Limits

Definitions

Precise Definition : We say $\lim\limits_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

"Working" Definition : We say $\lim\limits_{x \to a} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x sufficiently close to a (on either side of a) without letting $x = a$.

Right hand limit : $\lim\limits_{x \to a^+} f(x) = L$. This has the same definition as the limit except it requires $x > a$.

Left hand limit : $\lim\limits_{x \to a^-} f(x) = L$. This has the same definition as the limit except it requires $x < a$.

Limit at Infinity : We say $\lim\limits_{x \to \infty} f(x) = L$ if we can make $f(x)$ as close to L as we want by taking x large enough and positive.

There is a similar definition for $\lim\limits_{x \to -\infty} f(x) = L$ except we require x large and negative.

Infinite Limit : We say $\lim\limits_{x \to a} f(x) = \infty$ if we can make $f(x)$ arbitrarily large (and positive) by taking x sufficiently close to a (on either side of a) without letting $x = a$.

There is a similar definition for $\lim\limits_{x \to a} f(x) = -\infty$ except we make $f(x)$ arbitrarily large and negative.

Relationship between the limit and one-sided limits

$$\lim_{x \to a} f(x) = L \quad \Rightarrow \quad \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \qquad\qquad \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L \quad \Rightarrow \quad \lim_{x \to a} f(x) = L$$

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x) \quad \Rightarrow \quad \lim_{x \to a} f(x) \text{Does Not Exist}$$

Properties

Assume $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ both exist and c is any number then,

1. $\lim\limits_{x \to a} [c f(x)] = c \lim\limits_{x \to a} f(x)$
2. $\lim\limits_{x \to a} [f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$
3. $\lim\limits_{x \to a} [f(x) g(x)] = \lim\limits_{x \to a} f(x) \; \lim\limits_{x \to a} g(x)$
4. $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ provided $\lim\limits_{x \to a} g(x) \neq 0$
5. $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$
6. $\lim\limits_{x \to a} \left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim\limits_{x \to a} f(x)}$

Basic Limit Evaluations at $\pm\infty$

1. $\lim\limits_{x \to \infty} \mathbf{e}^x = \infty$ & $\lim\limits_{x \to -\infty} \mathbf{e}^x = 0$
2. $\lim\limits_{x \to \infty} \ln(x) = \infty$ & $\lim\limits_{x \to 0^+} \ln(x) = -\infty$
3. If $r > 0$ then $\lim\limits_{x \to \infty} \dfrac{b}{x^r} = 0$
4. If $r > 0$ and x^r is real for negative x then $\lim\limits_{x \to -\infty} \dfrac{b}{x^r} = 0$
5. n even : $\lim\limits_{x \to \pm\infty} x^n = \infty$
6. n odd : $\lim\limits_{x \to \infty} x^n = \infty$ & $\lim\limits_{x \to -\infty} x^n = -\infty$
7. n even : $\lim\limits_{x \to \pm\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
8. n odd : $\lim\limits_{x \to \infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
9. n odd : $\lim\limits_{x \to -\infty} a\,x^n + \cdots + c\,x + d = -\operatorname{sgn}(a)\infty$

Note : $\operatorname{sgn}(a) = 1$ if $a > 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$.

Evaluation Techniques

Continuous Functions

If $f(x)$is continuous at a then $\lim_{x\to a} f(x) = f(a)$

Continuous Functions and Composition

$f(x)$ is continuous at b and $\lim_{x\to a} g(x) = b$ then

$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b)$

Factor and Cancel

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} = \lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}$$
$$= \lim_{x\to 2}\frac{x+6}{x} = \frac{8}{2} = 4$$

Rationalize Numerator/Denominator

$$\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81} = \lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}\,\frac{3+\sqrt{x}}{3+\sqrt{x}}$$
$$= \lim_{x\to 9}\frac{9-x}{(x^2-81)(3+\sqrt{x})} = \lim_{x\to 9}\frac{-1}{(x+9)(3+\sqrt{x})}$$
$$= \frac{-1}{(18)(6)} = -\frac{1}{108}$$

Combine Rational Expressions

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right)$$
$$= \lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{x(x+h)}\right) = \lim_{h\to 0}\frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

L'Hospital's/L'Hôpital's Rule

If $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$ then,

$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, a is a number, ∞ or $-\infty$

Polynomials at Infinity

$p(x)$ and $q(x)$ are polynomials. To compute $\lim_{x\to\pm\infty}\frac{p(x)}{q(x)}$ factor largest power of x in $q(x)$ out of both $p(x)$ and $q(x)$ then compute limit.

