9

Network Theorems

9.1 INTRODUCTION

This chapter will introduce the important fundamental theorems of network analysis. Included are the **superposition, Thévenin's, Norton's, maximum power transfer, substitution, Millman's,** and **reciprocity theorems.** We will consider a number of areas of application for each. A thorough understanding of each theorem is important because a number of the theorems will be applied repeatedly in the material to follow.

9.2 SUPERPOSITION THEOREM

The **superposition theorem,** like the methods of the last chapter, can be used to find the solution to networks with two or more sources that are not in series or parallel. The most obvious advantage of this method is that it does not require the use of a mathematical technique such as determinants to find the required voltages or currents. Instead, each source is treated independently, and the algebraic sum is found to determine a particular unknown quantity of the network.

The superposition theorem states the following:

The current through, or voltage across, an element in a linear bilateral network is equal to the algebraic sum of the currents or voltages produced independently by each source.

When one is applying the theorem, it is possible to consider the effects of two sources at the same time and reduce the number of networks that have to be analyzed, but, in general,

$$\boxed{\begin{array}{c}\text{Number of networks}\\\text{to be analyzed}\end{array} = \begin{array}{c}\text{Number of}\\\text{independent sources}\end{array}} \tag{9.1}$$

To consider the effects of each source independently requires that sources be removed and replaced without affecting the final result. To remove a voltage source when applying this theorem, the difference in potential between the terminals of the voltage source must be set to zero (short circuit); removing a current source requires that its terminals be opened (open circuit). Any internal resistance or conductance associated with the displaced sources is not eliminated but must still be considered.

Figure 9.1 reviews the various substitutions required when removing an ideal source, and Figure 9.2 reviews the substitutions with practical sources that have an internal resistance.

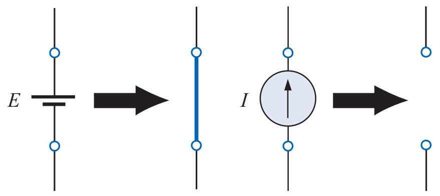


FIG. 9.1
Removing the effects of ideal sources.

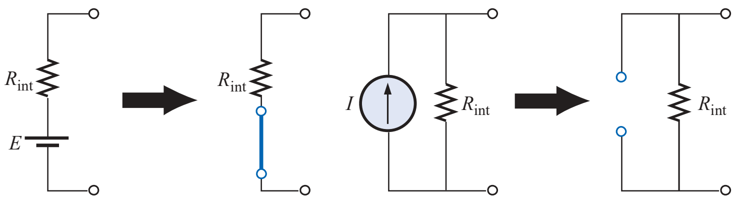


FIG. 9.2
Removing the effects of practical sources.

The total current through any portion of the network is equal to the algebraic sum of the currents produced independently by each source. That is, for a two-source network, if the current produced by one source is in one direction, while that produced by the other is in the opposite direction through the same resistor, *the resulting current is the difference of the two and has the direction of the larger.* If the individual currents are in the same direction, *the resulting current is the sum of two in the direction of either current.* This rule holds true for the voltage across a portion of a network as determined by polarities, and it can be extended to networks with any number of sources.

The superposition principle is not applicable to power effects since the power loss in a resistor varies as the square (nonlinear) of the current or voltage. For instance, the current through the resistor R of Fig. 9.3(a) is I_1 due to one source of a two-source network. The current through the same resistor due to the other source is I_2 as shown in Fig. 9.3(b). Applying the superposition theorem, the total current through the resistor due to both sources is I_T, as shown in Fig. 9.3(c) with

$$I_T = I_1 + I_2$$

The power delivered to the resistor in Fig. 9.3(a) is

$$P_1 = I_1^2 R$$

while the power delivered to the same resistor in Fig. 9.3(b) is

$$P_2 = I_2^2 R$$

If we assume that the total power delivered in Fig. 9.3(c) can be obtained by simply adding the power delivered due to each source, we find that

$$P_T = P_1 + P_2 = I_1^2 R + I_2^2 R = I_T^2 R$$

or $$I_T^2 = I_1^2 + I_2^2$$

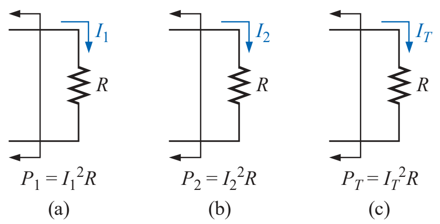


FIG. 9.3
Demonstration of the fact that superposition is not applicable to power effects.

This final relationship between current levels is incorrect, however, as can be demonstrated by taking the total current determined by the superposition theorem and squaring it as follows:

$$I_T^2 = (I_1 + I_2)^2 = I_1^2 + I_2^2 + 2I_1I_2$$

which is certainly different from the expression obtained from the addition of power levels.

In general, therefore,

the total power delivered to a resistive element must be determined using the total current through or the total voltage across the element and cannot be determined by a simple sum of the power levels established by each source.

EXAMPLE 9.1 Determine I_1 for the network of Fig. 9.4.

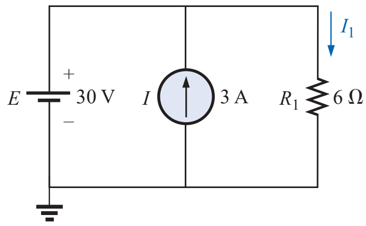


FIG. 9.4
Example 9.1.

Solution: Setting $E = 0$ V for the network of Fig. 9.4 results in the network of Fig. 9.5(a), where a short-circuit equivalent has replaced the 30-V source.

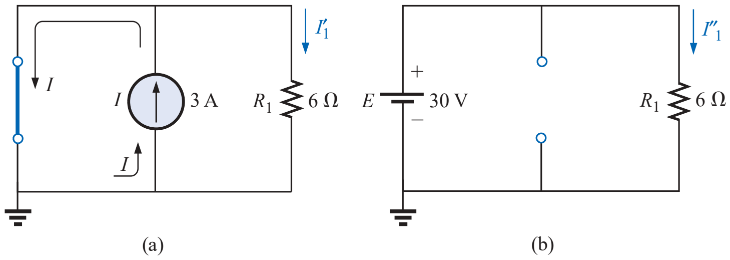


FIG. 9.5
(a) The contribution of I to I_1; (b) the contribution of E to I_1.

As shown in Fig. 9.5(a), the source current will choose the short-circuit path, and $I'_1 = 0$ A. If we applied the current divider rule,

$$I'_1 = \frac{R_{sc}I}{R_{sc} + R_1} = \frac{(0\ \Omega)I}{0\ \Omega + 6\ \Omega} = 0\text{ A}$$

Setting I to zero amperes will result in the network of Fig. 9.5(b), with the current source replaced by an open circuit. Applying Ohm's law,

$$I''_1 = \frac{E}{R_1} = \frac{30\text{ V}}{6\ \Omega} = 5\text{ A}$$

Since I'_1 and I''_1 have the same defined direction in Fig. 9.5(a) and (b), the current I_1 is the sum of the two, and

$$I_1 = I'_1 + I''_1 = 0\text{ A} + 5\text{ A} = \mathbf{5\ A}$$

Note in this case that the current source has no effect on the current through the 6-Ω resistor. The voltage across the resistor must be fixed at 30 V because they are parallel elements.

EXAMPLE 9.2 Using superposition, determine the current through the 4-Ω resistor of Fig. 9.6. Note that this is a two-source network of the type considered in Chapter 8.

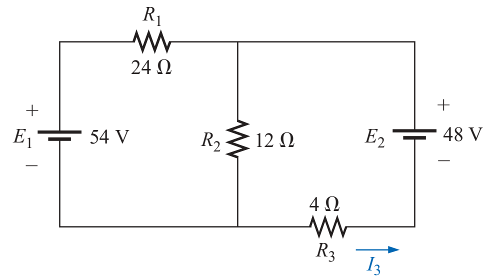


FIG. 9.6
Example 9.2.

Solution: *Considering the effects of a 54-V source (Fig. 9.7):*

$$R_T = R_1 + R_2 \parallel R_3 = 24\ \Omega + 12\ \Omega \parallel 4\ \Omega = 24\ \Omega + 3\ \Omega = 27\ \Omega$$

$$I = \frac{E_1}{R_T} = \frac{54\text{ V}}{27\ \Omega} = 2\text{ A}$$

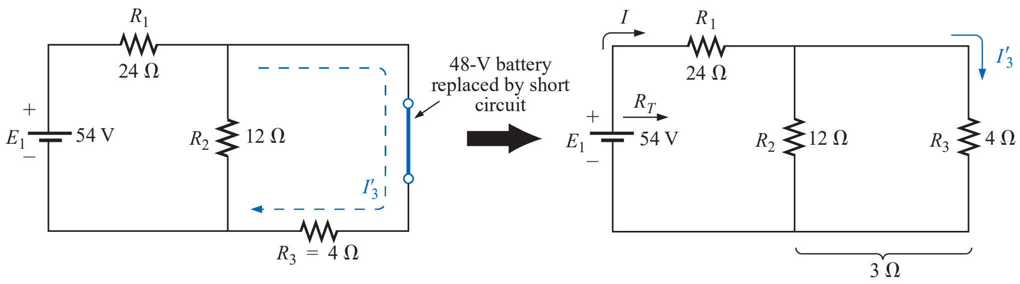


FIG. 9.7
The effect of E_1 on the current I_3.

Using the current divider rule,

$$I'_3 = \frac{R_2 I}{R_2 + R_3} = \frac{(12\ \Omega)(2\text{ A})}{12\ \Omega + 4\ \Omega} = \frac{24\text{ A}}{16} = 1.5\text{ A}$$

Considering the effects of the 48-V source (Fig. 9.8):

$$R_T = R_3 + R_1 \parallel R_2 = 4\ \Omega + 24\ \Omega \parallel 12\ \Omega = 4\ \Omega + 8\ \Omega = 12\ \Omega$$

$$I''_3 = \frac{E_2}{R_T} = \frac{48\text{ V}}{12\ \Omega} = 4\text{ A}$$

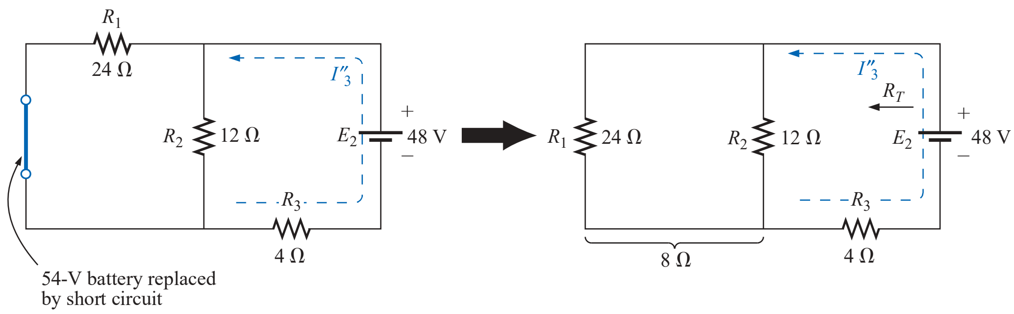


FIG. 9.8
The effect of E_2 on the current I_3.

The total current through the 4-Ω resistor (Fig. 9.9) is

$$I_3 = I''_3 - I'_3 = 4\text{ A} - 1.5\text{ A} = \mathbf{2.5\text{ A}} \qquad \text{(direction of } I''_3)$$

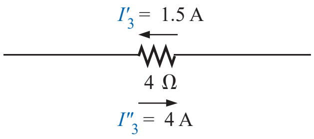


FIG. 9.9
The resultant current for I_3.

EXAMPLE 9.3

a. Using superposition, find the current through the 6-Ω resistor of the network of Fig. 9.10.

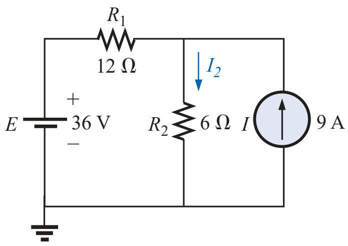


FIG. 9.10
Example 9.3.

b. Demonstrate that superposition is not applicable to power levels.

Solutions:

a. *Considering the effect of the 36-V source (Fig. 9.11):*

$$I'_2 = \frac{E}{R_T} = \frac{E}{R_1 + R_2} = \frac{36\text{ V}}{12\ \Omega + 6\ \Omega} = 2\text{ A}$$

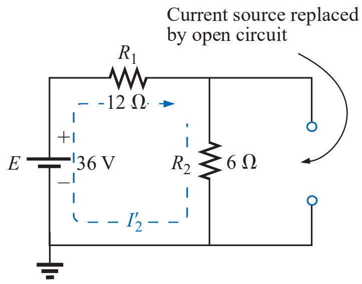


FIG. 9.11
The contribution of E to I_2.

Considering the effect of the 9-A source (Fig. 9.12):
Applying the current divider rule,

$$I''_2 = \frac{R_1 I}{R_1 + R_2} = \frac{(12\ \Omega)(9\text{ A})}{12\ \Omega + 6\ \Omega} = \frac{108\text{ A}}{18} = 6\text{ A}$$

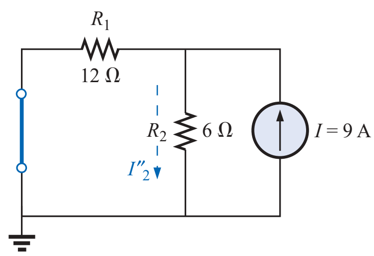


FIG. 9.12
The contribution of I to I_2.

The total current through the 6-Ω resistor (Fig. 9.13) is

$$I_2 = I'_2 + I''_2 = 2\text{ A} + 6\text{ A} = \mathbf{8\text{ A}}$$

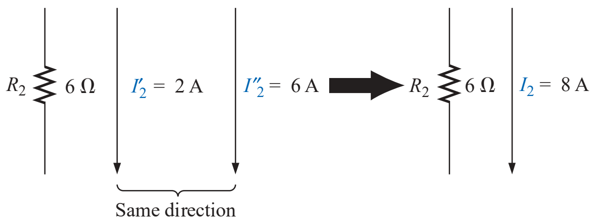


FIG. 9.13
The resultant current for I_2.

b. The power to the 6-Ω resistor is

$$\text{Power} = I^2R = (8\text{ A})^2(6\ \Omega) = \mathbf{384\text{ W}}$$

The calculated power to the 6-Ω resistor due to each source, *misusing* the principle of superposition, is

$$P_1 = (I'_2)^2R = (2\text{ A})^2(6\ \Omega) = 24\text{ W}$$
$$P_2 = (I''_2)^2R = (6\text{ A})^2(6\ \Omega) = 216\text{ W}$$
$$P_1 + P_2 = 240\text{ W} \neq 384\text{ W}$$

This results because 2 A + 6 A = 8 A, but

$$(2\text{ A})^2 + (6\text{ A})^2 \neq (8\text{ A})^2$$

As mentioned previously, the superposition principle is not applicable to power effects since power is proportional to the square of the current or voltage (I^2R or V^2/R).

Figure 9.14 is a plot of the power delivered to the 6-Ω resistor versus current.

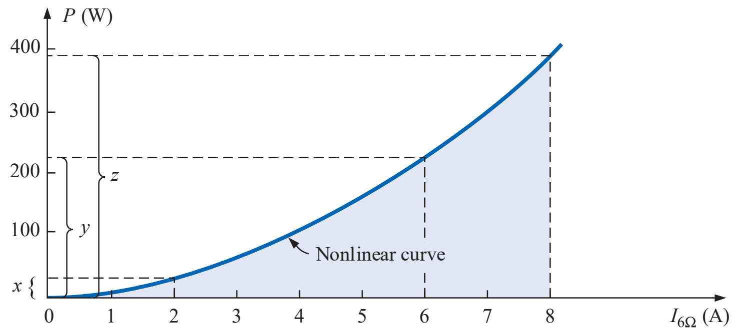


FIG. 9.14
Plotting the power delivered to the 6-Ω resistor versus current through the resistor.

