Vector Analysis: Previous Year Questions and Solve
Year-2020

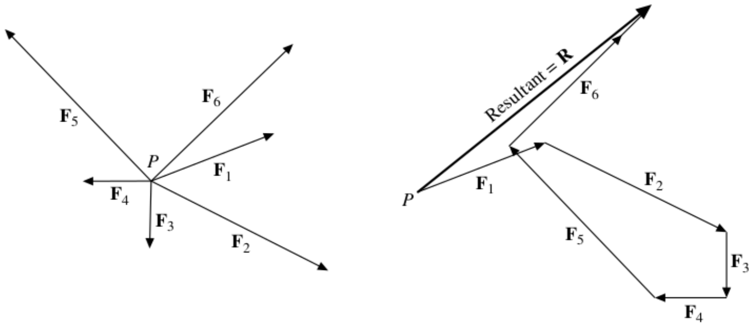
4.(a)

Forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_6$ act on an object P as shown in Fig. 1-10(a). Find the force that is needed to prevent P from moving.

Solution

Since the order of addition of vectors is immaterial, we may start with any vector, say $\mathbf{F}_1$. To $\mathbf{F}_1$ add $\mathbf{F}_2$, then $\mathbf{F}_3$, and so on as pictured in Fig. 1-10(b). The vector drawn from the initial point of $\mathbf{F}_1$ to the terminal point of $\mathbf{F}_6$ is the resultant $\mathbf{R}$, that is, $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_6$.

The force needed to prevent P from moving is $-\mathbf{R}$, sometimes called the *equilibrant*.



4.(b)

Prove that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$.

Solution

By Problem 2.37, $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$.

By a theorem of determinants which states that interchange of two rows of a determinant changes its sign, we have

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$$

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

4.(c)

Prove: $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) = (\mathbf{A} \cdot \mathbf{B} \times \mathbf{C})^2$.

Solution

By Problem 2.47(a), $\mathbf{X} \times (\mathbf{C} \times \mathbf{A}) = \mathbf{C}(\mathbf{X} \cdot \mathbf{A}) - \mathbf{A}(\mathbf{X} \cdot \mathbf{C})$. Let $\mathbf{X} = \mathbf{B} \times \mathbf{C}$; then

$$\begin{aligned} (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) &= \mathbf{C}(\mathbf{B} \times \mathbf{C} \cdot \mathbf{A}) - \mathbf{A}(\mathbf{B} \times \mathbf{C} \cdot \mathbf{C}) \\ &= \mathbf{C}(\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C} \times \mathbf{C}) \\ &= \mathbf{C}(\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}) \end{aligned}$$

Thus

$$\begin{aligned} (\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) &= (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}(\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}) \\ &= (\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}) \\ &= (\mathbf{A} \cdot \mathbf{B} \times \mathbf{C})^2 \end{aligned}$$

5.(a)

A particle moves so that its position vector is given by $\mathbf{r} = \cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}$ where ω is a constant. Show that (a) the velocity $\mathbf{v}$ of the particle is perpendicular to $\mathbf{r}$, (b) the acceleration $\mathbf{a}$ is directed toward the origin and has magnitude proportional to the distance from the origin, (c) $\mathbf{r}\times\mathbf{v} =$ a constant vector.

Solution

(a) $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -\omega\sin\omega t\mathbf{i} + \omega\cos\omega t\mathbf{j}$. Then

$$\begin{aligned}\mathbf{r}\cdot\mathbf{v} &= [\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}]\cdot[-\omega\sin\omega t\mathbf{i} + \omega\cos\omega t\mathbf{j}]\\ &= (\cos\omega t)(-\omega\sin\omega t) + (\sin\omega t)(\omega\cos\omega t) = 0\end{aligned}$$

and $\mathbf{r}$ and $\mathbf{v}$ are perpendicular.

(b) $$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt} &= -\omega^2\cos\omega t\mathbf{i} - \omega^2\sin\omega t\mathbf{j}\\ &= -\omega^2[\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}] = -\omega^2\mathbf{r}\end{aligned}$$

Then the acceleration is opposite to the direction of $\mathbf{r}$, that is, it is directed toward the origin. Its magnitude is proportional to $|\mathbf{r}|$, which is the distance from the origin.

(c) $$\begin{aligned}\mathbf{r}\times\mathbf{v} &= [\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}]\times[-\omega\sin\omega t\mathbf{i} + \omega\cos\omega t\mathbf{j}]\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\omega t & \sin\omega t & 0\\ -\omega\sin\omega t & \omega\cos\omega t & 0\end{vmatrix}\\ &= \omega(\cos^2\omega t + \sin^2\omega t)\mathbf{k}\\ &= \omega\mathbf{k},\ \text{a constant vector.}\end{aligned}$$

Physically, the motion is that of a particle moving on the circumference of a circle with constant angular speed ω. The acceleration, directed toward the center of the circle, is the *centripetal acceleration*.

