Discrete Mathematics

Rafy

CT-01

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Logic

Logic is the science of reasoning that helps us understand and reason about different mathematical statements.

Proposition

Proposition is a statement that can be either true or false but can not be both.

Example:

1) Yondea is a student
2) $2+2=3$

Not proposition

1) $x+1=2$
2) What time is it ?
3) Do it.
4) I request you to please allow me a day off

Propositional variables & Logic

Propositional logic is an area of logic that studies ways of joining and modifying propositions to form more complicated propositions and it studies the logical relationships and properties derived from these combined propositions.

Adam is good in playing football and this time he is representing his college.

logical connective

Logical connectives connects two or more logical statements.

I enjoy watching TV.

It is not the case that I enjoy watching TV.

Negation

Propositional variables

Variables that are used to represent propositions which holds only two values (truth and false) are called propositional variables.

Logical Operators

There are six logical operators.

1. Negation
2. Conjunction
3. Disjunction
4. Exclusive OR
5. Implication
6. Biconditional

Negation

Let p be a proposition. $\neg p$ is called negation of p which simply states that "It is not the case that p". if p is true than $\neg p$ is false, if p is false than $\neg p$ is true.

Truth table:

p	$\neg p$
T	F
F	T

Example:

Adam and Eve lived together for many years. (Negation)
— It is not the case that

Adam and Eve lived together for many years.

Or,

Adam and Eve haven't lived together for many years.

Conjuntion:

Let p and q be two proposition conjunction of p and q is denoted by $p \wedge q$ when both p and q are true then $p \wedge q$ is true.

Truth Table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example:

12 is divisible by 3 **and** 3 is a prime number.

N.B

Sometimes **"but"** is used instead of and.

72 is a non-prime but it is divisible by 2.

Disjunction

Let p and q be two propositions. Disjunction of p and q is denoted by $p \vee q$. When both p and q are false then only the compound proposition $p \vee q$ is false.

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example:

16 - 4 = 10 **or** 4 is an even number.

Exclusive OR

Let p and q be two proposition The exclusive OR of p and q is denoted by $p \oplus q$. is a proposition that simply means exactly one of p and q will be true but both can't be true.

Truth Table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Example:

Coffee or Tea comes with dinner. but not both.

N.B: Exclusive OR can be written as:

$p \oplus q = (p \vee q) \wedge \neg(p \wedge q)$

Conditional Statement or Implication

Let p and q be two propositions. The conditional statement $p \rightarrow q$ is the proposition "if p, then q". The conditional statement $p \rightarrow q$ is true when p is true and q is false, and true otherwise.

$p \rightarrow q$ here p is called hypotheses or antecedent or premise. q is called conclusion or consequence.

N.B.
A conditional statement is also called an implication.

Example:

If today is friday then today is my birthday!

N.B: $p \longrightarrow q = \neg p \vee q$

$p \rightarrow q$ means

if p, then q
if p, q
p is sufficient for q
a necessary condition for p is q.
p implies q
p only if q
q if p
q when p
q unless $\neg p$
q whenver p
q is necessary for p
q follows from p
a sufficient condition for q is p.

Truth Table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Converse, contrapositive, Inverse

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$q \to p$	$\neg q \to \neg p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

conditional

inverse

converse

contrapositive

same truth value

Example:

The home team wins, whenever it is raining.

Let q = "It is raining"

and p = " The home team wins".

Thus it froms a conditional $(p \to q)$ statement.

Inverse of this conditional is,

If it is raining, then the home team wins.

Inverse:

If it is not raining then the home team does not win.

Converse

If the home team wins, then it is raining.

Contrapositive

If the home team does not win then the it is not raining.

Biconditional

Let p and q be two propositions The biconditional statement denoted as $p \leftrightarrow q$ is the proposition "p if and only if q". The biconditional statement is true when p and q have the same truth values, and is false otherwise

$p \leftrightarrow q$ means

p if and only if q
p is necessary and sufficient for q
if p then q and conversely
p iff q

Truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

NB $p \leftrightarrow q = (p \to q) \wedge (q \to p)$

It has the opposite truth value of exclusive OR.

$p \leftrightarrow q = \neg (p \oplus q)$

Precedence of Logical Operators

Precedence Order

$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \longleftrightarrow$

Propositional equivalance

A logical equivalence that indicates the two sides always have the same truth values.

- It is denoted by $\equiv$ or $\Leftrightarrow$

Compound Proposition

Compound proposition refers to an expression formed from propositional variables using logical operators.

Tautology

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.

For example, $p \vee \neg p$ will always be true

Contradiction

A compound proposition that is always false is called a contradiction.

For example, $p \wedge \neg p$ is a contradiction.

Contingency

A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Logical equivalence

The compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

De Morgan's Law

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

*Show that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Truth table for $\neg(p \wedge q)$ and $\neg p \vee \neg q$.

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

The truth tables of compound propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ agree for all possible combination of the truth values of p and q. it follows $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$ tautology so these compound propositions are logically equivalent.

