Lecture-1 Date: 7.07.21

Numerical method → Approximate result

• Numerical Methods :

→ Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithematic operations.

→ Numerical methods involve large numbers of tedious arithmetic calculations.

→ These methods have gained popularity due to the advancements in efficient computational tools such as digital computers and calculators.

• Noncomputer methods

Analytical or exact methods

Graphical solutions used to solve complex problems but the result are not very precise. They are extremly tedious without computers → limited problems.

• Analytical vs. Numerical methods

Examples : Analytical methods

- Differentiation

$$\frac{dy}{dx}\left(x^2 - \sin(x)\right) = 2x - \cos(x)$$

- Integration

$$\int x^3 + x - e^x = \frac{x^4}{4} + \frac{x^2}{2} - e^x + c$$

- Root(s) of an Equation

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Problem description
Mathematical model
sol^n of math. model
using the solution

Need for Numerical Methods

• In general, there are few analytical (closed-form) solutions for many practical engineering problems.

• Numerical methods can handle :

→ Large systems of equations

→ Non-linearity

→ complicated geometries that are common in engineering practice and that are often impossible to solve automatically.

Examples :

$$F = \int_0^{30} \left(\frac{\cos(z)+z}{5+z} \right) e^{-2z/30}\, dz \qquad \frac{x}{1+\sin(x)} + e^x = 0$$

• Reasons to study numerical Analysis

→ powerful problem solving techniques and can be used to handle large systems of equations

→ Enables to use the commercial software packages as well as designing algo.

→ Efficient vehicles in learning to use computers

→ Reinforce our understanding of mathematics ; where it reduces higher math to basic arithmetic operation.

• Mathematical Model

→ A formulation or equation that expresses the essential features of a physical system or process in mathematical terms.

→ Generally, It can be represented as a functional relationship of the form

Dependent variable = f (independent variable , parameters , forcing function)

Dependent variable =	A characteristic that usually reflects the behavior or state of the system
Independent variable =	Are usually dimensions, such as time and space
parameters =	Are reflective of system's properties or composition
Forcing functions =	Are external influences acting on the system

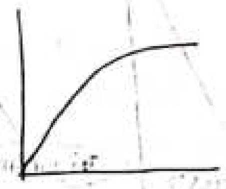
• Typical characteristics of Math. Model

→ It describes a natural process or system of mathematical way ✓

→ It represents the idealization and simplification of reality.

→ It yields reproducible results purpose.

$$V(t) = \frac{gm}{c}\left[1 - e^{-(c/m)t}\right]$$



$m = 68.1$

$c = 12.5$

$v =$

Numerical solution to Newton's second law

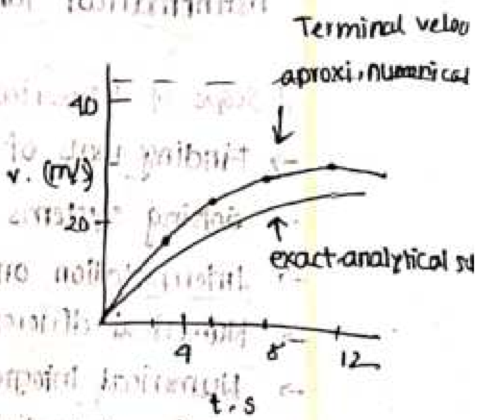
• Numerical solution : approximates the exact solution by ~~ari~~ arithematic operation

• Suppose $\frac{dv}{dt} \approx \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$

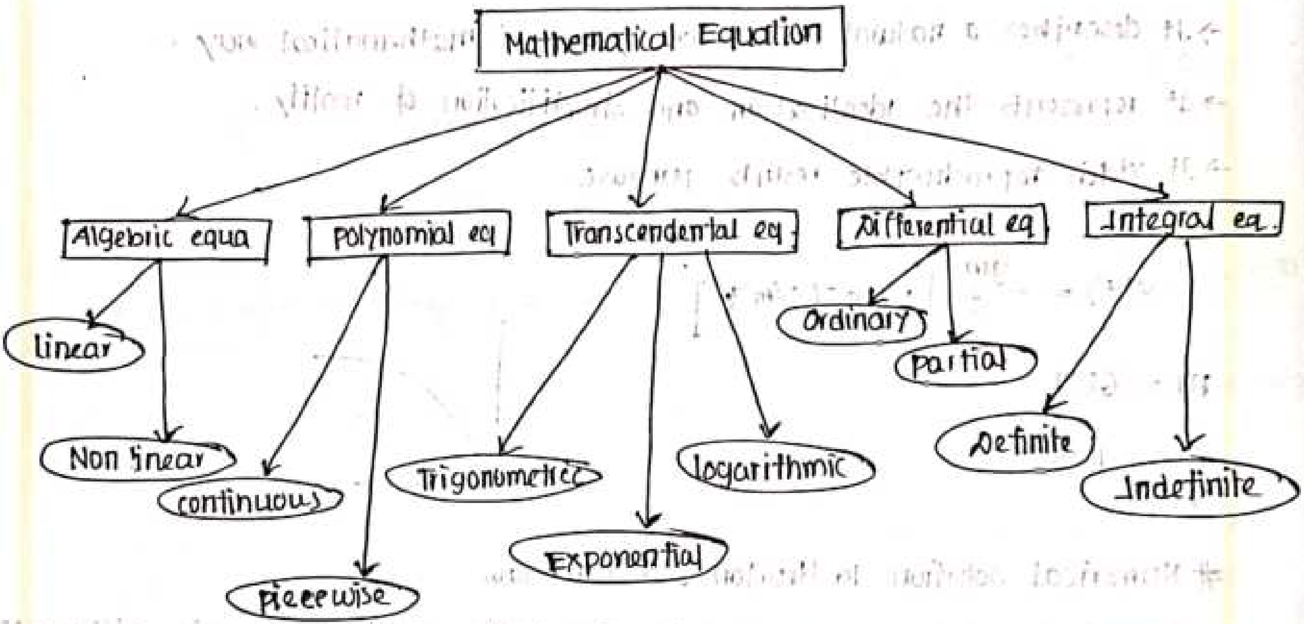
$$\left| \frac{dv}{dt} = g - \frac{c}{m} v \right| \Rightarrow \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c}{m} v$$

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c}{m} v(t_i)\right](t_{i+1} - t_i) \quad (1)$$

New Value = old value + slope × step size



Different forms of mathematical equations:



Advantages of Numerical Methods.

→ A numerical value can be obtained even when the problem has no analytical solution.

→ The mathematical operations required are essentially addition, subtraction, multiplication, and division plus making comparisons.

arithmetic

→ It is important to realize that solution by numerical analysis is always numerical

→ Analytical methods, on the other hand usually give a result in terms of mathematical functions that can then be evaluated for specific instances.

Scope of Numerical Analysis

→ Finding roots of equations

→ Solving systems of linear algebric equations

→ Interpolation and regression analysis

→ Numerical differentiation

→ Numerical Integration

→ Solution of ordinary differential equations (ODE)

→ Boundary value problem

→ Solution of matrix problem

steps of solving a practical problem :

step #1:

- State the problem clearly, including any simplifying assumptions.

step #2:

- Develop a mathematical statement of the problem in a form that can be solved for a numerical answer.
- This process may involve the use of calculus.
- In some situations, other mathematical procedures may be employed.
- When this statement is a ~~different~~ differential equation, appropriate initial conditions and/or boundary conditions must be specified.

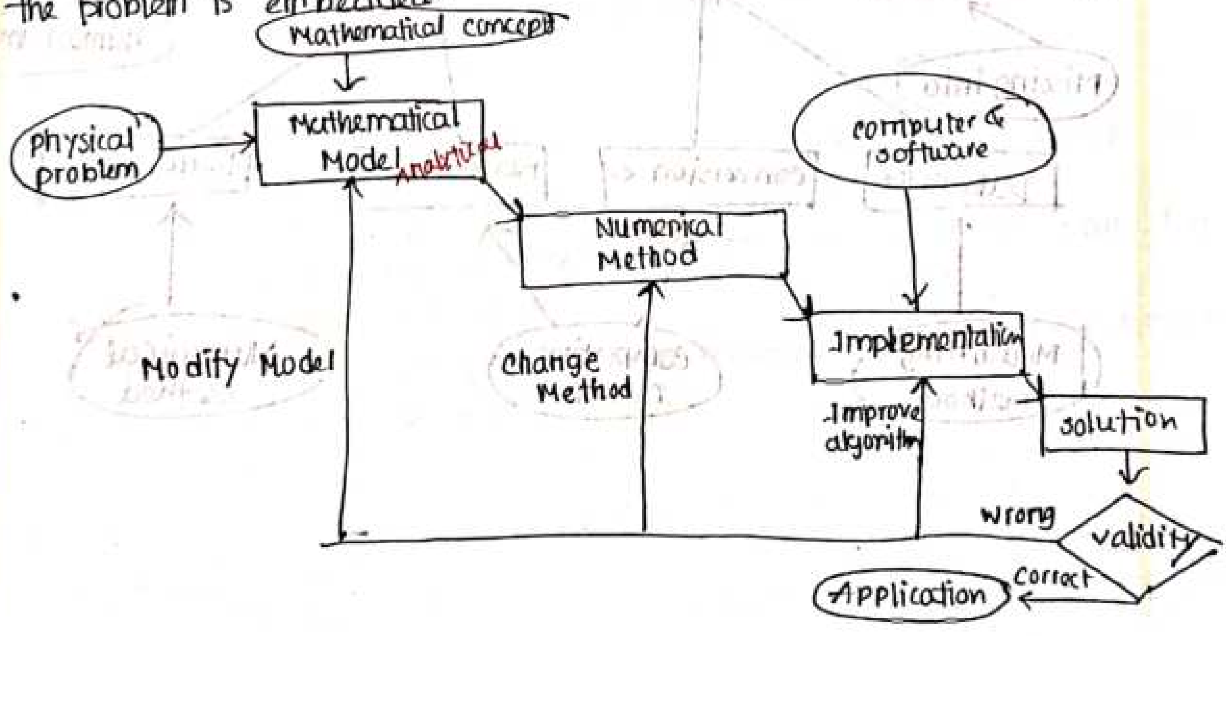
step #3:

- Solve the equations that are obtained from step #2
- Sometimes the method will be algebric
- But frequently more advanced methods will be needed
- The result of this step is a numerical answer or set of answers

step #4:

- Interpret the numerical result to arrive at a decision.
- This will require experience and understanding of the situation in which the problem is embedded.

Numerical computing Process



Accuracy in Numerical Analysis

• Numerical analysis is an approximation, but results can be made as accurately as desired.

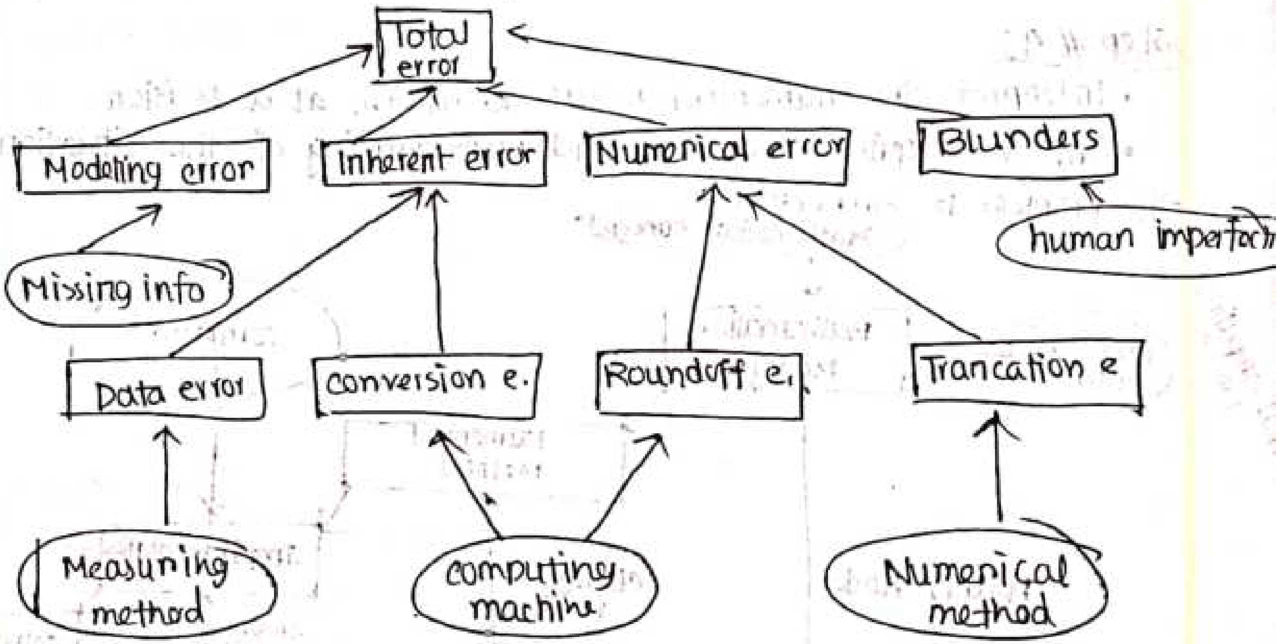
• Errors come in a variety forms and sizes ; some are avoidable and some are not.

• For example, data conversion and roundoff errors can not be avoided, but human errors can be eliminated.

• Although certains errors can not be eliminated complettly, we must atleast know the bounds of these errors to make use of our final selection.

• It is therefore essential to know that how errors arise, how they grow during numerical process and how they ~~effect~~ affect the accuracy of a solution.

Taxonomy of errors



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Modelling error : • force acting on a falling body, air resistance coefficient (drag coefficient) cannot able to estimate properly / determine the direction and magnitude of wind ~~model~~ force

This model is
• basic input to the numerical process

Inherent error : also known as input error / contain two components
• data errors
• conversion errors

Data error : also k.a - emperical error / aries when data for a problem are obtained by some experimental means and are, therefore, of limited accuracy and precision.

(2)

conversion error : also known as representational error / aries due to the limitations of the computer to store the data exactly

Numerical errors : procedural error ;/ introduced during the process of implementation of a numerical method

Roundoff error : when a fixed number of digits are used to represent exact numbers. 42.7893 → 42.79

Trancation error : arise from using an approximation in place of an exact mathematical procedure
• Typically it is the error resulting from the trancation of numerical process.
• We often use finite number of terms to estimate the sum of infinite series.

