LIMITS

2.23 (a) If $f(z) = z^2$. Prove that $\lim\limits_{z \to z_0} f(z) = z_0^2$.

(b) Find $\lim\limits_{z \to z_0} f(z)$ if $f(z) = \left[\begin{array}{ll} z^2 & z \neq z_0 \\ 0 & z = z_0 \end{array}\right.$.

Solution

(a) We must show that, given any $\epsilon > 0$, we can find δ (depending in general on ϵ) such that $\left|z^2 - z_0^2\right| < \sum \epsilon$ whenever $0 < |z - z_0| < \delta$.

If $\delta \leq 1$, then $0 < |z - z_0| < \delta$ implies that

$$\left|z^2 - z_0^2\right| = |z - z_0||z + z_0| < \delta|z - z_0 + 2z_0| < \delta(|z - z_0| + |2z_0|) < \delta(1 + 2|z_0|)$$

Take δ as 1 or $\epsilon/(1 + 2|z_0|)$, whichever is smaller. Then, we have $|z^2 - z_0^2| < \epsilon$ whenever $|z - z_0| < \delta$, and the required result is proved.

(b) There is no difference between this problem and that in part (a), since in both cases we exclude $z = z_0$ from consideration. Hence, $\lim\limits_{z \to z_0} f(z) = z_0^2$. Note that the limit of $f(z)$ as $z \to z_0$ has nothing whatsoever to do with the value of $f(z)$ at z_0.

2.30 Prove that $\lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist.

Solution If the limit is to exis,t it must be independent of the manner in which z approaches the point 0. Let $z \to 0$ along the x axis. Then $y = 0$, and $z = x + iy = x$ and $\bar{z} = x - iy = x$, so that the required limit is

$$\lim_{x\to 0} \frac{x}{x} = 1 .$$

Let $z \to 0$ along the y axis. Then $x = 0$, and $z = x + iy = iy$ and $\bar{z} = x - iy = -iy$, so that the required limit is

$$\lim_{x\to 0} \frac{-iy}{iy} = -1$$

Since the two approaches do not give the same answer, the limit does not exist.

CONTINUITY

2.31 (a) Prove that $f(z) = z^2$ is continuous at $z = z_0$.

(b) Prove that $f(z) = \begin{cases} z^2 & z \neq z_0 \\ 0 & z = z_0 \end{cases}$, where $z_0 \neq 0$, is discontinuous at $z = z_0$.

Solution

(a) By Problem 2.23(a), $\lim_{z\to z_0} f(z) = f(z_0) = z_0^2$ and so $f(z)$ is continuous at $z = z_0$.

Another method. We must show that given any $\epsilon > 0$, we can find $\delta > 0$ (depending on ϵ) such that $|f(z) - f(z_0)| = |z^2 - z_0^2| < \epsilon$ when $|z - z_0| < \delta$. The proof patterns that given in Problem 2.23(a).

(b) By Problem 2.23(b), $\lim_{z\to z_0} f(z) = z_0^2$, but $f(z_0) = 0$. Hence $\lim_{z\to z_0} f(z) \neq f(z_0)$ and so $f(z)$ is discontinuous at $z = z_0$ if $z_0 \neq 0$.

If $z_0 = 0$, then $f(z) = 0$; and since $\lim_{z\to z_0} f(z) = 0 = f(0)$, we see that the function is continuous.

3.2 Show that $\frac{d}{dz}\bar{z}$ does not exist anywhere, i.e. $f(z) = \bar{z}$ is non-analytic anywhere.

Solution By definition, $\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

if this limit exists independent of the manner in which $\Delta z = \Delta x + i\,\Delta y$ approaches zero.

Then $$\frac{d}{dz}\bar{z} = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x+iy+\Delta x+i\,\Delta y} - \overline{x+iy}}{\Delta x + i\,\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x-iy+\Delta x - i\,\Delta y - (x-iy)}{\Delta x + i\,\Delta y} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\,\Delta y}{\Delta x + i\,\Delta y}$$

If $\Delta y = 0$, the required limit is

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

If $\Delta x = 0$, the required limit is

$$\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

Then, since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist, i.e. $f(z) = \bar{z}$ is *non-analytic* anywhere.

3.3 Given $w = f(z) = \frac{(1+z)}{\ldots}$ ... $\frac{dw}{\ldots}$

3.6 Given $f(z) = u + iv$ is analytic in a region $\mathfrak{R}$. Prove that u and v are harmonic in $\mathfrak{R}$ if they have continuous second partial derivatives in $\mathfrak{R}$.