Obviously, $x + y \neq z$, or 24 W + 216 W ≠ 384 W, and superposition does not hold. However, for a linear relationship, such as that between the voltage and current of the fixed-type 6-Ω resistor, superposition can be applied, as demonstrated by the graph of Fig. 9.15, where $a + b = c$, or 2 A + 6 A = 8 A.

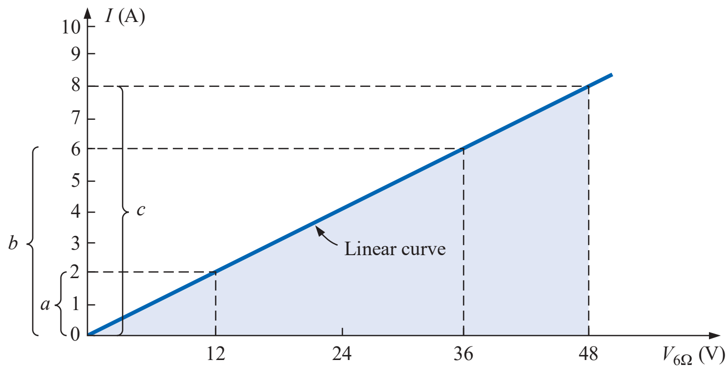


FIG. 9.15
Plotting I versus V for the 6-Ω resistor.

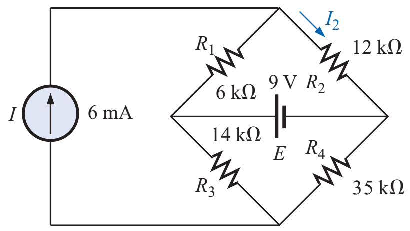


FIG. 9.16
Example 9.4.

EXAMPLE 9.4 Using the principle of superposition, find the current I_2 through the 12-kΩ resistor of Fig. 9.16.

Solution: *Considering the effect of the 6-mA current source (Fig. 9.17):*

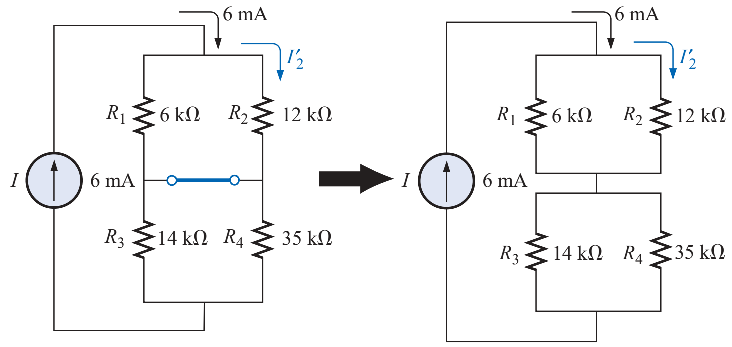


FIG. 9.17
The effect of the current source I on the current I_2.

Current divider rule:

$$I'_2 = \frac{R_1 I}{R_1 + R_2} = \frac{(6\ \text{k}\Omega)(6\ \text{mA})}{6\ \text{k}\Omega + 12\ \text{k}\Omega} = 2\ \text{mA}$$

Considering the effect of the 9-V voltage source (Fig. 9.18):

$$I''_2 = \frac{E}{R_1 + R_2} = \frac{9\ \text{V}}{6\ \text{k}\Omega + 12\ \text{k}\Omega} = 0.5\ \text{mA}$$

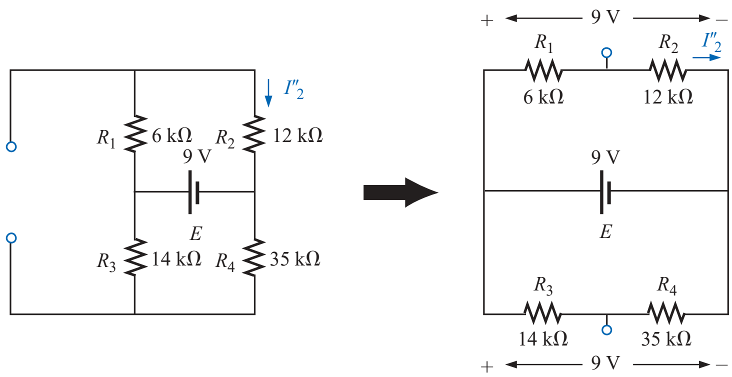


FIG. 9.18
The effect of the voltage source E on the current I_2.

Since I'_2 and I''_2 have the same direction through R_2, the desired current is the sum of the two:

$$\begin{aligned} I_2 &= I'_2 + I''_2 \\ &= 2\ \text{mA} + 0.5\ \text{mA} \\ &= \mathbf{2.5\ mA} \end{aligned}$$

EXAMPLE 9.5 Find the current through the 2-Ω resistor of the network of Fig. 9.19. The presence of three sources will result in three different networks to be analyzed.

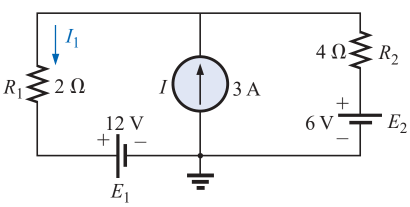


FIG. 9.19
Example 9.5.

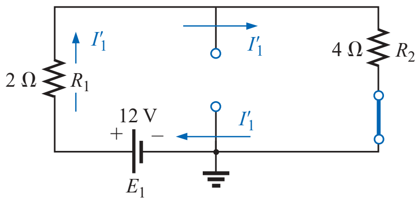


FIG. 9.20
The effect of E_1 on the current I.

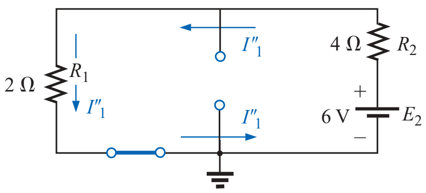


FIG. 9.21
The effect of E_2 on the current I_1.

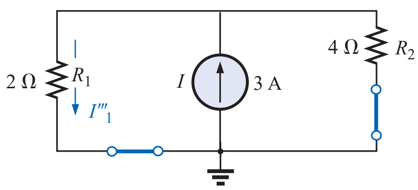


FIG. 9.22
The effect of I on the current I_1.

Solution: *Considering the effect of the 12-V source (Fig. 9.20):*

$$I'_1 = \frac{E_1}{R_1 + R_2} = \frac{12\ \text{V}}{2\ \Omega + 4\ \Omega} = \frac{12\ \text{V}}{6\ \Omega} = 2\ \text{A}$$

Considering the effect of the 6-V source (Fig. 9.21):

$$I''_1 = \frac{E_2}{R_1 + R_2} = \frac{6\ \text{V}}{2\ \Omega + 4\ \Omega} = \frac{6\ \text{V}}{6\ \Omega} = 1\ \text{A}$$

Considering the effect of the 3-A source (Fig. 9.22):
Applying the current divider rule,

$$I'''_1 = \frac{R_2 I}{R_1 + R_2} = \frac{(4\ \Omega)(3\ \text{A})}{2\ \Omega + 4\ \Omega} = \frac{12\ \text{A}}{6} = 2\ \text{A}$$

The total current through the 2-Ω resistor appears in Fig. 9.23, and

$$I_1 = \overbrace{I''_1 + I'''_1}^{\text{Same direction as } I_1 \text{ in Fig. 9.19}} \overset{\text{Opposite direction to } I_1 \text{ in Fig. 9.19}}{-\, I'_1}$$
$$= 1\text{A} + 2\text{A} - 2\text{A} = 1\textbf{A}$$

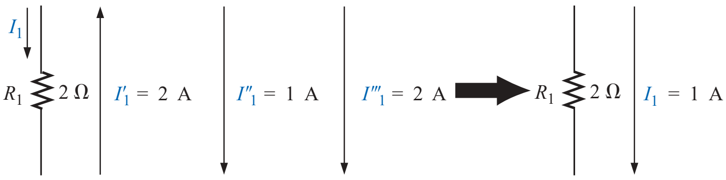


FIG. 9.23
The resultant current I_1.

9.3 THÉVENIN'S THEOREM

Thévenin's theorem states the following:

Any two-terminal, linear bilateral dc network can be replaced by an equivalent circuit consisting of a voltage source and a series resistor, as shown in Fig. 9.24.

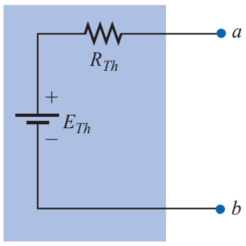


FIG. 9.24
Thévenin equivalent circuit.

In Fig. 9.25(a), for example, the network within the container has only two terminals available to the outside world, labeled *a* and *b.* It is possible using Thévenin's theorem to replace everything in the container with one source and one resistor, as shown in Fig. 9.25(b), and maintain the same terminal characteristics at terminals *a* and *b.* That is, any load connected to terminals *a* and *b* will not know whether it is hooked up to the network of Fig. 9.25(a) or Fig. 9.25(b). The load will receive the same current, voltage, and power from either configuration of Fig. 9.25. Throughout the discussion to follow, however, always keep in mind that

the Thévenin equivalent circuit provides an equivalence at the terminals only—the internal construction and characteristics of the original network and the Thévenin equivalent are usually quite different.

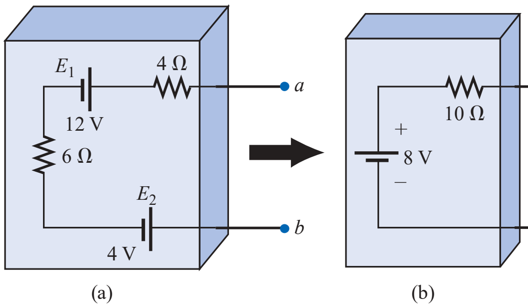


FIG. 9.25
The effect of applying Thévenin's theorem.

For the network of Fig. 9.25(a), the Thévenin equivalent circuit can be found quite directly by simply combining the series batteries and resistors. Note the exact similarity of the network of Fig. 9.25(b) to the Thévenin configuration of Fig. 9.24. The method described below will allow us to extend the procedure just applied to more complex configurations and still end up with the relatively simple network of Fig. 9.24.

In most cases, other elements will be connected to the right of terminals *a* and *b* in Fig. 9.25. To apply the theorem, however, the network to be reduced to the Thévenin equivalent form must be isolated as shown in Fig. 9.25, and the two "holding" terminals identified. Once the proper Thévenin equivalent circuit has been determined, the voltage, current, or resistance readings between the two "holding" terminals will be the same whether the original or the Thévenin equivalent circuit is connected to the left of terminals *a* and *b* in Fig. 9.25. Any load connected to the right of terminals *a* and *b* of Fig. 9.25 will receive the same voltage or current with either network.

This theorem achieves two important objectives. First, as was true for all the methods previously described, it allows us to find any particular voltage or current in a linear network with one, two, or any other number of sources. Second, we can concentrate on a specific portion of a network by replacing the remaining network with an equivalent circuit. In Fig. 9.26, for example, by finding the Thévenin equivalent circuit for the network in the shaded area, we can quickly calculate the change in current through or voltage across the variable resistor R_L for the various values that it may assume. This is demonstrated in Example 9.6.

French (Meaux, Paris)
(1857–1927)
Telegraph Engineer, Commandant and Educator
École Polytechnique and École Supérieure de Télégraphie

Courtesy of the Bibliothèque École Polytechnique, Paris, France

Although active in the study and design of telegraphic systems (including underground transmission), cylindrical condensers (capacitors), and electromagnetism, he is best known for a theorem first presented in the French *Journal of Physics—Theory and Applications* in 1883. It appeared under the heading of "Sur un nouveau théorème d'électricité dynamique" ("On a new theorem of dynamic electricity") and was originally referred to as the *equivalent generator theorem.* There is some evidence that a similar theorem was introduced by Hermann von Helmholtz in 1853. However, Professor Helmholtz applied the theorem to animal physiology and not to communication or generator systems, and therefore he has not received the credit in this field that he might deserve. In the early 1920s AT&T did some pioneering work using the equivalent circuit and may have initiated the reference to the theorem as simply Thévenin's theorem. In fact, Edward L. Norton, an engineer at AT&T at the time, introduced a current source equivalent of the Thévenin equivalent currently referred to as the Norton equivalent circuit. As an aside, Commandant Thévenin was an avid skier and in fact was commissioner of an international ski competition in Chamonix, France, in 1912.

LEON-CHARLES THÉVENIN

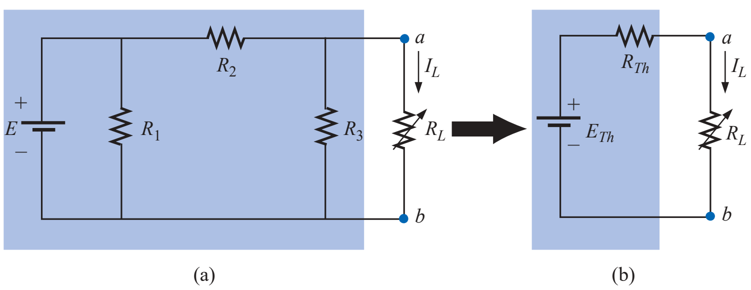


FIG. 9.26
Substituting the Thévenin equivalent circuit for a complex network.

Before we examine the steps involved in applying this theorem, it is important that an additional word be included here to ensure that the implications of the Thévenin equivalent circuit are clear. In Fig. 9.26, the entire network, except R_L, is to be replaced by a single series resistor and battery as shown in Fig. 9.24. The values of these two elements of the Thévenin equivalent circuit must be chosen to ensure that the resistor R_L will react to the network of Fig. 9.26(a) in the same manner as to the network of Fig. 9.26(b). In other words, the current through or voltage across R_L must be the same for either network for any value of R_L.

The following sequence of steps will lead to the proper value of R_{Th} and E_{Th}.

Preliminary:

1. ***Remove that portion of the network across which the Thévenin equivalent circuit is to be found.* *In Fig. 9.26(a), this requires that the load resistor R_L be temporarily removed from the network.***
2. ***Mark the terminals of the remaining two-terminal network. (The importance of this step will become obvious as we progress through some complex networks.)***

R_{Th}:

3. ***Calculate R_{Th} by first setting all sources to zero (voltage sources are replaced by short circuits, and current sources by open circuits) and then finding the resultant resistance between the two marked terminals. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.)***

E_{Th}:

4. ***Calculate E_{Th} by first returning all sources to their original position and finding the open-circuit voltage between the marked terminals. (This step is invariably the one that will lead to the most confusion and errors. In all cases, keep in mind that it is the open-circuit potential between the two terminals marked in step 2.)***

Conclusion:

5. ***Draw the Thévenin equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit. This step is indicated by the placement of the resistor R_L between the terminals of the Thévenin equivalent circuit as shown in Fig. 9.26(b).***

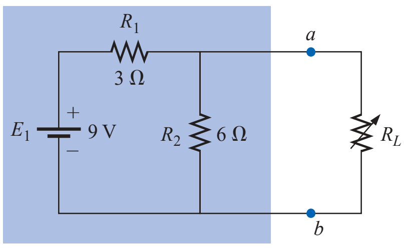


FIG. 9.27
Example 9.6.

EXAMPLE 9.6 Find the Thévenin equivalent circuit for the network in the shaded area of the network of Fig. 9.27. Then find the current through R_L for values of 2 Ω, 10 Ω, and 100 Ω.

