Student's t-distribution

Let $x_1, x_2, \ldots, x_n$ be a random Sample of Size n from a normal population with mean μ and variance σ^2. Then Student's t is defined by the statistic

$$t = \frac{\bar{x}-\mu}{S/\sqrt{n}}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the sample mean and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is an unbiased estimate of the population variance σ^2, and it follows Student's t-distribution with $\nu = (n-1)$ degree of freedom with probability density function

$$f(t) = \frac{1}{\sqrt{\nu}\,\beta\left(\frac{1}{2}, \frac{\nu}{2}\right)\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}, \quad -\infty < t < \infty$$

Derivation of Student's t-distribution

t is defined by the statistic

$$t = \frac{\bar{x}-\mu}{S/\sqrt{n}} = \frac{\sqrt{n}(\bar{x}-\mu)}{S}$$

$$\text{or } t^2 = \frac{n(\bar{x}-\mu)^2}{S^2}$$

$$\text{or } t^2 = \frac{n(\bar{x}-\mu)^2}{ns^2/(n-1)} \qquad (\text{Since } ns^2 = (n-1)S^2)$$

$$\text{or } t^2 = \frac{n(n-1)(\bar{x}-\mu)^2}{ns^2}$$

$$\text{or } \frac{t^2}{(n-1)} = \frac{n(\bar{x}-\mu)^2}{ns^2}$$

$$\text{or } \frac{t^2}{(n-1)} = \frac{(\bar{x}-\mu)^2/(\sigma^2/n)}{ns^2/\sigma^2}$$

Since $x_1, x_2, \ldots, x_n$ is a random Sample from the normal population with mean μ and variance σ^2, then the Sample mean

$$\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Then

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$ is a standard normal variate. Hence $(\bar{x}-\mu)^2/(\sigma^2/n)$ being the Square of a Standard normal variate is a χ^2-variate with 1 degree of freedom. Also ns^2/σ^2 is a χ^2-variate with $(n-1)$ degree of freedom. Further since $\bar{x}$ and s^2 are independently distributed, $\frac{t^2}{(n-1)}$ being the ratio of two independent χ^2-variates with 1 and $(n-1)$ degree of freedom respectively, is a

β_2 variate with $\frac{1}{2}$ and $\frac{\nu}{2}$ parameters, i.e. $\beta_2\left(\frac{1}{2}, \frac{\nu}{2}\right)$. Its distribution is given by

$$dF(t) = \frac{1}{\beta\left(\frac{1}{2}, \frac{\nu}{2}\right)} \cdot \frac{(t^2/\nu)^{\frac{1}{2}-1}}{\left(1+\frac{t^2}{\nu}\right)^{\frac{1}{2}+\frac{\nu}{2}}} d(t^2/\nu) \quad ; 0 < t^2 < \infty$$

$$= \frac{1}{\sqrt{\nu}\, \beta\left(\frac{1}{2}, \frac{\nu}{2}\right) \left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}} dt \quad ; -\infty < t < \infty$$

which is the required probability differential function of Student's t-distribution with $\nu = (n-1)$ degree of freedom.

Fisher's t-distribution

It is the ratio of a standard normal variate to the square root of an independent χ^2-variate divided by its degree of freedom. If X is a standard normal variate $N(0,1)$ and Y is an independent χ^2-variate with n degree of freedom then Fisher's t is given by the statistic

$$t = X\Big/\sqrt{\frac{Y}{n}}$$

and it follows Student's t-distribution with n degree of freedom and probability density function is given by

$$f(t) = \frac{1}{\sqrt{n}\, \beta\left(\frac{1}{2}, \frac{n}{2}\right) \left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}, \quad -\infty < t < \infty$$

Derivation of Fisher's t-distribution

If X is a standard normal variate, i.e. $X \sim N(0,1)$, then the probability differential function of X is given by

$$dF_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \quad ; -\infty < x < \infty$$

and Y is a χ^2-variate with n degree of freedom, then the probability differential function of Y is given by

$$dF_2(y) = \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} e^{-y/2} (y)^{\frac{n}{2}-1} dy \quad ; 0 \le y < \infty$$