5.(b)

Suppose $\mathbf{A}$ has constant magnitude. Show that $A \cdot dA/dt = 0$ and that $\mathbf{A}$ and $d\mathbf{A}/dt$ are perpendicular provided $|d\mathbf{A}/dt| \neq 0$.

Solution

Since $\mathbf{A}$ has constant magnitude, $\mathbf{A}\cdot\mathbf{A} =$ constant.

Then $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{A}) = \mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} + \dfrac{d\mathbf{A}}{dt}\cdot\mathbf{A} = 2\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} = 0.$

Thus $\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt} = 0$ and $\mathbf{A}$ is perpendicular to $\dfrac{d\mathbf{A}}{dt}$ provided $\left|\dfrac{d\mathbf{A}}{dt}\right| \neq 0.$

6.(a)

Let $\phi = x^2yz - 4xyz^2$. Find the directional derivative of ϕ at $P(1, 3, 1)$ in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Solution

First find $\nabla\phi = (2xyz - 4yz^2)\mathbf{i} + (x^2z - 4xz^2)\mathbf{j} + (x^2y - 8xyz)\mathbf{k}$. Then $\nabla\phi(1, 3, 1) = -6\mathbf{i} - 3\mathbf{j} - 21\mathbf{k}$. The unit vector in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is

$$\mathbf{a} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}.$$

Thus the required directional derivative is

$$\nabla\phi(1, 3, 1)\cdot\mathbf{a} = (-6\mathbf{i} - 3\mathbf{j} - 21\mathbf{k})\cdot\left(\tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}\right) = -4 + 1 + 14 = 11.$$

6.(b) Divergence:

Suppose $\mathbf{V}(x, y, z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ is defined and differentiable at each point (x, y, z) in a region of space. (That is, $\mathbf{V}$ defines a differentiable vector field.) Then the *divergence* of $\mathbf{V}$, written $\nabla\cdot\mathbf{V}$ or div $\mathbf{V}$ is defined as follows:

$$\nabla\cdot\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$
$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Given $\phi = 6x^3y^2z$. (a) Find $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi$ (or div grad ϕ).

(b) Show that $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}^2\phi$ where $\boldsymbol{\nabla}^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ denotes the Laplacian operator.

Solution

(a) $\boldsymbol{\nabla}\phi = \dfrac{\partial}{\partial x}(6x^3y^2z)\mathbf{i} + \dfrac{\partial}{\partial y}(6x^3y^2z)\mathbf{j} + \dfrac{\partial}{\partial z}(6x^3y^2z)\mathbf{k} = 18x^2y^2z\mathbf{i} + 12x^3yz\mathbf{j} + 6x^3y^2\mathbf{k}.$

Then
$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(18x^2y^2z\mathbf{i} + 12x^3yz\mathbf{j} + 6x^3y^2\mathbf{k})$$
$$= \frac{\partial}{\partial x}(18x^2y^2z) + \frac{\partial}{\partial y}(12x^3yz) + \frac{\partial}{\partial z}(6x^3y^2) = 36xy^2z + 12x^3z.$$

6.(c) Curl:

Suppose $\mathbf{V}(x, y, z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ is a differentiable vector field. Then the *curl* or *rotation* of $\mathbf{V}$, written $\nabla\times\mathbf{V}$, curl $\mathbf{V}$ or rot $\mathbf{V}$, is defined as follows:

$$\nabla\times\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\times(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$
$$= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix}\mathbf{k}$$
$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\mathbf{k}$$

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}\cdot\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$
$$= \boldsymbol{\nabla}\cdot\left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}\right]$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$
$$= \frac{\partial^2 A_3}{\partial x\partial y} - \frac{\partial^2 A_2}{\partial x\partial z} + \frac{\partial^2 A_1}{\partial y\partial z} - \frac{\partial^2 A_3}{\partial y\partial x} + \frac{\partial^2 A_2}{\partial z\partial x} - \frac{\partial^2 A_1}{\partial z\partial y} = 0$$

assuming that $\mathbf{A}$ has continuous second partial derivatives.

Note the similarity between the above results and the results $(\mathbf{C}\times\mathbf{C}m) = (\mathbf{C}\times\mathbf{C})m = \mathbf{0}$, where m is a scalar and $\mathbf{C}\cdot(\mathbf{C}\times\mathbf{A}) = (\mathbf{C}\times\mathbf{C})\cdot\mathbf{A} = \mathbf{0}$.