Solution:

Steps 1 and 2 produce the network of Fig. 9.28. Note that the load resistor R_L has been removed and the two "holding" terminals have been defined as *a* and *b*.

Step 3: Replacing the voltage source E_1 with a short-circuit equivalent yields the network of Fig. 9.29(a), where

$$R_{Th} = R_1 \parallel R_2 = \frac{(3\ \Omega)(6\ \Omega)}{3\ \Omega + 6\ \Omega} = \mathbf{2\ \Omega}$$

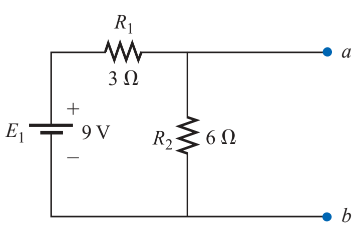


FIG. 9.28
Identifying the terminals of particular importance when applying Thévenin's theorem.

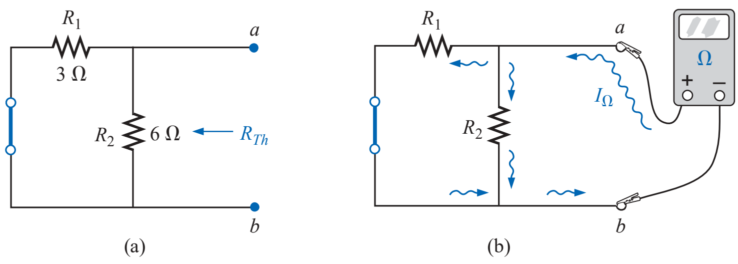


FIG. 9.29
Determining R_{Th} for the network of Fig. 9.28.

The importance of the two marked terminals now begins to surface. They are the two terminals across which the Thévenin resistance is measured. It is no longer the total resistance as seen by the source, as determined in the majority of problems of Chapter 7. If some difficulty develops when determining R_{Th} with regard to whether the resistive elements are in series or parallel, consider recalling that the ohmmeter sends out a trickle current into a resistive combination and senses the level of the resulting voltage to establish the measured resistance level. In Fig. 9.29(b), the trickle current of the ohmmeter approaches the network through terminal *a,* and when it reaches the junction of R_1 and R_2, it splits as shown. The fact that the trickle current splits and then recombines at the lower node reveals that the resistors are in parallel as far as the ohmmeter reading is concerned. In essence, the path of the sensing current of the ohmmeter has revealed how the resistors are connected to the two terminals of interest and how the Thévenin resistance should be determined. Keep the above in mind as you work through the various examples of this section.

Step 4: Replace the voltage source (Fig. 9.30). For this case, the open-circuit voltage E_{Th} is the same as the voltage drop across the 6-Ω resistor. Applying the voltage divider rule,

$$E_{Th} = \frac{R_2 E_1}{R_2 + R_1} = \frac{(6\ \Omega)(9\ \text{V})}{6\ \Omega + 3\ \Omega} = \frac{54\ \text{V}}{9} = \mathbf{6\ V}$$

It is particularly important to recognize that E_{Th} is the open-circuit potential between points *a* and *b.* Remember that an open circuit can have any voltage across it, but the current must be zero. In fact, the current through any element in series with the open circuit must be zero also. The use of a voltmeter to measure E_{Th} appears in Fig. 9.31. Note that it is placed directly across the resistor R_2 since E_{Th} and V_{R_2} are in parallel.

Step 5 (Fig. 9.32):

$$I_L = \frac{E_{Th}}{R_{Th} + R_L}$$

$$R_L = 2\ \Omega: \quad I_L = \frac{6\ \text{V}}{2\ \Omega + 2\ \Omega} = \mathbf{1.5\ A}$$

$$R_L = 10\ \Omega: \quad I_L = \frac{6\ \text{V}}{2\ \Omega + 10\ \Omega} = \mathbf{0.5\ A}$$

$$R_L = 100\ \Omega: \quad I_L = \frac{6\ \text{V}}{2\ \Omega + 100\ \Omega} = \mathbf{0.059\ A}$$

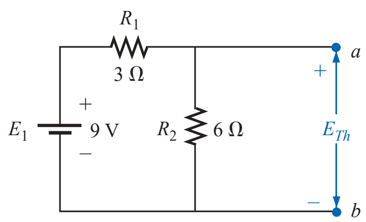


FIG. 9.30
Determining E_{Th} for the network of Fig. 9.28.

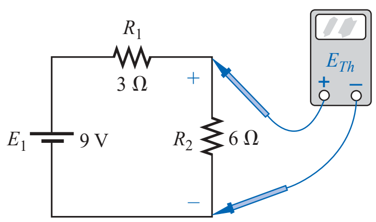


FIG. 9.31
Measuring E_{Th} for the network of Fig. 9.28.

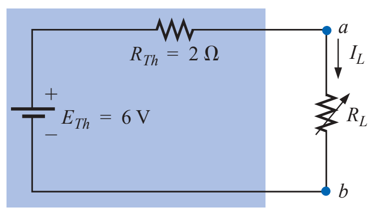


FIG. 9.32
Substituting the Thévenin equivalent circuit for the network external to R_L in Fig. 9.27.

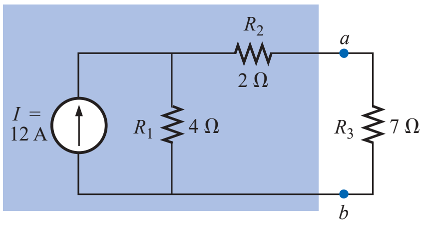


FIG. 9.33
Example 9.7.

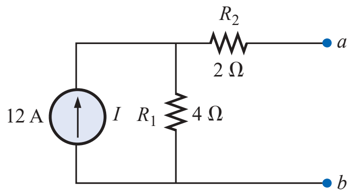


FIG. 9.34
Establishing the terminals of particular interest for the network of Fig. 9.33.

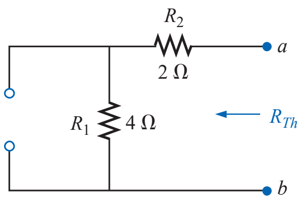


FIG. 9.35
Determining R_{Th} for the network of Fig. 9.34.

If Thévenin's theorem were unavailable, each change in R_L would require that the entire network of Fig. 9.27 be reexamined to find the new value of R_L.

EXAMPLE 9.7 Find the Thévenin equivalent circuit for the network in the shaded area of the network of Fig. 9.33.

Solution:

Steps 1 and 2 are shown in Fig. 9.34.

Step 3 is shown in Fig. 9.35. The current source has been replaced with an open-circuit equivalent, and the resistance determined between terminals *a* and *b*.

In this case an ohmmeter connected between terminals *a* and *b* would send out a sensing current that would flow directly through R_1 and R_2 (at the same level). The result is that R_1 and R_2 are in series and the Thévenin resistance is the sum of the two.

$$R_{Th} = R_1 + R_2 = 4\ \Omega + 2\ \Omega = \mathbf{6\ \Omega}$$

Step 4 (Fig. 9.36): In this case, since an open circuit exists between the two marked terminals, the current is zero between these terminals and through the 2-Ω resistor. The voltage drop across R_2 is, therefore,

$$V_2 = I_2R_2 = (0)R_2 = 0\ \text{V}$$

and
$$E_{Th} = V_1 = I_1R_1 = IR_1 = (12\ \text{A})(4\ \Omega) = \mathbf{48\ V}$$

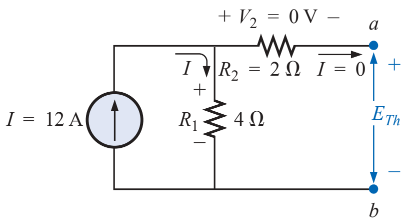


FIG. 9.36
Determining E_{Th} for the network of Fig. 9.34.

Step 5 is shown in Fig. 9.37.

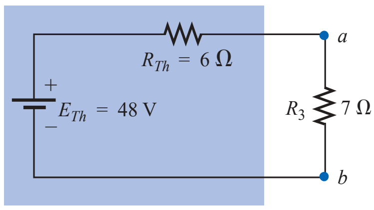


FIG. 9.37
Substituting the Thévenin equivalent circuit in the network external to the resistor R_3 of Fig. 9.33.

EXAMPLE 9.8 Find the Thévenin equivalent circuit for the network in the shaded area of the network of Fig. 9.38. Note in this example that

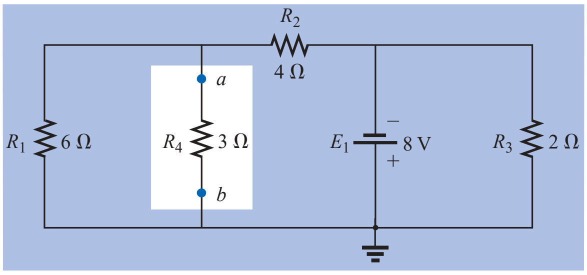


FIG. 9.38
Example 9.8.

there is no need for the section of the network to be preserved to be at the "end" of the configuration.

Solution:

Steps 1 and 2: See Fig. 9.39.

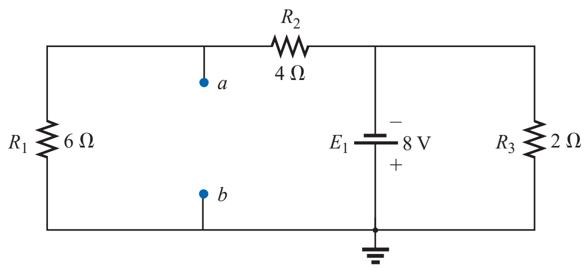


FIG. 9.39
Identifying the terminals of particular interest for the network of Fig. 9.38.

Step 3: See Fig. 9.40. Steps 1 and 2 are relatively easy to apply, but now we must be careful to "hold" onto the terminals a and b as the Thévenin resistance and voltage are determined. In Fig. 9.40, all the remaining elements turn out to be in parallel, and the network can be redrawn as shown.

$$R_{Th} = R_1 \parallel R_2 = \frac{(6\,\Omega)(4\,\Omega)}{6\,\Omega + 4\,\Omega} = \frac{24\,\Omega}{10} = \mathbf{2.4\,\Omega}$$

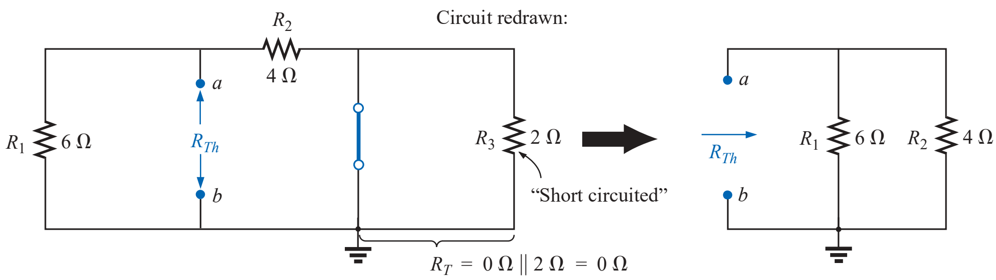


FIG. 9.40
Determining R_{Th} for the network of Fig. 9.39.

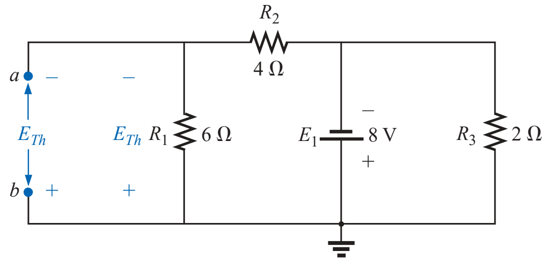


FIG. 9.41
Determining E_{Th} for the network of Fig. 9.39.

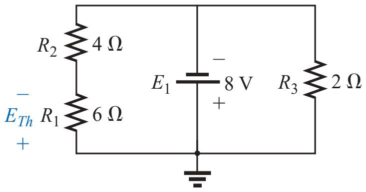


FIG. 9.42
Network of Fig. 9.41 redrawn.

Step 4: See Fig. 9.41. In this case, the network can be redrawn as shown in Fig. 9.42, and since the voltage is the same across parallel elements, the voltage across the series resistors R_1 and R_2 is E_1, or 8 V. Applying the voltage divider rule,

$$E_{Th} = \frac{R_1 E_1}{R_1 + R_2} = \frac{(6\ \Omega)(8\ \text{V})}{6\ \Omega + 4\ \Omega} = \frac{48\ \text{V}}{10} = \mathbf{4.8\ V}$$

Step 5: See Fig. 9.43.

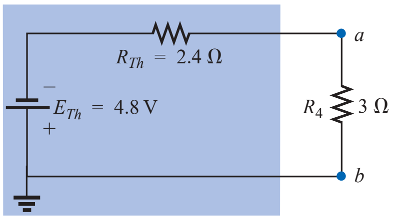


FIG. 9.43
Substituting the Thévenin equivalent circuit for the network external to the resistor R_4 of Fig. 9.38.

The importance of marking the terminals should be obvious from Example 9.8. Note that there is no requirement that the Thévenin voltage have the same polarity as the equivalent circuit originally introduced.

EXAMPLE 9.9 Find the Thévenin equivalent circuit for the network in the shaded area of the bridge network of Fig. 9.44.

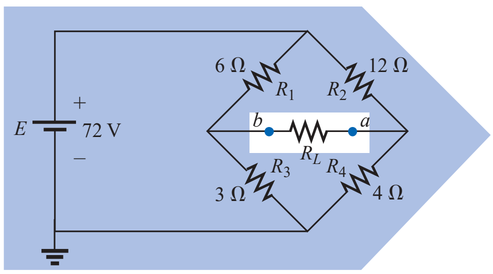


FIG. 9.44
Example 9.9.

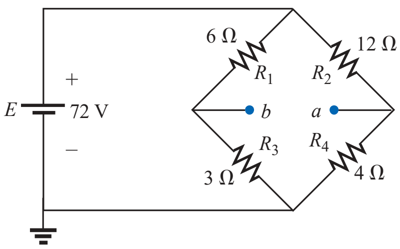


FIG. 9.45
Identifying the terminals of particular interest for the network of Fig. 9.44.

Solution:

Steps 1 and 2 are shown in Fig. 9.45.

Step 3: See Fig. 9.46. In this case, the short-circuit replacement of the voltage source E provides a direct connection between c and c' of Fig. 9.46(a), permitting a "folding" of the network around the horizontal line of a-b to produce the configuration of Fig. 9.46(b).

$$\begin{aligned} R_{Th} = R_{a\text{-}b} &= R_1 \parallel R_3 + R_2 \parallel R_4 \\ &= 6\ \Omega \parallel 3\ \Omega + 4\ \Omega \parallel 12\ \Omega \\ &= 2\ \Omega + 3\ \Omega = \mathbf{5\ \Omega} \end{aligned}$$

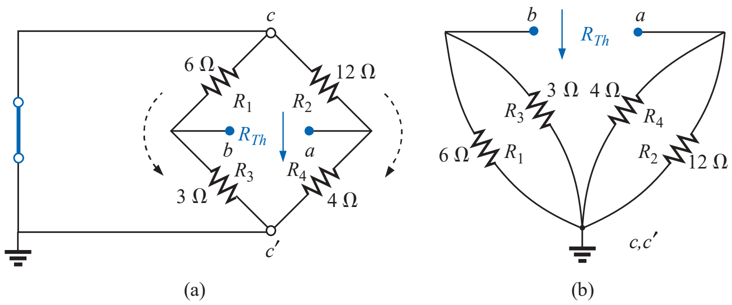


FIG. 9.46
Solving for R_{Th} for the network of Fig. 9.45.

