Introduction

The general problem of numerical integration may be stated as follows:

- Given a set of data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ of a function $y = f(x)$, where $f(x)$ is not known explicitly.
- It is required to compute the value of the definite integral

$$I = \int_a^b y \, dx$$

- In this case we have to replace $f(x)$ by an interpolating polynomial $\varphi(x)$ and obtain an approximate value of the definite integral by integrating $\varphi(x)$.
- Thus, different integration formulae can be obtained depending upon the type of the interpolation formula used.

Introduction

Definition:

Numerical differentiation is the process of calculating the derivatives of a function from a set of given values of that function.

How to Solve:

- The problem is solved by
  - Representing the function by an interpolation formula.
  - Then differentiating this formula as many times as desired.

Differentiation for Equidistant and Non-equidistant Values

- If the function is given by equidistant values, it should be represented by an interpolation formula employing differences, such as Newton's formula.

- If the given values of the function are not equidistant, we must represent the formula by Lagrange's formula.

Numerical Differentiation

- Consider Newton's Forward difference formula, putting $u = (x - x_0)/h$, we get

$$y = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

- Then,

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{dy}{du}.\frac{d}{dx}\left(\frac{x}{h} - \frac{x_0}{h}\right)$$

$$= \frac{dy}{du}.\frac{d}{dx}\left(\frac{x}{h}\right) - \frac{dy}{du}.\frac{d}{dx}\left(\frac{x_0}{h}\right)$$

$$= \frac{dy}{du}.\frac{1}{h} = \frac{1}{h}.\frac{dy}{du}$$

Numerical Differentiation

- Therefore,

$$\frac{dy}{dx}=\frac{1}{h}.\frac{dy}{du}$$

$$=\frac{1}{h}.\frac{d}{du}\left[y_0+u\,\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0+\ldots\right]$$

$$=\frac{1}{h}.\left[\begin{array}{c}\frac{d}{du}\left(y_0\right)+\frac{d}{du}\left(u\,\Delta y_0\right)+\frac{d}{du}\left(\frac{u(u-1)}{2!}\Delta^2 y_0\right)\\ +\frac{d}{du}\left(\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0\right)+\ldots\end{array}\right]$$

$$=\frac{1}{h}\left[\Delta y_0+\frac{2u-1}{2}\Delta^2 y_0+\frac{3u^2-6u+2}{6}\Delta^3 y_0+\cdots\right] \quad (1.1)$$

Numerical Differentiation

For tabular values of x, the formula takes a simpler form, by setting $x = x_0$ we obtain $u = 0$ [since $u = (x - x_0)/h$] and hence (1.1) gives

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \cdots\right] \quad (1.1)$$

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right] \quad (1.2)$$

Numerical Differentiation: Double Derivatives

We know,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \cdots\right] \quad (1.1)$$

Differentiating (1.1) again, we obtain,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6u-6}{6}\Delta^3 y_0 + \frac{12u^2-36u+22}{24}\Delta^4 y_0 + \cdots\right] \quad (1.3)$$

At $x = x_0$, $u = 0$ and we obtain

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 \cdots\right] \quad (1.4)$$

Formulae for computing higher derivatives may be obtained by successive differentiation.

Numerical Differentiation: Higher Derivatives

Different formulae can be derived by starting with other interpolation formulae.

(a) Newton's backward difference formula gives

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \ldots\right] \quad (1.5)$$

and

$$\left[\frac{d^2 y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n \ldots\right] \quad (1.6)$$

Numerical Differentiation: Higher Derivatives

If a derivative is required near the start of a table the following formulae may be used

$$hy_0^{/} = \left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \frac{1}{5}\Delta^5 - \frac{1}{6}\Delta^6 + \frac{1}{7}\nabla^7 - \frac{1}{8}\nabla^8 + \ldots\right] y_0 \quad (1.7)$$

$$hy_0^{/} = \left[\Delta + \frac{1}{2}\Delta^2 - \frac{1}{6}\Delta^3 + \frac{1}{12}\Delta^4 - \frac{1}{20}\Delta^5 + \frac{1}{30}\Delta^6 - \ldots\right] y_{-1} \quad (1.7b)$$

$$h^2 y_0^{//} = \left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \frac{137}{180}\Delta^6 - \frac{7}{10}\Delta^7 + \frac{363}{560}\Delta^8 + \ldots\right] y_0 \quad (1.8)$$

