# Hodge theory of $p$-adic varieties: a survey

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**Abstract.** $p$-adic Hodge theory is one of the most powerful tools in modern arithmetic geometry. In this survey, we will review $p$-adic Hodge theory of algebraic varieties, present current developments in $p$-adic Hodge theory of analytic varieties, and discuss some of its applications to problems in number theory. This is an extended version of a talk at the Jubilee Congress for the 100th anniversary of the Polish Mathematical Society, Kraków, 2019.

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**1. Introduction.** $p$-adic Hodge theory is one of the most powerful tools in modern arithmetic geometry. In this survey, we will review $p$-adic Hodge theory of algebraic varieties, present current developments in $p$-adic Hodge

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**2. Algebraic varieties.** We will first consider algebraic varieties, starting with the classical situation over complex numbers.

**2.1. A comparison theorem over C.** Let $X$ be a smooth and projective algebraic variety over the rational numbers $\mathbf{Q}$. Recall that, by the classical de Rham theorem, there exists a nondegenerate pairing ($n \geq 0$ is an arbitrary integer)

$$H_{\mathrm{dR}}^n(X_{\mathbf{C}}) \times H_n(X(\mathbf{C}), \mathbf{C}) \to \mathbf{C}, \quad (\omega, \gamma) \mapsto \int_{\gamma} \omega.$$

Here $H_n(X(\mathbf{C}), \mathbf{C})$ is the singular homology with complex coefficients of the topological space $X(\mathbf{C})$ and $H_{\mathrm{dR}}^n(X_{\mathbf{C}})$ is the algebraic de Rham cohomology of $X_{\mathbf{C}}$:

$$H_{\mathrm{dR}}^n(X_{\mathbf{C}}) := H^n(X_{\mathbf{C}}, \mathscr{O}_{X_{\mathbf{C}}} \to \Omega_{X_{\mathbf{C}}}^1 \to \Omega_{X_{\mathbf{C}}}^2 \to \cdots).$$

The pairing is obtained by integrating differential forms along cycles. Using resolutions of singularities, and changing de Rham cohomology and singular homology appropriately, this pairing can be extended to all algebraic varieties. The values obtained are classically called *periods*. Hence, we can say that:

> $\mathbf{C}$ *contains all periods of algebraic varieties over* $\mathbf{Q}$.

EXAMPLE 2.1. Here are a couple of examples of periods:

(1) Let $\gamma$ denote the unit circle in the complex plane. We have

$$\int_{\gamma} \frac{\mathrm{d}z}{z} = 2\pi i.$$

To put this integral in the context discussed above, take $X = \mathbb{G}_{m,\mathbf{Q}} = \mathbb{A}_{\mathbf{Q}}^1 \setminus \{0\}$, the punctured affine space. Then $\gamma \in H_1(X(\mathbf{C}), \mathbf{C})$, $\omega = \mathrm{d}z/z \in H_{\mathrm{dR}}^1(X_{\mathbf{C}})$, and $(\omega, \gamma) = 2\pi i$.

(2) Consider now the integral in the complex plane

$$2 \int_{1}^{+\infty} \frac{dx}{\sqrt{x^3 - x}} = \frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)},$$

where $\Gamma(-)$ denotes Euler's $\Gamma$-function. This is a period of the elliptic curve $E$ with equation $y^2 = x^3 - x$. (Modulo a sign) this is the integral of $\omega = \mathrm{d}x/y$ along the path in $E(\mathbf{C})$ whose projection in the projective space $\mathbb{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$ (by the map $(x, y) \mapsto x$) consists of the segment

$(\infty, 1 + \varepsilon]$, followed by the circle of center 1 and radius $\varepsilon$, followed by the segment $[1 + \varepsilon, \infty)$. This integral does not depend on $\varepsilon$ and when $\varepsilon \to 0$ the contribution from the circle tends towards 0. Since $\sqrt{x^3 - x}$ changes the sign going around the circle with center 1, the integrals $\int_1^\infty$ and $\int_\infty^1$ are equal. Hence the integral along the whole path in $E(\mathbf{C})$ is $2 \int_1^{+\infty} \frac{dx}{\sqrt{x^3 - x}}$.

We note that we started here with an algebraic variety $X$ over $\mathbf{Q}$, then we completed $\mathbf{Q}$ with respect to its natural Archimedean valuation, i.e. the usual absolute value, to obtain the field of real numbers $\mathbf{R}$ and the base changed variety $X_{\mathbf{R}}$, and then we passed to the algebraic closure $\mathbf{C}$ of $\mathbf{R}$ and the corresponding variety $X_{\mathbf{C}}$. We can illustrate this process by the picture:

$$\mathbf{Q} \mapsto \widehat{\mathbf{Q}} \simeq \mathbf{R} \hookrightarrow \overline{\mathbf{R}} \simeq \mathbf{C}.$$

**2.2. Étale cohomology and associated Galois representations.** Grothendieck defined étale cohomology as an algebraic replacement of singular cohomology. Before we proceed to review its properties let us make a small digression.

**2.2.1.** *Digression: non-Archimedean completions.* Besides the Archimedean valuation on $\mathbf{Q}$, we also have non-Archimedean valuations [1] indexed by prime numbers. The relevant picture is now

$$\mathbf{Q} \mapsto \widehat{\mathbf{Q}} =: \mathbf{Q}_p \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \widehat{\overline{\mathbf{Q}}}_p =: \mathbf{C}_p.$$

Here

(1) $p$ is a prime number. If $|\bullet| = |\bullet|_p$ is the $p$-adic norm on $\mathbf{Q}$, normalized with $|p| = p^{-1}$, we have $|xy| = |x||y|$ and $|x + y| \le \max(|x|, |y|)$.

(2) $\mathbf{Q}_p$ is the completion of $\mathbf{Q}$ for the $p$-adic norm $|\bullet|$ (introduced by Hensel, 1897). We have

$$x \in \mathbf{Q}_p \implies x = \sum_{n \ge n_0} x_n p^n, \; x_n \in \{0, \ldots, p-1\}, \quad \mathbf{Q}_p = \mathbf{Z}_p[1/p],$$

$$\mathbf{Z}_p := \{x \in \mathbf{Q}_p \mid |x| \le 1\}, \quad \mathbf{Z}_p \simeq \varprojlim_n \mathbf{Z}/p^n,$$

$$\mathbf{Z}_p \text{ “=” } \{0, 1, \ldots, p-1\}[[p]].$$

(3) $\overline{\mathbf{Q}}_p$ is the algebraic closure of $\mathbf{Q}_p$; the norm $|\bullet|$ extends uniquely to $\overline{\mathbf{Q}}_p$; the Galois group $G_{\mathbf{Q}_p} := \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts via isometries. $\overline{\mathbf{Q}}_p$ is infinite-dimensional (the polynomial $x^n - p$ is irreducible in $\mathbf{Q}_p[x]$), and not complete for $|\bullet|$.

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[1] By Ostrowski's theorem these valuations, taken together with the natural Archimedean valuation, constitute all nontrivial valuations on $\mathbf{Q}$ (taken up to equivalence).

(4) $\mathbf{C}_p$ is the completion of $\overline{\mathbf{Q}}_p$ and we have $G_{\mathbf{Q}_p} = \mathrm{Aut}_{\mathrm{cont}}(\mathbf{C}_p)$; $\dim_{\mathbf{Q}_p} \mathbf{C}_p$ is not countable. The axiom of choice produces isomorphisms of abstract fields $\mathbf{C}_p \simeq \mathbf{C}$.

