Matrices and Differential Equations

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Course MATH-2111

Section-A (Matrices)

1 Algebra of Matrices

Definition 1.1. *A set of mn numbers, real or complex, arranged in a rectangular array of m rows and n columns such as* $\begin{pmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots a_{mn} \end{pmatrix}$ *is called a matrix of order $m \times n$. If $m = n$, then the matrix is called a square matrix of order n. The element in the ith row and jth column of the matrix is usually denoted by small letter a_{ij} and the matrix is usually denoted by capital letters like A or $[a_{ij}]$ or (a_{ij}), where $i = 1, 2, \dots, m; j = 1, 2, \dots, n, i \neq j$.*

Definition 1.2. *Any matrix obtained by deleting any number of rows and any number of columns from a given matrix is called a sub-matrix of the given matrix.*

Definition 1.3. *Let $A = (a_{ij}), i = j$ be a square matrix. Then*

(i) *the elements a_{ii} are called diagonal elements of A;*

(ii) *the sum of the elements a_{ii} is called the trace of A and is denoted by* $\text{tr}.A = \sum_{i=1}^{n} a_{ii}$.

Types of Matrices

Definition 1.4. *Definition of different types of matrices are given below:*

(i) *A matrix having all of its elements zero is said to be a Null matrix and is denoted by* $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(ii) *A square matrix of order n having all its diagonal elements unity and zero elements everywhere else is called a unit matrix or an identity matrix and is denoted by I_n. Thus,*

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

(iii) *A matrix which has only one row is called a row matrix. Thus, a row matrix can be written as* $\begin{pmatrix} a_1 & a_2 & \dots & a_p \end{pmatrix}$.

(iv) *A matrix which has only one column is called a column matrix. Thus the column matrix can be defined as follows:* $\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_q \end{pmatrix}$.

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(v) *A square matrix $A = (a_{ij})$ of order n is said to be an upper triangular if its elements $a_{ij} = 0$ for $i > j$. Thus, an upper triangular matrix of order n can be defined as follows:*
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

(vi) *A square matrix $A = (a_{ij})$ of order n is said to be a lower triangular if its elements $a_{ij} = 0$ for $i < j$. Thus, a lower triangular matrix of order n can be defined as follows:*
$$\begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

(vii) *A square matrix $A = (a_{ij})$ of order n which is both upper and lower triangular is called a diagonal matrix. Thus, a diagonal matrix of order n can be defined as follows:*
$$\begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

(viii) *A square matrix $A = (a_{ij})$ of order n is said to be a scalar matrix if all the diagonal elements are equal to a scalar λ (say), i.e., $a_{11} = a_{22} = \ldots = a_{nn}$. Thus, a scalar matrix of order n can be defined as follows:*
$$\begin{pmatrix} \lambda & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix}.$$

(ix) *Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be equal if both are of the same order $m \times n$ and each element a_{ij} of A is equal to the corresponding element b_{ij} of B, i.e. $a_{ij} = b_{ij}$ for each pair of subscripts i and j.*

Addition and Subtraction of Matrices

Definition 1.5. *Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be conformable for addition (subtraction) if they have the same number of rows and the same number of columns. Thus the sum (difference) of the two matrices A and B is then a matrix each of whose elements is the sum (difference) of corresponding elements of A and B, i.e., $A+B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$ and $A - B = (a_{ij}) - (b_{ij}) = (a_{ij} - b_{ij})$. For example, if*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}.$$

Then
$$A + B == \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{pmatrix}.$$

The subtraction can be defined as $A - B = A + (-B)$.

Multiplication of Matrices

Definition 1.6. *Two matrices A and B are conformable for multiplication if and only if the number of columns of A is equal to the number of rows of B. The product of the two matrices A and B is denoted by AB and then it is defined as the matrix whose elements in the ith row of A and jth column is the algebraic sum of the products of the elements in the ith row of A by the corresponding elements in the jth column of B. For example, if* $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$ *and* $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$. *Then the product of A and B is*

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \\ a_{41}b_{11} + a_{42}b_{21} + a_{43}b_{31} & a_{41}b_{12} + a_{42}b_{22} + a_{43}b_{32} \end{pmatrix}.$$

Note that $AB \neq BA$ in general.

Problem 1.1. *Show that the commutative law for multiplication does not hold in general.*

Proof. Consider the matrices

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 3 \\ -3 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}.$$

These are conformable for multiplication and so

$$AB = \begin{pmatrix} 1+0 & 0-2 & 2-6 \\ 2+0 & 0+3 & 4+9 \\ -3+0 & 0+1 & -6+3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -4 \\ 2 & 3 & 13 \\ -3 & 1 & -3 \end{pmatrix} \text{ and}$$

$$BA = \begin{pmatrix} 1+0-6 & -2+0+2 \\ 0+2-9 & 0+3+3 \end{pmatrix} = \begin{pmatrix} -5 & 0 \\ -7 & 6 \end{pmatrix}.$$

It is apparent that the product matrix AB is of order 3×3 while the product matrix BA is of order 2×2 and therefore the two matrices are quite different i.e. $AB \neq BA$. It seems that the commutative law of multiplication does not hold. □

Problem 1.2. *Prove that* $\text{tr}\,(AB) = \text{tr}\,(BA)$, *all matrices being square of order n.*

Definition 1.7. *Let A and B be the n-square matrices. Then A and B are said to be*

(i) *Commute if $AB = BA$.*

(ii) *Anti-commute if $AB = -BA$.*

Definition 1.8. *Let A be a square matrix. Then*

(i) *A is said to be idempotent if $A^2 = A$.*

(ii) *A is said to be periodic if k is the positive integer such that $A^{k+1} = A$.*

(iii) *A is said to be of period k if k is the least positive integer such that $A^{k+1} = A$.*

Definition 1.9. *Let A be a square matrix. Then*

(i) *A is said to be nilpotent if ρ is some positive integer such that $A^\rho = 0$.*

(ii) *A is said to be nilpotent of index ρ if ρ is the least positive integer such that $A^\rho = 0$.*

Problem 1.3. *Let A and B are n-square matrices. Show that*

(i) *If $AB = A$ and $BA = B$, then A and B are idempotent.*

(ii) *If A and B are idempotent matrices, then $A+B$ is idempotent if and only if $AB = BA = 0$.*

Problem 1.4. *If A and B are n-square matrices, then show that A and B are commute if and only if $A - \lambda I$ and $B - \lambda I$ commute for every scalar λ.*

Problem 1.5. *Derive a rule for forming the product BA of an $m \times n$ matrix B and $A = \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn})$.*

The Transpose of a Matrix

Definition 1.10. *The matrix of order $n \times m$ obtained from any matrix A of order $m \times n$ by interchanging its rows and columns is called the transpose of A and is denoted by A' or A^T. Thus, if $A = (a_{ij})$, then $A' = (b_{ij})$, where $b_{ij} = a_{ji}$, i.e. the (j,i)th element of A' is the (i,j)th element of A. For instance, if*

$$\begin{pmatrix} 5 & 4 & -2 \\ 1 & 0 & 2 \\ 3 & 4 & -5 \\ 6 & -4 & 2 \end{pmatrix}, \text{ then } A' = \begin{pmatrix} 5 & 1 & 3 & 6 \\ 4 & 0 & 4 & -4 \\ -2 & 2 & -5 & 2 \end{pmatrix}.$$

Remark 1.1. *The following are the properties of the transpose of a matrix:*

(i) *The transpose of a matrix coincides with itself, i.e., if A is matrix, then $(A')' = A$.*

(ii) *The determinant of the transpose of a square matrix is the same as the determinant of the matrix, i.e., if A is a matrix, then $|A'| = |A|$.*

(iii) *If k is any scalar and A is a matrix, then $(kA)' = kA'$.*

(iv) *The transpose of the sum of two matrices A and B (conformable for addition) is the sum of their transpose, i.e., $(A+B)' = A' + B'$.*

(v) *The transpose of the product of two matrices A and B (conformable for multiplication) is the product of their transpose taken in reverse order, i.e., $(AB)' = B'A'$. This property is also called the reversal law for a transpose.*

Definition 1.11. *Let A be a square matrix. Then A is said to be*

(i) *Symmetric if its transpose coincides with itself, i.e., $A' = A$.*

(ii) *Skew-symmetric if $A' = -A$. It is clear that $a_{ij} = -a_{ji}$ for all integral values of i and j. Thus, for $i = j$, we have that $a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$, which shows that all the diagonal elements of a skew-symmetric matrix are zero.*

For instances, let

$$A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}, \text{ then } A' = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} 0 & -1 & -2 & 3 \\ 1 & 0 & 4 & -5 \\ 2 & -4 & 0 & -6 \\ -3 & 5 & 6 & 0 \end{pmatrix}, \text{ then } B' = \begin{pmatrix} 0 & 1 & 2 & -3 \\ -1 & 0 & -4 & 5 \\ -2 & 4 & 0 & 6 \\ 3 & -5 & -6 & 0 \end{pmatrix}.$$

Problem 1.6. *Prove that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.*

Proof. Let A be any square matrix. Then we have that

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A').$$

Denote $P = \dfrac{1}{2}(A + A')$ and $Q = \dfrac{1}{2}(A - A')$. Then $A = P + Q$. Now,

$$\begin{aligned} P' &= \left\{\frac{1}{2}(A + A')\right\}' = \frac{1}{2}\{A' + (A')'\} \\ &= \frac{1}{2}(A' + A) = P. \end{aligned}$$

This follows that P is a symmetric matrix. Also, we have that

$$\begin{aligned} Q' &= \left\{\frac{1}{2}(A - A')\right\}' = \frac{1}{2}\{A' - (A')'\} \\ &= \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A') = -Q. \end{aligned}$$

This implies that Q is skew-symmetric matrix. Thus, the square matrix A is expressible as the sum of a symmetric matrix P and skew-symmetric matrix Q. □

Problem 1.7. *If A and B are symmetric matrices, then show that AB is symmetric matrix if and only if A and B are commute.*

Problem 1.8. *If A is a m-square matrix and P is a matrix of order $m \times n$, then show that $B = P'AP$ will be symmetric or skew-symmetric according as A is symmetric or skew-symmetric.*

The Adjoint of a Matrix or Adjugate Matrix

Definition 1.12. *Let $A = (a_{ij})$ be a square matrix of order n and C_{ij} represents the cofactor of the element a_{ij} in the determinant $|A|$. Then the transpose of the matrix $C = (C_{ij})$ is called the adjoint of A and is denoted by* adj A. *Thus, if*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \text{ then adj } A = \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}.$$

As an illustrative example if $A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$, then, in the determinant $|A|$, we have that

$$\begin{aligned} \text{cofactor of } a, C_{11} &= (-1)^{1+1}\begin{vmatrix} b & f \\ f & c \end{vmatrix} = bc - f^2. \\ \text{Cofactor of } h, C_{12} &= (-1)^{1+2}\begin{vmatrix} h & f \\ g & c \end{vmatrix} = fg - ch. \\ \text{Cofactor of } g, C_{13} &= (-1)^{1+3}\begin{vmatrix} h & b \\ g & f \end{vmatrix} = hf - bg. \end{aligned}$$

etc.

Therefore, $C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \begin{pmatrix} bc - f^2 & gf - ch & hf - bg \\ gf - ch & ac - g^2 & gh - af \\ hf - bg & gh - af & ab - h^2 \end{pmatrix}$.

Hence, adj $A = C' = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} bc - f^2 & gf - ch & hf - bg \\ gf - ch & ac - g^2 & gh - af \\ hf - bg & gh - af & ab - h^2 \end{pmatrix}$.

Problem 1.9. *Let A be a square matrix of order n and I be the unit matrix of the same order, then* $A(\text{adj } A) = |A|I = (\text{adj } A)A$.

Problem 1.10. *Let A be a square matrix of order n. Then*

(i) $|\text{adj } A| = |A|^{n-1}$ *if* $|A| \neq 0$.

(ii) $\text{adj } (\text{adj}A) = |A|^{n-2}A$ *if* $|A| \neq 0$.

(iii) $A(\text{adj})A = (\text{adj } A)A = 0$ *if* $|A| = 0$.

Problem 1.11. *Let A and B be two n-square matrices. Then prove that* $\text{adj}(AB) = \text{adj } B \cdot \text{adj } A$.

Problem 1.12. *Compute the adjoint of* $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$.

Definition 1.13. *A square matrix $A = (a_{ij})$ is known as singular if its determinant $|A| = 0$. On the other hand, if $|A| \neq 0$, then the matrix A is called the non-singular matrix.*

As an illustrative example, $A = \begin{pmatrix} 12 & 3 & 7 \\ 27 & 7 & 17 \\ 36 & 9 & 22 \end{pmatrix}$. Then A is non-singular, since $|A| = 3 \neq 0$.

The Inverse of a Matrix or Reciprocal Matrix

Definition 1.14. *Let A be a square matrix of order n. If there exists a matrix B of order n such that $AB = BA = I$, where I is the n order unit matrix, then B is said to be the inverse of A and is denoted by A^{-1}. Thus, by definition, $AA^{-1} = A^{-1}A = I$. When the inverse of a matrix A exists, then A is said to be invertible.*

Problem 1.13. *Let A be a square matrix of order n which is invertible. Establish the formula for finding the inverse of A in terms of adjoint of A.*

Solution: According to the definition of a inverse matrix A, we have that

$$AA^{-1} = A^{-1}A = I. \tag{1.1}$$

Moreover, we know that

$$A(\text{adj } A) = |A|I = (\text{adj } A)A \Rightarrow A\Big(\frac{1}{|A|}\text{adj } A\Big) = \Big(\frac{1}{|A|}\text{adj } A\Big)A = I. \tag{1.2}$$

Comparing (1.1) and (1.2), we have that

$$AA^{-1} = A\Big(\frac{1}{|A|}\text{adj } A\Big) \text{ and } A^{-1}A = \Big(\frac{1}{|A|}\text{adj } A\Big)A.$$

Either of the above equality lead to

$$A^{-1} = \frac{1}{|A|}\text{adj } A,$$

which is the required formula for finding the inverse of a matrix in terms of the adjoint of the given matrix.

Problem 1.14. *Find the inverse of the matrices* (a) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & -1 & 3 \\ -1 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$.

Problem 1.15. *If $D = \text{diag}(d_1, d_2, \ldots, d_n), d_1, d_2, \ldots, d_n \neq 0$, then prove that*

$$D^{-1} = \text{diag}(d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1}).$$

Definition 1.15. *A real matrix A is said to be orthogonal if $AA' = I = A'A$. The determinant of an orthogonal matrix is $+1$ or -1. The orthogonal matrix is said to be proper or improper according as its determinant is $+1$ or -1.*

Problem 1.16. *If A is real skew-symmetric matrix such that $A^2 + I = 0$, then show that A is orthogonal.*

Proof. Given that

$$A^2 + I = 0. \tag{1.3}$$

Moreover, it is given that A is real skew-symmetric, so we have that $A = -A'$. Thus, $A^2 = -AA' \Rightarrow -A^2 = AA'$. This adding with (1.3), we have that $AA' = I$, which follows that the matrix A is orthogonal. □

2 Elementary Matrix and Elementary Operations

Definition 2.1. *A square matrix of order n is said to be elementary matrix if it is obtained from a unit matrix I_n by subjecting it to any of the following elementary operations (transformations):*

(i) *Interchanging of any two rows (columns) to be denoted by R_{ij} (C_{ij}) for the interchange of ith and jth rows (columns) and the elementary matrix obtained may be denoted by E_{ij}.*

(ii) *Multiplying of elements of any row (column) by any non-zero scalar to be denoted by $R_{i(\lambda)}$ ($C_{i(\lambda)}$) for the multiplication of ith row (column) by $\lambda =\neq 0$ and the elementary matrix obtained may be denoted by $E_{i(\lambda)}$.*

(iii) *Addition to the elements of any row (column) the corresponding elements of another row (column) multiplied by non-zero scalar to be denoted by $R_{ij(\lambda)}$ ($C_{ij(\lambda)}$) for the addition to ith row (column) of the jth row (column) multiplied by $\lambda \neq 0$ and the elementary matrix obtained may be denoted by $E_{ij(\lambda)}$ for row operations and by $E'_{ij(\lambda)}$ for column operations, since $E'_{ij(\lambda)}$ is the transpose of $E_{ij(\lambda)}$.*

As illustrative examples, the elementary matrices obtained from $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *are*

$$E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_{2(\lambda)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } E_{23(\lambda)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix}.$$

Definition 2.2. *An elementary operation is said to be the Row operation or Column operation according as it is obtained to rows or columns.*

Theorem 2.1. *If a square matrix A is reduced to an identity matrix I by a series of elementary row operations, then the same series of row operations applied to I yields the inverse of A, i.e., A^{-1}.*

Definition 2.3. *A matrix B is called the equivalent to a matrix A if B cab be obtained from A by a sequence of elementary transformations and it is denoted by $B \sim A$. As an illustrative example, if* $A = \begin{pmatrix} -6 & -2 & -4 & 5 \\ 3 & 4 & 5 & -1 \\ 6 & 2 & 4 & -3 \end{pmatrix}$, *then by $R_{13(1)}$, we obtain that* $B = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 3 & 4 & 5 & -1 \\ 6 & 2 & 4 & -3 \end{pmatrix}$.

Definition 2.4. *Two matrices A and B are equivalent if and only if there exist non-singular matrices P and Q such that $B = PAQ$.*

Definition 2.5. *Every non-zero matrix A of order $m \times n$ can be reduced by application of elementary row and column operations into equivalent matrix of one of the following forms:* (i) $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$; (ii) $\begin{pmatrix} I_r \\ O \end{pmatrix}$; (iii) $\begin{pmatrix} I_r & O \end{pmatrix}$; (iv) $\begin{pmatrix} I_r \end{pmatrix}$, *where I_r is $r \times r$ identity matrix and O is null matrix of any order. These four forms are called normal or canonical form of A.*

In another way, one can identify that canonical matrix is a non-zero matrix in which

(i) the first few rows have non-zero elements while the elements of succeeding rows may be all zero.

(ii) the first non-zero row is unity and

(iii) all the other elements of a column which contains the first non-zero element as unity are zero. As an illustrative example, the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is canonical.

Problem 2.1. *Reduce the matrix A to its normal form, where* $A = \begin{pmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{pmatrix}$.

Solution: Given that

$$\begin{aligned}
A &= \begin{pmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{pmatrix} \text{ by } C_{12} \\
&\sim \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{pmatrix} \text{ by } R_{31(-1)} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{pmatrix} \text{ by } C_{21(-2)} \text{ and } C_{31(-2)} \\
&\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 2 & 1 & 3 \end{pmatrix} \text{ by } R_{2(\frac{1}{4})} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ by } R_{32(-2)} \\
&\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ by } C_{32(-\frac{1}{2})} \text{ and } C_{42(\frac{3}{2})} \\
&\sim \begin{pmatrix} I_2 & O \\ O & O \end{pmatrix}, \text{ which gives the normal form of } A.
\end{aligned}$$

Problem 2.2. *Applying elementary transformations, find the inverse of the matrix* $A == \begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -3 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{pmatrix}$.

Solution: We may write, $A = IA$. This implies that

$$\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -3 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} A$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 3 & 2 & 7 \\ -1 & 1 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} A \text{ by } R_{12(1)}, R_{24(1)} \text{ and } R_{34(1)}$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 3 & 2 & 7 \\ 0 & 4 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} A \text{ by } R_{41(1)}$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 2 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix} A \text{ by } R_{32(-1)} \text{ and } R_{42(-2)}$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 4 & -3 & 0 & -3 \\ 3 & -4 & 1 & -3 \\ 1 & -1 & 0 & -1 \end{pmatrix} A \text{ by } R_{14(1)}, R_{24(4)} \text{ and } R_{34(3)}$$

$$\Rightarrow \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 3 & -4 & 1 & -3 \\ -1 & 1 & 0 & 1 \end{pmatrix} A \text{ by } R_{23(-1)} \text{ and } R_{4(-1)}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{pmatrix} A \text{ by } R_{12(-3)} \text{ and } R_{32(-1)}.$$

This follows that $A^{-1} = \begin{pmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{pmatrix}$.

Rank of a Matrix

Definition 2.6. *If any r rows and any r columns from an $m \times n$ matrix A are retained and remaining $(m - r)$ rows and $(n - r)$ columns removed, then the determinant of the remaining $r \times r$ submatrix of A is called the minor of A of order r.*

Definition 2.7. *If in an $m \times n$ matrix A, at least one of its $r \times r$ minors is different from zero while all the minors of order $(r + 1)$ are zero, then r is defined as the rank of the matrix A.*

Remark 2.1. (i) *The rank of a matrix A is denoted by $\rho(A)$.*

(ii) *The rank of zero matrix is 0 i.e. $\rho(O) = 0$.*

(iii) *The rank of a matrix remains unaltered by the application of elementary row or column operations.*

Definition 2.8. *(***Echelon Form of a Matrix***) If in a matrix*

(i) *all the non-zero rows, if any, precede the zero rows;*

(ii) *the number of zero preceding the first non-zero element in a row is less than the number of such zero in the succeeding row.*

(iii) *the first non-zero element in a row is unity,*

then it is called the echelon form of a matrix.

Remark 2.2. *The number of non-zero rows of a matrix given in the echelon form is its rank.*

Example 2.1. *Find the rank of the matrix* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{pmatrix}$.

Solution: The height order minor is $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{vmatrix} = 1(40-42)-2(20-21)+3(12-12) = 0.$

The second order minors are $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$; $\begin{vmatrix} 2 & 3 \\ 6 & 10 \end{vmatrix}$; etc. Thus, $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 4 - 4 = 0$ and $\begin{vmatrix} 2 & 3 \\ 6 & 10 \end{vmatrix} = 20 - 18 = 2 \neq 0$. Hence, the rank of the matrix A is 2.

Example 2.2. *Find the rank of the matrix* $A = \begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{pmatrix}$.

Solution: Here given that

$$\begin{aligned} A &= \begin{pmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ -1 & -3 & -4 & -3 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 3 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ by } R_2' = 3R_1 - R_2, R_3' = R_1 + R_3. \end{aligned}$$

Since there is one non-zero row after reducing the matrix A by echelon form, then the rank of the matrix A is 1.

Example 2.3. *Reduce the matrix A to the normal (canonical) form and hence find its rank, where* $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{pmatrix}$.

Solution: Given that

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & -2 & 0 \\ 1 & -3 & -5 & -6 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ by } C_{21(-1)}, C_{31(-2)} \text{ and } C_{41(-3)} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & -5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ by } R_{21(-1)}, R_{31(-1)} \text{ and } R_{41(-1)} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ by } R_{2(\frac{1}{2})} \text{ and } R_{3(-1)} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 8 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ by } C_{32(1)} \text{ and } C_{4(\frac{1}{4})} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ by } C_{24(-3)} \text{ and } C_{3(\frac{1}{8})} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ by } C_{43(-1)} \sim \begin{pmatrix} I_3 & O \\ O & O \end{pmatrix}, \end{aligned}$$

which is the required normal form and its rank is 3.

