Bisection method [Finding root]

→ Solution of Algebric & Transcedental Equation:

($\log$, $\sin$, $\cos$, e^x ....)

This method is based on the repeated application of intermediate value property.

Let $f(x)$ be continuous between a & b

& let $f(a)$ is $-$ive

let $f(b)$ is $+$ive

Then the first approximation of the root is

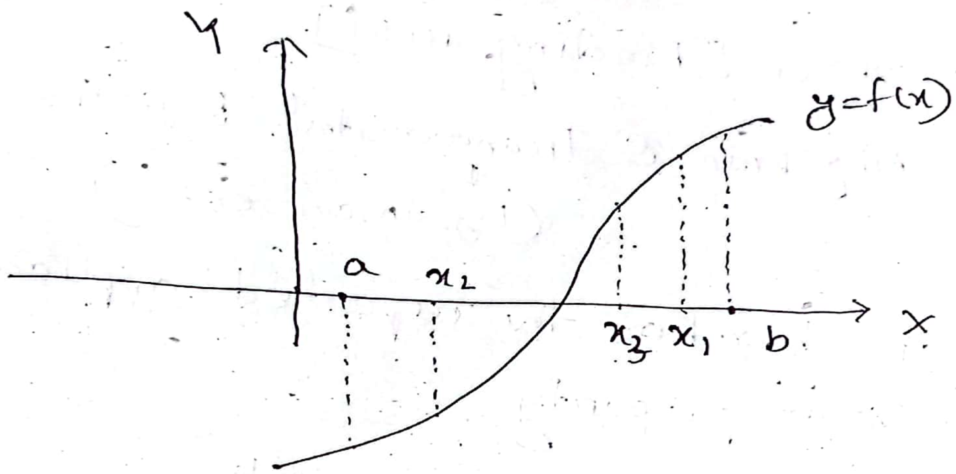
$x_1 = \frac{a+b}{2}$. if $f(x_1) = 0$ then x_1 is root of

$f(x) = 0$

Otherwise the root lies between a & x_1 or x_1 & b

according as $f(x_1)$ is positive or negetive.

Then we bisect the interval as before & continue the process untill the root is found accuracy

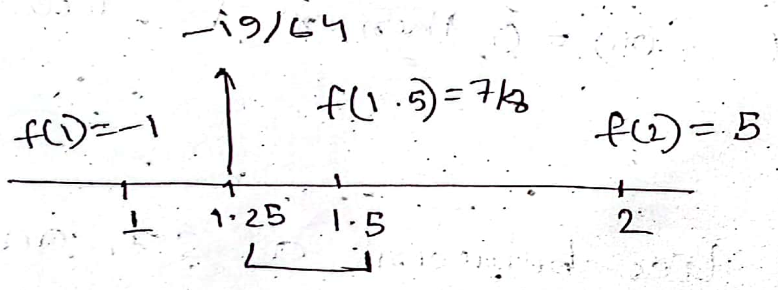


Q. find the real root of the equation $f(x) = x^3 - x - 1$

⇒ Sol^n:

$f(1) = 1 \times 1 - 1 = -1$

$f(2) = 2^3 - 2 - 1 = 5$



$x_1 = \frac{1+2}{2} = 1.5$

$\therefore f(x_1) = (1.5)^3 - 1.5 - 1 = 7/8$

$x_2 = \frac{1.5+1}{2} = 1.25$

$f(x_2) = (1.25)^3 - 1.25 - 1 = -19/64$

$$x_3 = \frac{1.25+1.5}{2} = 1.375$$

$$f(x_3) = (1.375)^3 - 1.375 - 1 = +ive$$

$$x_4 = \frac{1.25+1.375}{2} = 1.3125$$

Q. find the real root of the equation $x^3 - 2x - 5 = 0$

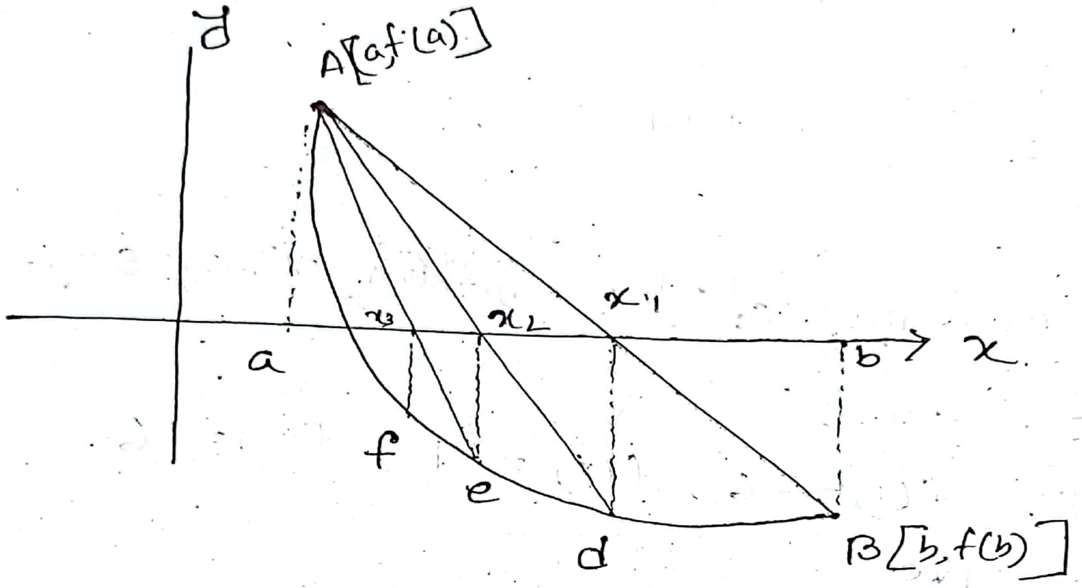
Soln: $f(x) = x^3 - 2x - 5$

$f(0) = -5$, $f(1) = -6$, $f(2) = -1$, $f(3) = 16$

-ve +ve

i	a	b	$x = \frac{a+b}{2}$	$f(x)$
1	2	3	2.5	5.625
2	2	2.5	2.25	1.8906 > 0
3	2	2.25	2.125	0.3457 > 0
4	2	2.125	2.0625	-0.3513 < 0
5	2.0625	2.125	2.09375	-0.0089 < 0
6	2.09375	2.125	2.10938	0.1668 > 0
7	2.09375	2.10938	2.10156	0.07856 > 0
8				
9				
10				

[4] False Position Method:



$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

first approximation:

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

if $f(x_1) = 0$ then x_1 is the root of $f(x)$

if $f(x_1) \neq 0$ and if $f(x_1)$ & $f(a)$ have oppsite sign then we can write the second approximation

$$x_2 = \frac{a f(x_1) - x_1 f(a)}{f(x_1) - f(a)}$$

→ Working procedure

① Find the interval $[a,b]$ so that $f(a)f(b) < 0$

② find $c = \dfrac{a f(b) - b f(a)}{f(b) - f(a)}$

③ $f(a)f(c) < 0$ root in $[a,c]$

$f(b)f(c) < 0$ root in $[c,b]$

④ repeat ① & ②

Q: Find a real root of $x^3 - 2x - 5 = 0$ using the method False position upto four iterations.

Soln: $f(2) = -1$

$f(3) = 16$

$a = 2, \quad b = 3$

$f(a) = -1, \quad f(3) = 16$

1st iteration

$$c = \frac{2 \times 16 - 3(-1)}{16 - (-1)}$$

$$= \frac{35}{17} = 2.0588$$

$f(c) = -0.3908 < 0$

2nd iteration:

$b = 3, \ f(b) = 16$

$a = 2.0588, \ f(a) = -0.3908$

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$= 2.0812$

$f(c) = -0.1479 < 0$

3rd Iteration:

$a = 2.0812 \ ; \ f(a) = -0.1479$

$b = 3, \ f(b) = 16$

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)} = 2.0896$$

$f(c) = -0.0551 < 0$

4th iteration:

$a = 2.0896, \ f(a) = -0.0551$

$b = 3, \ f(b) = 16$

$c = 2.0927$, Hence the required root is 2.0927

Interpolation: The technique or method of estimating unknown values from given set of observation is known as interpolation

(1973)

x	$f(x)$
1971	1000
1981	1025
1991	1080

equal interval
- Newton forward
- Newton backward

unequal interval
- langrange interpolation

Newton Forward:

Qn: Estimate the population in 1985 & 1925 from following statistics

year	1891	1901	1911	1921	1931
population	46	66	81	93	101

Soln:

$$\Delta f(x) = f(x+h) - f(x)$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1891	46				
		20			
1901	66		-5		
		15		2	-3
1911	81		-3		
		12		-1	
1921	93		-4		
		8			
1931	101				

Newton forward Formula:

$$f(a+hu) = f(a) + \frac{u}{1!}\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 f(a)$$

Here, $a = \cancel{1981}$ 1891

$h = 1901 - 1891 = 10$

$f(1895) = ?$

$\therefore a + hu = 1895$

$\Rightarrow 1891 + 10 \cdot u = 1895$

$u = 0.4$

$$\therefore f(1895) = 46 + \frac{0.4}{1!}(20) + \frac{0.4(0.4-1)}{2!}(-8) + \frac{0.4(0.4-1)(0.4-2)}{3!}(-2) + \frac{0.4(0.4-1)(0.4-2)(0.4-3)}{4!}(-3)$$

$$= 54.8528$$

$$\boxed{\nabla f(x) = f(x) - f(x-h)}$$

Newton backward:

$$f(a+hu) = f(a) + \frac{u}{1!}\nabla f(a) + \frac{u(u+1)}{2!}\nabla^2 f(a) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(a) + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 f(a)$$

$a + hu = 1925$

$\Rightarrow 1931 + 10 \cdot u = 1925$

$\Rightarrow u = -0.6$

Here,

$a = 1931$

$h = 10$

$$f(1925) = 101 + \frac{(-0.6)}{1!} \times 8 + \frac{(-0.6)(-0.6+1)}{2!}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!} \times (-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!} \times (-3)$$