**2.2.2** *Étale cohomology.* Let us now come back to the nondegenerate pairing
$$H_{\mathrm{dR}}^n(X_{\mathbf{C}}) \times H_n(X(\mathbf{C}), \mathbf{C}) \to \mathbf{C}, \quad (\omega, \gamma) \mapsto \int_\gamma \omega.$$

Using the singular cohomology $H_B^n(X(\mathbf{C}), \mathbf{Q})$ (known in the subject as *Betti cohomology*), which is dual to singular homology, this can be written as a $\mathbf{C}$-linear isomorphism

(2.2) $$H_{\mathrm{dR}}^n(X) \otimes_{\mathbf{Q}_p} \mathbf{C} \simeq H_B^n(X(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}.$$

Fix a prime $p$. It is known that, for $p$-adic coefficients $\mathbf{Q}_p$, we have the isomorphism

(2.3) $$H_B^n(X(\mathbf{C}), \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}_p \simeq H_{\mathrm{\acute{e}t}}^n(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$$

of Betti cohomology with (geometric) étale cohomology. The latter cohomology has the following properties:

(1) The vector space $H_{\mathrm{\acute{e}t}}^n(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ is finite-dimensional over $\mathbf{Q}_p$; it inherits a continuous action of $G_{\mathbf{Q}_p}$ (from its natural action on $X_{\overline{\mathbf{Q}}_p}$). The $\mathbf{Q}_p$-dimension of $H_{\mathrm{\acute{e}t}}^n(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ is equal to the $\mathbf{Q}$-dimension of Betti cohomology, hence also to the $\mathbf{Q}$-dimension of de Rham cohomology (see (2.3) and (2.2)).
(2) Locally in the Zariski topology, étale cohomology is the group cohomology of the fundamental group: $H_{\mathrm{\acute{e}t}}^n(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p) \simeq H^n(\pi_1(X_{\overline{\mathbf{Q}}_p}), \mathbf{Q}_p)$, where $\pi_1(X_{\overline{\mathbf{Q}}_p})$ is the algebraic fundamental group (which is the profinite completion of the classical fundamental group).

**2.2.3.** *Galois representations coming from étale cohomology.* The Galois action on étale cohomology $H_{\mathrm{\acute{e}t}}^n(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ carries information about

(1) finite field extensions of $\mathbf{Q}_p$,
(2) the arithmetic of $X$, for example, its rational points $X(\mathbf{Q})$ (see Section 2.7).

Let us look at some examples of such representations.

EXAMPLE 2.4. *Galois representations on* $H_{\mathrm{\acute{e}t}}^n(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$:

(1) *Tate twists*: We have the cyclotomic character
$$\chi : G_{\mathbf{Q}_p} \to \mathbf{Z}_p^* : \quad \sigma(e^{\frac{2\pi i}{p^n}}) = e^{\chi(\sigma)\frac{2\pi i}{p^n}}, \ n \geq 1.$$

If $i \in \mathbf{Z}$, denote by $\mathbf{Q}_p(i)$ the $i$th Tate twist: it is $\mathbf{Q}_p$ with the action of $G_{\mathbf{Q}_p}$ given by $\chi^r$. We have

$$(2.5) \qquad \mathbf{Q}_p(1) \simeq \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \varprojlim_n \mathbb{G}_m(\overline{\mathbf{Q}}_p)_{p^n}, \qquad \mathbf{Q}_p(1) \simeq H^2_{\text{ét}}(\mathbb{P}^1_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)^*.$$

Here, the subscript $p^n$ refers to the $p^n$-torsion elements and $\mathbb{P}^1$ denotes the projective line.

(2) *Tate modules*: Let $E$ be an elliptic curve over $\mathbf{Q}$. Then $E$ is a group variety: there is a map of varieties $E \times E \to E$ making the rational points $E(\overline{\mathbf{Q}}_p)$ an abelian group. We have the integral and rational Tate modules of $E$,

$$T_p E := \varprojlim_n E(\overline{\mathbf{Q}}_p)_{p^n}, \quad V_p E := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p E,, \quad \dim_{\mathbf{Q}_p} V_p E = 2.$$

Cohomologically,

$$V_p E \simeq H^1_{\text{ét}}(E_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)^*.$$

We note that we could add to (2.5) the isomorphism

$$(2.6) \qquad\qquad \mathbf{Q}_p(1) \simeq V_p \mathbb{G}_m \simeq H^1_{\text{ét}}(\mathbb{G}_{m,\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)^*.$$

(3) *Curves*: Let $X$ be a smooth projective geometrically connected curve of genus $g$ over $\mathbf{Q}$. If $g > 1$, then $X$ itself does not have a group law but its Jacobian $J$ does. Proceeding as in the case of an elliptic curve $E$, we get

$$T_p J := \varprojlim_n J(\overline{\mathbf{Q}}_p)_{p^n}, \quad V_p J := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p J,, \quad \dim_{\mathbf{Q}_p} V_p J = 2g.$$

Cohomologically,

$$V_p J \simeq H^1(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)^*.$$

**2.3. The main question of geometric $p$-adic Hodge theory.** Let us pass now to the local setting. That is, $X$ will be now a variety over a finite field extension $(^2)$ $K$ of $\mathbf{Q}_p$. The main question of the classical geometric $p$-adic Hodge theory was the following:

Does there exist a $p$-adic period ring $\mathbf{B}$ that contains all periods of algebraic varieties $X$ over $K$, for all $[K : \mathbf{Q}_p] < \infty$, and a pairing $(\omega, \gamma) \mapsto (\int_\gamma \omega) \in \mathbf{B}$ such that

(1) $H^n_{\text{dR}}(X) \otimes_K \mathbf{B} \simeq H^n_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}$,
(2) using the period isomorphism (1) we can recover the Galois representation on the étale cohomology $H^n_{\text{ét}}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$ from the de Rham cohomology $H^n_{\text{dR}}(X)$?

REMARK 2.7. It was observed by Tate (circa 1966) that $\mathbf{B}$ cannot be $\mathbf{C}_p$ because the latter does not contain a $p$-adic analog $(2\pi i)_p$ of $2\pi i$:

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$(^2)$ What we will describe holds more generally for complete discrete valuation fields of mixed characteristic with perfect residue fields.

(1) Naively, we would like to have $(2\pi i)_p = p^n \log e^{2\pi i/p^n}$, $n \geq 1$. But the $p$-adic logarithm defined on the the open unit ball $B(1, 1^-)$ with center 1 and radius 1, by the usual formula $\log(x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}(x-1)^n$, is a group homomorphism (as opposed to its complex analog). It follows that we have $\log e^{2\pi i/p^n} = 0$ and an exact sequence

(2.8) $$0 \to \mu_{p^\infty} \to B(1, 1^-) \xrightarrow{\log} \mathbf{C}_p \to 0.$$

(2) Perhaps less naively, $(2\pi i)_p$ should be a period of $\mathbb{G}_{m,\overline{\mathbf{Q}}_p}$ and as such transformed (see (2.6)) under the Galois action by $\sigma((2\pi i)_p) = \chi(\sigma)(2\pi i)_p$ for all $\sigma \in G_K$. But we have

$$\{x \in \mathbf{C}_p \mid \sigma(x) = \chi(\sigma)x, \forall \sigma \in G_K\} = 0,$$

which is a special case of the following fundamental theorem of Tate:

THEOREM 2.9 (Tate, [74]). *Let $k \in \mathbf{Z}$. Then*

$$\{x \in \mathbf{C}_p \mid \sigma(x) = \chi(\sigma)^k x, \forall \sigma \in G_K\} = \begin{cases} \mathbf{Q}_p & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

$$\boxed{\mathbf{C}_p \text{ does not contain } (2\pi i)_p.}$$

**2.3.1.** *History of $p$-adic Hodge theory of algebraic varieties: the beginnings*

(1) 1958–65: Grothendieck defines étale cohomology (as an analog of singular cohomology), algebraic de Rham cohomology, and, together with Berthelot, its refinement: crystalline cohomology.
(2) 1967–70: Tate and Grothendieck discover that, for an elliptic curve over $K$ (or, more generally, an abelian variety), its de Rham cohomology and its $p$-adic étale cohomology both determine and are determined by its $p^\infty$-torsion (more precisely, its $p$-divisible group).
(3) 1970: Grothendieck asks whether there exists an abstract "mysterious period functor" relating directly $p$-adic étale cohomology and de Rham cohomology.
(4) 1979–87: Fontaine constructs complicated period rings and formulates precise conjectures, which are now theorems, concerning this "mysterious functor".

**2.4. The first approximation: the de Rham period ring $\mathbf{B}_{dR}$.** We start with the least complicated of $p$-adic period rings.