Step 4: The circuit is redrawn in Fig. 9.47. The absence of a direct connection between a and b results in a network with three parallel branches. The voltages V_1 and V_2 can therefore be determined using the voltage divider rule:

$$V_1 = \frac{R_1E}{R_1 + R_3} = \frac{(6\ \Omega)(72\ \text{V})}{6\ \Omega + 3\ \Omega} = \frac{432\ \text{V}}{9} = 48\ \text{V}$$

$$V_2 = \frac{R_2E}{R_2 + R_4} = \frac{(12\ \Omega)(72\ \text{V})}{12\ \Omega + 4\ \Omega} = \frac{864\ \text{V}}{16} = 54\ \text{V}$$

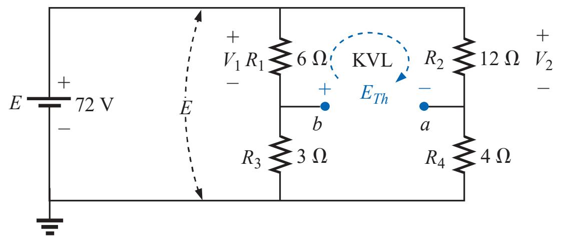


FIG. 9.47
Determining E_{Th} for the network of Fig. 9.45.

Assuming the polarity shown for E_{Th} and applying Kirchhoff's voltage law to the top loop in the clockwise direction will result in

$$\Sigma_{\circlearrowright} V = +E_{Th} + V_1 - V_2 = 0$$

and $$E_{Th} = V_2 - V_1 = 54\ \text{V} - 48\ \text{V} = \mathbf{6\ V}$$

Step 5 is shown in Fig. 9.48.

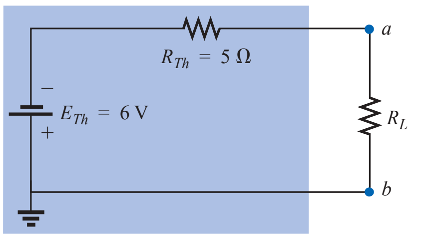


FIG. 9.48
Substituting the Thévenin equivalent circuit for the network external to the resistor R_L of Fig. 9.44.

Thévenin's theorem is not restricted to a single passive element, as shown in the preceding examples, but can be applied across sources, whole branches, portions of networks, or any circuit configuration, as shown in the following example. It is also possible that one of the methods previously described, such as mesh analysis or superposition, may have to be used to find the Thévenin equivalent circuit.

EXAMPLE 9.10 (Two sources) Find the Thévenin circuit for the network within the shaded area of Fig. 9.49.

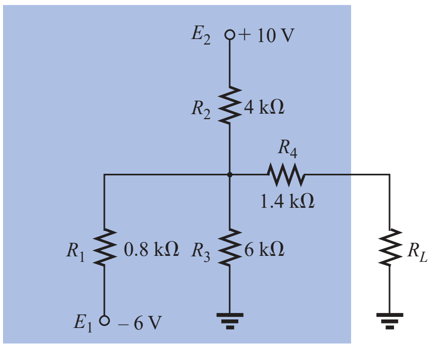


FIG. 9.49
Example 9.10.

Solution: The network is redrawn and *steps 1 and 2* are applied as shown in Fig. 9.50.

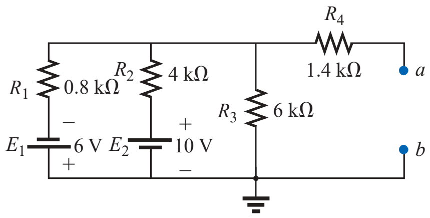


FIG. 9.50
Identifying the terminals of particular interest for the network of Fig. 9.49.

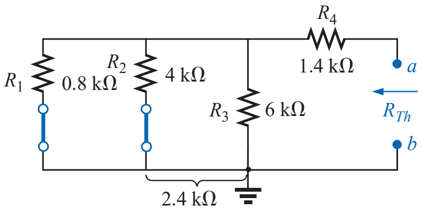


FIG. 9.51
Determining R_{Th} for the network of Fig. 9.50.

Step 3: See Fig. 9.51.

$$\begin{aligned} R_{Th} &= R_4 + R_1 \parallel R_2 \parallel R_3 \\ &= 1.4\ \text{k}\Omega + 0.8\ \text{k}\Omega \parallel 4\ \text{k}\Omega \parallel 6\ \text{k}\Omega \\ &= 1.4\ \text{k}\Omega + 0.8\ \text{k}\Omega \parallel 2.4\ \text{k}\Omega \\ &= 1.4\ \text{k}\Omega + 0.6\ \text{k}\Omega \\ &= \mathbf{2\ k\Omega} \end{aligned}$$

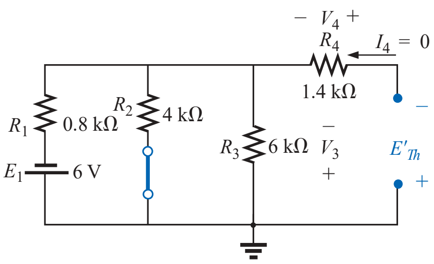


FIG. 9.52
Determining the contribution to E_{Th} from the source E_1 for the network of Fig. 9.50.

Step 4: Applying superposition, we will consider the effects of the voltage source E_1 first. Note Fig. 9.52. The open circuit requires that $V_4 = I_4R_4 = (0)R_4 = 0$ V, and

$$E'_{Th} = V_3$$
$$R'_T = R_2 \parallel R_3 = 4\ \text{k}\Omega \parallel 6\ \text{k}\Omega = 2.4\ \text{k}\Omega$$

Applying the voltage divider rule,

$$V_3 = \frac{R'_T E_1}{R'_T + R_1} = \frac{(2.4\ \text{k}\Omega)(6\ \text{V})}{2.4\ \text{k}\Omega + 0.8\ \text{k}\Omega} = \frac{14.4\ \text{V}}{3.2} = 4.5\ \text{V}$$
$$E'_{Th} = V_3 = 4.5\ \text{V}$$

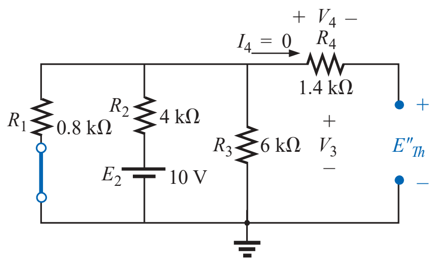


FIG. 9.53
Determining the contribution to E_{Th} from the source E_2 for the network of Fig. 9.50.

For the source E_2, the network of Fig. 9.53 will result. Again, $V_4 = I_4R_4 = (0)R_4 = 0$ V, and

$$E''_{Th} = V_3$$
$$R'_T = R_1 \parallel R_3 = 0.8\ \text{k}\Omega \parallel 6\ \text{k}\Omega = 0.706\ \text{k}\Omega$$

and $$V_3 = \frac{R'_T E_2}{R'_T + R_2} = \frac{(0.706\ \text{k}\Omega)(10\ \text{V})}{0.706\ \text{k}\Omega + 4\ \text{k}\Omega} = \frac{7.06\ \text{V}}{4.706} = 1.5\ \text{V}$$
$$E''_{Th} = V_3 = 1.5\ \text{V}$$

Since E'_{Th} and E''_{Th} have opposite polarities,

$$\begin{aligned} E_{Th} &= E'_{Th} - E''_{Th} \\ &= 4.5\ \text{V} - 1.5\ \text{V} \\ &= \mathbf{3\ V} \qquad (\text{polarity of } E'_{Th}) \end{aligned}$$

Step 5: See Fig. 9.54.

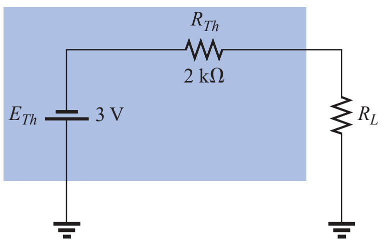


FIG. 9.54
Substituting the Thévenin equivalent circuit for the network external to the resistor R_L of Fig. 9.49.

Experimental Procedures

There are two popular experimental procedures for determining the parameters of a Thévenin equivalent network. The procedure for measuring the Thévenin voltage is the same for each, but the approach for determining the Thévenin resistance is quite different for each.

Direct Measurement of E_{Th} and R_{Th} For any physical network, the value of E_{Th} can be determined experimentally by measuring the open-circuit voltage across the load terminals, as shown in Fig. 9.55; $E_{Th} = V_{oc} = V_{ab}$. The value of R_{Th} can then be determined by completing the network with a variable R_L such as the potentiometer of Fig. 9.56(b). R_L can then be varied until the voltage appearing across the load is one-half the open-circuit value, or $V_L = E_{Th}/2$. For the series circuit of Fig. 9.56(a), when the load voltage is reduced to one-half the open-circuit level, the voltage across R_{Th} and R_L must be the same. If we read the value of R_L [as shown in Fig. 9.56(c)] that resulted in the preceding calculations, we will also have the value of R_{Th}, since $R_L = R_{Th}$ if V_L equals the voltage across R_{Th}.

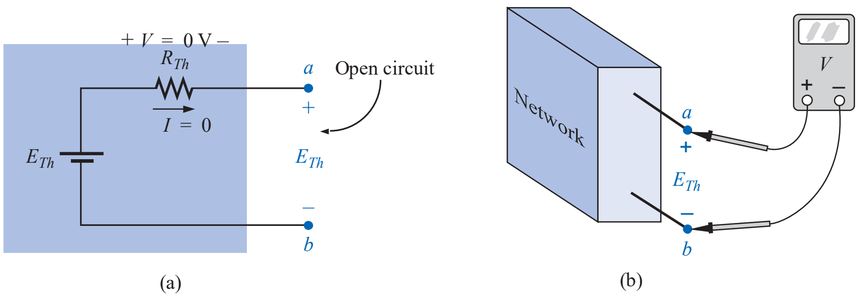


FIG. 9.55
Determining E_{Th} experimentally.

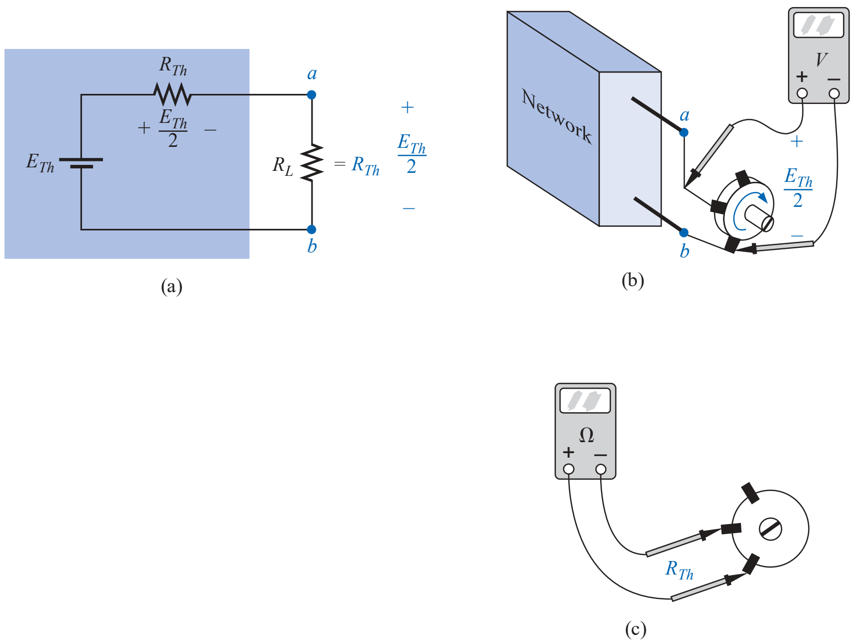


FIG. 9.56
Determining R_{Th} experimentally.

Measuring V_{oc} and I_{sc} The Thévenin voltage is again determined by measuring the open-circuit voltage across the terminals of interest;

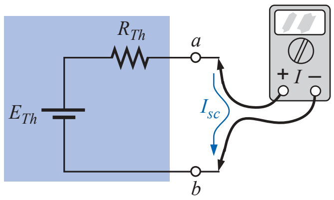


FIG. 9.57
Measuring I_{sc}.

American (Rockland, Maine; Summit, New Jersey)
(1898–1983)
Electrical Engineer, Scientist, Inventor
Department Head: Bell Laboratories
Fellow: Acoustical Society and Institute of Radio Engineers

Courtesy of AT&T Archives

Although interested primarily in communications circuit theory and the transmission of data at high speeds over telephone lines, Edward L. Norton is best remembered for development of the dual of Thévenin's equivalent circuit, currently referred to as *Norton's equivalent circuit*. In fact, Norton and his associates at AT&T in the early 1920s are recognized as some of the first to perform pioneering work applying Thévenin's equivalent circuit and who referred to this concept simply as Thévenin's theorem. In 1926 he proposed the equivalent circuit using a current source and parallel resistor to assist in the design of recording instrumentation that was primarily current driven. He began his telephone career in 1922 with the Western Electric Company's Engineering Department, which later became Bell Laboratories. His areas of active research included network theory, acoustical systems, electromagnetic apparatus, and data transmission. A graduate of MIT and Columbia University, he held nineteen patents on his work.

EDWARD L. NORTON

that is, $E_{Th} = V_{oc}$. To determine R_{Th}, a short-circuit condition is established across the terminals of interest, as shown in Fig. 9.57, and the current through the short circuit is measured with an ammeter. Using Ohm's law, we find that the short-circuit current is determined by

$$I_{sc} = \frac{E_{Th}}{R_{Th}}$$

and the Thévenin resistance by

$$R_{Th} = \frac{E_{Th}}{I_{sc}}$$

However, $E_{Th} = V_{oc}$ resulting in the following equation for R_{Th}:

$$\boxed{R_{Th} = \frac{V_{oc}}{I_{sc}}} \tag{9.2}$$

9.4 NORTON'S THEOREM

It was demonstrated in Section 8.3 that every voltage source with a series internal resistance has a current source equivalent. The current source equivalent of the Thévenin network (which, you will note, satisfies the above conditions), as shown in Fig. 9.58, can be determined by **Norton's theorem.** It can also be found through the conversions of Section 8.3.

The theorem states the following:

Any two-terminal linear bilateral dc network can be replaced by an equivalent circuit consisting of a current source and a parallel resistor, as shown in Fig. 9.58.

The discussion of Thévenin's theorem with respect to the equivalent circuit can also be applied to the Norton equivalent circuit. The steps leading to the proper values of I_N and R_N are now listed.

Preliminary:

1. ***Remove that portion of the network across which the Norton equivalent circuit is found.***
2. ***Mark the terminals of the remaining two-terminal network.***

R_N:

3. ***Calculate R_N by first setting all sources to zero (voltage sources are replaced with short circuits, and current sources with open circuits) and then finding the resultant resistance between the two marked terminals. (If the internal resistance of the voltage and/or current sources is included in the original network, it must remain when the sources are set to zero.) Since $R_N = R_{Th}$, the procedure and value obtained using the approach described for Thévenin's theorem will determine the proper value of R_N.***

I_N:

4. ***Calculate I_N by first returning all sources to their original position and then finding the short-circuit current between the marked terminals. It is the same current that would be measured by an ammeter placed between the marked terminals.***

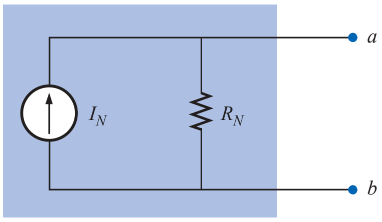


FIG. 9.58
Norton equivalent circuit.

Conclusion:

5. ***Draw the Norton equivalent circuit with the portion of the circuit previously removed replaced between the terminals of the equivalent circuit.***

The Norton and Thévenin equivalent circuits can also be found from each other by using the source transformation discussed earlier in this chapter and reproduced in Fig. 9.59.