Numerical Differentiation: Higher Derivatives

If a derivative is required near the end of a table the following formulae may be used

$$hy_n' = \left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \frac{1}{5}\nabla^5 + \frac{1}{6}\nabla^6 + \frac{1}{7}\nabla^7 + \frac{1}{8}\nabla^8 + \dots\right] y_n \quad (1.9)$$

$$hy_n' = \left[\nabla - \frac{1}{2}\nabla^2 - \frac{1}{6}\nabla^3 - \frac{1}{12}\nabla^4 - \frac{1}{20}\nabla^5 - \frac{1}{30}\nabla^6 - \frac{1}{42}\nabla^7 - \frac{1}{56}\nabla^8 - \dots\right] y_{n+1} \quad (1.9b)$$

$$h^2 y_n'' = \left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \frac{5}{6}\nabla^5 + \frac{137}{180}\nabla^6 + \frac{7}{10}\nabla^7 + \frac{363}{560}\nabla^8 + \dots\right] y_n \quad (1.10)$$

Example

From the following table of values of x and y, obtain

$$\frac{dy}{dx} \quad and \quad \frac{d^2y}{dx^2} \quad for \quad x = 1.2$$

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.953	6.0496	7.3891	9.025

Solution

The difference table is in the next slide:

Solution

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183						
		0.6081					
1.2 (x_0)	3.3201 (y_0)		0.1333				
		Δy_0 0.7351		0.0294			
1.4	4.0552		$\Delta^2 y_0$ 0.1627		0.0067		
		0.8978		$\Delta^3 y_0$ 0.0361		0.0013	
1.6	4.9530		0.1988		$\Delta^4 y_0$ 0.0080		0.0001
		1.0966		0.0441		$\Delta^5 y_0$ 0.0014	
1.8	6.0496		0.2429		0..0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.0250						

Solution

Here $x_0 = 1.2$, $y_0 = 3.3201$ and $h = 0.2$

$$\left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014)\right]$$

$$= 3.3205$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{0.04}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right]$$

$$= 3.318$$

Alternative Solution

Here $x_0 = 1.2$, $y_0 = 3.3201$ and $h = 0.2$

Then, $x_{-1} = 1.0$, $y_{-1} = 2.7183$ and $h = 0.2$

$$\left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.6018 + \frac{1}{2}(0.1333) - \frac{1}{6}(0.0294) + \frac{1}{12}(0.0067) - \frac{1}{20}(0.0013)\right]$$

$$= 3.3205$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{0.04}\left[0.1333 - \frac{1}{12}(0.0067) + \frac{1}{12}(0.0013)\right]$$

$$= 3.32$$

Class Work

From the following table of values of x and y, obtain

$$\frac{dy}{dx} \quad for \quad x = 2.0$$

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.953	6.0496	7.3891	9.025

Answer: 7.3896

Class Work

Find $\frac{d}{dx}(J_0)$ $x = 0.1$ from the following table:

x	0.0	0.1	0.2	0.3	0.4
$J_0(x)$	1.0000	0.9975	0.9900	0.9776	0.9604

Class Work

The following table gives the angular displacements θ (radians) at different intervals of time t (seconds).

Calculate the angular velocity at the instant $x = 0.408$.

θ	0.052	0.105	0.168	0.242	0.327	0.408	0.489
t	0	0.02	0.04	0.06	0.08	0.10	0.12

Errors in Numerical Differentiation

In the given example,

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.953	6.0496	7.3891	9.025

$$when\ x = 1.2, \text{ then we get } \frac{dy}{dx} = 3.3205 \text{ and } \frac{d^2y}{dx^2} = 3.318$$

$$But, \text{ here } y = \mathrm{e}^{\mathrm{x}}, \text{ therefore, } \frac{dy}{dx} = \frac{d}{dx}(\mathrm{e}^{\mathrm{x}}) = \mathrm{e}^{\mathrm{x}} \text{ and } \frac{d^2y}{dx^2} = \mathrm{e}^{\mathrm{x}}$$

- Therefore, here we can see with each differentiation, some error occurs in the derivatives.
- The error increases with higher derivatives.
- This is because, in interpolation the new polynomial would agree at the set of points.
- But, their slopes at these points may vary considerably.