**2.4.1.** *The de Rham period ring.* Fontaine (circa 1980) constructed a ring $\mathbf{B}_{dR}^+$, the de Rham period ring, that he conjectures should contain all periods of $p$-adic algebraic varieties. In $\mathbf{B}_{dR}^+$ he distinguished an element $t$ (for Tate) that will become the $p$-adic analog of $2\pi i$. In particular, the Galois group

$G_K$ acts on $\mathbf{B}_{\mathrm{dR}}^+$ and does act on $t$ via the cyclotomic character: $\sigma(t) = \chi(\sigma)t$ for $\sigma \in G_K$.

We list the following properties of $\mathbf{B}_{\mathrm{dR}}^+$:

(1) $\mathbf{B}_{\mathrm{dR}}^+ \simeq \mathbf{C}_p[[t]]$ but not in any reasonable way ([3]); we do, however, have a short exact sequence

$$0 \to t\mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{B}_{\mathrm{dR}}^+ \xrightarrow{\theta} \mathbf{C}_p \to 0,$$

where $\theta$ is a Galois equivariant continuous ring homomorphism.

(2) $\mathbf{B}_{\mathrm{dR}}^+$ is equipped with a descending filtration by the powers of $t$:

$$\mathbf{B}_{\mathrm{dR}}^+ \supset F^n\mathbf{B}_{\mathrm{dR}}^+ := (t^n), \quad \mathrm{gr}_F^n \mathbf{B}_{\mathrm{dR}}^+ \simeq \mathbf{C}_p(n).$$

(3) $\mathbf{B}_{\mathrm{dR}}^+$ is a completion of $\overline{\mathbf{Q}}_p$ involving "higher derivatives" [18].

EXAMPLE 2.10. We will illustrate (3). Define the norm

$$x \in \overline{\mathbf{Q}}_p, \quad |x|_{p,1} := \sup\left( |x|_p, \left| \frac{dx}{dp} \right|_p \right).$$

It is a submultiplicative norm: $|xy|_{p,1} \leq |x|_{p,1}|y|_{p,1}$. To define $\left| \frac{dx}{dp} \right|_p$, choose a uniformizer $\pi$ of $\mathbf{Q}_p(x)$, and write $x = Q(\pi)$ for a polynomial $Q \in \mathbf{Q}_p(\mu_N)[X]$, $(N, p) = 1$, of minimal degree. Let $P \in \mathbf{Q}_p(\mu_N)[X]$ be the minimal polynomial of $\pi$ (it is an Eisenstein polynomial). Set $\frac{dx}{dp} := \frac{-Q'(\pi)}{P'(\pi)}$. Then $\frac{dx}{dp}$ depends on the choices made but not too much, and $\left| \frac{dx}{dp} \right|_p$ does not depend on the choices made.

The ring $\mathbf{B}_{\mathrm{dR}}^+/t^2$ is the completion of $\overline{\mathbf{Q}}_p$ for the norm $|\bullet|_{p,1}$. In particular, $\overline{\mathbf{Q}}_p$ is dense in $\mathbf{B}_{\mathrm{dR}}^+/t^2$. Hence $\mathbf{B}_{\mathrm{dR}}^+/t^2$ is not a $\mathbf{C}_p$-Banach space: it would be of dimension 1, but we have a filtration

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & t\mathbf{B}_{\mathrm{dR}}^+/t^2 & \longrightarrow & \mathbf{B}_{\mathrm{dR}}^+/t^2 & \longrightarrow & \mathbf{B}_{\mathrm{dR}}^+/t & \longrightarrow & 0 \\
& & \downarrow \wr & & & & \downarrow \wr & & \\
& & \mathbf{C}_p & & & & \mathbf{C}_p & &
\end{array}
$$

whose graded pieces are isomorphic to $\mathbf{C}_p$. We can say that $\mathbf{B}_{\mathrm{dR}}^+/t^2$ looks like a $\mathbf{C}_p$-vector space of dimension 2; we will write this as $\mathbf{B}_{\mathrm{dR}}^+/t^2 \sim \mathbf{C}_p^2$. More generally, we have

$$\mathbf{B}_{\mathrm{dR}}^+/t^n\mathbf{B}_{\mathrm{dR}}^+ \sim \mathbf{C}_p^n.$$

**2.4.2.** *De Rham and Hodge–Tate comparison theorems.* Define the period ring $\mathbf{B}_{\mathrm{dR}} := \mathbf{B}_{\mathrm{dR}}^+[1/t]$.

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([3]) There exist $K$-linear continuous sections $\mathbf{C}_p \to \mathbf{B}_{\mathrm{dR}}^+$ of the surjective map $\theta : \mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{C}_p$ but none preserving the ring structure. The axiom of choice gives existence of sections preserving the ring structure but they cannot be continuous.

Theorem 2.11 (de Rham comparison theorem, Faltings, 1989 [32]). *Let X be a proper and smooth variety over K with $[K : \mathbf{Q}_p] < \infty$. There exists a period isomorphism*

$$(2.12) \qquad \alpha_{\mathrm{dR}} : \quad H^n_{\mathrm{dR}}(X) \otimes_K \mathbf{B}_{\mathrm{dR}} \simeq H^n_{\mathrm{\acute et}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}$$

*compatible with the Galois action and filtration, where the Hodge filtration on de Rham cohomology is defined by*

$$F^i H^n_{\mathrm{dR}}(X) := \mathrm{Im}(H^n(X, \Omega^{\geq i}_X) \to H^n_{\mathrm{dR}}(X)), \quad i \geq 0.$$

Using alterations and Deligne's de Rham cohomology this theorem can be extended to all varieties over $K$ (see [4]). Hence

> $\mathbf{B}_{\mathrm{dR}}$ *contains all p-adic periods of algebraic varieties over K.*

The above theorem yields a filtered isomorphism (take $G_K$-fixed points of both sides of (2.12) and use the fact that $\mathbf{B}^{G_K}_{\mathrm{dR}} = K$):

$$H^n_{\mathrm{dR}}(X) \simeq (H^n_{\mathrm{\acute et}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}})^{G_K}.$$

However we are not able to recover the Galois action on $H^n_{\mathrm{\acute et}}(X_{\overline{K}}, \mathbf{Q}_p)$ from the left hand side of the period isomorphism (2.12). The reason is that the structure both on de Rham cohomology and on $\mathbf{B}_{\mathrm{dR}}$ is too coarse for that. All we have at our disposal is the Hodge filtration. Using it, that is, taking $\mathrm{gr}^0_F$ of both sides of (2.12), we obtain the corollary below which describes not the Galois representation $H^n_{\mathrm{\acute et}}(X_{\overline{K}}, \mathbf{Q}_p)$ itself but its twist by $\mathbf{C}_p$ (with its natural Galois action). It was conjectured by Tate [74] and was the starting point for all $p$-adic comparison theorems.

Corollary 2.13 (Hodge–Tate decomposition). *We have a Galois equivariant Hodge–Tate decomposition*

$$(2.14) \qquad H^n_{\mathrm{\acute et}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{C}_p \simeq \bigoplus_{i \geq 0} H^{n-i}(X, \Omega^i_{X/K}) \otimes_K \mathbf{C}_p(-i).$$

This corollary says that the twist of the Galois representation $H^n_{\mathrm{\acute et}}(X_{\overline{K}}, \mathbf{Q}_p)$ by $\mathbf{C}_p$ splits as a direct sum of cyclotomic characters with multiplicities given by the Hodge numbers of the variety. It is reminiscent of the classical Hodge decomposition for projective varieties over $\mathbf{C}$.

**2.5. Refined period rings.** We will now introduce more sophisticated $p$-adic period rings, present the associated comparison theorems and discuss their history.

**2.5.1.** *The crystalline and semistable period rings.* To recover the Galois representations on étale cohomology, Fontaine defined more refined period rings:

$$\mathbf{B}_{\mathrm{cr}} \subset \mathbf{B}_{\mathrm{st}} \subset \overline{K} \cdot \mathbf{B}_{\mathrm{st}} \subset \mathbf{B}_{\mathrm{dR}}.$$

They have the following properties:

(1) The crystalline period ring $\mathbf{B}_{\mathrm{cr}}$ is equipped with (commuting) $G_K$-action and Frobenius operator $\phi$.