Solution

If $f(z)$ is analytic in $\mathcal{R}$ then the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \tag{1}$$

and

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \tag{2}$$

are satisfied in $\mathcal{R}$. Assuming u and v have continuous second partial derivatives, we can differentiate both sides of (1) with respect to x and (2) with respect to y to obtain

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\,\partial y} \tag{3}$$

and

$$\frac{\partial^2 v}{\partial y\,\partial x}=-\frac{\partial^2 u}{\partial y^2} \tag{4}$$

from which

$$\frac{\partial^2 u}{\partial x^2}=-\frac{\partial^2 u}{\partial y^2} \text{ or } \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$$

i.e. u is harmonic.

Similarly, by differentiating both sides of (1) with respect to y and (2) with respect to x, we find

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

and v is harmonic.

It will be shown later (Chapter 5) that if $f(z)$ is analytic in $\mathcal{R}$, all its derivatives exist and are continuous in $\mathcal{R}$. Hence, the above assumptions will not be necessary.

3.7 (a) Prove that $u=e^{-x}(x\sin y-y\cos y)$ is harmonic.
(b) Find v such that $f(z)=u+iv$ is analytic.

Solution

(a) $$\frac{\partial u}{\partial x}=(e^{-x})(\sin y)+(-e^{-x})(x\sin y-y\cos y)=e^{-x}\sin y-xe^{-x}\sin y+ye^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}(e^{-x}\sin y-xe^{-x}\sin y+ye^{-x}\cos y)=-2e^{-x}\sin y+xe^{-x}\sin y-ye^{-x}\cos y \tag{1}$$

$$\frac{\partial u}{\partial y}=e^{-x}(x\cos y+y\sin y-\cos y)=xe^{-x}\cos y+ye^{-x}\sin y-e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}(xe^{-x}\cos y+ye^{-x}\sin y-e^{-x}\cos y)=-xe^{-x}\sin y+2e^{-x}\sin y+ye^{-x}\cos y \tag{2}$$

Adding (1) and (2) yields $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ and u is harmonic.

(b) From the Cauchy–Riemann equations,

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=e^{-x}\sin y-xe^{-x}\sin y+ye^{-x}\cos y \tag{3}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}\cos y - xe^{-x}\cos y - ye^{-x}\sin y \tag{4}$$

Integrate (3) with respect to y, keeping x constant. Then

$$\begin{aligned} v &= -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x) \\ &= ye^{-x}\sin y + xe^{-x}\cos y + F(x) \end{aligned} \tag{5}$$

where $F(x)$ is an arbitrary real function of x.

Substitute (5) into (4) and obtain

$$-ye^{-x}\sin y - xe^{-x}\cos y + e^{-x}\cos y + F'(x) = -ye^{-x}\sin y - xe^{-x}\cos y - ye^{-x}\sin y$$

or $\quad F'(x) = 0$ and $F(x) = c$, a constant. Then from (5),

$$v = e^{-x}(y\sin y + x\cos y) + c$$

For another method, see Problem 3.55.

3.12 Construct an analytic function $f(z)$ whose real part is $e^x \cos y$.

Solution

Let $f(z) = u(x, y) + iv(x, y)$

It is given that $u(x, y) = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

We know $$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy, \text{ (using C-R Equation)}$$

$$= e^x \sin y\, dx + e^x \cos y\, dy$$

This is an exact differential equation.

$$v = \int e^x \sin y.\, dx + \int e^x \cos y\, dy$$

Ignoring the term containing x

$$v = e^x \sin y$$

$$f(z) = u + iv = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y)$$

$$= e^x \cdot e^{iy} = e^{x+iy} = e^z.$$

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$$= i(r^2 e^{i2\theta} - re^{-i\theta}) + C + 2i$$

3.16 Find the values of constants a, b, c and d such that the function $f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2)$ is analytic.

Solution $$f(z) = x^2 + axy + by^2 + i(cx^2 + dxy + y^2) = u + iv(\text{say})$$

where $$u = x^2 + axy + by^2, \quad v = cx^2 + dxy + y^2$$

$\therefore$ $$u_x = 2x + ay, \quad u_y = ax + 2by$$
$$v_x = 2cx + dy, \quad v_y = dx + 2y$$

Since $f(z) = u + iv$ is analytic, so Cauchy-Riemann equation must be satisfied.

i.e., $u_x = v_y$ and $u_y = -v_x$

Now, $$u_x = v_y \Rightarrow 2x + ay = dx + 2y \qquad (1)$$

and $$u_y = -v_x \Rightarrow ax + 2by = -2cx - dy \qquad (2)$$

$$(1) \Rightarrow 2x - dx + ay - 2y = 0 \Rightarrow (2-d)x + (a-2)y = 0$$

$$(2) \Rightarrow ax + 2cx + 2by + dy = 0 \Rightarrow (a+2c)x + (2b+d)y = 0$$

(1) and (2) will hold good if

$$2 - d = 0, a - 2 = 0$$

and $$a + 2c = 0, 2b + d = 0$$

i.e., $$a = 2, d = 2, c = -1, b = -1$$

3.20 Show that the function $u = \cos x \cosh y$ is harmonic and find its harmonic conjugate.

Solution It is given that $u = \cos x \cosh y$

then $\dfrac{\partial u}{\partial x} = -\sin x \cosh y$, $\quad \dfrac{\partial u}{\partial y} = \cos x \sinh y$

Now, $\dfrac{\partial^2 u}{\partial x} + \dfrac{\partial^2 u}{\partial y^2} = -\cos x \cosh y + \cos x \cosh y = 0 \Rightarrow u$ is a harmonic function.