Blunders : are the errors that are caused due to human imperfection.

- Some common type of this error are :

→ lack of understanding of the problem

→ wrong assumption

→ Overlooking of some basic assumptions required for formulating the model

→ ~~Error in~~ selecting a wrong numerical method for solving the mathematical model

→ selecting a wrong algorithm for implementing the numerical method

→ Making mistakes in the computer program / mistake in data input

→ wrong guessing the initial value.

Significant Digits

- All non-zero digits are significant
- All zeros occuring between non-zero digits are significant digits.
- Trailing zero 3·50, 65·0, 0·230
- Zeros between the decimal point and preceeding a non-zero digit are not significant

  0·0001234 (1234×10^{-7}) 1234 – 4 sig.dig

- When the decimal point is not written, trailing zeros are not considered to be significant, 5600 (56×10^2) has two significant digit.

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$0.0459 \rightarrow 3$

$4.590 \rightarrow 4$

$4008 \rightarrow 4$

$4008.0 \rightarrow 5$

$1.079 \times 10^3 \rightarrow 4$

$1.0790 \times 10^3 \rightarrow 5$

$1.07900 \times 10^3 \rightarrow 6$

$\boxed{1,000,000 = 1 \times 10^6 \rightarrow 1}$

$1,079,587 = 1.079587 \times 10^6 \rightarrow 7$

Relation between accuracy and precision

- Accuracy: refers to the number of significant digits in a value. 57.396 is accurate to five significant digits.

- Precision: refers to the number of decimal positions, i.e. the order of magnitude of the last digit in a value. 57.396 has a precision of 0.001 or 10^{-3}.

Error:

An error is defined as the difference between the actual value and the approximate value obtained from the experimental observation or from numerical computation.

$$\text{Error} = \text{Actual value} - \text{Approximate value}$$

$$= x - \bar{x}$$

$$= x - x_a$$

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Types of Errors

- Absolute error
- Relative error
- Percentage error

(1) Absolute error :

If x is the true value of a quantity and $\bar{x}$ is its approximate value, then Absolute error is denoted by E_a :

$$E_a = |\text{Exact value} - \text{Approximate value}|$$

$$= |x - \bar{x}|$$

(2) Relative error :

The relative error is defined by ,

$$E_r = \left| \frac{\text{Exact value} - \text{Approximate value}}{\text{Exact value}} \right|$$

$$E_r = \left| \frac{x - \bar{x}}{x} \right|$$

(3) percentage error :

$$E_p = E_r \times 100$$

$$= \left| \frac{x - \bar{x}}{x} \right| \times 100\%$$

Sources of errors

(1) Inherent error → which are already present in the statement of problem before its sol^n is obtained

(2) Truncation error → are caused by using approx results or on replacing an infinite process by a finite one

(3) Rounding-off error → arises from the process of rounding off the numbers during the computation.

Example-1 : Find error and relative error in the following cases :

(a) $x = 3.141592$, $\bar{x} = 3.14$

Solution :

(a) $x = 3.141592$

$\bar{x} = 3.14$

Absolute error : $E_a = |x - \bar{x}|$

$= |3.141592 - 3.14|$

$= 0.001592$

Relative error : $E_r = \dfrac{x - \bar{x}}{x}$

$= \left| \dfrac{3.141592 - 3.14}{3.141592} \right|$

$= 0.0005067$

$= \cancel{0}\ 5.067 \times 10^{-4}$

Example-2: Find the relative error is the computation of $x-y$ for $x = 12.05$ and $y = 8.02$ having absolute error $\delta x = 0.005$, $\delta y = 0.001$

Solution:

$x = 12.05$

$y = 8.02$

$\delta x = 0.005$

$\delta y = 0.001$

Relative error in x

$$= \frac{\text{absolute error in } x}{x} \quad \left[E_r = \left|\frac{x-\bar{x}}{x}\right|\right]$$

$$= \frac{\delta x}{x} = \frac{0.005}{12.05}$$

$$= 4.15 \times 10^{-4}$$

$$\frac{\delta y}{y} = \frac{0.001}{8.02} = 1.25 \times 10^{-4}$$

$\therefore$ Relative error in $x-y = 4.15 \times 10^{-4} - 1.25 \times 10^{-4} = 2.903 \times 10^{-4}$

Example 3: Find (i) A.E (ii) R.E (iii) P.E, if $2/3$ is approximate to four significant digits.

Solution: $x = \frac{2}{3}$, $\bar{x} = 0.6666$

$$A.E = |\overset{2/3}{x} - \overset{\bar{x}}{\bar{x}}| = 6.6666 \times 10^{-5}$$

$$R.E = \frac{6.6666 \times 10^{-5}}{\frac{2}{3}}$$

$$= 0.1 \times 10^{-3}$$

$$P.E = 0.01$$

Example-4: The solution of a problem is given as 3.436 (approximate value). It is known that the absolute error in the solution is less than 0.01. Find the interval within which the exact value must lie.

Solution :

Here , $\bar{x} = 3.436$

$|x - \bar{x}| < 0.01$

$\therefore |x - 3.436| < 0.01$

$\therefore -0.01 < x - 3.436 < 0.01$

$\therefore -0.01 + 3.436 < x < 0.01 + 3.436$

$\therefore 3.426 < x < 3.446$

Example 5 : Given the solution of a problem $x_a = 35.25$ (approx.) with relative error in the solution at most 2% (0.02). Find , to four decimal digits , the range of values within which the exact value of the solution must lie.

Solution : $\bar{x} = 35.25$

$$\left| \frac{x - \bar{x}}{x} \right| < 0.02$$

$$\text{or, } \left| \frac{x - 32.25}{x} \right| < 0.02$$

$$\text{or, } -0.02 < \frac{x - 35.25}{x} < 0.02$$

$$\text{or, } -0.02 < \frac{x - 35.25}{x}$$

or, $-0.02x < x - 35.25$

or, $35.25 < x + 0.02x$

or, $35.25 < 1.02x$

$$\therefore \frac{35.25}{1.02} < x$$

$\therefore 34.5588 < x$

$$\frac{x - 35.25}{x} < 0.02$$

$\therefore x - 35.25 < 0.02x$

$\therefore x - 0.02x < 35.25$

$\therefore 0.98x < 35.25$

$$\therefore x < \frac{35.25}{0.98}$$

$\therefore x < 35.9694$

Solution : $34.5588 < x < 35.9694$

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~~Steps~~ of problem solving:

1. Data Analysis
2. Designing ~~to~~ of Mathematical model
3. computer simulation and post processing or graphic ~~resource~~ result
4. validation/verification
5. Implementation

Alhamdulillah
End of lecture 1
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Lecture-2

Date: 22.07.21

Theorem 1: The maximum absolute error $|x-\bar{x}|$ of an algebric sum or difference of several approximate numbers does not exceed to the sum of absolute error of the numbers, that is, if x_i $(i=1,2,\cdots,n)$ be the n approximate numbers and u is their algebric sum or difference, then

$$\Delta u \leq \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n$$

we prove the results for two numbers:

(a) For sum:

• Let n_1, n_2 be approximate numbers to N_1, N_2 with errors E_1, E_2 so that $N_1 = n_1 + E_1$, $N_2 = n_2 + E_2$, then $N_1 + N_2 = (n_1 + E_1) + (n_2 + E_2)$

$$= (n_1 + n_2) + (E_1 + E_2)$$

Let $N_1 + N_2 = N$, $n_1 + n_2 = n$ and $N = n + E$, then $N = n + (E_1 + E_2)$

$N - n = E_1 + E_2$

$E = E_1 + E_2$

or, $\boxed{|E| = |E_1 + E_2|}$

or, $\underline{|E| \leq |E_1| + |E_2|}$ ——— (A)

(b) For difference:

$$N_1 - N_2 = (n_1 - n_2) + (E_1 - E_2)$$

let $N_1 - N_2 = N$, $n_1 - n_2 = n$ and $N = n + E$,

then, $N = n + (E_1 - E_2)$

$$N - n = E_1 - E_2$$

$$E = E_1 + (-E_2)$$

$$|E| \leq |E_1| + |-E_2|$$

$$|E| \leq |E_1| + |E_2| \quad \text{——— (B)}$$

(A) and (B) show that the absolute errors in the sum or difference of two numbers does not exceed the sum of their absolute errors. Similarly the result can be extended for three and more approximate numbers.

Theorem 2: The maximum relative error for both multiplication and division does not exceed to the algebric sum of their relative errors.

(a) For Multiplication: Let n_1, n_2 be approximate numbers to N_1, N_2 with errors E_1, E_2, so that $N_1 N_2 = N$, $n_1 n_2 = n$, $N = n + E$. Then,

$$N_1 N_2 = (n_1 + E_1)(n_2 + E_2)$$

$$= n_1 n_2 + E_1 n_2 + n_1 E_2 + E_1 E_2$$

$$N_1 N_2 - n_1 n_2 = E_1 n_2 + E_2 n_1 + E_1 E_2$$

Neglecting the second order term $E_1 E_2$ and dividing both sides by $n_1 n_2$ i.e. n, one gets,

$$\frac{E}{n} = \frac{E_1}{n_1} + \frac{E_2}{n_2}$$

$$\Rightarrow \left|\frac{E}{n}\right| \leq \left|\frac{E_1}{n_1}\right| + \left|\frac{E_2}{n_2}\right| \quad \text{——— (I)}$$

(b) For division : let $\frac{N_1}{N_2} = N$; $\frac{n_1}{n_2} = n$ and $N - n = E$, then

$$\frac{N_1}{N_2} = \frac{(n_1 + E_1)}{(n_2 + E_2)} = \frac{n_1}{n_2}\left(1 + \frac{E_1}{n_1}\right)\left(1 + \frac{E_2}{n_2}\right)^{-1}$$

$$N \cong n\left(1 + \frac{E_1}{n_1}\right)\left(1 - \frac{E_2}{n_2}\right) = n\left(1 + \frac{E_1}{n_1} - \frac{E_2}{n_2}\right)$$

where we have used the binomial theorem and neglected the second the higher order terms (assuming they are small). Then

$$\frac{N-n}{n} \equiv \frac{E}{n} = \frac{E_1}{n_1} - \frac{E_2}{n_2}$$

$$\Rightarrow \left|\frac{E}{n}\right| = \left|\frac{E_1}{n_1} + \left(-\frac{E_2}{n_2}\right)\right| \leq \left|\frac{E_1}{n_1}\right| + \left|\frac{E_2}{n_2}\right| \quad \text{——— (II)}$$

(I) and (II) shows that relative error in the product of two approximate numbers is always less than the algebric sum of their relative errors.

Example 1.3 If the number X is rounded to N decimal places, then

$$\boxed{\Delta X = \frac{1}{2}(10^{-N})}$$

relative error of a number

If $X = 0.51$ and is correct to 2-decimal places, then $\Delta X = 0.005$. and then relative accuracy is given by $0.005/0.51 = 0.98\%$

Example 1.4 : Three approximation values of the number $1/3$ are given as 0.30, 0.33, 0.34. which of these three is the best approximation ? we have

$$\left|\frac{1}{3} - 0.30\right| = \frac{1}{30}$$

$$\left|\frac{1}{3} - 0.33\right| = \frac{0.01}{3} = \frac{1}{300}$$

least number is the best Approximation

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$$\left|\frac{1}{3} - 0.34\right| = \frac{0.02}{3} = \frac{1}{150}$$

It follows that 0.33 is the best approximation for $1/3$.

Example 1·5 : Find the relative error of the number 8.6 if both of its digits are correct

$\frac{1}{2} \times 10^{-1} = 0.05$

Here, $E_A = 0.05$

Hence, $E_R = \frac{0.05}{8.6} = 0.0058$

Example 1·6 : Evaluate the sum $S = \sqrt{3} + \sqrt{5} + \sqrt{7}$ to 4 significant digits ~~digests~~ and find its absolute and relative errors.

we have

$-\frac{1}{2} \times 10^{-3} + \frac{1}{2} \times 10^{-3} + \frac{1}{2} \times 10^{-3}$

$\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$, $\sqrt{7} = 2.646$

Hence $S = 6.614$ Then

$$E_A = 0.0005 + 0.0005 + 0.0005 = 0.0015$$

The total absolute error shows that the sum is correct to 3 significant figures only. Hence we take $S = 6.61$ and then

$$E_R = \frac{0.0015}{6.61} = 0.0002$$

Mathematical Preliminaries :

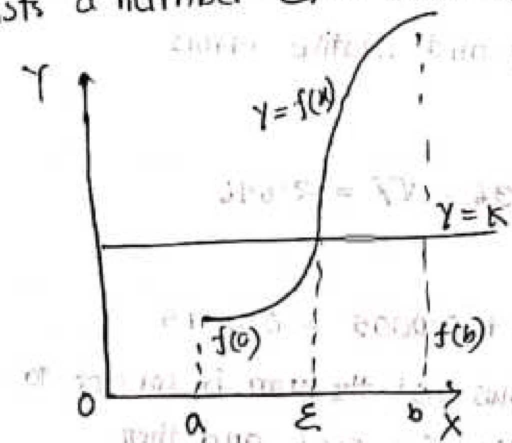
Theorem 1·1 : If $f(x)$ is continuous in $a \leq x \leq b$, and if $f(a)$ and $f(b)$ are of opposite signs, then $f(\xi) = 0$ for at least one number ξ such that $a < \xi < b$.

(eg)

Theorem 1·2 : (Rolle's theorem) If $f(x)$ is continuous in $a \le x \le b$, $f'(x)$ exists in $a < x < b$ and $f(a) = f(b) = 0$, then there exists at least one value of x, say ξ, such that $f'(\xi) = 0$, $a < \xi < b$.

Theorem 1·3 : (Generalized Rolle's Theorem) Let $f(x)$ be a function which is n times differentiable on $[a,b]$. If $f(x)$ vanishes at the $(n+1)$ distinct points $x_0, x_1, \cdots, x_n$ in (a,b), then there exists a number ξ in (a,b) such that $f^{(n)}(\xi) = 0$.

