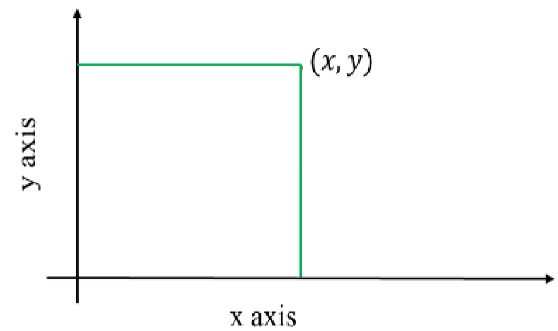
Complex Number: A complex number have of the form $a + ib$, where a and b are the real numbers and i is the imaginary unit having the property that $i^2 = -1$. If $z = a + ib$, the a is called the real part of z denoted by $a = Re(z)$ and b is called the imaginary part of z denoted by $b = Im(z)$. If $a = 0$, the complex number $0 + ib$ or ib is called a pure imaginary number.
Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$.

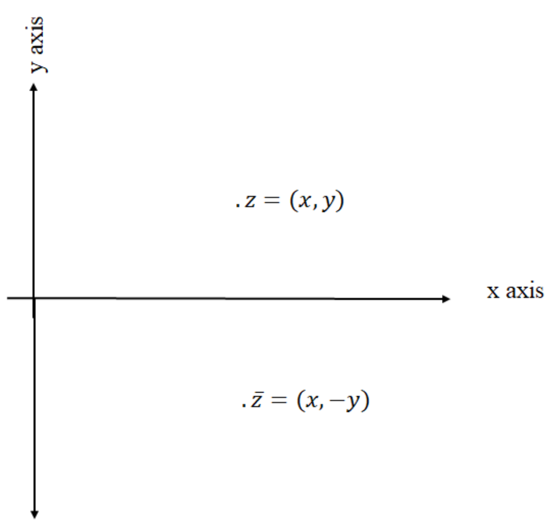
Graphical Representation of Complex Numbers

A complex number $z = x + iy$ can be considered as an ordered pair of real numbers (x, y) called rectangular coordinates of the point. Now this point can be represented in rectangular coordinates system. That is, we can represent such number by a point in the xy plane called the complex plane or Argand diagram.



Complex Conjugate

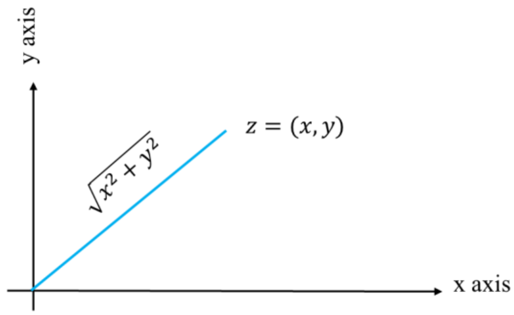
The complex conjugate, or briefly conjugate, of a complex number $x + iy$ is $x - iy$. The complex conjugate of a complex number z is often indicated by $\bar{z}$ or z^*.



Absolute Value

The absolute value or modulus of a complex number $x + iy$ is defined as $|x + iy| = \sqrt{x^2 + y^2}$.

This represents the distance between origin and (x, y).



Fundamental Operations with Complex Numbers

In performing operations with complex numbers, we can proceed as in the algebra of real numbers, replacing i^2 by 1 when it occurs.

(1) *Addition*

$$(a + bi) + (c + di) = a + bi + c + di = (a + c) + (b + d)i$$

(2) *Subtraction*

$$(a + bi) - (c + di) = a + bi - c - di = (a - c) + (b - d)i$$

(3) *Multiplication*

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

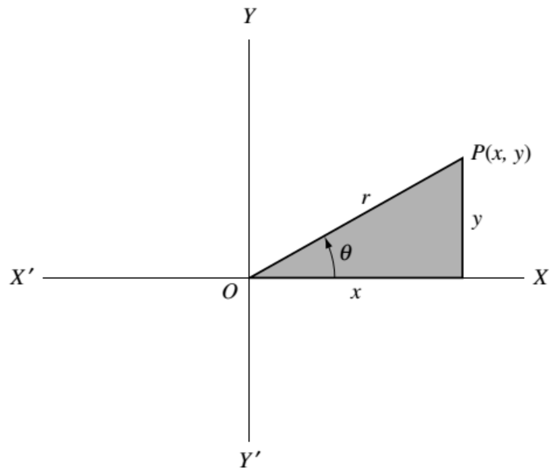
(4) *Division*

If $c \neq 0$ and $d \neq 0$, then

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac - adi + bci - bdi^2}{c^2 - d^2 i^2} \\ &= \frac{ac + bd + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i\end{aligned}$$

Polar Form of Complex Numbers

Let P be a point in the complex plane corresponding to the complex number (x, y) or $x + iy$.



Then we see from Figure that

$$x = r\cos\theta, \quad y = r\sin\theta$$

Where $r = \sqrt{x^2 + y^2} = |x + iy|$ is called the modulus or absolute value of $z = x + iy$ and θ, called the amplitude or argument of $z = x + iy$, is the angle that line OP makes with the positive x axis. It follows that

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

which is called the polar form of the complex number, and r and θ are called polar coordinates.

If $z_1, z_2, z_3, \ldots, z_m$ are complex numbers, the following properties hold.

(1) $|z_1 z_2| = |z_1||z_2|$ or $|z_1 z_2 \cdots z_m| = |z_1||z_2| \cdots |z_m|$

(2) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ if $z_2 \neq 0$

(3) $|z_1 + z_2| \leq |z_1| + |z_2|$ or $|z_1 + z_2 + \cdots + z_m| \leq |z_1| + |z_2| + \cdots + |z_m|$

(4) $|z_1 \pm z_2| \geq |z_1| - |z_2|$

Prove: (a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, (b) $|z_1 z_2| = |z_1||z_2|$.

Solution

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then

(a) $\overline{z_1 + z_2} = \overline{x_1 + iy_1 + x_2 + iy_2} = \overline{x_1 + x_2 + i(y_1 + y_2)}$

$= x_1 + x_2 - i(y_1 + y_2) = x_1 - iy_1 + x_2 - iy_2 = \overline{x_1 + iy_1} + \overline{x_2 + iy_2} = \bar{z}_1 + \bar{z}_2$

(b) $|z_1 z_2| = |(x_1 + iy_1)(x_2 + iy_2)| = |x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)|$

$= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} = |z_1||z_2|$

Prove (a) $|z_1 + z_2| \le |z_1| + |z_2|$, (b) $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$, (c) $|z_1 - z_2| \ge |z_1| - |z_2|$

Proof:

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then we must show that

$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \le \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$

Squaring both sides, this will be true if

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 \le x_1^2 + y_1^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} + x_2^2 + y_2^2$$

$$x_1^2 + 2x_1x_2 + x_2^2 + y_1^2 + 2y_1y_2 + y_2^2 \le x_1^2 + y_1^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} + x_2^2 + y_2^2$$

i.e., if $$x_1x_2 + y_1y_2 \le \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$$

if (squaring both sides again)

$$x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 \le x_1^2x_2^2 + x_1^2y_2^2 + y_1^2x_2^2 + y_1^2y_2^2$$

$$2x_1x_2y_1y_2 \le x_1^2y_2^2 + y_1^2x_2^2$$

$$0 \le x_1^2y_2^2 - 2x_1x_2y_1y_2 + y_1^2x_2^2$$

$$x_1^2y_2^2 - 2x_1x_2y_1y_2 + y_1^2x_2^2 \ge 0$$

$$(x_1y_2 - x_2y_1)^2 \ge 0.$$

which is true. Reversing the steps, which are reversible, proves the result.