Let v be its conjugate harmonic function, then we have

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = -\cos x \sinh y \, dx - \sin x \cosh y \, dy$$

$$= -(\cos x \sinh y \, dx + \sin x \cosh y \, dy).$$

Integrating, we obtain

$$v = -\sin x \sinh y + c, \text{ where } c \text{ is a real constant.}$$

SOLVED PROBLEMS

LINE INTEGRALS

4.1 Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ along: (a) the parabola $x = 2t$, $y = t^2 + 3$, (b) straight lines from (0, 3) to (2, 3) and then from (2, 3) to (2, 4); (c) a straight line from (0, 3) to (2, 4).

Solution

(a) The points (0, 3) and (2, 4) on the parabola correspond to $t = 0$ and $t = 1$, respectively. Then, the given integral equals

$$\int_{t=0}^{1} [2(t^2 + 3) + (2t)^2]\, 2\,dt + [3(2t) - (t^2 + 3)]\, 2t\,dt = \int_0^1 (24t^2 + 12 - 2t^3 + 6t)\,dt = \frac{33}{2}$$

(b) Along the straight line from (0, 3) to (2, 3), $y = 3$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{2} (6 + x^2)\,dx + (3x - 3)0 = \int_{x=0}^{2} (6 + x^2)\,dx = \frac{44}{3}$$

Along the straight line from (2, 3) to (2, 4), $x = 2$, $dx = 0$ and the line integral equals

$$\int_{y=3}^{4} (2y + 4)0 + (6 - y)\,dy = \int_{y=3}^{4} (6 - y)\,dy = \frac{5}{2}$$

Then, the required value = 44/3 + 5/2 = 103/6.

(c) An equation for the line joining (0, 3) and (2, 4) is $2y - x = 6$. Solving for x, we have $x = 2y - 6$. Then, the line integral equals

$$\int_{y=3}^{4} [2y + (2y - 6)^2]\, 2\,dy + [3(2y - 6) - y]\,dy = \int_3^4 (8y^2 - 39y + 54)\,dy = \frac{97}{6}$$

The result can also be obtained by using $y = \frac{1}{2}(x + 6)$.

4.2 Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by: (a) $z = t^2 + it$, (b) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

Solution

(a) The points $z = 0$ and $z = 4 + 2i$ on C correspond to $t = 0$ and $t = 2$, respectively. Then the line integral equals

$$\int_{t=0}^{2} \overline{(t^2 + it)}\, d(t^2 + it) = \int_0^2 (t^2 - it)(2t + i)\,dt = \int_0^2 (2t^3 - it^2 + t)\,dt = 10 - \frac{8i}{3}$$

Another method. The given integral equals

$$\int_C (x - iy)(dx + i\,dy) = \int_C x\,dx + y\,dy + i\int_C x\,dy - y\,dx$$

The parametric equations of C are $x = t^2$, $y = t$ from $t = 0$ to $t = 2$. Then, the line integral equals

$$\int_{t=0}^{2} (t^2)(2t\,dt) + (t)(dt) \; + \; i\int_{t=0}^{2} (t^2)(dt) - (t)(2t\,dt)$$

The parametric equations of C are $x = t^2$, $y = t$ from $t = 0$ to $t = 2$. Then, the line integral equals

$$\int_{t=0}^{2} (t^2)(2t\,dt) + (t)(dt) + i \int_{t=0}^{2} (t^2)(dt) - (t)(2t\,dt)$$

$$= \int_{0}^{2} (2t^3 + t)\,dt + i \int_{0}^{2} (-t^2)\,dt = 10 - \frac{8i}{3}$$

(b) The given line integral equals

$$\int_C (x - iy)(dx + i\,dy) = \int_C x\,dx + y\,dy + i \int_C x\,dy - y\,dx$$

The line from $z = 0$ to $z = 2i$ is the same as the line from (0, 0) to (0, 2) for which $x = 0$, $dx = 0$ and the line integral equals

$$\int_{y=0}^{2} (0)(0) + y\,dy + i \int_{y=0}^{2} (0)(dy) - y(0) = \int_{y=0}^{2} y\,dy = 2$$

The line from $z = 2i$ to $z = 4 + 2i$ is the same as the line from (0, 2) to (4, 2) for which $y = 2$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{4} x\,dx + 2 \cdot 0 + i \int_{x=0}^{4} x \cdot 0 - 2\,dx = \int_{0}^{4} x\,dx + i \int_{0}^{4} -2\,dx = 8 - 8i$$

Then, the required value $= 2 + (8 - 8i) = 10 - 8i$.

