Chapter 3

Nominal and Effective Interest Rates

LEARNING OUTCOMES

1. **Understand interest rate statements**
2. **Use formula for effective interest rates**
3. **Determine interest rate for any time period**
4. **Determine payment period (PP) and compounding period (CP) for equivalence calculations**
5. **Make calculations for single cash flows**
6. **Make calculations for series and gradient cash flows with PP ≥ CP**
7. **Perform equivalence calculations when PP < CP**
8. **Use interest rate formula for continuous compounding**
9. **Make calculations for varying interest rates**

Interest Rate Statements

The terms 'nominal' and 'effective' enter into consideration when the interest period is *less than one year*.

New time-based definitions to understand and remember

Interest period (t) – period of time over which interest is expressed. For example, 1% *per month.*

Compounding period (CP) – Shortest time unit over which interest is charged or earned. For example,10% per year *compounded monthly.*

Compounding frequency (m) – Number of times compounding occurs within the interest period t. For example, at i = 10% per year, compounded monthly, interest would be *compounded 12 times* during the one year interest period.

Understanding Interest Rate Terminology

- **A *nominal interest rate (r)* is obtained by multiplying an interest rate that is expressed over a short time period by the number of compounding periods in a longer time period: That is:**

r = interest rate per period x number of compounding periods

Example: If i = 1% per month, nominal rate per year is
r = (1)(12) = 12% per year)

- ***Effective interest rates (i)* take compounding into account (effective rates can be obtained from nominal rates via a formula to be discussed later).**

IMPORTANT: Nominal interest rates are essentially simple interest rates. Therefore, they can *never* be used in interest formulas.

Effective rates must *always* be used hereafter in all interest formulas.

$$\text{Effective rate per CP} = \frac{r\% \text{ per time period } t}{m \text{ compounding periods per } t} = \frac{r}{m} \qquad [4.2]$$

EXAMPLE 4.1

Three different bank loan rates for electric generation equipment are listed below. Determine the effective rate on the basis of the compounding period for each rate.

(*a*) 9% per year, compounded quarterly.
(*b*) 9% per year, compounded monthly.
(*c*) 4.5% per 6 months, compounded weekly.

Solution

Apply Equation [4.2] to determine the effective rate per CP for different compounding periods. The graphic in Figure 4–1 indicates the effective rate per CP and how the interest rate is distributed over time.

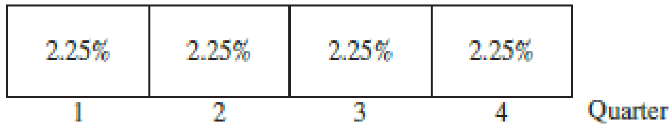
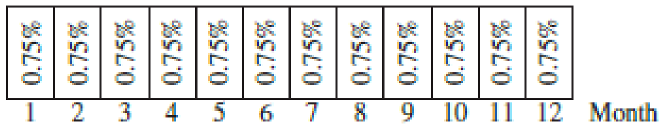
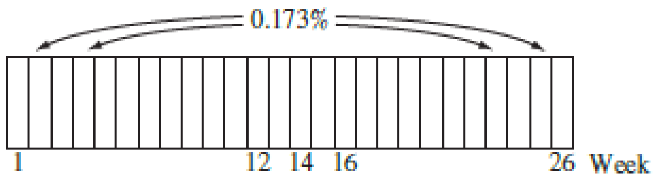
Nominal r% per t	Compounding Period (CP)	Compounding Frequency (m)	Effective Rate per CP $\left(\frac{r}{m}\right)$	Distribution over Time Period t
(*a*) 9% per year	Quarter	4	2.25%	
(*b*) 9% per year	Month	12	0.75%	
(*c*) 4.5% per 6 months	Week	26	0.173%	

Figure 4–1
Relations between interest period t, compounding period CP, and effective interest rate per CP.