$$= 96.8368$$

Q. Find Number of men getting wages between Rs 10 & Rs. 15 from following data

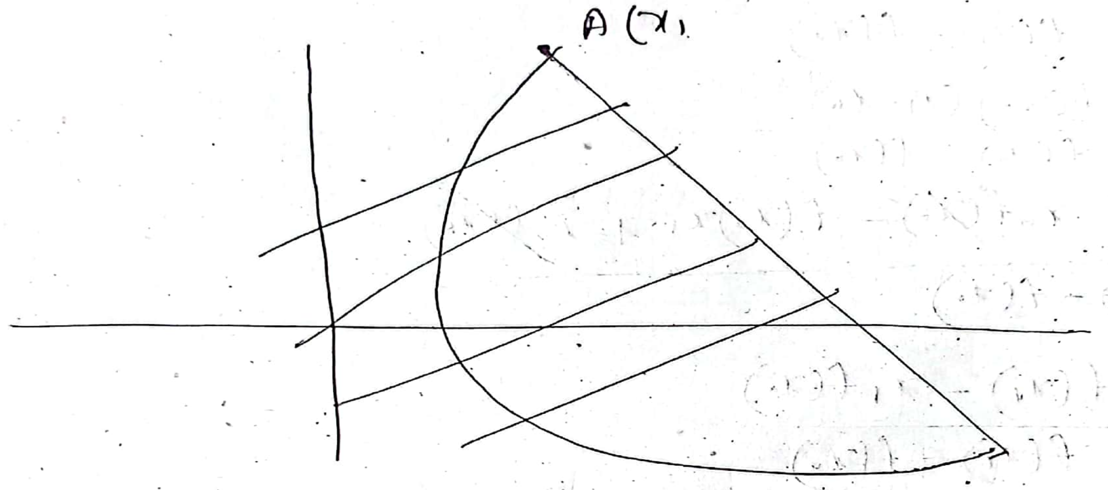
wages	0-10	10-20	20-30	30-40
frequency	9	30	35	42

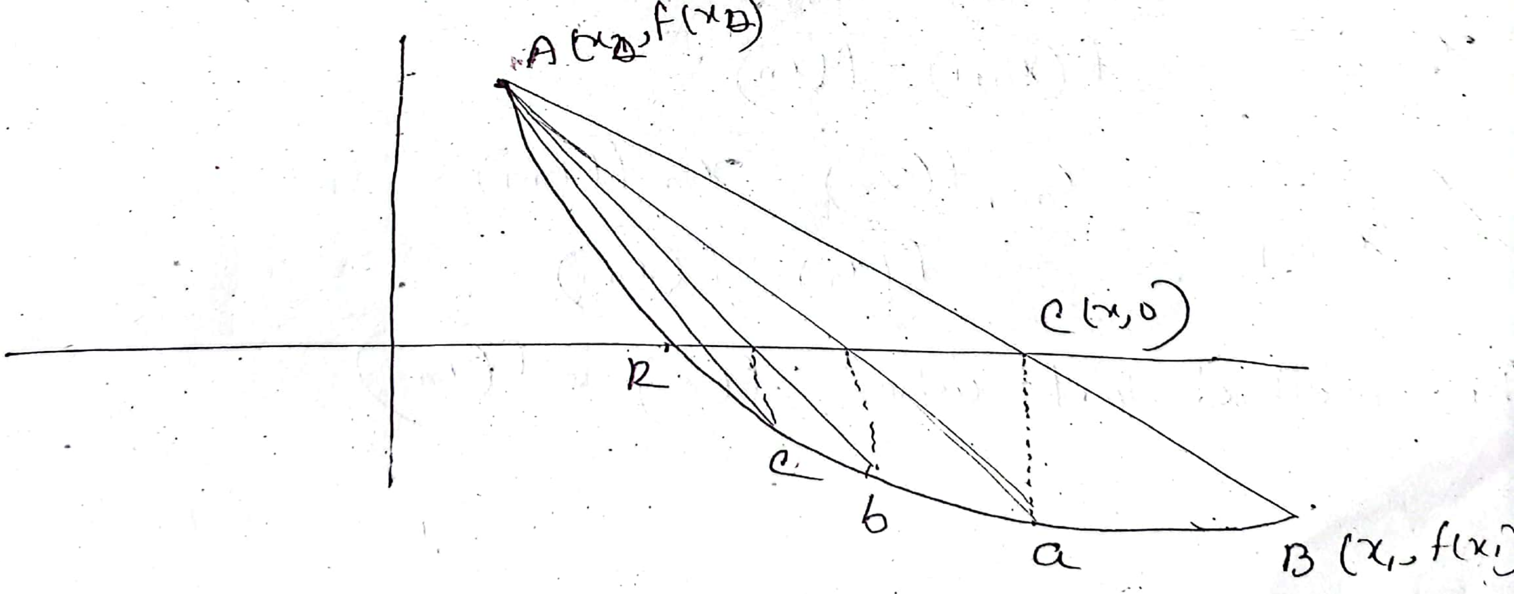
Soln:

x wages	$f(x)$ cf	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
bellow 10	9	30		
			5	
bellow 20	9+30 = 39	35		2
			7	
bellow 30	39+35 = 74	42		
bellow 40	74+42 = 116			

[Ⅱ] secant method : (chord method)

This method is quit similar to Regular falsi method except for the condition $f(x_1)(x_2) < 0$





slope of AB = slope of AC

$$\Rightarrow \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{0 - f(x_0)}{x - x_0}$$

$$\Rightarrow x - x_0 = \frac{-f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x = x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{x_0 f(x_1) - x_0 \cancel{f(x_0)} - f(x_0) x_1 + \cancel{x_0 f(x_0)}}{f(x_1) - f(x_0)}$$

$$= \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}$$

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

This method fails when $f(x_n) = f(x_{n-1})$

Q. A real root of the equation $x^3 \; 5x+1 = 0$ in the interval $(0,1)$. perform four iteration of the secant method.

Soln: $f(x) = x^3 - 5x + 1$

$f(0) = 1, \quad f(1) = -3$

$x_0 = 0 \qquad f(x_0) = 1$

$x_1 = 1 \qquad f(x_1) = -3$

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

put, $n=1$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{0 \times 3 - 1 \times 1}{-3 - 1} = 1/4 = 0.25$$

