Differential Equation and Optimization Classtest-01

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N.B: I am only including the maths Harun sir showed in the class and the necessary formulas and definition to solve them.

Formation of Differential Equations by Eliminating the Arbitrary Constants

Working Rule:

1. Write the equation(1) of the family of curves.
2. Differentiate the equation(1).
3. Eliminate the arbitrary constant from getting equation.

N.B: The number of times we need to differentiate the differential equation is equal to the number or arbitrary constants in it.

and eliminating the arbitrary constants involved in it.

Problem 4.1. *Form the differential equation of which $c(y+c)^2 = x^3$ is the complete integral.*

Solution: The given equation is

$$c(y+c)^2 = x^3. \tag{4.1}$$

Differentiating (4.1), we get

$$2c(y+c)\frac{dy}{dx} = 3x^2. \tag{4.2}$$

Now, dividing (4.1) by (4.2), we obtain that

$$\frac{y+c}{2\frac{dy}{dx}} = \frac{x}{3} \Rightarrow 3(y+c) = 2x\frac{dy}{dx} \Rightarrow c = \frac{1}{3}(2x\frac{dy}{dx} - 3y).$$

Thus, $y + c = y + \frac{1}{3}(2x\frac{dy}{dx} - 3y) = \frac{2}{3}x\frac{dy}{dx}$. Now, substituting the value of c in (4.2), we have that

$$\frac{4}{9}(2x\frac{dy}{dx} - 3y)\left(\frac{dy}{dx}\right)^2 = 3x \Rightarrow 8x\left(\frac{dy}{dx}\right)^3 - 12y\left(\frac{dy}{dx}\right)^2 = 27x,$$

which is the required differential equation of first order and third degree.

Problem 4.3. *Find the differential equation of all circles passing through the origin and having their centres on the x-axis.*

Problem 4.4. *Find the differential equation of the family of parabolas with foci at the origin and axis along the x-axis.*

Solution Verified by Toppr

General equation of a circle can be given as

$x^2 + y^2 + 2gx + 2fy + c = 0$

Since, it is given that this circle passes through origin and centre lies on x−axis

$\therefore$ Center of circle is $(-g, 0)$

So, the equation of a circle passing through origin and centre on x−axis is

$x^2 + y^2 + 2gx = 0$ ....(1)

$\Rightarrow g = -\frac{x^2 + y^2}{2x}$

Differentiating (1) w.r.t x

$2x + 2y\frac{dy}{dx} + 2g = 0$ ....(2)

$x^2 + y^2 + 2gx = 0$ ....(1)

$\Rightarrow g = -\frac{x^2 + y^2}{2x}$

Differentiating (1) w.r.t x

$2x + 2y\frac{dy}{dx} + 2g = 0$ ....(2)

Substituting g in (2), we get

$\Rightarrow 2x + 2y\frac{dy}{dx} - \frac{x^2 + y^2}{x} = 0$

$\Rightarrow 2x + 2y\frac{dy}{dx} = \frac{x^2 + y^2}{x}$

$\Rightarrow 2x^2 + 2xy\frac{dy}{dx} = x^2 + y^2$

$y^2 = x^2 + 2xy\frac{dy}{dx}$

their centres on the x-axis.

Problem 4.4. *Find the differential equation of the family of parabolas with foci at the origin and axis along the x-axis.*

Solution: Let the directrix be $x = -2a$ and latus rectum be $4a$. Then, the equation of the parabola is

$$\begin{aligned} &\text{distance from focus} = \text{ distance from directrix} \\ \Rightarrow\ & x^2 + y^2 = (2a + x)^2 \Rightarrow y^2 = 4a(a + x). \end{aligned} \tag{4.3}$$

Differentiating, we get

$$y\frac{dy}{dx} = 2a \Rightarrow a = \frac{1}{2}y\frac{dy}{dx}.$$

It follows from (4.3) that

$$y^2 = 2y\frac{dy}{dx}\left(\frac{1}{2}y\frac{dy}{dx} + x\right) \Rightarrow y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0.$$

Separable Equations and Equations Reducible to this Form

5.1.1 Separable Equations

Definition 5.2. *An equation of the form*

$$F(x)G(y)dx + f(x)g(y)dy = 0$$

is called an equation with variables separable or simply a separable equations.