Theorem 1·4 : (Intermediate value Theorem) Let $f(x)$ be continuous in $[a,b]$ and let k be any number between $f(a)$ and $f(b)$. Then there exists a number ξ in (a,b) such that $f(\xi) = k$



For derivative

Theorem 1·5 (Mean-value theorem for derivates) If $f(x)$ is continuous in $[a,b]$ and $f'(x)$ exists in (a,b), then there exists at least one value of x, say ξ, between a and b such that

$$\boxed{f'(\xi) = \frac{f(b)-f(a)}{b-a}}, \quad a < \xi < b.$$

Setting $b = a+h$, this theorem takes the form

$$f(a+h) = f(a) + hf'(a+\theta h), \quad 0 < \theta < 1.$$

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Theorem 1·6 (Taylor's series for a function of one variable) If $f(x)$ is continuous and possess continuous derivatives of order n in an interval that includes $x = a$, then in that interval

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n(x),$$

Reminder

where $R_n(x)$, the ~~reminder~~ ~~remember~~ remainder term, can be expressed in the form

$$R_n(x) = \frac{(x-a)^n}{n!} f^{(n)}(\varepsilon), \quad a < \varepsilon < x$$

Roots of Equation

- Graphical method
- Incremental search
- Bracketing method
  - → Bisection Method
  - → False - Position method
- open method
  - → Newton Raphson method
  - → secant method

Numerical method

- Analytical
- Graphical
- Numerical

Equation of one variable can be formulated as

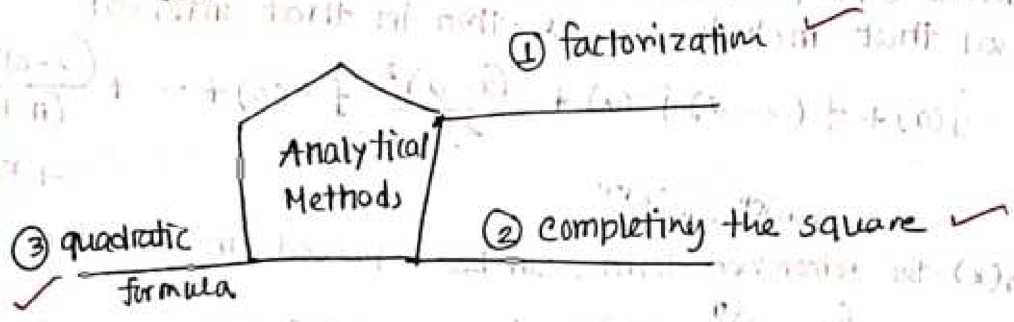
$$f(x) = 0 \quad \text{———} (1)$$

- Eq. (1) can be in the form of linear and nonlinear.
- solving equation (1) means that finding the values of x that satisfying equation (1)

- $4x - 3x^2y - 15 = 0 \longrightarrow$ algebric
- $x^2 + 2x - 4 = 0 \longrightarrow$ polynomial
- $\sin(2x) - 3x = 0 \longrightarrow$ Transcendental

Finding Roots for quadratic Equations

$f(x) = ax^2 + bx + c$



- Solution obtained by using analytical methods is called exact solution.
- Solution that obtained by using numerical methods is called numerical solution.

Three types of numerical methods can be used to find roots of eq.

(1) Incremental search (1 initial guess)

(2) Bracketing Method (2 initial)

- Bisection method
- False Position method

(3) Open method (1 initial guess)

- Newton Raphson method
- Secant method

→ Prior to the numerical methods, a graphical method of finding roots of the equations are presented.

- Incremental Search:

→ Incremental search is a technique of calculating $f(x)$ for incrementing values of x over the interval where the root lies.

→ It starts with an initial value, x_0.

→ The next value x_n for $n = 1, 2, 3, \ldots$, is calculate by using

$$x_n = x_{n-1} + h$$

where h is referred to a step size.

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→ If the sign of two f(x) changes or if

$$f(x_n) \cdot f(x_{n-1}) < 0$$

then the root exist over the prescribed interval of the lower bound, x_l and upper bound, x_u.

→ The root is estimated by using

$$\boxed{x_r = \frac{x_l + x_u}{2}}$$

Example 6 : Find the first root of $f(x) = 4.15x^2 - 16x + 8$ by using incremental search. start the procedure with the initial value, $x_0 = 0$ and step size, $h = 0.1$. perform three iterations of the increment search to achieve the best approximate root.

Solution :

start the estimation with initial value $x_0 = 0$ and step size, $h = 0.1$.

x	f(x)
0	8
0.1	6.4415
0.2	4.966
0.3	3.5735
0.4	2.264
0.5	1.0375
0.6	-0.106

$$f(0.5) \cdot f(0.6) < 0$$

$$x_r = \frac{0.5 + 0.6}{2} = 0.55$$

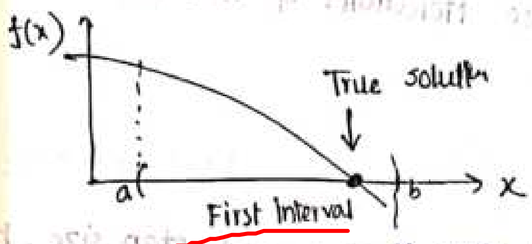
- Bracketing methods

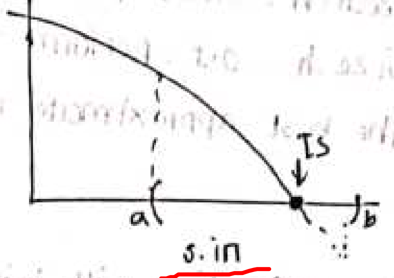
→ guessing an interval containing the root(s) of a function.

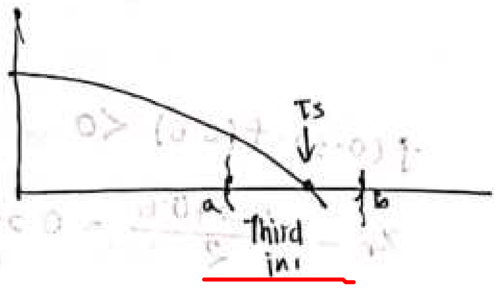
→ starting point of the interval is a lower bound, x_l. End point of the interval is an upper bound, x_u.

→ By using bracketing methods, the interval will split into two subintervals and the size of the interval is successively reduced to a smaller interval.

→ The subintervals will reduce the range of intervals until its distance is less than the desired accuracy of the solution.







→ Bracketing methods always converge to the true solution.

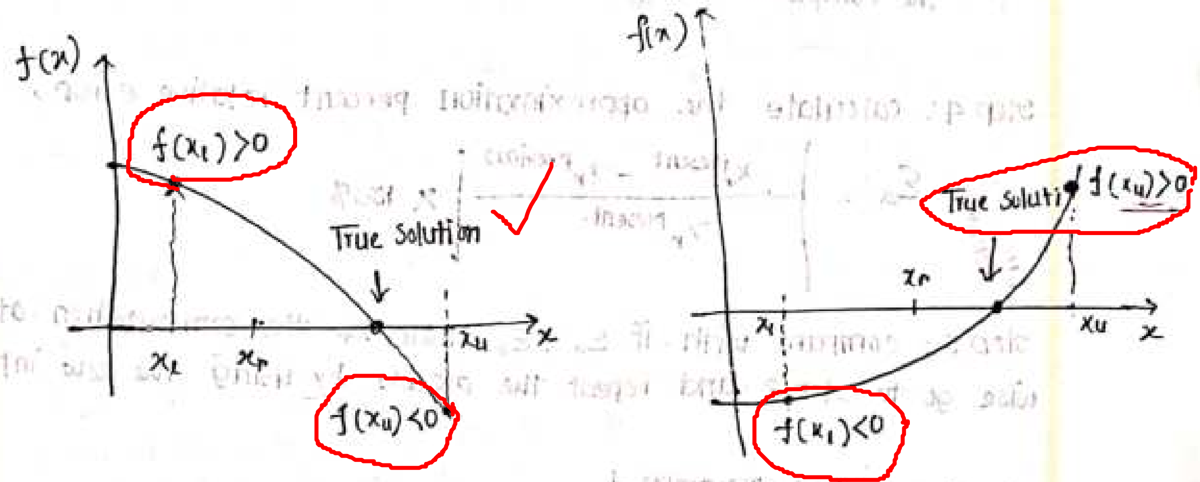
→ There are two types bracketing methods; bisection method and false position method.

Date: 23.07.21

Bisection method:

- Bisection method is the simplest bracketing method.
- The lower value, x_l and upper value, x_u which bracket the root(s) are required.
- The procedure starts by finding the interval $[x_l, x_u]$ where the solution exist.
- As shown in fig, at least one root exist in the interval $[x_l, x_u]$ if

$$\boxed{f(x_l) \cdot f(x_u) < 0}$$



Solution of $f(x) = 0$

Algorithm:

For the continuous eq. of one variable, $f(x) = 0$.

step 1: choose the lower guess, x_l and the upper bound, x_u that bracket the root such that the function has opposite sign over the interval. $x_l \leq x \leq x_u$.

step 2: The estimation root, x_r is computed by using

$$\boxed{x_r = \frac{x_l + x_u}{2}}$$

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Step 3: Use the following evaluations to identify the subinterval that the root lies.

✓ if $f(x_l) \cdot f(x_u) < 0$, then the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and repeat step 2.

✓ if $f(x_l) \cdot f(x_u) > 0$, upper, $x_l = x_r$ and repeat step 2.

✓ if $f(x_l) \cdot f(x_u) = 0$, then the root is equal to x_r. Terminate the computation.

Step 4: calculate the approximation percent relative error,

Error of tolerance

$$\varepsilon_a = \left| \frac{x_r^{present} - x_r^{previous}}{x_r^{present}} \right| \times 100\%$$

step 5: compare with. if $\varepsilon_a < \varepsilon_s$, then stop the computation. Otherwise go to step 2 and repeat the process by using the new interval.

Bisection Method : Example 1

Find the root of the equation $x^3 + 4x^2 - 1 = 0$.

Solution :

Let, $a = 0$ and $b = 1$

Now, $f(0) = (0)^3 + 4(0)^2 - 1$
$= -1 < 0$ and

$f(1) = (1)^3 + 4(1)^2 - 1 = 4 > 0$

i.e. $f(a)$ and $f(b)$ has opposite signs.

Therefore, $f(x)$ has a root in the interval $[a, b] = [0, 1]$

$x_c = (0+1)/2 = 0.5$,

$f(0.5) = 0.125$ Now $f(a)$ and $f(x_c)$ has opposite signs

so, the next interval is $[0, 0.5]$

a	b	$x_c = (a+b)/2$	$f(a)$	$f(b)$	$f(x_c)$
0	1	0.5	-1	1	0.125
0	0.5	0.25	-1	0.125	-0.73938
0.25	0.5	0.375	-0.73938	0.125	-0.38477
0.375	0.5	0.4375	-0.38477	0.125	-0.15063
0.4375	0.5	0.46875	-0.15063	0.125	-0.0181
0.46875	0.5	0.484375	-0.0181	0.125	0.05212
0.46875	0.484375	0.476563	-0.0181	0.05212	0.01668

Advantages

- simple and easy to implement
- One function evaluation per iteration
- The size of the interval containing the 0 is reduced by 50% after each iteration
- The no. of iterations can be determined a priori
- No knowledge of the derivative is needed
- The function ~~doesn~~ does not have to be differentiable.

Disadvantages

- slow to converge
- Good intermediate approximation may be discarded.
- Need two initial guesses a and b which bracket the root
- It is among the slowest methods to find the root
- When an interval contains more than one root, the bisection method can find only one of them.

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Class Work :

(1) Find the real root of the equation $f(x) = x^3 - x - 1 = 0$ correct to 2 decimal places ($\varepsilon = 0.01$).

Answer : 1.328125

(2) Find the real root of the eq. $f(x) = x^4 - \cos(x) + x = 0$ correct to 2 decimal places. ($\varepsilon = 0.01$)

Answer : 0.637695

Iteration Method :

- Let, we have an equation in the form $g(x) = 0$
- Rewrite the eq. in the form $x = f(x)$
- Start with an initial guess x_0, which is an approximation of the root.
- Calculate $x_1, \dots x_n, \dots$ such that

$$x_1 = f(x_0)$$
$$x_2 = f(x_1)$$
$$x_3 = f(x_2)$$

- Iterate the same ~~opt~~ process until $(x_n - x_{n-1})$ smaller than some specified tolerance.
- Geometrically, where the two graphs, x and $f(x)$ intersects, that the real roots of the equation.

Date : 24.07.21

Iteration Method : convergence conditions

• Any arbitrary approximation x_0, x_1, x_2 does not assure that it will converge to the actual root x of the equation .

• E.g. $x = 10^x + 1$

• if $x_0 = 0$, $x_1 = 2$, $x_2 = 101$, ..., that does not converge to the actual root x

• As n increase, x_n increase without limit !