<u>*Year-2021*</u>

2.(a) Dot Product:

The dot or scalar product of two vectors **A** and **B**, denoted by $\mathbf{A} \cdot \mathbf{B}$ (read: **A** dot **B**), is defined as the product of the magnitudes of **A** and **B** and the cosine of the angle θ between them. In symbols,

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos \theta, \qquad 0 \leq \theta \leq \pi$$

Cross Product:

The cross product of vectors **A** and **B** is a vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ (read: **A** cross **B**) defined as follows. The magnitude of $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is equal to the product of the magnitudes of **A** and **B** and the sine of the angle θ between them. The direction of $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is perpendicular to the plane of **A** and **B** so that **A**, **B**, and **C** form a right-handed system. In symbols,

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin \theta \, \mathbf{u} \qquad 0 \leq \theta \leq \pi$$

where **u** is a unit vector indicating the direction of $\mathbf{A} \times \mathbf{B}$. [Thus **A**, **B**, and **u** form a right-handed system.]

Solution

Since $\mathbf{PQ} = \mathbf{B} - \mathbf{r}$ is perpendicular to **A**, we have $(\mathbf{B} - \mathbf{r}) \cdot \mathbf{A} = 0$ or $\mathbf{r} \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{A}$ is the required equation of the plane in vector form. In rectangular form this becomes

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$$

or

$$2x - 3y + 6z = 2 - 6 + 18 = 14$$

2.(b)

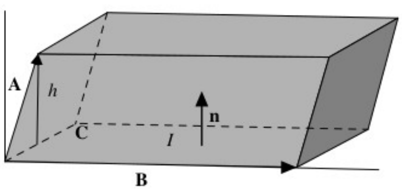
Show that the absolute value of the triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is the volume of a parallelepiped with sides **A**, **B**, and **C**.

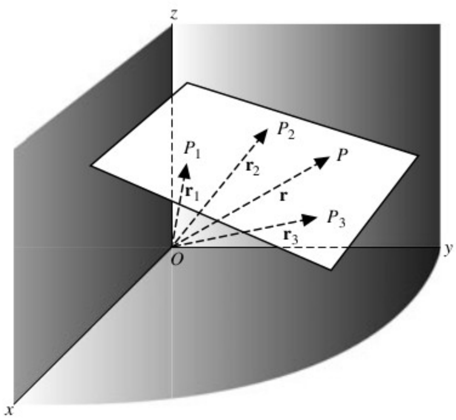
Solution

Let **n** be a unit normal to a parallelogram I, having the direction of $\mathbf{B} \times \mathbf{C}$, and let h be the height of the terminal point of **A** above the parallelogram I. [See Fig. 2-16.]

$$\begin{aligned} \text{Volume of parallelepiped} &= (\text{height } h)(\text{area of parallelogram } I) \\ &= (\mathbf{A} \cdot \mathbf{n})(|\mathbf{B} \times \mathbf{C}|) \\ &= \mathbf{A} \cdot \{|\mathbf{B} \times \mathbf{C}|\mathbf{n}\} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \end{aligned}$$

If **A**, **B**, and **C** do not form a right-handed system, $\mathbf{A} \cdot \mathbf{n} < 0$ and the volume $= |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$.





2.(c)

A particle moves along the curve $x = 2t^2$, $\quad y = t^2 - 4t$, $\quad z = -t - 5$ where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

Solution

$$\text{Velocity} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}[(2t^2)\mathbf{i} + (t^2 - 4t)\mathbf{j} + (-t-5)\mathbf{k}]$$
$$= (4t)\mathbf{i} + (2t-4)\mathbf{j} - \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k} \quad \text{at } t = 1.$$

$$\text{Unit vector in direction } \mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \text{ is } \frac{\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}}{\sqrt{(1)^2 + (-2)^2 + (2)^2}} = \tfrac{1}{3}\mathbf{i} - \tfrac{2}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}.$$

Then the component in the given direction is $(4\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\tfrac{1}{3}\mathbf{i} - \tfrac{2}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}\right) = 2$

$$\text{Acceleration} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}[(4t)\mathbf{i} + (2t-4)\mathbf{j} - \mathbf{k}] = 4\mathbf{i} + 2\mathbf{j}.$$

Then the component of the acceleration in the given direction is $(4\mathbf{i} + 2\mathbf{j}) \cdot \left(\tfrac{1}{3}\mathbf{i} - \tfrac{2}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}\right) = 0.$

4.(a)

Let $\phi(x, y, z)$ be a scalar function defined and differentiable at each point (x, y, z) in a certain region of space. [That is, ϕ defines a differentiable scalar field.] Then the gradient of ϕ, written $\nabla\phi$ or grad ϕ is defined as follows:

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

Note that $\nabla\phi$ defines a vector field.

Find the angle between the surfaces $z = x^2 + y^2$ and $z = \left(x - \frac{\sqrt{6}}{6}\right)^2 + \left(y - \frac{\sqrt{6}}{6}\right)^2$ at the point $P = \left(\frac{\sqrt{6}}{12}, \frac{\sqrt{6}}{12}, \frac{1}{12}\right)$.