For example, the equation $(x-4)y^4dx - x^3(y^2-3)dy = 0 \Rightarrow \frac{x-4}{x^3}dx = \frac{y^2-3}{y^4}dy$ is a separable equation.

Working Rule: Separate the variables and integrate i.e.

$$\begin{aligned} &\frac{F(x)}{f(x)}dx + \frac{g(y)}{G(y)}dy = 0 \Rightarrow \int \frac{F(x)}{f(x)}dx + \int \frac{g(y)}{G(y)}dy = c \\ \Rightarrow & \int M(x)dx + N(y)dy = c. \end{aligned}$$

Problem 5.1. *Solve the differential equation $(x-4)y^4dx - x^3(y^2-3)dy = 0.$*

Solution: The given equation is separable. Now, separating the variables by dividing by x^3y^4, we obtain

$$\frac{x-4}{x^3}dx = \frac{y^2-3}{y^4}dy \Rightarrow (x^{-2} - 4x^{-3})dx - (y^{-2} - 3y^{-4})dy = 0.$$

Integrating, we have the one-parameter family of solutions

$$-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c,$$

where c is the arbitrary constant.

Problem 5.2. *Solve the initial-value problem that consists of the differential equation* $x \sin y\, dx + (x^2+1) \cos y\, dy = 0$ *and* $y(1) = \frac{\pi}{2}$.

Solution: Given that $x \sin y\, dx + (x^2+1) \cos y\, dy = 0$. Now Separating the variables dividing by $(x^2+1) \sin y$, we obtain that

$$\frac{x}{x^2+1}dx + \frac{\cos y}{\sin y}dy = 0$$
$$\Rightarrow \int \frac{x\, dx}{x^2+1} + \int \frac{\cos y}{\sin y}dy = \log c, \text{ where } \log c \text{ is an arbitrary constant}$$
$$\Rightarrow \frac{1}{2}\log(x^2+1) + \log \sin y = \log c$$
$$\Rightarrow \sqrt{x^2+1}\, \sin y = c.$$

Now, we apply the initial condition $y(1) = \frac{\pi}{2}$. Then, $\sqrt{1^1+1}\, \sin \frac{\pi}{2} = c \Rightarrow c = \sqrt{2}$. Therefore, the solution of the initial-value problem under consideration is $\sqrt{x^2+1}\, \sin y = \sqrt{2}$.

Problem 5.3. *Solve the differential equation* $\sec^2 x \tan y\, dx + \sec^2 \tan x\, dy = 0$.

Problem 5.4. *Solve the differential equation* $(y - px)x = y$, *where* $p = \frac{dy}{dx}$.

Solution

Verified by Toppr

$\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$

On separating the variables (dividing the equation by \tan x \tan y)

$\Rightarrow \frac{\sec^2 x}{\tan x}dx = -\frac{\sec^2 y}{\tan y}dy$

On integrating both sides, we get

$\int \frac{\sec^2 x}{\tan x}dx = -\int \frac{\sec^2 y}{\tan y}dy$

Put $\tan x = u \Rightarrow \sec^2 x. dx = du$ and $\tan y = v \Rightarrow \sec^2 y. dy = dv$

$\therefore \int \frac{du}{u} = -\int \frac{dv}{v}$

$\Rightarrow \log u = -\log v + \log c$

$\Rightarrow u = \frac{c}{v} \Rightarrow u. v = c$

$\therefore \tan x. \tan y = c$

Problem 5.4. *Solve the differential equation* $(y - px)x = y$, *where* $p = \frac{dy}{dx}$.

