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Barnouli equation:

Q. $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

put, $v = \log y$

$\frac{dv}{dx} = \frac{1}{y}\cdot\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} v = \frac{y}{x^2} v^2$

$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} + \frac{v}{x} = \frac{v^2}{x^2}$

$\Rightarrow \frac{dv}{dx} + \frac{v}{x} = \frac{v^2}{x^2}$

$\Rightarrow \frac{1}{v^2}\frac{dv}{dx} + \frac{1}{vx} = \frac{1}{x^2}$

Again,

$z = \frac{1}{v}$

$\frac{dz}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$

$\therefore \frac{1}{v^2}\frac{dv}{dx} = -\frac{dz}{dx}$

$\Rightarrow -\frac{dz}{dx} + z\frac{1}{x} = \frac{1}{x^2}$

$\therefore \frac{dz}{dx} - z\frac{1}{x} = -\frac{1}{x^2}$

Here, $p(x) = -\frac{1}{x}$

$Q(x) = -\frac{1}{x^2}$

I.F, $e^{\int p(x)\,dx}$

$= e^{\int -\frac{1}{x}dx}$

$= e^{-\ln x}$

$= -x$

multiplying and integrating by,

$$z(I.F) = \int (I.F)\, q(x)\, dx$$

$$\Rightarrow \frac{1}{v}(-x) = \int (-x)\left(-\frac{1}{x^2}\right) dx$$

$$\Rightarrow -\frac{x}{\log y} = \int \frac{1}{x}\, dx$$

$$\Rightarrow -\frac{x}{\log y} = \cancel{-\frac{1}{x^2} + c}\ \ln x + c$$

$$\Rightarrow \frac{1}{\log y} = -\frac{\ln x}{x} - \frac{c}{x}$$

which is required solution.

(Ans)

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Operator method:-

Q) $(D^2 - 5D + 6)y = e^{4x} + x$

Solve:

Axiliary equation,

$$m^2 - 5m + 6 = 0$$

$$\Rightarrow m^2 - 3m - 2m + 6 = 0$$

$$\Rightarrow m(m-3) - 2(m-3) = 0$$

$$\Rightarrow (m-3)(m-2) = 0$$

$$\Rightarrow m = 2, 3$$

$$\therefore y_c = c_1 e^{2x} + c_2 e^{3x}$$

$$y_p = \frac{1}{D^2 - 5D + 6} e^{4x} + \frac{1}{D^2 - 5D + 6} x$$

$$= \frac{1}{4^2 - 5.4 + 6} e^{4x} + \frac{1}{6\left(1 - \frac{5D - D^2}{6}\right)} x$$

$= \frac{1}{22-20} e^{4x} + \frac{1}{6}\left(1 - \frac{5D-D^2}{6}\right)^{-1} x$

$= \frac{1}{2} e^{4x} + \frac{1}{6}\left(1 + \frac{5D-D^2}{6} + \cdots\right) x \quad [(1-x)^{-1} = 1 + x + x^2 + \cdots]$

$= \frac{e^{4x}}{2} + \frac{1}{6}\left(1 + \frac{5D}{6} - \frac{D^2}{6} + \cdots\right) x$

$= \frac{e^{4x}}{2} + \frac{1}{6}\left(x + \frac{5\cancel{x}.Dx}{6}\right)$

$= \frac{e^{4x}}{2} + \frac{1}{6}\left(x + \frac{5}{6}\right)$

$\Rightarrow \cancel{\frac{e^{4x}}{2}} +$

$\therefore Y = Y_c + Y_p$

$= c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^{4x} + \frac{1}{6}\left(x + \frac{5}{6}\right)$ (Ans)

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Undetermind Method

Q. $(D^2 - 5D + 6)Y = e^{4x} + x$ —— ①

A.E, $m^2 - 5m + 6 = 0$

$\Rightarrow m^2 - 3m - 2m + 6 = 0$

$\Rightarrow m(m-3) \cancel{(m} - 2(m-3) = 0$

$\Rightarrow (m-3)(m-2) = 0$

$\therefore m = 2, 3$

$Y_c = c_1 e^{2x} + c_2 e^{3x}$

Solution set of $S = \{e^{2x}, e^{3x}\}$

Here, $\{e^{4x}, x, 1\}$ is not part of x.

$\therefore y_p = c_3 e^{4x} + c_4 x + c_5$

$y_p' = 4c_3 e^{4x} + c_4$

$y_p'' = 16 c_3 e^{4x}$

$\Rightarrow$ From equation — (1)

$16c_3 e^{4x} - 5(4c_3 e^{4x} + c_4) + 6(c_3 e^{4x} + c_4 x + c_5)$
$= e^{4x} + x$

$\Rightarrow 16c_3 e^{4x} - 20c_3 e^{4x} - 5c_4 + 6c_3 e^{4x} + 6c_4 x + 6c_5$
$= e^{4x} + x$

$\Rightarrow (16c_3 - 20c_3 + 6c_3) e^{4x} + (6c_5 - 5c_4) + 6c_4 x = e^{4x} + x$