$f(x_2) = -0.23475$

put $n=2$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{1 \times (-0.23475) - 0.25 \times (-3)}{-0.23475 - (-3)}$$

$= 0.18649$

put $n=3$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= 0.20174$$

$\therefore f(x_4) = -0.00048$

put $n=4$

$$\boxed{x_5 = 0.20081}$$

* inherent error
* round off error
* truncation error
* significant figure, rules of significant figure
* Discuss briefly the different types of error encountered in performing numerical calculations.
* if x = 0.5L and correct to 2 decimal places, Find the percentage error
* what is accuracy & precision
* why do occur numerical error

Error analysis:

The numerical error is the difference between the exact solution and the approximation solution type of error

Inherent error: Inherent error exist in the problem either due to approximation given data, limitation of the computing aids, such as mathematical ~~aids~~ tables, desk calculator etc.

Such type of error can be reduced by taking better data & using high precision computing aids

The round off error : These error are due to rounding off the number

Ex : $1.24372 \approx 1.244$

Now error $1.244 - 1.24372 = 0.00028$

Truncation Error : The error arise due to truncating the infinite series to some approximate terms

Ex . $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - - -$

actual value of e^x at $x = 1$ is 2.71828

$e_a' = 1+1 = 2$

then error $= 2.71828 - 2 = 0.71828$

Absolute error : Absolute error is the positive difference between the actual value and the approximation value

$E_A = |x - x_1| = \delta x$

5. Relative error: Is the ratio of Absolute error & the exact value

$$E_R = \frac{\delta x}{X}$$

Percentage error: $E_p = 100 E_R$

Accuracy: Accuracy refers to how close a computed or measured value is to the true or exact value

precision: precision refers to how consistently a method produces the same result or how detailed the result is in terms of decimal places.

trailing zero

$\underbrace{0.00}_{\text{leading zero}}\underline{50830}$

5 significant figures

Significant figures refers to the digits in a number that contribute to its precision. They indicate accuracy of a measurement ~~and~~ or calculation & help ~~how~~ convey how precise the data is.

Rules of significant figures

1. Non-zero numbers are always significant
2. zeroes between non zero digits are significant
3. Leading zeros are not significant
4. Trailing zero in a decimal number are significant
5. Trailing zeros in a whole number without a decimal point are not significant
6. Exact numbers have infinite significant figures

Exact numbers are values that are known with complete certainity and can be expressed with infinite precision. Approximation ~~via~~ number on the otherhand, are values that have been rounded or estimated, thus they have some degree of uncertainity.

Exact numbers are results of counting. on defined values where there is no uncertainity. They have considered to have an infinite number of significant figure

example:

1. The number of people in a room
2. The " of inches in a foot → this is defined conversion and does not involve any estimation
3. Mathematical constant.

Approximation number: come from measurement, calculation on estimation where the result have some degree of uncertainity and it may be rounded to a certain number of decimal places on sig. figures.

Example: Length of an object → measurement might have some degree of error

* weight of an apple → ~~scale~~ scale shows 150 gram, scale may have a margin of error
* The value of 2/3 → repeating decimal (0.66...)

Lagrange's Interpolation for unequal interval

Suppose $y = f(x)$ be a function and $f(x_0), f(x_1), \ldots f(x_n)$ are value corresponding to points $x_0, x_1 \ldots x_n$ then

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} \times f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} \times f(x_n)$$

It's called Lagrange's interpolation formula

Example: Using Lagrange's interpolation formula, find the value of $f(5)$ from the following data

x	1	2	3	4	7
$f(x)$	2	4	8	16	128

Ans: 32.43

$\boxed{\text{II}}$ Newton Divided Difference formula

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0)$$

Example:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

find $f(8)$ & $f(15)$

448

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
4	48					
		52				
5	100		15			
		97		1		
7	294		21		0	
		202		1		0
10	900		27		0	
		310		1		
11	1210		33			
		404				
13	2028					

4.1 Curve Fitting:

4.2 fitting straight line by least square method

$$y = \cancel{ax+}\ a+bx$$

$$\boxed{\begin{aligned} &\Rightarrow \Sigma y = na + b\Sigma x \\ &\Rightarrow \Sigma xy = a\Sigma x + b\Sigma x^2 \end{aligned}}$$

→ fitting second degree parabola by least square method

$$y = a + bx + cx^2$$

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Qn: fid a (straight line) curve $y = a+bx$ for following data using least square method

x	1	2	3	4	6	8
y	2.4	3	3.6	4	5	6

x	y	xy	x^2
1	2.9	2.9	1
2	3	6	4
3	3.6	10.8	9
4	4	16	16
6	5	30	36
8	6	48	64
Σx	Σy	Σxy	Σx^2
24	24	113.2	130

Here, $n = 6$

$$6a + 24b = 24$$

$$24a + 130b = 113.2$$

$$(a, b) = 1.9765,\ 0.5059$$

$$\therefore\ y = 1.9765 + 0.5059x$$

Qn fit the curve $y = ab^x$ by using least square method to the following data & hence find y at $x = 8$

x	1	2	3	4	5	6	7
y	87	97	113	124	202	195	193

Soln:

$y = ab^x$

$\Rightarrow \log y = \log ab^x = \log a + \log b^x$

$\Rightarrow \log y = \log a + x \log b$

$Y = A + BX$

$\log y = Y$

$\log a = A$

$\log b = B$

$x = X$

$\sum Y = nA + B\sum X$

$\Rightarrow \sum XY = A\sum X + B\sum X^2$

$X = x$	y	$Y = \log y$	XY	X^2
1	87	4.4659	4.659	1
2	97	4.5747	9.149	4
3	113	4.7274	14.1822	9
4	129	4.8598	19.4392	16
5	202	5.3083	26.5413	25
6	195	5.2730	31.6380	36
7	193	5.2627	36.8388	49

$\Sigma X = 28$ $\quad \Sigma Y = 34.4718$ $\quad \Sigma XY = 142.2549$ $\quad \Sigma X^2 = 142$

$n = 7$

$7A + 28B = 34.4718 \longrightarrow (i)$

$28A + 142B = 142.2549 \longrightarrow (ii)$

$(A, B) = (4.3, 0.15598)$

$(a, b) = (e^A, e^B)$

$= (73.6998, 1.1688)$

$\therefore y = (73.6998)(1.1688)^x$

* What is Interpolation? Why is it needed?
* Derive Newtons forward difference interpolation formula for equidistant data points.
* describe the least square curve fitting procedure for straight line
* State any two properties of divided Difference
* Derive lagrange interpolation formula
* Derive Newton backward interpolation formula for equal spaced data.
* What is curve fitting. Explain the purpose of it
* Describe the least square curve fitting procedure for power function $f(x) = ax^c$
* Describe the least square curve fitting procedure for a polynomial of degree n

$$\boxed{y = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n}$$

curve fitting polynomial & ~~expo~~ এর (2) প্র

Interpolation: is a technique used to estimate the value of a function at a point within a known range of data points

Why it is needed

Data Estimation : It helps to find intermediate values between measured or known data points.

modeling & prediction : Creates a smooth curve or function for better analysis, simulation or prediction

□ Derivation of Newton forward interpolation :

Let $y = f(x)$ is a trancedental function, $y_0, y_1, y_2 \cdots y_n$ are value corresponding to point $x_0, x_0+h, x_0+2h \cdots$ we want to find $f(x) = y$ at point $x_0 + nh$,

$x = x_0 + uh$

Since, $y_n(x)$ is polynomial of nth degree

It may be written as

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$+ \cdots a_n(x - x_0) \cdots (x - x_{n-1})$$

y & $y_n(x)$ should agree at the set of tabulated points, we obtain

$$a_0 = y_0, \quad a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h},$$

$$a_2 = \frac{\Delta^2 y_0}{h^2 2!} \cdots a_n = \frac{\Delta^n y_0}{h^n n!}$$

setting,

$x = x_0 + uh$ & substituting for $a_0 \cdots a_n$

$$y_n(x) = y_0 + u\Delta y_0 + \frac{u(u-1)\Delta^2 y_0}{2!} + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \cdots \frac{u(u-1)(u-2)\cdots(u-n+1)}{n!}\Delta^n y_0$$

Newton backward:

If $x_0, x_1, \ldots x_n$ be a given set of observations with common difference h. and $y_0, y_1, y_2 \ldots y_n$ be the corresponding values where, $y = f(x)$ be the given function. We have to find the value of $f(x) = y$ at the point $x = x_n + uh$

Let us consider the n'th degree polynomial

$$f(x) = A_0 + A_1(x - x_n) + A_2(x - x_n)(x - x_{n-1}) + \ldots$$
$$A_n(x - x_n)(x - x_{n-1}) \ldots (x - x_0) \quad \text{---} (1)$$

substituting,

$x = x_n$ in (1) we get

$f(x_n) = A_0 \Rightarrow A_0 = y_n$.

Again substituting $x = x_{n-1}$ in (1) we get

$$A_1 = \frac{\Delta y_n}{1!\,h}$$

similarly $\cancel{A_n = \frac{\Delta y_n}{n!\,h^n}}$ $A_2 = \frac{\Delta^2 y_n}{2!\,h^2} \ldots A_n = \frac{\Delta^n y_n}{n!\,h^n}$

putting those values in (i).

$$f(x) = y_n + \frac{u \nabla y_n}{1!} + \frac{u(u+1)}{2!} \nabla^2 y_n + \cdots + \frac{u(u+1)(u+2)\cdots(u+n-1)}{n!} \times \nabla^n y_n$$

$\boxed{\square}$ curve fitting: Is a process of constructing a mathematical function (a curve) that best fits a set of data points, usually by minimizing the the differences between the data points and the curve

$\boxed{\square}$ purpose of curve fitting

(i) Data Representation: to provide a simplified mathematical model that represents the relationship between variables in the data

(ii) Prediction: to use the fitted curve to predict values for future data points.

(iii) Data interpolation: to estimate values within the range of observed data points

4. Pattern recognition: to identy ~~on~~ patterns in the data

5. Optimization: to optimize systems by understanding the relationship between variables.

III Least square curve fitting procedure for straight Line:

Let (x_i, y_i), $i = 1, 2, \dots n$ be the given data and

$$y = \cancel{bx}\ a + bx \quad \text{——— (i)}$$

is the straight line to be fitted.