Re-write the equation as $yx - px^2 = y$ or, $px^2 = y(x-1)$

Since $p = \frac{dy}{dx}$ one gets $\frac{dy}{dx} = \frac{y(x-1)}{x^2}$. Separating the variables

$\frac{1}{y}dy = \frac{x-1}{x^2}dx = \frac{1}{x}dx - \frac{1}{x^2}dx$. Integrate on both sides

$log(y) = log(x) + \frac{1}{x} + C$ where C is the constant of integration.

Problem 5.5. *Solve the differential equation* $y - x\frac{dy}{dx} = a(y^2 + \frac{dy}{dx})$.

Problem 5.6. *Solve the differential equation* $(3 + 2\sin x + \cos x)\, dy = (1 + 2\sin y + \cos y)\, dx$.

$(3+2\sin x+\cos x)\cdot \mathrm{d}y=(1+2\sin y+\cos y)\cdot \mathrm{d}x$

$$\implies \frac{\mathrm{d}y}{1+2\sin y+\cos y}=\frac{\mathrm{d}x}{3+2\sin x+\cos x} \tag{E01}$$

$$\implies \int\frac{\mathrm{d}y}{1+2\sin y+\cos y}=\int\frac{\mathrm{d}x}{3+2\sin x+\cos x} \tag{E02}$$

Let $t=\tan\frac{y}{2} \implies \mathrm{d}y=\dfrac{2\cdot \mathrm{d}t}{1+t^2}$

$$\text{LHS} = \int\frac{\frac{2\cdot \mathrm{d}t}{1+t^2}}{1+\frac{2\cdot(2t)}{1+t^2}+\frac{1-t^2}{1+t^2}}$$

$$=\int\frac{\mathrm{d}t}{1+2t}=\frac{\ln(2t)}{2}=\frac{1}{2}\cdot\ln\left(2\cdot\tan\frac{y}{2}\right)$$

Let $p=\tan\frac{x}{2} \implies \mathrm{d}x=\dfrac{2\cdot \mathrm{d}p}{1+p^2}$

$$\text{RHS} = \int\frac{\frac{2\cdot \mathrm{d}p}{1+p^2}}{3+\frac{2\cdot(2p)}{1+p^2}+\frac{1-p^2}{1+p^2}}$$

$$=\int\frac{\mathrm{d}p}{p^2+2p+2}=\int\frac{\mathrm{d}p}{(p+1)^2+1}=\tan^{-1}(p+1)$$

$$=\tan^{-1}\left(1+\tan\tfrac{x}{2}\right)$$

$$\therefore (E02):\ \frac{1}{2}\cdot\ln\left(2\cdot\tan\tfrac{y}{2}\right)=\tan^{-1}\left(1+\tan\tfrac{x}{2}\right)+C$$

$$\implies \ln\left(2\cdot\tan\tfrac{y}{2}\right)=2\cdot\tan^{-1}\left(1+\tan\tfrac{x}{2}\right)+C'$$

Equations Reducible to the Form in which Variables are Separable

5.2 Homogeneous Differential Equations

Definition 5.4. *A function $f(x,y)$ is called a homogeneous of degree n if it can be expressed as in the form $x^n\varphi(\frac{y}{x})$ or $y^n\varphi(\frac{x}{y})$ or if $f(\lambda x,\lambda y)=\lambda^n f(x,y)$.*

Definition 5.5. *An equation of the form $M(x,y)dx+N(x,y)dy=0$ or in the form $\dfrac{dy}{dx}=-\dfrac{M(x,y)}{N(x,y)}$ is called a homogeneous differential equation if both $M(x,y)$ and $N(x,y)$ are homogeneous and of the same degree.*

For instance, $(x^2-3y^2)dx+2xy\ dy=0$ is a homogeneous differential equation of degree 2 but $(x^2+y^2)dx-(xy^3-y^3)dy=0$ is not a homogeneous equation.