4.5 Verify Green's theorem in the plane for

$$\oint_C (2xy - x^2)\,dx + (x + y^2)\,dy$$

where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

Solution The plane curves $y = x^2$ and $y^2 = x$ intersect at (0, 0) and (1, 1). The positive direction in traversing C is as shown in Fig. 4.8.

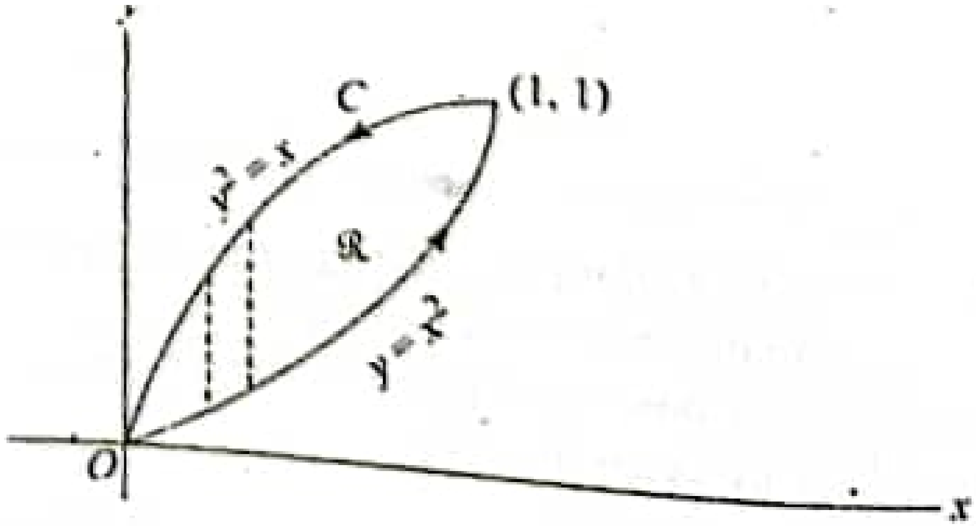

Fig. 4.8

Along $y = x^2$, the line integral equals

$$\int_{x=0}^{1}\{(2x)(x^2)-x^2\}\,dx+\{x+(x^2)^2\}\,d(x^2) = \int_0^1 (2x^3+x^2+2x^5)\,dx = \frac{7}{6}$$

Along $y^2 = x$, the line integral equals

$$\int_{y=1}^{0}\{2(y^2)(y)-(y^2)^2\}\,d(y^2)+\{y^2+y^2\}\,dy = \int_1^0 (4y^4-2y^5+2y^2)\,dy = -\frac{17}{15}$$

Then the required integral $= 7/6 - 17/15 = 1/30$. On the other hand,

$$\iint_{\mathscr{R}}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy = \iint_{\mathscr{R}}\left\{\frac{\partial}{\partial x}(x+y^2)-\frac{\partial}{\partial y}(2xy-x^2)\right\}dx\,dy$$

$$= \iint_{\mathscr{R}}(1-2x)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}}(1-2x)\,dy\,dx$$

$$= \int_{x=0}^{1}(y-2xy)\Big|_{y=x^2}^{\sqrt{x}}\,dx = \int_0^1 (x^{1/2}-2x^{3/2}-x^2+2x^3)\,dx = \frac{1}{30}$$

5.2 If $f(z)$ be analytic inside and on the boundary C of a simply-connected region $\mathfrak{R}$, prove that

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz$$

Solution

From Problem 5.1, if a and $a+h$ lie in $\mathfrak{R}$, we have

$$\frac{f(a+h)-f(a)}{h} = \frac{1}{2\pi i}\oint_C \frac{1}{h}\left\{\frac{1}{z-(a+h)}-\frac{1}{z-a}\right\} f(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)}$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2} + \frac{h}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$

The result follows on taking the limit as $h \to 0$ if we can show that the last term approaches zero.