Identity laws

$p \wedge T \equiv p$

$p \vee F \equiv p$

Domination Laws

$p \vee T \equiv T$

$p \wedge F \equiv F$

Idempotent Laws

$p \vee p \equiv p$

$p \wedge p \equiv p$

Double Negation Laws

$\neg(\neg p) \equiv p$

Commutative Laws

$p \wedge q \equiv q \wedge p$

$p \vee q \equiv q \vee p$

Associative Laws

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Distributive Laws

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Absorption Laws

$p \vee (p \wedge q) \equiv p$

$p \wedge (p \vee q) \equiv p$

Negation Laws

$p \vee \neg p \equiv T$

$p \wedge \neg p \equiv F$

Definition of implication

$p \rightarrow q \equiv \neg p \vee q$

Definition of Biconditional

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg(p \oplus q)$

Proof:

$(p \rightarrow q) \wedge (q \rightarrow p)$

$\equiv (\neg p \vee q) \wedge (\neg q \vee p)$ [Definition of implication]

$\equiv \neg p (\neg q \vee p) \vee q (\neg q \vee p)$ [Distributive Law]

$\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge \neg q) \vee (q \wedge p)$ [Distributive Law]

$\equiv (\neg p \wedge \neg q) \vee F \vee F \vee (q \wedge p)$ [Negation Law]

$\equiv (\neg p \wedge \neg q) \vee (q \wedge p)$ [identity Law]

$\equiv \neg(p \vee q) \vee (q \wedge p)$ [De morgan]

$\equiv (p \wedge q) \vee \neg(p \vee q)$ [Commutative]

$\equiv p \oplus q$

Logical Equivalences involving conditional statements

1. $p \to q \equiv \neg p \vee q$
2. $p \to q \equiv \neg q \to \neg p$
3. $p \vee q \equiv \neg p \to q$
4. $p \wedge q \equiv \neg(p \to \neg q)$
5. $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
6. $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
7. $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
8. $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Logical equivalence involving Biconditional

1. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
2. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
3. $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
4. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Q.

At a trial —

Bill says: "Sue is guilty and Fred is innocent".

Sue says: "If Bill is guilty, then so is Fred".

Fred says: "I am innocent, but at least one of the others is guilty".

Let b = Bill is innocent.
s = Sue is innocent.
f = Fred is innocent.

Statements are:

- $\neg s \wedge f$
- $\neg b \to \neg f$
- $f \wedge (\neg b \vee \neg s)$

To all of their statements to be true their conjunction has to be Tautology.

Tenaxit

$(\neg s \wedge f) \wedge (\neg b \rightarrow \neg f) \wedge f \wedge (\neg b \vee \neg s)$

$\equiv (\neg s \wedge f) \wedge f \wedge (\neg b \rightarrow \neg f) \wedge (\neg b \vee \neg s)$ [associative law]

$\equiv (\neg s \wedge f) \wedge (\neg b \rightarrow \neg f) \wedge (\neg b \vee \neg s)$ [idempotent law]

$\equiv (\neg s \wedge f) \wedge (\neg(\neg b) \vee \neg f) \wedge (\neg b \vee \neg s)$ [Definition of implication]

$\equiv (\neg s \wedge f) \wedge (b \vee \neg f) \wedge (\neg b \vee \neg s)$ [Double negation Law]

$\equiv (b \vee \neg f) \wedge (\neg s \wedge f) \wedge (\neg b \vee \neg s)$ [commutative Law]

$\equiv (b \vee \neg f) \wedge (\neg b \wedge (\neg s \wedge f) \vee \neg s (\neg s \wedge f))$ [Distributive Law]

$\equiv (b \vee \neg f) \wedge (\neg b \wedge \neg s \wedge f \vee \neg s \wedge f)$ [idempotent]

$\equiv ((\neg b \wedge \neg s \wedge f) \wedge (b \vee \neg f)) \vee ((b \vee \neg f) \wedge (\neg s \wedge f))$ [distributive

$\equiv (\neg b \wedge b \wedge \neg s \wedge f) \vee (\neg b \wedge \neg s \wedge f \neg f) \vee (b \wedge \neg s \wedge f) \vee (\neg f \wedge \neg s \wedge f)$

$\equiv F \vee F \vee (b \wedge \neg s \wedge f) \vee F$ [negation] [Distributive

$\equiv (b \wedge \neg s \wedge f)$ [identity]

As we can see conjunction of those three statements are neither a tautology nor a contradiction. Thus all of their statements are not true.

Predicates

Predicates are the statements involving variables which are neither true or false untill or unless the values of the variables are specified.

$$\underset{\uparrow \atop \text{subject}}{x} \text{ is } \underbrace{\text{an animal.}}_{\text{predicate}}$$

Quantifiers

Quantifiers are worlds that refer to quantities such as "some" or "all". It tells for how many elements a given predicate is true.

Propositional function

$$\underset{\text{variable}}{P(x)} = \underbrace{x+5>x}_{\text{predicate}}$$

N.B Quantifiers are used to express the quantities without giving an exact number.

Example: all, some, many none, few etc.

There are two kinds of Quantifiers.

(i) Universal

(ii) Existential

Universal Quantifiers

— Denoted by $\forall$

— means "for all"

Example:

Let, $P(x) = x+1>x$

for all values of x, $P(x)$ is true.

$\therefore \forall_x P(x)$ is true.

Universe of Discourse

What values x can represent called the domain or Universe of discourse.

Universal Quantification

Given some propositional function $P(x)$ and values in the universe $x_1 \ldots x_n$, the universal quantification implies:

$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$.

Existential Quantifiers

- Denoted by $\exists$
- means "there exists"

Existential Quantification

Given some propositional function $P(x)$ and values in the universe $x_1, \ldots x_n$. the existential quantification $\exists_x P(x)$ implies:

$$P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots \vee P(x_n)$$

Equivalences involving quantifiers

$$\forall x (P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall_x Q(x)$$

$$\cancel{\forall} \exists_x (P(x) \vee Q(x)) \equiv \exists_x \cancel{\forall} P(x) \vee \forall_x Q(x)$$