Solution of System of Linear Equations

To find the solution of any system of linear equations, homogeneous or non-homogeneous, we will make an attempt to apply the concepts and consequences of matrices.

Definition 2.9. *Let the system of m simultaneous equations in n unknowns $x_1, x_2, \ldots, x_n$ be*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots & \quad\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

or written in a compact form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \text{ where } i = 1, 2, \ldots, m. \tag{2.1}$$

Then the matrix $A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$ *of order $m \times n$ is known as the matrix of coefficient of the system of equations given by* (2.1).

If there be an n equations in (2.1), we write

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix},$$

which is called the determinant of the coefficient matrix A of the system of equations given by (2.1).

Remark 2.3. *If all b's are zero, then the system of equations given by* (2.1) *is said to be homogeneous and if at least one of b's is not zero, then the system of equations defined by* (2.1) *is said to be non-homogenous.*

Solution of Homogeneous Simultaneous Linear Equations

Definition 2.10. *Suppose that all b's are zero in the system* (2.1)*. Then the set of equations may be expressed as follows:*

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \end{pmatrix},$$

or, in contracted form it may be written as

$$AX = O, \tag{2.2}$$

where $A = (a_{ij})$ is the coefficient matrix of order $m \times n$, X is the matrix of order $n \times 1$ and O is the null matrix of order $m \times 1$.

Now, suppose that r is the rank of the matrix A of order $m \times n$. As regards the solution of the set of homogeneous linear equations, the following results can be summarised:

(i) A system of homogenous linear equations always have one or more solutions. In this case either $r = n$ or $r < n$. If $r = n$, then the (2.2) will have no linearly independent solutions and hence zero is the only solution. On the other hand, if $r < n$, there will be $n - r$ independent solutions and therefore the system (2.2) will have more than one solution.

(ii) The number of linearly independent solutions of (2.2) is $(n-r)$, i.e., if we assign arbitrary values to $(n-r)$ of the variables, then the values of the others can be uniquely determined.

Example 2.4. *Solve by matrix method*

$$\begin{aligned} 2x + 3y - z &= 0 \\ x - y - 2z &= 0 \\ 4x + y - 5z &= 0. \end{aligned}$$

Solution: The given system of equations is equivalent to the matrix equation

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & -1 & -2 \\ 4 & 1 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 5 & 3 \\ 1 & -1 & -2 \\ 0 & 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ by } R_{12(-1)} \text{ and } R_{32(-4)}$$

$$\sim \begin{pmatrix} 0 & 5 & 3 \\ 1 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ by } R_{31(-1)},$$

which is equivalent to

$$\begin{aligned} 5y + 3z &= 0 \\ x - y - 2z &= 0. \end{aligned}$$

This implies that $x = \frac{7z}{5}, y = -\frac{3z}{5}$. This gives the solution for arbitrary values of z. The general solution is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{7z}{5} \\ -\frac{3z}{5} \\ z \end{pmatrix} = z \begin{pmatrix} \frac{7}{5} \\ -\frac{3}{5} \\ 1 \end{pmatrix} = k \begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}, \text{ where } z = 5k \text{ is an arbitrary parameter.}$$

Example 2.5. *Solve by matrix method*

$$\begin{aligned} 4x+2y+z+3u &= 0 \\ 6x+3y+4z+7u &= 0 \\ 2x+y+u &= 0. \end{aligned}$$

Example 2.6. *Solve by matrix method*

$$\begin{aligned} x+2y+3z &= 0 \\ 3x+4y+4z &= 0 \\ 7x+10y+12z &= 0. \end{aligned}$$

Solution of Non-homogeneous Simultaneous Linear Equations

Definition 2.11. *The system of equations given in* (2.1) *can be written in the matrix form as follows:*

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix},$$

or, in more compact form it may be written as

$$AX = B, \tag{2.3}$$

where $A = (a_{ij})$ is the coefficient matrix of the system of equations given by (2.1)*, X is the transpose matrix of $(x_1, x_2, \dots, x_n)$ and B is the transpose matrix of $(b_1, b_2, \dots, b_m)$.*

Definition 2.12. *If a set of values of $x_1, x_2, x_3, \dots, x_m$ satisfy m simultaneous equations in* (2.1)*, i.e. if the system* (2.1) *has a solution, then the equations are said to be consistent, otherwise the equations are said to be inconsistent.*

A consistent system of equations has either one solution or infinitely many solutions.

Let the coefficient matrix A in the system (2.3) be non-singular. By multiplying both sides of (2.3) by A^{-1}, we have that

$$A^{-1}(AX) = A^{-1}B \quad \Rightarrow \quad (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B,$$

which is the required solution of the given system of equations and is unique.

Definition 2.13. *The matrix* $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ *augmented by the matrix* $B = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$ *is called the augmented matrix of A and it is written as A^* or $[A, B]$ and defined by*

$$[A, B] = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}.$$

Theorem 2.2. *The system of non-homogeneous equations $AX = B$ possesses a solution if* Rank of $A =$ Rank of $[A, B]$*, i.e., if r and ρ be the ranks of the matrices A and $[A, B]$ respectively, then the given equations are consistent when $r = \rho$ and inconsistent when $r < \rho$.*

Problem 2.3. *State the conditions under which a system of non-homogeneous linear equations will have* (i) *no solution,* (ii) *a unique solution and* (iii) *an infinitely many solutions.*

Solution: Let the system of non-homogeneous linear equations be equivalent to the matrix equation $AX = B$, where A is of order $m \times n$, X be of order $n \times 1$ and B be of order $m \times 1$. Then

(i) The system $AX = B$ has no solution if Rank of $A \neq$ Rank of $[A, B]$.

(ii) The system $AX = B$ has a unique solution if Rank of $A =$ Rank of $[A, B] =$ number of unknown variables. In particular, if A is a square matrix, then $AX = B$ will have a unique solution if $|A| \neq 0$.

(iii) The system $AX = B$ will have an infinite number of solutions if Rank of $A =$ Rank of $[A, B] <$ number of unknown variables.

Example 2.7. *Solve the equations by using matrix methods*

$$\begin{aligned} x + y + z &= 6 \\ x - y + z &= 2 \\ 2x + y - z &= 1. \end{aligned}$$

Solution:
The given system of equations is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 3 \end{pmatrix} \text{ by } R_{21(-1)} \text{ and } R_{32(1)}.$$

Set corresponding elements are equal. Then we have that

$$\begin{aligned} x + y + z &= 6 \\ -2y &= -4 \\ 3x &= 3. \end{aligned}$$

This gives that $x = 1, y = 2, z = 3$.

Problem 2.4. *Show that the equations*

$$\begin{aligned} x + 2y - z &= 3 \\ 3x - y + 2z &= 1 \\ 2x - 2y + 3z &= 2 \\ x - y + z &= -1 \end{aligned}$$

are consistent and solve them.

Solution: The given system of equations is equivalent to the matrix equation

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ -1 \end{pmatrix}. \tag{2.4}$$

Denoting the coefficient matrix by A and augmented matrix by $[A, B]$ we have

$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix}$; and $[A, B] = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{pmatrix}$. Here, we observe that fourth order minor of A is zero but none of the third order minor of A is not zero. Therefore, the rank of A is 3.

Again, fourth order minor of $[A, B]$ is

$$\begin{vmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 4 & -1 & 2 & 3 \\ 4 & -2 & 3 & 1 \\ 4 & -4 & 5 & 2 \\ 0 & 0 & 0 & -1 \end{vmatrix} \text{ by } C_1' = C_1 + C_4, C_2' = C_2 - C_4, C_3' = C_3 + C_4$$

$$= \begin{vmatrix} 4 & -1 & 2 \\ 4 & -2 & 3 \\ 4 & -4 & 5 \end{vmatrix} \text{ by expanding along 4th row}$$

$$= -4 \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{vmatrix} = -4 \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{vmatrix} \text{ by } R_2' = R_2 - R_1, R_3' = R_3 - R_1$$

$$= -4 \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0.$$

This shows that the 4th order minor of $[A, B]$ vanishes but none of the 3rd order minors are not vanishes, which can be seen easily. Thus, the rank of $[A, B]$ is also 3.

This show that rank of A= rank of $[A, B]$. Hence the given system of equations is consistent.

Now, we need to solve the given system of equations. From (2.4) we have that

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \\ 0 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ -4 \\ -4 \end{pmatrix} \text{ by } R_{21(-3)}, R_{31(-2)} \text{ and } R_{41(-1)}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & \frac{5}{7} \\ 0 & 0 & -\frac{1}{7} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ \frac{20}{7} \\ -\frac{4}{7} \end{pmatrix} \text{ by } R_{32(-\frac{6}{7})} \text{ and } R_{42(-\frac{3}{7})}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & \frac{5}{7} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ \frac{20}{7} \\ 0 \end{pmatrix} \text{ by } R_{43(\frac{1}{5})},$$

which is equivalent to

$$\begin{aligned} x + 2y - z &= 3 \\ -7y + 5z &= -8 \\ \frac{5}{7}z &= \frac{20}{7}. \end{aligned}$$

Solving these equations we get $x = -1, y = 4, z = 4$.

Problem 2.5. *Apply rank test to examine if the following system of equations is consistent and if consistent, then find the complete solution.*

$$\begin{aligned} 2x - y + 3z &= 8 \\ -x + 2y + z &= 4 \\ 3x + y - 4z &= 0. \end{aligned}$$

Problem 2.6. *Apply rank test to examine if the following system of equations is consistent and if consistent, then find the complete solution.*

$$\begin{aligned} 2x + 4y - z &= 9 \\ 3x - y + 5z &= 5 \\ 8x + 2y + 9z &= 19. \end{aligned}$$

Problem 2.7. *Show that the following system of equations*

$$\begin{aligned} 5x + 3y + 7z &= 4 \\ 3x + 26y + 2z &= 9 \\ 7x + 2y + 10z &= 5 \end{aligned}$$

is consistent and solve them.

Problem 2.8. *For what values of λ, the following system of equations*

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 4z &= \lambda \\ x + 4y + 10z &= \lambda^2 \end{aligned}$$

have a solution and solve completely in each case.

Problem 2.9. *Show the system of equations*

$$\begin{aligned} -2x + y + z &= a \\ x - 2y + z &= b \\ x + y - 2z &= c \end{aligned}$$

have no solution unless $a + b + c = 0$ in which case they have infinitely solutions. Find the solution when $a = 1, b = 1, c = -2$.

Problem 2.10. *Prove that if the system of equations*

$$\begin{aligned} x &= ay + z \\ y &= z + ax \\ z &= x + y \end{aligned}$$

is consistent, then $a + 1 = 0$.

3 Characteristic Matrix and Characteristic Equation of a Matrix

Definition 3.1. *Let A be an $n \times n$ matrix and I be an $n \times n$ identity matrix.*

(i) *Then the matrix polynomial $A - \lambda I$ is called the characteristic matrix of A.*

(ii) *The determinant $|A - \lambda I|$ is called the characteristic polynomial of the matrix A.*

(iii) *The equation $|A - \lambda I| = 0$ is said to be the characteristic equation of the matrix A.*

(iv) *The roots of the equation $|A - \lambda I| = 0$ are called the characteristic roots or latent roots or eigenvalues or characteristic values or latent values or proper values of A.*

(v) *The set of all eigenvalues of the matrix A is called the spectrum of A.*

(vi) *The problem of finding the eigenvalues of a matrix is known as an eigenvalue problem.*

Problem 3.1. *Find the eigenvalues of the matrix* $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix}$.

Solution: The characteristic equation is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 2-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$
$$\Rightarrow \quad (1-\lambda)(2-\lambda)^2 = 0 \Rightarrow \lambda = 1, 2, 2.$$

Therefore the eigenvalues are 1, 2 and 2.

Problem 3.2. *Find the eigenvalues of the matrix* $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

Problem 3.3. *If* $a_1, a_2, \ldots, a_n$ *are the eigenvalues of the square matrix* A *and* μ *is a scalar, then show that the eigenvalues of* $A - \mu I$ *are* $a_1 - \mu, a_{-}\mu, \ldots, a_n - \mu$.

Problem 3.4. *If* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the eigenvalues of the square matrix* A*, then show that* A' *has also the same eigenvalues.*

Definition 3.2. *Let* λ *be an eigenvalue of a matrix* A*. Then a non-zero vector* v *is called a eigenvector of a matrix* A *if there is a number* λ *such that* $Av = \lambda v$*. Eigenvector is also known as characteristic vector or latent vector or invariant vector.*

Theorem 3.1. v *is an eigenvector of a matrix* A *if and only if every root* λ *of its characteristic equation is an eigenvalue of* A.

Proof. First suppose that v is an eigenvector of the matrix A. We need to show that every root λ of its characteristic equation is an eigenvalue of A. Since v is an eigenvector, we have that $Av = \lambda v = \lambda I v \Rightarrow (A - \lambda I)v = 0$ As $v \neq 0$, this implies that $|A - \lambda I| = 0$. This shows that every root λ of its characteristic equation is an eigenvalue of A.

Conversely, suppose that every root λ of its characteristic equation $|A - \lambda I| = 0$ is an eigenvalue of A. We have to prove that v is an eigenvector of A. If λ is a root of the characteristic equation $|A - \lambda I| = 0$, then the matrix equation $(A - \lambda I)v = 0$ possesses non-zero solution for v so that there exists a vector $v \neq 0$ such that $Av = \lambda I v = \lambda v$. This implies that v is an eigenvector of the matrix A. This completes the proof. □

Problem 3.5. *Find the eigenvectors of the matrix* $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

Solution: The characteristic equation is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$
$$\Rightarrow \quad (\lambda-1)^2(\lambda-5) = 0 \Rightarrow \lambda = 1, 1, 5.$$

Let $v = (v_1, v_2, v_3)'$. Then the equation $(A - \lambda I)v = O$ is

$$\begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.1}$$

Putting $\lambda = 1$ in (3.1), we have that

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The corresponding eigenvector is given by the equation

$$v_1 + 2v_2 + v_3 = 0.$$

There have three unknowns and one equation. Thus the number of unknowns are more than equations. So, the system has infinite number solutions.

Choose $v_2 = -1, v_3 = 1$, then $v_1 = 1$. Hence the one set of eigenvector corresponding to $\lambda = 1$ is $v = (1, -1, 1)'$.

On the other hand, put $\lambda = 5$ in (3.1), then we obtain that

$$\begin{pmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The corresponding eigenvector is given by the equation

$$-3v_1 + 2v_2 + v_3 = 0, v_1 - 2v_2 + v_3 = 0 \text{ and } v_1 + 2v_2 - 3v_3 = 0.$$

There have three unknowns and three equations. Thus the number of unknowns and number of equations are equal. So, the system has unique solutions.

Solving the above equations we obtain that $v_1 = 1, v_2 = 1$ and $v_3 = 1$. The eigenvector corresponding to $\lambda = 5$ is $u = (1, 1, 1)'$. Therefore, the eigenvectors are

$$(v, v, u) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Problem 3.6. *Find the eigenvalues and eigenvectors of the matrix* $A = \begin{pmatrix} 3 & -5 & -4 \\ -5 & -6 & -5 \\ -4 & -5 & 3 \end{pmatrix}$.

Problem 3.7. *Find the eigenvalues and eigenvectors of the matrix* $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.

Theorem 3.2. *(Cayley-Hamilton Theorem) Every square matrix satisfies its characteristic equation or if* $|A - \lambda I| = (-1)^n[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n] = 0$ *is the characteristic equation of* $n \times n$ *matrix* A*, then the matrix equation is* $A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_nI = O$.

Proof. We know that $|\text{adj } A| = |A|^{n-1}$. Also, since the elements of $(A - \lambda I)$ are of degree n in λ, the elements of $\text{adj}(A - \lambda I)$ are of degree $(n - 1)$ in λ. Therefore, we suppose that

$$\text{adj}(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-1},$$

where $B_0, B_1, \ldots, B_{n-1}$ are $n \times n$ matrices.

We know that $A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = |A| \cdot I$. Therefore, we have that

$$\begin{aligned} & (A - \lambda I) \cdot \text{ adj}(A - \lambda I) = |A - \lambda I| \cdot I \\ \Rightarrow \quad & (A - \lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-1}) = (-1)^n[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n]I. \end{aligned}$$

Comparing the coefficients of like powers of λ, we get

$$\begin{aligned} -IB_0 &= (-1)^nI \\ AB_0 - IB_1 &= (-1)^na_1I \\ AB_1 - IB_2 &= (-1)^na_2I \\ \cdots \quad \cdots & \quad \cdots \quad \cdots \\ AB_{n-1} &= (-1)^na_nI. \end{aligned}$$

Now, pre-multiplying the above equations by $A^n, A^{n-1}, \ldots, A, I$ respectively and adding we get

$$O = (-1)^n[A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_n.I] \Rightarrow A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_n.I = O.$$

This completes the proof of the theorem. □

Problem 3.8. *Find the formula for finding the inverse of the matrix A from the Cayley-Hamilton theorem.*

Solution: The Cayley-Hamilton equation is

$$A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots + a_n.I = O.$$

This equation can be written as

$$\begin{aligned}
-a_nI &= A[A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \ldots + a_{n-1}I] \\
\Rightarrow -a_nA^{-1}I &= A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \ldots + a_{n-1}I \\
\Rightarrow A^{-1} &= -\frac{1}{a_n}[A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \ldots + a_{n-1}I].
\end{aligned}$$

Problem 3.9. *Find the characteristic equation of the matrix* $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{pmatrix}$ *and verify Cayley-Hamilton theorem for it. Hence or otherwise find* A^{-1}.

Solution: The Characteristic matrix is

$$\begin{aligned}
A - \lambda I &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & 1-\lambda \end{pmatrix}.
\end{aligned}$$

Therefore, the characteristic polynomial is

$$\begin{aligned}
|A - \lambda I| &= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & 1-\lambda \end{vmatrix} \\
&= -\lambda^3 + \lambda^2 + 18\lambda + 30.
\end{aligned}$$

Hence, the characteristic equation is $\lambda^3 - \lambda^2 - 18\lambda - 30 = 0$.

Now, in order to verify the Cayley-Hamilton theorem, we have to show that

$$A^3 - A^2 - 18A - 30I = O.$$

Here,

$$A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{pmatrix}$$

and $A^3 = A^2A = \begin{pmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{pmatrix}$.

Thus,

$$\begin{aligned}
& A^3 - A^2 - 18A - 30I \\
&= \begin{pmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{pmatrix} - \begin{pmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{pmatrix} - 18 \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{pmatrix} - 30 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 62-14-18-30 & 39-3-36-0 & 68-14-54-0 \\ 48-12-36-0 & 21-9+18-30 & 78-6-72-0 \\ 62-8-54-0 & 24-6-12-0 & 62-14-18-30 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O.
\end{aligned}$$

Hence, Cayley-Hamilton theorem is verified.

Now, $A^3 - A^2 - 18A - 30I = O \Rightarrow A^{-1} = \frac{1}{30}A^2 - \frac{1}{30}A - \frac{18}{30}I$ Hence

$$\begin{aligned} A^{-1} &= \frac{1}{30}\begin{pmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{pmatrix} - \frac{1}{30}\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{pmatrix} - \frac{18}{30}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{5}{30} & \frac{1}{30} & \frac{11}{30} \\ \frac{10}{30} & -\frac{8}{30} & \frac{2}{30} \\ \frac{5}{30} & \frac{5}{30} & -\frac{5}{30} \end{pmatrix}. \end{aligned}$$

Problem 3.10. *Show that the matrix* $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{pmatrix}$ *satisfies Cayley-Hamilton theorem and hence compute* A^{-1}.

Problem 3.11. (i) *Find the eigenvalues for the matrix* $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ *and verify Cayley-Hamilton theorem for the matrix.*

(ii) *Find the eigenvalues and inverse of the matrix* $A = \begin{pmatrix} 5 & 6 \\ 1 & 2 \end{pmatrix}$.

Definition 3.3. *Two square matrices* A *and* B *are said to be similar if there exists a non-singular matrix* P *such that* $B = P^{-1}AP$.

Problem 3.12. *Show that two similar matrices have the same eigenvalues.*

Proof. Let A and B be similar matrices. Then there exists a non-singular matrix P such that $B = P^{-1}AP$. Now

$$\begin{aligned} |B - \lambda I| &= |P^{-1}AP - \lambda I| = |P^{-1}AP - P^{-1}\lambda IP| \\ &= |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P| = |A - \lambda I|. \end{aligned}$$

Thus, A and B have the same characteristic equation and the same eigenvalues. □

Problem 3.13. *Show that two similar matrices* $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ *and* $B = \begin{pmatrix} 5 & 14 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *have the same eigenvalues.*

Solution: The characteristic equation of A is

$$\begin{aligned} |A - \lambda I| = 0 \Rightarrow \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = 0 \\ \Rightarrow \begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} = 0 \Rightarrow (\lambda - 5)(\lambda - 1)^2 = 0 \\ \Rightarrow \lambda = 1, 1, 5. \end{aligned}$$

The characteristic equation of B is

$$\begin{aligned} |B - \lambda I| = 0 \Rightarrow \begin{pmatrix} 5 & 14 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = 0 \\ \Rightarrow \begin{pmatrix} 5-\lambda & 14 & 13 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = 0 \Rightarrow (5 - \lambda)(1 - \lambda)^2 = 0 \\ \Rightarrow \lambda = 1, 1, 5. \end{aligned}$$

Thus, the matrices A and B have the same eigenvalues.

Problem 3.14. *Show that* $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ *and* $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{pmatrix}$ *have the same eigenvalues but are not similar.*

Definition 3.4. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n distinct eigenvalues of a matrix A and $v_1, v_2, \ldots, v_n$ be the corresponding n eigenvectors. Also, let v_i be the column vector given by*

$$v_i = \begin{pmatrix} v_{1i} \\ v_{2i} \\ \ldots \\ v_{ni} \end{pmatrix}.$$

Consider a matrix P whose column vectors are the n eigenvectors such that

$$\begin{pmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ v_{n1} & v_{n2} & \ldots & v_{nn} \end{pmatrix}.$$

Suppose that D is a diagonal matrix such that

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}.$$

Then

$$\begin{aligned} PD &= \begin{pmatrix} \lambda_1 v_{11} & \lambda_2 v_{12} & \ldots & \lambda_n v_{1n} \\ \lambda_1 v_{21} & \lambda_2 v_{22} & \ldots & \lambda_n v_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_1 v_{n1} & \lambda_2 v_{n2} & \ldots & \lambda_n v_{nn} \end{pmatrix} \\ &= \begin{pmatrix} Av_1 & Av_2 & \ldots & Av_n \end{pmatrix} = A \begin{pmatrix} v_1 & v_2 & \ldots & v_n \end{pmatrix} \\ &= AP. \end{aligned}$$

Since P is non-singular, we have that $P^{-1}AP = D$. This shows that pre-multiplying A by P^{-1} and post-multiplying by P, we get diagonal matrix whose diagonal elements are the eigenvalues. This process is called the diagonalization of the matrix A.