5.2.1 Equation reducible to separable form

Working Rule: If $M(x,y)dx + N(x,y)dy = 0$ is a homogeneous equation, then the change of variable $y = vx$ transform the homogeneous equation into a separable equation in the variables v and x. Thus, $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Example 5.2. *Solve the differential equation* $(x^2 - 3y^2)dx + 2xy\,dy = 0$.

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy},$$

which is a homogeneous equation of degree 2. Let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Then the given equation reduces to

$$v + x\frac{dv}{dx} = \frac{3v^2-1}{2v} \Rightarrow x\frac{dv}{dx} = \frac{v^2-1}{2v}$$
$$\Rightarrow \frac{2v}{v^2-1}dv = \frac{dx}{x} \Rightarrow \int \frac{2v}{v^2-1}dv = \int \frac{dx}{x} + \log c$$
$$\Rightarrow \log(v^2-1) = \log x + \log c \Rightarrow v^2 - 1 = cx \Rightarrow y^2 - x^2 = cx^3.$$

Equation reducible to homogeneous form but linear

An equation of the form $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ can be reduced to homogeneous form in the following two cases.

Case-1: $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ when $\frac{a}{a'} \neq \frac{b}{b'}$;

Case-2: $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$ when $\frac{a}{a'} = \frac{b}{b'}$.

Below we will describe the above two cases.

Case-1 When $\frac{a}{a'} \neq \frac{b}{b'}$. Put $x = x_1 + h$, $y = y_1 + k$. Then $\frac{dy}{dx} = \frac{dy_1}{dx_1}$. Therefore,

$$\frac{dy_1}{dx_1} = \frac{ax_1 + by_1 + (ah + bk + c)}{a'x_1 + b'y_1 + (a'h + b'k + c')},$$

we choose the constants h and k in such a way that $ah + bk + c = 0$ and $a'h + b'k + c' = 0$. With this substitution, the given differential equation reduces to

$$\frac{dy_1}{dx_1} = \frac{ax_1 + by_1}{a'x_1 + b'y_1},$$

which is a homogeneous equation in x_1 and y_1 and can be solved by putting $y_1 = vx_1$.

Example 5.4. *Solve* $(6x - 2y - 7)dx - (2x + 3y - 6)dy = 0$.

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{6x - 2y - 7}{2x + 3y - 6}.$$

Put $x = x_1 + h$ and $y = y_1 + k$. Then

$$\frac{dy_1}{dx_1} = \frac{6x_1 - 2y_1 + (6h - 2k - 7)}{2x_1 + 3y_1 + (2h + 3k - 6)}, \tag{5.2}$$

where

$$6h - 2k - 7 = 0 \text{ and } 2h + 3k - 6 = 0. \tag{5.3}$$

Solving these two equations, we obtain that $h = \frac{3}{2}, k = 1$. Then, $x_1 = x - \frac{3}{2}, y_1 = y - 1$. Therefore, (5.2) can be written as

$$\frac{dy_1}{dx_1} = \frac{6x_1 - 2y_1}{2x_1 + 3y_1}.$$

Put $y = vx_1$, then $\frac{dy_1}{dx_1} = v + x_1\frac{dv}{dx_1}$. Thus,

$$\begin{aligned}
&\frac{6x_1 - 2y_1}{2x_1 + 3y_1} = v + x_1\frac{dv}{dx_1} \Rightarrow \frac{6 - 4v - 3v^2}{2 + 3v} = x_1\frac{dv}{dx_1}\\
&\Rightarrow \frac{dx_1}{x_1} + \frac{2 + 3v}{3v^2 + 4v - 6}dv = 0 \Rightarrow \int \frac{dx_1}{x_1} + \int \frac{2 + 3v}{3v^2 + 4v - 6}dv = \log c\\
&\Rightarrow \log x_1 + \frac{1}{2}\log\,(3v^2 + 4v - 6) = \log c \Rightarrow x_1\sqrt{3v^2 + 4v - 6} = c\\
&\Rightarrow \sqrt{3y_1^2 + 4x_1y_1 - 6x_1^2} = c \Rightarrow 3y_1^2 + 4x_1y_1 - 6x_1^2 = c_1\\
&\Rightarrow 3(y - 1)^2 + 4(y - 1)(x - \frac{3}{2}) - 6(x - \frac{3}{2})^2 = c_1\\
&\Rightarrow 12(y - 1)^2 + 8(y - 1)(2x - 3) - 6(2x - 3)^2 = c_2.
\end{aligned}$$

Example 5.6. *Solve* $\frac{dy}{dx} = \frac{3x - 4y - 2}{3x - 4y - 3}$.