Effective Annual Interest Rate

$$i_a = (1 + i)^m - 1 \qquad [4.3]$$

TABLE 4–2 Effective Annual Interest Rates Using Equation [4.3]

r = 18% per year, compounded CP-ly

Compounding Period, CP	Times Compounded per Year, m	Rate per Compound Period, $i\%$	Distribution of i over the Year of Compounding Periods	Effective Annual Rate, $i_a = (1 + i)^m - 1$
Year	1	18	18% 1	$(1.18)^1 - 1 = 18\%$
6 months	2	9	9% | 9% 1 2	$(1.09)^2 - 1 = 18.81\%$
Quarter	4	4.5	4.5% | 4.5% | 4.5% | 4.5% 1 2 3 4	$(1.045)^4 - 1 = 19.252\%$
Month	12	1.5	1.5% in each 1 2 3 4 5 6 7 8 9 10 11 12	$(1.015)^{12} - 1 = 19.562\%$
Week	52	0.34615	0.34615% in each 1 2 3 24 26 28 50 52	$(1.0034615)^{52} - 1 = 19.684\%$

More About Interest Rate Terminology

There are 3 general ways to express interest rates as shown below

Sample Interest Rate Statements		Comment
(1)	i = 2% per month i = 12% per year	When no compounding period is given, rate is *effective*
(2)	i = 10% per year, comp'd semiannually i = 3% per quarter, comp'd monthly	When compounding period is given and it is *not the same* as interest period, it is *nominal*
(3)	i = effective 9.4%/year, comp'd semiannually i = effective 4% per quarter, comp'd monthly	When compounding period is given and rate is *specified as effective*, rate *is effective* over stated period

Effective Annual Interest Rates

Nominal rates are converted into effective annual rates via the equation:

$$i_a = (1 + i)^m - 1$$

where i_a = effective annual interest rate

i = effective rate for one compounding period

m = number times interest is compounded per year

Example: For a nominal interest rate of 12% per year, determine the nominal and effective rates per year for (a) quarterly, and (b) monthly compounding

Solution:

(a) Nominal r / year = 12% per year

Nominal r / quarter = 12/4 = 3.0% per quarter

Effective i / year = $(1 + 0.03)^4 - 1$ = 12.55% per year

(b) Nominal r /month = 12/12 = 1.0% per year

Effective i / year = $(1 + 0.01)^{12} - 1$ = 12.68% per year

Effective Interest Rates

Nominal rates can be converted into effective rates for any time period via the following equation:

$$i = (1 + r / m)^m - 1$$

where i = effective interest rate for any time period
r = nominal rate for same time period as i
m = no. times interest is comp'd in period specified for i

Spreadsheet function is = EFFECT(r%,m) where r = nominal rate per period specified for i

Example: For an interest rate of 1.2% per month, determine the nominal and effective rates (a) per quarter, and (b) per year

Solution:

(a) Nominal r / quarter = (1.2)(3) = 3.6% per quarter

Effective i / quarter = $(1 + 0.036/3)^3 - 1$ = 3.64% per quarter

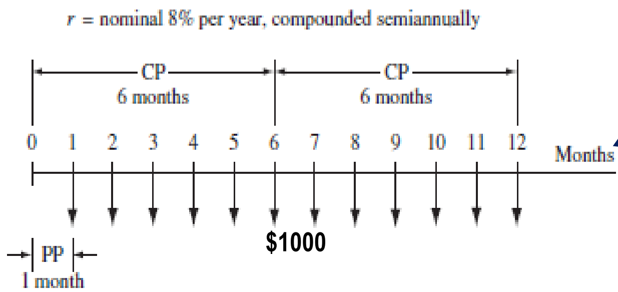
(b) Nominal i /year = (1.2)(12) = 14.4% per year

