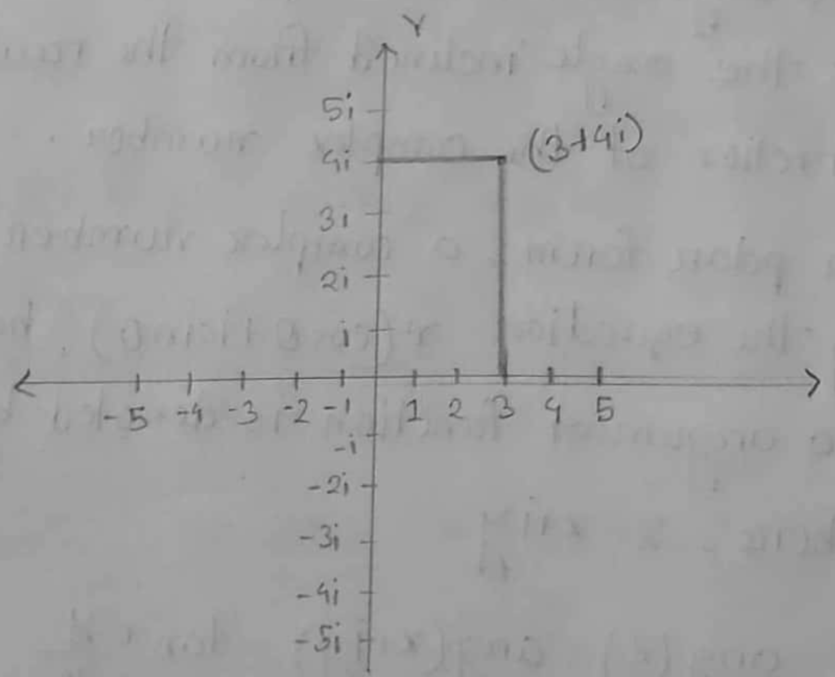
1. Complex number and its properties:

A complex number is a number that can be expre-ssed in the form $a+ib$, where a is the real number and ib is the imaginary number. Also a, b belongs to real number & $i=\sqrt{-1}$.

Graphical representation:

In the graph bellow, check the representation of complex number along the axes, where the x axes represents real part and Y axis represents the imaginary part. Let's consider a complex number $3+4i$, then



Mod or absolute value:

The absolute value of a real number is itself.
But in case of complex number $z = x+iy$, then the mod of z will be, $|z| = \sqrt{x^2+y^2}$

Conjugate:

let, a complex number $z = x+iy$. The conjugate of z is denoted by $\bar{z}$. mathematically,

$$\bar{z} = x - iy$$

Argument:

The argument of a complex number is defined as the angle inclined from the real axis in direction of the complex number.

In polar form, a complex number is represent by the equation $r(\cos\theta + i\sin\theta)$, here, θ is the argument function is denoted by $\arg(z)$, where, $z = x+iy$

$$\arg(z) = \arg(x+iy) = \tan^{-1}\frac{y}{x}$$

* prove

(i) $|z_1+z_2| \leq |z_1| + |z_2|$

Solⁿ: Analytically, let

$$z_1 = x_1 + iy_1$$
$$z_2 = x_2 + iy_2$$

then we must show that

$$\sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} \leq \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$$

squaring both of side

$$(x_1+x_2)^2 + (y_1+y_2)^2 \leq x_1^2+y_1^2 + 2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2} + x_2^2+y_2^2$$

$$\Rightarrow x_1x_2 + y_1y_2 \leq \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

$$\Rightarrow x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 \leq x_1^2x_2^2 + x_2^2y_1^2 + x_1^2y_2^2 + y_1^2y_2^2$$

[squaring both side]

$$\Rightarrow 2x_1x_2y_1y_2 \leq x_1^2y_2^2 + y_1^2x_2^2$$

$$\Rightarrow (x_1y_2 - x_2y_1)^2 \geq 0$$

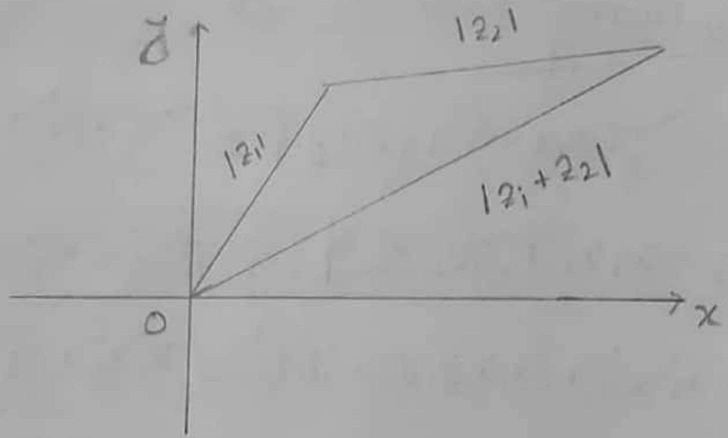
which is true

Reversing the steps which are reversible proves the result.