To show this we use the fact that if Γ is a circle of radius ϵ and centre a which lies entirely in $\mathfrak{R}$ (see Fig. 5.3), then

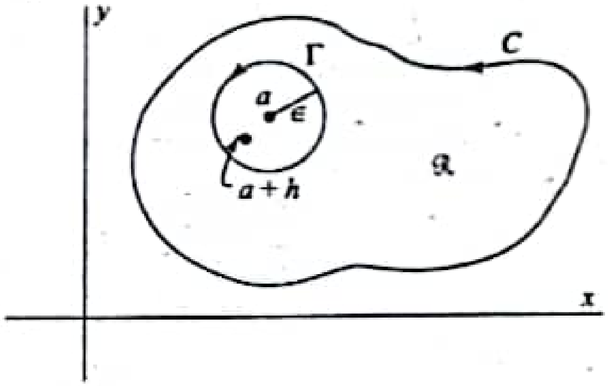

Fig. 5.3

$$\frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a-h)(z-a)^2} = \frac{h}{2\pi i}\oint_\Gamma \frac{f(z)\,dz}{(z-a-h)(z-a)^2}$$

Choosing h so small in absolute value that $a+h$ lies in Γ and $|h| < \epsilon/2$, we have by Problem 5.7(c), and the fact that Γ has equation $|z-a| = \epsilon$,

$$|z-a-h| \geqq |z-a| - |h| > \epsilon - \epsilon/2 = \epsilon/2$$

Also since $f(z)$ is analytic in $\mathfrak{R}$, we can find a positive number M such that $|f(z)| < M$.

Then, since the length of Γ is $2\pi\epsilon$, we have

$$\left|\frac{h}{2\pi i}\oint_\Gamma \frac{f(z)\,dz}{(z-a-h)(z-a)^2}\right| \leq \frac{|h|}{2\pi}\frac{M(2\pi\epsilon)}{(\epsilon/2)(\epsilon^2)} = \frac{2|h|M}{\epsilon^2}$$

and it follows that the left side approaches zero as $h \to 0$, thus completing the proof.

It is of interest to observe that the result is equivalent to

$$\frac{d}{da}f(a) = \frac{d}{da}\left\{\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz\right\} = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left\{\frac{f(z)}{z-a}\right\}dz$$

which is an extension to contour integrals of *Leibnitz's rule* for differentiating under the integral sign.

5.3 Prove that under the conditions of Problem 5.2

Solution

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz \quad n = 0, 1, 2, 3, \ldots$$

The cases where $n = 0$ and 1 follow from Problems 5.1 and 5.2, respectively provided we define $f^{(0)}(a) = f(a)$ and $0! = 1$.

To establish the case where $n = 2$, we use Problem 5.2 where a and $a+h$ lie in $\mathfrak{R}$ to obtain

$$\frac{f'(a+h)-f'(a)}{h} = \frac{1}{2\pi i}\oint_C \frac{1}{h}\left\{\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2}\right\} f(z)\,dz$$

$$= \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\,dz + \frac{h}{2\pi i}\oint_C \frac{3(z-a)-2h}{(z-a-h)^2(z-a)^3}\,f(z)\,dz$$

The result follows on taking the limit as $h \to 0$ if we can show that the last term approaches zero. The proof is similar to that of Problem 5.2, for using the fact that the integral around C equals the integral around Γ, we have

$$\left|\frac{h}{2\pi i}\oint_\Gamma \frac{3(z-a)-2h}{(z-a-h)^2(z-a)^3}\,f(z)\,dz\right| \le \frac{|h|}{2\pi}\,\frac{M(2\pi\epsilon)}{(\epsilon/2)^2(\epsilon^3)} = \frac{4|h|M}{\epsilon^4}$$

Since M exists such that $|\{3(z-a)-2h\}\,f(z)| < M$.

In a similar manner, we can establish the result for $n = 3, 4, \ldots$ (see Problems 5.45 and 5.46).

The result is equivalent to (see last paragraph of Problem 5.2)

$$\frac{d^n}{da^n}f(a) = \frac{d^n}{da^n}\left\{\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz\right\} = \frac{1}{2\pi i}\oint_C \frac{\partial^n}{\partial a^n}\left[\frac{f(z)}{z-a}\right]dz$$

5.4 If $f(z)$ is analytic in a region $\mathfrak{R}$, prove that $f'(z), f''(z), \ldots$ are analytic in $\mathfrak{R}$.

Solution This follows from Problems 5.2 and 5.3.

5.5 Evaluate (a) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$,

(b) $\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$ where C is the circle $|z| = 3$.

Solution

(a) Since $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz$$

By Cauchy's integral formula with $a = 2$ and $a = 1$ respectively, we have

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz = 2\pi i\{\sin \pi(2)^2 + \cos \pi(2)^2\} = 2\pi i$$

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\,dz = 2\pi i\{\sin \pi(1)^2 + \cos \pi(1)^2\} = -2\pi i$$

since $z = 1$ and $z = 2$ are inside C and $\sin \pi z^2 + \cos \pi z^2$ is analytic inside C. Then the required integral has the value $2\pi i - (-2\pi i) = 4\pi i$.