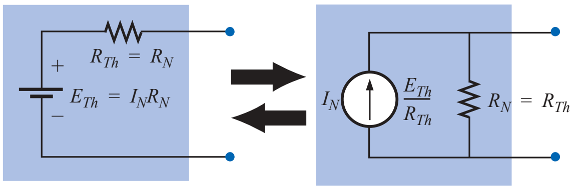


FIG. 9.59
Converting between Thévenin and Norton equivalent circuits.

EXAMPLE 9.11 Find the Norton equivalent circuit for the network in the shaded area of Fig. 9.60.

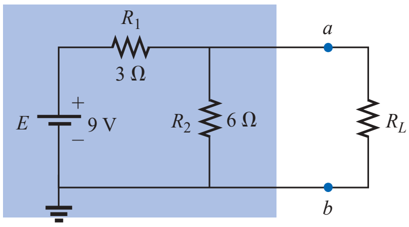


FIG. 9.60
Example 9.11.

Solution:

Steps 1 and 2 are shown in Fig. 9.61.

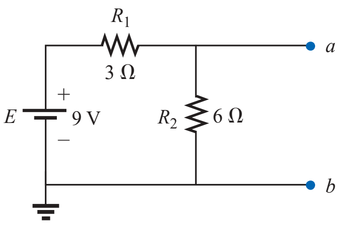


FIG. 9.61
Identifying the terminals of particular interest for the network of Fig. 9.60.

Step 3 is shown in Fig. 9.62, and

$$R_N = R_1 \parallel R_2 = 3\ \Omega \parallel 6\ \Omega = \frac{(3\ \Omega)(6\ \Omega)}{3\ \Omega + 6\ \Omega} = \frac{18\ \Omega}{9} = \mathbf{2\ \Omega}$$

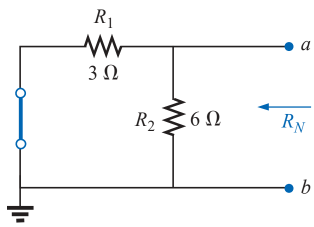


FIG. 9.62
Determining R_N for the network of Fig. 9.61.

Step 4 is shown in Fig. 9.63, clearly indicating that the short-circuit connection between terminals *a* and *b* is in parallel with R_2 and eliminates its effect. I_N is therefore the same as through R_1, and the full battery voltage appears across R_1 since

$$V_2 = I_2 R_2 = (0)6\ \Omega = 0\ \text{V}$$

Therefore,

$$I_N = \frac{E}{R_1} = \frac{9\ \text{V}}{3\ \Omega} = \mathbf{3\ A}$$

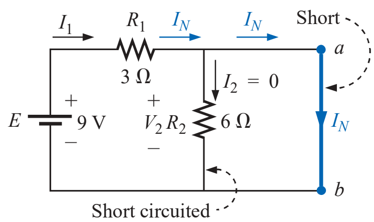


FIG. 9.63
Determining I_N for the network of Fig. 9.61.

Step 5: See Fig. 9.64. This circuit is the same as the first one considered in the development of Thévenin's theorem. A simple conversion indicates that the Thévenin circuits are, in fact, the same (Fig. 9.65).

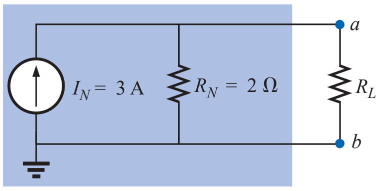


FIG. 9.64
Substituting the Norton equivalent circuit for the network external to the resistor R_L of Fig. 9.60.

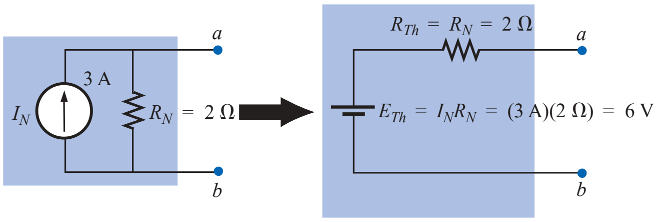


FIG. 9.65
Converting the Norton equivalent circuit of Fig. 9.64 to a Thévenin equivalent circuit.

EXAMPLE 9.12 Find the Norton equivalent circuit for the network external to the 9-Ω resistor in Fig. 9.66.

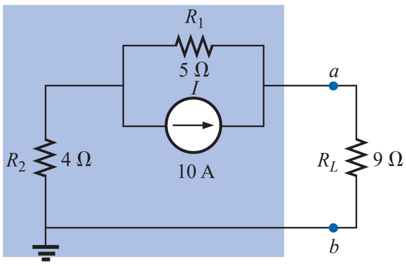


FIG. 9.66
Example 9.12.

Solution:

Steps 1 and 2: See Fig. 9.67.

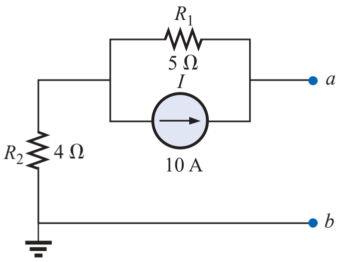


FIG. 9.67
Identifying the terminals of particular interest for the network of Fig. 9.66.

Step 3: See Fig. 9.68, and

$$R_N = R_1 + R_2 = 5\ \Omega + 4\ \Omega = \mathbf{9\ \Omega}$$

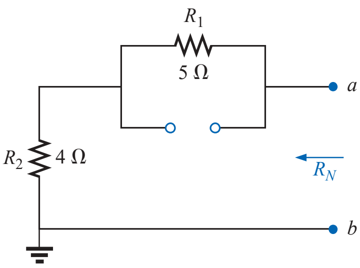


FIG. 9.68
Determining R_N for the network of Fig. 9.67.

Step 4: As shown in Fig. 9.69, the Norton current is the same as the current through the 4-Ω resistor. Applying the current divider rule,

$$I_N = \frac{R_1 I}{R_1 + R_2} = \frac{(5\ \Omega)(10\text{ A})}{5\ \Omega + 4\ \Omega} = \frac{50\text{ A}}{9} = \mathbf{5.556\ A}$$

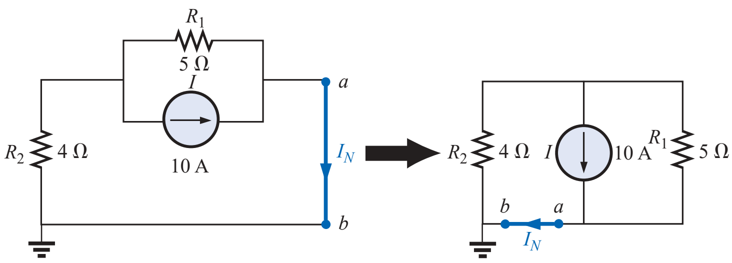


FIG. 9.69
Determining I_N for the network of Fig. 9.67.

Step 5: See Fig. 9.70.

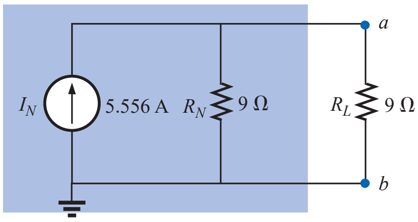


FIG. 9.70
Substituting the Norton equivalent circuit for the network external to the resistor R_L of Fig. 9.66.

EXAMPLE 9.13 (Two sources) Find the Norton equivalent circuit for the portion of the network to the left of *a-b* in Fig. 9.71.

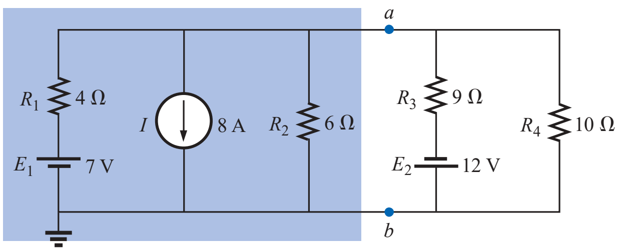


FIG. 9.71
Example 9.13.

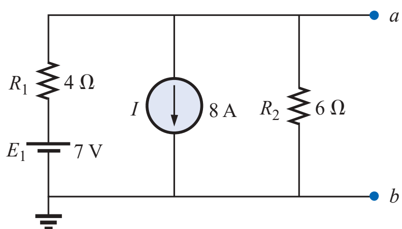


FIG. 9.72
Identifying the terminals of particular interest for the network of Fig. 9.71.

Solution:

Steps 1 and 2: See Fig. 9.72.

Step 3 is shown in Fig. 9.73, and

$$R_N = R_1 \parallel R_2 = 4\ \Omega \parallel 6\ \Omega = \frac{(4\ \Omega)(6\ \Omega)}{4\ \Omega + 6\ \Omega} = \frac{24\ \Omega}{10} = \mathbf{2.4\ \Omega}$$

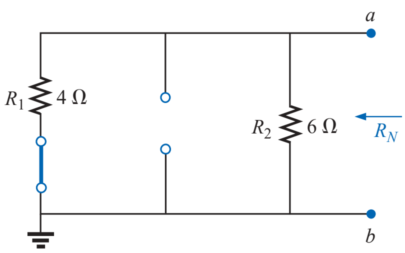


FIG. 9.73
Determining R_N for the network of Fig. 9.72.

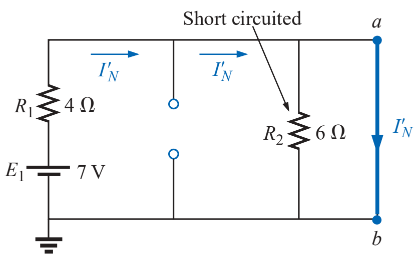


FIG. 9.74
Determining the contribution to I_N from the voltage source E_1.

Step 4: (Using superposition) For the 7-V battery (Fig. 9.74),

$$I'_N = \frac{E_1}{R_1} = \frac{7\text{ V}}{4\ \Omega} = 1.75\text{ A}$$

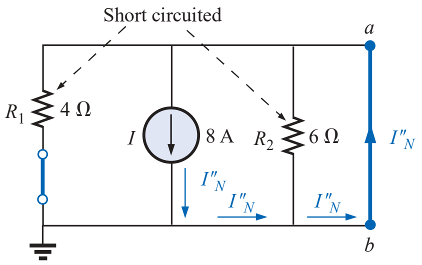


FIG. 9.75
Determining the contribution to I_N from the current source I.

For the 8-A source (Fig. 9.75), we find that both R_1 and R_2 have been "short circuited" by the direct connection between *a* and *b,* and

$$I''_N = I = 8\text{ A}$$

The result is

$$I_N = I''_N - I'_N = 8\text{ A} - 1.75\text{ A} = \mathbf{6.25\ A}$$

Step 5: See Fig. 9.76.

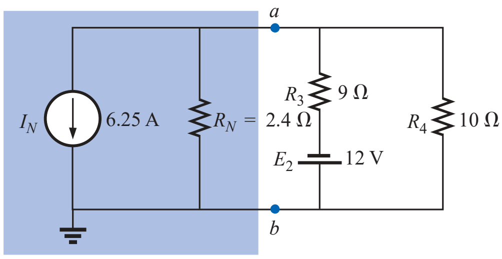


FIG. 9.76
Substituting the Norton equivalent circuit for the network to the left of terminals a-b in Fig. 9.71.

Experimental Procedure

The Norton current is measured in the same way as described for the short-circuit current for the Thévenin network. Since the Norton and Thévenin resistances are the same, the same procedures can be employed as described for the Thévenin network.

9.5 MAXIMUM POWER TRANSFER THEOREM

The **maximum power transfer theorem** states the following:

A load will receive maximum power from a linear bilateral dc network when its total resistive value is exactly equal to the Thévenin resistance of the network as "seen" by the load.

For the network of Fig. 9.77, maximum power will be delivered to the load when

$$\boxed{R_L = R_{Th}} \qquad \textbf{(9.3)}$$

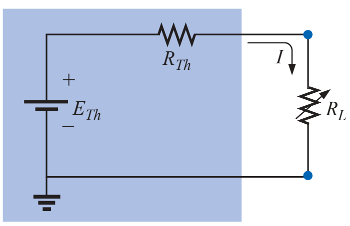


FIG. 9.77
Defining the conditions for maximum power to a load using the Thévenin equivalent circuit.

From past discussions, we realize that a Thévenin equivalent circuit can be found across any element or group of elements in a linear bilateral dc network. Therefore, if we consider the case of the Thévenin equivalent circuit with respect to the maximum power transfer theorem, we are, in essence, considering the *total* effects of any network across a resistor R_L, such as in Fig. 9.77.

For the Norton equivalent circuit of Fig. 9.78, maximum power will be delivered to the load when

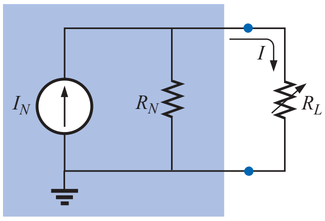


FIG. 9.78
Defining the conditions for maximum power to a load using the Norton equivalent circuit.

$$\boxed{R_L = R_N} \qquad \textbf{(9.4)}$$

This result [Eq. (9.4)] will be used to its fullest advantage in the analysis of transistor networks, where the most frequently applied transistor circuit model employs a current source rather than a voltage source.

For the network of Fig. 9.77,

$$I = \frac{E_{Th}}{R_{Th} + R_L}$$

and
$$P_L = I^2R_L = \left(\frac{E_{Th}}{R_{Th} + R_L}\right)^2 R_L$$

so that
$$P_L = \frac{E_{Th}^2 R_L}{(R_{Th} + R_L)^2}$$

Let us now consider an example where $E_{Th} = 60$ V and $R_{Th} = 9\ \Omega$, as shown in Fig. 9.79.

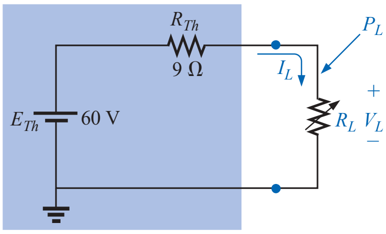


FIG. 9.79
Thévenin equivalent network to be used to validate the maximum power transfer theorem.

The power to the load is determined by

$$P_L = \frac{E_{Th}^2 R_L}{(R_{Th} + R_L)^2} = \frac{3600R_L}{(9\ \Omega + R_L)^2}$$

with
$$I_L = \frac{E_{Th}}{R_{Th} + R_L} = \frac{60\text{ V}}{9\ \Omega + R_L}$$

and
$$V_L = \frac{R_L(60\text{ V})}{R_{Th} + R_L} = \frac{R_L(60\text{ V})}{9\ \Omega + R_L}$$

A tabulation of P_L for a range of values of R_L yields Table 9.1. A plot of P_L versus R_L using the data of Table 9.1 will result in the plot of Fig. 9.80 for the range $R_L = 0.1\ \Omega$ to $30\ \Omega$.