Solution

The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

Let $\phi_1 = x^2 + y^2 - z$ and $\phi_2 = \left(x - \frac{\sqrt{6}}{6}\right)^2 + \left(y - \frac{\sqrt{6}}{6}\right)^2 - z$.

A normal to $z = x^2 + y^2$ is

$$\nabla\phi_1 = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \quad \text{and} \quad \nabla\phi_1(P) = \frac{\sqrt{6}}{6}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} - \mathbf{k}.$$

A normal to $z = \left(x - \frac{\sqrt{6}}{6}\right)^2 + \left(y - \frac{\sqrt{6}}{6}\right)^2$ is

$$\nabla\phi_2 = 2\left(x - \frac{\sqrt{6}}{6}\right)\mathbf{i} + 2\left(y - \frac{\sqrt{6}}{6}\right)\mathbf{j} - \mathbf{k} \quad \text{and} \quad \nabla\phi_2(P) = -\frac{\sqrt{6}}{6}\mathbf{i} - \frac{\sqrt{6}}{6}\mathbf{j} - \mathbf{k}.$$

Now $(\nabla\phi_1(P)) \cdot (\nabla\phi_2(P)) = |\nabla\phi_1(P)||\nabla\phi_2(P)|\cos\theta$ where θ is the required angle.

$$\left(\frac{\sqrt{6}}{6}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} - \mathbf{k}\right) \cdot \left(-\frac{\sqrt{6}}{6}\mathbf{i} - \frac{\sqrt{6}}{6}\mathbf{j} - \mathbf{k}\right) = \left|\frac{\sqrt{6}}{6}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} - \mathbf{k}\right|\left|-\frac{\sqrt{6}}{6}\mathbf{i} - \frac{\sqrt{6}}{6}\mathbf{j} - \mathbf{k}\right|\cos\theta$$

$$-\frac{1}{6} - \frac{1}{6} + 1 = \sqrt{\frac{1}{6} + \frac{1}{6} + 1}\sqrt{\frac{1}{6} + \frac{1}{6} + 1}\cos\theta \quad \text{and} \quad \cos\theta = \frac{2/3}{4/3} = \frac{1}{2}.$$

Thus the acute angle is $\theta = \arccos\left(\frac{1}{2}\right) = 60°$.

4.(b)

Let $\phi = x^2yz - 4xyz^2$. Find the directional derivative of ϕ at $P(1, 3, 1)$ in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

Solution

First find $\nabla\phi = (2xyz - 4yz^2)\mathbf{i} + (x^2z - 4xz^2)\mathbf{j} + (x^2y - 8xyz)\mathbf{k}$. Then $\nabla\phi(1, 3, 1) = -6\mathbf{i} - 3\mathbf{j} - 21\mathbf{k}$. The unit vector in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is

$$\mathbf{a} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}.$$

Thus the required directional derivative is

$$\nabla\phi(1, 3, 1) \cdot \mathbf{a} = (-6\mathbf{i} - 3\mathbf{j} - 21\mathbf{k}) \cdot \left(\tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}\right) = -4 + 1 + 14 = 11.$$

4.(c) Divergence:

Suppose $\mathbf{V}(x, y, z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ is defined and differentiable at each point (x, y, z) in a region of space. (That is, $\mathbf{V}$ defines a differentiable vector field.) Then the *divergence* of $\mathbf{V}$, written $\nabla \cdot \mathbf{V}$ or div $\mathbf{V}$ is defined as follows:

$$\begin{aligned}\nabla \cdot \mathbf{V} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) \\ &= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\end{aligned}$$

Given $\phi = 6x^3y^2z$. (a) Find $\nabla \cdot \nabla\phi$ (or div grad ϕ).

(b) Show that $\nabla \cdot \nabla\phi = \nabla^2\phi$ where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ denotes the Laplacian operator.

Solution

(a) $\nabla\phi = \dfrac{\partial}{\partial x}(6x^3y^2z)\mathbf{i} + \dfrac{\partial}{\partial y}(6x^3y^2z)\mathbf{j} + \dfrac{\partial}{\partial z}(6x^3y^2z)\mathbf{k} = 18x^2y^2z\mathbf{i} + 12x^3yz\mathbf{j} + 6x^3y^2\mathbf{k}.$

Then

$$\begin{aligned}\nabla \cdot \nabla\phi &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (18x^2y^2z\mathbf{i} + 12x^3yz\mathbf{j} + 6x^3y^2\mathbf{k}) \\ &= \frac{\partial}{\partial x}(18x^2y^2z) + \frac{\partial}{\partial y}(12x^3yz) + \frac{\partial}{\partial z}(6x^3y^2) = 36xy^2z + 12x^3z.\end{aligned}$$