Now, Error at $X = X_i$ is

$$E_i = y_i - Y_i$$

$$E_i = y_i - a - bX_i \qquad [\because Y_i = a + bX_i]$$

let,

$$E = \sum_{i=1}^{n} E_i^2$$

$$= \sum_{i=1}^{n} (y_i - a - bX_i)^2$$

By the method of least square ; E should be minimum.

For min value of E, $\frac{\partial E}{\partial a} = 0$ $\frac{\partial E}{\partial b} = 0$

Now,

$$\frac{\partial E}{\partial a} = 2\sum_{i=1}^{n}(y_i - a - bx_i)\cdot(-1) = 0$$

$$\Rightarrow \sum_{i=1}^{n}(y_i) - a\sum_{i=1}^{n}1 - b\sum_{i=1}^{n}x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n}y_i = an + b\sum_{i=1}^{n}x_i$$

$$\boxed{\Rightarrow \sum y = an + b\sum x} \quad \text{—— (i)}$$

again,

$$\frac{\partial E}{\partial b} = 2\sum_{i=1}^{n}(y_i - a - bx_i)(-x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n}(x_i y_i - ax_i - bx_i^2) = 0$$

$$\Rightarrow \sum_{i=1}^{n}x_i y_i = a\sum_{i=1}^{n}x_i + b\sum_{i=1}^{n}x_i^2$$

$$\Rightarrow \boxed{\sum xy = a\sum x + b\sum x^2} \quad \text{—— (ii)}$$

(ii) & (iii) called normal equation of a, b

On solving Normal equ. we get a & b. put these value of a & b in eq (1)

$$y = \cancel{bx}\ a + bx$$

we get best fit straight line to the given data

III. Lagrange interpolation derivation:

Let, $y = f(x)$ be a polynomial of the nth degree which takes the values $y_0, y_1, \ldots y_n$ corresponding the argument $x_0, x_1, \ldots x_n$ respectively

x	x_0	x_1	x_2	$x_3 \ldots$	x_n
$y = f(x)$	y_0	y_1	y_2	$y_3 \ldots$	y_n

Let us consider that

$$y = f(x) = (x-x_1)(x-x_2)\cdots(x-x_n)A_0 + (x-x_0)(x-x_2)\cdots(x-x_n)A_1$$
$$+ \cdots (x-x_0)(x-x_1)\cdots(x-x_{n-1})A_n \quad \text{--- (1)}$$

where $A_0, A_1 \ldots A_n$ are some arbitrary constant putting $x = x_0, x_1 \cdots x_n \longrightarrow (1)$

$$A_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$$

$$A_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}$$

$\vdots$

$$A_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}$$

putting the value of $A_0, A_1 \cdots A_n$ into (1)

$$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1)$$

$$+ \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} \times f(x_n)$$

* Ordinary differential equation [Euler's method + modified Euler method]
* tylor method
* Eylar modified
* simpson 1/3 & trapezoidal

[* Derive Euler method & Modified Euler's method for solution of ordinary differential equation

* Solve by Euler method the equation $\frac{dy}{dx} = x+y$, $y(0)=0$, choose $h=0.2$ and compute $y(0.6)$

V.Vg

* Major drawbacks of Tylor series method
* From the Tylor series for $y(x)$, find $y(0.1)$ correct to 4 decimal places if $y(x)$ satisfies $y' = x - y^2$ & $y(0)=1$

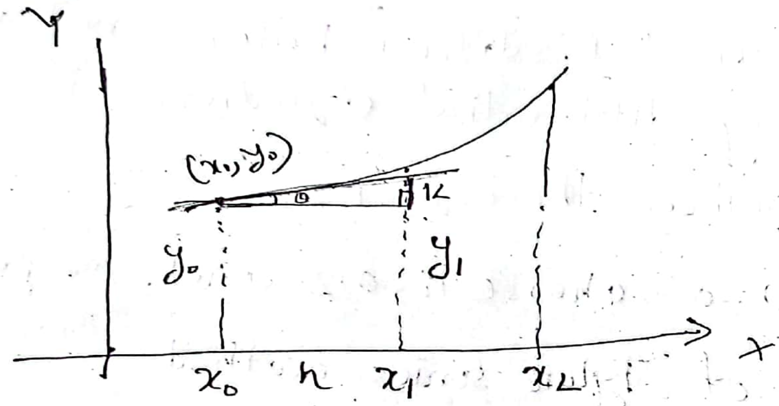
[* Determine the value of y using modified Euler's ~~method~~ formula when $x=0.1$ given that $y(0)=1$ and $y' = x^2+y$ and $h=0.05$.