The equation $x = f(x)$ ~~increase~~ converges to the real root x,

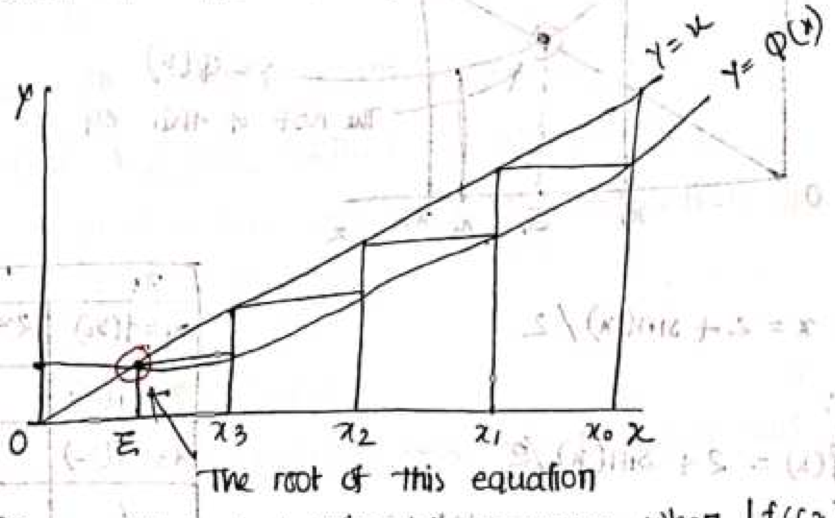
• if $f(x)$ is continuous

• if $|f'(x)| < 1$ (derivative)

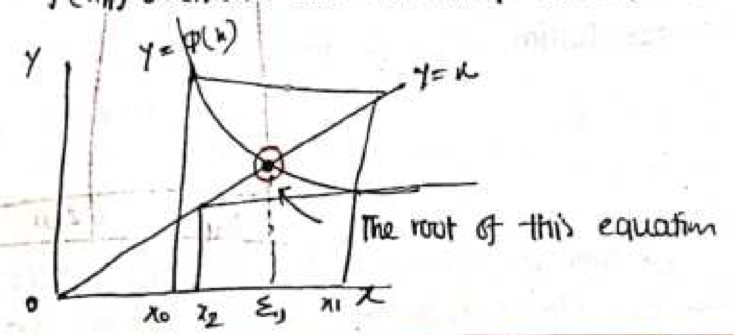
• The equation $x = f(x)$ does not converge to the real root x if $|f'(x)| > 1$

• Therefore, $g(x) = 0$ has to be re-written as $x = f(x)$ in such a way that $\boxed{|f'(x)| < 1}$

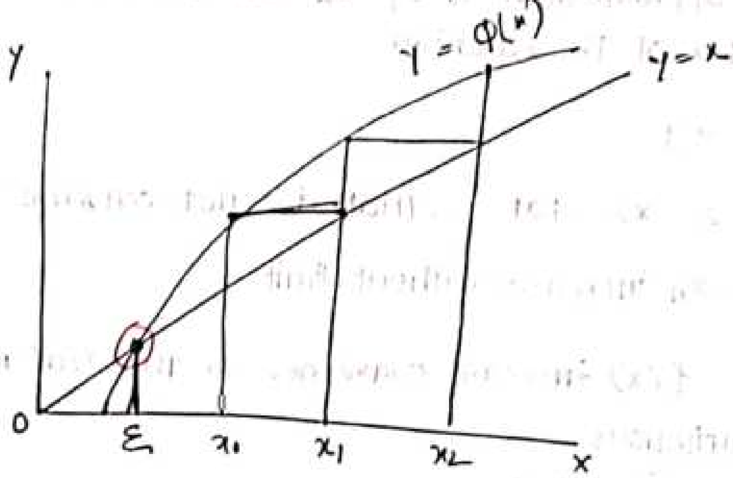
convergence of $x_{n+1} = f(x_n)$, where $|f'(x)| < 1$



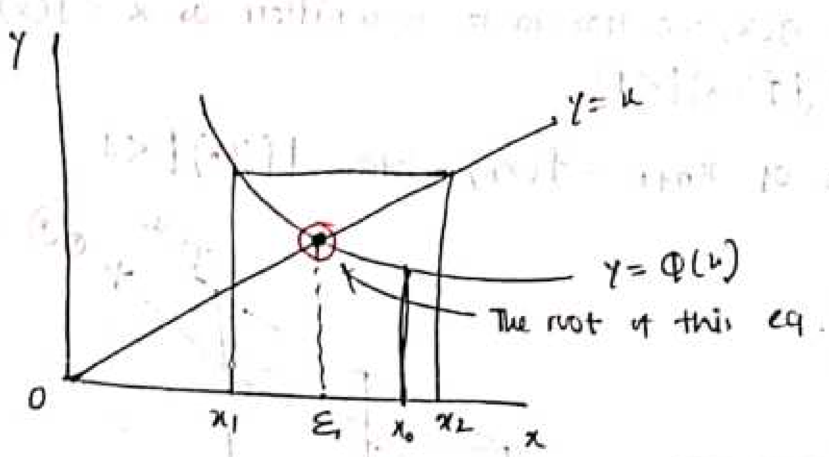
• $x_{n+1} = f(x_n)$ oscillates but ultimately converges, when $|f'(x)| < 1$, but $f'(x) < 0$



$x_{n+1} = f(x_n)$ diverges, when $f'(x) > 1$



$x_{n+1} = f(x_n)$ diverges, when $f'(x) > 1$



$\Rightarrow$ solve $x = 2 + \sin(x)/2$

solution :

Here $f(x) = 2 + \sin(x)/2$

starting with $x_0 = 2$ we calculate $x_1, x_2 \ldots$

x_0	2
$x_1 = f(x_0)$	2.4546...
$x_2 = f(x_1)$	
$x_3 = f(x_2)$	
x_{11}	2.35

Find the real root of the equation

$$g(x) = x^3 + x^2 - 1 = 0$$

Rewrite $g(x)$

$$x^3 + x^2 - 1 = 0$$

or, $x^3 + x^2 = 1$

or, $x^2(x+1) = 1$

or, $x^2 = 1/(x+1)$

or, $x = 1/\sqrt{(x+1)}$

let, $x_0 = 0.75$

$x_0 = 0.7500000$
$x_1 = 0.7559289$
⋮
$x_{10} = 0.7548777$

Class work

Find the real root of the equation using iterative method (till 4 decimal places).

$$e^{-x} = 10x$$

Answer : 0.0912765 27

Drawbacks

- We need an approximate initial guesses x_0.
- It is also a slower method to find the root.
- If the equation has more than one roots, then this method can find ~~ea~~ only one of them.

Newton - Raphson method

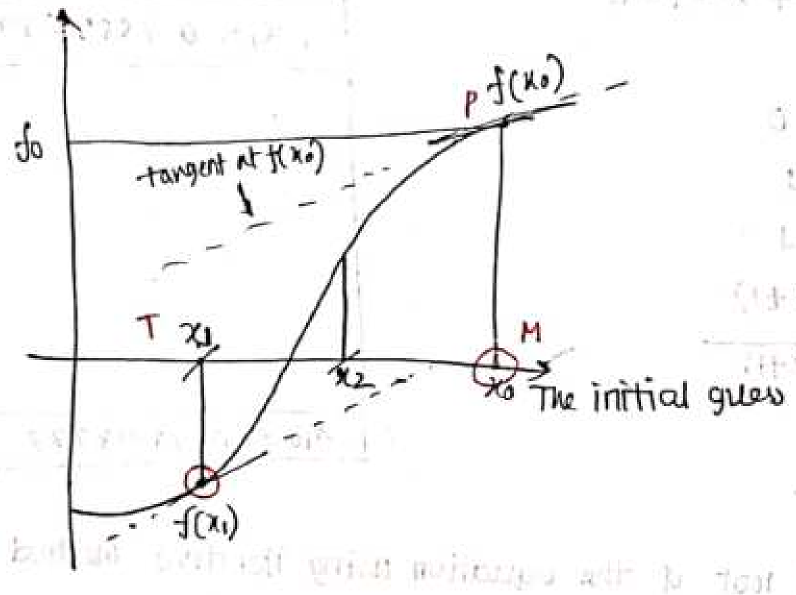
- This method is more efficient than the Bisection and Iteration methods.

If

- x is the real root and x_0 is an initial approximation of the real root of an equation $f(x) = 0$,
- $f'(x_0) \neq 0$
- $f(x)$ has the same sign between x_0 and x,

Then, the tangent at $f(x_0)$ can lead to the real root x.

Geometric significance



Here,

→ The slope at x_1 is $\tan(PTM)$

→ $\tan(PTM) = PM/TM$

→ $\tan(PTM) = f(x_0)/h$

→ Again, $\tan(PTM) = f'(x_0)$

→ Therefore, $f'(x_0) = f(x_0)/h$

→ or, $h = f(x_0)/f'(x_0)$

→ $x_1 = x_0 - h$

→ Therefore, $x_1 = \underline{x_0 - f(x_0)/f'(x_0)}$

→ similarly, $x_2 = x_1 - f(x_1)/f'(x_1)$

Methodology.

→ Let x_0 be an approximate root of $f(x) = 0$ and

→ Let, x_1 be is the correct root such that $x_1 = x_0 + h$ and $f(x_1) = 0$

→ Expanding, $f(x_0+h)$ by Taylor's series, we obtain

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0$$

→ Neglecting the second and higher order derivatives, we have

$$f(x_0) + hf'(x_0) = 0$$

→ which gives

$$h = -\frac{f(x_0)}{f'(x_0)}$$

→ A better approximation than x_0 is therefore given by x_1 where

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

→ successive approximation are given by $x_2, x_3, \ldots, x_n, x_{n+1}$ where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This formula is known as the Newton - Raphson formula

⇒ Find the real root of the equation using Newton - Raphson's Method

$$f(x) = x^3 + 4x^2 - 1 = 0$$

$$f'(x) = 3x^2 + 8x - 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 4x_n^2 - 1}{3x_n^2 + 8x}$$

x_0	0.5
x_1	0.473684211
x_2	0.472834787
x_3	0.472833909
x_4	0.472833909

Drawbacks

- The Newton - Raphson method requires the calculation of the derivative of a function, which is not always easy.
- If f' vanishes at an iteration point, then the method will fail to converge.
- When the step is too large or the value is oscillating, other more conservative methods should take over the case.

End of lecture-2
24.07.21
2:21 pm

class work:

use Newton-Raphson's Method to find a root of the equation correct to 2 decimal places. ($\varepsilon = 0.01$)

$$x^3 - 2x - 5 = 0$$

$$f(x) = x^3 - 2x - 5$$

$$f'(x) = 3x^2 - x$$

Result 2.0945514x2

The method of False Position or Regular Falsi

- Like the bisection method, Method of False Position requires two initial guesses x_a and x_b such that $f(x) = 0$ and $f(x_a)$ and $f(x_b)$ has opposite signs.

- Since the graph of $y = f(x)$ crosses the x-axis between these two points, a root must lie in between these points.

- The difference between these two methods is, instead of simply dividing the region in two, it obtains a new point x_1 which is (hopefully, but not necessarily) closer to the root.

Date : 29.12.21

Error
Roots finding
~~Interpretation~~ Interpolation
#
Matrices
solution of system of equ
ordinary diff. equation

ERROS

Numerial → approximate
↳ errors

→ sources of error (errors from formula method) instrument
→ Types of errors
→ Inherant
→ Trancation
→ significant number / figure **
→ Theorem
→ Minvalue theorem
→ Taylor series
→ Newton Raphson

Algeb + Trigonometric

~~Alberic~~ Algebric and Transcendent Eq.

→ Explain
→ solution
→ Method
→ Bisection
→ Kalf
→ Bracketing
opening
iteration
Newton-Raph.

#Interpolation

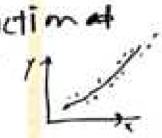
→ Tabulated value x, y
→ Newton forward Interpolat
backward " (for equi-distance 'x' variable)
→ "

10, 20, 30, ...

→ non-equi dis
langrance
→ general formular
→ Newton

curve ~~Kalf~~ fitting

tabulated value $\xrightarrow{to}$ function of



. st line
. Minimum min sq. error

$(0.1)^2 = 0.01$

$y_1 - y_0 = \varepsilon_0$

. derivative = 0 (exact line)

straight line
→ 2 Equation s $\frac{dy}{dx} = a$
$a_0 - a_{n-1}$

polynomial n-deg
systems of equation

Weighted

Matrices

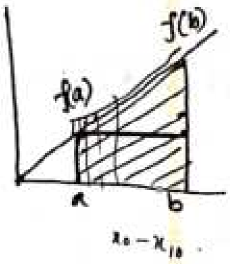
Systems of equ. value of x, y

. Algo
→ gauss elemination upper trianow
→ gauss ~~gendan jorda~~ jordan
direct . diagonal → upper tria = 0

Iterative
gauss & cdel →

Integration
Newton trapizum rule
simson 3/8 rule
Rombark widels
Integration formula

area →



ODE

$y' = \cancel{f(x)}\ f(x,y) \quad \left(\frac{dy}{dx} = y'\right)$

$y(x_0) = y_0$

$y(x) =$ series of power of x → Taylor method

from tabulated set of ... → Picard method → successive approx.

$$y(x) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \cdots$$

$$x = x_0 + h$$
$$h = x - x_0$$

~~tabat~~

at $x = x_0$

$$y(x) = f(x_0) + (x - x_0)y_0' + \frac{(x-x_0)^2}{2} y_0'' + \cdots$$

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x-x_0)^2}{2} y_0'' + \cdots$$

→ $y' = x - y^2$; $y(0) = 1$, $x_0 = 0$

$y(0.1) = ?$

$$y(x) = 1 + \cancel{x} \ xy_0' + \frac{x^2}{2} y_0'' + \frac{x^3}{6} y_0''' + \frac{x^4}{24} y_0'''' + \cdots$$

$y'(x) = x - y^2$, $y_0' = 0 - (1)^2 = -1$

$y''(x) = 1 - 2yy'$, $y_0'' = 1 + 2 = 3$

~~$y_0'''(x) = 1 + 2 = 3$~~

$y'''(x) = -2yy'' - 2y'^2$, $y_0''' = -8$

$y''''(x) = 34$

$y_0^{5} = -186$

Taylor series

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 - \frac{17}{12}x^4 - \frac{31}{20}x^5 - \cdots$$

Picard successive Method

$$y' = f(x,y)$$

$$\text{or, } \frac{dy}{dx} = f(x,y)$$

$$\text{or, } dy = f(x,y)\,dx$$

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

1st approxim

$$y_1^{(1)} = y_0 + \int_{x_0}^{x_1} f(x,y_0)\,dx$$

$$\cancel{y^{(1)} - y_0}$$

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{n-1})\,dx$$

Example

$y' = x + y^2$, $y = 1$ at $x = 0$

$$y' = f(x,y)$$

$$f(x,y) = x + y^2 = x + 1$$

$$y^{(0)} = 1$$

$$y^{(1)} = y^{(0)} + \int_{x_0}^{x} f(x, y^{(0)})\,dx$$

$$= 1 + \int_{0}^{x} (x+1)\,dx = 1 + x + \frac{x^2}{2}$$

$$y^{(2)} = 1 + \int_{0}^{x} \left[x + \left(1 + x + \frac{x^2}{2}\right)^2\right] dx$$

Euler's Method

$$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx$$

$x = x_0 + rh$; $n = 1, 2, 3, \dots$

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x_0, y_0)\,dx$$

$$= y_0 + hf(x_0, y_0)$$

$$y_2 = y_1 + \int_{x_1}^{x_2} f(x_1, y_1)\,dx$$

$$= y_1 + hf(x_1, y_1)\,dx$$

$$y_{n+1} = y_n + hf(x_n, y_n)\,dx$$

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$y_1 = y_0 + \boxed{hf(x_0, y_0)}$

Modified Euler's method

$y_1 = \underline{y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right]}$

$y_1 = f(x_0, y_0)$

$y' = -y \quad y(0) = 1$

$f(x_0, y_0) = y_0' = -y_0$

$= -1$

$h = 0.01$

$y(0.01) = ?$

$y_1 = y_0 + hf(x_0, y_0)$

$y(0.01) = 1 + 0.01(-1) = 1 - 0.01 = 0.99$

$y(0.02) = 0.99 + 0.01(-0.99) = 0.9801$

$y(0.03) = 0.9801 + 0.01(-0.9801) =$

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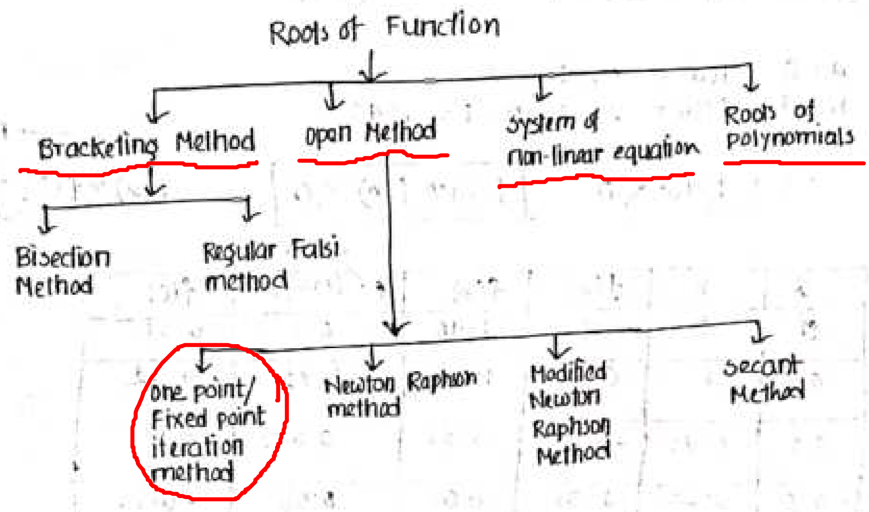
Roots of Function



$f(x) = 3x - \cos x - 1$ Root. 0.61

$= 3 \times 0.61 - \cos(0.61) - 1$

$= 0.01$

Bisection Method

Algorithm

(1) choose 2 real numbers a and b, such that $f(a) * f(b) < 0$.