Effective i / year = $(1 + 0.144 / 12)^{12} - 1$ = 15.39% per year

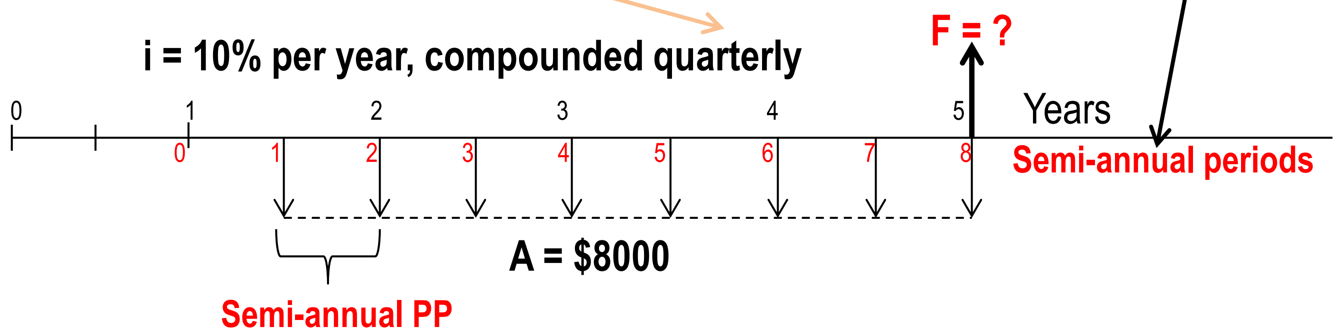
Equivalence Relations: PP and CP

New definition: **Payment Period (PP)** – Length of time between cash flows

In the diagram below, the compounding period (CP) is semiannual and the payment period (PP) is monthly



Similarly, for the diagram below, the CP is quarterly and the payment period (PP) is semiannual



Single Amounts with PP > CP

For problems involving single amounts, the payment period (PP) is usually longer than the compounding period (CP). For these problems, there are an infinite number of i and n combinations that can be used, with only two restrictions:

(1) The i must be an *effective* interest rate, and

(2) The time units on n must be *the same* as those of i
(i.e., if i is a rate per quarter, then n is the number of quarters between P and F)

There are two equally correct ways to determine i and n

Method 1: Determine effective interest rate over the compounding period CP, and set n equal to the number of compounding periods between P and F

Method 2: Determine the effective interest rate for any time period t, and set n equal to the total number of those **same time periods**.

Example: Single Amounts with PP ≥ CP

How much money will be in an account in 5 years if $10,000 is deposited now at an interest rate of 1% per month? Use three different interest rates: (a) monthly, (b) quarterly , and (c) yearly.

(a) For monthly rate, 1% is effective [n = (5 years)×(12 CP per year = 60]

F = 10,000(F/P,1%,60) = $18,167

months
effective i per month

i and n must *always* have same time units

(b) For a quarterly rate, effective i/quarter = $(1 + 0.03/3)^3 - 1 = 3.03\%$

F = 10,000(F/P,3.03%,20) = $18,167

quarters
effective i per quarter

i and n must *always* have same time units

(c) For an annual rate, effective i/year = $(1 + 0.12/12)^{12} - 1 = 12.683\%$

F = 10,000(F/P,12.683%,5) = $18,167

years
effective i per year

i and n must *always* have same time units

Series with PP ≥ CP

For series cash flows, *first step* is to determine *relationship* between PP and CP

Determine if PP ≥ CP, or if PP < CP

When PP ≥ CP, the *only* procedure (2 steps) that can be used is as follows:

(1) First, find effective i per PP

Example: if PP is in quarters, *must* find effective *i/quarter*

(2) Second, determine n, the number of A values involved

Example: quarterly payments for 6 years yields n = 4×6 = 24

Note: Procedure when PP < CP is discussed later

Example: Series with PP ≥ CP

How much money will be accumulated in 10 years from a deposit of $500 every 6 months if the interest rate is 1% per month?

Solution: First, find relationship between PP and CP

PP = ***six months***, CP = ***one month***; Therefore, **PP > CP**

Since PP > CP, find effective i per PP of six months

Step 1. $i/6\text{ months} = (1 + 0.06/6)^6 - 1 = 6.15\%$

Next, determine n (number of 6-month periods)

Step 2: **n = 10(2) = 20 six month periods**

Finally, set up equation and solve for F

F = 500(F/A,6.15%,20) = $18,692 (by factor or spreadsheet)

Series with PP < CP

Two policies: (1) interperiod cash flows earn *no interest* (most common)
(2) interperiod cash flows earn *compound interest*