Graphically:

The result follows graphically from the fact that

The results follows graphically from the fact that $|z_1|$, $|z_2|$, $|z_1+z_2|$ represent the length of the sides of a triangle and that the sum of the length of two sides of a triangle is greater than or equal to the length of the third side



(ii) $|z_1 - z_2| \geq |z_1| - |z_2|$

Analytically by part (1)

$$|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$$

then

$$|z_1 - z_2| \geq |z_1| - |z_2|$$

An equivalent result obtain on replacing z_2 by $-z_2$ is $|z_1 - z_2| \geq |z_1| - |z_2|$

Graphically the result is equivalent to the statement that a side of a triangle has length greater than or equal to the difference in the length of the other two sides.

2. Limit of complex function:

Let $f(z)$ be a complex function and let z_0 be an accumulation point of the domain A of $f(z)$. The Limit of $f(z)$ a z approaches z_0 is L denoted

$$\lim_{z \to z_0} f(z) = L$$

if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$z \in A$$

$$|z - z_0| < \delta$$

then

$$|f(z) - L| < \varepsilon$$

continuity:

Let $f(z)$ be defined and single valued in neighborhood of $z = z_0$ as well as at $z = z_0$ (i.e. in a δ neighborhood of z_0)

The function $f(z)$ is said to be continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$

Note that this implies three conditions that must be met in order that $f(z)$ be continuous at $z = z_0$

1. $\lim_{z \to z_0} f(z) = l$ must be exist

2. $f(z_0)$ must exist, i.e. $f(z)$ is defined at z_0

3. $l = f(z_0)$

Equivalent, if $f(z)$ is continuous at z_0, we can write this in the suggestive from

$$\lim_{z \to z_0} f(z) = f\left(\lim_{z \to z_0} z\right)$$

Analyticity:

If the derivative $f'(z)$ exist at all points z of a region R, then $f(z)$ is said to be analytic in R and is Refered to as an analytic function in R or a function analytic in R. The terms regular and holomorphic are some times used as synonyms for analytic

A function is called analytic at a point z_0 if there exist a neighborhood $|z_1 - z_0| < \delta$ at all points at which $f'(z)$ exists.

3. Cauchy- Riemann Equation:

A necessary condition that $w = f(z)$

$$f(z) = u(x,y) + iv(x,y)$$

be analytic in a region R is that in R, u and v satisfy the cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

CAUCHY'S Integral Formula:

If $f(z)$ is analytic function within & on a closed contour C, and if a is any point within c, then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} \, dz$$

CAUCHY'S Integral theorem:

If $f(z)$ is analytic at every point within & on a closed curve c, then:

$$\int_C f(z) \, dz = 0$$

9.

Singularity:

A point at which $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$. various types of singularities exist.

1. Isolated Singularity: The point $z = z_0$ is called isoloted singularity point of $f(z)$

if we find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0

2. Pole: If z_0 is an isoloted singularity and we can find a positive integer n such that

$$\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0,$$ then $z = z_0$ is called a pole of order n.

3. Removal singularity: An isoloted singular point z_0 is called a removable singularity of $f(z)$ if

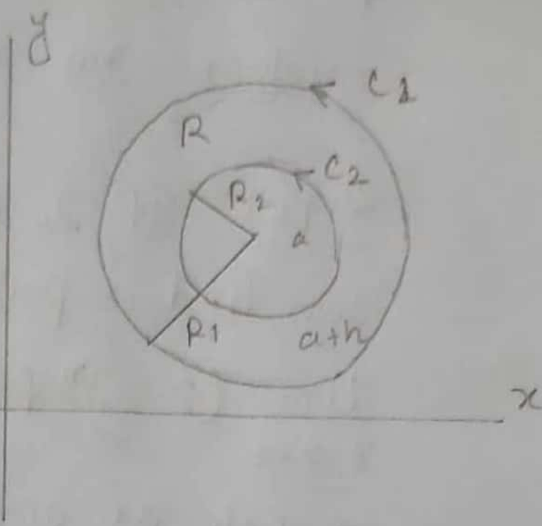
$$\lim_{z \to z_0} f(z) \text{ exists.}$$

4. Essential singularity: An isoloted singularity that is not pole or removable singularity is called an essential singularity

Ex. $f(z) = e^{\frac{1}{z-2}}$ has a essential singularity at $z = 2$

Laurent's theorem:

let C_1 and C_2 be concentric circles of radii R_1 & R_2 respectively and center at a. Suppose that $f(z)$ is single valued and analytic on C_1 and C_2



and in the ring shaped region R [also called the annulus for annular region] between C_1 and C_2 is shown shaded in Fig. let a+h be any point in R. Then we have

$$f(a+h) = a_0 + a_1 h + a_2 h^2 + \dots \frac{a_{-1}}{h} + \frac{a_{-2}}{h^2} + \frac{a_{-3}}{h^3} + \dots$$

where,

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz \quad n = 0, 1, 2 \dots$$

$$a_{-n} = \frac{1}{2\pi i} \oint_{C_1} (z-a)^{n-1} f(z)\, dz \quad n = 1, 2, 3 \dots$$

C_1 and C_2 being traversed in the positive direction with respect to their interiors. In the above integration, we can replace C_1 and by any concentric cycle c between C_1 and C_2

Then, the coefficients can be written in a single formula,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \quad n = 0, \pm 1, \pm 2 \cdots$$

With an appropriate change of notation, we can write the above as

$$f(z) = a_0 + a_1 (z-a) + a_2 (z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}} dz \quad n = 0, \pm 1, \pm 2 \cdots$$

This coefficient is called Laurent series or expansion.