$$\lim_{x\to-\infty}\frac{3x^2-4}{5x-2x^2} = \lim_{x\to-\infty}\frac{x^2\left(3-\frac{4}{x^2}\right)}{x^2\left(\frac{5}{x}-2\right)}$$
$$= \lim_{x\to-\infty}\frac{3-\frac{4}{x^2}}{\frac{5}{x}-2} = -\frac{3}{2}$$

Piecewise Function

$\lim_{x\to-2} g(x)$ where $g(x) = \begin{cases} x^2+5 & \text{if } x<-2 \\ 1-3x & \text{if } x\geq -2 \end{cases}$

Compute two one sided limits,

$$\lim_{x\to-2^-} g(x) = \lim_{x\to-2^-} x^2+5 = 9$$
$$\lim_{x\to-2^+} g(x) = \lim_{x\to-2^+} 1-3x = 7$$

One sided limits are different so $\lim_{x\to-2} g(x)$ doesn't exist. If the two one sided limits had been equal then $\lim_{x\to-2} g(x)$ would have existed and had the same value.

Some Continuous Functions

Partial list of continuous functions and the values of x for which they are continuous.

1. Polynomials for all x.
2. Rational function, except for x's that give division by zero.
3. $\sqrt[n]{x}$ (n odd) for all x.
4. $\sqrt[n]{x}$ (n even) for all $x \geq 0$.
5. **e**^{x} for all x.
6. ln(x) for $x > 0$.
7. cos(x) and sin(x) for all x.
8. tan(x) and sec(x) provided $x \neq \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$
9. cot(x) and csc(x) provided $x \neq \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

Intermediate Value Theorem

Suppose that $f(x)$ is continuous on $[a,b]$ and let M be any number between $f(a)$ and $f(b)$. Then there exists a number c such that $a < c < b$ and $f(c) = M$.

Derivatives

Definition and Notation

If $y = f(x)$ then the derivative is defined to be $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

If $y = f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}\left(f(x)\right) = Df(x)$$

If $y = f(x)$ all of the following are equivalent notations for derivative evaluated at $x = a$.

$$f'(a) = y'\big|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a} = Df(a)$$

Interpretation of the Derivative

If $y = f(x)$ then,

1. $m = f'(a)$ is the slope of the tangent line to $y = f(x)$ at $x = a$ and the equation of the tangent line at $x = a$ is given by $y = f(a) + f'(a)(x - a)$.
2. $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x = a$.
3. If $f(t)$ is the position of an object at time t then $f'(a)$ is the velocity of the object at $t = a$.

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $\dfrac{d}{dx}\left(c\right) = 0$
2. $\left(c\,f(x)\right)' = c\,f'(x)$
3. $\dfrac{d}{dx}\left(x^n\right) = n\,x^{n-1}$ – **Power Rule**
4. $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$
5. $\left(f(x)\,g(x)\right)' = f'(x)\,g(x) + f(x)\,g'(x)$ – **Product Rule**
6. $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{\left(g(x)\right)^2}$ – **Quotient Rule**
7. $\dfrac{d}{dx}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right)\,g'(x)$ – **Chain Rule**

Common Derivatives

$$\frac{d}{dx}\left(x\right) = 1$$

$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x)$$

$$\frac{d}{dx}\left(\cos(x)\right) = -\sin(x)$$

$$\frac{d}{dx}\left(\tan(x)\right) = \sec^2(x)$$

$$\frac{d}{dx}\left(\sec(x)\right) = \sec(x)\tan(x)$$

$$\frac{d}{dx}\left(\csc(x)\right) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}\left(\cot(x)\right) = -\csc^2(x)$$

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(a^x\right) = a^x \ln(a)$$

$$\frac{d}{dx}\left(\mathbf{e}^x\right) = \mathbf{e}^x$$

$$\frac{d}{dx}\left(\ln(x)\right) = \frac{1}{x},\quad x > 0$$

$$\frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x},\quad x \neq 0$$

$$\frac{d}{dx}\left(\log_a(x)\right) = \frac{1}{x\ln(a)},\quad x > 0$$

Chain Rule Variants

The chain rule applied to some specific functions.