TABLE 9.1

R_L (Ω)	P_L (W)		I_L (A)		V_L (V)	
0.1	4.35		6.59		0.66	
0.2	8.51		6.52		1.30	
0.5	19.94		6.32		3.16	
1	36.00		6.00		6.00	
2	59.50		5.46		10.91	
3	75.00		5.00		15.00	
4	85.21		4.62		18.46	
5	91.84	Increase	4.29	Decrease	21.43	Increase
6	96.00		4.00		24.00	
7	98.44		3.75		26.25	
8	99.65 ↓		3.53 ↓		28.23 ↓	
9 (R_{Th})	**100.00 (Maximum)**		**3.33 ($I_{max}/2$)**		**30.00 ($E_{Th}/2$)**	
10	99.72		3.16		31.58	
11	99.00		3.00		33.00	
12	97.96		2.86		34.29	
13	96.69		2.73		35.46	
14	95.27		2.61		36.52	
15	93.75		2.50		37.50	
16	92.16		2.40		38.40	
17	90.53	Decrease	2.31	Decrease	39.23	Increase
18	88.89		2.22		40.00	
19	87.24		2.14		40.71	
20	85.61		2.07		41.38	
25	77.86		1.77		44.12	
30	71.00		1.54		46.15	
40	59.98		1.22		48.98	
100	30.30		0.55		55.05	
500	6.95		0.12		58.94	
1000	3.54 ↓		0.06 ↓		59.47 ↓	

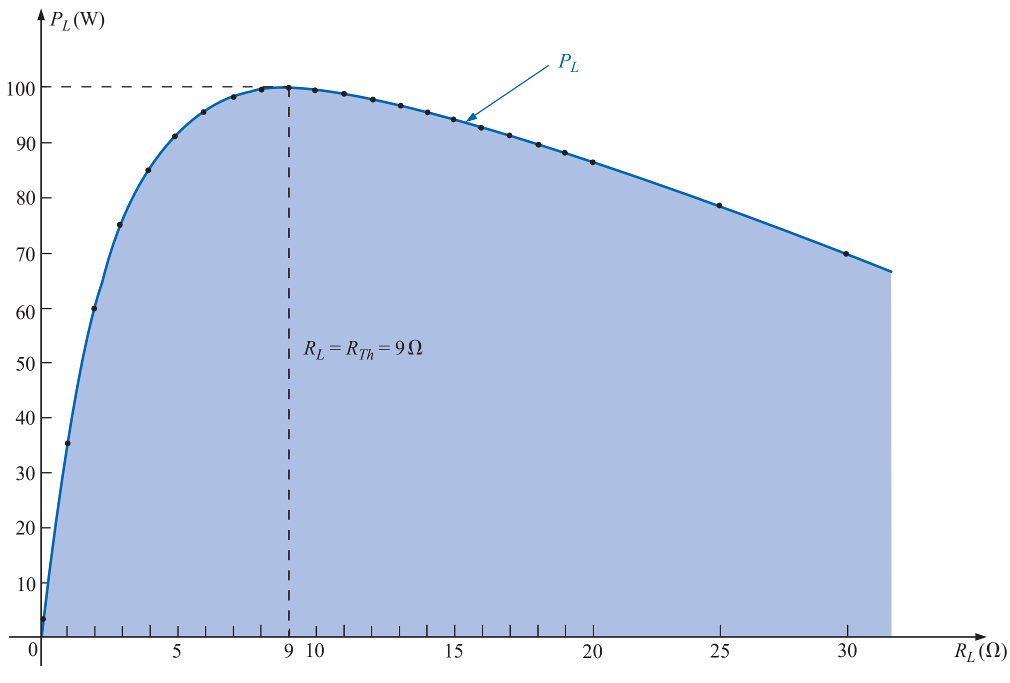


FIG. 9.80
P_L versus R_L for the network of Fig. 9.79.

Note, in particular, that P_L is, in fact, a maximum when $R_L = R_{Th} = 9\ \Omega$. The power curve increases more rapidly toward its maximum value than it decreases after the maximum point, clearly revealing that a small change in load resistance for levels of R_L below R_{Th} will have a more dramatic effect on the power delivered than similar changes in R_L above the R_{Th} level.

If we plot V_L and I_L versus the same resistance scale (Fig. 9.81), we find that both change nonlinearly, with the terminal voltage increasing with an increase in load resistance as the current decreases. Note again that the most dramatic changes in V_L and I_L occur for levels of R_L less than R_{Th}. As pointed out on the plot, when $R_L = R_{Th}$, $V_L = E_{Th}/2$ and $I_L = I_{max}/2$, with $I_{max} = E_{Th}/R_{Th}$.

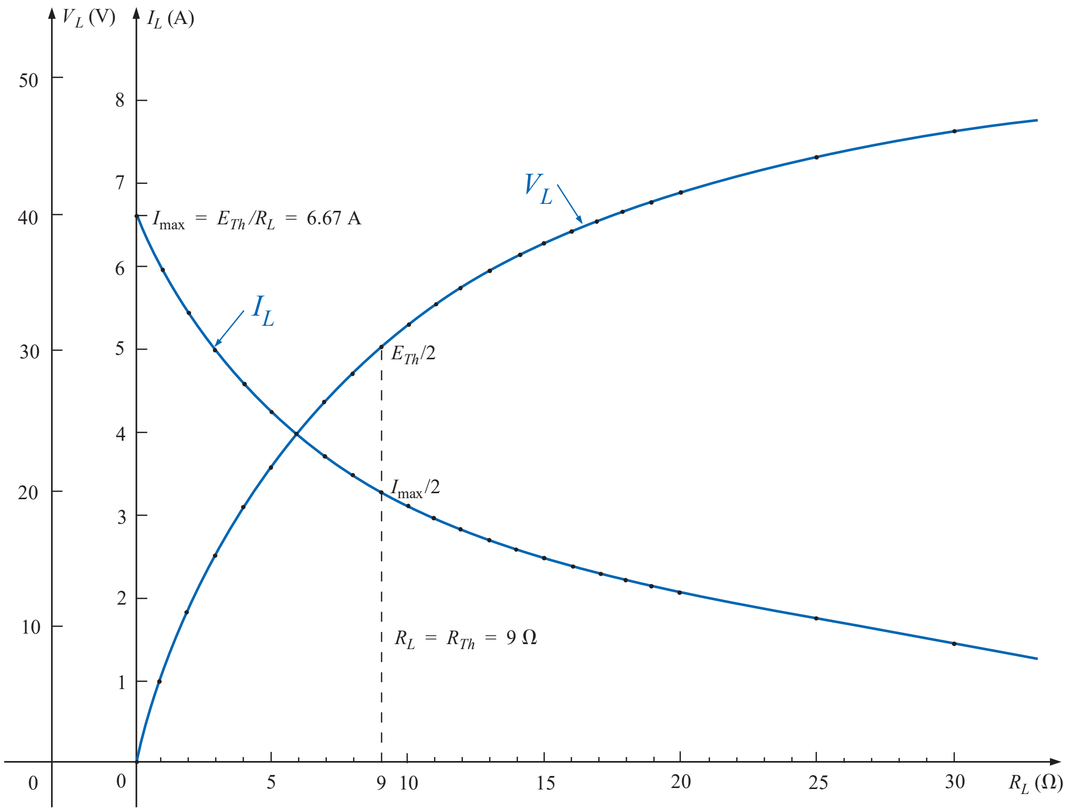


FIG. 9.81
V_L and I_L versus R_L for the network of Fig. 9.79.

The dc operating efficiency of a system is defined by the ratio of the power delivered to the load to the power supplied by the source; that is,

$$\boxed{\eta\% = \frac{P_L}{P_s} \times 100\%} \qquad \textbf{(9.5)}$$

For the situation defined by Fig. 9.77,

$$\eta\% = \frac{P_L}{P_s} \times 100\% = \frac{I_L^2 R_L}{I_L^2 R_T} \times 100\%$$

and $$\eta\% = \frac{R_L}{R_{Th} + R_L} \times 100\%$$

For R_L that is small compared to R_{Th}, $R_{Th} \gg R_L$ and $R_{Th} + R_L \cong R_{Th}$, with

$$\eta\% \cong \frac{R_L}{R_{Th}} \times 100\% = \underbrace{\left(\frac{1}{R_{Th}}\right)}_{\text{Constant}} R_L \times 100\% = kR_L \times 100\%$$

The resulting percentage efficiency, therefore, will be relatively low (since k is small) and will increase almost linearly as R_L increases.

For situations where the load resistance R_L is much larger than R_{Th}, $R_L \gg R_{Th}$ and $R_{Th} + R_L \cong R_L$.

$$\eta\% = \frac{R_L}{R_L} \times 100\% = 100\%$$

The efficiency therefore increases linearly and dramatically for small levels of R_L and then begins to level off as it approaches the 100% level for very large values of R_L, as shown in Fig. 9.82. Keep in mind, however, that the efficiency criterion is sensitive only to the ratio of P_L to P_s and not to their actual levels. At efficiency levels approaching 100%, the power delivered to the load may be so small as to have little practical value. Note the low level of power to the load in Table 9.1 when $R_L = 1000\ \Omega$, even though the efficiency level will be

$$\eta\% = \frac{R_L}{R_{Th} + R_L} \times 100\% = \frac{1000}{1009} \times 100\% = 99.11\%$$

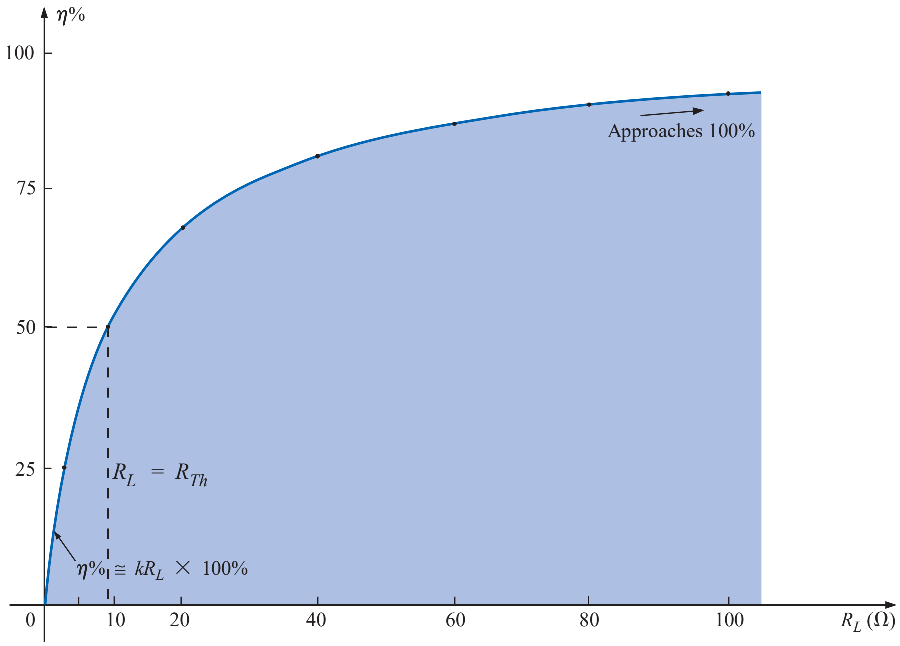


FIG. 9.82
Efficiency of operation versus increasing values of R_L.

When $R_L = R_{Th}$,

$$\eta\% = \frac{R_L}{R_{Th} + R_L} \times 100\% = \frac{R_L}{2R_L} \times 100\% = \mathbf{50\%}$$

Under maximum power transfer conditions, therefore, P_L is a maximum, but the dc efficiency is only 50%; that is, only half the power delivered by the source is getting to the load.

A relatively low efficiency of 50% can be tolerated in situations where power levels are relatively low, such as in a wide variety of electronic systems. However, when large power levels are involved, such as at generating stations, efficiencies of 50% would not be acceptable. In fact, a great deal of expense and research is dedicated to raising power-generating and transmission efficiencies a few percentage points. Raising an efficiency level of a 10-mega-kW power plant from 94% to 95% (a 1% increase) can save 0.1 mega-kW, or 100 million watts, of power—an enormous saving!

Consider a change in load levels from 9 Ω to 20 Ω. In Fig. 9.80, the power level has dropped from 100 W to 85.61 W (a 14.4% drop), but the efficiency has increased substantially to 69% (a 38% increase), as shown in Fig. 9.82. For each application, therefore, a balance point must be identified where the efficiency is sufficiently high without reducing the power to the load to insignificant levels.

Figure 9.83 is a semilog plot of P_L and the power delivered by the source $P_s = E_{Th}I_L$ versus R_L for $E_{Th} = 60$ V and $R_{Th} = 9\ \Omega$. A semilog graph employs one log scale and one linear scale, as implied by the prefix *semi,* meaning *half.* Log scales are discussed in detail in Chapter 23. For the moment, note the wide range of R_L permitted using the log scale compared to Figs. 9.80 through 9.82.

It is now quite clear that the P_L curve has only one maximum (at $R_L = R_{Th}$), whereas P_s decreases for every increase in R_L. In particular, note that for low levels of R_L, only a small portion of the power delivered by the source makes it to the load. In fact, even when $R_L = R_{Th}$, the source is generating twice the power absorbed by the load. For values of R_L greater than R_{Th}, the two curves approach each other until eventually they are essentially the same at high levels of R_L. For the range $R_L = R_{Th} = 9\ \Omega$ to $R_L = 100\ \Omega$, P_L and P_s are relatively close in magnitude, suggesting that this would be an appropriate range of operation, since a majority of the power delivered by the source is getting to the load and the power levels are still significant.

The power delivered to R_L under maximum power conditions ($R_L = R_{Th}$) is

$$I = \frac{E_{Th}}{R_{Th} + R_L} = \frac{E_{Th}}{2R_{Th}}$$

$$P_L = I^2R_L = \left(\frac{E_{Th}}{2R_{Th}}\right)^2 R_{Th} = \frac{E_{Th}^2 R_{Th}}{4R_{Th}^2}$$

and

$$\boxed{P_{L_{\text{max}}} = \frac{E_{Th}^2}{4R_{Th}}} \quad \text{(watts, W)} \qquad \mathbf{(9.6)}$$

For the Norton circuit of Fig. 9.78,

$$\boxed{P_{L_{\text{max}}} = \frac{I_N^2 R_N}{4}} \quad \text{(W)} \qquad \mathbf{(9.7)}$$