Since X and Y are independent variates, then their joint probability differential function is given by

$$dF(x,y) = dF_1(x) \cdot dF_2(y)$$

$$= \left\{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx\right\}\left\{\frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} e^{-y/2} (y)^{\frac{n}{2}-1} dy\right\}$$

$$= \frac{1}{\sqrt{\pi}\, 2^{\frac{n+1}{2}} \Gamma\frac{n}{2}} e^{-\frac{1}{2}(x^2+y)} (y)^{\frac{n}{2}-1} dx\,dy \;; -\infty<x<\infty, 0\le y<\infty$$

Let us make the transformation

$$t = x/\sqrt{\frac{y}{n}} \quad \text{and} \quad u = y.$$

then

$$x = t\sqrt{\frac{u}{n}} \quad \text{and} \quad y = u.$$

Jacobian of transformation J is given by

$$|J| = \left|\frac{\partial(x,y)}{\partial(t,u)}\right|$$

$$= \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} \sqrt{\frac{u}{n}} & \frac{1}{2} t \left(\frac{u}{n}\right)^{-\frac{1}{2}} \left(\frac{1}{n}\right) \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{n}}$$

The joint probability differential function of t and u becomes

$$dG(t,u) = dF(t,u)\,|J|$$

$$= \frac{1}{\sqrt{\pi}\, 2^{\frac{n+1}{2}} \Gamma\frac{n}{2}} e^{-\frac{1}{2}\left(t^2\frac{u}{n} + u\right)} (u)^{\frac{n}{2}-1} dt\,du \cdot \sqrt{\frac{u}{n}}$$

$$= \frac{1}{\sqrt{\pi}\, 2^{\frac{n+1}{2}} \Gamma\frac{n}{2} \sqrt{n}} e^{-\frac{1}{2}\left(1+\frac{t^2}{n}\right)u} (u)^{\frac{n-1}{2}} dt \cdot du \;;$$

$$-\infty<t<\infty, 0\le u<\infty$$

Integrating out u over the range 0 to ∞, then the marginal probability differential function of t becomes

$$dG_1(t) = \frac{dt}{\sqrt{\pi}\, 2^{\frac{n+1}{2}} \Gamma\frac{n}{2} \sqrt{n}} \int_{u=0}^{\infty} e^{-\frac{1}{2}\left(1+\frac{t^2}{n}\right)u} (u)^{\frac{n-1}{2}} du$$

put

$$\frac{1}{2}\left(1+\frac{t^2}{n}\right)u = T$$

$$u = \frac{2T}{\left(1+\frac{t^2}{n}\right)}$$

$$du = \frac{2\,dT}{\left(1+\frac{t^2}{n}\right)}$$

If $u \to 0$ then $T \to 0$ and $u \to \infty$ then $T \to \infty$, so

$$dG_1(t) = \frac{dt}{\sqrt{\pi}\, 2^{\frac{n+1}{2}} \Gamma\frac{n}{2} \sqrt{n}} \int_0^{\infty} e^{-T} \left(\frac{2T}{1+\frac{t^2}{n}}\right)^{\frac{n-1}{2}} \left(\frac{2\,dT}{1+\frac{t^2}{n}}\right)$$

$$= \frac{dt}{\sqrt{\pi}\, 2^{\frac{n+1}{2}} \Gamma\frac{n}{2} \sqrt{n}} \int_0^{\infty} e^{-T} \left(\frac{2T}{1+\frac{t^2}{n}}\right)^{\frac{n+1}{2}-1} \left(\frac{2\,dT}{1+\frac{t^2}{n}}\right)$$

$$= \frac{\cancel{2^{\frac{n+1}{2}}}\, dt}{\sqrt{\pi}\, \cancel{2^{\frac{n+1}{2}}}\, \Gamma\frac{n}{2}\, \sqrt{n}\, \left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \int_0^\infty e^{-T} (T)^{\frac{n+1}{2}-1}\, dT$$

$$= \frac{dt}{\sqrt{\pi}\, \Gamma\frac{n}{2}\, \sqrt{n}\, \left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \Gamma\frac{n+1}{2}$$

$$= \frac{dt}{\sqrt{n}\, \frac{\Gamma\frac{1}{2}\, \Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}} \left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}$$

$$= \frac{dt}{\sqrt{n}\, \beta\left(\frac{1}{2},\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}, \quad -\infty < t < \infty$$

Which is the required probability differential function of Student's t-distribution with n degree of freedom.