Maximum Value of a Tabulated Function

- It is known that the maximum values of a function can be found by equating the first derivative to zero and solving for the variable.
- The same procedure can be applied to determine the maxima of a tabulated function.
- Consider Newton's forward difference formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots$$

$$where \quad x = x_0 + uh$$

$$Then, \frac{dy}{du} = \Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \dots$$

Maximum Value of a Tabulated Function

- For maxima, $dy/dx = 0$.
- Hence, terminating the right-hand side after the third difference (for simplicity) and equating it to zero.
- We obtain the quadratic for u.

$$c_0 + c_1 u + c_2 u^2 = 0$$

where

$$c_0 = \Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0$$

$$c_1 = \Delta^2 y_0 - \Delta^3 y_0$$

$$c_2 = \frac{1}{2}\Delta^3 y_0$$

The values of x can then be found from the relation $x = x_0 + uh$

Example

From the following table, find x, correct to two decimal places, for which y the function has the maximum value and find the value of y.

x	1.2	1.3	1.4	1.5	1.6
y	0.9320	0.9636	0.9855	0.9975	0.9996

Solution

The difference table is in the next slide:

Solution

x	y		
1.2	0.9320		
		0.0316	
1.3	0.9636		-0.0097
		0.0219	
1.4	0.9855		-0.0099
		0.0120	
1.5	0.9975		-0.0099
		0.0021	
1.6	0.9996		

Solution

Let, $x_0 = 1.2$ and we can terminate the formula after the second difference (since the difference is very negligible).

Now we have,

$$0.0316+(2u-1)(-0.0097)/2 = 0$$

Therefore, $u = 3.8$ and $x = x_0 + uh = 1.2+(3.8)(0.1) = 1.58$

For $x = 1.58$, we have the maximum value of y.

Using Newton's backward difference formula at $x_n = 1.6$ gives,

$y(1.58) = 1.0$ (CLASS WORK)

That is the maximum value of y in the function.

Introduction

The general problem of numerical integration may be stated as follows:

- Given a set of data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ of a function $y = f(x)$, where $f(x)$ is not known explicitly.
- It is required to compute the value of the definite integral

$$I = \int_a^b y\, dx$$

- In this case we have to replace $f(x)$ by an interpolating polynomial $\varphi(x)$ and obtain an approximate value of the definite integral by integrating $\varphi(x)$.
- Thus, different integration formulae can be obtained depending upon the type of the interpolation formula used.

Numerical Integration

- Let, the interval $[a, b]$ be divided into n equal subintervals such that $a = x_0 < x_1 < \ldots < x_n = b$.
- Then, $x_n = x_0 + nh$.
- Hence, the integral becomes $$I = \int_{x_0}^{x_n} y\, dx$$
- Integrating Newton's forward difference formula, we obtain

$$I = \int_{x_0}^{x_n} \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \right] dx$$

$$= \int_{x_0}^{x_0+nh} \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \right] dx$$

Numerical Integration

- Since $x = x_0 + hu$ from which we get $dx = hdu$.
- The limit of integration for x are x_0 and x_0+nh
- We know, $u = (x - x_0)/h$
- Therefore, for u, the corresponding lower limit is $(x_0 - x_0)/h = 0$.
- For u, the corresponding upper limit is $(x_n - x_0)/h = (x_0 + hn - x_0)/h = n$.
- We therefore have,

$$I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \right] du$$

Numerical Integration

- Now,

$$I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{\Delta^2 y_0}{2}(u^2 - u) + \frac{\Delta^3 y_0}{3!}(u^3 - 3u^2 + 2u) + \cdots \right] du$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + (\frac{n^3}{3} - \frac{n^2}{2})\frac{\Delta^2 y_0}{2} + (\frac{n^4}{4} - n^3 + n^2)\frac{\Delta^3 y_0}{3!} + \cdots \right]$$

- Which gives on simplification

$$I = \int_{x_0}^{x_n} y\, dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}.\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right] \qquad (1)$$

- From this general formula we can obtain different integration formulae by putting $n = 1, 2, 3, \ldots$ etc.