(2) The semistable period ring, besides the $G_K$-action and the Frobenius $\phi$, has also an action of a $\mathbf{B}_{\mathrm{cr}}$-linear monodromy operator $N$ which is a $\mathbf{B}_{\mathrm{cr}}$-derivation (we have $\mathbf{B}_{\mathrm{st}}^{N=0} = \mathbf{B}_{\mathrm{cr}}$) that commutes with the action of $G_K$ and satisfies $N\varphi = p\varphi N$.

REMARK 2.15 (Geometric interpretation of period rings). It is instructive to look at the geometric interpretation of period rings: it explains the origins of the extra structures. It is this interpretation that is often used in the proofs of comparison theorems (see Section 2.5). Roughly speaking:

> All period rings are cohomologies of geometric points.

Let us look at some examples:

(1) We have $\mathbf{B}_{\mathrm{dR}}^{+}/F^m = (\mathrm{R}\Gamma_{\mathrm{dR}}(\mathrm{Spec}(\mathscr{O}_{\overline{K}}))\otimes\mathbf{Q}_p)/F^m$, the derived log-de Rham cohomology.

(2) We have $\mathbf{B}_{\mathrm{cr}}^{+} = \mathrm{R}\Gamma_{\mathrm{cr}}(\mathrm{Spec}(\mathscr{O}_{\overline{K},1})) \otimes \mathbf{Q}_p$, the absolute (i.e., over $\mathbf{Q}_p$) crystalline cohomology [4]. The Frobenius $\phi$ comes from the geometric Frobenius acting on $\mathscr{O}_{\overline{K},1} = \mathscr{O}_{\overline{K}}/p$.

(3) We have $\widehat{\mathbf{B}}_{\mathrm{st}}^{+} = \mathrm{R}\Gamma_{\mathrm{cr}}(\mathrm{Spec}(\mathscr{O}_{\overline{K},1}^{\times})/E) \otimes \mathbf{Q}_p$, the log-crystalline cohomology of $\mathrm{Spec}(\mathscr{O}_{\overline{K},1})$—equipped with the logarithmic structure induced by the uniformizers—over the log-crystalline affine line $E$. The Frobenius $\phi$ is induced by the geometric Frobenius as before, the monodromy $N$ is the Gauss–Manin connection, and the equality $N\varphi = p\varphi N$ comes from the fact that the action of monodromy can be integrated to a Frobenius equivariant action of the PD-group scheme $\mathbb{G}_m^{\flat}$ [5, 1.16]. We have $\mathbf{B}_{\mathrm{st}}^{+} = (\widehat{\mathbf{B}}_{\mathrm{st}}^{+})^{N\text{-nilp}}$.

**2.5.2.** *Comparison theorems.* The most general $p$-adic comparison theorem for algebraic varieties takes the following form:

THEOREM 2.16 (Semistable comparison theorem). *Let $X$ be a variety over $K$. There exists a period isomorphism*

$$(2.17) \qquad \alpha_{\mathrm{st}}: \quad H_{\mathrm{HK}}^n(X_{\overline{K}}) \otimes_{K^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}} \simeq H_{\text{ét}}^n(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}}, \quad n \geq 0,$$

*compatible with the Frobenius, monodromy, the Galois action, and with the de Rham period isomorphism $\alpha_{\mathrm{dR}}$, i.e., $\alpha_{\mathrm{st}} \otimes \mathbf{B}_{\mathrm{dR}} \simeq \alpha_{\mathrm{dR}}$.*

Here, $H_{\mathrm{HK}}^n(X_{\overline{K}})$ is the Hyodo–Kato cohomology, a finite-dimensional $K^{\mathrm{nr}}$-vector space [5], equipped with a Frobenius, monodromy, and a locally

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[4] Here $\mathbf{B}_{\mathrm{cr}}^{+}$ is the $^{+}$-version of $\mathbf{B}_{\mathrm{cr}}$, i.e., $\mathbf{B}_{\mathrm{cr}} \simeq \mathbf{B}_{\mathrm{cr}}^{+}[1/t]$, and similarly for other period rings.

[5] $K^{\mathrm{nr}}$ is the maximal unramified extension of the Witt vectors of the residue field of $K$.

constant action of $G_K$. Moreover, we have a Hyodo–Kato isomorphism

$$\iota_{\mathrm{HK}} : H^n_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{K^{\mathrm{nr}}} \overline{K} \simeq H^n_{\mathrm{dR}}(X_{\overline{K}}).$$

Hyodo–Kato cohomology plays the role of limit Hodge structures in the $p$-adic world.

$$\boxed{\overline{K} \cdot \mathbf{B}_{\mathrm{st}} \text{ contains all } p\text{-adic periods of algebraic varieties over } K.}$$

The period isomorphism (2.17) allows one to go back and forth between de Rham cohomology and étale cohomology (with all the additional structures):

(2.18)
$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \simeq (H^n_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{K^{\mathrm{nr}}} \mathbf{B}_{\mathrm{st}})^{N=0,\,\phi=1} \cap F^0(H^n_{\mathrm{dR}}(X_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}_{\mathrm{dR}}),$$
$$H^n_{\mathrm{HK}}(X_{\overline{K}}) \simeq (H^n_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{st}})^{G_K\text{-sm}}.$$

Here $(-)^{G_K\text{-sm}}$ refers to taking vectors stable under the action of an open subgroup of $G_K$.

**2.5.3.** *History of p-adic Hodge theory of algebraic varieties: comparison theorems*

(A) *Early years: 1985–2010.*

(1) The first quite general proof of comparison theorems was given by Fontaine–Messing [42]. The proof worked for all large enough primes $p$ and also for some cohomologically simple varieties. It employed the syntomic technique, which was later refined by Hyodo and Kato [48], [49] (based on an earlier work of Bloch–Kato [13]), and finally by Tsuji [76] who proved the semistable comparison theorem in full generality. The word "syntomic" here refers to the topology used: it is generated by syntomic, i.e., flat and locally complete intersection, morphisms. Syntomic topology sits between flat and étale topologies; it is general enough so that it contains solutions of certain algebraic equations but restrictive enough so that it has good cohomological properties.

The modern version of the syntomic technique yields a computation of $p$-adic nearby cycles (which are, in general, nontrivial even for varieties having good reduction) via syntomic cohomology defined as a filtered Frobenius eigenspace of crystalline cohomology. This coupled with the Poincaré duality or the rigidity supplied by Banach–Colmez spaces (see Section 2.6) yields the comparison theorems for algebraic varieties. But, perhaps surprisingly, it has turned out also to be a powerful tool in studying comparison theorems for rigid analytic varieties (see Section 3.3). So the technique is seeing a major revival recently.

(2) The first completely general proof of the de Rham ([6]) and the crystalline comparison theorems was given by Faltings [32] who also proved the semistable comparison theorem [33] (a bit later than Tsuji). He uses the technique of almost étale extensions, which he invented for this purpose by refining the original ideas of Tate in dimension 0. He was also able to treat local systems. This technique, amplified by a strong almost purity result of Kedlaya–Liu [51] and Scholze [70], is very powerful and was also used recently to study comparison theorems for rigid analytic varieties (see Section 3.2).

(3) In my studies of Galois actions on geometric motivic cohomology, I found yet another proof of the comparison theorems [61], [63]: in it, both étale cohomology and syntomic cohomology are seen as incarnations of $p$-adic motivic cohomology, and the comparison isomorphism becomes an incarnation of the localization map in motivic cohomology. The relation between motivic cohomology and $p$-adic cohomologies was studied extensively by Geisser, Hesselholt, and Levine [45], [46]; a more recent study was initiated by Nikolaus–Scholze [59] and Bhatt–Morrow–Scholze [10].