(b) Let $f(z) = e^{2z}$ and $a = -1$ in the Cauchy integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad (1)$$

If $n = 3$, then $f'''(z) = 8e^{2z}$ and $f'''(-1) = 8e^{-2}$. Hence (1) becomes

$$8e^{-2} = \frac{3!}{2\pi i}\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$$

from which we see that the required integral has the value $8\pi i e^{-2}/3$.

5.7 Evaluate $\int_C \frac{3z^2+z}{z^2-1}\,dz$, where C is the circle $|z-1|=1$.

Solution The integrand has singularities, where $z^2-1=0$ i.e. at $z=1$ and $z=-1$. The circle $|z-1|=1$ has center at $z=1$, $f(z)=3z^2+z$, is an analytic function.

Also,
$$\frac{1}{z^2-1}=\frac{1}{(z-1)(z+1)}=\frac{1}{2}\left(\frac{1}{z-1}-\frac{1}{z+1}\right)$$

$$\therefore \quad \int_C \frac{3z^2+z}{z^2-1}\,dz=\frac{1}{2}\int_C \frac{3z^2+z}{z-1}\,dz-\frac{1}{2}\int_C \frac{3z^2+z}{z+1}\,dz \qquad (1)$$

By Cauchy's integral formula,
$$\int_C \frac{3z^2+z}{z-1}\,dz=2\pi i f(1)=8\pi i \qquad \text{where } f(z)=3z^2+z$$

By Cauchy's theorem, $\int_C \frac{3z^2+z}{z+1}\,dz=0$

$\therefore$ From (1), we have $\int_C \frac{3z^2+z}{z^2-1}\,dz=4\pi i$.

last function has n zeros all located at $z = 0$, $f(z) + g(z)$ also has n zeros and the proof is complete.

5.26 Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Solution Consider the circle C_1: $|z| = 1$. Let $f(z) = 12$, $g(z) = z^7 - 5z^3$. On C_1 we have

$$|g(z)| = |z^7 - 5z^3| \leqq |z^7| + |5z^3| \leqq 6 < 12 = |f(z)|$$

Hence by Rouché's theorem $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z| = 1$ as $f(z) = 12$, i.e. there are no zeros inside C_1.

Consider the circle C_2: $|z| = 2$. Let $f(z) = z^7$, $g(z) = 12 - 5z^3$. On C_2 we have

$$|g(z)| = |12 - 5z^3| \leqq |12| + |5z^3| \leqq 60 < 2^7 = |f(z)|$$

Hence by Rouché's theorem $f(z) + g(z) = z^7 - 5z^3 + 12$ has the same number of zeros inside $|z| = 2$ as $f(z) = z^7$, i.e. all the zeros are inside C_2.

Hence all the roots lie inside $|z| = 2$ but outside $|z| = 1$, as required.

6.26 Find Laurent series about the indicated singularity for each of the following functions. Name the singularity in each case and give the region of convergence of each series.

(a) $\dfrac{e^{2z}}{(z-1)^3}$; $z = 1$. (b) $(z-3)\sin\dfrac{1}{z+2}$; $z=-2$. (c) $\dfrac{z-\sin z}{z^3}$; $z=0$.

(d) $\dfrac{z}{(z+1)(z+2)}$; $z=-2$. (e) $\dfrac{1}{z^2(z-3)^2}$; $z=3$.

Solution

(a) Let $z - 1 = u$. Then $z = 1 + u$ and

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2+2u}}{u^3} = \frac{e^2}{u^3}\cdot e^{2u} = \frac{e^2}{u^3}\left\{1+2u+\frac{(2u)^2}{2!}+\frac{(2u)^3}{3!}+\frac{(2u)^4}{4!}+\cdots\right\}$$

$$= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \frac{4e^2}{3} + \frac{2e^2}{3}(z-1) + \cdots$$

$z = 1$ is a *pole of order* 3, or *triple pole*.
The series converges for all values of $z \neq 1$.

(b) Let $z + 2 = u$ or $z = u - 2$. Then

$$(z-3)\sin\frac{1}{z+2} = (u-5)\sin\frac{1}{u} = (u-5)\left\{\frac{1}{u} - \frac{1}{3!u^3} + \frac{1}{5!u^5} - \cdots\right\}$$

$$= 1 - \frac{5}{u} - \frac{1}{3!u^2} + \frac{5}{3!u^3} + \frac{1}{5!u^4} - \cdots$$

$$= 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \cdots$$

$z = -2$ is an *essential singularity*.
The series converges for all values of $z \neq -2$.

(c)
$$\frac{z - \sin z}{z^3} = \frac{1}{z^3}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\right\}$$

$$= \frac{1}{z^3}\left\{\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots\right\} = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots$$

$z = 0$ is a *removable singularity*.
The series converges for all values of z.