$\Rightarrow 2c_3 e^{4x} + 6c_4 x + 6c_5 - 5c_4 = e^{4x} + x$

Compairing from

$2c_3 = 1$

$\therefore c_3 = 1/2$

Again, $6c_4 = 1$

$\therefore c_4 = 1/6$

Also Again, $6c_5 - 5c_4 = 0$

$6c_5 = 5c_4$

$= 5 \cdot \frac{1}{6}$

$\therefore c_5 = \frac{5}{36}$

$\therefore y_p = \frac{1}{2} e^{4x} + \frac{1}{6} x + 5/36$

$\therefore y = y_c + y_p = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^{4x} + \frac{1}{6} x + \frac{5}{36}$. (Ans)

Variation of parameters:

Q. $(D^2+1)y = \tan x$

A.E., $m^2+1=0$

$m^2=-1$

$m = \pm i$

$\therefore y_c = e^{0.x}(c_1 \sin x + c_2 \cos x)$

$= c_1 \sin x + c_2 \cos x$

$\therefore y_p = u \sin x + v \cos x$

$y_p' = u \cos x - v \sin x + u' \sin x + v' \cos x$

$= u \cos x - v \sin x$

$y_p'' = -u \sin x - v \cos x + u' \cos x - v' \sin x$

$\therefore (D^2+1)y = \tan x$

$\Rightarrow -u \cancel{\sin x} - v \cancel{\cos x} + u' \cos x - v' \sin x + u \cancel{\sin x} + v \cancel{\cos x} = \tan x$

$\Rightarrow u' \cos x - v' \sin x = \tan x$ —— (i)

and, $u' \sin x + v' \cos x = 0$ —— (ii)

(i) $\times \cos x$ + (ii) $\times \sin x$, we get,

$u' \cos x . \cos x - v' \sin x . \cancel{\cos x} = \tan x . \cos x$

$u' \sin x . \sin x - v' \cancel{\cos x} \sin x = 0$

$u' \cos^2 x + u' \sin^2 x = \tan x . \cos x$

$\Rightarrow u'(\cos^2 x + \sin^2 x) = \frac{\sin x}{\cos x} . \cos x$

$\Rightarrow u' . 1 = \sin x$

$\therefore u' = \sin x$

Again,

(i) × $\sin x$ − (ii) × $\cos x$ we get,

$$u'\cos x\cdot\sin x - v'\sin x\cdot\sin x = \tan x\cdot\sin x$$
$$u'\sin x\cdot\cos x + v'\cos x\cdot\cos x = 0$$
$$\overline{\quad - \quad \pm \quad\quad -\quad}$$

$$-v'(\sin^2 x + \cos^2 x) = -\tan x\cdot\sin x$$

$$-v'\cdot 1 = -\frac{\sin x}{\cos x}\cdot\sin x$$

$$v' = \frac{\sin^2 x}{\cos x}$$
$$= \frac{1-\cos^2 x}{\cos x}$$
$$= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}$$
$$= \sec x - \cos x$$

$$u = \int u'\,dx = \int \sin x\,dx$$
$$= -\cos x$$

$$v = \int v'\,dx = \int(\sec x - \cos x)\,dx$$
$$= \int \sec x\,dx - \int \cos x\,dx$$
$$= \ln(\sec x + \tan x) - \sin x$$
$$= \sin x - \ln(\sec x + \tan x)$$

$$\therefore y_p = -\cos x\cdot\sin x + [\sin x - \ln(\sec x + \tan x)]\cos x$$
$$= -\cos x\sin x + \cos x\sin x - \ln(\sec x + \tan)\cos x$$
$$= -\ln(\sec x + \tan x)\cos x$$

$$\therefore y = y_c + y_p = c_1\sin x + c_2\cos x - \ln(\sec x + \tan x)\cos x$$

which is required solution. (Ans)

Exact:

Q. $(3x^2y+2)dx+(x3+y)dy=0$

Now,

$$\frac{\partial M}{\partial y}=3x^2 \quad ; \frac{\partial N}{\partial x}=3x^2$$

$$\therefore \frac{\partial M}{\partial \cancel{x}}=\frac{\partial N}{\partial x}$$

Therefore it is an exact Differential equation.

Thus there exist a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x}=3x^2y+2 \text{ —— (I)}$$

$$\frac{\partial f}{\partial y}=x^3+y \text{ —— (II)}$$

Let we ~~it~~ integrate eq (I) with respect to x

$$\int\frac{\partial f}{\delta x}\delta x=\int(3x^2y+2)\delta x$$

$$f=3.\frac{x^3}{3}y+2x+h(y)$$

$$=x^3y+2x+h(y) \text{ —— (III)}$$

$$\frac{\partial f}{\delta y}=x^3+h'(y) \text{ —— (IV)}$$

Now Compairing eq (IV) and eq (II)

$$x^3+h'(y)=x^3+y$$

$$h'(y)=y$$

$$\int h'(y)=\int y\,dy$$

$$h(y)=\cancel{h(y)}\frac{y^2}{2}+c$$

$$\therefore f(x,y)=x^3y+2x+\frac{y^2}{2}+c$$

which is required solution.