Trapezoidal Rule

- Setting $n = 1$ in the general formula (1) and neglecting all differences above the first we obtain for the first interval $[x_0, x_1]$

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}[y_0 + y_1]$$

- For the next interval $[x_1, x_2]$, we deduce similarly … (and so on) …

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}[y_1 + y_2]$$

- Similarly, for the last interval $[x_{n-1}, x_n]$, we have

$$\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}[y_{n-1} + y_n]$$

- Combining all these expressions, we obtain the rule

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$

- This rule is known as the Trapezoidal Rule.

Trapezoidal Rule: Geometric Significance

- The geometrical significance of this rule is that
  - The curve $y = f(x)$ is replaced by n straight lines joining the points (x_0, y_0) and (x_1, y_1); (x_1, y_1) and (x_2, y_2); ... (x_{n-1}, y_{n-1}), and (x_n, y_n).
  - The area bounded by the curve $y = f(x)$, within the x-coordinates $x = x_0$, and $x = x_n$, and the x-axis is then approximately equivalent to the sum of the areas of the n trapeziums obtained.

Example

Evaluate $I = \int_0^1 \frac{1}{1+x}\,dx,$

for h = 0.5, 0.25 and 0.125 using Trapezoidal rule (correct to three decimal places).

Solution

The values of x and y are tabulated below $h = 0.5$

x	0	0.5	1.0
y	1.0000	0.6667	0.5

Trapezoidal rule gives

$$I = \int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right) + y_n\right]$$

$$I = \frac{0.5}{2}\left[1.0000 + 2(0.6667) + 0.5\right] = 0.7084$$

Example (Cont.)

Solution

The values of x and y are tabulated below $h = 0.25$

x	0	0.25	0.5	0.75	1
y	1	0.8	0.6667	0.5714	0.5

Trapezoidal rule gives

$$I = \int_{x_0}^{x_n} y\, dx = \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \cdots + y_{n-1}\right) + y_n\right]$$

$$I = \frac{0.25}{2}\left[1 + 2(0.8 + 0.6667 + 0.5714) + 0.5\right] = 0.6970$$

Example (Cont.)

Solution

The values of x and y are tabulated below $h = 0.125$ (CLASS WORK)

Answer: $I = 0.6941$

Class Work

A solid of revolution is formed by rotating about the x-axis the area between the x-axis, the lines $x = 0$ and $x = 1$, and a curve through the points with the following coordinates

x	0.00	0.25	0.50	0.75	1.00
y	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Trapezoidal rule, giving the answer to three decimal places.

Answer: 0.9447625

Simpson's 1/3-Rule

- Setting $n = 2$ in the general formula (1) and neglecting all differences above the second we obtain for the first interval $[x_0, x_2]$

$$I = \int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \quad (1)$$

$$\int_{x_0}^{x_2} y\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right] = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

- For the next interval $[x_2, x_4]$, we deduce similarly … (and so on) …

$$\int_{x_2}^{x_4} y\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

- Finaly, for the last interval $[x_{n-2}, x_n]$, we have

$$\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

- Combining all these expressions, we obtain the rule

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}\left[y_0 + 4\left(y_1 + y_3 + y_5 + \cdots + y_{n-1}\right) + 2\left(y_2 + y_4 + y_6 + \cdots + y_{n-2}\right) + y_n\right]$$

- This rule is known as the Simson's 1/3 Rule (or Simpson's Rule).

Simpson's 1/3 Rule: Geometric Significance

- The geometrical significance of this rule is that
  - Replacing the curve $y = f(x)$ is by n/2 arcs of second degree polynomials or parabolas joining the points (x_0, y_0) and (x_2, y_2); (x_2, y_2) and (x_4, y_4); ... (x_{n-2}, y_{n-2}), and (x_n, y_n).
  - It should be noted that this rule requires the division of the whole range into an even number of subintervals of width h.

Example

Evaluate $I = \int_0^1 \frac{1}{1+x}\,dx,$
correct to three decimal places for h = 0.5, 0.25 and 0.125 using Simpson's 1/3 rule.

Solution

The values of x and y are tabulated below $h = 0.5$

x	0	0.5	1.0
y	1.0000	0.6667	0.5

Simpson's rule gives

$$I = \frac{1}{6}\left[1.0000 + 4(0.6667) + 0.5\right] = 0.6945$$

Example (Cont.)