(b) Now

$$|z_1 + z_2 + z_3| = |z_1 + (z_2 + z_3)|$$

Using the formula $|z_1 + z_2| \le |z_1| + |z_2|$, we get

$$|z_1 + z_2 + z_3| \le |z_1| + |z_2 + z_3|$$
$$\le |z_1| + |z_2| + |z_3|$$

(c) We have

$$|z_1| = |z_1 - z_2 + z_2|$$
$$\le |z_1 - z_2| + |z_2|$$
$$|z_1| - |z_2| \le |z_1 - z_2|$$
$$\text{Then } |z_1 - z_2| \ge |z_1| - |z_2|$$

An equivalent result obtained on replacing z_2 by $-z_2$ is

$$|z_1 + z_2| \ge |z_1| - |z_2|.$$

Functions of a Complex Variable :

Let D be a nonempty set in C. A single-valued complex function or, simply, a complex function $f: D \to C$ is a map that assigns a unique complex number $w = u + iv$ to each complex argument $z = x + iy$ in D. We write $w = f(z)$. The set D is called the domain of the function f and the set $f(D)$ is the range or the image of f. So, a complex-valued function f of a complex variable z is a rule that assigns to each complex number z in a set D one and only one complex number w. We call w as the image of z under f.

If $z = x + iy \in D$, we shall write $f(z) = u(x, y) + iv(x, y)\ or\ f(z) = u(z) + iv(z)$. The real functions u and v are called the real and, respectively, the imaginary part of the complex function f. Therefore, we can describe a complex function with the aid of two real functions depending on two real variables.

Example1. The function $f: C \to C$, defined by $f(z) = z^3$, can be written as $f(z) = u(x, y) + iv(x, y)$, with $u, v: R^2 \to R$ given by $u(x, y) = x^3 - 3xy^2, v(x, y) = 3x^2y - y^3$.

Example 2. For the function $f: C \to C$, defined by $f(z) = e^z$, we have $u(x, y) = e^x cosy$, $v(x, y) = e^x siny$, for any $(x, y) \in R^2$.

Limits

Let $f(z)$ be defined and single-valued in a neighborhood of $z = z_0$ with the possible exception of $z = z_0$itself. We say that the number l is the limit of $f(z)$ as z approaches z_0 and write $\lim_{z\to z_0} f(z) = l$ if for any positive number ϵ (however small), we can find some positive number δ (usually depending on ϵ) such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$. In such a case, we also say that $f(z)$ approaches l as z approaches z_0 and write $f(z) \to l$ as $z \to z_0$. The limit must be independent of the manner in which z approaches z_0. Geometrically, if z_0 is a point in the complex plane, then $\lim_{z\to z_0} f(z) = l$ if the difference in absolute value between $f(z)$ and l can be made as small as we wish by choosing points z sufficiently close to z_0.

Example: Show that if $f(z) = i\frac{\bar{z}}{2}$ in the disk $|z| < 1$, then $\lim_{z\to 1} f(z) = \frac{i}{2}$.

Solution: Observe that when z is in the disk $|z| < 1$, Now

$$\left|f(z) - \frac{i}{2}\right| = \left|i\frac{\bar{z}}{2} - \frac{i}{2}\right|$$

$$= \left|i\frac{x - iy}{2} - \frac{i}{2}\right| = \left|\frac{ix - i^2y}{2} - \frac{i}{2}\right| = \left|\frac{ix + y}{2} - \frac{i}{2}\right| = \left|\frac{i(x-1) + y}{2}\right|$$

$$= \frac{|i(x-1) + y|}{2} = \frac{\sqrt{(x-1)^2 + y^2}}{2} = \frac{|(x-1) + iy|}{2} = \frac{|x + iy - 1|}{2} = \frac{|z-1|}{2}$$

Thus,

$$\left|f(z) - \frac{i}{2}\right| = \frac{|z-1|}{2}$$

Hence, for any such z and each positive number ε.

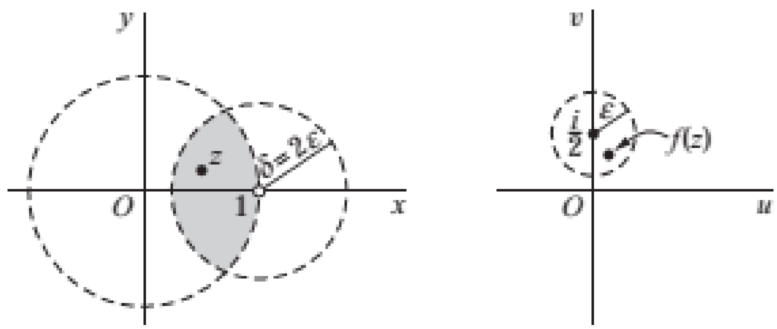
$$\left|f(z) - \frac{i}{2}\right| < \epsilon$$

whenever

$$0 < \frac{|z-1|}{2} < \epsilon$$

$$0 < |z-1| < 2\epsilon$$

$$0 < |z-1| < \delta$$



Thus conditions of limit is satisfied by points in the region $|z| < 1$ when δ is equal to 2ε or any smaller positive number.

Example: Show that if $f(z) = \frac{z}{\bar{z}}$, the limit $\lim\limits_{z \to 0} f(z)$ does not exist.

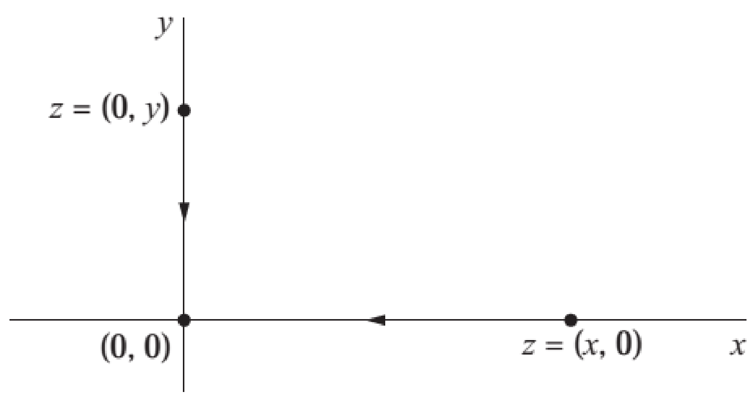
Solution: For, if it did exist, it could be found by letting the point $z = (x, y)$ approach the origin in any manner. But when $z = (x, 0)$ is a nonzero point on the real axis

$$f(z) = \frac{x + i0}{x - i0} = 1;$$

and when $z = (0, y)$ is a nonzero point on the imaginary axis,

$$f(z) = \frac{0 + iy}{0 - iy} = -1$$

Thus, by letting z approach the origin along the real axis, we would find that the desired limit is 1. An approach along the imaginary axis would, on the other hand, yield the limit -1. Since a limit is unique, we must conclude that limit does not exist.