Problem 3.15. *Show that the matrix* $A = \begin{pmatrix} \frac{4}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} \end{pmatrix}$ *is diagonalizable.*

Solution: The Characteristic equation is

$$\begin{aligned} |A - \lambda I| = 0 &\Rightarrow \begin{vmatrix} \frac{4}{3} - \lambda & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} - \lambda \end{vmatrix} = 0 \\ &\Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda - 1)(\lambda - 2) = 0 \\ &\Rightarrow \lambda = 1, 2. \end{aligned}$$

The diagonal matrix is $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

Then the equation $(A - \lambda I)v = O$ is

$$\begin{pmatrix} \frac{4}{3} - \lambda & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} - \lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 1$, then we have that

$$\begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The corresponding eigenvectors are given by the equation

$$v_1 + \sqrt{2}v_2 = 0, \sqrt{2}v_1 + 2v_2 = 0,$$

which gives that $v_1 + \sqrt{2}v_2 = 0$. Here number of unknowns are more than number of equations. So, the system has infinitely many solutions. Thus, if we consider v_2 is free variable and choose $v_2 = -1$, then $v_1 = \sqrt{2}$. Hence, the eigenvector corresponding to $\lambda = 1$ is $v = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$.

Similarly, the eigenvector corresponding to $\lambda = 2$ is $v' = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$.

Therefore, $P = \begin{pmatrix} v & v' \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix}$.

Here, adj $P = \begin{pmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{pmatrix}$ and $|P| = 3$. Thus, $P^{-1} = \dfrac{\text{adj } P}{|P|} = \begin{pmatrix} \frac{\sqrt{2}}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{\sqrt{2}}{3} \end{pmatrix}$

Therefore,

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} \frac{\sqrt{2}}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{\sqrt{2}}{3} \end{pmatrix} \begin{pmatrix} \frac{4}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{5}{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{2}}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{\sqrt{2}}{3} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 2 \\ -1 & 2\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}. \end{aligned}$$

This shows that A is diagonalizable matrix.

Problem 3.16. *Diagonalize the matrix* $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$.

Section-B (Differential Equations)

4 Differential Equations

Definition 4.1. *An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation. For examples,* $\frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0$, $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin\, t$, $\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ *etc.*

Definition 4.2. *A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variables is called an ordinary differential equation (ODE). As illustrations,* $\frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0$, $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin\, t$ *etc.*

Definition 4.3. *A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called a partial differential equation (PDE). As illustrations,* $\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ *etc.*

Definition 4.4. *The order of the highest ordered derivatives involved in a differential equation is called the order of the differential equation. For examples,*

(i) $\frac{d^3y}{dx^3} + 2\frac{dy}{dx} = 3x$, *which is the third order ordinary differential equation.*

(ii) $\frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0$, *which is the second order ordinary differential equation.*

(iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$, *which is the second order partial differential equation.*

Definition 4.5. *The degree of an differential equation is the power (or degree) of the highest differential coefficient when the equation has been made rational. As for examples,*

(i) $\frac{d^3y}{dx^3} + 2\frac{dy}{dx} = 3x$, *which is the first degree ordinary differential equation.*

(ii) $(y'')^3 - 3y' + 5y = \sin\, x$, *which is the third degree ordinary differential equation.*

(iii) $(1 + y'^2)^3 = 9(y'')^2$, *which is the second degree differential equation of* y''.

Definition 4.6. *A linear ordinary differential equation of order* n *in the dependent variable* y *and independent variable* x *is an equation that can be expressed in the following form:*

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}\frac{dy}{dx} + a_n(x)y = f(x),$$

where $a_0(x) \neq 0$. *For examples,* $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$, $\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = xe^x$.

Definition 4.7. *A nonlinear ordinary differential equation is an ordinary differential equation that is not linear. As for examples,* $y''+5y'+6y^2 = 0, y''+5(y')^3+6y = 0$ *and* $y''+5yy'+6y = 0$.

Definition 4.8. *A problem involving one or more differential equations with no supplementary conditions is called a general problem (GP). For example,* $y'' + 4y = 0$.

Definition 4.9. *A problem involving one or more differential equations with one or more supplementary conditions related to one value of* x *is called an initial value problem (IVP). a general problem (GP). For example,* $y'' + y = 0, y(1) = 3, y'(1) = -4$.

Definition 4.10. *A problem involving one or more differential equations with one or more supplementary conditions related to two different values of* x *is called a boundary value problem (BVP). For examples,* $y'' + y = 0, y'(0) = 1, y'(\frac{\pi}{2}) = 5$, *which has a unique solution;* $y'' + y = 0, y(0) = 1, y(\pi) = 5$, *which has no solution.*

Definition 4.11. *The solution of a nth order ODE containing n arbitrary constants is called a general solution (GS). For example, $y' = 2 \Rightarrow dy = 2dx \Rightarrow y = 2x + c$.*

Definition 4.12. *Any solution derived from the general solution by choosing particular value to the arbitrary constants is called a particular solution (PS). For example, $y' = 2 \Rightarrow dy = 2dx \Rightarrow y = 2x + c$. Now, if we choose $c = 1$, then this equation reduces to $y = 2x + 1$.*

Definition 4.13. *The solution of the ODE which can not be obtained/derived from the general solution by any choice of the arbitrary constants is called a singular solution (SS). For example, $(x-4)y^4dx - x^3(y^2-3)dy = 0$. The GS is $x^2(y^2-1) + y^3(2-x) = cx^2y^3$. Then, the singular solution is $y = 0$.*

Formation of Differential Equations by Eliminating the Arbitrary Constants

Definition 4.14. *A differential equation is formed by differentiating a function of equation and eliminating the arbitrary constants involved in it.*

Problem 4.1. *Form the differential equation of which $c(y+c)^2 = x^3$ is the complete integral.*

Solution: The given equation is

$$c(y+c)^2 = x^3. \tag{4.1}$$

Differentiating (4.1), we get

$$2c(y+c)\frac{dy}{dx} = 3x^2. \tag{4.2}$$

Now, dividing (4.1) by (4.2), we obtain that

$$\frac{y+c}{2\frac{dy}{dx}} = \frac{x}{3} \Rightarrow 3(y+c) = 2x\frac{dy}{dx} \Rightarrow c = \frac{1}{3}(2x\frac{dy}{dx} - 3y).$$

Thus, $y + c = y + \frac{1}{3}(2x\frac{dy}{dx} - 3y) = \frac{2}{3}x\frac{dy}{dx}$. Now, substituting the value of c in (4.2), we have that

$$\frac{4}{9}(2x\frac{dy}{dx} - 3y)\Big(\frac{dy}{dx}\Big)^2 = 3x \Rightarrow 8x\Big(\frac{dy}{dx}\Big)^3 - 12y\Big(\frac{dy}{dx}\Big)^2 = 27x,$$

which is the required differential equation of first order and third degree.

Problem 4.2. *Form the differential equation corresponding to the family of curves $y = c(x-c)^2$, where c is an arbitrary constant.*

Problem 4.3. *Find the differential equation of all circles passing through the origin and having their centres on the x-axis.*

Problem 4.4. *Find the differential equation of the family of parabolas with foci at the origin and axis along the x-axis.*

Solution: Let the directrix be $x = -2a$ and latus rectum be $4a$. Then, the equation of the parabola is

$$\begin{aligned} &\text{distance from focus} = \text{ distance from directrix} \\ \Rightarrow\ & x^2 + y^2 = (2a+x)^2 \Rightarrow y^2 = 4a(a+x). \end{aligned} \tag{4.3}$$

Differentiating, we get

$$y\frac{dy}{dx} = 2a \Rightarrow a = \frac{1}{2}y\frac{dy}{dx}.$$

It follows from (4.3) that

$$y^2 = 2y\frac{dy}{dx}\Big(\frac{1}{2}y\frac{dy}{dx} + x\Big) \Rightarrow y\Big(\frac{dy}{dx}\Big)^2 + 2x\frac{dy}{dx} - y = 0.$$

Problem 4.5. *Form the differential equation that represents all parabolas each of which has a latus rectum $4a$ and whose axes are parallel to x-axis.*

Problem 4.6. *Form the differential equation of all parabolas whose axes are parallel to y-axis.*

Problem 4.7. *Form the differential equation of all circles of radius a.*

Problem 4.8. *Form the differential equation of all parabolas which have their centres on x-axis and have a given radius.*

5 Equations of First Order and First Degree

Definition 5.1. *A differential equation of the form*

$$M(x, y)dx + N(x, y)dy = 0, \tag{5.1}$$

where $M(x, y)$ and $N(x, y)$ are functions of x and y or constants, is called a differential equation of the first order and first degree.

Examples of first order and first degree differential equations:

$$(x^2 + y^2)dx + (y - x)dy = 0, \frac{dy}{dx} = \frac{\sin x + y}{x + 3y}.$$

5.1 Separable Equations and Equations Reducible to this Form

5.1.1 Separable Equations

Definition 5.2. *An equation of the form*

$$F(x)G(y)dx + f(x)g(y)dy = 0$$

is called an equation with variables separable or simply a separable equations.

For example, the equation $(x - 4)y^4dx - x^3(y^2 - 3)dy = 0 \Rightarrow \frac{x-4}{x^3}dx = \frac{y^2-3}{y^4}dy$ is a separable equation.

Working Rule: Separate the variables and integrate i.e.

$$\frac{F(x)}{f(x)}dx + \frac{g(y)}{G(y)}dy = 0 \Rightarrow \int \frac{F(x)}{f(x)}dx + \int \frac{g(y)}{G(y)}dy = c$$
$$\Rightarrow \int M(x)dx + N(y)dy = c.$$

Problem 5.1. *Solve the differential equation $(x - 4)y^4dx - x^3(y^2 - 3)dy = 0$.*

Solution: The given equation is separable. Now, separating the variables by dividing by x^3y^4, we obtain

$$\frac{x-4}{x^3}dx = \frac{y^2-3}{y^4}dy \Rightarrow (x^{-2} - 4x^{-3})dx - (y^{-2} - 3y^{-4})dy = 0.$$

Integrating, we have the one-parameter family of solutions

$$-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c,$$

where c is the arbitrary constant.

Problem 5.2. *Solve the initial-value problem that consists of the differential equation $x \sin y\, dx + (x^2 + 1) \cos y\, dy = 0$ and $y(1) = \frac{\pi}{2}$.*

Solution: Given that $x \sin y\, dx + (x^2 + 1) \cos y\, dy = 0$. Now Separating the variables dividing by $(x^2 + 1) \sin y$, we obtain that

$$\frac{x}{x^2+1}dx + \frac{\cos y}{\sin y}dy = 0$$
$$\Rightarrow \int \frac{x\, dx}{x^2+1} + \int \frac{\cos y}{\sin y}dy = \log c, \text{ where } \log c \text{ is an arbitrary constant}$$
$$\Rightarrow \frac{1}{2}\log(x^2 + 1) + \log \sin y = \log c$$
$$\Rightarrow \sqrt{x^2+1} \sin y = c.$$

Now, we apply the initial condition $y(1) = \frac{\pi}{2}$. Then, $\sqrt{1^1 + 1} \sin \frac{\pi}{2} = c \Rightarrow c = \sqrt{2}$. Therefore, the solution of the initial-value problem under consideration is $\sqrt{x^2+1} \sin y = \sqrt{2}$.

Problem 5.3. *Solve the differential equation* $\sec^2 x \tan y \, dx + \sec^2 \tan x \, dy = 0$.

Problem 5.4. *Solve the differential equation* $(y - px)x = y$, *where* $p = \dfrac{dy}{dx}$.

Problem 5.5. *Solve the differential equation* $y - x\dfrac{dy}{dx} = a(y^2 + \dfrac{dy}{dx})$.

Problem 5.6. *Solve the differential equation* $(3+2 \sin x+\cos x) \, dy = (1+2 \sin y+\cos y) \, dx$.

5.1.2 Equations Reducible to the Form in which Variables are Separable

Definition 5.3. *Equations of the form*

$$\frac{dy}{dx} = f(ax + by + c)$$

can be reduced to an equation in which variables can be separated.

In this case it is required to put $ax + by + c = v$. Then, $a + b\dfrac{dy}{dx} = \dfrac{dv}{dx}$. This implies that $\dfrac{dy}{dx} = \dfrac{1}{b}\Big(\dfrac{dv}{dx} - a\Big)$. Then the equation becomes

$$\frac{1}{b}\Big(\frac{dv}{dx} - a\Big) = f(v) \Rightarrow \frac{dv}{dx} = a + bf(v),$$

which shows that variables are separable.

Problem 5.7. *Solve* $\dfrac{dy}{dx} = (4x + y + 1)^2$.

Solution: Put $4x + y + 1 = v$, so that $4 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$. The given equation then becomes

$$\frac{dv}{dx} - 4 = v^2 \Rightarrow \frac{dv}{dx} = v^2 + 4 \Rightarrow dx = \frac{dv}{v^2 + 4}.$$

Integrating, we obtain that $x = \dfrac{1}{2}\tan^{-1}\dfrac{v}{2} + c \Rightarrow x = \dfrac{1}{2}\tan^{-1}\dfrac{(4x + y + 1)}{2} + c$, which is the solution of the given differential equation.

Example 5.1. *Solve the following differential equations:*

(i) $\dfrac{dy}{dx} = \sin(x + y) + \cos(x + y)$ (ii) $(x - y)^2\dfrac{dy}{dx} = a^2$ (iii) $(x + y)^2\dfrac{dy}{dx} = a^2$

(iv) $\dfrac{x \, dx + y \, dy}{x \, dy - y \, dx} = \sqrt{\dfrac{a^2 - x^2 - y^2}{x^2 + y^2}}$ (v) $x\dfrac{dy}{dx} - y = x\sqrt{(x^2 + y^2)}$ (vi) $\dfrac{dy}{dx} = (x + y)^2$.

5.2 Homogeneous Differential Equations

Definition 5.4. *A function* $f(x, y)$ *is called a homogeneous of degree* n *if it can be expressed as in the form* $x^n\varphi(\frac{y}{x})$ *or* $y^n\varphi(\frac{x}{y})$ *or if* $f(\lambda x, \lambda y) = \lambda^n f(x, y)$.

Definition 5.5. *An equation of the form* $M(x, y)dx + N(x, y)dy = 0$ *or in the form* $\dfrac{dy}{dx} = -\dfrac{M(x, y)}{N(x, y)}$ *is called a homogeneous differential equation if both* $M(x, y)$ *and* $N(x, y)$ *are homogeneous and of the same degree.*

For instance, $(x^2 - 3y^2)dx + 2xy \, dy = 0$ is a homogeneous differential equation of degree 2 but $(x^2 + y^2)dx - (xy^3 - y^3)dy = 0$ is not a homogeneous equation.

5.2.1 Equation reducible to separable form

Working Rule: If $M(x,y)dx + N(x,y)dy = 0$ is a homogeneous equation, then the change of variable $y = vx$ transform the homogeneous equation into a separable equation in the variables v and x. Thus, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Example 5.2. *Solve the differential equation* $(x^2 - 3y^2)dx + 2xy\ dy = 0$.

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy},$$

which is a homogeneous equation of degree 2. Let $y = vx$. Then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Then the given equation reduces to

$$\begin{aligned} & v + x\frac{dv}{dx} = \frac{3v^2 - 1}{2v} \Rightarrow x\frac{dv}{dx} = \frac{v^2 - 1}{2v} \\ & \Rightarrow \frac{2v}{v^2 - 1}dv = \frac{dx}{x} \Rightarrow \int \frac{2v}{v^2 - 1}dv = \int \frac{dx}{x} + \log c \\ & \Rightarrow \log\,(v^2 - 1) = \log x + \log c \Rightarrow v^2 - 1 = cx \Rightarrow y^2 - x^2 = cx^3. \end{aligned}$$

Example 5.3. *Solve the following differential equations:*

(i) $(y + \sqrt{x^2 + y^2})dx - x\ dy$ (ii) $(x^2 + y^2)dx + 2xy\ dy = 0$ (iii) $(x^3 + y^3)dx - 3xy^2\ dy = 0$

(iv) $x\ dy - y\ dx - \sqrt{x^2 - y^2}dx = 0$ (v) $x^2y\ dx - (x^3 + y^3)dy = 0$ (vi) $(2x + 3y)dx + (y - x)dy = 0$

(vii) $(x^2 + 2xy - y^2)dx + (y^2 + 2xy - x^2)dy = 0$ (viii) $2y^3dx + (x^2 - 3y^2)x\ dy = 0$.

5.2.2 Equation reducible to homogeneous form but linear

An equation of the form $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$ can be reduced to homogeneous form in the following two cases.

Case-1: $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$ when $\dfrac{a}{a'} \neq \dfrac{b}{b'}$;

Case-2: $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$ when $\dfrac{a}{a'} = \dfrac{b}{b'}$.

Below we will describe the above two cases.

Case-1 When $\dfrac{a}{a'} \neq \dfrac{b}{b'}$. Put $x = x_1 + h$, $y = y_1 + k$. Then $\dfrac{dy}{dx} = \dfrac{dy_1}{dx_1}$. Therefore,

$$\frac{dy_1}{dx_1} = \frac{ax_1 + by_1 + (ah + bk + c)}{a'x_1 + b'y_1 + (a'h + b'k + c')},$$

we choose the constants h and k in such a way that $ah + bk + c = 0$ and $a'h + b'k + c' = 0$. With this substitution, the given differential equation reduces to

$$\frac{dy_1}{dx_1} = \frac{ax_1 + by_1}{a'x_1 + b'y_1},$$

which is a homogeneous equation in x_1 and y_1 and can be solved by putting $y_1 = vx_1$.

Example 5.4. *Solve* $(6x - 2y - 7)dx - (2x + 3y - 6)dy = 0$.

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{6x - 2y - 7}{2x + 3y - 6}.$$

Put $x = x_1 + h$ and $y = y_1 + k$. Then

$$\frac{dy_1}{dx_1} = \frac{6x_1 - 2y_1 + (6h - 2k - 7)}{2x_1 + 3y_1 + (2h + 3k - 6)}, \tag{5.2}$$

where

$$6h - 2k - 7 = 0 \text{ and } 2h + 3k - 6 = 0. \tag{5.3}$$

Solving these two equations, we obtain that $h = \frac{3}{2}, k = 1$. Then, $x_1 = x - \frac{3}{2}, y_1 = y - 1$. Therefore, (5.2) can be written as

$$\frac{dy_1}{dx_1} = \frac{6x_1 - 2y_1}{2x_1 + 3y_1}.$$

Put $y = vx_1$, then $\frac{dy_1}{dx_1} = v + x_1\frac{dv}{dx_1}$. Thus,

$$\begin{aligned}
&\frac{6x_1 - 2y_1}{2x_1 + 3y_1} = v + x_1\frac{dv}{dx_1} \Rightarrow \frac{6 - 4v - 3v^2}{2 + 3v} = x_1\frac{dv}{dx_1}\\
&\Rightarrow \frac{dx_1}{x_1} + \frac{2 + 3v}{3v^2 + 4v - 6}dv = 0 \Rightarrow \int \frac{dx_1}{x_1} + \int \frac{2 + 3v}{3v^2 + 4v - 6}dv = \log c\\
&\Rightarrow \log x_1 + \frac{1}{2}\log (3v^2 + 4v - 6) = \log c \Rightarrow x_1\sqrt{3v^2 + 4v - 6} = c\\
&\Rightarrow \sqrt{3y_1^2 + 4x_1y_1 - 6x_1^2} = c \Rightarrow 3y_1^2 + 4x_1y_1 - 6x_1^2 = c_1\\
&\Rightarrow 3(y - 1)^2 + 4(y - 1)(x - \frac{3}{2}) - 6(x - \frac{3}{2})^2 = c_1\\
&\Rightarrow 12(y - 1)^2 + 8(y - 1)(2x - 3) - 6(2x - 3)^2 = c_2.
\end{aligned}$$

Example 5.5. *Solve the following differential equations:*

(i) $(2x - 5y + 3)dx - (2x + 4y - 6)dy = 0$ (ii) $(x - y - 1)dx + (4y + x - 1)dy = 0$ (iii) $(2x + 9y - 20)dx - (6x + 2y - 10)dy = 0$

(iv) $(3x - 7y - 3)\frac{dy}{dx} = 3y - 7x + 7$ (v) $(2x + y + 3)\frac{dy}{dx} = x + 2y + 3$.

Case-2: When $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ (say). Then the given differential equation can be written as

$$\frac{dy}{dx} = \frac{ax + by + c}{m(ax + by) + c}. \tag{5.4}$$

Put $ax + by = v$, so that $a + b\frac{dy}{dx} = \frac{dv}{dx}$. Then (5.4) becomes

$$\frac{1}{b}\Big(\frac{dv}{dx} - a\Big) = \frac{v + c}{mv + c},$$

in which variables can be separated.

Example 5.6. *Solve* $\frac{dy}{dx} = \frac{3x - 4y - 2}{3x - 4y - 3}$.

Solution: The given equation is

$$\frac{dy}{dx} = \frac{3x - 4y - 2}{3x - 4y - 3}. \tag{5.5}$$

The transformation $3x - 4y = v$, since $\frac{a}{a'} = \frac{b}{b'}$. Then $3 - 4\frac{dy}{dx} = \frac{dv}{dx}$. Hence from (5.5), we have that

$$\begin{aligned}
&\frac{1}{4}\frac{dv}{dx} - \frac{3}{4} = \frac{v - 2}{v - 3} \Rightarrow \frac{1}{4}\frac{dv}{dx} = \frac{v - 2}{v - 3} + \frac{3}{4}\\
&\Rightarrow \frac{dv}{dx} = -\frac{v + 1}{v - 3} \Rightarrow \frac{v - 3}{v + 1}dv + dx = 0\\
&\Rightarrow (1 - \frac{4}{v + 1})dv + dx = 0 \Rightarrow \int (1 - \frac{4}{v + 1})dv + \int dx = c\\
&\Rightarrow (v - 4 \log (v + 1) + x = c \Rightarrow x - y - \log (3x - 4y + 1) = c_1.
\end{aligned}$$

Example 5.7. *Solve the following differential equations:*

(i) $(2x - 2y + 5)\dfrac{dy}{dx} = x - y + 3$ (ii) $(x + y)dx + (3x + 3y - 4)dy = 0$

(iii) $(3x + 2y + 1)dx - (3x + 2y - 1)dy = 0$ (iv) $\dfrac{dy}{dx} = \dfrac{6x - 2y - 7}{3x - y + 4}$

(vi) $(3y + 2x + 4)dx - (4x + 6y + 5)dy = 0$.

6 Exact Differential Equations

Definition 6.1. *Let u be a function of two real variables x and y such that u has continuous first order partial derivatives in the domain D. The total differential du is defined by*

$$du(x, y) = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy = u_x\, dx + u_y\, dy. \tag{6.1}$$

As illustrations, $u(x, y) = xy^2 + 2x^3y$, $du(x, y) = (y^2 + 6x^2y)\, dx + (2xy + 2x^3)\, dy$.