(2) Define root, $c = \frac{a+b}{2}$

(3) Find $f(c)$

(4) If $f(a) * f(c) \leq 0$ then set $b = c$
else $a = c$
Return to step 1 until finding the root matched

$\Rightarrow f(x) = 3x - \cos x - 1 \qquad [0,1]$

$a = 0 \quad f(a) = -1 - 1 = -2$
$b = 1 \quad f(b) = 3 - \cos(1) - 1 = 1.46$

$-2 \times 1.46 < 0 \qquad \boxed{f(a) \times f(b) < 0}$

$f(a) \times f(c) < 0 \rightarrow b = c$
pos $\rightarrow a = c$

root

(6)

a	b	f(a)	f(b)	c = (a+b)/2	f(c)
0	1	-2	1.46	0.5	-0.38
0.5	1	-0.38	1.46	0.75	0.52
0.5	0.75	-0.38	0.52	0.625	0.06
0.5	0.625	-0.38	0.06	0.56	-0.16
0.56	0.625	-0.16	0.06	0.59	-0.05
0.59	0.625	-0.05	0.06	0.61	0.00
0.59	0.61	-0.05	0.00	0.66	-0.02
0.60	0.61	-0.02	0.00	0.61	-0.04

pos

Bisection ↳ slower

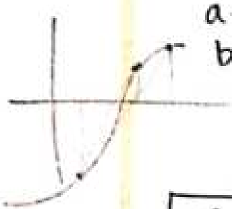
$\therefore$ Root $= 0.61 \qquad 0.01$

$x^3 + x^2 - 1$

$$\frac{0 \times 1.46 - 1 \times (-2)}{1.46 + 2} = 0.58$$

Regular Falsi Method

$f(x) = 3x - \cos x - 1$



$a = 0 \quad f(a) = 3 \cdot 0 - \cos 0 - 1 = -2 \qquad [0, 1]$
$b = 1 \quad f(b) = ~~3 - 0~~\ 3 - \cos 1 - 1 = 1.46$

$f(a) \times f(b) < 0$

$$\boxed{c = \frac{a \times f(b) - b \times f(a)}{f(b) - f(a)}}$$

(3)

a	b	f(a)	f(b)	c	f(c)
0	1	-2	1.46	0.58	-0.10
0.58	~~0.58~~ 1	-0.10	1.46	0.61	-0.00
0.61	1	-0.00	1.46	0.61	0.01

positive

Root $= 0.61$

Newton - Raphson Method

$f(x) = 3x - \cos x - 1$

Algorithm

1. find $f'(x)$
2. Find a, b so that $f(a) * f(b) < 0$
3. Assume $x_0 = a$
4. Find out $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
5. Find $x_1, x_2, x_3, \ldots x_n$ until any 2 successive values are equal

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$f(x) = 3x - \cos x - 1$

① $\boxed{f'(x_n) = 3 + \sin x}$

② $a = 0 \quad f(a) = -2$
$b = 1 \quad f(b) = 1.46$

③ $x_0 = 0$

root

x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	-2	3	$0 - \frac{-2}{3} = 0.67$
0.67	0.23	3.62	$0.67 - \frac{0.23}{3.62} = 0.61$
0.61	0.01	3.57	$0.61 - \frac{0.01}{3.57} = 0.61$

value
2

Secant Method

$$\boxed{x_{i+1} = x_i - \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} * f(x_i)}$$

$x_0 = 0 \quad f(x_0) = -2$
$x_1 = 1 \quad f(x_1) = 1.46$

$f(x_0) * f(x_1) < 0$

$x_{i+1} = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} * f(x_1)$

$\therefore x_2 = 1 - \frac{1 - 0}{1.46 - (-2)} * 1.46$

$= 0.58$

$x_3 = 0.58 - \frac{0.58 - 1}{f(0.58) - f(1)} * f(0.58)$

$= 0.64$

$x_4 = x_3 - \frac{x_3 - x_2}{f(x_3) - f(x_2)} * f(x_3) = 0.57$

$x_5 = 0.61$
$x_6 = 0.61$ ← root

one Point

Iteration Method

$f(x) = 3x - \cos x - 1$

Algorithm

(1) Given an equation $f(x) = 0$

(2) convert $f(x)=0$ into the form of $x = g(x)$

(3) let the initial guess be 0.5 / 3.2 → exponential e^x

log / trigonometric — sin, cos

one int Guys

Do

$x_{i+1} = g(x_i)$

while (none of the convergence iteration is matched)

⇒

(1) $f(x) = 0$

$\Rightarrow 3x - \cos x - 1 = 0$

$\Rightarrow 3x = \cos x + 1$

$\Rightarrow x = (\cos x + 1)/3 = g(x)$

(2) $x_0 = g(3.2)$

$= \frac{\cos(3.2)+1}{3}$

$= 0$

$x_1 = g(x_0)$

$= \frac{\cos 0 + 1}{3}$

$= 0.67$

$x_2 = g(.67)$

$= 0.59$

$x_3 = g(0.59)$

$= 0.61$ ✓

$x_4 = .61$ ✓

Practice

Raphson

$\Rightarrow f(x) = x^3 + 4x^2 - 1 = 0$

$f'(x) = 3x^2 + 8x$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0.5	0.125	4.75	~~0.125~~ 0.4736
0.125	0.50	0.0058	

0 / 1

Regular falsi

$\Rightarrow f(x) = x^3 - 2x^2 + 3x = 5$

$\Rightarrow f(x) = x^3 - 2x^2 + 3x - 5 = 0$

$f(1) = 1 - 2.1 + 3.1 - 5 = -3$

$f(2) = 8 - 8 + 6 - 5 = 1$

a, 1

$c = \frac{a * f(b) - b * f(a)}{f(b) - f(a)}$

$= \frac{1 - 2.(-3)}{1+3}$

$= \frac{1+6}{4}$

root

a	b	f(a)	f(b)	c	f(c)
1	2	-3	1	1.75	-0.515
1.75	2	-0.515	1	1.834	

posi

$f(a) * f(b) < 0$

b = c

or

a = c

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Difference

Bracketing Method	Open Method
1) Need two initial guesses	1) can involve 1/or more initial guess
2) The root is located within an interval prescribed by a lower and an upper bound	2) Not necessarily bracket the root
3) Always work but converge slowly	3) Do not always work (can converge) but they do they usually converge much more quickly

Advantages / Disad

Method	Ad	Dis
Bisection	- Easy, Reliable, convergent - one func evaluation per itera - No knowledge of derivative is need	- slow - needs interval $[a,b]$ $f(a)\cdot f(b)<0$
Newton	- Fast (if near the root) - 2 func. evaul per iter	- May diverge - needs derivati and an initial guess $f'(x_0)$ is non zero
secant	- Fast (slower than newton) - 1 func/iter - no knowledge of deriva	- May diverge - 2 initial guess $f(x_0)-f(x_1)\neq 0$

Interpolation

Interpolation is a process of computing intermediate values of an unknown function $f(x)$ from a set of given values of that function.

→ Let $y = f(x)$ be a function of given by the values of $y_0, y_1, y_2, \ldots, y_n$ which it takes for the values $x_0, x_1, x_2, \ldots x_n$ of the independent variable x.

$f(x) \longrightarrow$ totally unknown/complicated

→ it is desirable to replace the given function by another which can be easily handled.

→ Let $\phi(x)$ denotes an arbitrary simpler function so constructed that it takes the same values as $f(x)$ for the values $x_0, x_1, x_2, \ldots x_n$.

If

- $\phi(x)$ is polynomial, the process of representing $f(x)$ by $\phi(x)$ is called parabolic/polynomial interpolation.
- $\phi(x)$ is finite trigonometric series. → trigonometric interpolation
- $\phi(x)$: exponential function ⟶ Legendre polymomial, Bessel function etc.

~~Foward~~ Forward Differences

If $y_0, y_1, y_2, \ldots y_n$ denote a set of values of any function $y = f(x)$, then $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots y_n - y_{n-1}$ are called the differences of the function y.

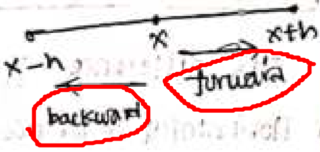
$$\Delta f(x) = f(x+h) - f(x)$$

$\Delta y_0, \Delta y_1, \Delta y_2$

1st $\quad \Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \ldots\ldots \Delta y_n = y_{n+1} - y_n$

$\Delta \longrightarrow$ forward difference operation

$\Delta y_0, \Delta y_1, \Delta y_2, \ldots$ first forward differe.



2nd $\left\{ \begin{aligned} \Delta^2 y_0 &= \Delta y_1 - \Delta y_0 \\ \Delta^2 y_1 &= \Delta y_2 - \Delta y_1 \end{aligned} \right\}$ second " "

$$\begin{aligned} \Delta^2 y_0 &= \Delta y_1 - \Delta y_0 \\ &= y_2 - y_1 - y_1 + y_0 \\ &= y_2 - 2y_1 + y_0 \end{aligned}$$

Class work

$\boxed{\Delta y_{10},\ \Delta^2 y_{20},\ \Delta^3 y_{15},\ \Delta^5 y_{10} = ?}$

forw.

Given the set of values

	y_{10}	y_{15}	y_{20}	y_{25}	y_{30}	y_{35}
x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

$\Delta y_{10} = (21.51 - 19.97)$, $\Delta y_{15} = (22.47 - 21.51)$; $\Delta y_{20} = (23.52 - 22.47)$

~~$\Delta^2 y_{20}$~~ =

diagonal

$\Delta^2 y_{20} = \Delta y_{25} - \Delta y_{20} = 0.08$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
10	19.97					
		1.54				
15	21.51		−0.58			
		0.96		0.67		
20	22.47		0.09		−0.68	
		1.05		−0.01		0.72
25	23.52		0.08		0.04	
		1.13		0.03		
30	24.65		0.11			
		1.24				
35	25.89					

Backward differences

The differences $y_1 - y_0,\ y_2 - y_1, \dots, y_n - y_{n-1}$ are called Backward or Horizontal differences, if they are denoted by $\nabla y_1, \nabla y_2, \dots \nabla y_n$.

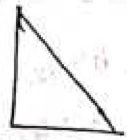
$\nabla y_1 = y_1 - y_0$

$\nabla y_2 = y_2 - y_1$

$\nabla y_n = y_n - y_{n-1}$

→ ∇ is called the backward difference operator.