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Non exact:

Q. $(x^2+y^2+x)\,dx + xy\,dy = 0$ ——— (1)

here, $M = x^2+y^2+x$ and $N = xy$

$\frac{\partial M}{\partial y} = 2y$ $\qquad$ $\frac{\partial N}{\partial x} = y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

This is a Non-exact D.equation.

Now, $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{y}{xy} = \frac{1}{x}$ $\therefore$ a function of x along

Here,
I.F $e^{\int \frac{1}{x} dx}$
$= e^{\ln x}$
$= x$

(1) × I.F we get,
$(x^3+xy^2+x^2)\,dx + x^2y\,dy = 0$

Again, $M = x^3+xy^2+x^2$ and, $N = x^2y$

$\frac{\partial M}{\partial y} = 2xy$ $\qquad$ $\frac{\partial N}{\partial x} = 2xy$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ this is an exact D.E.

Thus general equation

$\int M\,dx\,(y \text{ is constant}) + \int N\,dy\,(\text{terms free from } x) = c$

$\Rightarrow \int (x^3+xy^2+x^2)\,dx + \int 0 \cdot dy = c$

$\Rightarrow \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c$

which is required solution.

Variable separable

Q. $(x+1)\frac{dy}{dx} = x(y^2+1)$

Solve, $(x+1)\,dy = x(y^2+1)\,dx$

$\Rightarrow \frac{1}{y^2+1}\,dy = \frac{x}{x+1}\,dx$

$\Rightarrow \int \frac{1}{y^2+1}\,dy = \int \frac{x}{x+1}\,dx$

$\Rightarrow \tan^{-1}y = \int\left(1-\frac{1}{1+x}\right)dx$

$= \int 1\,dx - \int \frac{1}{1+x}\,dx$

$= x - \ln(1+x) + c$

$\therefore \tan^{-1}y = x - \ln(1+x) + c$

which is required solution.

Linear equation:

Q. $\frac{dy}{dx} + \frac{3y}{x} = 6x^2$

Solⁿ: $\frac{dy}{dx} + \frac{3}{x}y = 6x^2$

Here,
$P(x) = \frac{3}{x}$
$Q(x) = 6x^2$

I.F $e^{\int P(x)\,dx}$

$= e^{\int \frac{3}{x}dx}$

$= e^{3\ln x}$

$= e^{\ln x^3}$

$= x^3$

multiplying and Integrating.

$$y \cdot x^3 = \int 6x^2 \cdot x^3 dx + C$$

$$x^3 y = \int 6x^5 dx + C$$

$$x^3 y = 6 \cdot \frac{x^6}{6} + C$$

$$y = \frac{x^6}{x^3} + \frac{C}{x^3}$$

$$= x^3 + \frac{C}{x^3}$$

which is required solution.

Homogeneous:

Q. $(x^2+y^2)\,dx + 2xy\,dy = 0$

Solve: $\Rightarrow 2xy\,dy = -(x^2+y^2)\,dx$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^2+y^2}{2xy} \quad \text{--- (1)}$$

putting $y = vx$

$$\frac{dy}{dx} = v \cdot 1 + x\frac{dv}{dx}$$

$$= v + \frac{dv}{dx}$$

Equation (1) becomes

$$v + dx\frac{dv}{dx} = -\frac{(x^2+v^2x^2)}{2x \cdot vx}$$

$$= \frac{-x^2(1+v^2)}{2x^2 v}$$

$$= \frac{-(1+v^2)}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1-v^2}{2v} - v$$

$$= \frac{-1-v^2-2v^2}{2v}$$

$$= \frac{1-3v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-(1+3v^2)}{2v}$$

$$\Rightarrow \frac{dx}{x\,dv} = \frac{-2v}{1+3v^2}$$

$$\Rightarrow \frac{dx}{x} = -\frac{2v}{1+3v^2}dv$$

$$\Rightarrow \int\frac{dx}{x} = -2\int\frac{v}{1+3v^2}dv$$

$\therefore \left[\int\frac{f'(x)}{f(x)}dx = \ln f(x)\right]$

$$\Rightarrow \cancel{\ln(x)}\int\frac{dx}{x} = -\frac{6}{3}\int\frac{v}{1+3v^2}dv$$

$$\Rightarrow \ln(x) = -\frac{1}{3}\int\frac{6v}{1+3v^2}dv$$

$$\Rightarrow \ln x = -\frac{1}{3}\ln(1+3v^2) + \ln(c)$$

$$\Rightarrow \ln(x) = -\ln(1+3v^2)^{1/3} + \ln(c)$$

$$\Rightarrow \ln(x) + \ln(1+3v^2)^{1/3} = \ln(c)$$

$$\Rightarrow \ln\left[x + (1+3v^2)^{1/3}\right] = \ln c$$

~~(Ans)~~

$$\Rightarrow x * (1+3v^2)^{1/3} = c$$

$$\therefore x * \left(1+3\cdot\frac{y^2}{x^2}\right)^{1/3} = c$$

which is required solution.

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