proved

Properties of t- distribution

(1) Moments

Since $f(t)$ is symmetrical about the line $t=0$, all moments of odd order about origin vanish, i.e.,

$$\mu'_{2r+1}\text{(about Origin)} = 0 \qquad r = 0,1,2,\cdots\cdots$$

In particular

$$\mu'_1\text{(about origin)} = 0 = \text{Mean}$$

$$\mu'_3\text{(about origin)} = 0.$$

Hence central moments coincide with moments about origin, then

$$\mu'_{2r+1}\text{(about origin)} = \mu_{2r+1}\text{(about mean)} = 0$$

The moments of even order are given by

$$\mu_{2r}\text{(about mean)} = \mu'_{2r}\text{(about origin)} \qquad r = 0,1,2,\cdots$$

$$= E(t^{2r})$$

$$= \int_{-\infty}^{\infty} t^{2r} f(t)\, dt$$

$$= \int_{-\infty}^{\infty} t^{2r} \frac{1}{\sqrt{n}\, \beta\left(\frac{1}{2},\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} dt$$

$$= \frac{1}{\sqrt{n}\, \beta\left(\frac{1}{2},\frac{n}{2}\right)} \int_{-\infty}^{\infty} \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} dt$$

$$= \frac{2}{\sqrt{n}\, \beta\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^\infty \frac{t^{2r}}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} dt$$

put $1+\frac{t^2}{n} = \frac{1}{y}$

or, $\frac{t^2}{n} = \frac{1}{y} - 1 = \frac{1-y}{y}$

or, $t^2 = \frac{n(1-y)}{y} = n\left(\frac{1}{y}-1\right)$

$2t \cdot dt = -\frac{n}{y^2} dy$

$dt = -\frac{n}{2ty^2} dy$

If $t \to 0$ then $y \to 1$ and $t \to \infty$ then $y \to 0$, so

$$\mu_{2r} = \frac{\cancel{2}}{\cancel{\sqrt{n}}\, B\left(\frac{1}{2},\frac{n}{2}\right)} \int_1^0 \frac{t^{2r}}{\left(\frac{1}{y}\right)^{\frac{n+1}{2}}} \left(-\frac{\cancel{n}}{\cancel{2}ty^2} dy\right)$$

$$= \frac{\sqrt{n}}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 t^{2r-1} (y)^{\frac{n+1}{2}-2} dy$$

$$= \frac{\sqrt{n}}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 (t^2)^{\frac{2r-1}{2}} (y)^{\frac{n+1}{2}-2} dy$$

$$= \frac{\sqrt{n}}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 \left[\frac{n(1-y)}{y}\right]^{\frac{2r-1}{2}} (y)^{\frac{n+1}{2}-2} dy$$

$$= \frac{\sqrt{n}\,(n)^{\frac{2r-1}{2}}}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 (1-y)^{\frac{2r-1}{2}} (y)^{\frac{n+1}{2}-2-\frac{2r-1}{2}} dy$$

$$= \frac{(n)^{\frac{2r-1}{2}+\frac{1}{2}}}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 (1-y)^{\frac{2r-1}{2}} (y)^{\frac{n}{2}+\frac{1}{2}-2-r+\frac{1}{2}} dy$$

$$= \frac{(n)^r}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 (1-y)^{r-\frac{1}{2}} (y)^{\frac{n}{2}-r-1} dy$$

$$= \frac{n^r}{B\left(\frac{1}{2},\frac{n}{2}\right)} \int_0^1 (y)^{\frac{n}{2}-r-1} (1-y)^{r+\frac{1}{2}-1} dy$$

$$= \frac{n^r}{B\left(\frac{1}{2},\frac{n}{2}\right)} B\left(\frac{n}{2}-r, r+\frac{1}{2}\right)$$

$$= \frac{n^r}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2} \big/ \cancel{\Gamma\frac{n+1}{2}}} \cdot \Gamma\left(\frac{n}{2}-r\right)\Gamma\left(r+\frac{1}{2}\right) \big/ \cancel{\Gamma\frac{n+1}{2}}$$

$$= \frac{n^r \cdot \Gamma\left(\frac{n}{2}-r\right) \Gamma\left(r+\frac{1}{2}\right)}{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}$$