(d) Let $z + 2 = u$. Then

$$\frac{z}{(z+1)(z+2)} = \frac{u-2}{(u-1)u} = \frac{2-u}{u} \cdot \frac{1}{1-u} = \frac{2-u}{u}(1 + u + u^2 + u^3 + \cdots)$$

$$= \frac{2}{u} + 1 + u + u^2 + \cdots = \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots$$

$z = -2$ is a *pole of order* 1, or *simple pole*.
The series converges for all values of z such that $0 < |z+2| < 1$.

(e) Let $z - 3 = u$. Then by the binomial theorem,

$$\frac{1}{z^2(z-3)^2} = \frac{1}{u^2(3+u)^2} = \frac{1}{9u^2(1+u/3)^2}$$

$$= \frac{1}{9u^2}\left\{1 + (-2)\left(\frac{u}{3}\right) + \frac{(-2)(-3)}{2!}\left(\frac{u}{3}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{u}{3}\right)^3 + \cdots\right\}$$

$$= \frac{1}{9u^2} - \frac{2}{27u} + \frac{1}{27} - \frac{4}{243}u + \cdots$$

$$= \frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4(z-3)}{243} + \cdots$$

$z = 3$ is a *pole of order* 2 or *double pole*.
The series converges for all values of z such that $0 < |z-3| < 3$.

6.27 Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent series valid for: (a) $1 < |z| < 3$, (b) $|z| > 3$, (c) $0 < |z+1| < 2$, (d) $|z| < 1$.

Solution

(a) Resolving into partial fractions,

$$\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$$

If $|z| > 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2z(1+1/z)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| < 3$,

$$\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right) = \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| < 3$, i.e. $1 < |z| < 3$, is

$$\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

(b) If $|z| > 1$, we have as in part (a),

$$\frac{1}{2(z+1)} = \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

If $|z| > 3$,

$$\frac{1}{2(z+3)} = \frac{1}{2z(1+3/z)} = \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right) = \frac{1}{2z} - \frac{3}{2z^2} + \frac{9}{2z^3} - \frac{27}{2z^4} + \cdots$$

Then the required Laurent expansion valid for both $|z| > 1$ and $|z| > 3$, i.e. $|z| > 3$, is by subtraction

$$\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \cdots$$

(c) Let $z + 1 = u$. Then

$$\frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)} = \frac{1}{2u}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \cdots$$

valid for $|u| < 2$, $u \neq 0$ or $0 < |z+1| < 2$.

(d) If $|z| < 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2(1+z)} = \frac{1}{2}(1 - z + z^2 - z^3 + \cdots) = \frac{1}{2} - \frac{1}{2}z + \frac{1}{2}z^2 - \frac{1}{2}z^3 + \cdots$$

If $|z| < 3$, we have by part (a),

$$\frac{1}{2(z+3)}=\frac{1}{6}-\frac{z}{18}+\frac{z^2}{54}-\frac{z^3}{162}+\cdots$$

Then the required Laurent expansion, valid for both $|z|<1$ and $|z|<3$, i.e. $|z|<1$, is by subtraction

$$\frac{1}{3}-\frac{4}{9}z+\frac{13}{27}z^2-\frac{40}{81}z^3+\cdots$$

This is a *Taylor series.*

the required integral has the

7.10 Show that $\int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+1)^2(x^2+2x+2)}=\frac{7\pi}{50}$.

Solution The poles of $z^2/(z^2+1)^2(z^2+2z+2)$ enclosed by the contour C of Fig. 7.5 are $z = i$ of order 2 and $z = -1 + i$ of order 1.

Residue at $z = i$ is

$$\lim_{z\to i}\frac{d}{dz}\left\{(z-i)^2\frac{z^2}{(z+1)^2(z-1)^2(z^2+2z+2)}\right\}=\frac{9i-12}{100}$$

Residue at $z = -1 + i$ is

$$\lim_{z\to -1+i}(z+1-i)\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)}=\frac{3-4i}{25}$$

Then

$$\oint_C\frac{z^2dz}{(z^2+1)^2(z^2+2z+2)}=2\pi i\left\{\frac{9i-12}{100}+\frac{3-4i}{25}\right\}=\frac{7\pi}{50}$$

or

$$\int_{-R}^{R}\frac{x^2dx}{(x^2+1)^2(x^2+2x+2)}+\int_\Gamma\frac{z^2dz}{(z^2+1)^2(z^2+2z+2)}=\frac{7\pi}{50}$$

Taking the limit as $R\to\infty$ and noting that the second integral approaches zero by Problem 7.7, we obtain the required result.