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$
$$= y_2 - y_1 - (y_1 - y_0)$$
$$= y_2 - y_1 - y_1 + y_0$$
$$= y_2 - 2y_1 + y_0$$



class work :

∇y_{20} $\quad \nabla^2 y_{25}$ $\quad \nabla^3 y_{30}$ $\quad \nabla^5 y_{35}$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
10	19.97					
15	21.51	1.54				
20	22.47	0.96	-0.58			
25	23.52	1.05	0.09	0.67		
30	24.65	1.13	0.08	-0.01	-0.68	
35	25.89	1.24	0.11	0.03	0.04	0.72

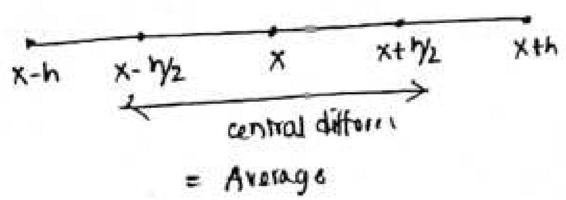
central Differences

- central operator is denoted by (small delta) [δ]
- Formula of central operator :-

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} \quad (\because E^n f(x) = f(x + nh))$$

E → shift operator



central differences Table

$2-\frac{1}{2}$

forward

$y_1 - y_0 = \delta y_{1/2}$ ← $(1+0)/2$

$y_2 - y_1 = \delta y_{3/2}$ $(2+1)/2$

$\boxed{y_n - y_{n-1} = \delta y_{n-1/2}}$ $(n+n-1)/2$

$\frac{2n-1}{2}$

$n - 1/2$

prev table

~~δ_{10}~~ δy_{10}, $\delta^2 y_{20}$, $\delta^3 y_{15}$, $\delta^5 y_{10}$

Relation

$\Delta^2 y_3 - \Delta^2 y_2$

$\Delta^3 y_2 = y_5 - 3y_4 + 3y_3 - y_2$

$\Delta y_0 = y_1 - y_0$

$\nabla^2 y_5 - \nabla^2 y_4$

$\nabla^3 y_5 = y_5 - 3y_4 + 3y_3 - y_2$

(nabla) $\nabla y_1 = y_1 - y_0$

$\delta y_{1/2} = y_1 - y_0$

$\Delta y_0 = \nabla y_1 = \delta_{1/2}$

$\Delta^3 y_2 = \nabla^3 y_5 = \delta^3 y_{7/2}$

$$\boxed{\Delta^m y_k = \nabla^m y_{k+m} = \delta^m y_{(2k+m)/2}}$$

⇒ # shift operator E, $n=1$ $\quad E^n f(x) = f(x+nh)$

~~$E^1 f(x) = f(x+h)$~~

$E^1 f(x) = f(x+h)$

operators
$(\Delta, \nabla, E, \delta, \mu, D)$
central average

Average operator:

- denoted by (Mu) μ
- $\mu f(x) = \frac{1}{2}\left[f(x+h/2) + f(x-h/2)\right]$

$$\mu = \frac{E^{1/2} + E^{-1/2}}{2} \quad (\because E^n f(x) = f(x+nh))$$

Differential operator (D)

$$Df(x) = \frac{d}{dx} f(x) = f'(x) \quad (\because D = d/dx)$$

** Prove that, $E = 1+\Delta$

$$\begin{aligned}
&(1+\Delta) f(x) \\
&= f(x) + \Delta f(x) \\
&= f(x) + f(x+h) - f(x) \qquad (\because \Delta f(x) = f(x+h) - f(x)) \\
&= f(x+h) \\
&= E^1 f(x) \qquad [E^1 f(x) = f(x+h)] \\
&= E f(x)
\end{aligned}$$

$\Rightarrow (1+\Delta) = E$ ✓

$\Rightarrow E\nabla = \Delta$

$f(x-h+h)$

$$\begin{aligned}
E\nabla(f(x)) &= E[\nabla f(x)] \\
&= E[f(x) - f(x-h)] \qquad f(x+nh) = E^n f(x) \\
&= Ef(x) - Ef(x-h) \qquad Ef(x-h) = f(x+h-h) = f(x) \\
&= f(x+h) - f(x) \\
&= \Delta f(x)
\end{aligned}$$

$\therefore E\nabla = \Delta$

$E\nabla f(x) = \Delta f(x) \; ; \; \forall f(x)$ ✓

$\Delta\nabla = \Delta - \nabla$

$$\Delta\nabla(f(x)) = \Delta[\nabla f(x)]$$
$$= \Delta[f(x) - f(x-h)] \quad (\because \nabla f(x) = f(x) - f(x-h))$$
$$= \Delta f(x) - \Delta f(x-h)$$
$$= \Delta f(x) - [f(x) - f(x-h)] \quad (\because \Delta f(x) = f(x+h) - f(x))$$
$$= \Delta f(x) - \nabla f(x) \quad (\because \nabla f(x) = f(x) - f(x-h))$$
$$= (\Delta - \nabla) f(x)$$

$$\Rightarrow \Delta\nabla(f(x)) = (\Delta - \nabla) f(x) \; ; \; \forall f(x)$$

$$\Rightarrow \Delta\nabla = \Delta - \nabla$$

$E = e^{hD}$

increament — differ

$$Ef(x) = f(x+h)$$
$$= f(x) + \cancel{hf} \; hf'(x) + \frac{h^2}{2!} f''(x) + \cdots \quad \text{(By Taylor's expansion)}$$
$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \cdots \quad (\because f'(x) = Df(x))$$
$$= \left[1 + hD + \frac{h^2}{2!} D^2 + \cdots\right] f(x)$$
$$\Rightarrow Ef(x) = e^{hD} f(x)$$
$$\Rightarrow E = e^{hD} \qquad \left(\because e^x = 1 + x + \frac{x^2}{2!} + \cdots\right)$$

Effect of an Error in a Tabular value:

Let $y_0, y_1, y_2, \ldots, y_n$ be the true values of a function, and suppose the value y_3 to be effected with an error ε, so that its erroneous value is $y_3 + \varepsilon$. Then the successive differences of the y's are shown below:

y	Δ	$Δ^2$	$Δ^3$
y_0			
	$Δy_0$		
y_1		$Δ^2y_0$	
	$Δy_1$		$Δy^3_0+ε$
y_2		$Δ^2y_1+ε$	
	$Δy_2+ε$		$Δ^3y_1-3ε$
$y_3+ε$		$Δ^2y_2-2ε$	
	$Δy_3-ε$		$Δ^3y_2+3ε$
y_4		$Δ^2y_3+ε$	
	$Δy_4$		$Δ^3y_3-ε$
y_5		$Δ^2y_4$	
	$Δy_5$		
y_6			

Suppose that there is an error of +1 unit in a certain tabular value. As higher differences are formed, the error spreads out fanwise, and is at the same time, considerably magnified as show below:

y	Δ	$Δ^2$	$Δ^3$	$Δ^4$	$Δ^5$
0					
	0				
0		0			
	0		0		
0		0		0	
	0		0		1
0		0		1	
	0		1		-5
0		1		-4	
	1		-3		10
1		$\cancel{0}$-2		6	
	-1		3		-10
0		1		-4	
	0		-1		5
0		0		1	
	0		0		-1
0		0		0	
	0		0		
0		0			
	0				
0					

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

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The table shows the following characteristics:

- The effect of an error increases with the successive differences.
- The coefficients of the ε's are the binomial coefficients with alternating signs.
- The algebric sum of the errors in any difference column is zero.
- The max. error in the differences is in the same horizontal line as the erroneous value.

→ the effect of Horizontal difference:

y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	
3010						
	414					
3424		-36				$-4\varepsilon = 178$
	378		-39			$\varepsilon = -45$ (approx)
3802		-75		178		
	368		139		-449	
4105		64		-271		
	~~299~~ 367		-132		452	
4472		-68		181		$6\varepsilon = -271$
	~~280~~ 299		49		-229	$\varepsilon = -45$ (appr
4771		-19		-46		
	~~264~~ 280		3			
5051		-16				
	264					
5315						

∴ the actual entry is $4105 - \varepsilon = 4105 - (-45)$

$= 4150$

Paskel's Triangle

0: 1

1: 1 1

2: 1 2 1

3: 1 3 3 1

4: 1 4 6 4 1

5: 1 5 10 10 5 1

6: 1 6 15 20 15 6 1

7: 1 7 21 35 35 21 7 1

8: 1 8 28 56 70 56 28 8

Effect of an Error in a Tabular Value of Backward Interpolation

same horizontal line

x	y	∇	∇^2	∇^3	∇^4
x_0	y_0				
x_1	y_1	∇y_1	$\nabla^2 y_2$		
x_2	y_2	∇y_2	$\nabla^2 y_3$		
x_3	y_3	∇y_3	$\nabla^3 y_4$		
x_4	y_4	∇y_4	$\nabla^3 y_{5+\varepsilon}$		
x_5	y_5	$\nabla y_{5+\varepsilon}$	$\nabla^3 y_{6-3\varepsilon}$		
x_6	y_6	$\nabla y_{6-\varepsilon}$	$\nabla^3 y_{7+3\varepsilon}$		
x_7	y_7	∇y_7	$\nabla^3 y_{8-\varepsilon}$		
x_8	y_8	∇y_8	$\nabla^3 y_9$		
x_9	y_9	∇y_9	∇y_{10}		
x_{10}	y_{10}	∇y_{10}			

Newton's formula for Forward Interpolation:

→ Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2, \cdots y_n$ for the ~~equal~~ equidistance values $x_0, x_1, x_2 \cdots x_n$ of the independent variable x.

→ It is required to find $\phi(x)$, a polynomial of the n-th degree such that y and $\phi(x)$ agree at the tabulated points (i.e. they have the same values

→ let $\phi(x)$ denote a polynomial of the n-th degree.

This polynomial can be written in the form

$$\phi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_n) \quad (1)$$

⇒ determine the co-efficients $a_0, a_1, a_2, \cdots a_n$ so that we can get

$$\phi(x_0) = y_0 \qquad \phi(x_n) = y_n$$
$$\phi(x_1) = y_1$$
$$\phi(x_2) = y_2$$

we know that,

$$\phi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_n) \quad (1)$$

→ we can substitute the given successive values $x_0, x_1, x_2, \cdots x_n$ in equation (1) ; at the same time we can put $\phi(x_0) = y_0$, $\phi(x_1) = y_1$, $\phi(x_2) = y_2 \cdots \phi(x_n) = y_n$

And let, $x_1 - x_0 = h$, $x_2 - x_1 = 2h$ etc. (since the values of x are equidistance) in equation (1) we have.

$$\phi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \dots + a_n(x-x_0)(x-x_{n-1}) \quad -(1)$$

That is, at $x = x_0$ (substituting x with x_0 in eq.(1)) we have

ϕ

Newton's formula for forward interpolation:

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2, \dots y_n$ for the equidistant values $x_0, x_1, x_2, \dots x_n$ of the independent variable x.

→ find $\phi(x)$, a polynomial of the n-th degree such that y and $\phi(x)$ agree at the tabulated points (i.e. they have the same values)

Let $\phi(x)$ denote a polynomial of the nth degree

$$\phi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \dots$$
$$+ a_n(x-x_0)(x-x_1)(x-x_{n-1}) \quad \text{——} (1)$$

$\phi(x_0) = y_0$ $\qquad x_1 - x_0 = h$

$\phi(x_1) = y_1$ $\qquad x_2 - x_0 = 2h$

$\phi(x_n) = y_n$

$$a_4 = \frac{\Delta^4 y_0}{4!\,h^4}$$

$x = x_0$

$\phi(x_0) = a_0 = y_0$

$$a_5 = \frac{\Delta^5 y_0}{5!\,h^5}$$

$x = x_1$

$$\phi(x_1) = y_1 = a_0 + a_1(x_1 - x_0)$$
$$= y_0 + a_1 h.$$

$$a_n = \frac{\Delta^n y_0}{n!\,h^n}$$

$x = x_2$

$$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$= y_0 + \frac{y_1 - y_0}{h}(2h) + a_2(2h)(h)$$
$$\Rightarrow a_2 = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

$y_3 =$

$$a_3 = \frac{y_3 - 3y_2 + 3y_1 - y_0}{6h^3} = \frac{\Delta^3 y_0}{3!h^3}$$

interpolating value near the begining of a set

$$\Rightarrow \frac{x - x_0}{h} = u \qquad x = x_0 + hu$$

$x_1 = x_0 + h$

$x_2 = x_0 + 2h$

$$\frac{x - x_1}{h} = \frac{x - (x_0 + h)}{h} = \frac{x - x_0}{h} - \frac{h}{h} = u - 1$$

$$\frac{x - x_2}{h} = \cancel{\ \ } \; u - 2$$

$$\frac{x - x_{n-1}}{h} = u - n + 1$$

$$\phi(x) = \phi(x_0 + hu) = g(u)$$

$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$+ \frac{u(u-1)(u-2)\cdots(u-n+1)}{n!}\Delta^2 y_0 \ldots$$

Example 1:

Find the cubic polynomial which takes the following values:

$y(0) = 1 \qquad y(1) = 0 \qquad y(2) = 1 \qquad y(3) = 10$

Hence, or otherwise obtain $y(0.5)$

Solution:

x	y	Δ	Δ^2	Δ^3
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

Here $h = 1$

$$y(x) = y_0 + \frac{\Delta y_0}{1!\,h}(x-x_0) + \frac{\Delta^2 y_0}{2!\,h^2}(x-x_0)(x-x_1) + \frac{\Delta^3 y_0}{3!\,h^3}(x-x_0)(x-x_1)(x-x_2) + \cdots$$

$$+ \frac{\Delta^n y_0}{n!\,h^n}(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$= 1 + \frac{(-1)(x-0)}{1} + \frac{(x-0)(x-1)}{2(1)^2}(2) + \frac{(x-0)(x-1)(x-2)}{6(1)^3}(6)$$

$$= 1 - x + x(x-1) + x(x-1)(x-2)$$

$$= 1 - x + x^2 - x + x^3 - 3x^2 + 2x$$

$$= \boxed{x^3 - 2x^2 + 1}$$

$\therefore$ The polynomial we obtained for the given tabular values is,

$$y = x^3 - 2x^2 + 1$$

Now, $y(0.5) = (0.5)^3 - 2(0.5)^2 + 1 = 0.625$

which is the same value as that obtained by substituting $x = 0.5$ in the cubic polynomial above

$\tan(x)$ for $0.10 \le x \le 0.30$ Find $\tan(0.12)$

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Ans: 0.120537

Euler and Euler modified $\xrightarrow{}$ second order

Euler method is called Runge Kutta Method of first order

Solution of linear Algebric method

i) Direct method

- Gauss Elimination
- Gauss Jordan method

(ii) Indirect method

Gauss Elimination Method

$$x - y + 2z = 3$$
$$x + 2y + 3z = 5$$
$$3x - 4y - 5z = -13$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix}$$

$\boxed{AX = B}$

$$(A:B) = \left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 1 & 2 & 3 & 5 \\ 3 & -4 & -5 & -13 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$= \left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & -1 & -11 & -22 \end{array}\right] \quad R_3 \to 3R_3 + R_2$$

$$= \left[\begin{array}{ccc:c} 1 & -1 & 2 & 3 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & -32 & -64 \end{array}\right]$$

$$\Rightarrow \quad x - y + 2z = 3$$
$$3y + z = 2$$
$$-32z = -64$$

$$\left.\begin{array}{l} z = 2 \\ y = 0 \\ x = -1 \end{array}\right\}$$

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Gauss ~~Jordan~~ elimination Method

upper triangular

$$x + 4y + 9z = 16$$
$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$

$$\begin{bmatrix} 1 & 4 & 9 \\ 2 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \\ 18 \end{bmatrix}$$

$AX = B$

$(A : B)$ $\left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \end{array}\right]$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$= \begin{array}{c} \\ 10 \\ 7 \end{array}\left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & -7 & -17 & -22 \\ 0 & -10 & -24 & -30 \end{array}\right]$$

$$= \left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & -7 & -17 & -22 \\ 0 & 0 & 2 & 10 \end{array}\right]$$ $R_3 \to 7R_3 - 10R_2$ or ~~$R_2 \to R_2/-7$~~ ~~R_3~~