(B) *Recent years: 2010–*

(1) More recently, Beilinson has found a new approach to comparison theorems [4], [5] (Bhatt gave a similar proof of the semistable conjecture in [8]). He uses $h$-topology (built from proper and Zariski open maps) to prove a Poincaré Lemma: the complex of differentials modulo $p^n$ becomes just constants in the $h$-topology. In the proof, it is easy to see that mod $p^n$ differentials can be killed by $p$-covers; it is more difficult to kill cohomology mod $p^n$ in nontrivial degrees of the sheaf of regular functions: this fact was proved in Bhatt's thesis. Beilinson's approach is particularly simple in the case of the de Rham comparison theorem.

(2) Scholze has rewritten the foundations of the almost étale approach to relative $p$-adic Hodge theory [70]. The backbone of it is almost purity, which Scholze, and independently Kedlaya–Liu [51], proved in a very general form. Weaker versions of almost purity were proved earlier by Faltings and Gabber–Ramero [43]. The general almost purity allows one to do $p$-adic Hodge theory on algebraic varieties over $K$, and to extend it to rigid analytic varieties over $K$ (see below). It also allows one to treat local systems.

(3) *Integral p-adic Hodge theory.* We focus in this survey on rational $p$-adic Hodge theory. But there also exists integral $p$-adic Hodge theory which is very important in applications to arithmetic. From the beginning of $p$-adic Hodge theory we knew that we can do integral computations

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([6]) This was preceded by a proof of the Hodge–Tate decomposition [31].

for large primes $p$ (Fontaine–Lafaille theory) and also that the denominators that appear in general can be universally bounded (in terms of $p$ and the dimension of the varieties). However, perhaps no expert expected the result proved recently by Bhatt–Morrow–Scholze [9], [10] (see Bhatt–Scholze [11] for a different treatment via prismatic cohomology and Česnavičius–Koshikawa [16] for generalizations): it is enough to "twist" by the element $\mu \in \mathbf{B}_{\mathrm{cr}}$ (a lift of $\zeta_p - 1$, for a primitive $p$th root of unity $\zeta_p$, from $\mathbf{C}_p$) to obtain optimal integral $p$-adic comparison theorems.

(C) *Uniqueness of p-adic comparison morphisms.* The comparison morphisms are normalized to be compatible with Chern classes (crystalline, de Rham, and étale). Hence, whatever the technique used for their construction, one would expect them to be all equal. In [64], [65], I proved this to be indeed the case for the syntomic, the old almost étale, and the Beilinson approaches; the case of the (rational) comparison morphism of Bhatt–Morrow–Scholze and Česnavičius–Koshikawa was treated in the PhD thesis of Sally Gilles [47]. The proof in [47] is direct, using the fact that the comparison morphism constructed in [47] (a global geometric version of the morphism constructed by Colmez–Nizioł [22]) is very similar ([7]) to that of [9] and [16] and can be shown directly to be the same as the comparison morphism of Fontaine–Messing. My proofs exploit the motivic approach (A3) in [64] and the $h$-topology approach (B1) in [65]. These approaches allow one to reduce all comparison morphisms to the motivic localization morphism or to the tautological morphism in the Fundamental Exact Sequence of $p$-adic Hodge theory (see (2.21) below) ([8]), respectively.

**2.6. Banach–Colmez spaces.** Banach–Colmez spaces are omnipresent in modern $p$-adic Hodge theory. We will illustrate how they enter the picture, discuss their properties, and give some examples.

**2.6.1.** *The fundamental exact sequences.* Twisting the isomorphism (2.18) by $\mathbf{Q}_p(r), r \geq 0$, we obtain the short exact sequence

$$(2.19) \quad 0 \to H^r_{\mathrm{ét}}(X_{\overline{K}}, \mathbf{Q}_p(r)) \to (H^r_{\mathrm{HK}}(X_{\overline{K}}) \otimes_{K^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{\mathbf{N}=0,\,\phi=p^r}$$
$$\to (H^r_{\mathrm{dR}}(X_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}^+_{\mathrm{dR}})/F^r \to 0$$

We can ask:

> *In which category does the above sequence live?*

The first term of the sequence is a finite-dimensional $\mathbf{Q}_p$-vector space; the

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([7]) Morally speaking, $L\eta_\mu$ from [9] gets replaced with $L\eta_t$ in [47].

([8]) The Fundamental Exact Sequence can be seen as an isomorphism between a Tate twist and a syntomic complex.

last term, by Example 2.10, looks like $\mathbf{C}_p^n$. What about the middle term? Let us look at an example.

EXAMPLE 2.20. Let $U = \{x = (x_0, x_1, \ldots) \mid x_n \in B(1, 1^-),\ x_{n+1}^p = x_n\}$. Consider the commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\ \widetilde{\log}\ } & \mathbf{B}_{\mathrm{dR}}^+ \\
{\scriptstyle x \mapsto x_0}\downarrow & & \downarrow{\scriptstyle \theta} \\
B(1, 1^-) & \xrightarrow{\ \log\ } & \mathbf{C}_p
\end{array}
$$

where $\widetilde{\log}$ is a lifting of the logarithm such that $\widetilde{\log}\big((e^{2\pi i/p^n})_{n \in \mathbf{N}}\big) = t$. It induces an isomorphism $\widetilde{\log} : U \xrightarrow{\sim} \mathbf{B}_{\mathrm{cr}}^{+,\phi=p}$. Hence the above diagram and the exact sequence (2.8) yield the short exact sequence

$$0 \to \mathbf{Q}_p t \to \mathbf{B}_{\mathrm{cr}}^{+,\phi=p} \to \mathbf{C}_p \to 0.$$

We have shown that $\mathbf{B}_{\mathrm{cr}}^{+,\phi=p} \sim \mathbf{C}_p \oplus \mathbf{Q}_p$.

More generally, for $m \geq 0$, we have the *Fundamental Exact Sequence of p-adic Hodge theory*:

$$(2.21) \qquad 0 \to \mathbf{Q}_p t^m \to \mathbf{B}_{\mathrm{cr}}^{+,\phi=p^m} \to \mathbf{B}_{\mathrm{dR}}^+/t^m \mathbf{B}_{\mathrm{dR}}^+ \to 0.$$

Hence $\mathbf{B}_{\mathrm{cr}}^{+,\phi=p^m} \sim \mathbf{C}_p^m \oplus \mathbf{Q}_p$.

Coming back to the exact sequence (2.19) we see that the terms are $\mathbf{Q}_p$- or $\mathbf{C}_p$-vector spaces, or extensions of such. The most naive guess for the category which contains such objects and in which one can do homological algebra would be the category of (locally convex) topological vector spaces over $\mathbf{Q}_p$. But this category is too "flabby": there exists an isomorphism $\mathbf{C}_p \oplus \mathbf{Q}_p \simeq \mathbf{C}_p$ (to define this isomorphism consider both sides as $\mathbf{Q}_p$-Banach spaces and recall that in the $p$-adic world those behave like Hilbert spaces: they have orthonormal bases [73, Lemma 1]), and this we definitely do not want.

**2.6.2.** *Banach–Colmez spaces.* We have however:

THEOREM 2.22 (Colmez, Fontaine, [17], [40]). *There exists an abelian category of Banach–Colmez vector spaces $\mathbb{W}$ which are finite-dimensional $\mathbf{C}_p$-vector spaces $\pm$ finite-dimensional $\mathbf{Q}_p$-vector spaces. Moreover,*

(1) *such a space has a total dimension* $\mathrm{Dim}(\mathbb{W}) := (\dim_{\mathbf{C}_p} \mathbb{W}, \dim_{\mathbf{Q}_p} \mathbb{W}) \in (\mathbf{N}, \mathbf{Z})$,
(2) $\mathrm{Dim}(\mathbb{W})$ *is additive on short exact sequences.*

Foir example $\mathrm{Dim}(\mathbf{C}_p) = (1, 0)$ and $\mathrm{Dim}(\mathbf{C}_p \oplus \mathbf{Q}_p) = (1, 1)$. It follows that in this category $\mathbf{C}_p$ cannot be isomorphic to $\mathbf{C}_p \oplus \mathbf{Q}_p$.

We will now discuss the above theorem in more detail.