Continuity

A function f is *continuous* at a point $z0$ if all three of the following conditions are satisfied:

$$(1) \quad \lim_{z \to z_0} f(z) \quad \text{exists.}$$

$$(2) \quad f(z_0) \quad \text{exists,}$$

$$(3) \quad \lim_{z \to z_0} f(z) = f(z_0)$$

Observe that statement (3) actually contains statements (1) and (2), since the existence of the quantity on each side of the equation there is needed. Statement (3) says, of course, that for each positive number ε, there is a positive number δ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \textbf{whenever} \quad |z - z_0| < \delta.$$

Example: the function $f(z) = \begin{cases} \frac{z^2+3iz-2}{z+i} & for\ \ z \neq -i \\ 5 & for\ \ z = -i \end{cases}$ continuous at $z = -i$?

Solution: Continuity at $z = -i$ that is, $(x, y) = (0, -1)$:

Now

$$\lim_{z \to -i} f(z) = \lim_{z \to -i} \frac{z^2 + 3iz - 2}{z + i}$$

$$\lim_{(x,y) \to (0,-1)} f(z) = \lim_{(x,y) \to (0,-1)} \frac{(x + iy)^2 + 3i(x + iy) - 2}{(x + iy) + i}$$

$$= \lim_{x \to 0} \left[\lim_{y \to -1} \frac{(x + iy)^2 + 3i(x + iy) - 2}{(x + iy) + i} \right]$$

$$= \lim_{x \to 0} \left[\frac{(x - i)^2 + 3i(x - i) - 2}{(x - i) + i} \right]$$

$$= \lim_{x \to 0} \left[\frac{x^2 - 2ix + i^2 + 3ix - 3i^2 - 2}{(x - i) + i} \right]$$

$$= \lim_{x \to 0} \left[\frac{x^2 - 2ix - 1 + 3ix + 3 - 2}{x} \right]$$

$$= \lim_{x \to 0} \left[\frac{x^2 + ix}{x} \right]$$

$$= \lim_{x \to 0} [x + i]$$

$$= i$$

Again

$$\lim_{z\to -i} f(z) = \lim_{z\to -i} \frac{z^2+3iz-2}{z+i}$$

$$= \lim_{y\to -1}\left[\lim_{x\to 0} \frac{(x+iy)^2+3i(x+iy)-2}{(x+iy)+i}\right]$$

$$= \lim_{y\to -1}\left[\frac{(iy)^2+3i(iy)-2}{(iy)+i}\right]$$

$$= \lim_{y\to -1}\left[\frac{-y^2-3y-2}{(iy)+i}\right]$$

$$= \lim_{y\to -1}\left[\frac{-(y+2)(y+1)}{i(y+1)}\right]$$

$$= \lim_{y\to -1}\left[\frac{-(y+2)}{i}\right]$$

$$= \lim_{y\to -1}\left[\frac{-i^2(-1+2)}{i}\right]$$

$$= i$$

Thus, limit exist.

But $f(z) = 5$ at $z = -i$ from given data.

Hence,

$$\lim_{z\to -i} f(z) \neq f(-i)$$

Therefore $f(z)$ is not continuous at $z = -i$.

Theorem 1. A composition of continuous functions is itself continuous.

Given $f(z)$ and $g(z)$ are continuous at $z = z_0$. Then so are the functions $f(z) + g(z)$, $f(z) - g(z)$, $f(z)g(z)$ and $f(z)/g(z)$, the last if $g(z_0) = 0$. Similar results hold for continuity in a region.

Theorem 2. If a function $f\ (z)$ is continuous and nonzero at a point z_0, then $f\ (z) \neq 0$ throughout some neighborhood of that point.

Theorem 3. Suppose $f(z)$ is continuous in a closed and bounded region. Then it is bounded in the region; i.e., there exists a constant M such that $|f(z)| < M$, for all points z of the region.

Derivatives

If $f(z)$ is single-valued in some region R of the z plane, the derivative of $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

provided that the limit exists independent of the manner in which $\Delta z \to 0$. In such a case, we say that $f(z)$ is differentiable at z. In the definition (3.1), we sometimes use h instead of Δz.

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

Although differentiability implies continuity, the reverse is not true.

Analytic Functions

If the derivative $f'(z)$ exists at all points z of a region R, then $f(z)$ is said to be analytic in R and is referred to as an analytic function in R or a function analytic in R. The terms regular and holomorphic are sometimes used as synonyms for analytic.

Problem: Show that $\frac{d\bar{z}}{dz}$ does not exist anywhere, i.e., $f(z) = \bar{z}$ is non-analytic anywhere.

Solution:

By definition,

$$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

if this limit exists independent of the manner in which $\Delta z = \Delta x + i\Delta y$ approaches zero.

Then

$$\frac{d}{dz}\bar{z} = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x + iy + \Delta x + i\Delta y} - \overline{x + iy}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x - iy + \Delta x - i\Delta y - (x - iy)}{\Delta x + i\Delta y} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$, the required limit is

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

If $\Delta x = 0$, the required limit is

$$\lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

Then, since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist, i.e., $f(z) = \bar{z}$ is *non-analytic* anywhere.

Cauchy–Riemann Equations

A necessary condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region R is that, in R, u and v satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If the partial derivatives in the above equations are continuous in R, then the Cauchy–Riemann equations are sufficient conditions that $f(z)$ be analytic in R. The functions $u(x, y)$ and $v(x, y)$ are sometimes called conjugate functions.

Proof:

Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region R. That is, the derivative $f'(z)$ exists at all points z of a region . Now

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)\} - \{u(x, y) + iv(x, y)\}}{\Delta x + i\Delta y} \tag{1}$$

must exist independent of the manner in which Δz (or Δx and Δy) approaches zero. We consider two possible approaches.

Case 1. $\Delta y = 0$, $\Delta x \to 0$. In this case, (1) becomes

$$\lim_{\Delta x \to 0} \left\{ \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \left[\frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right] \right\} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

provided the partial derivatives exist.

Case 2. $\Delta x = 0$, $\Delta y \to 0$. In this case, (1) becomes

$$\lim_{\Delta y \to 0} \left\{ \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} \right\} = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Now $f(z)$ cannot possibly be analytic unless these two limits are identical. Thus, a necessary condition that $f(z)$ be analytic is

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts, we get

$$\text{or} \quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

Theorem. Suppose that $f(z) = u(x,y) + iv(x,y)$ and that $f(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy–Riemann equations $u_x = v_y, u_y = -v_x$ there. Also, $f'(z_0)$ can be written $f'(z_0) = u_x + iv_x$, where these partial derivatives are to be evaluated at (x_0, y_0).