Definition 6.2. *The expression $M(x, y)\, dx + N(x, y)\, dy$ is called an exact differential if there exists a function $u(x, y)$ such that*

$$du(x, y) = M(x, y)\, dx + N(x, y)\, dy. \tag{6.2}$$

Comparing (6.1) and (6.2), we have seen that

$$u_x = M(x, y) \text{ and } u_y = N(x, y).$$

Definition 6.3. *If $M(x, y)\, dx+N(x, y)\, dy$ is an exact differential, then the differential equation*

$$M(x, y)\, dx + N(x, y)\, dy = 0 \text{ i.e. } du(x, y) = 0,$$

is called an exact differential equation.

For example, $y^2\, dx + 2xy\, dy = 0$ is an exact differential equation, because $u = xy^2$, then $du = y^2\, dx + 2xy\, dy$ is an exact differential.

Theorem 6.1. *The necessary and sufficient condition for the exact differential equation $M(x, y)\, dx+ N(x, y)\, dy = 0$ is $M_y = N_x$ or $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.*

Proof. Suppose that

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{6.3}$$

is an exact differential equation. Then $M(x, y)\, dx + N(x, y)\, dy$ is an exact differential and hence by the definition of exact differential, there exists a function $u(x, y)$ such that

$$du = M(x, y)\, dx + N(x, y)\, dy.$$

But we know that

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy.$$

Comparing the above two relations, we have that

$$\frac{\partial u}{\partial x} = M(x, y) \text{ and } \frac{\partial u}{\partial y} = N(x, y).$$

Thus, we obtain that

$$M_y = \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x}\Big) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\Big) = N_x.$$

Conversely, suppose that $M_y = N_x$. We need to show that $M(x,y)\,dx + N(x,y)\,dy = 0$ is an exact differential equation.

Let $\int M\,dx = u$. Then $\dfrac{\partial u}{\partial x} = M$. Hence,

$$
\begin{aligned}
& \frac{\partial^2 u}{\partial y \partial x} = M_y = N_x \Rightarrow N_x = \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\Big) \\
\Rightarrow \quad & \int N_x \partial x = \int \frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\Big)\partial x + f(y), \\
& \text{where } f(y) \text{ is a function of } y \text{ free from } x \\
\Rightarrow \quad & N = \frac{\partial u}{\partial y} + f(y).
\end{aligned}
$$

$$
\begin{aligned}
\text{Now, } M + N\frac{dy}{dx} &= \frac{\partial u}{\partial x} + \Big[\frac{\partial u}{\partial y} + f(y)\Big]\frac{dy}{dx} \\
&= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} + f(y)\frac{dy}{dx} \\
&= \frac{d}{dx}\Big[u + \int f(y)\frac{dy}{dx}dx\Big] \\
&= \frac{d}{dx}[u + F(y)] = \frac{du}{dx}.
\end{aligned}
$$

This implies that $M\,dx + N\,dy = du$ and hence $u_x = M$ and $u_y = N$. This shows that $M\,dx + N\,dy = 0$ is exact differential equation. This completes the proof. □

The solution for the given differential equation obtained by the method described in the following theorem is known as grouping method.

Theorem 6.2. *If $M(x,y)\,dx + N(x,y)\,dy = 0$ is an exact differential equation such that there exists a function $u(x,y)$ with $u_x = M(x,y)$ and $u_y = N(x,y)$, then a one parameter family of solutions of the differential equation is given by $u(x,y) = c$, where c is an arbitrary constant.*

Proof. Let $u(x,y)$ be a function of two real variables x and y such that u has a continuous first order partial derivatives. Since $M(x,y)\,dx + N(x,y)\,dy = 0$ is an exact differential equation, then $M(x,y)\,dx + N(x,y)\,dy$ is an exact differential with $u_x = M(x,y)$ and $u_y = N(x,y)$. Then,

$$
\begin{aligned}
& du(x,y) = u_x\,dx + u_y\,dy = M(x,y)\,dx + N(x,y)\,dy = 0 \\
\Rightarrow \quad & \int du(x,y) = c \Rightarrow u(x,y) = c,
\end{aligned}
$$

where c is an arbitrary constant. □

Standard Method : The equation $M\,dx + N\,dy = 0$ is exact means that there exists a function $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = M(x,y) \tag{6.4}$$

$$\text{and } \frac{\partial u}{\partial y} = N(x,y). \tag{6.5}$$

Integrating (6.4), we obtain that

$$u(x,y) = \int M(x,y)\partial x + \phi(y), \tag{6.6}$$

where $\int M(x,y)\partial x$ indicates partial integration with respect to x holding y constant and ϕ is an arbitrary function of y only.

Now, differentiating (6.6) with respect to y, we obtain

$$\begin{aligned}
&\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\partial x + \frac{d\phi(y)}{dy} \\
&\Rightarrow N(x,y) = \frac{\partial}{\partial y}\int M(x,y)\partial x + \frac{d\phi(y)}{dy} \\
&\Rightarrow \frac{d\phi(y)}{dy} = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x.
\end{aligned} \tag{6.7}$$

Since ϕ is a function of y only, the derivative $\dfrac{d\phi}{dy}$ must also be independent of x. We need to show that

$$N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x \tag{6.8}$$

must be independent of x. To do this, we have to show that

$$\frac{\partial}{\partial x}\Big[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\Big] = 0.$$

Since

$$\frac{\partial^2}{\partial x \partial y}\int M(x,y)\partial x = N_x = M_y = \frac{\partial^2}{\partial y \partial x}\int M(x,y)\partial x,$$

we have that

$$\begin{aligned}
& \frac{\partial}{\partial x}\Big[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\Big] \\
= \; & \frac{\partial N(x,y)}{\partial x} - \frac{\partial^2}{\partial y \partial x}\int M(x,y)\partial x \\
= \; & \frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y} \\
= \; & 0, \text{ since (6.4) and (6.5) are true.}
\end{aligned}$$

Hence (8.20) is independent of x. Thus, we can write

$$\phi(y) \;=\; \int \Big[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\Big]dy.$$

Thus, from (6.6) we have that

$$u(x,y) = \int M(x,y)\partial x + \int \Big[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\Big]dy.$$

Example 6.1. *Solve the equation*

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0.$$

Solution: Here we have that

$$M = 3x^2 + 4xy,\ N = 2x^2 + 2y \text{ and } M_y = 4x,\ N_x = 4x.$$

Since $M_y = N_x$, the given equation is exact. Thus, we must find a function $u(x,y)$ such that

$$\begin{aligned}
u(x,y) &= \int M(x,y)\partial x + \int \Big[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\partial x\Big]dy \\
&= \int (3x^2 + 4xy)\partial x + \int \Big[(2x^2 + 2y) - \frac{\partial}{\partial y}\int (3x^2 + 4xy)\partial x]dy \\
&= x^3 + 2x^2y + \int \Big[2x^2 + 2y - \frac{\partial}{\partial y}(x^3 + 2x^2y)]dy \\
&= x^3 + 2x^2y + \int (2x^2 + 2y - 2x^2)dy = x^3 + 2x^2y + y^2.
\end{aligned}$$

Hence, a one parameter family of solution is $u(x, y) = c$. Thus, the general solution is

$$x^3 + 2x^2y + y^2 = c.$$

Alternative Proof:(Grouping Method)

$$\begin{aligned} & (3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0 \\ \Rightarrow \quad & 3x^2\,dx + (4xy\,dx + 2x^2\,dy) + 2y\,dy = 0. \end{aligned}$$

This can be written as

$$\begin{aligned} & d(x^3) + d(2x^2y) + d(y^2) = d(c), \text{ where } c \text{ is an arbitrary constant} \\ \Rightarrow \quad & d(x^3 + 2x^2y + y^2) = d(c) \\ \Rightarrow \quad & x^3 + 2x^2y + y^2 = c, \text{ by integrating.} \end{aligned}$$

Example 6.2. *Solve the following differential equations:*

(1) $(y^4 + 4x^3y + 3x)\,dx + (x^4 + 4xy^3 + y + 1)\,dy = 0$;

(2) $(x^2 - 2xy + 3y^2)\,dx + (4y^3 + 6xy - x^2)\,dy = 0$;

(3) $x\,dx + y\,dy = \dfrac{a^2(x\,dy - y\,dx)}{x^2 + y^2}$; (4) $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$;

(5) $[\cos x \tan y + \cos(x + y)]dx + [\sin x \sec^2 y + \cos(x + y)]dy = 0$;

(6) $(2x \cos y + 3x^2y)dx + (x^3 - x^2 \sin y - y)dy = 0$;

(7) $(x - y)dx - (x + y)dy = 0$; (8) $(xy^2 - 1)dx + (x^2y - 1)dy = 0$.

Definition 6.4. *If the differential equation*

$$M(x, y)dx + N(x, y)dy = 0 \tag{6.9}$$

is not exact but the differential equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0 \tag{6.10}$$

is exact, then $\mu(x, y)$ is called an integrating factor of the differential equation (6.9).

For example, $y\,dx + 2x\,dy = 0$ is not exact because $M_y \neq N_x$ as $M_y = 1$ and $N_x = 2$. But $y^2\,dx + 2xy\,dy = 0$ is exact. Since this resulting exact equation is integrable, we say that y is an integrating factor of the equation $y\,dx + 2x\,dy = 0$.

Example 6.3. *Test whether the following equations are exact or not:*

(1) $(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$;

(2) $(y^2 + 2xy)dx - x^2\,dy = 0$;

(3) $(4x + 3y^2)dx + 2xy\,dy = 0$.

6.1 Rules for finding the integrating factors

Rule-1. If $\dfrac{M_y - N_x}{N} = f(x)$, a function of x only, then the integrating factor is

$$\mu = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int f(x) dx}.$$

Rule-2. If $\dfrac{N_x - M_y}{M} = g(y)$, a function of y only, then the integrating factor is

$$\mu = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int g(y) dy}.$$

Example 6.4. *Solve the differential equation* $y\,dx + (3 + 3x - y)dy = 0$.

Solution: Here, $M_y = 1$ and $N_x = 3$. Hence the given equation is not exact. Now,

$$\frac{M_y - N_x}{N} = \frac{1-3}{3+3x-3} = -\frac{2}{3+3x-y} \neq f(x).$$

$$\frac{N_x - M_y}{M} = \frac{2}{y} = g(y).$$

Therefore, the integrating factor is

$$\mu = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int \frac{2}{y} dy} = e^{\log y^2} = y^2.$$

Now, multiplying the given equation by y^2, we obtain

$$\begin{aligned}
& y^3\,dx + (3y^2 + 3xy^2 - y^3)dy = 0 \\
\Rightarrow\;& (y^3\,dx + 3xy^2 dy) + 3y^2 dy - y^3 dy = 0 \\
\Rightarrow\;& d(xy^3) + d(y^3) - d(\frac{y^4}{4})dy = d(c) \\
\Rightarrow\;& d\Big(xy^3 + y^3 - \frac{y^4}{4}\Big) = d(c) \\
\Rightarrow\;& xy^3 + y^3 - \frac{y^4}{4} = c.
\end{aligned}$$

Example 6.5. *Solve the following differential equation:*

(1) $(2x^2 + y)dx + (x^2 y - x)dy = 0$; (2) $(x^2 + y^2 + x)dx + xy\,dy = 0$;

(3) $(x^2 + y^2)dx - 2xy\,dy = 0$; (4) $(x^3 - 2y^2)dx + 2xy\,dy = 0$;

(5) $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$.

Rule-3. If $M\,dx + N\,dy = 0$ is a homogeneous and $Mx + Ny \neq 0$, then

$$\frac{1}{Mx + Ny}$$

is an integrating factor.

Example 6.6. *Solve* $x^2y\,dx - (x^3 + y^3)\,dy = 0$.

Solution: The given equation is homogeneous and $Mx + Ny = x^3y - x^3y - y^4 = -y^4 \neq 0$. Hence, the integrating factor is

$$\frac{1}{Mx + Ny} = \frac{1}{x^3y - (x^3y + y^4)} = -\frac{1}{y^4}.$$

Multiplying the given equation by $-\frac{1}{y^4}$, then it becomes

$$-\frac{x^2}{y^3}dx + \frac{x^3 + y^3}{y^4}dy = 0 \Rightarrow -\frac{x^2}{y^3}dx + \frac{x^3}{y^4}dy + \frac{1}{y}dy = 0.$$

This equation is exact. Now,

$$\begin{aligned}
& d\Big(-\frac{x^3}{3y^3}\Big) + d(\log\, y) = d(\log\, c) \\
\Rightarrow\;& \int d\Big(-\frac{x^3}{3y^3}\Big) + \int d(\log\, y) = \int d(\log\, c) \\
\Rightarrow\;& -\frac{x^3}{3y^3} + \log\, y = \log\, c \\
\Rightarrow\;& \log\, y = \log\, c + \frac{x^3}{3y^3} \\
\Rightarrow\;& y = ce^{\frac{x^3}{3y^3}}.
\end{aligned}$$

Example 6.7. *Solve the following differential equation:*

(1) $(x^4+y^4)dx - xy^3dy = 0$; (2) $y^2dx + (x^2 - xy - y^2)dy = 0$;

(3) $y^3dx + (x^2y - 1)dy = 0$;, (4) $(y^4 - 2x^3y)dx + (x^4 - 2xy^3)dy = 0$;

Rule-4. If the equation can be written in the form

$$yf(xy)dx + xg(xy)dy = 0, f(xy) \neq g(xy), \text{ then}$$

$$\frac{1}{xy[f(xy) - g(xy)]} = \frac{1}{Mx - Ny}$$

is an integrating factor.

Example 6.8. *Solve the following differential equation:*

(1) $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$; (2) $y(x^2y^2 + 2)dx + x(2 - 2x^2y^2)dy = 0$;

(3) $y(2xy+1)dx + x(1+2xy-x^3y^3)dy = 0$, (4)$(x^2y^2+xy+1)ydx + (x^2y^2-xy+1)xdy = 0$

(5) $(xy\sin xy + \cos xy)ydx + (xy\sin xy - \cos xy)xdy = 0$.

Solution:(1) The given equation is $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$. This equation is of the form

$$yf(xy)dx + xg(xy)dy = 0.$$

Thus, the integrating factor is

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{xy(xy + 2x^2y^2) - xy(xy - x^2y^2)} = \frac{1}{3x^3y^3}.$$

Multiplying the given equation by $\frac{1}{3x^2y^2}$, then the equation becomes

$$\Big(\frac{1}{3x^2y} + \frac{2}{3x}\Big)dx + \Big(\frac{1}{3xy^2} - \frac{1}{3y}\Big)dy = 0,$$

which is exact. Thus, we have that

$$\begin{aligned}
&\Big(\frac{dx}{3x^2y} + \frac{dy}{3xy^2}\Big) + \frac{2dx}{3x} - \frac{dy}{3y} = 0\\
\Rightarrow\ & d\Big(-\frac{1}{3xy}\Big) + \frac{2}{3}d\Big(\log x\Big) - \frac{1}{3}d\Big(\log y\Big) = 0\\
\Rightarrow\ & \int d\Big(-\frac{1}{3xy}\Big) + \frac{2}{3}\int d\Big(\log x\Big) - \frac{1}{3}\int d\Big(\log y\Big) = \log c\\
\Rightarrow\ & -\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = \log c\\
\Rightarrow\ & \log\Big(\frac{x^2}{y}\Big) = 3\log c + \log^{e^{\frac{1}{xy}}}\\
\Rightarrow\ & \frac{x^2}{y} = c_1e^{\frac{1}{xy}},\ \text{where } c^3 = c_1.
\end{aligned}$$

Rule-5. Let the equation be of the form

$$x^ay^b(mydx + nxdy) + x^cy^d(\mu ydx + \nu xdy) = 0,$$

where $a, b, c, d, m, n, \mu, \nu$ are all constants. Then it has an integrating factor $x^\alpha y^\beta$, where α, β are so chosen that after multiplying by $x^\alpha y^\beta$ the equation becomes exact.

Example 6.9. *Solve the following differential equation:*

(1) $(y^3 - 3x^2y)dx + x(2xy^2 - x^3)dy = 0$; (2) $(8ydx + 8xdy) + x^2y^2(4ydx + 5xdy) = 0$;

(3) $x^3y^2(2ydx + xdy) - (5ydx + 7xdy) = 0$, (4)$(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$

(5) $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$, (6) $3ydx - 2xdy + x^2y^{-1}(10ydx - 6xdy) = 0$;

(7) $(20x^2 + 8xy + 4y^2 + 3y^3)ydx + 4(x^2 + xy + y^2 + y^3)xdy = 0$.

Solution:(1) Given the differential equation $(y^3 - 3x^2y)dx + x(2xy^2 - x^3)dy = 0$. This equation can be written as

$$y^2(ydx + 2xdy) - x^2(2ydx + xdy) = 0.$$

Let $x^\alpha y^\beta$ be an integrating factor of the equation. Multiplying the given equation by $x^\alpha y^\beta$, then it becomes

$$\left(y^{3+\beta}x^\alpha - 2y^{\beta+1}x^{\alpha+2}\right)dx + \left(2x^{\alpha+1}y^{\beta+2} - x^{\alpha+3}y^\beta\right)dy = 0.$$

In this (exact) equation,

$$M = y^{3+\beta}x^\alpha - 2y^{\beta+1}x^{\alpha+2},\ N = 2x^{\alpha+1}y^{\beta+2} - x^{\alpha+3}y^\beta.$$

Hence α and β are such that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This implies that

$$\begin{aligned}
& (3+\beta)y^{2+\beta}x^\alpha - 2(\beta+1)y^\beta x^{2+\alpha} = 2(1+\alpha)x^\alpha y^{2+\beta} - (\alpha+3)x^{\alpha+2}y^\beta \\
\Rightarrow\ & (3+\beta) = 2(1+\alpha) \text{ and } 2(\beta+1) = (\alpha+3) \\
\Rightarrow\ & \alpha = 1 \text{ and } \beta = 1.
\end{aligned}$$

Hence, xy is an integrating factor. Now multiplying by xy, the equation becomes

$$\begin{aligned}
& (xy^4 - 2x^3y^2)dx + (2x^2y^3 - x^4y)dy = 0 \Rightarrow \frac{1}{2}d(x^2y^4) - \frac{1}{2}d(x^4y^2) = d(c) \\
\Rightarrow\ & \frac{1}{2}\int d(x^2y^4) - \frac{1}{2}\int d(x^4y^2) = \int d(c) \\
\Rightarrow\ & x^2y^4 - x^4y^2 = 2c \Rightarrow x^2y^2(y^2 - x^2) = c_1.
\end{aligned}$$

Example 6.10. *Given that for some constant α, $(x+y)^\alpha$ is an integrating factor of*

$$(4x^2 + 2xy + 6y)dx + (2x^2 + 9y + 3x)dy = 0,$$

find α and solve the differential equation.

Example 6.11. *Prove that $\dfrac{1}{(x+y+1)^4}$ is an integrating factor of*

$$(2xy - y^2 - y)dx + (2xy - x^2 - x)dy = 0$$

and hence integrate the equation.

7 Linear and Bernoulli differential equation

Definition 7.1. *A differential equation of the form*

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{7.1}$$

where $P(x)$ and $Q(x)$ are functions of x or constants, is called the first order linear differential equation in y.

Equation (7.1) can be written in the form

$$[P(x)y - Q(x)]dx + dy = 0.$$

Here, $M = P(x)y - Q(x)$ and $N = 1$. Thus, $\dfrac{M_y - N_x}{N} = P(x)$. Therefore,

$$\text{I.F.} = e^{\int P(x)dx}.$$

Multiplying (7.1) by $e^{\int P(x)dx}$, then we obtain

$$\begin{aligned}
& e^{\int P(x)dx}\left[\frac{dy}{dx} + P(x)y\right] = Q(x)e^{\int P(x)dx} \\
\Rightarrow\ & \frac{d}{dx}[e^{\int P(x)dx}y] = Q(x)e^{\int P(x)dx} \\
\Rightarrow\ & \int \frac{d}{dx}[e^{\int P(x)dx}y]dx = \int Q(x)e^{\int P(x)dx} + c \\
\Rightarrow\ & ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx} + c.
\end{aligned}$$

Example 7.1. *Solve the differential equation* $\dfrac{dy}{dx} + \Big(\dfrac{2x+1}{x}\Big)y = e^{-2x}$.

Solution: Here

$$P(x) = \frac{2x+1}{x}.$$

Hence, the integrating factor

$$\begin{aligned}
\text{I.F.} &= e^{\int P(x)dx} = e^{\int \frac{2x+1}{x}dx} \\
&= e^{2x+\log x} = e^{2x}e^{\log x} = xe^{2x}.
\end{aligned}$$

Now, multiplying the given equation by xe^{2x}, then it becomes

$$\begin{aligned}
& xe^{2x}\left[\frac{dy}{dx} + e^{2x}(2x+1)y\right] = x \\
\Rightarrow\ & \frac{d}{dx}[xe^{2x}y] = x \\
\Rightarrow\ & \int \frac{d}{dx}[xe^{2x}y]dx = \int xdx + c \\
\Rightarrow\ & xye^{2x} = \frac{1}{2}x^2 + c \\
\Rightarrow\ & y = \frac{1}{2}xe^{-2x} + \frac{c}{x}e^{-2x}.
\end{aligned}$$

Example 7.2. *Solve the following differential equations:*

(i) $(x^2+1)\dfrac{dy}{dx} + 4xy = x,\ y(2) = 1$; (ii) $y^2dx + (3xy-1)dy = 0$;

(iii) $x\dfrac{dy}{dx} + 2y = x^2\log x$; (iv) $x\dfrac{dy}{dx} + 2y = x^4$;

(v) $\dfrac{dy}{dx} + 2y\tan x = \sin x,\ \ y(\frac{\pi}{3}) = 0.$

Definition 7.2. *An equation of the form*

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{7.2}$$

is called a Bernoulli differential equation.

It is remarked that if $n = 0$ or 1, (7.2) reduces to a linear differential equation.