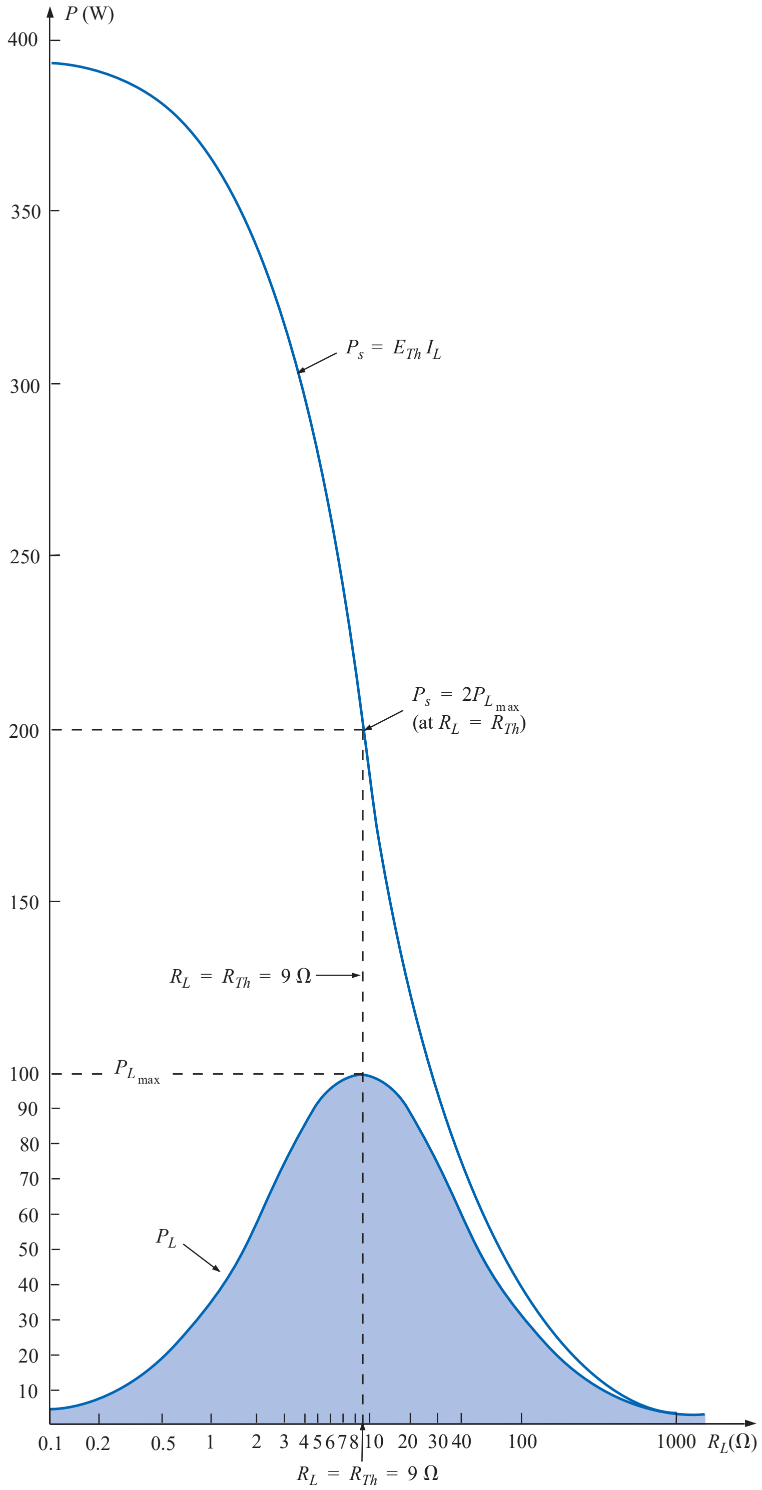
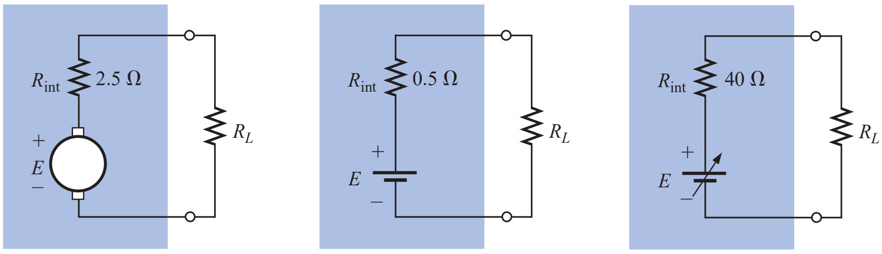


FIG. 9.83

P_s and P_L versus R_L for the network of Fig. 9.79.

EXAMPLE 9.14 A dc generator, battery, and laboratory supply are connected to a resistive load R_L in Fig. 9.84(a), (b), and (c), respectively.



(a) dc generator (b) Battery (c) Laboratory supply

FIG. 9.84
Example 9.14.

a. For each, determine the value of R_L for maximum power transfer to R_L.
b. Determine R_L for 75% efficiency.

Solutions:

a. For the dc generator,

$$R_L = R_{Th} = R_{int} = \mathbf{2.5\ \Omega}$$

For the battery,

$$R_L = R_{Th} = R_{int} = \mathbf{0.5\ \Omega}$$

For the laboratory supply,

$$R_L = R_{Th} = R_{int} = \mathbf{40\ \Omega}$$

b. For the dc generator,

$$\eta = \frac{P_o}{P_s} \qquad (\eta \text{ in decimal form})$$

$$\eta = \frac{R_L}{R_{Th} + R_L}$$

$$\eta(R_{Th} + R_L) = R_L$$

$$\eta R_{Th} + \eta R_L = R_L$$

$$R_L(1 - \eta) = \eta R_{Th}$$

and

$$\boxed{R_L = \frac{\eta R_{Th}}{1 - \eta}} \qquad \textbf{(9.8)}$$

$$R_L = \frac{0.75(2.5\ \Omega)}{1 - 0.75} = \mathbf{7.5\ \Omega}$$

For the battery,

$$R_L = \frac{0.75(0.5\ \Omega)}{1 - 0.75} = \mathbf{1.5\ \Omega}$$

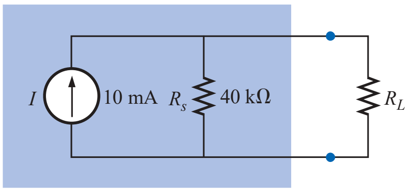


FIG. 9.85
Example 9.15.

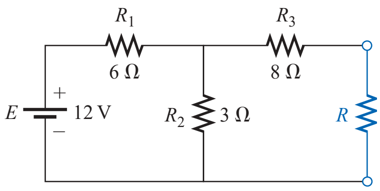


FIG. 9.86
Example 9.16.

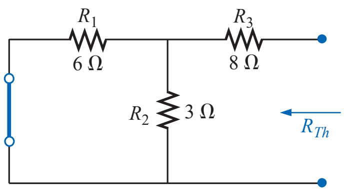


FIG. 9.87
Determining R_{Th} for the network external to the resistor R of Fig. 9.86.

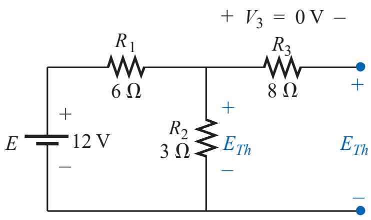


FIG. 9.88
Determining E_{Th} for the network external to the resistor R of Fig. 9.86.

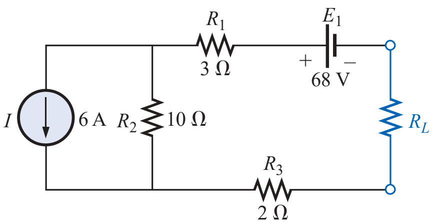


FIG. 9.89
Example 9.17.

For the laboratory supply,

$$R_L = \frac{0.75(40\ \Omega)}{1 - 0.75} = \mathbf{120\ \Omega}$$

The results of Example 9.14 reveal that the following modified form of the **maximum power transfer theorem** is valid:

For loads connected directly to a dc voltage supply, maximum power will be delivered to the load when the load resistance is equal to the internal resistance of the source; that is, when

$$\boxed{R_L = R_{\text{int}}} \qquad \textbf{(9.9)}$$

EXAMPLE 9.15 Analysis of a transistor network resulted in the reduced configuration of Fig. 9.85. Determine the R_L necessary to transfer maximum power to R_L, and calculate the power of R_L under these conditions.

Solution: Eq. (9.4):

$$R_L = R_s = \mathbf{40\ k\Omega}$$

Eq. (9.7):

$$P_{L_{\max}} = \frac{I_N^2 R_N}{4} = \frac{(10\ \text{mA})^2(40\ \text{k}\Omega)}{4} = \mathbf{1\ W}$$

EXAMPLE 9.16 For the network of Fig. 9.86, determine the value of R for maximum power to R, and calculate the power delivered under these conditions.

Solution: See Fig. 9.87.

$$R_{Th} = R_3 + R_1 \parallel R_2 = 8\ \Omega + \frac{(6\ \Omega)(3\ \Omega)}{6\ \Omega + 3\ \Omega} = 8\ \Omega + 2\ \Omega$$

and $$R = R_{Th} = \mathbf{10\ \Omega}$$

See Fig. 9.88.

$$E_{Th} = \frac{R_2 E}{R_2 + R_1} = \frac{(3\ \Omega)(12\ \text{V})}{3\ \Omega + 6\ \Omega} = \frac{36\ \text{V}}{9} = \mathbf{4\ V}$$

and, by Eq. (9.6),

$$P_{L_{\max}} = \frac{E_{Th}^2}{4R_{Th}} = \frac{(4\ \text{V})^2}{4(10\ \Omega)} = \mathbf{0.4\ W}$$

EXAMPLE 9.17 Find the value of R_L in Fig. 9.89 for maximum power to R_L, and determine the maximum power.

Solution: See Fig. 9.90.

$$R_{Th} = R_1 + R_2 + R_3 = 3\ \Omega + 10\ \Omega + 2\ \Omega = 15\ \Omega$$

and $$R_L = R_{Th} = \mathbf{15\ \Omega}$$

Note Fig. 9.91, where

$$V_1 = V_3 = 0\text{ V}$$

and $$V_2 = I_2R_2 = IR_2 = (6\text{ A})(10\ \Omega) = 60\text{ V}$$

Applying Kirchhoff's voltage law,

$$\Sigma_{\circlearrowright} V = -V_2 - E_1 + E_{Th} = 0$$

and $$E_{Th} = V_2 + E_1 = 60\text{ V} + 68\text{ V} = 128\text{ V}$$

Thus, $$P_{L_{\max}} = \frac{E_{Th}^2}{4R_{Th}} = \frac{(128\text{ V})^2}{4(15\ \Omega)} = \mathbf{273.07\ W}$$

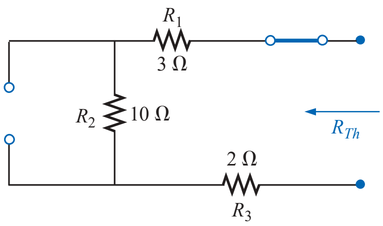


FIG. 9.90
Determining R_{Th} for the network external to the resistor R_L of Fig. 9.89.

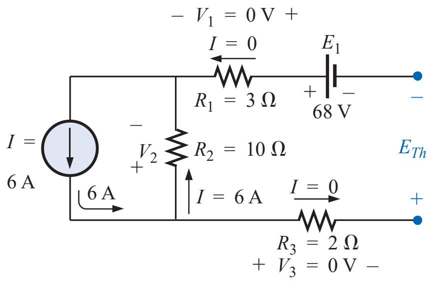


FIG. 9.91
Determining E_{Th} for the network external to the resistor R_L of Fig. 9.89.

9.6 MILLMAN'S THEOREM

Through the application of **Millman's theorem,** any number of parallel voltage sources can be reduced to one. In Fig. 9.92, for example, the three voltage sources can be reduced to one. This would permit finding the current through or voltage across R_L without having to apply a method such as mesh analysis, nodal analysis, superposition, and so on. The theorem can best be described by applying it to the network of Fig. 9.92. Basically, three steps are included in its application.

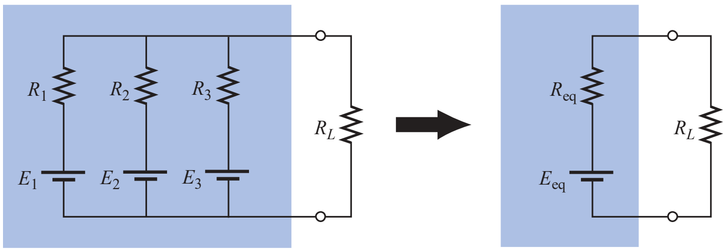


FIG. 9.92
Demonstrating the effect of applying Millman's theorem.

Step 1: Convert all voltage sources to current sources as outlined in Section 8.3. This is performed in Fig. 9.93 for the network of Fig. 9.92.

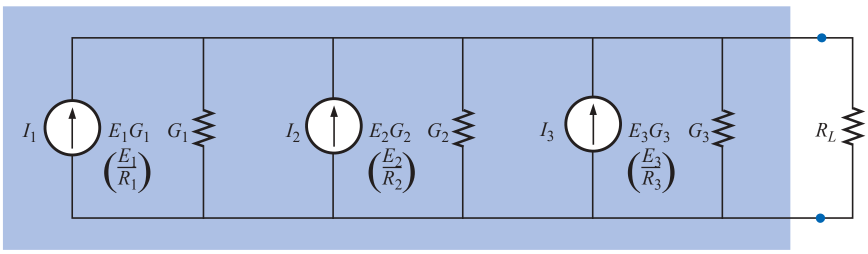


FIG. 9.93
Converting all the sources of Fig. 9.92 to current sources.

Step 2: Combine parallel current sources as described in Section 8.4. The resulting network is shown in Fig. 9.94, where

$$I_T = I_1 + I_2 + I_3 \quad \text{and} \quad G_T = G_1 + G_2 + G_3$$

Step 3: Convert the resulting current source to a voltage source, and the desired single-source network is obtained, as shown in Fig. 9.95.

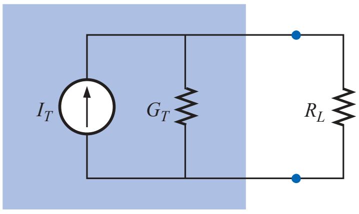


FIG. 9.94
Reducing all the current sources of Fig. 9.93 to a single current source.

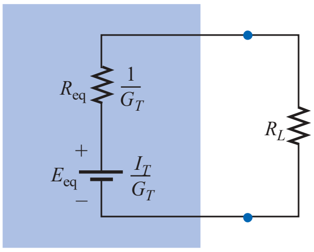


FIG. 9.95
Converting the current source of Fig. 9.94 to a voltage source.

In general, Millman's theorem states that for any number of parallel voltage sources,

$$E_{eq} = \frac{I_T}{G_T} = \frac{\pm I_1 \pm I_2 \pm I_3 \pm \cdots \pm I_N}{G_1 + G_2 + G_3 + \cdots + G_N}$$

or

$$\boxed{E_{eq} = \frac{\pm E_1G_1 \pm E_2G_2 \pm E_3G_3 \pm \cdots \pm E_NG_N}{G_1 + G_2 + G_3 + \cdots + G_N}} \qquad \textbf{(9.10)}$$

The plus-and-minus signs appear in Eq. (9.10) to include those cases where the sources may not be supplying energy in the same direction. (Note Example 9.18.)

The equivalent resistance is

$$\boxed{R_{eq} = \frac{1}{G_T} = \frac{1}{G_1 + G_2 + G_3 + \cdots + G_N}} \qquad \textbf{(9.11)}$$

In terms of the resistance values,

$$\boxed{E_{eq} = \frac{\pm \dfrac{E_1}{R_1} \pm \dfrac{E_2}{R_2} \pm \dfrac{E_3}{R_3} \pm \cdots \pm \dfrac{E_N}{R_N}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \cdots + \dfrac{1}{R_N}}} \qquad \textbf{(9.12)}$$

and

$$\boxed{R_{eq} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \cdots + \dfrac{1}{R_N}}} \qquad \textbf{(9.13)}$$

The relatively few direct steps required may result in the student's applying each step rather than memorizing and employing Eqs. (9.10) through (9.13).

EXAMPLE 9.18 Using Millman's theorem, find the current through and voltage across the resistor R_L of Fig. 9.96.

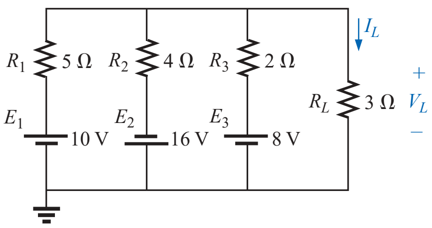


FIG. 9.96
Example 9.18.