7.11 Show that $\int_0^{\infty}\frac{dx}{(1+x^2)^2}=\frac{\pi}{2}$.

Solution Consider $\int_C\frac{dz}{(1+z^2)^2}=\int_C f(z)dz$, where $f(z)=\frac{1}{(1+z^2)^2}$. C is the contour consisting of a large semi-circle Γ of radius R together with the part of real axis from $x = -R$ to $x = R$.

By Cauchy's residue theorem,

$$\int_C f(z)dz=\int_{-R}^{R}\frac{dx}{(1+x^2)^2}+\int_\Gamma\frac{dz}{(1+z^2)^2}=2\pi i\sum R^+ \qquad (1)$$

$$\lim_{z\to\infty} zf(z)=\lim_{z\to\infty} z.\frac{1}{(1+z^2)^2}=\lim_{z\to\infty}\frac{z}{z^4\left(1+\frac{1}{z^2}\right)^2}=0$$

$$\therefore \lim_{R\to\infty}\int_\Gamma\frac{dz}{(1+z^2)^2}=0 \text{ and } \lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{(1+x^2)^2}=\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2}$$

Taking $R \to \infty$ in (1), we get

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} + 0 = 2\pi i \sum R^+ \qquad (2)$$

Poles of $f(z) = \dfrac{1}{(1+z^2)^2}$ are given by $(1 + z^2)^2 = 0 \Rightarrow z = \pm i$ (twice) out of which only pole $z = i$ (order 2) lies inside C.

Therefore, residue at $i = \dfrac{\varphi'(i)}{\lfloor 1} = \varphi'(i)$

Here $\phi(z) = (z-i)^2 f(z) = (z-i)^2 \cdot \dfrac{1}{(1+z^2)^2} = \dfrac{(z-i)^2}{(z-i)^2(z+1)^2} = \dfrac{1}{(z+i)^2}$

$\therefore$ $$\phi'(z) = -\frac{2}{(z+i)^3}$$

Therefore, residue at $i = \phi'(i) = -\dfrac{2}{(2i)^3} = -\dfrac{2}{8i^3} = \dfrac{1}{4i}$

Therefore, from (2), $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = 2\pi i \times \frac{1}{4i} = \frac{\pi}{2}$

$$\Rightarrow 2\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2} \Rightarrow \int_0^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}.$$

7.16 Show that $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \frac{\pi}{12}$.

Solution Let $z = e^{i\theta}$. Then $\cos\theta = (z+z^{-1})/2$, $\cos 3\theta = (e^{3i\theta}+e^{-3i\theta})/2 = (z^3+z^{-3})/2$, $dz = iz\,d\theta$ so that

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \oint_C \frac{(z^3+z^{-3})/2}{5-4(z+z^{-1})/2}\frac{dz}{iz} = -\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)} dz$$

where C is the contour of Fig. 7.6

The integrand has a pole of order 3 at $z = 0$ and a simple pole $z = \frac{1}{2}$ inside C.

Residue at $z = 0$ is

$$\lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left\{z^3 \cdot \frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = \frac{21}{8}$$

Residue at $z = \frac{1}{2}$ is

$$\lim_{z\to 1/2}\left\{\left(z-\frac{1}{2}\right)\cdot\frac{z^6+1}{z^3(2z-1)(z-2)}\right\} = -\frac{65}{24}$$

Then

$$-\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)} dz = -\frac{1}{2i}(2\pi i)\left\{\frac{21}{8}-\frac{65}{24}\right\} = \frac{\pi}{12} \text{ as required.}$$

7.17 Show that $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \frac{5\pi}{32}$.

Solution Letting $z = e^{i\theta}$, we have $\sin\theta = (z-z^{-1})/2i$, $dz = ie^{i\theta}d\theta = iz\,d\theta$ and so

$$\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \oint_C \frac{dz/iz}{\{5-3(z-z^{-1})/2i\}^2} = -\frac{4}{i}\oint_C \frac{z\,dz}{(3z^2-10iz-3)^2}$$

where C is the contour of Fig. 7.6.

The integrand has poles of order 2 at $z = (10i \pm \sqrt{-100+36})/6 = (10i \pm 8i)/6 = 3i, i/3$. Only the pole $i/3$ lies inside C.

Residue at

$$z = i/3 \text{ is } \lim_{z\to i/3}\frac{d}{dz}\left\{(z-i/3)^2\cdot\frac{z}{(3z^2-10iz-3)^2}\right\}$$

$$= \lim_{z\to i/3}\frac{d}{dz}\left\{(z-i/3)^2\cdot\frac{z}{(3z-i)^2(z-3i)^2}\right\} = -\frac{5}{256}$$

Then

$$-\frac{4}{i}\oint_C \frac{z\,dz}{(3z^2-10iz-3)^2} = -\frac{4}{i}(2\pi i)\left(\frac{-5}{256}\right) = \frac{5\pi}{32}$$