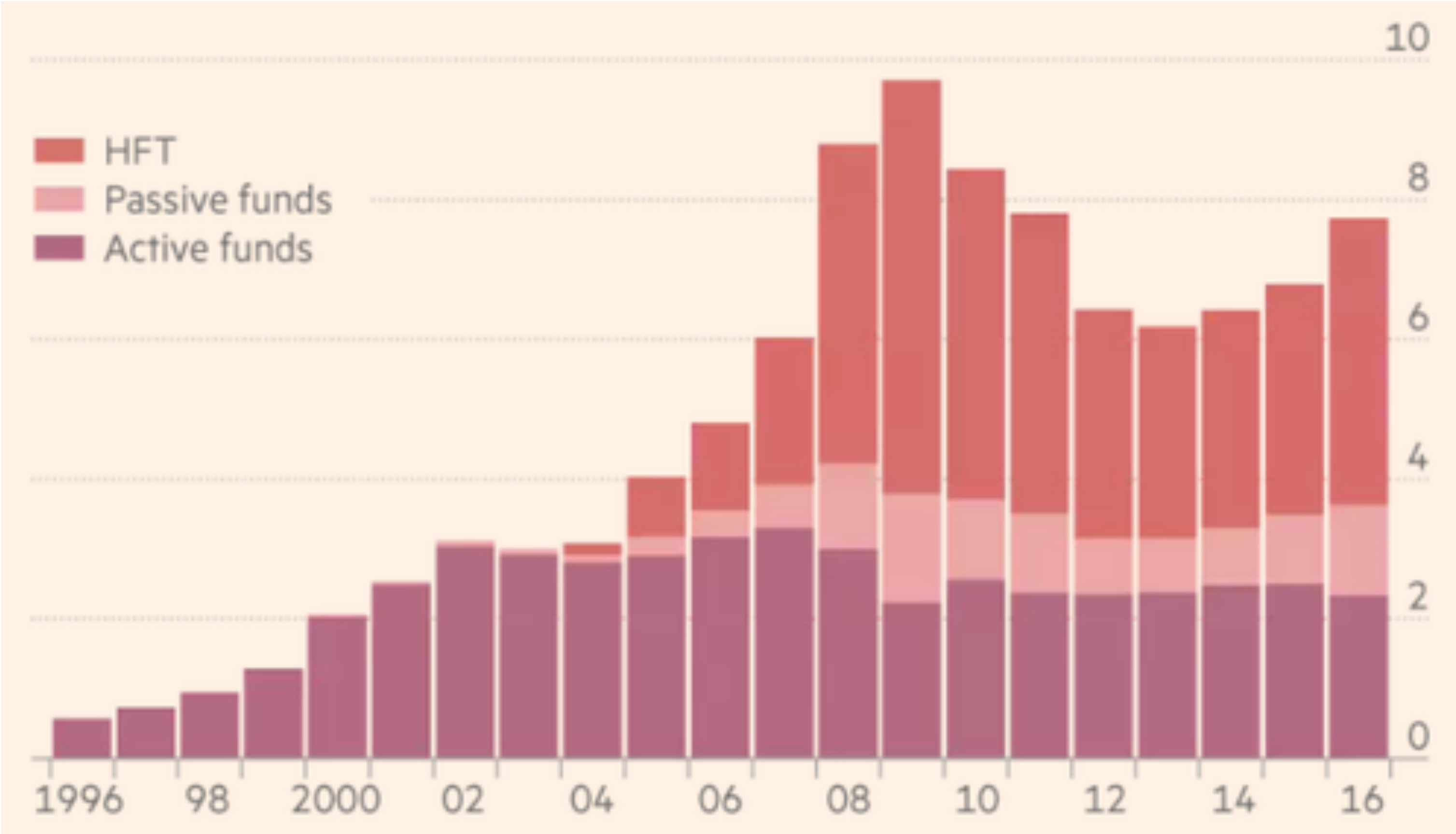
2. The performance of professional managers is broadly consistent with market efficiency (semi-strong form). The amounts by which professional managers as a group beat or are beaten by the market fall within the margin of statistical uncertainty.

3. Markets are very efficient, but especially diligent, intelligent, or creative investors may probably make more money than the average investor.

4. Having access to private information is valuable. In general, it is prohibited (insider trading). What about high frequency trading?