Problem: Show that the function $f(z) = z^2 = x^2 - y^2 + i2xy$ is differentiable everywhere and that $f'(z) = 2z$. To verify that the Cauchy–Riemann equations are satisfied everywhere.
Solution: Given function

$$f(z) = z^2 = x^2 - y^2 + i2xy$$

$$f(z) = u(x,y) + iv(x,y) = x^2 - y^2 + i2xy$$

Equation real and imaginary parts from above equation, we get

$$u(x,y) = x^2 - y^2$$

$$v(x,y) = 2xy$$

Thus, $\quad u_x = 2x = v_y$ and $u_y = -2y = -v_x$ (verified Cauchy–Riemann equations)

We know $\quad f'(z) = u_x + iv_x$

$$f'(z) = 2x + 2iy = 2(x + iy) = 2z \quad \text{(Verified)}$$

Prove that in polar form the Cauchy–Riemann equations can be written

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Solution:

We have $\quad x = r\cos\theta, \quad y = r\sin\theta$ or $r=\sqrt{x^2+y^2}, \quad \theta=\tan^{-1}(y/x)$. Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$=\frac{\partial u}{\partial r}\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\frac{\partial u}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{\partial u}{\partial r}\left(\frac{r\cos\theta}{r}\right) - \frac{\partial u}{\partial \theta}\left(\frac{r\sin\theta}{r}\right)$$

$$= \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta \qquad (1)$$

Again,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial u}{\partial \theta}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{\partial u}{\partial r}\left(\frac{r\sin\theta}{r}\right) + \frac{\partial u}{\partial \theta}\left(\frac{r\cos\theta}{r}\right)$$

$$= \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta \qquad (2)$$

Similarly,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta \qquad (3)$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta \qquad (4)$$

From the Cauchy–Riemann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ we have, using (1) and (4),

$$\frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta$$

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta - \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta = 0 \qquad (5)$$

From the Cauchy–Riemann equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ we have, using (2) and (3),

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta = 0 \qquad (6)$$

Multiplying (5) by $\cos\theta$ (6) by $\sin\theta$ and adding yields

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)(cos^2\theta + sin^2\theta) = 0$$

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right) = 0$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Multiplying (5) by $-\sin\theta$ (6) by $\cos\theta$ and adding yields

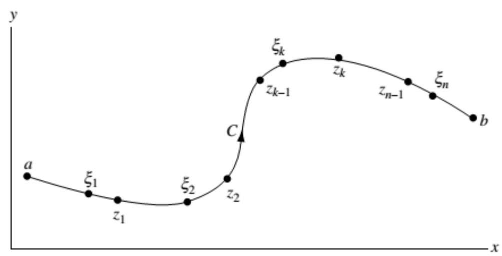
$$\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)(cos^2\theta + sin^2\theta) = 0$$

$$\left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right) = 0$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Complex Line Integrals

Let $f(z)$ be continuous at all points of a curve C [Fig], which we shall assume has a finite length



Subdivide C into n parts by means of points $z_1, z_2, \ldots, z_{n-1}$, chosen arbitrarily, and call $a = z_0$, $b = z_n$. On each arc joining z_{k-1} to z_k [where k goes from 1 to n], choose a point ξ_k. Form the sum

$$S_n = f(\xi_1)(z_1 - a) + f(\xi_2)(z_2 - z_1) + \cdots + f(\xi_n)(b - z_{n-1})$$

On writing $z_k - z_{k-1} = \Delta z_k$, this becomes

$$S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta z_k$$

Let the number of subdivisions n increase in such a way that the largest of the chord lengths $|\Delta z_k|$ approaches zero. Then, since $f(z)$ is continuous, the sum Sn approaches a limit that does not depend on
the mode of subdivision and we denote this limit by

$$\int_a^b f(z)\,dz \quad \text{or} \quad \int_C f(z)\,dz$$

called the complex line integral or simply line integral of $f(z)$ along curve C, or the definite integral of $f(z)$ from a to b along curve C.

Complex Integration:

let us consider $f(t) = u(t) + iv(t)$, where u and v are real valued functions of t in a closed interval a,b. We define

$$\int_a^b f(t)dt = \int_a^b u(t)\ dt + i\int_a^b v(t)\,dt$$

Thus $\int_a^b f(t)\,dt$ is a complex number in which $\int_a^b v(t)\,dt$ is a real part and $\int_a^b v(t)\,dt$ is an imaginary part.

Line integral:

Let $f(z)$ be a complex variable defined in a domain D. let C be an arc in a domain D then

$$\int f(z)\,dz = \int (u(x,y) + iv(x,y))(dx + idy)$$

$$= \int (u + iv)(dx + idy)$$

$$= \int (udx + ivdx + iudy + i^2vdy)$$

$$= \int (udx - vdy) + i\int (udy + vdx)$$

Problem: Evaluate

$$\int_c (x + y)dx + x^2ydy$$

along $y = 3x$ between points $(0,0), (3,9)$.

Solution: given points are $A(0,0), B(3,9)$. The equation of line passing through A and B is $y = 3x$, then $dy = 3dx$.

Now $\int_c (x + y)dx + x^2ydy = \int_0^3 (x + 3x)dx + (x^2 3x)3dx$

$$= \int_0^3 (4x)dx + 9x^3dx$$

$$= \int_0^3 ((4x) + 9x^3)dx$$

$$= \left[4\frac{x^2}{2} + 9\frac{x^4}{4}\right]_0^3$$

$$= 2(3)^2 + \frac{9}{4}(3)^4 - 0\text{-}0$$

=18+729/4

=801/4

Problem 2: Evaluate $\int_{i-1}^{2+i} (2x + 1 + iy)dz$ along the line joining $1 - i$ to $2 + i$.

Solution: given points $A(1, -1)$, $B(2,1)$. We know that equation of line joining two points $A(x_1, y_1), B(x_2, y_2\ )$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y + 1 = \frac{1+1}{2-1}(x - 1)$$

$$y = 2x - 2 - 1$$

$y = 2x - 3$, $dy = 2dx$ and x varies from 1 to 2.

$$\int_{i-1}^{2+i} (2x + 1 + iy)dz = \int_1^2 \big(2x + 1 + i(2x - 3)\big)(dx + i2dx)$$

$$= \int_1^2 (2x + 1 - 2(2x - 3) + i(2x - 3 + 4x + 2))dx$$

$$= \int_1^2 ((-2x + 7) + i(6x - 1))dx$$

$$= \left[-2\frac{x^2}{2} + 7x + i(6\frac{x^2}{2} - x)\right]_1^2$$

$$= -4 + 14 + i(12 - 2) + 1 - 7 - i(3 - 1)$$

$$= 10 + 10i - 6 - 2i$$

$$= 4 + 8i$$

Problem:

Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$ along: (a) the parabola $x = 2t,\ y = t^2 + 3$; (b) straight lines from (0, 3) to (2, 3) and then from (2, 3) to (2, 4); (c) a straight line from (0, 3) to (2, 4).

Solution

(a) The points (0, 3) and (2, 4) on the parabola correspond to $t = 0$ and $t = 1$, respectively. Then, the given integral equals

$$\int_{t=0}^{1} [2(t^2 + 3) + (2t)^2]2\,dt + [3(2t) - (t^2 + 3)]2t\,dt = \int_{0}^{1} (24t^2 + 12 - 2t^3 - 6t)\,dt = \frac{33}{2}$$

(b) Along the straight line from (0, 3) to (2, 3), $y = 3$, $dy = 0$ and the line integral equals

$$\int_{x=0}^{2} (6 + x^2)\,dx + (3x - 3)0 = \int_{x=0}^{2} (6 + x^2)\,dx = \frac{44}{3}$$

Along the straight line from (2, 3) to (2, 4), $x = 2$, $dx = 0$ and the line integral equals

$$\int_{y=3}^{4} (2y + 4)0 + (6 - y)\,dy = \int_{y=3}^{4} (6 - y)\,dy = \frac{5}{2}$$

Then, the required value $= 44/3 + 5/2 = 103/6$.