Backward substitution

$$x + 4y + 9z = 16$$
$$-7y - 17z = -22$$
$$2z = 10$$

$z = 5$
$y = -9$
$x = 7$

Gauss Jordan Method :

diagonal

$$\left[\begin{array}{ccc:c} 1 & 4 & 9 & 16 \\ 0 & -7 & -17 & -22 \\ 0 & -10 & -24 & -30 \end{array}\right]$$ $R_3 \to 7R_3 - 10R_2$, $R_1 \to 7R_1 + 4R_2$

$$\begin{array}{c} \\ 2 \\ 17 \end{array}\left[\begin{array}{ccc:c} 7 & 0 & -5 & 24 \\ 0 & -7 & -17 & -22 \\ 0 & 0 & 2 & 10 \end{array}\right]$$ $R_1 \to 2R_1 + 5R_3$, $R_2 \to 2R_2 + 17R_3$

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$$= \left[\begin{array}{ccc|c} 14 & 0 & 0 & 98 \\ 0 & -14 & 0 & 126 \\ 0 & 0 & 2 & 10 \end{array}\right]$$

$$\left.\begin{aligned} 14x &= 98 \\ -14y &= 126 \\ 2z &= 10 \end{aligned}\right\} \quad \begin{aligned} x &= 7 \\ y &= -9 \\ z &= 5 \end{aligned}$$

Gauss elimination

$a[i][i] = 0 \rightarrow$ error

$$\left[\begin{array}{ccc|c} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \end{array}\right]$$

$4/1 = 4$

Reducing lower triangular zero

zero under pivot ~~mat~~ entries

$a21$

$$m = \frac{a21}{a11}$$

$a31$

$$m = \frac{a31}{a21}$$

a_{n1}

$$m = \frac{a_{n1}}{a_{11}}$$

$$\boxed{m_{ik} = \frac{a_{ik}}{a_{kk}}}$$

(i) for 1st iteration $k = 1$, $i = 2, 3, 4, 5, n$

2nd $k = 2$ $i = 3, 4, 5$

$K = 1, 2, 3 \dots n-1$

$i = K+1, K+2, \dots n$

entries are made 0 by multiplying m_{ik} to pivot row and subtracting from ith row

$$a_{ij}^{(k+1)} = a_{ij}^{K} - m_{ik} * a_{kj}^{k}$$

Here, $j = K, K+1, K+2 \dots n$

$k = 1, 2, 3, \cdots n-1$

$i = k+1, k+2, k+3 \cdots n$

$j = k, k+1, k+2 \cdots n$

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 4 & -7 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$$

11	12	13
21	22	23
31	32	33

$m_{21} = \frac{2}{1} = 2 \qquad k=1, i=2, j=1$

$m_{31} = \frac{4}{1} = 4 \qquad k=1, i=3, j=1$

$a_{21} = a_{21} - m_{21} * a_{11}$

$= 2 - 2 * 1 = 0$

$a_{31} = a_{31} - m_{31} * a_{11}$

$= 4 - 4 * 1 = 0$

$a_{32} = -7 - (7 * 1) \qquad k=2, i=3, j=2$

$= -7 + 7$

$= 0$

Interpolation for unequal interval

→ Lagrange interpolation method

$f(x) =$

x	5	6	9	11
y	12	13	14	16

$x = 10$ find $y = ?$

94/7

$$f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$f(10) = \frac{4 \cdot 1 \cdot (-1)}{(-1)(-4)(-6)}$$

Newton Divided Difference Interpolation ~~fo~~

	x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	5	12	$\frac{(13-12)}{6-5}=1$	$\frac{1/3-1}{9-5} = -1/6$	$\frac{\frac{2}{15}+\frac{1}{6}}{11-5} = 1/20$
x_1	6	13	$\frac{14-13}{9-6} = 1/3$	$\frac{1-1/3}{11-6} = 2/15$	
x_2	9	14	$\frac{16-14}{11-9} = 1$		
x_3	11	16			

$$\boxed{f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + }$$

$$f(x) = 12 + (x-5)1 + (x-5)(x-6)(-1/6) + (x-5)(x-6)(x-9)\,1/20$$

$$f(10) = 12 + 5 + 5\cdot 4\cdot(-1/6) + 5\cdot 4\cdot 1\cdot 1/20$$

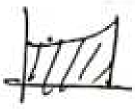
$$= 18 - \frac{10}{3}$$

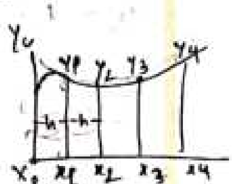
$$= 14.66$$

Numerical Integration:

Trapezoidal Rule:

The area bounded by the curve $f(x)$ and x-axis between limit a and b is denoted by,





$$I = \int_a^b f(x)\,dx \quad \text{——— (1)}$$

divide the interval (a,b) into n equal interval with length h

i.e $(a,b) = (a = x_0, x_1, x_2, \dots x_{n-1}, x_n = b)$

$a = x_0$

$x_1 = x_0 + h$

$x_2 = x_1 + h$

$x_n = x_{n-1} + h$

$$\boxed{n = \frac{b-a}{h}} \qquad h = \frac{b-a}{h}$$

$$\int_a^b f(x)dx = h\left(\frac{y_0+y_n}{2} + y_1 + y_2 + \cdots y_{n-1}\right)$$

→ it applicable on any no. of interval

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(ii) ~~Simpson~~ Simpson's 1/3 rule (even interval)

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots) + 2(y_2+y_4+\cdots)\right]$$

it is applicable when the total no. of interval is even

Example

evaluate $\int_0^1 \frac{dx}{1+x^2}$ By trapezoidal form

$h = \frac{1-0}{6} = 1/6$ $\quad y = \frac{1}{1+x^2}$ $\quad \int_0^1 \frac{1}{1+x^2}\,dx$

$x_0 = 0 \quad y_0 = \frac{1}{1+x_0^2} = 1$

$x_1 = 1/6 \quad y_1 = \frac{1}{1+x_1^2} = \frac{36}{37}$

$x_2 = 2/6 \quad y_2 = 0.9$

$x_3 = 3/6 \quad y_3 = 0.8$

$x_4 = 4/6 \quad y_4 = 9/13$

$x_5 = 5/6 \quad y_5 = 36/61$

$x_6 = 1 \quad y_6 = 0.5$

$$= h\left[\frac{y_0+y_6}{2} + y_1+y_2+y_3+y_4+y_5\right]$$

$$= \frac{1}{6}\left[\frac{1+0.5}{2} + \frac{36}{37} + 0.9 + 0.8 + 9/13 + 36/61\right]$$

$$= 0.785396$$

$O(n \cdot m)$

Pobin-Karp

(8) 1 2 3 4 5 6 7 8 $\quad$ n $\quad O(mn)$

(9) 1 2 2 3 $\quad$ m

1

2

Date : 20.2.22

What are Numerical Method?

→ In Numerical Analysis, numerical methods are mathematical tools designed to solve numerical problems.

What are the reasons to study Numerical method?

→ There are many things that you want to compute that cannot be computed exactly. The roots of higher degree polynomials. The solution of simple differential equations on irregular domains. Any calculation with real numbers that you do with a computer.

Numerical analysis is the branch of mathematics that is about approximate computing. It tells you how likely quickly you can get how close to the true solution. If you study any sort of engineering science you need inevitably learn some corner of numerical analysis.

III Numerical Errors

Define Accuracy and Precision.

→ Accuracy : Accuracy refers to the closeness of a measured value to a standard or known value.

→ Precision : The quality of being exact and accurate is called precision.

III Bisection Method :

Describe the bisection method for finding root of equation $f(x) = 0$.

→ If the function $f(x) = 0$ is continuous between a and b, and $f(a)$ and $f(b)$ are off opposite signs, then there exists at least one root between a and b. let $f(a)$ is negative and $f(b)$ is positive for definiteness. Then the root lies between a and b and let it approximate value ~~ca~~ be given by $x_0 = (a+b)/2$. If $f(x_0) = 0$, we conclude that x_0 is a root of the equation $f(x)$. otherwise, the root lies between either x_0 and b or between x_0 and a depending on whether $f(x_0)$ is negative or positive. Now we designate new interval $[a_1, b_1]$, whose length is $|a-b|/2$. As before this is bisected at x_1 and the new interval will be exactly half of the length of the previous one. We repeat that process until the latest interval is as small as desired. At the end of the process, which is the value is the root of the equation $f(x)=0$.

what are the merits and demerits of bisection method?

Merits :

- simple and easy to implement
- one function evaluation per iteration
- The size of the interval is reduced after each iteration
- The function does not have to be differentiable.

Demerits :

- Slow convergence rate
- It is unable to detect multiple roots.
- It takes so many iterations.
- It requires a fixed accuracy level called by tolerance.

Newton's Interpolation :

Find out Newton's Forward Difference Interpolation Formula

Given the set of $(n+1)$ values are $(x_0, y_0), (x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$ of x and y. It is required to find $y_n(x)$, a polynomial of nth degree such that y and $y_n(x)$ agree at the tabulated points. let the value of x be equidistant, i.e let.

$$x_i = x_0 + ih \qquad i = 0, 1, 2, 3 \dots n$$

Since $y_n(x)$ is a polynomial of nth degree, it may be written as,

$$y_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cdots$$
$$a_n(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}) \text{ ——— } (1)$$

Imposing now the condition that y and $y_n(x)$ should agree at the set of tabulated points, we obtain,

$$A_0 = y_0 \; ; \; a_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}, \; a_2 = \frac{\Delta^2 y_0}{h^2 2!} \; ; a_3 = \frac{\Delta^3 y_0}{h^3 3!} \; ; \cdots \; a_n = \frac{\Delta^n y_0}{h^n n!}$$

Setting $x = x_0 + ph$ and substituting $a_0, a_1, a_2 \cdots a_n$ from (1) we get,

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)\Delta^2 y_0}{2!} + \frac{p(p-1)(p-2)\Delta^3 y_0}{3!} + \cdots$$
$$+ \frac{p(p-1)(p-2)\cdots(p-n+1)\Delta^n y_0}{n!}$$

This is Newton's Forward difference Interpolation formula.

Curve Fitting for polynomial

Find out the equation of n degree for curve fitting by polynomial.

Let the polynomial of the nth degree.

$$Y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

Be fitted to the data points (x_i, y_i), $i = 1, 2, 3, \ldots, m$. Then we have,

$$S = [y_1 - (a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n)]^2 + [y_2 - (a_0 + a_1 x_2 + a_2 x_2^2 +$$
$$\cdots + a_n x_2^n]^2 + \cdots + [y_m - (a_0 + a_1 x_m + a_2 x_m^2 + \cdots a_n x_m^n]^2$$

Equating to zero the first partial derivatives and simplifying, we obtain the normal equation:

$$m a_0 + a_1 \Sigma x_i + a_2 \Sigma x_i^2 + \cdots + a_n \Sigma x_i^n = \Sigma y_i$$
$$a_0 \Sigma x_i + a_1 \Sigma x_i^2 + a_2 \Sigma x_i^3 + \cdots + a_n \Sigma x_i^n = \Sigma x_i y_i$$
$$\vdots$$
$$a_0 \Sigma x_i^n + a_1 \Sigma x_i^{n+1} + \cdots + a_n \Sigma x_i^{2n} = \Sigma x_i^n y_i$$

where the summations are performed from $i=1$ to $i=m$.

The system (ii) constitutes $(n+1)$ equations with $(n+1)$ unknown, and hence can be solved for $a_0, a_1 \ldots a_n$. Then equation (i) gives the required polynomial of n degree.

Gauss sheidal Method

Explain Gauss sheidal Method for solution of linear system.

we wish to solve laplace's equation

$$U_{xx} + U_{yy} = 0$$

in a bounded region R with boundary C. let R be a square region so that it can be divided into network of small squares of side h. Let the values of $u(x,y)$ on the boundary C be given by c_i. The approximate function values at the interior mesh points can now be computed according to the scheme, we first use the diagonal five-point formula and compute u_5, u_7, u_9, u_1 and u_3 in this order. Thus we obtain,

$$u_5 = 1/4\,(c_1 + c_5 + c_9 + c_{13}) \qquad u_7 = 1/4\,(c_{15} + u_5 + c_{11} + c_{13})$$

$$u_9 = 1/4\,(u_5 + c_7 + c_9 + c_{11}) \qquad u_1 = 1/4\,(c_1 + c_3 + c_{15} + u_{15})$$

$$u_3 = 1/4\,(c_3 + c_5 + c_7 + u_5)$$

we then compute u_8, u_4, u_6 and u_2 by the standard five point formula.

Thus we have,

$$u_8 = 1/4\,(u_5 + u_9 + c_{11} + u_7) \qquad u_4 = 1/4\,(u_1 + u_5 + u_7 + c_{15})$$

$$u_2 = 1/4\,(c_3 + u_3 + u_5 + u_1)$$

Now let $u_{i,j}^{(n)}$ denotes the nth iterative value of $u_{i,j}$. Then the iterative formula by Gauss sheidel Method is,

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right]$$

This method is also referred to as Liebmann's method.

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Gaussian Elimination

Solve the following system using Gaussian Elimination Method :

$$2x + y + z = 10 \qquad 3x + 2y + 3z = 18 \qquad x + 4y + 9z = 16$$

we first eliminate x from 2^{nd} and 3^{rd} equation. For this we multiply 2^{nd} and 3^{rd} equation by $(-2/3)$ and (-2) respectively and add to 1^{st} equation to get 4^{th} and 5^{th} equation.

$$-1/3\ y - z = -2 \qquad \text{and} \qquad -7y - 17z = -22$$

Now we eliminate y from 5th equation. For this we multiply 5th equation by $(-1/21)$ and add to 4th equation to get,

$$-4/21\ z = -20/21 \qquad \text{or}, \ 4z = 20 \qquad \text{or}, \ z = 5$$

The upper triangular form is therefore given by,

$$2x + y + z = 10$$
$$-1/3\ y - z = -2$$
$$z = 5$$

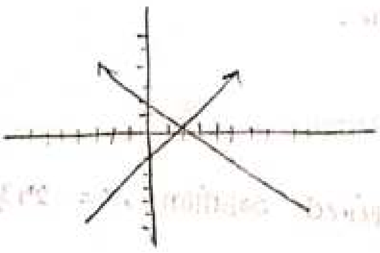
It follows the required solution, $x = 7$, $y = -9$, $z = 5$.