Efficient Market Hypothesis (EMH)

High frequency trading (avg daily volume of US equity)



Objectives of the lecture

- Investment Environment
- Return and Risk
- Optimal Allocation
- Capital Asset Pricing Model
- Market Efficiency

➔ **Predicting Risk Premia and the Covariance Matrix**

Estimating Risk Premia: Unconditional Mean

Given historical prices (including dividends) $\{P_0, P_1, \ldots, P_T\}$, we compute simple returns as $R_t = (P_t - P_{t-1})/P_{t-1}$ and log-returns as $r_t = \log(P_t) - \log(P_{t-1})$. Using the log-risk-free rate, r_f, we obtain excess log-returns, $\{r_1^e, \ldots, r_T^e\}$. We then estimate unconditional risk premia as:

$$\hat{\mu}^e = \frac{1}{T}\sum_{t=1}^{T} r_t^e$$

Properties:

- easy to compute and update
- assumes constant risk premia
- with iid normal returns, the central limit theorem states that $\sqrt{T}(\hat{\mu}^e - \mu^e) \sim N(0, \Sigma)$

Example: with $\hat{\mu}^e = 6\%$, $\sigma = 15\%$, and $T = 120$ (10 years of monthly data), the standard error of the sample mean is $std[\hat{\mu}^e] = 15\%/\sqrt{120} = 1.4\%$, implying a 95% confidence interval of $[3.25\%; 8.75\%]$

Risk premia are difficult to estimate precisely.

Remark: Most models are designed for daily returns. However, the asset allocation is often performed at the weekly or monthly frequency (→ temporal aggregation).

Estimating Risk Premia: Linear Conditional Mean

Conditional risk premia can be defined as linear functions of macro-economic variables or firm-specific variables:

- Time series:

$$\mu_{i,t}^e = a_i + b_i' Z_t \qquad t = 1, \ldots, T$$

where Z_t may include business cycle indicators, such as interest rate, inflation rate, or dividend yield.

- Cross section:

$$\mu_{i,t}^e = a_i + b_i' Z_{i,t} \qquad t = 1, \ldots, T, i = 1, \ldots, N$$

where $Z_{i,t}$ may be the market beta $\beta_{i,t}$.

Factor Fishing

- **Theory**
  - Market portfolio (CAPM) or Intertemporal hedge portfolios: portfolios maximally correlated with changes in investment opportunities (I-CAPM)

- **Fundamental factors (accounting-based)**
  - Size
  - Book to market
  - Earnings to market
  - Cash-flow to market
  - Dividend yield
  - Industry factors

- **Macroeconomic factors**
  - Default premium
  - Term premium
  - Industrial production
  - Inflation

- **Statistical procedure**
  - Factor analysis or Principal components

Estimating Covariance Matrix: Unconditional Moment

Given excess returns, $\{r_1^e, \dots, r_T^e\}$, sample covariance matrix can be estimated as:

$$V[r_i^e] = \frac{1}{T-1}\sum_{t=1}^{T}\left(r_{i,t}^e - \hat{\mu}_i^e\right)^2 \qquad \text{and} \qquad Cov\left[r_i^e, r_j^e\right] = \frac{1}{T-1}\sum_{t=1}^{T}\left(r_{i,t}^e - \hat{\mu}_i^e\right)\left(r_{j,t}^e - \hat{\mu}_j^e\right)$$

Properties:

- easy to compute and update
- assumes constant return distribution

However:

- the horizon of the allocation is not necessarily the same as the frequency of the data
- volatilities and correlations are time varying

Remark: For N assets, the number of parameters to estimate is $N(N+1)/2$.

- For 500 stocks and 600 months of data (50 years): 125’250 parameters and 300’000 observations, so that each parameter is estimated with 2.4 observations on average.

- The sample covariance matrix is singular when $N > T - 1$.

Estimating Time-Varying Moments

An alternative approach that accounts for time-variability is the use of **Rolling Windows**

- Consider the first τ observations ($\tau \ll T$)
- Estimate risk premia (μ_τ^e) and the covariance matrix (Σ_τ) using $t = 1, \ldots, \tau$
- Roll the sample by 1 observation
- Estimate risk premia ($\mu_{\tau+1}^e$) and the covariance matrix ($\Sigma_{\tau+1}$) using $t = 2, \ldots, \tau + 1$
- …
- Estimate risk premia (μ_T^e) and the covariance matrix (Σ_T) using $t = T - \tau + 1, \ldots, T$

A slightly better estimator is the **Exponentially Weighted Moving Average (EWMA)**

- Consider a memory parameter ϕ ($0 \leq \phi \leq 1$), an initial risk premium (μ_0^e), and an initial covariance matrix Σ_0. μ_0^e and Σ_0 can be based on a pre-sample.
- At date *t*, compute the risk premium and the covariance matrix as:

$$\mu_t^e = \phi \mu_{t-1}^e + (1 - \phi) r_t^e \qquad \text{and} \qquad \Sigma_t = \phi \Sigma_{t-1} + (1 - \phi)(r_t^e - \mu_t^e)(r_t^e - \mu_t^e)'$$

Remark: Problems related to the number of observations are even more severe.