Solution: The given equation is

$$\frac{dy}{dx} = \frac{3x - 4y - 2}{3x - 4y - 3}. \tag{5.5}$$

The transformation $3x - 4y = v$, since $\frac{a}{a'} = \frac{b}{b'}$. Then $3 - 4\frac{dy}{dx} = \frac{dv}{dx}$. Hence from (5.5), we have that

$$\begin{aligned}
&\frac{1}{4}\frac{dv}{dx} - \frac{3}{4} = \frac{v - 2}{v - 3} \Rightarrow \frac{1}{4}\frac{dv}{dx} = \frac{v - 2}{v - 3} + \frac{3}{4}\\
&\Rightarrow \frac{dv}{dx} = -\frac{v + 1}{v - 3} \Rightarrow \frac{v - 3}{v + 1}dv + dx = 0\\
&\Rightarrow (1 - \frac{4}{v + 1})dv + dx = 0 \Rightarrow \int (1 - \frac{4}{v + 1})dv + \int dx = c\\
&\Rightarrow (v - 4\log\,(v + 1) + x = c \Rightarrow x - y - \log\,(3x - 4y + 1) = c_1.
\end{aligned}$$

Exact Differential Equations

Theorem 6.1. *The necessary and sufficient condition for the exact differential equation* $M(x, y)\,dx + N(x, y)\,dy = 0$ *is* $M_y = N_x$ *or* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Example 6.1. *Solve the equation*

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0.$$

Solution: Here we have that

$$M = 3x^2 + 4xy, \ N = 2x^2 + 2y \text{ and } M_y = 4x, \ N_x = 4x.$$

Since $M_y = N_x$, the given equation is exact. Thus, we must find a function $u(x, y)$ such that

$$\begin{aligned} u(x,y) &= \int M(x,y)\partial x + \int \Big[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\Big] dy \\ &= \int (3x^2 + 4xy)\partial x + \int \Big[(2x^2 + 2y) - \frac{\partial}{\partial y}\int (3x^2 + 4xy)\partial x\Big] dy \\ &= x^3 + 2x^2y + \int \Big[2x^2 + 2y - \frac{\partial}{\partial y}(x^3 + 2x^2y)\Big] dy \\ &= x^3 + 2x^2y + \int (2x^2 + 2y - 2x^2) dy = x^3 + 2x^2y + y^2. \end{aligned}$$

Equations Reducible to Exact Differential Equation

If any differential equation is not exact, then we can reduce it to exact differential equation by multiplying it with a factor called Integrating Factor(μ).

Rules for finding the integrating factors

Rule-1. If $\dfrac{M_y - N_x}{N} = f(x)$, a function of x only, then the integrating factor is

$$\mu = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int f(x) dx}.$$

Rule-2. If $\dfrac{N_x - M_y}{M} = g(y)$, a function of y only, then the integrating factor is

$$\mu = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int g(y) dy}.$$

Rule-3. If $M\,dx + N\,dy = 0$ is a homogeneous and $Mx + Ny \neq 0$, then

$$\frac{1}{Mx + Ny}$$

is an integrating factor.

Rule-4. If the equation can be written in the form

$$yf(xy)dx + xg(xy)dy = 0, f(xy) \neq g(xy), \text{ then}$$

$$\frac{1}{xy[f(xy) - g(xy)]} = \frac{1}{Mx - Ny}$$

is an integrating factor.