For policy (1), ***positive cash flows*** are moved *to* ***beginning of the interest period*** in which they occur and ***negative cash flows*** *are moved to the* ***end of the interest period***

Note: The condition of PP < CP with no interperiod interest is the *only situation in which* the actual cash flow diagram is changed

For policy (2), cash flows are ***not moved*** and equivalent P, F, and A values are determined using the ***effective interest rate per payment period***

Example: Series with PP < CP

A person deposits $100 per month into a savings account for 2 years. If $75 is withdrawn in months 5, 7 and 8 (in addition to the deposits), construct the cash flow diagram to determine how much will be in the account after 2 years at i = 6% per year, compounded quarterly. Assume there is no interperiod interest.

Solution: Since PP < CP with no interperiod interest, the cash flow diagram must be *changed using quarters as the time periods*

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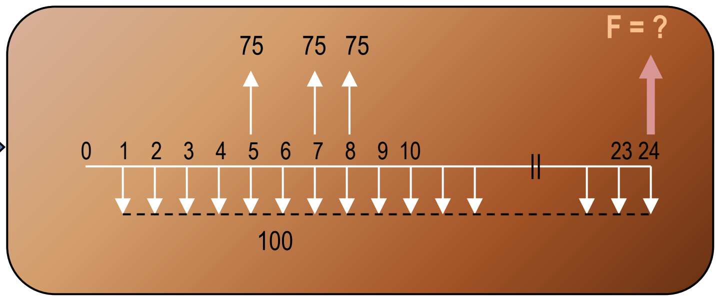

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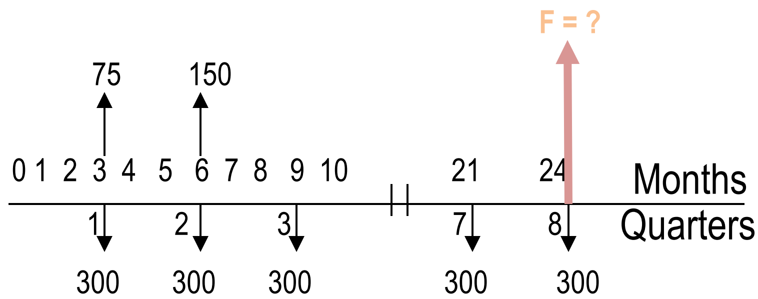

Continuous Compounding

When the interest period is infinitely small, interest is *compounded continuously.* Therefore, PP > CP and m increases.

Take limit as m → ∞ to find the effective interest rate equation

$$i = e^r - 1$$

Example: **If a person deposits \$500 into an account every 3 months at an interest rate of 6% per year, compounded continuously, how much will be in the account at the end of 5 years?**

Solution:

Payment Period: PP = 3 months

Nominal rate per *three months*: r = 6%/4 = 1.50%

Effective rate per 3 months: $i = e^{0.015} - 1 = 1.51\%$

F = 500(F/A,1.51%,20) = \$11,573

Varying Rates

When interest rates vary over time, use the interest rates associated with their respective time periods to find P

Example: Find the present worth of $2500 deposits in years 1 through 8 if the interest rate is 7% per year for the first five years and 10% per year thereafter.

Solution: P = 2,500(P/A,7%,5) + 2,500(P/A,10%,3)(P/F,7%,5)

= **$14,683**

An equivalent annual worth value can be obtained by replacing each cash flow amount with 'A' and setting the equation equal to the calculated P value

14,683 = A(P/A,7%,5) + A(P/A,10%,3)(P/F,7%,5)

A = **$2500 per year**

Summary of Important Points

Must understand: interest period, compounding period, compounding frequency, and payment period

Always use *effective rates* in interest formulas

$$i = (1 + r / m)^m - 1$$

Interest rates are stated different ways; must know how to get effective rates

For single amounts, make sure units on i and n are the same

Important Points (cont'd)

For uniform series with PP ≥ CP, find effective i over PP

For uniform series with PP < CP and no interperiod interest, move cash flows to match compounding period

For continuous compounding, use $i = e^r - 1$ to get effective rate

For varying rates, use stated i values for respective time periods