Theorem 7.1. *Suppose that $n \neq 0$ or 1. Then the transformation $v = y^{1-n}$ reduces the Bernoulli equation (7.2) to a linear differential equation.*

Example 7.3. *Solve the differential equation $\frac{dy}{dx} + y = xy^3$.*

Solution: The given equation is

$$\frac{dy}{dx} + y = xy^3 \Rightarrow y^{-3}\frac{dy}{dx} + y^{-2} = x.$$

Let $v = y^{1-n} = y^{-2}$. Then $\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$. Then the above equation reduces to

$$\begin{aligned} & -\frac{1}{2}\frac{dv}{dx} + v = x \\ \Rightarrow \quad & \frac{dv}{dx} - 2v = -2x, \end{aligned} \tag{7.3}$$

which is linear equation in v. Thus, the integrating factor is

$$\text{I.F.} = e^{\int -2dx} = e^{-2x}.$$

Multiplying (7.3) by e^{-2x} and integrate, we find

$$\begin{aligned} ve^{-2x} &= -2\int xe^{-2x}dx + c \Rightarrow ve^{-2x} = \frac{1}{2}e^{-2x}(2x+1) + c \\ &\Rightarrow \quad v = x + \frac{1}{2} + ce^{2x} \\ &\Rightarrow \quad \frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}. \end{aligned}$$

Example 7.4. *Solve the following differential equations:*

(i) $\frac{dy}{dx} = x^3y^3 - xy$, (ii) $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}$;

(iii) $\frac{dy}{dx}(x^2y^3 + xy) = 1$, (iv) $\frac{dy}{dx} + \frac{2}{x}y = \frac{y^2}{x^3}$;

(v) $x\frac{dy}{dx} + y = y^2 \log x$, (vi) $\frac{dy}{dx} - 2y\tan x = y^2 \tan^2 x$.

8 The Homogeneous and Nonhomogeneous Linear Equation with Constant Coefficients

Definition 8.1. *A differential equation of the form*

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = f(x), \tag{8.1}$$

where $a_0, a_1, a_2, \ldots, a_n$ and $f(x)$ are functions of x, is called a n^{th}-order linear differential equation (LDE).

If $a_0, a_1, a_2, \ldots, a_n$ are constants and $f(x)$ is a function of x, then (8.1) is called a n^{th}-order linear differential equation (LDE) with constant coefficients.

We will deal with LDE with constant coefficients and for our convenience we use the operators $D := \frac{d}{dx}, D^2 := \frac{d^2}{dx^2}, \ldots, D^n := \frac{d^n}{dx^n}$. Then (8.1) becomes

$$\big(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + a_3D^{n-2} + \ldots + a_{n-1}D + a_n\big)y = f(x), \tag{8.2}$$

which can be briefly written as,

$$F(D)y = f(x).$$

Definition 8.2. *If $f(x) = 0$, then the equation* (8.2) *becomes*

$$\big(a_0D^n + a_1D^{n-1} + \ldots + a_n\big)y = 0 \ \textit{i.e.} \ \ F(D)y = 0. \tag{8.3}$$

This equation is called the homogeneous linear differential equation with constant coefficients. Otherwise it is called nonhomogeneous linear differential equation with constant coefficients i.e. if $f(x) \neq 0$, then (8.2) *is called nonhomogeneous linear differential equation with constant coefficients.*

Theorem 8.1. *If $y = y_1, y = y_2, \ldots, y = y_n$ are linearly independent solutions of $F(D)y = 0$, then*

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$$

is the general or complete solution of the differential equation, where $c_1, c_2, \ldots, c_n$ are n arbitrary constants.

8.1 Solution of Homogeneous Equation (8.3)

Auxiliary Equations

Let $y = e^{mx}$ be the trial solution of (8.3). Then putting

$$y = e^{mx}, Dy = me^{mx}, D^2y = m^2e^{mx}, \ldots, D^ny = m^ne^{mx},$$

(8.3) becomes

$$(a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \ldots + a_n)e^{mx} = 0$$

Hence e^{mx} will be a solution of (8.3) if m is a root of the algebraic equation

$$a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \ldots + a_n = 0 \tag{8.4}$$

The equation (8.4) in m is called the Auxiliary equation.

Case-I: When the auxiliary equation has distinct roots

Let $m_1, m_2, \ldots, m_n$ be the distinct roots of (8.4). Then $y = e^{m_1x}, y = e^{m_2x}, \ldots, y = e^{m_nx}$ are all independent solution of (8.3). Therefore, the general solution of (8.3) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \ldots + c_ne^{m_nx}.$$

Example 8.1. *Solve the differential equation* $\dfrac{d^3y}{dx^3} - 13\dfrac{dy}{dx} - 12y = 0.$

Solution: The given equation is $(D^3 - 13D - 12)y = 0$.
Let $y = e^{mx}$ be the trial solution of the given equation. Then, the auxiliary equation is

$$m^3 - 13m - 12 = 0 \Rightarrow m = -1, -3, 4.$$

Hence, the complete solution is

$$y = c_1e^{-x} + c_2e^{-3x} + c_3e^{4x}.$$

Example 8.2. *Solve the differential equation* $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} + 6y = 0.$

Case-II: When the auxiliary equation has repeated roots

Consider the 2nd order differential equation having equal roots as follows

$$(D - m_1)^2y = 0. \tag{8.5}$$

Put $(D - m_1)y = v$. Then (8.5) becomes

$$(D - m_1)v = 0 \Rightarrow \frac{dv}{dx} = m_1v.$$

Separating the variables, we obtain that

$$\begin{aligned}
& \frac{dv}{v} = m_1 dx \Rightarrow \int \frac{dv}{v} = \int m_1 dx + \log c \\
\Rightarrow \quad & \log v = m_1 x + \log c \Rightarrow v = ce^{m_1 x} \\
\Rightarrow \quad & (D - m_1)y = ce^{m_1 x} \text{ as } v = (D - m_1)y \\
\Rightarrow \quad & \frac{dy}{dx} - m_1 y = ce^{m_1 x},
\end{aligned}$$

which is a first order linear differential equation in y. Its integrating factor

$$\text{I.F.} = e^{-\int m_1 dx} = e^{-m_1 x}.$$

Therefore,

$$ye^{-m_1 x} = \int ce^{m_1 x} e^{-m_1 x} dx + c_1 \Rightarrow ye^{-m_1 x} = \int c dx + c_1 \Rightarrow y = (c_1 + cx)e^{m_1 x}.$$

Therefore, if $m_1, m_2, \ldots, m_n$ are roots of the auxiliary equation for (8.3) with $m_1 = m_2$, then the most general solution of (8.3), when two roots of auxiliary equation are equal, is

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}.$$

If three roots of the auxiliary equation are equal i.e. $m_1 = m_2 = m_3$, the general solution is

$$y = (c_1 + c_2 x + c_3 x^2)e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}.$$

Example 8.3. *Solve* $\dfrac{d^4y}{dx^4} - \dfrac{d^3y}{dx^3} - 9\dfrac{d^2y}{dx^2} - 11\dfrac{dy}{dx} - 4y = 0.$

Solution: Let $y = e^{mx}$ be the trial solution of the given equation. Then the auxiliary equation is

$$m^4 - m^3 - 9m^2 - 11m - 4 = 0 \Rightarrow (m+1)^3(m-4) = 0 \Rightarrow m = -1, -1, -1, 4.$$

Hence the general solution is

$$y = (c_1 + c_2 x + c_3 x^2)e^{-x} + c_4 e^{4x}.$$

Example 8.4. *Solve* $(D^3 - 2D^2 - 4D + 8)y = 0.$

Case-III: When the auxiliary equation has imaginary roots

Let $\alpha \pm i\beta$ be the imaginary roots of a 2nd order differential equation. Then, its general solution is

$$\begin{aligned}
y &= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} \\
&= e^{\alpha x}[c_1 e^{i\beta x} + c_2 e^{-i\beta x}] \\
&= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] \\
&= e^{\alpha x}[(c_1 + c_2)\cos\beta x + (c_1 - c_2)i\sin\beta x] \\
&= (A\cos\beta x + B\sin\beta x)e^{\alpha x}.
\end{aligned}$$

If the auxiliary equation has two equal pairs of imaginary roots, then the general solution is obtained as follows

$$y = [(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]e^{\alpha x}.$$

Example 8.5. *Solve* $\dfrac{d^4y}{dx^4} + 5\dfrac{d^2y}{dx^2} + 6y = 0.$

Solution: The auxiliary equation of the given differential equation is

$$m^4 + 5m^2 + 6 = 0 \Rightarrow (m^2+3)(m^2+2) = 0 \Rightarrow m = \pm i\sqrt{3}, \pm i\sqrt{2}.$$

Hence, the complete solution is

$$y = c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x + c_3\cos\sqrt{2}x + c_4\sin\sqrt{2}x.$$

Example 8.6. *Solve the equation* $D^4 - D^3 - D + 1 = 0.$

8.2 Solution of Nonhomogeneous LDE

Consider the Nonhomogeneous LDE

$$\begin{aligned} &\big(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + a_3D^{n-2} + \ldots + a_{n-1}D + a_n\big)y = f(x) \qquad (8.6)\\ \Rightarrow\quad &F(D)y = f(x). \end{aligned}$$

Complementary function (C.F.) and particular integral (P.I.) are the two parts of the complete solution of a nonhomogeneous LDE with constant coefficients. The general solution of $F(D)y = 0$ is called the complementary function and the particular solution of $F(D)y = f(x)$ is called the particular integral. Hence the **complete solution** of (8.2) is **C.F.+P.I.**

8.2.1 Particular Integral

For finding the particular integral of non-homogenous differential equation, there have some certain methods such as method of undetermined coefficients, method of variation of parameters, and operator methods. We deal with operator method in this discussion.

Meaning of the symbol $\frac{1}{F(D)}$

Definition 8.3. $\frac{1}{F(D)}f(x)$ *is that function of x, free from arbitrary constants, which when operated by $F(D)$ gives $f(x)$. Thus,*

$$F(D)\cdot\frac{1}{F(D)}f(x) = f(x).$$

Therefore, $F(D)$ and $\frac{1}{F(D)}$ are inverse operators. Thus, the symbol $\frac{1}{F(D)}$ stands for integration.

Prove that $\frac{1}{D-\alpha}f(x) = e^{\alpha x}\frac{1}{D}(e^{-\alpha x}f(x))$

Suppose that $y = \frac{1}{D-\alpha}f(x)$. Then

$$(D-\alpha)y = f(x) \Rightarrow \frac{dy}{dx} - \alpha y = f(x),$$

which is linear in y. Now, integrating factor is

$$\text{I.F.} = e^{\int -\alpha x} = e^{-\alpha x}.$$

Therefore, the solution of this equation is

$$\begin{aligned} &ye^{-\alpha x} = \int e^{-\alpha x}f(x)dx\\ &\text{(constant is not adding as it is the particular solution)}\\ &y = e^{\alpha x}\frac{1}{D}(e^{-\alpha x}f(x)), \text{ as } \frac{1}{D} \equiv \text{ integration.} \end{aligned}$$

8.2.2 Working rule for finding particular integral of $F(D)y = f(x)$

The discussion on these cases can be given as follows:

Case-I. When $f(x) = x^n$

Let $F(D) = (D-\alpha_1)(D-\alpha_2)\ldots(D-\alpha_n)$. Then, resolving into partial fraction, we get

$$\frac{1}{F(D)} = \frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \ldots + \frac{A_n}{D-\alpha_n} \text{ (say).}$$

Now, the particular integral is

$$\begin{aligned}
\text{P.I.} &= \frac{1}{F(D)}f(x) = \Big\{\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \ldots + \frac{A_n}{D-\alpha_n}\Big\}f(x) \\
&= A_1\frac{1}{D-\alpha_1}f(x) + A_2\frac{1}{D-\alpha_2}f(x) + \ldots + A_n\frac{1}{D-\alpha_n}f(x) \\
&= A_1e^{\alpha_1 x}\int e^{-\alpha_1 x}f(x)dx + A_2e^{\alpha_2 x}\int e^{-\alpha_2 x}f(x)dx + \ldots + A_ne^{\alpha_n x}\int e^{-\alpha_n x}f(x)dx,
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\text{P.I.} &= \frac{1}{F(D)}f(x) = \Big\{\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + \ldots + \frac{A_n}{D-\alpha_n}\Big\}f(x) \\
&= A_1\frac{1}{D-\alpha_1}f(x) + A_2\frac{1}{D-\alpha_2}f(x) + \ldots + A_n\frac{1}{D-\alpha_n}f(x) \\
&= -\frac{A_1}{\alpha_1}\Big(1-\frac{D}{\alpha_1}\Big)^{-1}f(x) - \frac{A_2}{\alpha_2}\Big(1-\frac{D}{\alpha_2}\Big)^{-1}f(x) + \ldots - \frac{A_n}{\alpha_n}\Big(1-\frac{D}{\alpha_n}\Big)^{-1}f(x) \\
&= -\frac{A_1}{\alpha_1}\Big(1+\frac{D}{\alpha_1}+\frac{D^2}{\alpha_1^2}+\ldots\Big)f(x) - \frac{A_2}{\alpha_2}\Big(1+\frac{D}{\alpha_2}+\frac{D^2}{\alpha_2^2}+\ldots\Big)f(x) \\
&\quad -\frac{A_n}{\alpha_n}\Big(1+\frac{D}{\alpha_n}+\frac{D^2}{\alpha_n^2}+\ldots\Big)f(x)
\end{aligned}$$

which can be evaluated easily and thus the particular integral can be found.

Example 8.7. *Solve* $\frac{dy^2}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$.

Solution: The given differential equation is $\frac{dy^2}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$. The corresponding homogeneous equation of the given equation is

$$\frac{dy^2}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

The auxiliary equation is

$$m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3.$$

The complementary function is

$$y_c = c_1e^{2x} + c_2e^{3x}.$$

The particular integral is

$$\begin{aligned}
y_p &= \frac{1}{D^2-5D+6}x^2 = \frac{1}{(D-2)(D-3)}x^2 = \Big(\frac{1}{D-3} - \frac{1}{D-2}\Big)x^2 \\
&= -\frac{1}{3}\Big(1-\frac{D}{3}\Big)^{-1}x^2 + \frac{1}{2}\Big(1-\frac{D}{2}\Big)^{-1}x^2 \\
&= -\frac{1}{3}\Big(1+\frac{D}{3}+\frac{D^2}{3^2}+\ldots\Big)x^2 + \frac{1}{2}\Big(1+\frac{D}{2}+\frac{D^2}{2^2}+\ldots\Big)x^2 \\
&= -\frac{1}{3}\Big(x^2+\frac{2}{3}x+\frac{2}{9}\Big) + \frac{1}{2}\Big(x^2+x+\frac{1}{2}\Big) \\
&= \frac{1}{6}x^2 + \frac{5}{18}x + \frac{37}{108}.
\end{aligned}$$

Hence, the complete integral is $y = y_c + y_p = c_1e^{2x} + c_2e^{3x} + \frac{1}{6}x^2 + \frac{5}{18}x + \frac{37}{108}$.

Case-II. When $f(x) = e^{ax}$ but $F(a) \neq 0$

Consider the differential equation

$$F(D)y = (a_0D^n + a_1D^{n-1} + \ldots + a_n)y = f(x).$$

Let $y = e^{ax}$. Then,

$$Dy = ae^{ax},\ D^2y = a^2e^{ax},\ \ldots,\ D^ny = a^ne^{ax}.$$

Thus, we obtain that

$$F(D)y = (a_0a^n + a_1a^{n-1} + \ldots + a_n)e^{ax} = F(a)e^{ax} \Rightarrow F(D)e^{ax} = F(a)e^{ax}.$$

Now, operating on both sides by $\dfrac{1}{F(D)}$, we get

$$\frac{1}{F(D)}F(D)e^{ax} = \frac{1}{F(D)}F(a)e^{ax} \Rightarrow e^{ax} = F(a)\frac{1}{F(D)}e^{ax} \Rightarrow \frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax},$$

provided that $F(a) \neq 0$. Thus, the particular integral is

$$\begin{aligned} y_p &= \frac{1}{F(D)}f(x) = \frac{1}{F(D)}e^{ax} \\ &= \frac{1}{F(a)}e^{ax}, \text{ where } F(a) \neq 0. \end{aligned}$$

Example 8.8. *Solve* $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 2e^{4x}$.

Solution: The given differential equation can be written as

$$(D^2 - 2D - 3)y = 2e^{4x}.$$

The auxiliary equation is

$$m^2 - 2m - 3 = 0 \Rightarrow m = -1, 3.$$

Thus, the complementary function is $y_c = c_1e^{3x} + c_2e^{-x}$.

The particular integral is

$$\begin{aligned} y_p &= \frac{1}{D^2 - 2D - 3}(2e^{4x}) = \frac{1}{4^2 - 2\cdot 4 - 3}(2e^{4x}) \\ &= \frac{2}{5}e^{4x}. \end{aligned}$$

Hence, the complete solution is

$$y = y_c + y_p = c_1e^{3x} + c_2e^{-x} + \frac{2}{5}e^{4x}.$$

Case-III. When $f(x) = \sin ax$ but $F(-a^2) \neq 0$

Let $y = f(x) = \sin ax$. Then,

$$\begin{aligned} \sin ax &= \sin ax \\ D\sin ax &= a\cos ax \\ D^2\sin ax &= (-a^2)\sin ax \\ D^3\sin ax &= -a^3\cos ax \\ D^4\sin ax &= a^4\sin ax = (-a^2)^2\sin ax \\ \ldots\ldots & \quad \ldots\ldots\ldots \\ (D^2)^n\sin ax &= (-a^2)^n\sin ax. \end{aligned}$$

Thus,

$$F(D^2)\sin ax = F(-a^2)\sin ax.$$

Operating by $\dfrac{1}{F(D^2)}$ on both sides, we get

$$\begin{aligned} & \frac{1}{F(D^2)}F(D^2)\sin ax = \frac{1}{F(D^2)}\cdot F(-a^2)\sin ax \\ \Rightarrow\ & \sin ax = F(-a^2)\frac{1}{F(D^2)}\sin ax \\ \Rightarrow\ & \frac{1}{F(D^2)}\sin ax = \frac{1}{F(-a^2)}\sin ax, \text{ provided that } F(-a^2) \neq 0. \end{aligned}$$

Important Note: It follows from the above result that we have to put $-a^2$ in stead of D^2. We can not put anything in place of D.

Example 8.9. *Solve the differential equation* $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = \sin 2x.$

Solution: The given differential equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x.$$

The auxiliary equation of the corresponding homogeneous equation is

$$m^2 + m + 1 = 0 \Rightarrow m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

The complementary function is

$$y_c = \Big(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\Big)e^{-\frac{x}{2}}.$$

The particular integral is

$$\begin{aligned} y_p &= \frac{1}{F(D)}\sin 2x = \frac{1}{D^2+D+1}\sin 2x \\ &= \frac{1}{-4+D+1}\sin 2x = \frac{D+3}{D^2-9}\sin 2x \\ &= -\frac{1}{13}(D+3)\sin 2x = -\frac{1}{13}(2\cos 2x + 3\sin 2x). \end{aligned}$$

Hence, the complete solution is

$$y = \Big(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\Big)e^{-\frac{x}{2}} - \frac{1}{13}(D+3)\sin 2x - \frac{1}{13}(2\cos 2x + 3\sin 2x).$$

Example 8.10. *Solve the following differential equations:*

(i) $(D^2+1)^2y = \cos 3x$, (ii) $(D^3+D^2+D+1)y = \sin 2x$;

(iii) $(D^2+4)y = \sin ax$ *when* $a \neq 2$ *with* $y(0)=0$, $Dy = 0$ *when* $x = 0$;

(iv) $(D^2-2D-3)y = 2e^x - 10\sin x$.

Case-IV. When $f(x) = e^{ax}$ but $F(a) = 0$

Since $F(a) = 0$, then $(D-a)$ is a factor of $F(D)$. Therefore, let

$$F(D) = (D-a)\phi(D),\ \phi(a) \neq 0. \tag{8.7}$$

Then

$$\begin{aligned} \frac{1}{F(D)}e^{ax} &= \frac{1}{(D-a)\phi(D)}e^{ax} \\ &= \frac{1}{(D-a)}\cdot\frac{1}{\phi(a)}e^{ax}, \quad \text{as } \phi(a) \neq 0 \\ &= \frac{1}{\phi(a)}\frac{1}{(D-a)}e^{ax} = \frac{1}{\phi(a)}\cdot e^{ax}\int e^{-ax}\cdot e^{ax}dx \\ &= \frac{1}{\phi(a)}e^{ax}\int dx = \frac{xe^{ax}}{\phi(a)}. \end{aligned} \tag{8.8}$$

Now, differentiating (8.7) with respect to D, we obtain

$$F'(D) = (D-a)\phi'(D) + \phi(D) \Rightarrow F'(a) = 0 + \phi(a) \Rightarrow \phi(a) = F'(a).$$

It follows from (8.8) that

$$\frac{1}{F(D)}e^{ax} = \frac{xe^{ax}}{F'(a)}.$$

If $F'(a) = 0$ and $F''(a) \neq 0$, then applying the same procedure discussed above, we obtain that

$$\frac{1}{F(D)}e^{ax} = x^2\frac{e^{ax}}{F''(a)} \text{ and so on.}$$

Example 8.11. *Solve the following differential equations:*

(i) $(D^2-3D+2)y=e^x$, (ii) $(D^2+4D+3)y=e^{-3x}$;

(iii) $D^3+3D^2+3D+1)y=e^{-x}$, (iv) $2D^3-3D^2+1)y=e^x+1$;

(v) $(D^3-2D^2-5D+6)y=e^{3x}$, (vi) $D^2+4D+4)y=e^{2x}+e^{-2x}$.

Solution:(vi) The given differential equation is

$$D^2+4D+4)y=e^{2x}+e^{-2x}.$$

The auxiliary equation corresponding to the homogeneous equation is

$$m^2+4m+4=0 \Rightarrow m=-2,-2.$$

The complementary function is

$$y_c=(c_1+c_2x)e^{-2x}.$$

The particular integral is

$$\begin{aligned}
y_p &= \frac{1}{D^2+4D+4}(e^{2x}+e^{-2x}) \\
&= \frac{e^{2x}}{4+8+4}+\frac{1}{D^2+4D+4}e^{-2x} \\
&= \frac{1}{16}e^{2x}+x\frac{1}{2D+4}e^{-2x} \\
&= \frac{1}{16}e^{2x}+x^2\frac{1}{2}e^{-2x}.
\end{aligned}$$

Hence, the complete solution is

$$y=(c_1+c_2x)e^{-2x}+\frac{1}{16}e^{2x}+x^2\frac{1}{2}e^{-2x}.$$

Case-V. When $f(x)=\sin ax$ but $F(-a^2)=0$

Since $F(-a^2)=0$, then (D^2+a^2) is a factor of $F(D^2)$. Therefore, let

$$F(D^2)=(D^2+a^2)\phi(D^2),\ \phi(-a^2)\neq 0. \tag{8.9}$$

Then

$$\begin{aligned}
\frac{1}{F(D^2)}(\cos ax+i\sin ax) &= \frac{1}{F(D^2)}e^{iax} \\
&= x\frac{1}{F'(D^2)}e^{iax},\ \text{ as } F(-a^2)=0 \\
&= x\frac{1}{F'(D^2)}(\cos ax+i\sin ax). \qquad (8.10)
\end{aligned}$$

Now, equating real and imaginary parts from (8.9), we obtain that

$$\frac{1}{F(D^2)}\cos ax=x\frac{1}{F'(D^2)}\cos ax \text{ and } \frac{1}{F(D^2)}\sin ax=x\frac{1}{F'(D^2)}\sin ax.$$

In case $F'(-a^2)=0$ and $F''(-a^2)\neq 0$, (D^2+a^2) is a twice repeated factor of $F(D^2)$. Applying the above result once again, we obtain that

$$\frac{1}{F(D^2)}\cos ax=x^2\frac{1}{F''(D^2)}\cos ax \text{ and } \frac{1}{F(D^2)}\sin ax=x^2\frac{1}{F''(D^2)}\sin ax.$$

Example 8.12. *Solve the following differential equations:*

(i) $(D^6+1)y = \frac{1}{2}(\cos x - \cos 2x)$, (ii) $(D^3-3D^2+4D-2)y = e^x + \cos x$;

(iii) $(D^3-5D^2+7D-3)y = e^{2x}\cosh x$.