DEFINITION 2.23. A *Vector Space* (VS for short) $\mathbb{W}$ is a functor $\Lambda \mapsto \mathbb{W}(\Lambda)$ from nice $\mathbf{Q}_p$-algebras to $\mathbf{Q}_p$-vector spaces.

Here, a $\mathbf{Q}_p$-Banach algebra $\Lambda$ ($|x+y| \leq \sup(|x|,|y|)$, $|xy| \leq |x|\,|y|$) is *nice* ([9]) if $|x| = \sup |s(x)|_p$, for all $\mathbf{Q}_p$-Banach algebra maps $s : \Lambda \to \mathbf{C}_p$, and the map $x \mapsto x^p$ is surjective.

EXAMPLE 2.24.

(1) If $V$ is a finite-dimensional $\mathbf{Q}_p$-vector space, the associated VS is the functor
$$\Lambda \mapsto V(\Lambda) = V, \quad (\Lambda_1 \to \Lambda_2) \mapsto (\mathrm{Id} : V \to V).$$
We have $V(\mathbf{C}_p) = V$.

(2) Let $d \geq 1$. The VS $\mathbb{V}^d$ is the functor
$$\Lambda \mapsto \mathbb{V}^d(\Lambda) = \Lambda^d, \quad (\Lambda_1 \to \Lambda_2) \mapsto (\Lambda_1^d \to \Lambda_2^d).$$
We have $\mathbb{V}^d(\mathbf{C}_p) = \mathbf{C}_p^d$.

A Vector Space $\mathbb{W}$ is called finite-dimensional (nowadays called simply a *Banach–Colmez space*) if it admits a presentation

$$
\begin{array}{ccccccccc}
0 \longrightarrow V_2 & & & & & & & & \\
 & \searrow & & & & & & & \\
0 \longrightarrow V_1 & \longrightarrow & \mathbb{W}' & \longrightarrow & \mathbb{V}^d & \longrightarrow & 0 \\
 & & & \searrow & & & & & \\
 & & & & \mathbb{W} & \longrightarrow & 0
\end{array}
$$

where $V_1, V_2$ are finite-dimensional $\mathbf{Q}_p$-vector spaces and $\mathbb{V}^d$ is as defined in (2) above. We have $\mathrm{Dim}(\mathbb{W}) = (d, \dim_{\mathbf{Q}_p} V_1 - \dim_{\mathbf{Q}_p} V_2)$. This value is independent of the presentation chosen (a fact difficult to prove). The proof of this independence can also be used to show that every Banach–Colmez space $\mathbb{W}$ is a Banach Space (i.e., $\mathbb{W}(\Lambda)$ is a Banach space for any $\Lambda$).

EXAMPLE 2.25. We list some examples of Banach–Colmez spaces:

(1) There is a Banach–Colmez space $\mathbb{B}_m$ such that $\mathbf{B}_{\mathrm{dR}}^+/t^m = \mathbb{B}_m(\mathbf{C}_p)$; we have $\mathrm{Dim}(\mathbb{B}_m) = (m, 0)$.

(2) There is a Banach–Colmez space $\mathbb{U}_{a,b}$ such that $\mathbf{B}_{\mathrm{cr}}^{+,\phi^a=p^b} = \mathbb{U}_{a,b}(\mathbf{C}_p)$; we have $\mathrm{Dim}(\mathbb{U}_{a,b}) = (b, a)$. The exact sequence (2.21) can be lifted to the category of Banach–Colmez spaces, i.e., it is the set of $\mathbf{C}_p$-points of the exact sequence of Banach–Colmez spaces
$$0 \to \mathbf{Q}_p t^m \to \mathbb{U}_{1,m} \to \mathbb{B}_m \to 0.$$

(3) The exact sequence (2.19) can be lifted to the category of Banach–Colmez spaces.

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([9]) Colmez called them sympathetique. They are perfectoid in the terminology of Scholze.

(4) $\mathbf{C}_p/\mathbf{Q}_p$ is the set of $\mathbf{C}_p$-points of a Banach–Colmez space with Dim $= (1, -1)$.

REMARK 2.26. The category of Banach–Colmez spaces is quite rigid: If $\mathbb{W}_1$ is a successive extension of $\mathbb{V}^1$'s, and if $\mathbb{W}_2$ is of $\mathbf{C}_p$-dimension 0, then any morphism $\mathbb{W}_1 \to \mathbb{W}_2$ is the 0-map. This fact is very useful: it allows one to circumvent Poincaré duality arguments in the proofs of $p$-adic comparison theorems [22].

REMARK 2.27. The idea of defining a category akin to the category of Banach–Colmez spaces goes back to Fontaine. Fontaine studied similar structures equipped with Galois action ([10]) (he called them "almost $\mathbf{C}_p$-representations"). In Fontaine's category we can also distinguish between $\mathbf{C}_p$ and $\mathbf{C}_p \oplus \mathbf{Q}_p$—equipped with the natural action of $G_K$. Namely, if you believe that you can detect the presence of $\mathbf{C}_p$ in your space (because $\mathbf{C}_p$ is big) then, if you have a $G_K$-action, you can take a look at the associated Euler characteristic of both spaces. It kills $\mathbf{C}_p$ and allows you to detect the presence of $\mathbf{Q}_p$. For Banach–Colmez spaces, Colmez found an analytic replacement for this Galois-theoretical argument.

The theory of Banach–Colmez spaces was refined and extended by Plût [67], and, more recently, by Fargues–Fontaine [35], Fargues [34], and Le Bras [56]. Banach–Colmez spaces are a special case of Scholze's diamonds [72].

**2.7. Complements.** We will finish the survey of $p$-adic Hodge theory of algebraic varieties with a brief discussion of its arithmetic aspects and number-theoretical applications.

**2.7.1.** *Arithmetic p-adic Hodge theory.* What we have discussed above is the geometric side of $p$-adic Hodge theory. But $p$-adic Hodge theory also has its arithmetic side. In it Fontaine defined subcategories of $p$-adic Galois representations (i.e., continuous representations of $G_K$ on finite-dimensional vector spaces over $\mathbf{Q}_p$):

(2.28)
$$\mathrm{Rep}(G_K) \supsetneq \mathrm{Rep}_{\mathrm{HT}}(G_K) \supsetneq \mathrm{Rep}_{\mathrm{dR}}(G_K) = \mathrm{Rep}_{\mathrm{pst}}(G_K) \supsetneq \mathrm{Rep}_{\mathrm{geom}}(G_K).$$

The categories $\mathrm{Rep}_{\mathrm{HT}}(G_K), \mathrm{Rep}_{\mathrm{dR}}(G_K)$, and $\mathrm{Rep}_{\mathrm{pst}}(G_K)$ of Hodge–Tate, de Rham, and potentially semistable representations, respectively, consist of $p$-adic representations that satisfy an abstract form of the Hodge–Tate decomposition (2.14), the de Rham comparison isomorphism (2.12), or the potentially semistable comparison isomorphism (2.17), respectively. The category $\mathrm{Rep}_{\mathrm{geom}}(G_K)$ is the category of representations coming from geometry, i.e., from subquotients of $p$-adic étale cohomology of algebraic varieties over $K$.

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([10]) We think of the theory of Banach–Colmez spaces as the geometric theory and of that of almost $\mathbf{C}_p$-representations as the corresponding arithmetic theory.

REMARK 2.29. In the case of $\ell$-adic representations, i.e., on vector spaces over $\mathbf{Q}_\ell$, for $\ell \neq p$, the analog of (2.28) is

$$\mathrm{Rep}(G_K) = \mathrm{Rep}_{\mathrm{pst}}(G_K) \supsetneq \mathrm{Rep}_{\mathrm{geom}}(G_K).$$

The equality follows from the theorem of Grothendieck: any abstract $\ell$-adic representation is potentially unipotent.