In particular

$$\mu_2 = n \cdot \frac{\Gamma\left(\frac{n}{2}-1\right)\Gamma\left(1+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} = \frac{n \cdot \Gamma\left(\frac{n}{2}-1\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\left(\frac{n}{2}-1\right)\Gamma\left(\frac{n}{2}-1\right)} = \frac{\frac{n}{2}}{\frac{n-2}{2}} = \frac{n}{(n-2)} = Var$$

and

$$\mu_4 = \frac{n^2\,\Gamma\left(\frac{n}{2}-2\right)\Gamma\left(2+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} = \frac{n^2\,\Gamma\left(\frac{n}{2}-2\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n}{2}\right)} = \frac{n^2\,\Gamma\left(\frac{n}{2}-2\right)\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\Gamma\left(\frac{n}{2}-2\right)}$$

$$= \frac{\frac{3n^2}{4}}{\frac{(n-2)(n-4)}{4}}$$

$$= \frac{3n^2}{(n-2)(n-4)}$$

hence

$$\beta_1 = \frac{\mu_3^2}{\mu_2^2} = 0$$

and
$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\frac{3n^2}{(n-2)(n-4)}}{\frac{n^2}{(n-2)^2}} = 3\left(\frac{n-2}{n-4}\right).$$

as $n\to\infty$ then $\beta_1 \to 0$ and

$$\beta_2 = \lim_{n\to\infty}\left[3\left(\frac{n-2}{n-4}\right)\right] = \lim_{n\to\infty}\left[3\left(\frac{1-\frac{2}{n}}{1-\frac{4}{n}}\right)\right] = 3\left[\frac{(1-0)}{(1-0)}\right] = 3$$

hence for large degree of freedom, i.e. $n\to\infty$, t-distribution tends to normal distribution.

(2) t-distribution becomes Cauchy's distribution for $n=1$.

Proof The probability density function of t-distribution with n degree of freedom is.

$$f(t) = \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}, \quad -\infty < t < \infty$$

put $n=1$, we get

$$f(t) = \frac{1}{\sqrt{1}\,\beta\left(\frac{1}{2},\frac{1}{2}\right)\left(1+\frac{t^2}{1}\right)^{\frac{1+1}{2}}}$$

$$= \frac{1}{\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)}(1+t^2)}$$

$$= \frac{1}{\sqrt{\pi}\sqrt{\pi}\,(1+t^2)} = \frac{1}{\pi(1+t^2)}, \quad -\infty < t < \infty$$

which is the probability density function of Cauchy distribution.
proved

(3) Limiting property

t - distribution tends to a normal distribution as $n \to \infty$.

Proof The probability density function of t - distribution is given by

$$f(t) = \frac{1}{\sqrt{n}\, \beta\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \quad , -\infty < t < \infty$$

as $n \to \infty$

$$\lim_{n\to\infty} f(t) = \lim_{n\to\infty}\left[\frac{1}{\sqrt{n}\, \beta\left(\frac{1}{2}, \frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{n}\, \frac{\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}{\Gamma\frac{n+1}{2}}\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\Gamma\frac{n+1}{2}}{\sqrt{n}\,\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}} \times \frac{1}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\Gamma\frac{n+1}{2}}{\sqrt{n}\,\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}\right] \times \lim_{n\to\infty}\left[\left(1+\frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}\right] \quad \text{—— ①}$$

Then from first factor of equation ①

$$\lim_{n\to\infty}\left[\frac{\Gamma\frac{n+1}{2}}{\sqrt{n}\,\Gamma\frac{1}{2}\,\Gamma\frac{n}{2}}\right] = \lim_{n\to\infty}\left[\frac{\Gamma\frac{n+1}{2}}{\sqrt{n}\,\sqrt{\pi}\,\Gamma\frac{n}{2}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\left\lfloor\frac{n-1}{2}\right.}{\sqrt{n\pi}\,\left\lfloor\frac{n-2}{2}\right.}\right]$$

using stirling's approximation

$$= \lim_{n\to\infty}\left[\frac{\cancel{\sqrt{2\pi}}\; e^{-\left(\frac{n-1}{2}\right)}\left(\frac{n-1}{2}\right)^{\left(\frac{n-1}{2}\right)+\frac{1}{2}}}{\sqrt{n\pi}\;\cancel{\sqrt{2\pi}}\; e^{-\left(\frac{n-2}{2}\right)}\left(\frac{n-2}{2}\right)^{\left(\frac{n-2}{2}\right)+\frac{1}{2}}}\right] ✓$$