* Derive trapezoidal rule for numerical integration
* Evalwate $\int_0^1 \frac{dx}{1+x^2}$ for $h=0.5$ & 0.125 using trapezoidal (correct to 3 decimal places).
* use simpson 1/3 rule to numerically evalute the integral $\int_0^4 f(x)dx$

Euler method

Suppose that first order ODE : $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$

slope at $(x_0, y_0) = \left\{ \frac{dy}{dx} \right\}_{(x_0,y_0)}$



$h = \frac{b-a}{n}$; $b = x_n$, $a = x_0$

$\tan\theta = f(x_0, y_0)$

$\Rightarrow \frac{K}{h} = f(x_0, y_0)$

$\Rightarrow K = h \cdot f(x_0, y_0)$

$x = x_0$; $y = y_0$

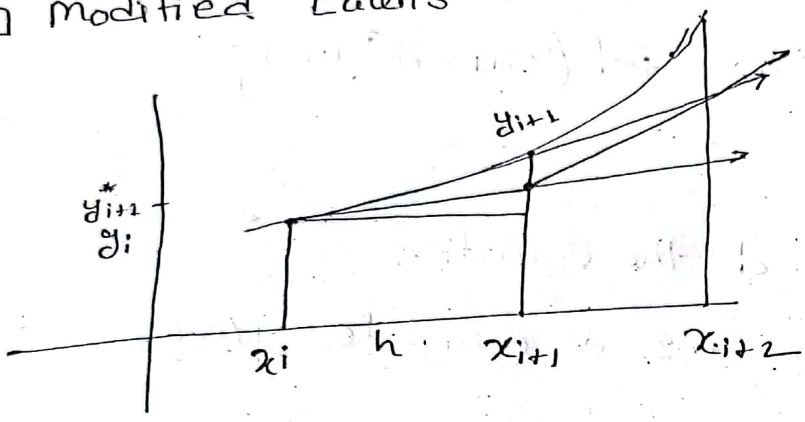
$x_1 = x_0 + h$ $\quad y_1 = y_0 + h \cdot f(x_0, y_0)$

$x_2 = x_1 + h$ $\quad y_2 = y_1 + hf(x_1, y_1)$

$\therefore x_n = x_{n-1} + h$

$\therefore y_n = y_{n-1} + h\, f(x_{n-1}, y_{n-1})$

[II] Modified Euler's Method



first order ODE : $\frac{dy}{dx} = f(x,y)$

slope at $(x_i, y_i) \Rightarrow y_i' = f(x_i, y_i)$

By Euler method,

$$y^*_{i+1} = y_i + f(x_i, y_i)h \longrightarrow \text{predicted equn}$$

correct the predicted equn

$$y'_{i+1} = f(x_{i+1}, y^*_{i+1})$$

Now, combine the two slopes to obtain an avg slope.

$$\overline{y'} = \frac{f(x_i, y_i) + f(x_{i+1}, y^*_{i+1})}{2}$$

By Euler's Method

$$y_{i+1} = y_i + \frac{h}{2}\{f(x_i, y_i) + f(x_{i+1}, y^*_{i+1})\}$$

Q. Solve by Euler method the equation

$\frac{dy}{dx} = x+y$; $y(0)=0$; $h=0.2$ & compute $y(0.6)$

Soln:

Here,

$x_0 = 0$ $\qquad$ $y_0 = 0$

$x_1 = x_0 + h$
$= 0+0.2$
$= 0.2$

$y_1 = y_0 + f(x_0, y_0)h$
$= y_0 + (x_0+y_0)h$
$= 0 + (0+0) \times 0.2 = 0$

$x_2 = x_1 + h$
$= 0.2+0.2$
$= 0.4$

$y_2 = y_1 + f(x_1, y_1)h$
$= y_1 + (x_1+y_1) \times h$
$= 0 + (0.2+0) \times 0.2$
$= 0.04$

$x_3 = x_2 + h$
$= 0.4+0.2$
$= 0.6$

$y_3 = y_2 + f(x_2, y_2)h$
$= 0.04 + (x_2+y_2) \times h$
$= 0.04 + (0.4+0.04) \times 0.2$
$= 0.128$ Ans

Q. Determine the value of y using modified Euler formula when $x=0.1$, given that $y(0)=1$ & $y'=x^2+y$; $h=0.05$

Soln:

Here, $\frac{dy}{dx}=x^2+y$; $x_0=0$, $y_0=0$, $h=0.05$

$x_0=0$

$x_1=x_0+h$
$=0+0.05$
$=0.05$

$y_1^*=y_0+h\times f(x_0,y_0)$
$=0+0.05\times(x_0^2+y_0)$
$=0+0.05\times(0^2+0)$
$=0$

$y_1=y_0+\frac{h}{2}\{f(x_0,y_0)+f(x_1,y_1^*)\}$
$=y_0+\frac{h}{2}\{(x_0^2+y_0)+(x_1^2+y_1^*)\}$
$=0+\frac{0.05}{2}\{(0^2+0)+(0.05)^2+0\}$

$=0.0000625$

$x_2=x_1+h$
$=0.05+0.05$
$=0.1$

$y_2^*=y_1+h\times f(x_1,y_1)$
$=y_1+h(x_1^2+y_1)$
$=0.000191$

$$y_2 = y_1 + \frac{h}{2}\{f(x_1, y_1) + f(x_2, y_2^*)\}$$

$$= y_1 + \frac{h}{2}\{(x_1^2 + y_1) + (x_2^2 + y_2^*)\}$$

$$= 0.00003813$$

□ Tylor Series:

consider the differential equation

$$y' = f(x, y)$$

initial condition,

$$y(x_0) = y_0$$

Taylor's series for $y(x)$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots$$

Ques^n: From the Tylor series for $y(x)$, find $y(0.1)$ correct to four decimal places if $y(x)$ satisfies

$$y' = x - y^2 \ \& \ y(0) = 1$$

Sol^n:

Here, $x_0 = 0$, $y_0 = 1$

Tylor series for $y(x)$

$$y(x) = y_0 + (x - x_0) y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \frac{(x-x_0)^3}{3!} y_0''' + \frac{(x-x_0)^4}{4!} y_0^{iv} + \cdots$$

$y' = x - y^2$ $\qquad y_0' = 0 - (1)^2 = -1$

$y'' = 1 - 2yy'$ $\qquad y_0'' = 1 - 2 \times 1 \times (-1) = 3$

$y''' = 0 - 2(yy'' + y'y')$ $\qquad y_0''' = -2\{1 \times 3 + (-1)^2\}$

$= -2\{y \cdot y'' + (y')^2\}$ $\qquad = -8$

$y^{iv} = -2(yy''' + y'' \cdot y' + 2y'y'')$ $\qquad y_0^{iv} = -2\{1 \times (-8) + 3(-1) + 2 \times (-1) \times 3\}$

$= 34$

$\therefore y(x) = y_0 + x y_0' + \frac{x^2}{2!} y_0'' + \frac{x^3}{3!} y_0''' + \frac{x^4}{4!} y_0^{iv}$

$= 1 + (-1) \times x + \frac{x^2}{2} 3 + (-8) \times \frac{x^3}{6} + 34 \times \frac{x^4}{24}$

$= 1 - x + \frac{3}{2} x^2 - \frac{4}{3} x^3 + \frac{17}{12} x^4$

$\therefore y(0.1) = 1 - (0.1) + \frac{3}{2}(0.1)^2 - \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4$

$= 0.9164$ Δ

Trapezoidal Rule

The definite integral over $[a, b]$ can be written as

$$I = \int_a^b f(x)\,dx$$

Let the interval $[a,b]$ be divided into n equal subin-tirval such that $a = x_0 < x_1 < x_2 < \cdots x_n = b$, $x_n = x_0 + nh$

Approximating $f(x)$ by Newto forward dif. formula

$$I = \int_{x_0}^{x_n} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \cdots \right] dx$$

since $x_i = x_0 + ph$,

$\therefore dx = h\,dp$

$$\therefore I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2}\Delta^2 y_0 + \cdots\right]dp$$

$$= h\int_0^n \left[py_0 + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \cdots\right]$$

$$= h\int_0^n p\left[\cancel{y_0 + \frac{\Delta y_0}{2} + \frac{p^2(2p-3)}{12}}\Delta^2 y_0 + \cdots\right]$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \cdots\right]$$

put $n = 1$

$$\int_{x_0}^{x_1} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right)$$

$$= \frac{h}{2}(y_0 + y_1) \quad [\Delta y_0 = y_1 - y_0]$$

next interval $[x_1, x_2]$

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}(y_1 + y_2)$$

$\vdots$

last interval $[x_{n-1}, x_n]$, $\int_{x_{n-1}}^{x_n} \frac{h}{2}(y_{n-1} + y_n)$

$$\therefore \int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[y_0 + y_n + 2(x_1 + \cdots x_{n-1})\right]$$

☐ geometric significane of this rule (Trapezoidal)

The geometric significance of this rule is that the curve $y=f(x)$ is replaced by ~~7~~ n straigh lines joining the points (x_0, y_0) & (x_1, y_1) ~~x~~ ... (x_n, y_n).

the area bounded the curve $y=f(x)$, the ordinates $x=x_0$ & $x=x_n$ & the x axis is then the equivalent to the sum of the areas of the n trapeziums obtained.

Qn: Evalwate $\int_0^1 \frac{dx}{1+x^2}$ for $h=0.5$ & 0.125 using trapezoidal

Solⁿ: Here, $a=0,\ b=1,\ h=0.5$ | $f(x)=\frac{1}{1+x^2}$

$\Rightarrow n = \frac{b-a}{0.5} = 2$

	x_0	x_1	x_2	~~x_3~~	~~x_4~~
x	0	0.5	1	~~1.5~~	~~2~~
$f(x)$	1	0.8	0.5	~~0.307~~	~~0.2~~

according to trapezoidal formula,

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{h}{2}\left\{y_0+y_2+2\left(y_1 \cancel{+y_2+y_3}\right)\right\}$$

$$= \frac{0.5}{2}\left\{1+\overset{0.5}{\cancel{0.2}}+2\left(0.8 \cancel{+0.5+0.307+}\right)\right\}$$

$$= \cancel{1.1035}\ 0.775$$

for $h = 0.125$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	1
$f(x)$	1	0.985	0.941	0.877	0.8	0.719	0.64	0.5
	y_0							y_7

$$\therefore \int_0^1 \frac{1}{1+x^2}dx = \frac{h}{2}\left(y_0+y_7+2\left(y_1+y_2+y_3+y_4+y_5+y_6\right)\right)$$

$$= 0.7765$$

* use simpson 1/3 rule to numerically evalute the integral

$$\int_0^4 f(x)\,dx$$

Soln: ~~a = 0, b = 4 let, n = 4~~

~~$\therefore h = \frac{b-a}{n} = \frac{4}{4} = 1$~~

$$\int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \cdots\right]$$

put $n = 2$

$$\int_{x_0}^{x_2} y\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right)$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

similarly $\int_{x_1}^{x_4} y\,dx = \frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$

$$\therefore \int_{x_0}^{x_4} y\,dx = \frac{h}{3}\left\{4(y_1 + y_3) + 2y_2 + (y_0 + y_4)\right\}$$