1. $\dfrac{d}{dx}\left(\left[f(x)\right]^n\right) = n\left[f(x)\right]^{n-1} f'(x)$
2. $\dfrac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\,\mathbf{e}^{f(x)}$
3. $\dfrac{d}{dx}\left(\ln\left[f(x)\right]\right) = \dfrac{f'(x)}{f(x)}$
4. $\dfrac{d}{dx}\left(\sin\left[f(x)\right]\right) = f'(x)\cos\left[f(x)\right]$
5. $\dfrac{d}{dx}\left(\cos\left[f(x)\right]\right) = -f'(x)\sin\left[f(x)\right]$
6. $\dfrac{d}{dx}\left(\tan\left[f(x)\right]\right) = f'(x)\sec^2\left[f(x)\right]$
7. $\dfrac{d}{dx}\left(\sec\left[f(x)\right]\right) = f'(x)\sec\left[f(x)\right]\tan\left[f(x)\right]$
8. $\dfrac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right) = \dfrac{f'(x)}{1+\left[f(x)\right]^2}$

Higher Order Derivatives

The 2^{nd} Derivative is denoted as $f''(x) = f^{(2)}(x) = \dfrac{d^2 f}{dx^2}$ and is defined as $f''(x) = \left(f'(x)\right)'$, *i.e.* the derivative of the first derivative, $f'(x)$.

The n^{th} Derivative is denoted as $f^{(n)}(x) = \dfrac{d^n f}{dx^n}$ and is defined as $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$, *i.e.* the derivative of the $(n-1)^{st}$ derivative, $f^{(n-1)}(x)$.

Implicit Differentiation

Find y' if $\mathbf{e}^{2x-9y} + x^3y^2 = \sin(y) + 11x$. Remember $y = y(x)$ here, so products/quotients of x and y will use the product/quotient rule and derivatives of y will use the chain rule. The "trick" is to differentiate as normal and every time you differentiate a y you tack on a y' (from the chain rule). Then solve for y'.

$$\begin{aligned}
&\mathbf{e}^{2x-9y}(2-9y') + 3x^2y^2 + 2x^3y\,y' = \cos(y)y' + 11 \\
&2\mathbf{e}^{2x-9y} - 9y'\mathbf{e}^{2x-9y} + 3x^2y^2 + 2x^3y\,y' = \cos(y)y' + 11 \qquad \Rightarrow \qquad y' = \frac{11 - 2\mathbf{e}^{2x-9y} - 3x^2y^2}{2x^3y - 9\mathbf{e}^{2x-9y} - \cos(y)} \\
&\left(2x^3y - 9\mathbf{e}^{2x-9y} - \cos(y)\right)y' = 11 - 2\mathbf{e}^{2x-9y} - 3x^2y^2
\end{aligned}$$

Increasing/Decreasing – Concave Up/Concave Down

Critical Points

$x = c$ is a critical point of $f(x)$ provided either

1. $f'(c) = 0$ or,
2. $f'(c)$ doesn't exist.

Increasing/Decreasing

1. If $f'(x) > 0$ for all x in an interval I then $f(x)$ is increasing on the interval I.
2. If $f'(x) < 0$ for all x in an interval I then $f(x)$ is decreasing on the interval I.
3. If $f'(x) = 0$ for all x in an interval I then $f(x)$ is constant on the interval I.

Concave Up/Concave Down

1. If $f''(x) > 0$ for all x in an interval I then $f(x)$ is concave up on the interval I.
2. If $f''(x) < 0$ for all x in an interval I then $f(x)$ is concave down on the interval I.

Inflection Points

$x = c$ is a inflection point of $f(x)$ if the concavity changes at $x = c$.

Integrals

Definitions

Definite Integral : Suppose $f(x)$ is continuous on $[a,b]$. Divide $[a,b]$ into n subintervals of width Δx and choose x_i^* from each interval. Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x.$$

Anti-Derivative : An anti-derivative of $f(x)$ is a function, $F(x)$, such that $F'(x) = f(x)$.

Indefinite Integral : $\int f(x)\,dx = F(x) + c$ where $F(x)$ is an anti-derivative of $f(x)$.

Fundamental Theorem of Calculus

Part I : If $f(x)$ is continuous on $[a,b]$ then $g(x) = \int_a^x f(t)\,dt$ is also continuous on $[a,b]$ and $g'(x) = \dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Part II : $f(x)$ is continuous on $[a,b]$, $F(x)$ is an anti-derivative of $f(x)$, *i.e.* $F(x) = \int f(x)\,dx$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Variants of Part I :

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x) f\big[u(x)\big]$$

$$\frac{d}{dx}\int_{v(x)}^{b} f(t)\,dt = -v'(x) f\big[v(x)\big]$$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\,dt = u'(x) f\big[u(x)\big] - v'(x) f\big[v(x)\big]$$

Properties

$$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$\int c f(x)\,dx = c\int f(x)\,dx$, c is a constant

$\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant

$\int_a^b c\,dx = c(b-a)$, c is a constant

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b \left|f(x)\right| dx$$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for any value c.