Solution:((ii)) The given differential equation is

$$(D^3-3D^2+4D-2)y = e^x + \cos x.$$

The auxiliary equation corresponding to the homogeneous part is

$$m^3-2m^2+4m-2=0 \Rightarrow (m-1)(m^2-2m+2)=0 \Rightarrow m=1, 1\pm i.$$

Thus, the complementary function is

$$y_c = c_1e^x + (c_2\cos x + c_3\sin x)e^x.$$

The particular solution is

$$\begin{aligned}
y_p &= \frac{1}{D^3-3D^2+4D-2}e^x + \frac{1}{D^3-3D^2+4D-2}\cos x \\
&= x\frac{1}{3D^2-6D+4}e^x + \frac{1}{(-1)D-3(-1)+4D-2}\cos x \\
&= xe^x + \frac{1}{3D+1}\cos x = xe^x + \frac{3D-1}{9D^2-1}\cos x \\
&= xe^x - \frac{1}{10}(3D-1)\cos x = xe^x + \frac{1}{10}(3\sin x + \cos x).
\end{aligned}$$

Hence, the complete solution is

$$y = c_1e^x + (c_2\cos x + c_3\sin x)e^x + xe^x + \frac{1}{10}(3\sin x + \cos x).$$

Case-VI. When $f(x) = e^{ax}V$, where V is a function of x

By successive differentiation, we get

$$D(e^{ax}V) = e^{ax}DV + ae^{ax}V = e^{ax}(D+a)V.$$

$$\begin{aligned}
D^2(e^{ax}V) &= e^{ax}D^2V + ae^{ax}V + a^2e^{ax}V + ae^{ax}DV \\
&= e^{ax}(D^2+2aD+a^2)V = e^{ax}(D+a)^2V.
\end{aligned}$$

Similarly, we obtain

$$D^3(e^{ax}V) = e^{ax}(D+a)^3V, \ldots, D^n(e^{ax}V) = e^{ax}(D+a)^nV.$$

Therefore,

$$F(D)(e^{ax}V) = e^{ax}F(D+a)V.$$

Taking the inverse operators, we have

$$\frac{1}{F(D)}(e^{ax}V) = e^{ax}\frac{1}{F(D+a)}V.$$

Example 8.13. *Solve the differential equations:*

(i) $(D^2-9)y = 6e^{3x} + xe^{3x}$, (ii) $(D^3-3D^2+3D-1)y = xe^x + e^x$;

(iii) $(D^2-7D-6)y = x^2e^{2x}$, (iv) $(D^3-2D+4)y = e^x\cos x$;

(v) $(D^2-2D+4)y = e^x\cos x$, (vi) $(D^2+2D+2)y = xe^{-x}$.

Solution:(iv) The given differential equation is

$$(D^3 - 2D + 4)y = e^x \cos x.$$

The auxiliary equation corresponding to the homogeneous equation is

$$m^3 - 2m + 4 = 0 \Rightarrow (m+2)(m^2 - 2m + 2) = 0 \Rightarrow m = -2, 1 \pm i.$$

Thus, the complementary function is

$$y_c = c_1 e^{-2x} + (c_2 \cos x + c_3 \sin x)e^x.$$

The particular integral is

$$\begin{aligned} y_p &= \frac{1}{D^3 - 2D + 4} e^x \cos x = e^x \frac{1}{(D+1)^3 - 2(D+1) + 4} \cos x \\ &= e^x \frac{1}{D^3 + 3D^2 + D + 3} \cos x = x e^x \frac{1}{3D^2 + 6D + 1} \cos x \\ &= x e^x \frac{1}{6D - 2} \cos x = \frac{1}{2} x e^x \frac{3D + 1}{9D^2 - 1} \cos x \\ &= -\frac{1}{20} x e^x (3D + 1) \cos x = \frac{1}{20} x e^x (3 \sin x - \cos x). \end{aligned}$$

Hence, the complete solution is

$$y = c_1 e^{-2x} + (c_2 \cos x + c_3 \sin x)e^x + \frac{1}{20} x e^x (3 \sin x - \cos x).$$

Example 8.14. *Solve the following differential equations:*

(1) $(D^4 - 1)y = x \sin x$, (2) $(D^2 - 4D + 4)y = 3x^2 e^{2x} \sin 2x$;

(3) $(D^2 + 1)y = 8x^2 e^{2x} \sin 2x$, (4)$(D^4 + 2D^2 + 1)y = x^2 \cos x$;

(4) $(D^2 - 6D + 13)y = 8e^{3x} \sin 2x$, (5) $(D^2 - 2D + 1)y = x^2 e^{2x}$.

Solution:(2) The given differential equation is

$$(D^2 - 4D + 4)y = 3x^2 e^{2x} \sin 2x.$$

The auxiliary equation corresponding homogeneous equation is

$$m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2.$$

The complementary function is

$$y_c = (c_1 + c_2 x)e^{2x}.$$

The particular integral is

$$\begin{aligned}
y_p &= \frac{1}{(D-2)^2}3x^2e^{2x}\sin 2x = 3e^{2x}\cdot\frac{1}{(D+2-2)^2}x^2\sin 2x\\
&= 3e^{2x}\frac{1}{D^2}x^2\sin 2x\\
&= \text{imaginary part of } 3e^{2x}\frac{1}{D^2}x^2e^{2ix}\\
&= \text{imaginary part of } 3e^{2x}\cdot e^{2ix}\frac{1}{(D+2i)^2}x^2\\
&= \text{imaginary part of } 3e^{2x}e^{2ix}\frac{1}{D^2+4iD-4}x^2\\
&= \text{imaginary part of } -\frac{3}{4}e^{2x}e^{2ix}\Big(1-(iD+\frac{1}{4}D^2)\Big)^{-1}x^2\\
&= \text{imaginary part of } -\frac{3}{4}e^{2x}e^{2ix}\Big(1+(iD+\frac{1}{4}D^2)+(iD+\frac{1}{4}D^2)^2+\dots\Big)x^2\\
&= \text{imaginary part of } -\frac{3}{4}e^{2x}e^{2ix}(1+iD+\frac{1}{4}D^2-D^2+\dots)x^2\\
&= \text{imaginary part of } -\frac{3}{4}e^{2x}e^{2ix}(x^2+2ix-\frac{3}{2})\\
&= \text{imaginary part of } -\frac{3}{4}e^{2x}(\cos 2x+i\sin 2x)\Big((x^2-\frac{3}{2})+i2x\Big)\\
&= -\frac{3}{4}e^{2x}\Big((x^2-\frac{3}{2})\sin 2x+2x\cos 2x\Big)\\
&= -\frac{3}{8}e^{2x}\Big((2x^2-3)\sin 2x+4x\cos 2x\Big).
\end{aligned}$$

Hence, the complete solution is

$$(c_1+c_2x)e^{2x}-\frac{3}{8}e^{2x}\Big((2x^2-3)\sin 2x+4x\cos 2x\Big).$$

8.2.3 The method of undetermined coefficients

To determine the particular integral, we need to present the method of undetermined coefficients. The method of undetermined coefficients applies when the nonhomogeneous function $f(x)$ in the differential equation is a finite linear combination of undetermined coefficients functions.

Definition 8.4. *A function $f(x)$ is said to be a undetermined coefficients (UC) function if it is either a function defined by one of the following:*

(i) x^n, *where n is a positive integer or zero,*

(ii) e^{ax}, *where a is a non-zero constant,*

(iii) $\sin(bx+c)$, *where b and c are constants, $b\neq 0$,*

(iv) $\cos(bx+c)$, *where b and c are constants, $b\neq 0$,*

or a function defined as a finite product of two or more functions of these four functions.

Definition 8.5. *Let $f(x)$ be the undetermined constants (UC) function. Then the UC set of $f(x)$ is the set of functions consisting of $f(x)$ itself and all linearly independent UC functions of which the successive derivatives of $f(x)$ are either constant multiple or linear combinations.*

Example 1: Let $f(x)=x^3$ is a UC function. Computing derivatives of $f(x)$, we find

$$f'(x)=3x^2,\ f''(x)=6x,\ f'''(x)=6=6\cdot 1,\ f^{iv}(x)=0.$$

The linearly independent UC functions of which the successive derivatives of $f(x)$ are either constant multiple or linear combinations are those given by

$$x^3,\ x^2,\ x,\ 1.$$

Thus, the UC set of x^3 is the set $S = \{x^3, x^2, x, 1\}$.

Example 2: Let $f(x) = \sin 2x$ is a UC function. Computing derivatives of $f(x)$, we find

$$f'(x) = 2\cos 2x,\ f''(x) = -4\sin 2x,\ \ldots.$$

The only linearly independent UC function of which the successive derivatives of $f(x)$ are either constant multiple or linear combinations is that given by $\cos 2x$.

Thus, the UC set of $\sin 2x$ is the set $S = \{\sin 2x, \cos 2x\}$.

Remark 8.1. *Let $f(x) = x^3 \cos 2x$, which is the product of two UC functions defined by x^3 and $\cos 2x$. Then the UC set of this product $x^3 \cos 2x$ is the set of all products obtained by multiplying the various members of the UC set of x^3 by the various members of the UC set of $\cos 2x$. Therefore, the UC set of x^3 is*

$$\{x^3, x^2, x, 1\} \text{ and}$$

the UC set of $\cos 2x$ is

$$\{\cos 2x, \sin 2x\}.$$

Thus, the UC set of the product $x^3 \cos 2x$ is the set of all products of each of x^3, x^2, x and 1 by each of $\sin 2x$ and $\cos 2x$ and so it is

$$\{x^3 \sin 2x, x^3 \cos 2x, x^2 \sin 2x, x^2 \cos 2x, x \sin 2x, x \cos 2x, \sin 2x, \cos 2x\}.$$

Remark 8.2. *If any member of the UC set S is a solution of the homogeneous equation corresponding to nonhomogeneous LDE, then we need to multiply each member of UC set S by the lowest positive integral power of x so that the resulting revised UC set will contain no members that are solutions of the homogeneous equation corresponding to nonhomogeneous LDE.*

As illustrations, consider the two two equations

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 e^x \tag{8.11}$$

and

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^x. \tag{8.12}$$

The auxiliary equation of the corresponding to homogeneous equation of (8.11) is

$$m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2.$$

The complementary function corresponding to homogeneous equation of (8.11) is

$$y_c = c_1 e^x + c_2 e^{2x}.$$

The UC set of (8.11) is

$$S = \{x^2 e^x, x e^x, e^x\}.$$

One sees that the member e^x of UC set S is a solution of the corresponding to homogeneous equation of (8.11), so multiply each member of S by x and then the revised UC set is

$$S' = \{x^3 e^x, x^2 e^x, x e^x\}.$$

This revised set has no members that satisfy the corresponding to homogeneous equation of (8.11).

In the same way, the auxiliary equation of the corresponding to homogeneous equation of (8.12) is

$$m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1.$$

The complementary function corresponding to homogeneous equation of (8.11) is

$$y_c = (c_1 + c_2 x)e^x.$$

The UC set of (8.12) is

$$S = \{x^2e^x, xe^x, e^x\}.$$

One sees that the two member e^x and xe^x of UC set S are the solutions of the corresponding to homogeneous equation of (8.12), so multiply each member of S by x^2 and then the revised UC set is

$$S' = \{x^4e^x, x^3e^x, x^2e^x\}.$$

This revised set has no members that satisfy the corresponding to homogeneous equation of (8.12).

Example 8.15. *Solve the differential equation* $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 2e^x - 10\sin x.$

Solution: The auxiliary equation corresponding to the homogeneous equation is

$$m^2 - 2m - 3 = 0 \Rightarrow m = 3, -1.$$

The complementary function is

$$y_c = c_1e^{3x} + c_2e^{-x}.$$

The nonhomogeneous term is the linear combination $2e^x - 10\sin x$ of the two UC functions given by e^x and $\sin x$. So, we need to form UC set for each of these two functions. Thus,

$$S_1 = \{e^x\},\ S_2 = \{\sin x, \cos x\}.$$

Moreover, we see that none of the functions e^x, $\sin x$, $\cos x$ in either of these sets is a solution of the corresponding homogeneous equation.

Thus, we form the linear combination

$$Ae^x + B\sin x + C\cos x$$

of the three elements e^x, $\sin x$, $\cos x$ of S_1 and S_2, with undetermined coefficients A, B, C.

Thus, we take the particular solution as

$$y_p = Ae^x + B\sin x + C\cos x.$$

Then

$$y_p' = Ae^x + B\cos x - C\sin x \text{ and } y_p'' = Ae^x - B\sin x - C\cos x.$$

Substituting these in the given equation, we find

$$\begin{aligned} & Ae^x - B\sin x - C\cos x - 2(Ae^x + B\cos x - C\sin x) - 3(Ae^x + B\sin x + C\cos x) \\ & \quad = 2e^x - 10\sin x \\ \Rightarrow\ & -4Ae^x + (-4B + 2C)\sin x + (-4C - 2B)\cos x = 2e^x - 10\sin x. \end{aligned}$$

Equating the coefficients of these like terms, we obtain the equations

$$-4A = 2,\ -4B + 2C = -10,\ -4C - 2B = 0.$$

Solving these equations, we find that

$$A = -\frac{1}{2},\ B = 2,\ C = -1.$$

Thus, the particular integral is

$$y_p = -\frac{1}{2}e^x + 2\sin x - \cos x.$$

Thus, the general solution of the differential equation is

$$y = c_1e^{3x} + c_2e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x.$$

Example 8.16. *Solve the differential equation* $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$.

Solution: The auxiliary equation corresponding to the homogeneous equation of the given equation is

$$m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2.$$

The complementary function is

$$y_c = c_1e^x + c_2e^{2x}.$$

The nonhomogeneous term is the linear combination

$$2x^2 + e^x + 2xe^x + 4e^{3x}$$

of the four UC functions given by x^2, e^x, xe^x, and e^{3x}. Thus, the UC sets are

$$S_1 = \{x^2, x, 1\},\ S_2 = \{e^x\},\ S_3 = \{xe^x, e^x\},\ S_4 = \{e^{3x}\}.$$

Note that S_2 is completely included in S_3, so S_2 is omitted from the UC set. On the other hand, S_3 contains e^x which is the solution of the corresponding homogeneous equation. Thus, we multiply each member of S_3 by x and then the revised set is

$$S_3' = \{x^2e^x, xe^x\}.$$

Therefore, the revised UC set is

$$S_1 = \{x^2, x, 1\},\ S_4 = \{e^{3x}\},\ S_3' = \{x^2e^x, xe^x\}.$$

Hence, we form the linear combination

$$Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x,$$

of the six elements $x^2, x, 1, e^3x, x^2e^x, xe^x$.

Let $y_p = Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x$ be the particular solution of the given equation. Then

$$\begin{aligned} y_p' &= 2Ax + B + 3De^{3x} + Ex^2e^x + 2Exe^x + Fxe^x + Fe^x, \\ y_p'' &= 2A + 9De^{3x} + Ex^2e^x + 4Exe^x + 2Ee^x + Fxe^x + 2Fe^x. \end{aligned}$$

Substituting the values of y_p, y_p', y_p'' into the given differential equation, we obtain that

$$\begin{aligned} & 2A + 9De^{3x} + Ex^2e^x + (4E+F)xe^x + (2E+2F)e^x - 3[2Ax + B + 3De^{3x} + Ex^2e^x \\ & \quad + (2E+F)xe^x + Fe^x] + 2(Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x) \\ & \quad = 2x^2 + e^x + 2xe^x + 4e^{3x} \\ \Rightarrow\ & (2A - 3B + 2C) + (2B - 6A)x + 2Ax^2 + 2De^{3x} + (-2E)xe^x \\ & \quad + (2E - F)e^x = 2x^2 + e^x + 2xe^x + 4e^{3x}. \end{aligned}$$

Equating the coefficietns of like terms, we have

$$2A - 3B + 2C = 0,\ 2B - 6A = 0,\ 2A = 2,\ 2D = 4,\ -2E = 2,\ 2E - F = 1.$$

Solving these equations, we obtain

$$A = 1,\ B = 3,\ C = \frac{7}{2},\ D = 2,\ E = -1,\ F = -3.$$

Thus, the particular integral is

$$y_p = x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x.$$

The general solution is therefore

$$y = c_1e^x + c_2e^{2x} + x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x.$$

Example 8.17. *Solve the following differential equations:*

(1) $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = 3x^2 + 4\sin x - 2\cos x;$

(2) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 2e^x - 10\sin x,\ y(0) = 2,\ y'(0) = 4;$

(3) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 4y = \cos 4x;$

(4) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y = 5xe^{-2x};$

(5) $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = 2x^2 + 4\sin x.$

8.2.4 Variation of Parameters

Consider the differential equation in the form

$$\frac{d^2y}{dx^2} + y = \tan x. \tag{8.13}$$

The method of undetermined coefficients would not apply to obtain particular integral for the differential equation (8.13). Thus, we seek a method for finding a particular integral in which the complementary function is known. Such a method is called the method of variation of parameters.

We develop this method in connection with the general second order linear differential equation with variable coefficients

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = f(x). \tag{8.14}$$

Suppose that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the corresponding homogeneous equation of (8.14). Then, the complementary function corresponding to the homogeneous equation of (8.13) is

$$y_c = c_1y_1(x) + c_2y_2(x),$$

where c_1 and c_2 are arbitrary constants.

The procedure in the method of variation of parameters is to replace the arbitrary constants C_1 and c_2 in the complementary function by respective functions $v_1(x)$ and $v_2(x)$ which will be determined so that the resulting function, which is defined by

$$v_1(x)y_1(x) + v_2(x)y_2(x),$$

will be a particular integral of (8.14) (hence the name, variation of parameters).

Thus, we assume that a particular solution of (8.14) is of the form

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x). \tag{8.15}$$

Now, differentiating (8.15), we have

$$y_p'(x) = v_1(x)y_1'(x) + v_2(x)y_2'(x) + v_1'(x)y_1(x) + v_2'(x)y_2(x). \tag{8.16}$$

Now, we simplify (8.16) by imposing the condition that

$$v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0. \tag{8.17}$$

With this condition imposed. (8.15) reduces to

$$y_p'(x) = v_1(x)y_1'(x) + v_2(x)y_2'(x). \tag{8.18}$$

Now, differentiating (8.18), we obtain

$$y_p''(x) = v_1(x)y_1''(x) + v_2(x)y_2''(x) + v_1'(x)y_1'(x) + v_2'(x)y_2'(x). \tag{8.19}$$

Now, substituting the values of y_p, y_p', y_p'' in (8.14), we obtain

$$\begin{aligned}
& a_0(x)[v_1(x)y_1''(x) + v_2(x)y_2''(x) + v_1'(x)y_1'(x) + v_2'(x)y_2'(x)] \\
& \quad + a_1(x)[v_1(x)y_1'(x) + v_2(x)y_2'(x)] + \\
& \quad a_2(x)[v_1(x)y_1(x) + v_2(x)y_2(x)] = f(x) \\
\Rightarrow\quad & v_1(x)[a_0(x)y_1''(x) + a_1(x)y_1'(x) + a_2(x)y_1(x)] \\
& \quad + v_2(x)[a_0(x)y_2''(x) + a_1(x)y_2'(x) + a_2(x)y_2(x)] \\
& \quad + a_0(x)[v_1'(x)y_1'(x) + v_2'(x)y_2'(x)] = f(x).
\end{aligned}$$

Since y_1 and y_2 are solutions of the corresponding homogeneous differential equation (8.14), the expression in the first two brackets in the above equation are identically zero. Thus, the above equation reduces to

$$v_1'(x)y_1'(x) + v_2'(x)y_2'(x) = \frac{f(x)}{a_0(x)}. \tag{8.20}$$

Thus, the two imposed conditions require that the functions v_1 and v_2 be chosen such that the system of equations

$$\begin{aligned}
v_1'(x)y_1(x) + v_2'(x)y_2(x) &= 0, \\
v_1'(x)y_1'(x) + v_2'(x)y_2'(x) &= \frac{f(x)}{a_0(x)},
\end{aligned} \tag{8.21}$$

is satisfied. The determinant of coefficients of this system is

$$W[y_1(x), y_2(x)] = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Since y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation (8.14), then $W[y_1(x), y_2(x)] \neq 0$. Hence, the system (8.21) has a unique solution.

Solving this system, we obtain

$$v_1'(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ \dfrac{f(x)}{a_0(x)} & y_2'(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = -\frac{f(x)y_2(x)}{a_0(x)W[y_1(x), y_2(x)]},$$

$$v_2'(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & \dfrac{f(x)}{a_0(x)} \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = \frac{f(x)y_1(x)}{a_0(x)W[y_1(x), y_2(x)]}.$$

Thus, we obtain the functions v_1 and v_2 defined by

$$\begin{aligned}
v_1(x) &= -\int \frac{f(x)y_2(x)}{a_0(x)W[y_1(x), y_2(x)]}dx, \\
v_2(x) &= \int \frac{f(x)y_1(x)}{a_0(x)W[y_1(x), y_2(x)]}dx.
\end{aligned}$$

Therefore, the particular integral of (8.14) is defined by

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x),$$

and the complete solution of (8.14) is

$$y = y_c + y_p.$$

Example 8.18. *Solve the differential equation* $\dfrac{d^2y}{dx^2} + y = \tan x.$

Solution: The auxiliary equation corresponding homogeneous equation of the given equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i.$$

The complementary function is

$$y_c = c_1 \sin x + c_2 \cos x.$$

Let $y_p = v_1 \sin x + v_2 \cos x$ be the particular solution of the given equation. Then

$$y_p' = v_1 \cos x - v_2 \sin x + v_1' \sin x + v_2' \cos x.$$

We impose the condition

$$v_1' \sin x + v_2' \cos x = 0, \tag{8.22}$$

leaving

$$y_p' = v_1 \cos x - v_2 \sin x.$$

From this

$$y_p'' = -v_1 \sin x - v_2 \cos x + v_1' \cos x - v_2' \sin x.$$

Thus, from the given equation we obtain that

$$v_1' \cos x - v_2' \sin x = \tan x. \tag{8.23}$$

Solving (8.22) and (8.23), we obtain that

$$v_1'(x) = \frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = -\frac{-\cos x \tan x}{-1} = \sin x,$$

$$\begin{aligned} v_2'(x) &= \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}} = \frac{\sin x \tan x}{-1} = \frac{-\sin^2 x}{\cos x} \\ &= \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x. \end{aligned}$$

Integrating, we find

$$v_1 = -\cos x + c_3,\ v_2 = \sin x - \log|\sec x + \tan x| + c_4.$$

Therefore,

$$\begin{aligned} y_p &= (-\cos x + c_3)\sin x + (\sin x - \log|\sec x + \tan x| + c_4)\cos x \\ &= -\sin x \cos x + c_3 \sin x + \sin x \cos x - \cos x \log|\sec x + \tan x| + c_4 \cos x \\ &= c_3 \sin x + c_4 \cos x - \cos x \log|\sec x + \tan x|. \end{aligned}$$

Since a particular integral is a solution free from arbitrary constants, we may assign any particular values A and B to c_3 and c_4, respectively and the resulting particular integral is

$$y_p = A \sin x + B \cos x - \cos x \log|\sec x + \tan x|.$$

Thus, the general solution is

$$\begin{aligned} y &= c_1 \sin x + c_2 \cos x + A \sin x + B \cos x - \cos x \log|\sec x + \tan x| \\ &= (c_1 + A)\sin x + (c_2 + B)\cos x - \cos x \log|\sec x + \tan x| \\ &= C_1 \sin x + C_2 \cos x - \cos x \log|\sec x + \tan x|, \end{aligned}$$

where $C_1 = c_1 + A, C_2 = c_2 + B$.