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{n\pi}}\; \frac{e^{-\left\{\frac{n-1}{2}-\frac{n-2}{2}\right\}}\left(\frac{n-1}{2}\right)^{\frac{n}{2}}}{\left(\frac{n-2}{2}\right)^{\frac{n-1}{2}}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{n\pi}}\; e^{-\left\{\cancel{\frac{n}{2}}-\frac{1}{2}-\cancel{\frac{n}{2}}+1\right\}}\,(2)^{\frac{n-1}{2}-\frac{n}{2}}\left(\frac{n-1}{n-2}\right)^{\frac{n-1}{2}}(n-1)^{\frac{1}{2}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{n\pi}}\; e^{-\frac{1}{2}}\,(2)^{-\frac{1}{2}}\left(\frac{n-2}{n-1}\right)^{-\left(\frac{n-1}{2}\right)}(n-1)^{\frac{1}{2}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\cancel{\sqrt{n\pi}}}\; e^{-\frac{1}{2}}\,(2)^{-\frac{1}{2}}\left\{1-\frac{1}{n-1}\right\}^{-\left(\frac{n-1}{2}\right)}\cancel{n^{\frac{1}{2}}}\left(1-\frac{1}{n}\right)^{\frac{1}{2}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{\sqrt{2\pi}}\; e^{-\frac{1}{2}}\left\{\left(1-\frac{1}{n-1}\right)^{n-1}\right\}^{-\frac{1}{2}}\left(1-\frac{1}{n}\right)^{\frac{1}{2}}\right]$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}} \lim_{n\to\infty} \left\{\left(1-\frac{1}{n-1}\right)^{n-1}\right\}^{-\frac{1}{2}} \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} \cancel{e^{-\frac{1}{2}}}\, \cancel{e^{\frac{1}{2}}} \cdot 1 = \frac{1}{\sqrt{2\pi}}$$

and taking log of second factor of equation ①

$$\lim_{n\to\infty} \log\left[\left(1+\frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}\right] = \lim_{n\to\infty}\left[-\left(\frac{n+1}{2}\right)\log\left(1+\frac{t^2}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[-\left(\frac{n+1}{2}\right)\left\{\frac{t^2}{n} - \frac{1}{2}\left(\frac{t^2}{n}\right)^2 + \frac{1}{3}\left(\frac{t^2}{n}\right)^3 - \cdots\right\}\right]$$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\left\{-\frac{t^2}{2} + \frac{1}{4}\left(\frac{t^4}{n}\right) - \frac{1}{6}\left(\frac{t^6}{n^2}\right) + \cdots\right\}\right]$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left\{-\frac{t^2}{2} + \frac{1}{4}\left(\frac{t^4}{n}\right) - \frac{1}{6}\left(\frac{t^6}{n^2}\right) + \cdots\right\}\right]$$

$$= \lim_{n\to\infty}\left[\left\{-\frac{t^2}{2} + \frac{1}{4}\left(\frac{t^4}{n}\right) - \frac{1}{6}\left(\frac{t^6}{n^2}\right) + \cdots\right\} + \frac{1}{n}\left\{-\frac{t^2}{2} + \frac{1}{4}\left(\frac{t^4}{n}\right) - \frac{1}{6}\left(\frac{t^6}{n^2}\right) + \cdots\right\}\right]$$

$$= -\frac{t^2}{2} + 0 + \cdots\cdots$$

$$= -\frac{t^2}{2}$$

$$\therefore \lim_{n\to\infty}\left[\left(1+\frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)}\right] = e^{-t^2/2}$$

Therefore from equation ①

$$\lim_{n\to\infty} f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

Hence for large degree of freedom, i.e, $n \to \infty$, the t-distribution tends to standard normal distribution.

Proved

(4) The t-distribution does not depend upon any population parameter, so it is called a non-parametric distribution.