CLASS WORK: Do the same for $h = 0.25$ and $h = 0.125$

Solution

For $h = 0.25$

Simpson's rule gives I = 0.6932

For $h = 0.125$

Simpson's rule gives I = 0.6932

Class Work

Apply trapezoidal and Simpson's 1/3 rules to the integral for 10, 20, 30, 40, and 50 subintervals.

$$I = \int_0^1 \sqrt{1-x^2}\,dx$$

Simpson's 3/8-Rule

$$I = \int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

The rule is obtained by putting $n = 3$ in the general equation (1) and neglecting all the differences above the third we have,

$$\int_{x_0}^{x_3} y\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly,

$$\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_6\right]$$

Simpson’s 3/8-Rule

$$\textit{And finally} \quad \cdots \cdots$$

$$\int_{x_{n-3}}^{x_n} y\, dx = \frac{3h}{3}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

Summing up we obtain

$$\int_{x_0}^{x_n} y\, dx = \frac{3h}{8}\left[\left(y_0 + 3y_1 + 3y_2 + y_3\right) + \left(y_3 + 3y_4 + 3y_5 + y_6\right) + \cdots\cdots + \left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \cdots\cdots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

This rule called Simpson’s 3/8-rule, is not so accurate as Simpson’s rule.

Class Work

Apply trapezoidal and Simpson's 3/8 rules to the integral for 3, 6 and 12subintervals.

$$I = \int_0^3 \sqrt{1+x^2}\,dx$$

Weddle's Rule

- The rule is obtained by putting $n = 6$ in the general equation i.e., and neglecting all the differences above the sixth we have,

$$\int_{x_0}^{x_6} y\,dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right]$$

- Here the coefficient of $\Delta^6 y_0$ differs from 3/10 by the small fraction 1/140 (i.e., 3/10 – 41/140 = 1/140, which is very negligible)

- Hence if we replace this coefficient by 3/10, we commit an error of only $\frac{h}{140}\Delta^6 y_0$

- If the value of h is such that the sixth differences are small, the error committed will be negligible.

- We therefore change the last term to

$$(3/10)\Delta^6 y_0$$

Weddle's Rule

- Then replace all differences by their values in terms of the given y's. The result reduces down to

$$\int_{x_0}^{x_6} y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

Similarly,

$$\int_{x_6}^{x_{12}} y\,dx = \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$$

- Adding all such expressions as these from x_0 to x_n, where n is now a multiple of six, we get Weddle's Rule

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{10}\begin{bmatrix} y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + \\ 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12} + \ldots \\ + 2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n \end{bmatrix}$$

Weddle's Rule: More

- Weddle's rule is more accurate, in general than Simpson's rule,
- It requires at least seven consecutive values of the function.
- The geometric meaning of Weddle's Rule is that we replace the graph of the function by n/6 arcs of fifth-degree polynomials.

Example

Compute the value of the definite integral for $h = 0.2$ using Weddle's rule

$$\int_{4}^{5.2} \ln x dx$$

Solution

The values of this function is computed for each point of subdivision.

x	$\ln x$
4.0	1.3863
4.2	1.4351
4.4	1.4816
4.6	1.5261
4.8	1.5686
5.0	1.6094
5.2	1.6487

By Weddle's rule we get

I=3(0.2)[1.3863+5(1.4351)+1.4816+6(1.5261)+1.5686+5(1.6094)+1.6487]/10

=1.827858

Home Work

Compute the value of the definite integral for $h = 0.1$ using Weddle's rule

$$I = \int_{0.2}^{1.4} \left(\sin x - \ln x + e^{x} \right) dx$$

Answer: 4.05095

Romberg Integration

- This method can be used to improve the approximate results obtained by the finite difference methods such as trapezoidal method.
- Let T_n be the approximation of the integral $I = \int_a^b y\,dx$ using trapezoid rule with 2^n subintervals.
- Let $I_{1,1} = T_1$. (here, I is calculated with 2^1 segments)
- Calculated $I_{1,n}$, $I_{2,n}$ ..., $I_{n,n}$ as follows:
  - Set $I_{1,n+1} = T_{n+1}$ (i.e., $I_{1,2} = T_2$, calculated with 2^2 segments, $I_{1,3} = T_3$, calculated with 2^3 segments, $I_{1,4} = T_4$, calculated with 2^4 segments)
  - Next, for $j = 2, 3, \ldots, n$

$$I_{j,k} = I_{j-1,k+1} + \frac{I_{j-1,k+1} - I_{j-1,k}}{4^{j-1} - 1}, \quad j \geq 2$$