To illustrate the strictness of the first two inclusions in (2.28) we will look at some extension groups. Let $\mathrm{Ext}^1_*(\mathbf{Q}_p, \mathbf{Q}_p(i))$, $* = G_K, \mathrm{HT}, \mathrm{dR}$, denote the extension groups of $\mathbf{Q}_p$ by $\mathbf{Q}_p(i)$ in the respective categories. We have (the second result was conjectured by Fontaine [36] and proved by Bloch–Kato [14])

$$\mathrm{Ext}^1_*(\mathbf{Q}_p, \mathbf{Q}_p) = \begin{cases} K \oplus \mathbf{Q}_p & \text{if } * = G_K, \\ \mathbf{Q}_p & \text{if } * = \mathrm{HT}, \end{cases}$$

$$\mathrm{Ext}^1_*(\mathbf{Q}_p, \mathbf{Q}_p(i)) = \begin{cases} K & \text{if } * = \mathrm{HT}, \\ 0 & \text{if } * = \mathrm{dR}, \end{cases} \quad \text{for } i < 0.$$

(For $i > 0$, all $\mathrm{Ext}^1_*(\mathbf{Q}_p, \mathbf{Q}_p(i))$ coincide.)

The fact that the inclusion $\mathrm{Rep}_{\mathrm{dR}}(G_K) \supset \mathrm{Rep}_{\mathrm{pst}}(G_K)$ is actually an equality is a $p$-adic version of the potentially unipotent monodromy theorem. Berger [6] reduced the proof of this theorem to a conjecture of Crew [26] on $p$-adic differential equations which was then immediately proved by André [2], Mebkhout [57], and Kedlaya [50]. The last inclusion in (2.28) follows from Theorem 2.4. This inclusion is strict: we did not put any restrictions on eigenvalues of Frobenius (we note however that these restrictions could be added, see [78] for an example) but, as shown in [77], even adding such restrictions would not have been sufficient to force the equality.

EXAMPLE 2.30. The characterization of geometric representations as potentially semistable ones is very powerful. Very early on in the development of $p$-adic Hodge theory it allowed proving a conjecture of Shafarevich saying that there are no abelian varieties over $\mathbf{Q}$ with good reduction everywhere [37, 1]. The argument goes as follows: assume that such an abelian variety exists. One looks at its first $p$-adic geometric étale cohomology group. By assumption, it is a crystalline representation—a version of potentially semistable representation abstracting the cohomological features of varieties with good reduction— corresponding to the first de Rham cohomology group. The Hodge weights of the latter then give an upper bound on the ramification of the largest field whose Galois group acts trivially on $H^1_{\mathrm{ét}}$ with $\mathbf{Z}/p$ coefficients. The lower bound is supplied by Odlyzko. The two bounds overlap, giving a contradiction.

REMARK 2.31. The development of arithmetic $p$-adic Hodge theory was followed by a quest for a $p$-adic local Langlands correspondence. This is a program which studies the relationship between $p$-adic Galois representations and $p$-adic unitary representations of $p$-adic reductive groups. It was initiated by Breuil [15] and major progress in dimension 2 was made by Colmez [19].

**2.7.2.** *Number-theoretical applications.* The inclusion

$$\mathrm{Rep}_{\mathrm{geom}}(G_K) \subset \mathrm{Rep}_{\mathrm{pst}}(G_K)$$

makes sense in the global setting, that is, over number fields, as well. There, amazingly, it tends to be an equality:

> *In a global situation, de Rham implies geometric.*

Fontaine–Mazur stated this as a conjecture:

CONJECTURE 2.32 (Fontaine–Mazur, [41], [38]). *Suppose that $\rho : G_{\mathbf{Q}} \to \mathrm{GL}(V)$ is an irreducible $p$-adic representation which is unramified at all but finitely many primes and $\rho|G_{\mathbf{Q}_p}$ is de Rham. Then there is a smooth projective variety $X/\mathbf{Q}$ and integers $i, j$ such that $V$ is a subquotient of $H^i_{\mathrm{\acute{e}t}}(X_{\overline{\mathbf{Q}}}, \overline{\mathbf{Q}}_p(j))$.*

A special case of this conjecture is easy to state:

CONJECTURE 2.33 (Unramified Fontaine–Mazur). *Let $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{Q}_p)$ be a continuous representation which is unramified at all but finitely many primes and the inertia at $p$ has finite image under $\rho$. Then $\rho$ has finite image.*

Combined with the conjectural global Langlands correspondence, Conjecture 2.32 implies that 2-dimensional representations of $G_{\mathbf{Q}}$ satisfying the stated conditions come from modular forms. This is exactly what Wiles [79] proved in 1994, in a special case, on his way to proving Fermat's Last Theorem (FLT); his proof makes heavy use of integral $p$-adic Hodge theory.

The Fontaine–Mazur conjecture is basically known in dimension 2 by the work of Emerton [29], Kisin [53], and Lue Pan [66].

EXAMPLE 2.34 (Diophantine applications). Recall that Fermat's Last Theorem states that there are no nontrivial positive integral solutions to the equation

$$(2.35) \qquad\qquad x^n + y^n = z^n, \quad n \geq 3.$$

This is an example of a Diophantine problem and $p$-adic Hodge theory has been used with great success to solve such problems. For example, before Wiles' proof of FLT we knew that there are a finite number of positive integral solutions to the equation (2.35) because this equation can be seen as defining a plane curve of genus $> 1$ and Faltings proved the Mordell Conjecture in 1983 [30] (see the wonderful sketch of the proof by Bloch [12]):

CONJECTURE 2.36 (Mordell, 1922). *A curve over a number field $K$ of genus $> 1$ has only finitely many $K$-rational points.*

Faltings' proof uses as one of the tools a very early version of $p$-adic Hodge theory. Let us sketch his argument. In the first step, Faltings reduced proving finiteness of rational points to proving finiteness of different mathematical objects: isomorphism classes of principally polarized abelian varieties of dimension $d$ defined over $K$ with good reduction outside a given finite set $S$ of primes. This problem, in turn, he subdivided into two problems: (a) finiteness of isogeny classes of abelian varieties of dimension $d$ with good reduction outside $S$, and (b) finiteness of isomorphism classes in every isogeny class. $p$-adic Hodge theory is used in part (b) via Raynaud's theorem about Galois actions on points of group schemes [69].

For a modern, more powerful, variation on Faltings' proof see the recent work of Lawrence–Venkatesh [55]. For a nonabelian version (studying iterated integrals instead of just integrals) see the work of Kim [52] for theoretical aspects and [3] for an application to counting rational points on a particularly stubborn curve. For a review of $p$-adic approaches to rational points on curves see the survey of Poonen [68].

EXAMPLE 2.37 (Selmer groups). The extension groups $\mathrm{Ext}^1_{\mathrm{dR}}(\mathbf{Q}_p, V)$ appearing above are usually denoted by $H^1_g(G_K, V)$ (where $g$ stands for "geometric", a notation introduced by Bloch and Kato [14]). The fact that they are often strictly smaller than $H^1(G_K, V)$ is crucial for many questions and made them omnipresent in modern algebraic number theory under the name of *Selmer groups*. They have been generalized in two ways:

(1) *geometrically, for varieties over local fields*: they can be defined for any algebraic variety over $K$ [58], [27] under the name of *syntomic cohomology groups* and are used as an approximation of $p$-adic motivic cohomology (a refinement of $p$-adic étale cohomology capturing classes coming from geometry);
(2) *globally* (*over number fields*): they can be globalized and extended to all $H^i$ (and not only $H^1$); see [60] for a direct construction and [44] for reinterpretations via derived Galois deformation rings.

Both generalizations come into play in the study of special values of (complex and $p$-adic) $L$-functions. They enter in the computation of the $p$-adic valuation of complex $L$-values, i.e., Tamagawa numbers [14] and in the definition of $p$-adic regulators that appear in the $p$-adic $L$-values [7].

**3. Rigid analytic varieties.** We now pass to the relatively new study of $p$-adic Hodge theory of $p$-adic analytic varieties (called rigid analytic varieties since the seminal work of Tate [75]).

**3.1. Rigid analytic varieties.** Rigid analytic varieties have, in general, very large cohomology groups. Moreover, even locally, while $\ell$-adic étale cohomology, for $\ell \neq p$, tends to behave like de Rham cohomology, $p$-adic étale cohomology exhibits a very different behavior.

**3.1.1.** *Balls and annuli.* We will illustrate what we said above with a couple of examples.

EXAMPLE 3.1.