Rule-5. Let the equation be of the form

$$x^a y^b(my\,dx + nx\,dy) + x^c y^d(\mu y\,dx + \nu x\,dy) = 0,$$

where $a, b, c, d, m, n, \mu, \nu$ are all constants. Then it has an integrating factor $x^\alpha y^\beta$, where α, β are so chosen that after multiplying by $x^\alpha y^\beta$ the equation becomes exact.

Example 6.6. *Solve* $x^2 y\, dx - (x^3 + y^3)\, dy = 0.$

Solution: The given equation is homogeneous and $Mx + Ny = x^3y - x^3y - y^4 = -y^4 \neq 0$. Hence, the integrating factor is

$$\frac{1}{Mx + Ny} = \frac{1}{x^3y - (x^3y + y^4)} = -\frac{1}{y^4}.$$

Multiplying the given equation by $-\frac{1}{y^4}$, then it becomes

$$-\frac{x^2}{y^3}dx + \frac{x^3 + y^3}{y^4}dy = 0 \Rightarrow -\frac{x^2}{y^3}dx + \frac{x^3}{y^4}dy + \frac{1}{y}dy = 0.$$

This equation is exact. Now,

$$\begin{aligned}
& d\Big(-\frac{x^3}{3y^3}\Big) + d(\log\, y) = d(\log\, c) \\
\Rightarrow\quad & \int d\Big(-\frac{x^3}{3y^3}\Big) + \int d(\log\, y) = \int d(\log\, c) \\
\Rightarrow\quad & -\frac{x^3}{3y^3} + \log\, y = \log\, c \\
\Rightarrow\quad & \log\, y = \log\, c + \frac{x^3}{3y^3} \\
\Rightarrow\quad & y = ce^{\frac{x^3}{3y^3}}.
\end{aligned}$$

Linear and Bernoulli differential equation

Linear Differential Equation:

There are two form

Form - 1 : $\frac{dy}{dx} + Py = Q$

, where P, Q are the function of x.

Integrating factor (I.F.) = $= e^{\int P dx}$

Then solution

$$y(I.F.) = \int Q(I.F)dx + c$$

Form - 2 : $\frac{dx}{dy} + Px = Q$

, where P, Q are the function of y.

Integrating factor (I.F.) = $= e^{\int P dy}$

Then solution

$$x(I.F.) = \int Q(I.F)dy + c$$

Example 7.1. *Solve the differential equation* $\dfrac{dy}{dx}+\left(\dfrac{2x+1}{x}\right)y=e^{-2x}$.

Solution: Here
$$P(x)=\frac{2x+1}{x}.$$
Hence, the integrating factor
$$\begin{aligned}\text{I.F.} &= e^{\int P(x)dx}=e^{\int \frac{2x+1}{x}dx}\\ &= e^{2x+\log x}=e^{2x}e^{\log x}=xe^{2x}.\end{aligned}$$
Now, multiplying the given equation by xe^{2x}, then it becomes
$$\begin{aligned}& xe^{2x}\left[\frac{dy}{dx}+e^{2x}(2x+1)y\right]=x\\ \Rightarrow\ & \frac{d}{dx}[xe^{2x}y]=x\\ \Rightarrow\ & \int\frac{d}{dx}[xe^{2x}y]dx=\int xdx+c\\ \Rightarrow\ & xye^{2x}=\frac{1}{2}x^2+c\\ \Rightarrow\ & y=\frac{1}{2}xe^{-2x}+\frac{c}{x}e^{-2x}.\end{aligned}$$

Bernoulli Differential Equation:

Definition 7.2. *An equation of the form*
$$\frac{dy}{dx}+P(x)y=Q(x)y^n \tag{7.2}$$
is called a Bernoulli differential equation.

It is remarked that if $n=0$ or 1, (7.2) reduces to a linear differential equation.

Theorem 7.1. *Suppose that* $n\neq 0$ *or* 1. *Then the transformation* $v=y^{1-n}$ *reduces the Bernoulli equation (7.2) to a linear differential equation.*

Example 7.3. *Solve the differential equation* $\dfrac{dy}{dx}+y=xy^3$.