(c) An equation for the line joining $(0, 3)$ and $(2, 4)$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or, $\quad y - 3 = \dfrac{4 - 3}{2 - 0}(x - 0)$

or, $\quad y - 3 = \dfrac{1}{2}x$

or, $\quad x = 2y - 6$

and $dx = 2dy$

Then, the line integral

$\int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$

Equals

$$\int_{y=3}^{4} [2y + (2y - 6)^2]2\,dy + [3(2y - 6) - y]\,dy$$

$$= \int_{3}^{4} (8y^2 - 39y + 54)\, dy$$

$$= \left[8\frac{y^3}{3} - 39\frac{y^2}{2} + 54y\right]_3^4$$

$$= \frac{97}{6}$$

Green's Theorem in the Plane

Let $P(x, y)$ and $Q(x, y)$ be continuous and have continuous partial derivatives in a region R and on its boundary C. Green's theorem states that

$$\oint_C P\, dx + Q\, dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

Cauchy–Goursat Theorem

Let $f(z)$ be analytic in a region R and on its boundary C. Then

$$\oint_C f(z)\, dz = 0$$

This fundamental theorem, often called Cauchy's integral theorem or simply Cauchy's theorem.

Cauchy's Theorem

Statement: let $f(z) = u(x, y) + iv(\, x, y)$ be analytic on and within a simple closed contour C and $f'(z)$ be continuous then $\int_c \, f\,(z\,)dz = 0$.

Proof:

Let $f(z) = u(x, y) + iv(\, x, y) = u + iv$ then

$$\int_c f(z)dz = \big(u(x, y) + iv(\, x, y)\big)(dx + idy)$$

$$= (u + iv)(dx + idy)$$

$$= (udx - vdy) + i(vdx + udy)$$

$$\int_c f(z)dz = \int_c (udx - vdy) + i\int_c (vdx + udy) \qquad \dots (1)$$

Since $f'(z)$ be continuous then the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region enclosed by C.

We know Greens theorem

$$\oint_C P\,dx + Q\,dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Therefore equation (1) can be written as

$$\int_c f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy$$

Since $f(z)$ is analytic and satisfy Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence

$$\int_c f(z)dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right)dxdy$$

$$= 0.$$

Thus

$$\int_c f(z)dz = 0$$

Evaluate $\int_C \frac{z^2-z+1}{z-1}dz$ where $c: |z| = \frac{1}{2}$ is taken in anticlock wise direction.

Solution : given $\int_C \frac{z^2-z+1}{z-1}dz$

Here $f(z) = \frac{z^2-z+1}{z-1}$ is analytic on and within the circle $|z| = \frac{1}{2}$

Since $z = 1$ lies outside the circle $|z| = \frac{1}{2}$

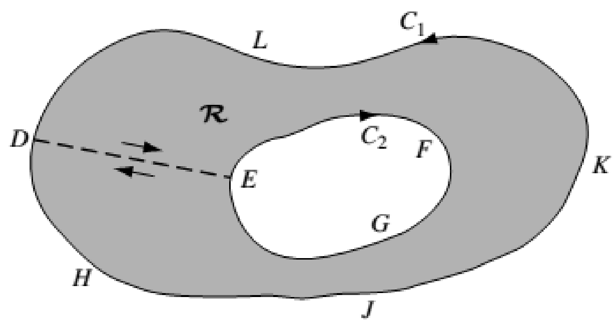
Now we can apply Cauchy integral theorem , $\int_c f(z)dz = 0$

$$\int_C \frac{z^2-z+1}{z-1}\,dz = 0$$

Let $f(z)$ be analytic in a region R bounded by two simple closed curves C_1 and C_2 and also on C_1 and C_2. Then

$$\int_{C_1} f(z)\ dz = \int_{C_2} f(z)\ dz$$

where C_1 and C_2 are both traversed in the positive sense relative to their interiors.



Cauchy integral formula:

Statement: Let $f(z)$ be analytic function within and on a closed contour , if $z = a$ is any point within C then

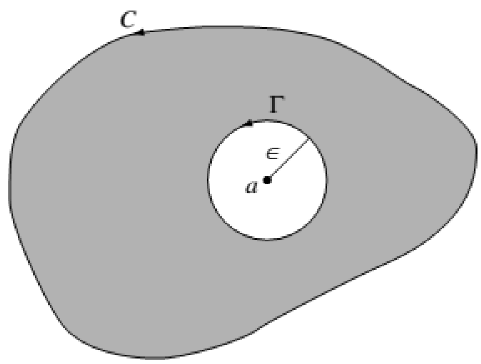
$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

where the integral is taken in the positive sense around C.

Proof: let f(z) be analytic function within and on a closed contour. Let is any point within C now we have to prove

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

Choose a suitably small positive number $\in$ and describes a circle Γ with center at a and radius $\in$ so that the circle entirely within C.



Since f(z) is analytic anywhere on C therefore $\frac{f(z)}{z-a}$ is also analytic except $z = a$, thus $\frac{f(z)}{z-a}$ is analytic in the region between C and Γ, therefore by generalization of Cauchy theorem we get

$$\int_C \frac{f(z)}{z-a} dz = \int_\Gamma \frac{f(z)}{z-a} dz$$

On the circle Γ, $z = a + \in e^{i\theta}$, $0 \le \theta \le 2\pi$.

$$= \int_{\theta=0}^{2\pi} \frac{f(a+\in e^{i\theta})}{a+\in e^{i\theta} - a} \in e^{i\theta}.\, id\theta$$

$$= i \int_{\theta=0}^{2\pi} \frac{f(a+\in e^{i\theta})}{\in e^{i\theta}} \in e^{i\theta}.\, d\theta$$

$$= i \int_{\theta=0}^{2\pi} f(a+\in e^{i\theta})\, d\theta$$

As $\in \to 0$ the circle Γ shrinks to the point a. Hence $\in \to 0$

$$\int_C \frac{f(z)}{z-a} dz = i \int_{\theta=0}^{2\pi} f(a)\, d\theta$$

$$\text{or,} \int_C \frac{f(z)}{z-a} dz = i\, f(a) \int_{\theta=0}^{2\pi} d\theta$$

$$\text{or,} \int_C \frac{f(z)}{z-a} dz = i\, f(a) 2\pi$$

$$\text{or,} f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

Generalization of Cauchy integral formula:

Statement: Let $f(z)$ be analytic function within and on a closed contour , if $z = a$ is any point within C then

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

where the integral is taken in the positive sense around C.

Proof: from the Cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Note that "a" is any point within C and by definition of analytic function

$$f'(z) = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$f'(a) = \lim_{\Delta a\to 0}\frac{f(a+\Delta a)-f(a)}{\Delta a}$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}[f(a+\Delta a)-f(a)]$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\left[\frac{1}{2\pi i}\int_C \frac{f(z)}{z-(a+\Delta a)}\,dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz\right]$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\frac{1}{2\pi i}\left[\int_C f(z)\left[\frac{1}{z=(a+\Delta a)}-\frac{1}{z-a}\right]dz\right]$$

$$= \lim_{\Delta a\to 0}\frac{1}{\Delta a}\frac{1}{2\pi i}\left[\int_C f(z)\left[\frac{\Delta a}{\big(z=(a+\Delta a)\big)(z-a)}\right]dz\right]$$

$$= \lim_{\Delta a\to 0}\frac{1}{2\pi i}\left[\int_C \left[\frac{f(z)}{\big(z=(a+\Delta a)\big)(z-a)}\right]dz\right]$$

$$f'(a) = \frac{1!}{2\pi i}\left[\int_C \left[\frac{f(z)}{(z-a)^2}\right]dz\right]$$

This is called Cauchy's integral formula for the derivative of analytic function.