Note: If we choose $c_3 = c_4 = 0$, the same result can be found.

Example 8.19. *Solve the differential equations:*

(1) $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = e^x$;

(2) $\dfrac{d^2y}{dx^2} + y = \cot x$, (3) $\dfrac{d^2y}{dx^2} + y = \sec x$;

(4) $\dfrac{d^2y}{dx^2} + y = \tan^2 x$, (5) $\dfrac{d^2y}{dx^2} + y = \tan x \sec x$.

8.2.5 The Cauchy-Euler Equation

In this section we will study how to solve the linear differential equation with variable coefficients. This can be done by Cauchy-Euler equation. The Cauchy-Euler equation is of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}x\frac{dy}{dx} + a_n y = f(x), \tag{8.24}$$

where $a_0, a_1, \ldots, a_n$ are constants.

Theorem 8.2. *The transformation $x = e^t$ reduces the equation* (8.24) *to a linear differential equation with constant coefficients.*

Put $x = e^t$. Then $t = \log x \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{x}$. Thus,

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dt}.$$

and

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{1}{x}\frac{d}{dx}\Big(\frac{dy}{dt}\Big) + \frac{dy}{dt}\frac{d}{dx}\Big(\frac{1}{x}\Big)\\
&= \frac{1}{x}\Big(\frac{d^2y}{dt^2}\frac{dt}{dx}\Big) - \frac{1}{x^2}\frac{dy}{dt}\\
&= \frac{1}{x}\Big(\frac{d^2y}{dt^2}\frac{1}{x}\Big) - \frac{1}{x^2}\frac{dy}{dt}\\
&= \frac{1}{x^2}\Big(\frac{d^2y}{dt^2} - \frac{dy}{dt}\Big)\\
\Rightarrow x^2\frac{d^2y}{dx^2} &= \frac{d^2y}{dt^2} - \frac{dy}{dt}.
\end{aligned}$$

Similarly,

$$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} \text{ and so on.}$$

Thus, if we put $x\dfrac{d}{dx} = \dfrac{d}{dt} = D$, then we obtain that

$$\begin{aligned}
x\frac{dy}{dx} &= Dy,\\
x^2\frac{d^2y}{dx^2} &= D(D-1)y,\\
x^3\frac{d^3y}{dx^3} &= D(D-1)(D-2)y,\\
\ldots\ldots & \quad \ldots\ldots\ldots\\
x^n\frac{d^ny}{dx^n} &= D(D-1)(D-2)\ldots(D-n+1)y.
\end{aligned}$$

Substituting these values in (8.24), we obtain that

$$\begin{aligned}
& [a_0\Big(D(D-1)(D-2)\ldots(D-n+1)\Big) + a_1\Big(D(D-1)(D-2)\ldots(D-n+2)\Big)\\
& \quad + \ldots + a_{n-1}D + a_n]y = f(t)\\
\Rightarrow \quad & F(D)y = f(t),
\end{aligned}$$

which is a linear differential equation with constant coefficients and it can be solved by the methods that we have discussed previously.

Example 8.20. *Solve the differential equation* $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3$.

Solution: Let $x = e^t$. Then, we have $t = \log x$. Then

$$x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y.$$

Hence, from the given equation we find that

$$[D(D-1) - 2D + 2]y = e^{3t} \Rightarrow (D^2 - 3D + 2)y = e^{3t}, \tag{8.25}$$

which is a linear differential equation in t with constant coefficients. Therefore, the auxiliary equation corresponding to the homogeneous equation of this equation is

$$m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2.$$

The complementary function of this equation is

$$y_c = c_1e^t + c_2e^{2t} = c_1x + c_2x^2.$$

Let $y_p = Ae^{3t}$. Then, $y_p' = 3Ae^{3t}$, $y_p'' = 9Ae^{3t}$. Hence, from (8.25) we obtain that

$$2Ae^{3t} = e^{3t} \Rightarrow A = \frac{1}{2}.$$

Hence,

$$y_p = \frac{1}{2}e^{3t} = \frac{1}{2}x^3.$$

Hence, the general solution of the given equation is

$$y = c_1x + c_2x^2 + \frac{1}{2}x^3.$$

Example 8.21. *Solve the following differential equations:*

(1) $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$, (2) $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 4\log x$;

(3) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 4\sin\log x$, (4) $x^3\frac{d^3y}{dx^3} - 4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} - 8y = 4\log x$.

9 Series Solution of Linear Differential Equations

Already we have learned that certain types of higher order linear differential equations have solutions that can be expressed as finite linear combinations of known elementary functions. In general, however, higher order linear differential equations have no solutions that can be expressed in such a simple manner. Thus, we must seek other means of expression for the solutions of these equations. One such means of expression is furnished by infinite series representations, which is known as the infinite series solution.

9.1 Power Series Solutions about an Ordinary Point

Consider the second-order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0. \tag{9.1}$$

Suppose that (9.1) has no solution that is expressible as a finite linear combinations of known elementary functions. Let us assume that it does have a solution that can be expressed in the form of infinite series as follows:

$$c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \ldots = \sum_{n=0}^{\infty} c_n(x - x_0)^n, \tag{9.2}$$

where $c_0, c_1, \ldots$ are constants. An expression (9.2) is called a power series in $x - x_0$. Thus, we have assumed that (9.1) has a so-called power series solution of the form (9.2).

In the equivalent normalized form of the differential equation (9.1) is

$$\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0, \tag{9.3}$$

where

$$P_1(x) = \frac{a_1(x)}{a_0(x)}, \text{ and } P_2(x) = \frac{a_2(x)}{a_0(x)}.$$

Definition 9.1. *A function f is said to be analytic at x_0 if its Taylor series about x_0,*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

exists and converges to $f(x)$ for all x in some open interval including x_0.

For example, let

$$f(x) = \frac{1}{x^2 - 3x + 2}.$$

This function is analytic everywhere in real line except $x = 1$ and $x = 2$.

Definition 9.2. *The point x_0 is called an ordinary point of* (9.1) *if both of the functions P_1 and P_2 in* (9.3) *are analytic at x_0. If either (or both) of these functions is not analytic at x_0, then x_0 is called a singular point of* (9.1).

For examples, consider the differential equation

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x^2 + 2)y = 0.$$

Here $P_1(x) = -x$ and $P_2(x) = x^2+2$. Both of the functions P_1 and P_2 are polynomial functions and so they are analytic everywhere. Thus, all points are ordinary points of this equation.

Again, consider the differential equation

$$(x - 1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \frac{1}{x}y = 0.$$

This equation can be written as

$$\frac{d^2y}{dx^2} - \frac{x}{x - 1}\frac{dy}{dx} + \frac{1}{x(x - 1)}y = 0.$$

Here,

$$P_1(x) = -\frac{x}{x - 1}, \text{ and } P_2(x) = \frac{1}{x(x - 1)}.$$

The function P_1 is analytic except $x = 1$ and P_2 is analytic except $x = 0$ and $x = 1$. Thus, $x = 0$ and $x = 1$ are singular points of the given differential equation.

Theorem 9.1. *Suppose that x_0 is an ordinary point of the differential equation* (9.1)*. Then* (9.1) *has two nontrivial linearly independent power series solutions of the form*

$$\sum_{n=0}^{\infty} c_n(x - x_0)^n,$$

and these power series converges in some interval $|x - x_0| < R$ (where $R > 0$) about x_0.

Example 9.1. *Find the power series solution of the differential equation*

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0 \tag{9.4}$$

in powers of x (that is, about $x_0 = 0$).

Solution: We observe that $x_0 = 0$ is an ordinary point of (9.4). Let

$$y = \sum_{n=0}^{\infty} c_n x^n. \tag{9.5}$$

Differentiating term by term, we obtain

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n c_n x^{n-1} \text{ and}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Substituting these values in (9.4), we obtain

$$\begin{aligned}
& \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} + x^2 \sum_{n=0}^{\infty} c_n x^n \\
& +2\sum_{n=0}^{\infty} c_n x^n = 0 \\
\Rightarrow \quad & \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} \\
& +2\sum_{n=0}^{\infty} c_n x^n = 0 \\
\Rightarrow \quad & \sum_{n=0}^{\infty}(n+1)(n+2)c_{n+2} x^n + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n \\
& +2\sum_{n=0}^{\infty} c_n x^n = 0 \\
\Rightarrow \quad & \sum_{n=0}^{\infty} \Big((n+1)(n+2)c_{n+2} + 2c_n\Big) x^n + \sum_{n=1}^{\infty} n c_n x^n \\
& +\sum_{n=2}^{\infty} c_{n-2} x^n \\
\Rightarrow \quad & (2c_0 + 2c_2) + (3c_1 + 6c_3)x + \sum_{n=2}^{\infty}[(n+1)(n+2)c_{n+2} \\
& +(n+2)c_n + c_{n-2}]x^n = 0.
\end{aligned} \tag{9.6}$$

The coefficients of each power of x must be equated to zero. This leads to the conditions

$$2c_0 + 2c_2 = 0,\ 3c_1 + 6c_3 = 0 \text{ and} \tag{9.7}$$

$$(n+1)(n+2)c_{n+2} + (n+2)c_n + c_{n-2} = 0,\ n \geq 2. \tag{9.8}$$

The equations (9.7) (i.e. the lowest degree terms of x) are called indicial equations and (9.8) is called the recurrence formula.

Therefore, (9.7) gives that

$$c_2 = -c_0 \text{ and } c_3 = -\frac{1}{2}c_1.$$

Now, the equation (9.8) can be written as

$$c_{n+2} = -\frac{(n+2)c_n + c_{n-2}}{(n+1)(n+2)},\ n \geq 2. \tag{9.9}$$

For $n = 2, 3, 4, 5, \ldots$, (9.9) gives that

$$c_4 = -\frac{4c_2 + c_0}{12} = \frac{1}{4}c_0.$$

$$c_5 = -\frac{5c_3 + c_1}{20} = \frac{3}{40}c_1.$$

In the same way we may express each even coefficients in terms of c_0 and each odd coefficients in terms od c_1.

Substituting the values of $c_2, c_3, c_4, c_5, \ldots$ in (9.5), we obtain that

$$\begin{aligned} y &= c_0 + c_1x - c_0x^2 - \frac{1}{2}c_1x^3 + \frac{1}{4}c_0x^4 + \frac{3}{40}c_1x^5 + \ldots \\ &= c_0(1 - x^2 + \frac{1}{4}x^4 + \ldots) + c_1(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \ldots). \end{aligned}$$

The two series in the parenthesis in the above equation are the power series expansions of two linearly independent solutions of the given differential equation and the constants c_0 and c_1 are arbitrary constants. Hence, the above equation represents the general solution of the given differential equation in powers of x.

Example 9.2. *Solve the differential equations:*

(1) $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$, (2) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (2x^2+1)y = 0$;

(3) $(x^2-1)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + xy = 0, y(0) = 4, y'(0) = 6$;

(4) $(2x^2-3)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + y = 0, y(0) = -1, y'(0) = 5.$

9.2 Solution about singular point; The method of Frobenius

If x_0 is a singular point of (9.1), then we are not assured of a power solution (9.2) of (9.1) in powers of $x - x_0$. In this case we must seek a different type solution. It happens that under certain conditions we are justified in assuming a solution of the form

$$y = |x - x_0|^r \sum_{n=0}^{\infty} c_n (x - x_0)^n, \tag{9.10}$$

where r is a certain (real or complex) constant. Such a solution is clearly a power series in $x - x_0$ multiplied by a certain power of $|x - x_0|$.

Definition 9.3. *Consider the differential equation* (9.1) *and assume that at least one of the functions P_1 and P_2 in* (9.3) *is not analytic at x_0 so that x_0 is a singular point of* (9.1). *If the functions defined by the products*

$$(x - x_0)P_1(x) \text{ and } (x - x_0)^2 P_2(x) \tag{9.11}$$

are both analytic at x_0, then x_0 is called a regular singular point of the differential equation (9.1). *If either (or both) of the functions defined by* (9.11) *is not analytic at x_0, then x_0 is called an irregular singular point of* (9.1).

Example-1: Consider the differential equation

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0. \tag{9.12}$$

This can be written as follows

$$\frac{d^2y}{dx^2} - \frac{1}{2x}\frac{dy}{dx} + \frac{x-5}{2x^2}y = 0.$$

Here

$$P_1(x) = -\frac{1}{2x} \text{ and } P_2(x) = \frac{x-5}{2x^2}.$$

Since both P_1 and P_2 fail to be analytic at $x = 0$, we conclude that $x = 0$ is a singular point of (9.12). Now,

$$xP_1(x) = -\frac{1}{2} \text{ and } x^2P_2(x) = \frac{x-5}{2}.$$

Both of these product functions are analytic at $x = 0$ and so $x = 0$ is a regular singular point of (9.12).

Example-2: Consider the differential equation

$$x^2(x-2)^2\frac{d^2y}{dx^2} + 2(x-2)\frac{dy}{dx} + (x+1)y = 0. \tag{9.13}$$

This can be written as follows

$$\frac{d^2y}{dx^2} + \frac{2}{x^2(x-2)}\frac{dy}{dx} + \frac{x+1}{x^2(x-2)^2}y = 0.$$

Here

$$P_1(x) = \frac{2}{x^2(x-2)} \text{ and } P_2(x) = \frac{x+1}{x^2(x-2)^2}.$$

Since both P_1 and P_2 fail to be analytic at $x = 0$, we conclude that $x = 0$ is a singular point of (9.13). Now,

$$xP_1(x) = \frac{2}{x(x-2)} \text{ and } x^2P_2(x) = \frac{x+1}{(x-2)^2}.$$

The product function defined by $x^2P_2(x)$ is analytic at $x = 0$ and but that defined by $xP_1(x)$ is not. Thus, $x = 0$ is an irregular point of (9.13).

On the other hand, consider $x = 2$. Then

$$(x-2)P_1(x) = \frac{2}{x^2} \text{ and } (x-2)^2P_2(x) = \frac{x+1}{x^2}.$$

Both of these product functions defined are analytic at $x = 2$ and hence $x = 2$ is a regular singular point of (9.13).

Theorem 9.2. *Suppose that x_0 is a regular singular point of* (9.1). *Then* (9.1) *has at least one nontrivial solution of the form*

$$y = |x-x_0|^r\sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

where r is a definite (real or complex) constant which may determined.

This solution is commonly called the method of Frobenius.

9.2.1 When $x = 0$ is aregular singular point

In this case we assume that the trial solution is

$$y = x^r \sum_{n=0}^{\infty} c_n x^n,$$

where all $c's$ are constants. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ and put their values in the given equation.

The index r will be determined by the quadratic equation which will be obtained by equating to zero the coefficients of the lowest power of x. This equation in r is called the Indicial equation.

The values of $c_1, c_2, \ldots$ are all determined in terms of c_0 by equating to zero the coefficients of other various powers of x.

Depending upon the nature of the roots of the indicial equation , there arise the following cases:

(i) The roots of indicial equation are unequal and not differing by an integer;

(ii) The roots of indicial equation are equal;

(iii) The roots of indicial equation are unequal and differing by integer.

We will discuss these cases one by one by taking examples of each other.

Case-I. Roots of indicial equation are unequal and not differing by an integer

Let α and β be two roots of the indicial equation. If α and β do not differ by integer, then two independent solutions are obtained by putting $r = \alpha$ and β in the series. Let u and v be these two solutions, the general solution is $y = cu + c'v$, where c and c' are arbitrary constants.

Example 9.3. *Use the method of Frobenius to find solutions of the differential equation*

$$2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0. \tag{9.14}$$

Solution: We observe that $x = 0$ is a regular singular point of (9.14). Let

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \text{ where } c_0 \neq 0. \tag{9.15}$$

Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}.$$

Substituting these values into (9.14), we find

$$\begin{aligned}
& 2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+1} \\
& -5\sum_{n=0}^{\infty} c_n x^{n+r} = 0 \\
\Rightarrow\ & \sum_{n=0}^{\infty}[2(n+r)(n+r-1) - (n+r) - 5]c_n x^{n+r} + \sum_{n=1}^{\infty} c_{n-1}x^{n+r} = 0 \\
\Rightarrow\ & [2r(r-1) - r - 5]c_0 x^r \\
& + \sum_{n=0}^{\infty}\Big([2(n+r)(n+r-1) - (n+r) - 5]c_n + c_{n-1}\Big)x^{n+r} = 0.
\end{aligned} \tag{9.16}$$

Equating to zero the coefficients of the lowest power of x (i.e. the coefficient of x^r), which leads to a quadratic equation (is called the indicial equation)

$$2r(r-1)-r-5=0 \Rightarrow 2r^2-3r-5=0 \Rightarrow r=-1, \frac{5}{2}.$$

Equating to zero the coefficients of the higher powers of x in (9.16), we obtain the recurrence formula

$$[2(n+r)(n+r-1)-(n+r)-5]c_n + c_{n-1} = 0,\ n \geq 1. \tag{9.17}$$

Letting $r = r_1 = \dfrac{5}{2}$ in (9.17), it reduces to

$$\begin{aligned} & n(2n+7)c_n + c_{n-1} = 0,\ n \geq 1 \\ \Rightarrow\quad & c_n = -\frac{c_{n-1}}{n(2n+7)},\ n \geq 1. \end{aligned} \tag{9.18}$$

Putting $n = 1, 2, 3, \ldots$, we obtain

$$c_1 = -\frac{c_0}{9}, c_2 = -\frac{c_1}{22} = \frac{c_0}{198}, c_3 = -\frac{c_2}{39} = -\frac{c_0}{7722}, \ldots$$

using these values with $r = \dfrac{5}{2}$, we obtain from (9.15) that

$$\begin{aligned} y &= c_0(x^{\frac{5}{2}} - \frac{1}{9}x^{\frac{7}{2}} + \frac{1}{198}x^{\frac{9}{2}} - \frac{1}{7722}x^{\frac{11}{2}} + \ldots) \\ &= c_0 x^{\frac{5}{2}}\Big(1 - \frac{1}{9}x + \frac{1}{198}x^2 - \frac{1}{7722}x^3 + \ldots\Big) = cu \text{ (say)}. \end{aligned} \tag{9.19}$$

Now, let $r = r_2 = -1$ in (9.17) to obtain the recurrence formula

$$[2(n-1)(n-2)-(n-1)-5]c_n + c_{n-1} = 0,\ n \geq 1.$$

Simplifying this equation, we obtain that

$$n(2n-7)c_n + c_{n-1} = 0 \Rightarrow c_n = -\frac{c_{n-1}}{n(2n-7)},\ n \geq 1.$$

Putting $n = 1, 2, 3, \ldots$ in the above equation, we find that

$$c_1 = \frac{1}{5}c_0, c_2 = \frac{1}{6}c_1 = \frac{1}{30}c_0, c_3 = \frac{1}{3}c_2 = \frac{1}{90}c_0, \ldots.$$

Using these values together with $r = -1$, we obtain from (9.15) that

$$\begin{aligned} y &= c_0(x^{-1} + \frac{1}{5} + \frac{1}{30}x + \frac{1}{90}x^2 - \ldots) \\ &= c_0 x^{-1}\Big(1 + \frac{1}{3}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 - \ldots\Big) = c'v \text{ (say)}. \end{aligned} \tag{9.20}$$

The general solution is $y = cu + c'v$.

Example 9.4. *Solve in series of the following equations:*

(1) $(2x + x^3)\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6xy = 0$;

(2) $2x\dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} + 3y = 0$;

(3) $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (x^2+1)y = 0$.

Case-II. Roots of indicial equation are equal

Let $r = \alpha$ be the repeated (equal) root of the indicial equation, then two independent solutions are obtained by putting $r = \alpha$ in y and $\dfrac{\partial y}{\partial r}$.

It will be seen that the second solution always consists of the product or a numerical multiple of the first solution and $\log x$ plus a series.