Solution: The given equation is
$$\frac{dy}{dx}+y=xy^3\Rightarrow y^{-3}\frac{dy}{dx}+y^{-2}=x.$$
Let $v=y^{1-n}=y^{-2}$. Then $\dfrac{dv}{dx}=-2y^{-3}\dfrac{dy}{dx}$. Then the above equation reduces to
$$\begin{aligned}& -\frac{1}{2}\frac{dv}{dx}+v=x\\ \Rightarrow\ & \frac{dv}{dx}-2v=-2x,\end{aligned}\tag{7.3}$$
which is linear equation in v. Thus, the integrating factor is
$$\text{I.F.}=e^{\int -2dx}=e^{-2x}.$$
Multiplying (7.3) by e^{-2x} and integrate, we find
$$\begin{aligned}ve^{-2x} &= -2\int xe^{-2x}dx+c\Rightarrow ve^{-2x}=\frac{1}{2}e^{-2x}(2x+1)+c\\ &\Rightarrow\ v=x+\frac{1}{2}+ce^{2x}\\ &\Rightarrow\ \frac{1}{y^2}=x+\frac{1}{2}+ce^{2x}.\end{aligned}$$

The Homogeneous and Nonhomogeneous Linear Equation with Constant Coefficients

Definition 8.1. *A differential equation of the form*

$$a_0(x)\frac{d^ny}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+a_2(x)\frac{d^{n-2}y}{dx^{n-2}}+\ldots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=f(x), \tag{8.1}$$

where $a_0, a_1, a_2, \ldots, a_n$ and $f(x)$ are functions of x, is called a n^{th}-order linear differential equation (LDE).

If $a_0, a_1, a_2, \ldots, a_n$ are constants and $f(x)$ is a function of x, then (8.1) is called a n^{th}-order linear differential equation (LDE) with constant coefficients.

We will deal with LDE with constant coefficients and for our convenience we use the operators $D := \frac{d}{dx}, D^2 := \frac{d^2}{dx^2}, \ldots, D^n := \frac{d^n}{dx^n}$. Then (8.1) becomes

$$\left(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + a_3D^{n-2} + \ldots + a_{n-1}D + a_n\right)y = f(x), \tag{8.2}$$

which can be briefly written as,

$$F(D)y = f(x).$$

Definition 8.2. *If $f(x) = 0$, then the equation (8.2) becomes*

$$(a_0D^n + a_1D^{n-1} + \ldots + a_n)y = 0 \;\; i.e. \;\; F(D)y = 0. \tag{8.3}$$

This equation is called the homogeneous linear differential equation with constant coefficients. Otherwise it is called nonhomogeneous linear differential equation with constant coefficients i.e. if $f(x) \neq 0$, then (8.2) is called nonhomogeneous linear differential equation with constant coefficients.

Theorem 8.1. *If $y = y_1, y = y_2, \ldots, y = y_n$ are linearly independent solutions of $F(D)y = 0$, then*

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$$

is the general or complete solution of the differential equation, where $c_1, c_2, \ldots, c_n$ are n arbitrary constants.

Solution of Homogeneous Equation:

Case-I: When the auxiliary equation has distinct roots

Let $m_1, m_2, \ldots, m_n$ be the distinct roots of (8.4). Then $y = e^{m_1x}, y = e^{m_2x}, \ldots, y = e^{m_nx}$ are all independent solution of (8.3). Therefore, the general solution of (8.3) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \ldots + c_ne^{m_nx}.$$

Example 8.1. *Solve the differential equation* $\frac{d^3y}{dx^3} - 13\frac{dy}{dx} - 12y = 0.$