Features of t- distribution Curve

The Curve

$$f(t) = \frac{1}{\sqrt{n}\,\beta\left(\frac{1}{2},\frac{n}{2}\right)\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} \quad ; \quad -\infty < t < \infty$$

has the following main features:

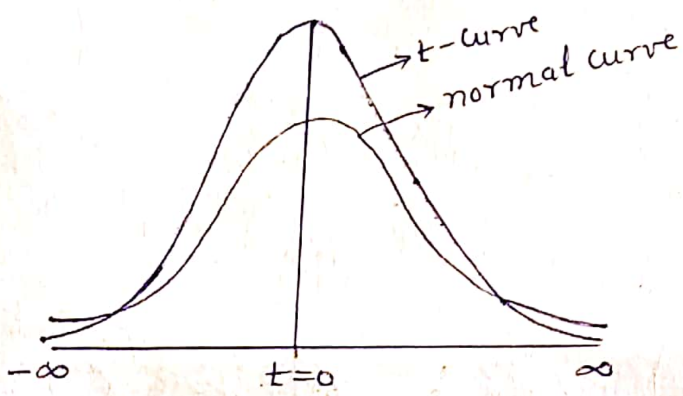
(1) The probability curve is symmetrical about the line $t=0$, since

$f(-t) = f(t)$.

(2) The curve is unimodal with maximum ordinate at $t=0$. Thus the mean and mode coincide at $t=0$.

(3) As t increases, $f(t)$ decreases rapidly and tends to zero as $t\to\infty$, so that t-axis is an asymptote to the curve.

(4) For $n>4$, the curve is pronouncedly peaked than the corresponding normal curve. However peakedness of the cuve goes on decreasing with the increase in the number of degrees of freedom, so much so that for large n it becomes approximately normal ($n>30$)



Problem

A Variate t is said to be a student t-distribution on n degree of freedom if its probability density function is given by-

$$f(t) = \frac{K}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}, \quad -\infty<t<\infty$$

Find the value of K and show that $f(t)$ is a probability density function of t-distribution.

Solution Since the total area under the probability curve is unity, i.e,

$$\int_{-\infty}^{\infty} f(t)dt = 1.$$

or $$\int_{-\infty}^{\infty} \frac{K}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} dt = 1$$

or $$K\int_{-\infty}^{\infty} \frac{dt}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}} = 1.$$

or $$K = \frac{1}{\int_{-\infty}^{\infty} \frac{dt}{\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}}$$

put $t^2 = n\tan^2\theta$

$t = \sqrt{n}\tan\theta$

$dt = \sqrt{n}\sec^2\theta\, d\theta$

If $t \to -\infty$ then $\theta \to -\frac{\pi}{2}$ and $t \to \infty$ then $\theta \to \frac{\pi}{2}$, So

$$K = \frac{1}{\int_{-\pi/2}^{\pi/2} \frac{1}{\left(1+\frac{\cancel{n}\tan^2\theta}{\cancel{n}}\right)^{\frac{n+1}{2}}} \sqrt{n}\sec^2\theta\, d\theta}$$

$$= \frac{1}{\int_{-\pi/2}^{\pi/2} \frac{\sqrt{n}\sec^2\theta}{(\sec^2\theta)^{\frac{n+1}{2}}} d\theta}$$

$$= \frac{1}{\sqrt{n}\int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta}{\sec^{n+1}\theta} d\theta}$$

$$= \frac{1}{\sqrt{n}\int_{-\pi/2}^{\pi/2} \sec^{-n+1}\theta\, d\theta}$$

$$= \frac{1}{\sqrt{n}\int_{-\pi/2}^{\pi/2} \cos^{n-1}\theta\, d\theta}$$

$$= \frac{1}{2\sqrt{n}\int_{0}^{\pi/2} \cos^{n-1}\theta\, d\theta}$$

$$= \frac{1}{\cancel{2}\sqrt{n}\, \frac{\Gamma\frac{n}{2}\,\Gamma\frac{1}{2}}{\cancel{2}\,\Gamma\frac{n+1}{2}}} \qquad \left(\text{Since } \int_0^{\pi/2} \cos^m\theta \sin^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}\right)$$

$$= \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})}$$

Therefore

$$f(t) = \frac{1}{\sqrt{n}\,\beta(\frac{1}{2},\frac{n}{2})\left(1+\frac{t^2}{n}\right)^{\frac{n+1}{2}}}, \quad -\infty < t < \infty$$

which is the probability density function of t-distribution.