If $f(x) \ge g(x)$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

If $f(x) \ge 0$ on $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$

If $m \le f(x) \le M$ on $a \le x \le b$ then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$

Common Integrals

$\int k\,dx = k\,x + c$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$

$\int \mathbf{e}^u\,du = \mathbf{e}^u + c$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$

$\int \ln(u)\,du = u\ln(u) - u + c$

$\int \cos(u)\,du = \sin(u) + c$

$\int \sec(u)\tan(u)\,du = \sec(u) + c$

$\int \tan(u)\,du = \ln|\sec(u)| + c$

$\int \sin(u)\,du = -\cos(u) + c$

$\int \csc(u)\cot(u)du = -\csc(u) + c$

$\int \tan(u)\,du = -\ln|\cos(u)| + c$

$\int \sec^2(u)\,du = \tan(u) + c$

$\int \sec(u)du = \ln\left|\sec(u) + \tan(u)\right| + c$

$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$

$\int \csc^2(u)\,du = -\cot(u) + c$

$\int \csc(u)du = -\ln\left|\csc(u) + \cot(u)\right| + c$

$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c$

Standard Integration Techniques

u Substitution : $\int_a^b f(g(x))\,g'(x)\,dx$ will convert the integral into $\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ using the substitution $u = g(x)$ where $du = g'(x)dx$. For indefinite integrals drop the limits of integration.

Example $\int_1^2 5x^2\cos\left(x^3\right)\,dx$

$u = x^3 \quad \Rightarrow \quad du = 3x^2dx \quad \Rightarrow \quad x^2dx = \frac{1}{3}du$

$x = 1 \quad \Rightarrow \quad u = 1^3 = 1 \quad :: \quad x = 2 \quad \Rightarrow \quad u = 2^3 = 8$

$$\int_1^2 5x^2\cos\left(x^3\right)\,dx = \int_1^8 \frac{5}{3}\cos(u)\,du = \frac{5}{3}\sin(u)\Big|_1^8 = \frac{5}{3}\Big(\sin(8) - \sin(1)\Big)$$

Products and (some) Quotients of Trig Functions

For $\int \sin^n(x)\cos^m(x)\,dx$ we have the following :

1. **n odd.** Strip 1 sine out and convert rest to cosines using $\sin^2(x) = 1 - \cos^2(x)$, then use the substitution $u = \cos(x)$.
2. **m odd.** Strip 1 cosine out and convert rest to sines using $\cos^2(x) = 1 - \sin^2(x)$, then use the substitution $u = \sin(x)$.
3. **n and m both odd.** Use either 1. or 2.
4. **n and m both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For $\int \tan^n(x)\sec^m(x)\,dx$ we have the following :

1. **n odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using $\tan^2(x) = \sec^2(x) - 1$, then use the substitution $u = \sec(x)$.
2. **m even.** Strip 2 secants out and convert rest to tangents using $\sec^2(x) = 1 + \tan^2(x)$, then use the substitution $u = \tan(x)$.
3. **n odd and m even.** Use either 1. or 2.
4. **n even and m odd.** Each integral will be dealt with differently.

Trig Formulas : $\sin(2x) = 2\sin(x)\cos(x)$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$, $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

Example $\int \tan^3(x)\sec^5(x)\,dx$

$$\begin{aligned}\int \tan^3 x\,\sec^5 x\,dx &= \int \tan^2 x\,\sec^4 x\,\tan x\,\sec x\,dx\\ &= \int \left(\sec^2(x) - 1\right)\sec^4(x)\,\tan(x)\sec(x)dx\\ &= \int \left(u^2 - 1\right)u^4du \qquad \Big[u = \sec(x)\Big]\\ &= \tfrac{1}{7}\sec^7(x) - \tfrac{1}{5}\sec^5(x) + c\end{aligned}$$

Example $\int \frac{\sin^5(x)}{\cos^3(x)}\,dx$

$$\begin{aligned}\int \frac{\sin^5 x}{\cos^3 x}\,dx &= \int \frac{\sin^4 x\sin x}{\cos^3 x}\,dx = \int \frac{(\sin^2 x)^2\sin x}{\cos^3 x}\,dx\\ &= \int \frac{(1-\cos^2(x))^2\sin(x)}{\cos^3(x)}\,dx \qquad \Big[u = \cos(x)\Big]\\ &= -\int \frac{(1-u^2)^2}{u^3}du = -\int \frac{1-2u^2+u^4}{u^3}du\\ &= \tfrac{1}{2}\sec^2(x) + 2\ln\left|\cos(x)\right| - \tfrac{1}{2}\cos^2(x) + c\end{aligned}$$