Now

$$f''(a) = [f'(a)]'$$

$$= \lim_{\Delta a \to 0} \frac{f'(a + \Delta a) - f'(a)}{\Delta a}$$

$$= \lim_{\Delta a \to 0} \frac{1}{\Delta a} [f'(a + \Delta a) - f'(a)]$$

$$= \lim_{\Delta a \to 0} \frac{1}{\Delta a} \left[\frac{1!}{2\pi i} \left[\int_C \left[\frac{f(z)}{\left(z - (a + \Delta a)\right)^2} \right] dz \right] - \frac{1!}{2\pi i} \left[\int_C \left[\frac{f(z)}{(z-a)^2} \right] dz \right] \right]$$

$$= \lim_{\Delta a \to 0} \frac{1}{\Delta a} \frac{1!}{2\pi i} \left[\int_C f(z) \left[\frac{1}{\left(z - (a + \Delta a)\right)^2} - \frac{1}{(z-a)^2} \right] dz \right]$$

$$= \lim_{\Delta a \to 0} \frac{1}{\Delta a} \frac{1!}{2\pi i} \left[\int_C f(z) \left[\frac{(z-a)^2 - \left((z-a) - \Delta a\right)^2}{\left(z - (a + \Delta a)\right)^2 (z-a)^2} \right] dz \right]$$

$$= \lim_{\Delta a \to 0} \frac{1}{\Delta a} \frac{1!}{2\pi i} \left[\int_C f(z) \left[\frac{(z-a)^2 - ((z-a)^2 - 2(z-a)\Delta a + (\Delta a)^2)}{\left(z - (a + \Delta a)\right)^2 (z-a)^2} \right] dz \right]$$

$$= \lim_{\Delta a \to 0} \frac{1}{\Delta a} \frac{1!}{2\pi i} \left[\int_C f(z) \left[\frac{2(z-a)\Delta a - (\Delta a)^2}{\left(z - (a + \Delta a)\right)^2 (z-a)^2} \right] dz \right]$$

$$= \lim_{\Delta a \to 0} \frac{1!}{2\pi i} \left[\int_C f(z) \left[\frac{2(z-a) - \Delta a}{\left(z - (a + \Delta a)\right)^2 (z-a)^2} \right] dz \right]$$

$$= \frac{1!}{2\pi i} \left[\int_C f(z) \left[\frac{2(z-a)}{(z-a)^2 (z-a)^2} \right] dz \right]$$

$$f''(a) = \frac{2!}{2\pi i} \left[\int_C \left[\frac{f(z)}{(z-a)^3} \right] dz \right]$$

Similarly

$$f'''(a) = \frac{3!}{2\pi i}\left[\int_C \left[\frac{f(z)}{(z-a)^4}\right] dz\right] = \frac{3!}{2\pi i}\left[\int_C \left[\frac{f(z)}{(z-a)^{3+1}}\right] dz\right]$$

Thus

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

Problem: Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)} dz$ where C is the circle $|z| = 3$.

Solution: Given integral

$$\int_C \frac{e^{2z}}{(z-1)(z-2)} dz = \int_C e^{2z} \frac{1}{(z-1)(z-2)}$$

Let $f(z) = e^{2z}$ which is analytic within on the circle C center at $z = 0$ with radius 3.

The integrand has two singular points $z = 1,2$ lies inside the circle. So we use partial fraction to split the integral

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

Put $z = 1$ in the above equation, we get

$$1 = -A \text{ implies } A = -1$$

Put $z = 2$ in the above equation, we get

$$1 = B$$

Thus

$$\frac{1}{(z-1)(z-2)} = -\frac{1}{(z-1)} - \frac{1}{(z-2)}$$

Now

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}dz = \int_C \frac{e^{2z}}{(z-2)}dz - \int_C \frac{e^{2z}}{(z-1)}dz$$

Now consider

$$\int_C \frac{e^{2z}}{(z-2)}dz$$

Here $z = 2$ is the singular point inside the circle, by Cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

We get

$$\int_C \frac{e^{2z}}{(z-2)}dz = 2\pi i f(2) = 2\pi i e^{2(2)} = 2\pi i e^4$$

Now consider

$$\int_C \frac{e^{2z}}{(z-1)}dz$$

Here $z = 2$ is the singular point inside the circle, by Cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

We get

$$\int_C \frac{e^{2z}}{(z-1)}dz = 2\pi i f(1) = 2\pi i e^{2(1)} = 2\pi i e^2$$

Thus,

$$\int_C \frac{e^{2z}}{(z-1)(z-2)}dz = \int_C \frac{e^{2z}}{(z-2)}dz - \int_C \frac{e^{2z}}{(z-1)}dz = 2\pi i e^4 - 2\pi i e^2$$

$$= 2\pi i(e^4 - e^4)$$

Problem: Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-4)}dz$ where C is the circle $|z| = 2$

Solution: Given integral

$$\int_C \frac{e^{2z}}{(z-1)(z-4)}dz$$

Since $\int_C \frac{e^{2z}}{(z-1)(z-4)}$ has two singular points $z = 1,4$ but $z = 4$ lies outside the circle $|z| = 2$ so we write the integral as

$$\int_C \frac{e^{2z}}{(z-1)(z-4)}dz = \int_C \frac{\frac{e^{2z}}{(z-4)}}{(z-1)}dz$$

This is of the form $\int_C \frac{f(z)}{z-a}dz$, where $f)z) = \frac{e^{2z}}{(z-4)}$, $a = 1$ lies inside the circle, by Cauchy integral formula

$$\int_C \frac{e^{2z}}{(z-1)(z-4)}dz = \int_C \frac{\frac{e^{2z}}{(z-4)}}{(z-1)}dz = \int_C \frac{f(z)}{z-a}dz$$

$$= 2\pi i f(1)$$

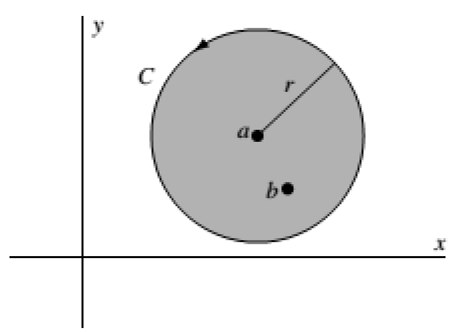
$$= 2\pi i \frac{e^{2(1)}}{(1-4)}$$

$$= -\frac{2}{3}\pi i e^2$$

Liouville's theorem

Suppose that for all z in the entire complex plane, (i) $f(z)$ is analytic and (ii) $f(z)$ is bounded, i.e., $|f(z)| < M$ for some constant M. Then $f(z)$ must be a constant.

Proof: Let a and b be any two points in the z plane. Suppose that C is a circle of radius r having center at a and enclosing point b.