Example 9.5. *Use the method of Frobenius to obtain a general solution in series of power of x of the differential equation*

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0. \tag{9.21}$$

Solution: We observe that $x = 0$ is a regular singular point of (9.21). Let

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \text{ where } c_0 \neq 0. \tag{9.22}$$

Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}.$$

Substituting these values into (9.21), we find

$$\begin{aligned}
& \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0 \\
\Rightarrow\ & \sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)]c_n x^{n+r-1} + \sum_{n=2}^{\infty} c_{n-2}x^{n+r-1} = 0 \\
\Rightarrow\ & \sum_{n=0}^{\infty}(n+r)^2 c_n x^{n+r-1} + \sum_{n=2}^{\infty} c_{n-2}x^{n+r-1} = 0 \\
\Rightarrow\ & c_0 r^2 x^{r-1} + c_1(r+1)^2 x^r + \sum_{n=2}^{\infty}\Big((n+r)^2 c_n + c_{n-2}\Big)x^{n+r-1} = 0. \qquad (9.23)
\end{aligned}$$

The indicial equation is

$$r^2 = 0 \Rightarrow r = 0 \text{ as } c_0 \neq 0.$$

Other indicial equation is

$$c_1(r+1)^2 = 0 \Rightarrow c_1 = 0.$$

Equating to zero the coefficients of the higher powers of x in (9.23), we obtain the recurrence formula

$$\begin{aligned}
& (n+r)^2 c_n + c_{n-2} = 0,\ n \geq 2 \\
\Rightarrow\ & c_n = -\frac{c_{n-2}}{(n+r)^2},\ n \geq 2. \qquad (9.24)
\end{aligned}$$

Putting $n = 2, 3, 4, \ldots$, we obtain

$$c_2 = -\frac{1}{(r+2)^2}c_0, c_3 = -\frac{1}{(r+3)^2}c_1 = 0, c_4 = -\frac{1}{(r+4)^2}c_2 = \frac{c_0}{(r+4)^2(r+2)^2}c_0, \ldots$$

Using these values, we obtain from (9.15) that

$$y = c_0 x^r\Big(1 - \frac{1}{(r+2)^2}x^2 + \frac{1}{(r+4)^2(r+2)^2}x^4 - \ldots\Big). \tag{9.25}$$

Differentiating (9.25) with respect to r, we get

$$\begin{aligned}
\frac{\partial y}{\partial r} &= c_0 x^r \log x\Big(1 - \frac{1}{(r+2)^2}x^2 + \frac{1}{(r+4)^2(r+2)^2}x^4 - \ldots\Big) \\
&\quad + c_0 x^r\Big\{\frac{2}{(r+2)^3}x^2 - \Big(\frac{2}{(r+2)^3(r+4)^2} + \frac{2}{(r+2)^2(r+4)^3}\Big)x^4 + \ldots\Big\} \\
&= y\log x + c_0 x^r\Big\{\frac{2}{(r+2)^3}x^2 - \Big(\frac{2}{(r+2)^3(r+4)^2} + \frac{2}{(r+2)^2(r+4)^3}\Big)x^4 + \ldots\Big\} \qquad (9.26)
\end{aligned}$$

Now putting $r = 0$ and $c_0 = c$ and c' in (9.25) and (9.26) respectively, the two independent solutions are

$$y = c\Big(1 - \frac{1}{2^2}x^2 + \frac{1}{2^2 \cdot 4^2}x^4 - \dots\Big) = cu \text{ (say)},$$

$$\frac{\partial y}{\partial r}\Big|_{r=0} = c'u \log x + c'\Big\{\frac{1}{2^2}x^2 - \frac{1}{2^2 \cdot 4^2}(1 + \frac{1}{2})x^4 + \dots\Big\} = c'v \text{ (say)}.$$

Hence the complete solution is $y = cu + c'v$.

Example 9.6. *Integrate in series the equations*

(1) $(x - x^2)\dfrac{d^2y}{dx^2} + (1 - 5x)\dfrac{dy}{dx} - 4y = 0$;

(2) $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + x^2y = 0$;

(3) $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0.$

Case-III. Roots of indicial equation are unequal and differing by an integer

This case can further be sub-divided into two subcases.

(Case-III.A) If one of the roots makes y infinite;

(Case-III.B) If one of the roots makes y indeterminate.

We shall discuss these two cases separately.

Case-III.A: The indicial equation has two roots α and β ($\alpha > \beta$) differing by an integer and some of the coefficients of x become infinite for $r = \beta$.

In this case put $c(r - \beta)$ for c_0. This would lead to two independent solutions for $r = \beta$, namely, modified y and $\dfrac{\partial y}{\partial r}$ as in **Case-II**.

Thus, we find the three solutions

(i) The solution by putting $r = \alpha$ in y;

(ii) The solution by putting $r = \beta$ in modified y;

(iii) The solution by putting $r = \beta$ in $\dfrac{\partial y}{\partial r}$.

But only two of these are independent as solution (i) is numerical multiple of (ii).

Example 9.7. *Use the method of Frobenius to find solutions of the differential equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 1)y = 0. \tag{9.27}$$

Solution: We observe that $x = 0$ is a regular singular point of (9.27). Let

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}, \text{ where } c_0 \neq 0. \tag{9.28}$$

Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n + r)c_n x^{n+r-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n + r)(n + r - 1)c_n x^{n+r-2}.$$

Substituting these values into (9.27), we find

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} \\
&-\sum_{n=0}^{\infty} c_n x^{n+r} = 0 \\
\Rightarrow\ &\sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)-1]c_n x^{n+r} + \sum_{n=2}^{\infty} c_{n-2}x^{n+r} = 0 \\
\Rightarrow\ &\sum_{n=0}^{\infty}[(n+r)^2-1]c_n x^{n+r} + \sum_{n=2}^{\infty} c_{n-2}x^{n+r} = 0 \\
\Rightarrow\ &c_0(r^2-1)x^r + c_1[(r+1)^2-1]x^{r+1} + \sum_{n=2}^{\infty}\Big([(n+r)^2-1]c_n + c_{n-2}\Big)x^{n+r} = 0. \quad (9.29)
\end{aligned}$$

The indicial equations are

$$r^2 - 1 = 0 \Rightarrow r = \pm 1 \text{ and } c_1[(r+1)^2-1] = 0 \Rightarrow c_1 = 0.$$

The recurrence formula is

$$[(n+r)^2-1]c_n + c_{n-2} = 0,\ n \geq 2 \Rightarrow c_n = -\frac{c_{n-2}}{(n+r)^2-1},\ n \geq 2. \quad (9.30)$$

Putting $n = 2, 3, 4, \ldots$ into (9.30), we obtain that

$$\begin{aligned}
c_2 &= -\frac{1}{(r+2)^2-1}c_0,\ c_3 = -\frac{1}{(r+3)^2-1}c_1 = 0, \\
c_4 &= -\frac{1}{(r+4)^2-1}c_2 = \frac{1}{\big((r+4)^2-1\big)\big((r+2)^2-1\big)}c_0, \\
\cdots &\quad \cdots\cdots\cdots \quad \cdots\cdots
\end{aligned}$$

This gives

$$y = c_0 x^r\Big[1 - \frac{1}{(r+1)(r+3)}x^2 + \frac{1}{(r+1)(r+3)^2(r+5)}x^4 - \ldots\Big] \quad (9.31)$$

If we put $r = -1$ in (9.31), then the coefficients become infinite owing to the factor $(r+1)$ in the denominator. So, we replace c_0 by $c(r+1)$ in (9.31). Thus,

$$y = cx^r\Big[(r+1) - \frac{1}{(r+3)}x^2 + \frac{1}{(r+3)^2(r+5)} - \ldots\Big]. \quad (9.32)$$

Differentiating (9.32) with respect to r partially, we obtain that

$$\begin{aligned}
\frac{\partial y}{\partial r} &= cx^r \log x\Big[(r+1) - \frac{1}{(r+3)}x^2 + \frac{1}{(r+3)^2(r+5)} - \ldots\Big] \\
&\quad + cx^r\Big[1 + \frac{1}{(r+3)^2}x^2 - \Big\{\frac{2}{(r+3)^3(r+5)} + \frac{1}{(r+3)^2(r+5)^2}\Big\} + \ldots\Big] \\
&= y\log x + cx^r\Big[1 + \frac{1}{(r+3)^2}x^2 - \Big\{\frac{2}{(r+3)^3(r+5)} \\
&\quad + \frac{1}{(r+3)^2(r+5)^2}\Big\}x^4 + \ldots\Big]. \quad (9.33)
\end{aligned}$$

Put $r = -1$ in (9.32) and (9.33), we get

$$cx^{-1}\Big[-\frac{1}{4}x^2 + \frac{1}{2^2\cdot 4}x^4 + \ldots\Big] = cu \text{ (say)},$$

and

$$cu\log x + cx^{-1}\Big[1 + \frac{1}{2^2}x^2 - \frac{1}{2^2\cdot 4}(1+\frac{1}{4})x^4 - \ldots\Big] = c'v \text{ (say)}.$$

Hence, the general solution is

$$y = cu + c'v.$$

Case-III.B: The indicial equation has two roots α and β ($\alpha > \beta$) differing by an integer making a coefficient of x indeterminate.

Let α and β ($\alpha > \beta$) be roots of the indicial equation differing by an integer. If one of the coefficient of x in y becomes indeterminate when $r = \beta$, the complete primitive is given by putting $r = \beta$ in y which then contains two arbitrary constants.

The result on putting $r = \alpha$ in y simply gives a numerical multiple of the series contained in the first solution.

Example 9.8. *Integrate the series*

$$(1-x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0. \tag{9.34}$$

Solution: Let

$$y = \sum_{n=0}^{\infty} c_n x^{n+r} \tag{9.35}$$

be the Frobenius type solution of (9.34). Then

$$\frac{dy}{dx} = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}.$$

Substituting these values into (9.34), we find

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} - \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + 2\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}$$
$$+\sum_{n=0}^{\infty} c_n x^{n+r}$$
$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} - \sum_{n=0}^{\infty}\Big[(n+r)(n+r-1) - 2(n+r) - 1\Big]c_n x^{n+r} = 0$$
$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} - \sum_{n=0}^{\infty}\Big[(n+r)(n+r-3) - 1\Big]c_n x^{n+r} = 0$$
$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} - \sum_{n=2}^{\infty}\Big[(n+r-2)(n+r-5) - 1\Big]c_{n-2} x^{n+r-2}$$
$$\Rightarrow c_0 r(r-1)x^{r-2} + c_1 r(r+1)x^{r-1} + \sum_{n=2}^{\infty}\Big((n+r)(n+r-1)c_n$$
$$-[(n+r-2)(n+r-5)-1]c_{n-2}\Big)x^{n+r-2} = 0. \tag{9.36}$$

The indicial equations are

$$c_0 r(r-1) = 0 \Rightarrow r = 0, 1 \text{ as } (c_0 \neq 0),$$

and

$$c_1 r(r+1) = 0,$$

which makes c_1 indeterminate when $r = 0$. The recurrence formula is

$$(n+r)(n+r-1)c_n - [(n+r-2)(n+r-5)-1]c_{n-2} = 0,\ n \geq 2$$
$$\Rightarrow \quad c_n = \frac{(n+r-2)(n+r-5)-1}{(n+r)(n+r-1)}c_{n-2},\ n \geq 2. \tag{9.37}$$

Put $n = 2, 3, 4, \ldots$ in (9.37), we get

$$c_2 = \frac{r(r-3)-1}{(r+2)(r+1)}c_0,\ c_3 = \frac{(r+1)(r-2)-1}{(r+3)(r+2)}c_1, \ldots.$$

Substituting these values with $r = 0$ in (9.35), we obtain that

$$[y]_{r=0} \quad = \quad c_0[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \ldots] + c_1(1 - \frac{1}{2}x^3 + \frac{1}{40}x^5 - \ldots],$$

which is general solution for containing two arbitrary constants c_0 and c_1.

Example 9.9. *Integrate the following series:*

(1) $(1 - x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$;

(2) $(2 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (1 + x)y = 0$;

(3) $\dfrac{d^2y}{dx^2} + x^2y = 0$;

(4) $x^2\dfrac{d^2y}{dx^2} + (x^2 - 3x)\dfrac{dy}{dx} + 3y = 0$.

10 Laplace Transform

Definition 10.1. *Let $F(t)$ be a function of t defined for all positive values of t. Then the Laplace transform of $F(t)$ denoted by $L\{F(t)\}$ or $f(s)$ is defined by the expression*

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st}F(t)dt,$$

where s is a parameter (real or complex).

Problem 10.1. *Find the Laplace transform of the following functions:*

(i) $F(t) = 1$, (ii) $F(t) = t$, (iii) $F(t) = t^n, n = 0, 1, 2, 3, \ldots$;

(iv) $F(t) = e^{at}$, (v) $F(t) = \sin at$, (vi) $F(t) = \cos at$;

(vii) $F(t) = t\sin at$, (viii) $F(t) = t\cos at$.

Solution: By the definition of Laplace transform, we have that

$$L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt. \tag{10.1}$$

(i) When $F(t) = 1$, (10.1) becomes

$$L\{1\} = \int_0^\infty e^{-st} \cdot 1dt = \Big[\frac{e^{-st}}{-s}\Big]_0^\infty = \frac{1}{s},\ s > 0.$$

(ii) When $F(t) = t$, then (10.1) gives

$$\begin{aligned} L\{t\} &= \int_0^\infty e^{-st} \cdot tdt \\ &= \Big[\frac{e^{-st}}{-s} \cdot t\Big]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}dt, \text{ (integrating by parts)} \\ &= 0 + \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s}\Big[\frac{e^{-st}}{-s}\Big]_0^\infty \\ &= \frac{1}{s^2},\ s > 0. \end{aligned}$$

(iii) When $F(t) = t^n$, the transformation (10.1) reduces to

$$\begin{aligned}
L\{t^n\} &= \int_0^\infty e^{-st} \cdot t^n dt \\
&= \left[\frac{e^{-st}}{-s} \cdot t^n\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} \cdot t^{n-1} dt, \text{ (integrating by parts)} \\
&= 0 + \frac{n}{s}\left[\frac{e^{-st}}{-s} \cdot t^{n-1}\right]_0^\infty + \frac{n(n-1)}{s^2}\int_0^\infty e^{-st} \cdot t^{n-2} dt, \text{ (integrating by parts)} \\
&= \frac{n(n-1)(n-2)}{s^3}\int_0^\infty e^{-st} \cdot t^{n-3} dt \\
&\quad \text{repating the process of integration by parts} \\
&= \dots\dots\dots\dots\dots \\
&= \frac{n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1}{s^n}\int_0^\infty e^{-st} dt \\
&= \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}, \ s > 0.
\end{aligned}$$

(iv) When $F(t) = e^{at}$, (10.1) gives that

$$\begin{aligned}
L\{e^{at}\} &= \int_0^\infty e^{-st} \cdot e^{at} dt \\
&= \int_0^\infty e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty \\
&= \frac{1}{s-a}, \ s > a.
\end{aligned}$$

(v) When $F(t) = \sin at$, then (10.1) gives that

$$\begin{aligned}
L\{\sin at\} &= \int_0^\infty e^{-st} \cdot \sin at dt \\
&= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty \\
&\quad \left[\because \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right] \\
&= \frac{a}{s^2+a^2}, \ s > 0.
\end{aligned}$$

(vi) When $F(t) = \cos at$, then (10.1) gives that

$$\begin{aligned}
L\{\cos at\} &= \int_0^\infty e^{-st} \cdot \cos at dt \\
&= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty \\
&\quad \left[\because \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right] \\
&= \frac{s}{s^2+a^2}, \ s > 0.
\end{aligned}$$

(vii) When $F(t) = t\sin at$, then (10.1) gives that

$$\begin{aligned}
L\{t\sin at\} &= \int_0^\infty e^{-st}\cdot t\sin at dt \\
&= \left[t\cdot \frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty \\
&\quad - \int_0^\infty \frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)dt \\
&\quad \Big[\text{integrating by parts by treating } t \text{ as first function and } e^{-st}\sin at \\
&\quad \text{as the second function and using the result} \\
&\quad \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\Big] \\
&= \frac{s}{s^2+a^2}\int_0^\infty e^{-st}\sin at\ dt + \frac{a}{s^2+a^2}\int_0^\infty e^{-st}\cos at\ dt \\
&= \frac{s}{s^2+a^2}\cdot\frac{a}{s^2+a^2} + \frac{a}{s^2+a^2}\cdot\frac{s}{s^2+a^2} \\
&= \frac{2as}{(s^2+a^2)^2},\ s>0.
\end{aligned}$$

(vii) When $F(t) = t\cos at$, then (10.1) gives that

$$\begin{aligned}
L\{t\cos at\} &= \int_0^\infty e^{-st}\cdot t\cos at dt \\
&= \left[t\cdot \frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty \\
&\quad - \int_0^\infty \frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)dt \\
&\quad \Big[\text{integrating by parts by treating } t \text{ as first function and } e^{-st}\cos at \\
&\quad \text{as the second function and using the result} \\
&\quad \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\Big] \\
&= \frac{s}{s^2+a^2}\int_0^\infty e^{-st}\cos at\ dt - \frac{a}{s^2+a^2}\int_0^\infty e^{-st}\sin at\ dt \\
&= \frac{s}{s^2+a^2}\cdot\frac{s}{s^2+a^2} - \frac{a}{s^2+a^2}\cdot\frac{a}{s^2+a^2} \\
&= \frac{s^2-a^2}{(s^2+a^2)^2},\ s>0.
\end{aligned}$$

Problem 10.2. *Find the Laplace transform of* $4e^{5t} + 6t^3 - 4\cos 3t + 3\ \sin 4t$.

Solution:

$$\begin{aligned}
L\{4e^{5t} &+ 6t^3 - 4\cos 3t + 3\ \sin 4t\} = 4L\{e^{5t}\} + 6L\{t^3\} - 4L\{\cos 3t\} + 3L\{\sin 4t\} \\
&= 4\Big(\frac{1}{s-5}\Big) + 6\Big(\frac{3!}{s^4}\Big) - 4\Big(\frac{s}{s^2+9}\Big) + 3\Big(\frac{4}{s^2+16}\Big) \\
&= \frac{4}{s-5} + \frac{36}{s^4} - \frac{4s}{s^2+9} + \frac{12}{s^2+16}.
\end{aligned}$$

Problem 10.3. *Find the Laplace Transform of* $e^{-2t}\sin 3t$.

Solution:

$$\begin{aligned}
L\{e^{-2t}\sin 3t\} &= \int_0^\infty e^{-st}\cdot e^{-2t}\sin 3t\ dt \\
&= \int_0^\infty e^{-(s+2)}t\sin 3t\ dt = \frac{3}{(s+2)^2+9} \\
&= \frac{3}{s+4s+13}.
\end{aligned}$$

10.1 Laplace Equation

Definition 10.2. *For steady-state heat flow, let u be independent of time i.e.* $\dfrac{\partial u}{\partial t} = 0$. *Then the equation of the form*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \tag{10.2}$$

is known as the three-dimensional cartesian Laplace's equation.

In the same way, two-dimensional cartesian Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and one-dimensional cartesian Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2} = 0.$$

The above Laplace's equation can be solved by the method of separation of variables.

10.1.1 Method of Separation of Variables or Product Method

Let us consider the following linear partial differential equation to illustrate the method of separation of variables for finding solutions of it:

$$a_0 \frac{\partial^2 u}{\partial x^2} + a_1 \frac{\partial^2 u}{\partial x \partial y} + a_2 \frac{\partial^2 u}{\partial y^2} + a_3 \frac{\partial u}{\partial x} + a_4 \frac{\partial u}{\partial y} + a_5 u = f(x, y), \tag{10.3}$$

where $a_0, a_1, \ldots, f$ are functions of x and y.

Let us assume that

$$u(x, y) = \Phi(x)\Psi(y), \tag{10.4}$$

be the trial solution of (10.3), where $\Phi(x)$ and $\Psi(y)$ are respectively functions of x and y alone.

Now, from (10.4) we get the partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2} \text{ in terms of } \Phi \text{ and } \Psi.$$

Substituting these values of partial derivatives in homogeneous part of (10.3), we obtain that

$$\begin{aligned} & a_0 \Phi''(x)\Psi(y) + a_1 \Phi'(x)\Psi'(y) + a_2 \Phi(x)\Psi''(y) + a_3 \Phi'(x)\Psi(y) \\ & + a_4 \Phi(x)\Psi'(y) + a_5 \Phi(x)\Psi(y) = 0 \\ \Rightarrow \quad & a_0 \frac{\Phi''(x)}{\Phi(x)} + a_1 \frac{\Phi'(x)}{\Phi(x)} \frac{\Psi'(y)}{\Psi(y)} + a_2 \frac{\Psi''(y)}{\Psi(y)} + a_3 \frac{\Phi'(x)}{\Phi(x)} + a_4 \frac{\Psi'(y)}{\Psi(y)} + a_5 = 0. \end{aligned}$$

It is possible to write in the form

$$\frac{\Phi''(x)}{\Phi(x)} = \frac{\Psi''(y)}{\Psi(y)}, \tag{10.5}$$

which is separable in the variables x and y.

Thus, we can find a relation between the two sides of (10.5), in which L.H.S is a function of x alone and R.H.S is a function of y alone. The two sides of equation (10.5) can be equal only if each is equal to a constant λ (say). Thus, we have

$$\frac{\Phi''(x)}{\Phi(x)} = \frac{\Psi''(y)}{\Psi(y)} = \lambda.$$

Therefore, the problem of finding solution of the form (10.4) of the partial differential equation (10.3) reduces to solving a pair of linear ordinary differential equations at least one of them being of order two which are given below:

$$\Phi''(x) = \lambda \Phi(x) \text{ and } \Psi''(y) = \lambda \Psi(y). \tag{10.6}$$

After having the solutions of the both equations of (10.6), the required solution is obtained by using (10.4).

Problem 10.4. *Solve the Laplace's equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

under the conditions $u(0, y) = 0$, $u(l, y) = 0$, $u(x, 0) = F(x)$ *and* $u(x, \infty) = 0$.

Solution: In order to apply the the method of separation of variables, assume that

$$u(x, y) = \Phi(x)\Psi(y), \tag{10.7}$$

so that

$$\frac{\partial^2 u}{\partial x^2} = \Psi(y)\Phi''(x) \text{ and } \frac{\partial^2 u}{\partial y^2} = \Phi(x)\Psi''(y).$$

Then from the given equation, we obtain that

$$\Psi(y)\Phi''(x) + \Phi(x)\Psi''(y) = 0 \Rightarrow \frac{\Phi''(x)}{\Phi(x)} + \frac{\Psi''(y)}{\Psi(y)} = 0$$

$$\Rightarrow \quad \frac{\Phi''(x)}{\Phi(x)} = -\frac{\Psi''(y)}{\Psi(y)} = -\lambda^2 \text{ (say)}.$$

This gives that

$$\Phi''(x) + \lambda^2\Phi(x) = 0 \text{ and } \Psi''(y) - \lambda^2\Psi(y) = 0.$$

Solving these equations, we obtain the complementary functions

$$\Phi(x) = A\cos\lambda x + B\sin\lambda x \text{ and } \Psi(y) == Ce^{\lambda y} + De^{-\lambda y}.$$

Hence, the general solution is

$$u(x, y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}). \tag{10.8}$$

When $u(x, \infty) = 0$, we obtain from (10.8) that $C = 0$. Again, when $u(0, y) = 0$, we obtain from (10.8) that $A = 0$. Moreover, when $u(x, 0) = F(x)$, (10.8) gives that $F(x) = BD\sin\lambda x \Rightarrow D = 1$, since $F(x) = \sin\lambda x$. Hence, (10.8) takes the form

$$u(x, y) = B\sin\lambda x \cdot e^{-\lambda y}. \tag{10.9}$$

Again, condition $u(l, y) = 0$ yields that $\sin\lambda l = 0 = \sin n\pi \Rightarrow \lambda = \dfrac{n\pi}{l}$. Hence, for all distinct n, the general solution of the given equation is

$$u(x, y) = \sum_{n=0}^{\infty} B_n \sin\frac{n\pi}{l} e^{-\frac{n\pi y}{l}}. \tag{10.10}$$

In the similar way we can solve the diffusion equations

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c}\frac{\partial u}{\partial t}, \text{ One-dimensional heat (diffusion) equation.}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c}\frac{\partial u}{\partial t}, \text{ Two-dimensional heat (diffusion) equation.}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c}\frac{\partial u}{\partial t}, \text{ Three-dimensional heat (diffusion) equation.}$$

$$\text{Helmhotz Equation, } \nabla^2 A + c^2 A = 0, \text{ where } c = \text{wave number}, A = \text{ amplitude}.$$