Solution: By Eq. (9.12),

$$E_{eq} = \frac{+\dfrac{E_1}{R_1} - \dfrac{E_2}{R_2} + \dfrac{E_3}{R_3}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}}$$

The minus sign is used for E_2/R_2 because that supply has the opposite polarity of the other two. The chosen reference direction is therefore that of E_1 and E_3. The total conductance is unaffected by the direction, and

$$E_{eq} = \frac{+\frac{10\text{ V}}{5\,\Omega} - \frac{16\text{ V}}{4\,\Omega} + \frac{8\text{ V}}{2\,\Omega}}{\frac{1}{5\,\Omega} + \frac{1}{4\,\Omega} + \frac{1}{2\,\Omega}} = \frac{2\text{ A} - 4\text{ A} + 4\text{ A}}{0.2\text{ S} + 0.25\text{ S} + 0.5\text{ S}}$$

$$= \frac{2\text{ A}}{0.95\text{ S}} = \mathbf{2.105\text{ V}}$$

with $$R_{eq} = \frac{1}{\frac{1}{5\,\Omega} + \frac{1}{4\,\Omega} + \frac{1}{2\,\Omega}} = \frac{1}{0.95\text{ S}} = \mathbf{1.053\,\Omega}$$

The resultant source is shown in Fig. 9.97, and

$$I_L = \frac{2.105\text{ V}}{1.053\,\Omega + 3\,\Omega} = \frac{2.105\text{ V}}{4.053\,\Omega} = \mathbf{0.519\text{ A}}$$

with $$V_L = I_L R_L = (0.519\text{ A})(3\,\Omega) = \mathbf{1.557\text{ V}}$$

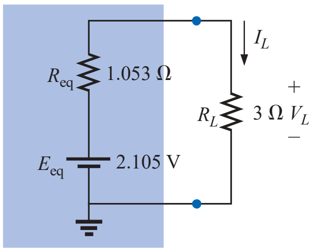


FIG. 9.97
The result of applying Millman's theorem to the network of Fig. 9.96.

EXAMPLE 9.19 Let us now consider the type of problem encountered in the introduction to mesh and nodal analysis in Chapter 8. Mesh analysis was applied to the network of Fig. 9.98 (Example 8.12). Let us now use Millman's theorem to find the current through the 2-Ω resistor and compare the results.

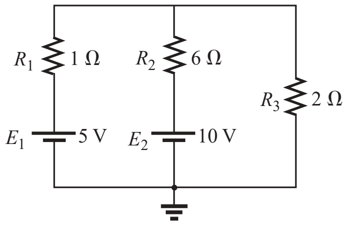


FIG. 9.98
Example 9.19.

Solutions:

a. Let us first apply each step and, in the (b) solution, Eq. (9.12). Converting sources yields Fig. 9.99. Combining sources and parallel conductance branches (Fig. 9.100) yields

$$I_T = I_1 + I_2 = 5\text{ A} + \frac{5}{3}\text{ A} = \frac{15}{3}\text{ A} + \frac{5}{3}\text{ A} = \frac{20}{3}\text{ A}$$

$$G_T = G_1 + G_2 = 1\text{ S} + \frac{1}{6}\text{ S} = \frac{6}{6}\text{ S} + \frac{1}{6}\text{ S} = \frac{7}{6}\text{ S}$$

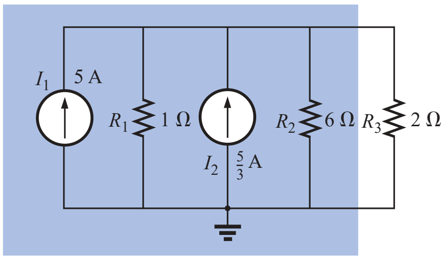


FIG. 9.99
Converting the sources of Fig. 9.98 to current sources.

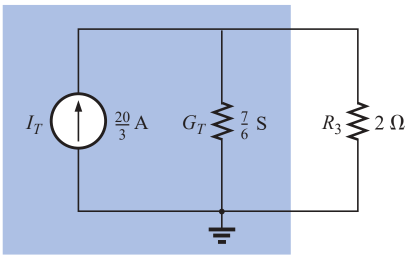


FIG. 9.100
Reducing the current sources of Fig. 9.99 to a single source.

Converting the current source to a voltage source (Fig. 9.101), we obtain

$$E_{eq} = \frac{I_T}{G_T} = \frac{\frac{20}{3}\text{ A}}{\frac{7}{6}\text{ S}} = \frac{(6)(20)}{(3)(7)}\text{ V} = \mathbf{\frac{40}{7}\text{ V}}$$

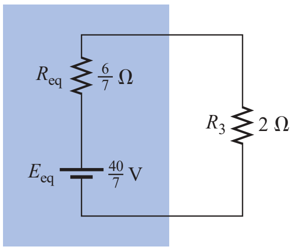


FIG. 9.101
Converting the current source of Fig. 9.100 to a voltage source.

and
$$R_{eq} = \frac{1}{G_T} = \frac{1}{\frac{7}{6}\text{S}} = \mathbf{\frac{6}{7}\ \Omega}$$

so that

$$I_{2\Omega} = \frac{E_{eq}}{R_{eq} + R_3} = \frac{\frac{40}{7}\text{V}}{\frac{6}{7}\Omega + 2\ \Omega} = \frac{\frac{40}{7}\text{V}}{\frac{6}{7}\Omega + \frac{14}{7}\Omega} = \frac{40\text{ V}}{20\ \Omega} = \mathbf{2\ A}$$

which agrees with the result obtained in Example 8.18.

b. Let us now simply apply the proper equation, Eq. (9.12):

$$E_{eq} = \frac{+\frac{5\text{ V}}{1\ \Omega} + \frac{10\text{ V}}{6\ \Omega}}{\frac{1}{1\ \Omega} + \frac{1}{6\ \Omega}} = \frac{\frac{30\text{ V}}{6\ \Omega} + \frac{10\text{ V}}{6\ \Omega}}{\frac{6}{6\ \Omega} + \frac{1}{6\ \Omega}} = \mathbf{\frac{40}{7}\ V}$$

and

$$R_{eq} = \frac{1}{\frac{1}{1\ \Omega} + \frac{1}{6\ \Omega}} = \frac{1}{\frac{6}{6\ \Omega} + \frac{1}{6\ \Omega}} = \frac{1}{\frac{7}{6}\text{S}} = \mathbf{\frac{6}{7}\ \Omega}$$

which are the same values obtained above.

The dual of Millman's theorem (Fig. 9.92) appears in Fig. 9.102. It can be shown that I_{eq} and R_{eq}, as in Fig. 9.102, are given by

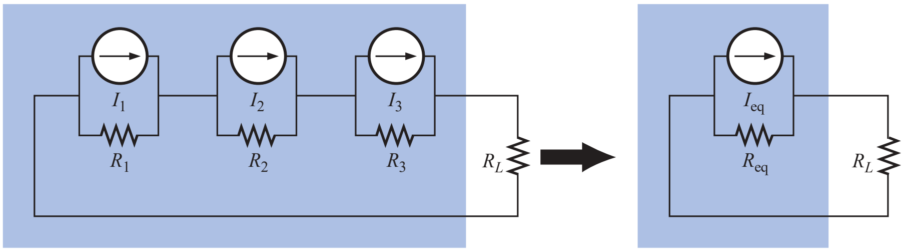


FIG. 9.102
The dual effect of Millman's theorem.

$$\boxed{I_{eq} = \frac{\pm I_1R_1 \pm I_2R_2 \pm I_3R_3}{R_1 + R_2 + R_3}} \qquad \textbf{(9.14)}$$

and
$$\boxed{R_{eq} = R_1 + R_2 + R_3} \qquad \textbf{(9.15)}$$

The derivation will appear as a problem at the end of the chapter.

9.7 SUBSTITUTION THEOREM

The **substitution theorem** states the following:

If the voltage across and the current through any branch of a dc bilateral network are known, this branch can be replaced by any

combination of elements that will maintain the same voltage across and current through the chosen branch.

More simply, the theorem states that for branch equivalence, the terminal voltage and current must be the same. Consider the circuit of Fig. 9.103, in which the voltage across and current through the branch *a-b* are determined. Through the use of the substitution theorem, a number of equivalent *a-a′* branches are shown in Fig. 9.104.

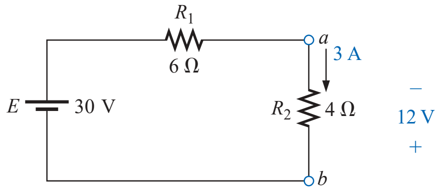


FIG. 9.103
Demonstrating the effect of the substitution theorem.

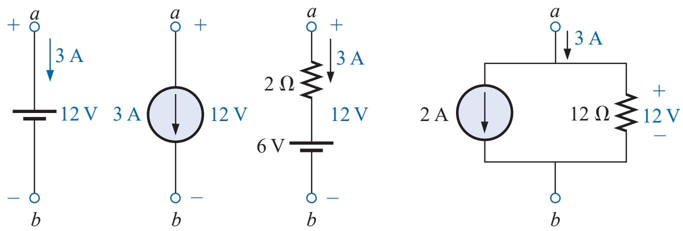


FIG. 9.104
Equivalent branches for the branch a-b of Fig. 9.103.

Note that for each equivalent, the terminal voltage and current are the same. Also consider that the response of the remainder of the circuit of Fig. 9.103 is unchanged by substituting any one of the equivalent branches. As demonstrated by the single-source equivalents of Fig. 9.104, *a known potential difference and current in a network can be replaced by an ideal voltage source and current source, respectively.*

Understand that this theorem cannot be used to *solve* networks with two or more sources that are not in series or parallel. For it to be applied, a potential difference or current value must be known or found using one of the techniques discussed earlier. One application of the theorem is shown in Fig. 9.105. Note that in the figure the known potential difference V was replaced by a voltage source, permitting the isolation of the portion of the network including R_3, R_4, and R_5. Recall that this was basically the approach employed in the analysis of the ladder network as we worked our way back toward the terminal resistance R_5.

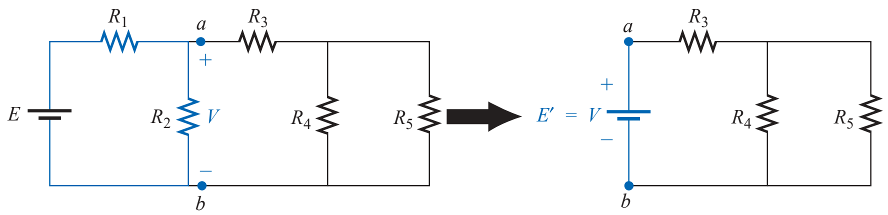


FIG. 9.105
Demonstrating the effect of knowing a voltage at some point in a complex network.

The current source equivalence of the above is shown in Fig. 9.106, where a known current is replaced by an ideal current source, permitting the isolation of R_4 and R_5.

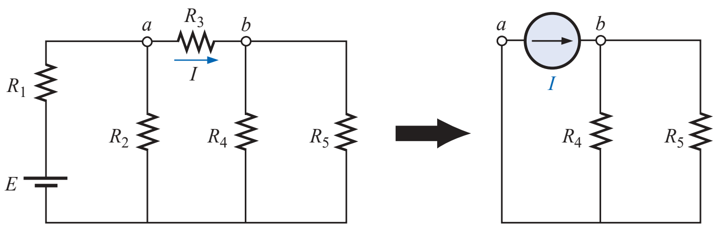


FIG. 9.106
Demonstrating the effect of knowing a current at some point in a complex network.

You will also recall from the discussion of bridge networks that $V = 0$ and $I = 0$ were replaced by a short circuit and an open circuit, respectively. This substitution is a very specific application of the substitution theorem.

9.8 RECIPROCITY THEOREM

The **reciprocity theorem** is applicable only to single-source networks. It is, therefore, not a theorem employed in the analysis of multisource networks described thus far. The theorem states the following:

The current I in any branch of a network, due to a single voltage source E anywhere else in the network, will equal the current through the branch in which the source was originally located if the source is placed in the branch in which the current I was originally measured.

In other words, the location of the voltage source and the resulting current may be interchanged without a change in current. The theorem requires that the polarity of the voltage source have the same correspondence with the direction of the branch current in each position.

In the representative network of Fig. 9.107(a), the current I due to the voltage source E was determined. If the position of each is interchanged as shown in Fig. 9.107(b), the current I will be the same value as indicated. To demonstrate the validity of this statement and the theorem, consider the network of Fig. 9.108, in which values for the elements of Fig. 9.107(a) have been assigned.

The total resistance is

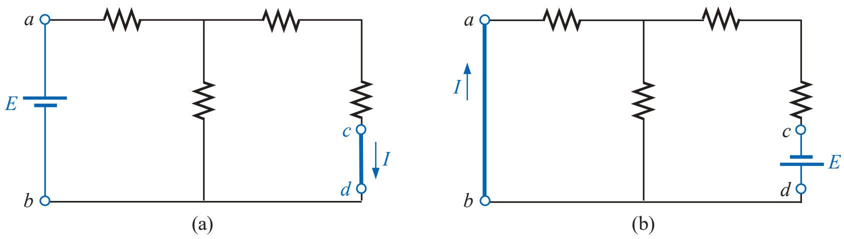


FIG. 9.107
Demonstrating the impact of the reciprocity theorem.

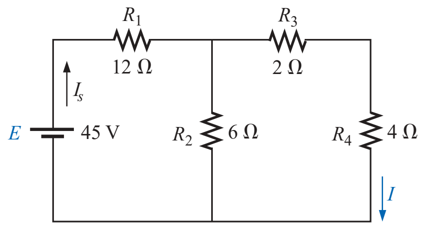


FIG. 9.108
Finding the current I due to a source E.

$$R_T = R_1 + R_2 \| (R_3 + R_4) = 12\ \Omega + 6\ \Omega \| (2\ \Omega + 4\ \Omega)$$
$$= 12\ \Omega + 6\ \Omega \| 6\ \Omega = 12\ \Omega + 3\ \Omega = 15\ \Omega$$

and $$I_s = \frac{E}{R_T} = \frac{45\text{ V}}{15\ \Omega} = 3\text{ A}$$

with $$I = \frac{3\text{ A}}{2} = \mathbf{1.5\ A}$$

For the network of Fig. 9.109, which corresponds to that of Fig. 9.107(b), we find

$$R_T = R_4 + R_3 + R_1 \| R_2$$
$$= 4\ \Omega + 2\ \Omega + 12\ \Omega \| 6\ \Omega = 10\ \Omega$$

and $$I_s = \frac{E}{R_T} = \frac{45\text{ V}}{10\ \Omega} = 4.5\text{ A}$$

so that $$I = \frac{(6\ \Omega)(4.5\text{ A})}{12\ \Omega + 6\ \Omega} = \frac{4.5\text{ A}}{3} = \mathbf{1.5\ A}$$

which agrees with the above.

The uniqueness and power of such a theorem can best be demonstrated by considering a complex, single-source network such as the one shown in Fig. 9.110.

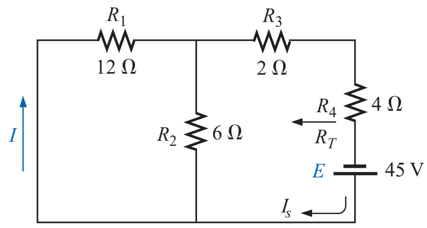


FIG. 9.109
Interchanging the location of E and I of Fig. 9.108 to demonstrate the validity of the reciprocity theorem.

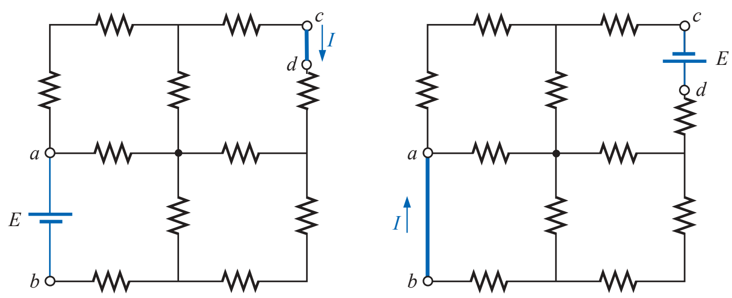


FIG. 9.110
Demonstrating the power and uniqueness of the reciprocity theorem.

9.9 APPLICATION

Speaker System

One of the most common applications of the maximum power transfer theorem introduced in this chapter is to speaker systems. An audio amplifier (amplifier with a frequency range matching the typical range