Romberg Integration

We have,

$$I_{j,k} = I_{j-1,k+1} + \frac{I_{j-1,k+1} - I_{j-1,k}}{4^{j-1} - 1}, \quad j \geq 2$$

- The index j represents the order of interpolation.
- For example, $j = 1$ represents the values obtained from the regular Trapezoidal rule.
- The index k represents the more or less accurate estimate of the integral.
- The value of the integral with $k + 1$ index is more accurate than with k index.
- With this notation the following table can be formed.

Romberg Integration

We have,

$$I_{j,k} = I_{j-1,k+1} + \frac{I_{j-1,k+1} - I_{j-1,k}}{4^{j-1} - 1}, \quad j \geq 2$$

With this notation the following table can be formed.

$$\begin{tikzcd}
I_{1,1} \arrow[dr] & & & \\
& I_{2,1} \arrow[dr] & & \\
I_{1,2} \arrow[ur] \arrow[dr] & & I_{3,1} \arrow[dr] & \\
& I_{2,2} \arrow[ur] \arrow[dr] & & I_{4,1} \\
I_{1,3} \arrow[ur] \arrow[dr] & & I_{3,2} \arrow[ur] & \\
& I_{2,3} \arrow[ur] & & \\
I_{1,4} \arrow[ur] & & &
\end{tikzcd}$$

An advantage of this method is that the accuracy of the computed value is known at each step.

Romberg Integration

We have,

$$I_{j,k} = I_{j-1,k+1} + \frac{I_{j-1,k+1} - I_{j-1,k}}{4^{j-1} - 1}, \quad j \geq 2$$

For j =2, k = 1,

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{4^{2-1} - 1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$

For j = 3, k = 1,

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{4^{3-1} - 1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

Example

Use Romberg method to compute the following integral correct to three decimal places.

$$I = \int_0^1 \frac{1}{1+x}\,dx,$$

Use 2, 4 and 8-segment Trapezoidal rule results.

Example: Solution

Here, we have to calculate I using $2 = 2^1$, $4 = 2^2$ and $8 = 2^3$ intervals. Therefore,

- $I_{1,1} = T_1$, that is calculate I using Trapezoidal rule with $2^1 = 2$ intervals.
- $I_{1,2} = T_2$, that is calculate I using Trapezoidal rule with $2^2 = 4$ intervals.
- $I_{1,3} = T_3$, that is calculate I using Trapezoidal rule with $2^3 = 8$ intervals.

With this notation the following table can be formed.

$$\begin{tikzcd}
I_{1,1} \arrow[dr] & & \\
& I_{2,1} \arrow[dr] & \\
I_{1,2} \arrow[ur] \arrow[dr] & & I_{3,1} \\
& I_{2,2} \arrow[ur] & \\
I_{1,3} \arrow[ur] & &
\end{tikzcd}$$

Example: Solution

Using Trapezoidal Rule, we get [Class Work]

$$I_{1,1} = 0.7084, \quad I_{1,2} = 0.6970, \quad I_{1,3} = 0.6941$$

Now,

$$I_{j,k} = I_{j-1,k+1} + \frac{I_{j-1,k+1} - I_{j-1,k}}{4^{j-1} - 1}, \quad j \geq 2$$

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{4^{2-1} - 1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} = 0.6970 + \frac{1}{3}(0.6970 - 0.7084) = 0.6932$$

$$I_{2,2} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{4^{2-1} - 1} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} = 0.6941 + \frac{1}{3}(0.6941 - 0.6970) = 0.6931$$

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{4^{3-1} - 1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} = 0.6931 + \frac{1}{15}(0.6931 - 0.6932) = 0.6931$$

Solution (Cont.)

The table of values is therefore

0.7084		
	0.6932	
0.6970		0.6931
	0.6931	
0.6941		

Therefore, $I = 0.6931$

Home Work

Compute the values of

$$I = \int_0^1 \frac{1}{1+x^2}\,dx,$$

by using the trapezoidal rule with h=0.5, 0.25 and 0.125. Then obtain a better estimate by using Romberg's method.