From Cauchy's integral formula, we have

$$f(b) - f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-b}dz - \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

$$= \frac{b-a}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-b)(z-a)}$$

Now we have

$$|z-a| = r, \quad |z-b| = |z-a+a-b| \geq |z-a| - |a-b| = r - |a-b| \geq r/2$$

if we choose r so large that $|a-b| < r/2$. Then, since $|f(z)| < M$ and the length of C is $2\pi r$, That is

$$\oint_C |dz| = 2\pi r$$

we have by

$$|f(b) - f(a)| = \frac{|b-a|}{2\pi}\left|\oint_C \frac{f(z)\,dz}{(z-b)(z-a)}\right|$$

$$\leq \frac{|a-b|}{2\pi}\oint_C \frac{|f(z)|}{|z-b||z-a|}|dz|$$

$$= \frac{|a-b|}{2\pi}\frac{M}{\frac{r}{2}\,r}2\pi r$$

$$= \frac{|a-b|M}{r}$$

Letting $r \to \infty$, we see that

$$|f(b) - f(a)| = 0$$

$$or, f(b) = f(a)$$

which shows that $f(z)$ must be a constant.

Taylor's Theorem

Let $f(z)$ be analytic inside and on a simple closed curve C. Let a and $a+h$ be two points inside C. Then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^n}{n!}f^{(n)}(a)+\cdots$$

or writing $z = a+h, h = z-a$,

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(z-a)^n+\cdots$$

This is called Taylor's theorem and the series is called a Taylor series or expansion for $f(a+h)$ or $f(z)$.

Problem:

Expand e^z as Taylor's about $z=1$.

Solution : Given $f(z)=e^z$

We have to find the Taylor series expansion of $f(z)=e^z$ about $z=1$

Let $z-1=w$

$$\begin{aligned} f(z) &= e^z \\ &= e^{1+w} \\ &= e.e^w \\ &= e\left(1+\frac{w}{1!}+\frac{w^2}{2!}+\frac{w^3}{3!}+\cdots\ldots\ldots\ldots\ldots\right) \text{ for all } w \end{aligned}$$

Substitute $w=z-1$

$f(z)=e\left(1+\frac{z-1}{1!}+\frac{(z-1)^2}{2!}+\frac{(z-1)^3}{3!}+\cdots\ldots\ldots\ldots\ldots\right)$ for all $z-1$.

Problem:

Expand $f(z)=sinz$ in Taylor series about $z=\frac{\pi}{4}$

The Taylor series expansion of $f(z)$ at a point is given by

$$f(z) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \ldots \ldots \ldots \ldots$$

Here $f(z) = sinz,\ a = \frac{\pi}{4}$

$$sinz = f\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)}{1!}f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}f''\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^3}{3!}f'''\left(\frac{\pi}{4}\right) + \cdots \ldots \ldots \ldots \ldots$$

$f(z) = sinz,\ f\left(\frac{\pi}{4}\right) = sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(z) = cosz,\ f'\left(\frac{\pi}{4}\right) = cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f''(z) = -sinz,\ f''\left(\frac{\pi}{4}\right) = -sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f'''(z) = -cosz,\ f'''\left(\frac{\pi}{4}\right) = -cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$, and so on

Then

$$sinz = \frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)}{1!}\frac{1}{\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(x-\frac{\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots \ldots \ldots \ldots \ldots$$

Laurent's Theorem

Let C_1 and C_2 be concentric circles of radii R_1 and R_2, respectively, and center at a [Fig. 6-1]. Suppose that $f(z)$ is single-valued and analytic on C_1 and C_2 and, in the ring-shaped region $\mathcal{R}$ [also called the *annulus* or *annular region*] between C_1 and C_2, is shown shaded in Fig. 6-1. Let $a+h$ be any point in $\mathcal{R}$. Then we have

$$f(a+h) = a_0 + a_1h + a_2h^2 + \cdots + \frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \frac{a_{-3}}{h^3} + \cdots \quad (6.5)$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, 1, 2, \ldots \\ a_{-n} &= \frac{1}{2\pi i}\oint_{C_1} (z-a)^{n-1} f(z)\,dz \quad n = 1, 2, 3, \ldots \end{aligned} \quad (6.6)$$

C_1 and C_2 being traversed in the positive direction with respect to their interiors.

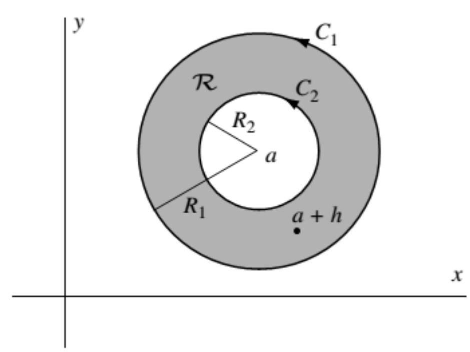


Fig. 6-1

In the above integrations, we can replace C_1 and C_2 by any concentric circle C between C_1 and C_2 [see Problem 6.100]. Then, the coefficients (6.6) can be written in a single formula,

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6.7}$$

With an appropriate change of notation, we can write the above as

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots \tag{6.8}$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6.9}$$

This is called Laurent's theorem and (6.5) or (6.8) with coefficients (6.6), (6.7), or (6.9) is called a Laurent series or expansion.

The part $a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$ is called the *analytic part* of the Laurent series, while the remainder of the series, which consists of inverse powers of $z-a$, is called the *principal part*. If the principal part is zero, the Laurent series reduces to a Taylor series.

Expand $\frac{1}{z^2-3z+2}$ in the region i) $0 < |z-1| < 1$ ii) $1 < |z| < 2$

Solution: let $f(z) = \frac{1}{z^2-3z+2}$

$$= \frac{1}{z^2-2z-z+2}$$

$$= \frac{1}{(z-1)(z-2)}$$

Consider $\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$

$$1 = A(z-2) + B(z-1)$$

Put $z = 1$ in the above equation

$$1 = -A, \ A = -1$$

Put $z = 2$ in the above equation

$$1 = B$$

Then

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

Now $f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$

Here the singular points of $f(z)$ are 1,2

i) The function $f(z)$ is analytic in the ring shaped region $0 < |z-1| < 1$

Put $z - 1 = w$ the region is $0 < |w| < 1$

Then $z = 1 + w$

$$f(z) = \frac{1}{(w-1)} - \frac{1}{w}$$

$$= \frac{-1}{(1-w)} - \frac{1}{w}$$

$$= -(1-w)^{-1} - \frac{1}{w}$$

$$= -(1 + w + w^2 + w^3 + \cdots \ldots) - \frac{1}{w} \quad \text{for } 0 < |w| < 1$$

$$= -(1 + (z-1) + (z-1)^2 + (z-1)^3 + \cdots \ldots) - \frac{1}{(z-1)}$$

The above series is valid for $0 < |(z-1)| < 1$

ii) The function $f(z)$ is analytic in the ring shaped region $1 < |z| < 2$

In the given region $1 < |z|$ implies $\left|\frac{1}{z}\right| < 1$ and $|z| < 2$ implies $\left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{-1}{(2-z)} - \frac{1}{(z-1)}$$

$$= \frac{-1}{2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots \ldots.\right) - \frac{1}{z}(1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots \ldots.)$$

Valid for $\left|\frac{z}{2}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$

$$= -\frac{1}{2}\left(\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) - \frac{1}{z}\left(\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right)$$

$$= -\sum_{n=0}^{\infty}\left(\frac{z^n}{2^{n+1}}\right) - \sum_{n=0}^{\infty}\left(\frac{1}{z^{n+1}}\right)$$

Singular Point: A point at which $f(z)$ fails to be analytic is called a singular point of $f(z)$. Example $f(z) = \frac{1}{z}$, It is not defined at $z = 0$. So $z = 0$ is the singular point of $f(z)$.