Solution: The given equation is $(D^3 - 13D - 12)y = 0$.
Let $y = e^{mx}$ be the trial solution of the given equation. Then, the auxiliary equation is

$$m^3 - 13m - 12 = 0 \Rightarrow m = -1, -3, 4.$$

Hence, the complete solution is

$$y = c_1e^{-x} + c_2e^{-3x} + c_3e^{4x}.$$

Case-II: When the auxiliary equation has repeated roots

Consider the 2nd order differential equation having equal roots as follows

$$(D-m_1)^2y=0. \tag{8.5}$$

Put $(D-m_1)y=v$. Then (8.5) becomes

$$(D-m_1)v=0 \Rightarrow \frac{dv}{dx}=m_1v.$$

Separating the variables, we obtain that

$$\begin{aligned} & \frac{dv}{v}=m_1dx \Rightarrow \int\frac{dv}{v}=\int m_1dx+\log c \\ \Rightarrow\ & \log v=m_1x+\log c \Rightarrow v=ce^{m_1x} \\ \Rightarrow\ & (D-m_1)y=ce^{m_1x} \text{ as } v=(D-m_1)y \\ \Rightarrow\ & \frac{dy}{dx}-m_1y=ce^{m_1x}, \end{aligned}$$

which is a first order linear differential equation in y. Its integrating factor

$$\text{I.F.}=e^{-\int m_1dx}=e^{-m_1x}.$$

Therefore,

$$ye^{-m_1x}=\int ce^{m_1x}e^{-m_1x}dx+c_1 \Rightarrow ye^{-m_1x}=\int cdx+c_1 \Rightarrow y=(c_1+cx)e^{m_1x}.$$

Therefore, if $m_1, m_2, \ldots, m_n$ are roots of the auxiliary equation for (8.3) with $m_1=m_2$, then the most general solution of (8.3), when two roots of auxiliary equation are equal, is

$$y=(c_1+c_2x)e^{m_1x}+c_3e^{m_3x}+c_4e^{m_4x}+\ldots+c_ne^{m_nx}.$$

If three roots of the auxiliary equation are equal i.e. $m_1=m_2=m_3$, the general solution is

$$y=(c_1+c_2x+c_3x^2)e^{m_1x}+c_4e^{m_4x}+\ldots+c_ne^{m_nx}.$$

Example 8.3. *Solve* $\frac{d^4y}{dx^4}-\frac{d^3y}{dx^3}-9\frac{d^2y}{dx^2}-11\frac{dy}{dx}-4y=0.$

Solution: Let $y=e^{mx}$ be the trial solution of the given equation. Then the auxiliary equation is

$$m^4-m^3-9m^2-11m-4=0 \Rightarrow (m+1)^3(m-4)=0 \Rightarrow m=-1,-1,-1,4.$$

Hence the general solution is

$$y=(c_1+c_2x+c_3x^2)e^{-x}+c_4e^{4x}.$$

Case-III: When the auxiliary equation has imaginary roots

Let $\alpha \pm i\beta$ be the imaginary roots of a 2nd order differential equation. Then, its general solution is

$$\begin{aligned} y &= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \\ &= e^{\alpha x}[c_1 e^{i\beta x} + c_2 e^{-i\beta x}] \\ &= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] \\ &= e^{\alpha x}[(c_1+c_2)\cos\beta x + (c_1-c_2)i\sin\beta x] \\ &= (A\cos\beta x + B\sin\beta x)e^{\alpha x}. \end{aligned}$$

If the auxiliary equation has two equal pairs of imaginary roots, then the general solution is obtained as follows

$$y = [(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]e^{\alpha x}.$$

Example 8.5. *Solve* $\dfrac{d^4y}{dx^4} + 5\dfrac{d^2y}{dx^2} + 6y = 0.$

Solution: The auxiliary equation of the given differential equation is

$$m^4 + 5m^2 + 6 = 0 \Rightarrow (m^2+3)(m^2+2) = 0 \Rightarrow m = \pm i\sqrt{3}, \pm i\sqrt{2}.$$

Hence, the complete solution is

$$y = c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x + c_3\cos\sqrt{2}x + c_4\sin\sqrt{2}x.$$