(1) Let $\mathbb{D}$ be the open unit disk in $\mathbf{C}_p$. We have

$$H^1_{\mathrm{dR}}(\mathbb{D}) = 0, \quad H^1_{\mathrm{\acute{e}t}}(\mathbb{D}, \mathbf{Q}_\ell) = 0, \quad \ell \neq p;$$
$$H^1_{\mathrm{pro\acute{e}t}}(\mathbb{D}, \mathbf{Q}_p) = \mathscr{O}(\mathbb{D})/\mathbf{C}_p.$$

Pro-étale cohomology used here is a version of étale cohomology defined by Scholze [71], in which infinite (but inverse limits of finite) étale covers are allowed. The cohomology group $H^1_{\mathrm{pro\acute{e}t}}(\mathbb{D}, \mathbf{Q}_p)$ is so big because of the existence of Artin–Schreier coverings: $y = x^p - x$ defines an étale covering of $\mathbb{A}^1_{\mathbf{F}_p}$ (we note that $\mathrm{d}y/\mathrm{d}x = -1$), and hence also of $\mathbb{D}$.

(2) Let $X = \{z \in \mathbf{C}_p \mid r < |z| < s\}$ be an open annulus in $\mathbf{C}_p$. We have

$$H^1_{\mathrm{dR}}(X) \simeq \mathbf{C}_p \simeq \mathbf{C}_p\langle \mathrm{d}z/z \rangle,$$
$$H^1_{\mathrm{\acute{e}t}}(X, \mathbf{Q}_\ell) \simeq \mathbf{Q}_\ell, \quad \ell \neq p,$$

and an exact sequence

$$0 \to \mathscr{O}(X)/\mathbf{C}_p \to H^1_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p) \to \mathbf{Q}_p\langle \mathrm{d}z/z \rangle \to 0.$$

**3.1.2.** *History of p-adic Hodge theory of rigid analytic varieties.*

(1) 1967: Tate [74] asked whether there is a Hodge theory of $p$-adic analytic varieties.
(2) 2010: Scholze responded positively to Tate's question by developing $p$-adic Hodge theory of adic spaces [70], [71].

The subject is now a very active area of research. For example, we have seen substantial progress in $p$-adic Hodge theory for proper or Stein analytic varieties (the two endpoints of the spectrum).

**3.2. Proper rigid analytic varieties** The $p$-adic Hodge theory of proper rigid analytic is very similar to the $p$-adic Hodge theory of algebraic varieties. In particular:

(1) $p$-adic geometric étale cohomology $H^n_{\mathrm{\acute{e}t}}(-, \mathbf{Q}_p)$ is finite-dimensional [71].
(2) There is a de Rham comparison theorem (allowing local systems) obtained by Scholze via the almost étale technique amplified by general almost purity [71]. See [28], [54] for generalizations.

(3) There is a potentially semistable comparison theorem for smooth and ("almost") proper varieties obtained by Colmez–Nizioł via the syntomic technique that works surprisingly well in the analytic context [22], [25].
(4) There are integral comparison theorems of Bhatt–Morrow–Scholze, Česnavičius–Koshikawa, and Bhatt–Scholze [9], [10], [16], [11] that introduce a new cohomology ($\mathbf{A}_{\mathrm{inf}}$-chomology or prismatic cohomology) as a go-between for étale and de Rham cohomologies.

**3.3. Stein rigid analytic varieties.** At the other end of the spectrum, one finds Stein varieties (analytic analogs of affine varieties). The key property of Stein rigid varieties is that coherent sheaves have no higher cohomology.

**3.3.1.** *Some examples.* We will start with some examples of computations in case the Stein space has a semistable formal model over the ring of integers $\mathscr{O}_K$ (in this case all the irreducible components of the special fiber of the formal model are proper).

EXAMPLE 3.2. (i) *Rigid analytic affine space* $\mathbb{A}_K^d$ (see [23]):
$$H^r_{\mathrm{pro\acute{e}t}}(\mathbb{A}^d_{\mathbf{C}_p}, \mathbf{Q}_p(r)) \simeq \Omega^{r-1}(\mathbb{A}^d_{\mathbf{C}_p})/\ker d, \quad r \geq 1,$$
where $d$ is the de Rham differential. The pro-étale cohomology is not finite-dimensional. Its Banach–Colmez total dimension is $\mathrm{Dim} = (\infty, 0)$.

(ii) *Torus* $\mathbb{G}^d_{m,K}$ *of dimension* $d$ *(and* $r \geq 1$): we have a short exact sequence of Banach–Colmez spaces
$$0 \to \Omega^{r-1}(\mathbb{G}^d_{m,\mathbf{C}_p})/\ker d \to H^r_{\mathrm{pro\acute{e}t}}(\mathbb{G}^d_{m,\mathbf{C}_p}, \mathbf{Q}_p(r)) \to \bigwedge{}^r \mathbf{Q}_p^d \to 0.$$
Here $\bigwedge^r \mathbf{Q}_p^d = \bigoplus_{i_1 < \cdots < i_r}(\mathrm{dlog}\, z_{i_1} \wedge \cdots \wedge \mathrm{dlog}\, z_{i_r})\mathbf{Q}_p$. The total dimension of $H^r_{\mathrm{pro\acute{e}t}}(\mathbb{G}^d_{m,\mathbf{C}_p}, \mathbf{Q}_p(r))$ is $\mathrm{Dim} = (\infty, \binom{d}{r})$.

(iii) *Drinfeld half-plane* $\Omega_K := \mathbb{P}_K \backslash \mathbb{P}(K)$: we have a short exact sequence of Banach–Colmez spaces
$$0 \to \mathscr{O}(\Omega_{\mathbf{C}_p})/\ker d \to H^1_{\mathrm{pro\acute{e}t}}(\Omega_{\mathbf{C}_p}, \mathbf{Q}_p(1)) \to \mathrm{Sp}(\mathbf{Q}_p)^* \to 0,$$
where $\mathrm{Sp}(\mathbf{Q}_p) = \mathscr{C}^\infty(\mathbb{P}(K), \mathbf{Q}_p)/\mathbf{Q}_p$ is the (smooth) Steinberg representation of $\mathrm{GL}_2(K)$. The total dimension of $H^1_{\mathrm{pro\acute{e}t}}(\Omega_{\mathbf{C}_p}, \mathbf{Q}_p(1))$ is $\mathrm{Dim} = (\infty, \infty)$.

REMARK 3.3. (i) We have a similar result [20] for $\Omega^d_K$, the Drinfeld symmetric space of any dimension $d > 1$.

(ii) We have similar results [21] for étale coverings of $\Omega_K$. This is used to show that, for $K = \mathbf{Q}_p$, the $p$-adic étale cohomology of these coverings encodes a part of the $p$-adic local Langlands correspondence, which yields a geometric realization for this correspondence (it is a classical result that the $\ell$-adic étale cohomology of these coverings can be used to provide a geometric realization of the classical local Langlands correspondence).

**3.3.2.** *A comparison theorem.* We finish with the following general result:

THEOREM 3.4 (Colmez–Dospinescu–Nizioł, [20], [24], [25]). *Let* $r \geq 0$ *and let* $X$ *be a Stein rigid analytic variety over* $K$. *There exists a* $G_K$-*equivariant exact sequence*

$$0 \to H^r_{\mathrm{pro\acute{e}t}}(X_{\mathbf{C}_p}, \mathbf{Q}_p(r)) \to \mathit{\Omega}^r(X_{\mathbf{C}_p})^{d=0} \oplus (H^r_{\mathrm{HK}}(X_{\mathbf{C}_p}) \widehat{\otimes}_{K^{\mathrm{nr}}} \mathbf{B}^+_{\mathrm{st}})^{N=0, \phi=p^r}$$

$$\xrightarrow{\mathrm{can} + \iota_{\mathrm{HK}}} H^r_{\mathrm{dR}}(X_{\mathbf{C}_p}) \to 0.$$

Hence, just as for algebraic varieties, in the world of Stein or proper rigid analytic varieties,

> Pro-étale cohomology can be recovered from de Rham data!

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