Isolated Singular Point: A singular point $z = z_0$ is said to be isolated singular point of $f(z)$ if there exists a neibourhood of z_0, in the neibourhood there is no another singular point.

Example: $f(z) = \frac{1}{(z-2)(z-3)}$ here $z = 2,3$ are isolated singular points

Poles. If $f(z)$ has the form in which the principal part has only a finite number of terms given by

$$\frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots + \frac{a_{-n}}{(z-a)^n}$$

where a

$a_{-n} \neq 0$, then $z = a$ is called a pole of order n. If $n = 1$, it is called a simple pole.

Removable singularities. If a single-valued function $f(z)$ is not defined at $z = a$ but $\lim_{z\to a} f(z)$ exists, then $z = a$ is called a *removable singularity*. In a such case, we define $f(z)$ at $z = a$ as equal to $\lim_{z\to a} f(z)$, and $f(z)$ will then be analytic at a.

If $f(z) = \sin z/z$, then $z = 0$ is a removable singularity since $f(0)$ is not defined but $\lim_{z\to 0} \sin z/z = 1$. We define $f(0) = \lim_{z\to 0} \sin z/z = 1$. Note that in this case

$$\frac{\sin z}{z} = \frac{1}{z}\left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right\} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

Essential singularities. If $f(z)$ is single-valued, then any singularity that is not a pole or removable singularity is called an essential singularity. If $z = a$ is an essential singularity of $f(z)$, the principal part of the Laurent expansion has infinitely many terms.

Since $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$, $z = 0$ is an essential singularity.

Residues

Let $f(z)$ be single-valued and analytic inside and on a circle C except at the point $z = a$ chosen as the center of C. Then $f(z)$ has a Laurent series about $z = a$ given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$= a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad n = 0, \pm 1, \pm 2, \ldots$$

In the special case $n = -1$, we have

$$\oint_C f(z)\, dz = 2\pi i a_{-1}$$

a_{-1} is call the residue of $f(z)$ at $z = a$.

Formula to calculate the residue of $f(z)$ at $= a$, where $z = a$ is a pole of order k, is

$$a_{-1} = \lim_{z \to a} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \{(z-a)^k f(z)\}$$

If $k = 1$ (simple pole), then the result is especially simple and is given by

$$a_{-1} = \lim_{z \to a} (z-a)\, f(z)$$

Problem: Find the residues of $f(z) = \frac{z}{(z-1)(z+1)^2}$

Solution:

To find the poles of $f(z)$ set

$$(z-1)(z+1)^2 = 0$$

$$(z-1) = 0 \text{ or } (z+1)^2 = 0$$

then $z = 1$ is a pole of order 1 and $z = -1$ is a pole of order 2.

Residue at $z = 1$ is

$$\lim_{z \to 1} \frac{1}{(1-1)!} \frac{d^{1-1}}{dz^{1-1}} \left\{ (z-1)^1 \frac{z}{(z-1)(z+1)^2} \right\}$$

$$= \lim_{z \to 1} \frac{z}{(z+1)^2} = \frac{1}{2^2} = \frac{1}{4}$$

Residue at $z = -1$ is

$$\lim_{z\to -1} \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left\{ (z-(-1))^2 \frac{z}{(z-1)(z+1)^2} \right\}$$

$$= \lim_{z\to -1} \frac{1}{1!} \frac{d}{dz} \left\{ (z+1))^2 \frac{z}{(z-1)(z+1)^2} \right\}$$

$$= \lim_{z\to -1} \frac{d}{dz} \left\{ \frac{z}{(z-1)} \right\}$$

$$= \lim_{z\to -1} \left\{ \frac{(z-1)\frac{d}{dz} z - z \frac{d}{dz}(z-1)}{(z-1)^2} \right\}$$

$$= \lim_{z\to -1} \left\{ \frac{(z-1) - z}{(z-1)^2} \right\}$$

$$= \lim_{z\to -1} \left\{ \frac{-1}{((-1)-1)^2} \right\}$$

$$= \lim_{z\to -1} \left\{ \frac{-1}{(-2)^2} \right\}$$

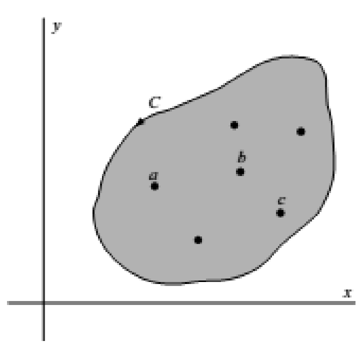
$$= -\frac{1}{4}$$

The Residue Theorem

Let $f(z)$ be single-valued and analytic inside and on a simple closed curve C except at the singularities $a, b, c, \ldots$ inside C, which have residues given by $a_{-1}, b_{-1}, c_{-1}, \ldots$ Then, the residue theorem states that

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

That is, the integral of $f(z)$ around C is $2\pi i$ times the sum of the residues of $f(z)$ at the singularities enclosed by C.



Prove the *residue theorem*. If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of points $a, b, c, \ldots$ inside C at which the residues are $a_{-1}, b_{-1}, c_{-1}, \ldots,$ respectively, then

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots)$$

i.e., $2\pi i$ times the sum of the residues at all singularities enclosed by C.

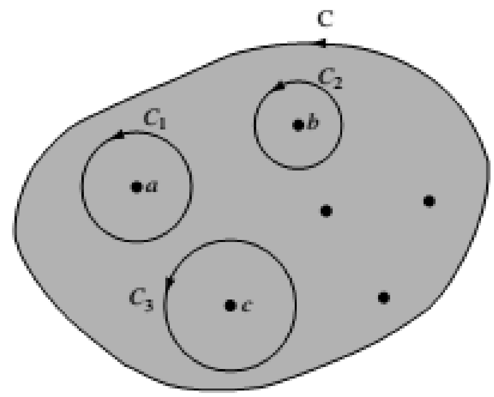


Fig. 7-4

Solution

With centers at $a, b, c, \ldots,$ respectively, construct circles $C_1, C_2, C_3, \ldots$ that lie entirely inside C as shown in Fig. 7-4. This can be done since $a, b, c, \ldots$ are interior points. By Theorem 4.5, page 118, we have

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \cdots \qquad (1)$$

But, by Problem 7.1,

$$\oint_{C_1} f(z)\,dz = 2\pi i a_{-1}, \qquad \oint_{C_2} f(z)\,dz = 2\pi i b_{-1}, \qquad \oint_{C_3} f(z)\,dz = 2\pi i c_{-1}, \ldots \qquad (2)$$

Then, from (1) and (2), we have, as required,

$$\oint_C f(z)\,dz = 2\pi i(a_{-1} + b_{-1} + c_{-1} + \cdots) = 2\pi i \text{ (sum of residues)}$$