Linear Systems :

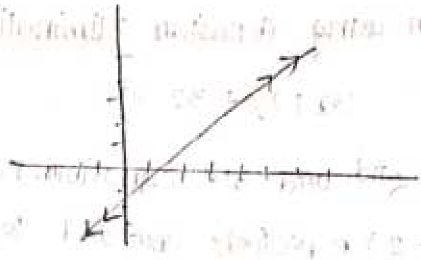
Classify system of linear equations and explain them based on graphical representation.

Inconsistant : A system of equation that has no solution.

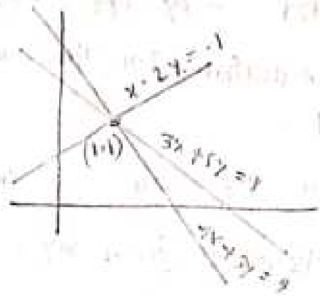
Consistant : A ..... that has one or more solutions.



Dependent : A system of equation that has infinitly many solution.



Independent : that has only one solution.



III Gaussian Elimination :

Solve the following set of simultaneous equations using Gauss Elimination

$2x - 4y + 6z = 5$ $\quad x + 3y - 7z = 2$ $\quad 7x + 5y + 9z = 4$

First multiply 2nd and 3rd equation by (-2) and (-2/7) respectively and add to 1st equation to get 4th and 5th equation.

$-10y + 20z = 1$ $\quad -38/7\, y + 24/7\, z = 27/7$

or, $-38y + 24z = 27$

Then we multiply 5th equation by (-5/19) and add to 4th equation to get,

$260/19\, z = -116/19$ $\quad$ or, $260z = 116$ $\quad$ or, $z = 29/65$

The upper triangular form therefore,

$2x - 4y + 6z = 5$

$-10y + 20z = 1$

$z = 29/65$

Solving these equation we get required solutions, $x = 293/130$, $y = 103/130$, $z = 29/65$

Explain Gaussian Elimination method to solve linear system of equation.

let the linear system of equations in n unknown be given by ;

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

There are two steps in gausian elimination.

step 1: Eliminate the unknowns to obtain upper triangular system. To eliminate x_1 from 2nd equation multiply it by $(-a_{11}/a_{21})$ and add to 1st equation to obtain.

$$(-a_{11}/a_{21})\, a_{22}x_2 + \cdots + (-a_{11}/a_{21})\, a_{2n}x_n = (-a_{11}/a_{21})\, b_2$$

let write it.

$$a'_{22}x_2 + \cdots + a'_{2n}x_n = b'_2$$

similarly eliminate x_1 from all equation except 1st equation. And by this way eliminate other variable from below equations and get upper triangular.

Now the upper triangle form.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a'_{22}x_2 + \cdots + a'_{2n}x_n = b'_2$$
$$\vdots$$
$$a'_{nn}x_n = b'_n$$

step 2: Now solve the equation to get required solutions. From the last equation of the system we obtain,

$$x_n = \frac{b_n^{(n-1)}}{a_n^{(n-1)}}$$

Similarly, we can solve for all unknown.

Trapezoidal Rule :

Describe the geometric meaning of Trapezoidal Rule.

The geometric significance of trapezoidal rule is that the curve $y = f(x)$ is replaced by n straight lines joining the points (x_0, y_0) and (x_1, y_1), (x_1, y_1) and (x_2, y_2), ... (x_{n-1}, y_{n-1}) and (x_n, y_n). The area bounded by the curve $y=f(x)$, the ordinates $x = x_0$ and $x = x_n$, and the x-axis is then approximately equivalent to the sum of the areas of the n trapeziums obtained.

Gauss sheidal Method :

Solve the following equations using Gauss Sheidal Method.

$$10x + 2y + z = 9$$
$$2x + 20y - 2z = -44$$
$$-2x + 3y + 10z = 22$$

we get from the equations,

$$x = 9/10 - 1/5y - 1/10z \quad \text{———} (i)$$

$$y = -\frac{11}{5} - \frac{1}{10}x + \frac{1}{10}z = \frac{11}{5} - \frac{1}{10}\left(\frac{9}{10} - \frac{1}{5}y - \frac{1}{10}z\right) + \frac{1}{10}z = \frac{211}{100} + \frac{1}{50}y + \frac{11}{100}z$$

$$\text{or, } \frac{49}{50}y = \frac{211}{100} + \frac{11}{100}z$$

$$\text{or, } y = \frac{211}{98} + \frac{11}{98}z \quad \text{———} (ii)$$

$$z = \frac{22}{10} + \frac{1}{5}x - \frac{3}{10}y = \frac{22}{10} + \frac{1}{5}\left(\frac{9}{10} - \frac{1}{5}y - \frac{1}{10}z\right) - \frac{3}{10}y$$
$$= \frac{119}{50} - \frac{17}{50}y - \frac{1}{50}z$$

or, $49/50\ z = \frac{119}{50} - \frac{17}{50} y$

or, $49z = 119 - 17y = 119 - 17\left(\frac{211}{98} + \frac{11}{98} z\right) = \frac{8075}{98} + \frac{187}{98} z$

or, $\frac{34015}{98} z = \frac{8075}{98}$

$z = \frac{8075}{34015} = 0.265$

From (i) and (ii) we get,

$x = 0.473$, $y = 2.183$, $z = 0.265$

Lagrange Interpolation :

Lagrange Interpolation formular for unequal distance.

Derive Lagrange Interpolation formular for unequal distance. Given

Let $y(x)$ be continuous and differentiable $(n+1)$ times in the interval (a,b). Given $(n+1)$ unequally distanced points $(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$. We wish to find a polynomial of degree n. say $L_n(x)$, such that,

$$L_n(x_i) = y(x_i) = y_i, \quad i = 0, 1, 2 \ldots n$$

Then the polynomial is,

$$L_n(x) = \sum_{i=0}^{n} l_i(x) y_i$$

where,

$$l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

which obviously satisfies the condition,

$$l_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

If we set,

$$\Pi_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)$$

$$\Pi'_{n+1}(x_i) = \frac{d}{dx}\left[\Pi_{n+1}(x)\right]_{x=x_i} = (x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)$$

so,

$$l_i(x) = \frac{\Pi_{n+1}(x)}{(x-x_i)\,\Pi'_{n+1}(x_i)}$$

hence,

$$L_n(x) = \sum_{i=0}^{n} \frac{\Pi_{n+1}(x)}{(x-x_i)\,\Pi'_{n+1}(x_i)} y_i$$

This is lagrange interpolation formula.

III Newton's Interpolation :

Find the value of tan (0·05) from the following data : (0·10, 0·1003), (0·15, 0·1511), (0·20, 0·2027), (0·25, 0·2553), (0·30, 0·3039)

Ans.
The table of difference is in right :

To find tan (0·05) we have,

$0{\cdot}05 = 0{\cdot}10 + p(0{\cdot}05)$, which gives, $p = -1$

hence, according to Newton's forward difference Interpolation formula,

$$\tan(0{\cdot}05) = 0{\cdot}1003 + (-1)\,0{\cdot}0508 + \frac{(-1)(-1-1)}{2}(0{\cdot}0008) + \frac{(-1)(-1-1)(-1-2)}{6}(0{\cdot}0002) + \frac{(-1)(-1-1)(-1-2)(-1-3)}{24}(0{\cdot}0002)$$

$$= 0{\cdot}0503$$

hence ~~tan~~ tan (0·05) = 0·0503

x	y	Δ	Δ²	Δ³	Δ⁴
0·10	0·1003				
		0·0508			
0·15	0·1511		0·0008		
		0·0516		0·0002	
0·20	0·2027		0·0010		0·0002
		0·0526		0·0004	
0·25	0·2553		0·0014		
		0·0540			
0·30	0·3039				

田 Curve Fitting :

Define curve fitting. Explain the purpose of it.

Ans.

curve fitting : curve fitting is the process of constructing a curve, or mathematical function, that has the best fit to a series of data points, possibly subject to constrains.

purpose : curve fitting, also known as regression analysis, is used to find the 'best-fit' line or curve for a series of data points. Most of the time, the curve fit will produce an equation that can be used to find points anywhere along the curve.

Describe the least square curve fitting procedure for a straight line.

Ans : let $Y = a_0 + a_1 x$ be a straight line to be fitted to the given data (x_i, y_i).

Then we have,

$$S = [Y_1 - (a_0 + a_1 x_1)]^2 + [Y_2 - (a_0 + a_1 x_2)]^2 + \cdots + [Y_m - (a_0 + a_1 x_m)]^2$$

For s to be minimum,

$$\frac{ds}{da_0} = 0 = -2[Y_1 - (a_0 + a_1 x_1)] - 2[Y_2 - (a_0 + a_1 x_2)] - \cdots -2[Y_m - (a_0 + a_1 x_m)]$$

and, $$\frac{ds}{da_1} = 0 = -2x_1[Y_1 - (a_0 + a_1 x_1)] - 2x_1[Y_2 - (a_0 + a_1 x_2)] - \cdots -2x_1[Y_m - (a_0 + a_1 x_m)]$$

The above equation simplify to,

$$m a_0 + a_1 (x_1 + x_2 + \cdots + x_m) = Y_1 + Y_2 + \cdots + Y_m$$

and, $$a_0 (x_1 + x_2 + \cdots + x_m) + a_1 (x_1^2 + x_2^2 + \cdots + x_m^2) = x_1 Y_1 + x_2 Y_2 + \cdots + x_m Y_m$$

or more compactly to,

$$m a_0 + a_1 \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i \quad \text{and}$$

Now we can easily solve for a_0 and a_1,

$$A^1 = \frac{m \sum_{i=1}^{m} x_i Y_i - \sum_{i=1}^{m} x_i \sum_{i=1}^{m} Y_i}{m \sum^m x_i^2 - (\sum^m x_i)^2} \quad \text{and}$$

7) The exponential function $y = ae^{bx}$ is fitted to the data : (1·0, 40·170), (1·2, 73·196), (1·4, 133·372), (1·6, 243·02). Find the value of a and b.

Ans. we have

$y = ae^{bx}$

Therefore,

$$\ln y = \ln a + bx \Rightarrow Y = A_0 + A_1 X$$

where, $Y = \ln y$, $A_0 = \ln a$, $A_1 = b$ and $X = x$

The table of values is given right :

we obtain, $m = \bar{X} = 1·3$, $\bar{Y} = 4·593$

X	Y = ln y	X^2	XY
1·0	3·693	1·0	3·693
1·2	4·293	1·44	5·152
1·4	4·893	1·96	6·850
1·6	5·493	2·56	8·789
5·2	18·372	6·96	24·484

~~we obtain,~~

~~m~~ Then, $$A_1 = \frac{m\sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} x_i \sum_{i=1}^{m} y_i}{m\sum_{i=1}^{m} x_i^2 - \left(\sum_{i=1}^{m} x_i\right)^2}$$

$$= \frac{1·13(24·484) - 5·2(18·372)}{1·3(6·96) - 27·04}$$

$$= 3·772$$

$$A_0 = \bar{Y} - A_1\bar{X} = 4·593 - 4·904$$

$$= -0·311$$

$$a = e^{A_0} = 0·733 \quad b = 3·772$$

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2

The exponential function $y = ae^{bx}$ is fitted to the data: $(0, 0\cdot10)$, $(0\cdot5, 0\cdot45)$, $(1\cdot0, 2\cdot15)$, $(1\cdot5, 9\cdot15)$, $(2\cdot0, 40\cdot35)$, $(2\cdot5, 180\cdot75)$. Find the value of a and b.

Ans.: we have

$$y = ae^{bx}$$

Therefore,

$$\ln y = \ln a + bx \quad \Rightarrow \quad Y = A_0 + A_1 X$$

where, $Y = \ln y$, $A_0 = \ln a$, $A_1 = b$ and $X = x$.

The table of values is given right:

x	Y=lny	X²	XY
0	-2·303	0	0
0·5	-0·898	0·25	-0·449
1·0	0·765	1·0	0·765
1·5	2·219	2·25	3·321
2·0	3·698	4·0	7·396
2·5	5·197	6·25	12·992
7·5	8·672	13·75	24·025

We obtain, $m = \bar{X} = 1\cdot25$ $\quad \bar{Y} = 1\cdot445$

$$A^1 = \frac{\sum_{i=1}^{m} x_i y_i - \sum_{i=1}^{m} x_i \sum_{i=1}^{m} y_i}{\sum_{i=1}^{m} x_i^2 - \left(\sum_{i=1}^{m} x_i\right)^2}$$

$$= \frac{1\cdot25\,(24\cdot025) - 7\cdot5\,(8\cdot672)}{1\cdot25\,(13\cdot75) - 56\cdot25} = 0\cdot896$$

$$A_0 = \bar{Y} - A_1\bar{X} = 1\cdot445 - 1\cdot12 = 0\cdot325$$

$$a = e^{A_0} = 1\cdot384 \qquad b = 0\cdot896$$

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Lecture 4 Date: 25.2.22

Newton's ~~forward~~ formula for Forward Interpolation :

→ let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2 \ldots y_n$ for the equidistant values $x_0, x_1, \ldots x_n$ of the independent variable x.

→ It is required to find $\phi(x)$, a polynomial of the n-th degree such that y and $\phi(x)$ agree at the tabulated points (i.e. they ~~th~~ have the same values).

→ let $\phi(x)$ denote a polynomial of the n-th degree.

→ This polynomial can be written in the form

$$\phi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad \text{——(1)}$$

→ we shall now determinate the coefficients $a_0, a_1, a_2 \ldots a_n$, so that we can get $\phi(x_0) = y_0$, $\phi(x_1) = y_1$, $\phi(x_2) = y_2 \ldots \phi(x_n) = y_n$

we ~~can subst~~ know that,

$$\phi(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}) \quad \text{——(1)}$$

we can substitute the given successive values $x_0, x_1, x_2, \ldots x_n$ in eq(1),

At the same time we can put,

$\phi(x_0) = y_0$ $\qquad$ $\phi(x_n) = y_n$

$\phi(x_1) = y_1$

And, let $x_1 - x_0 = h$. Then, $x_2 - x_1 = 2h$ etc. (since the values of x are equidistance).

if $x = x_0$ then, eq(1)

$$\phi(x_0) = a_0 + a_1(x_0-x_0) + a_2(x_0-x_0)(x_0-x_1) + \cdots + a_n(x_0-x_0)\cdots(x-x_{n-1})$$

or, $\phi(x_0) = a_0 = y_0$

$\therefore a_0 = y_0$

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similarly, substituting x_1 in eq. (1) we get

$$\phi(x_1) = y_1 = a_0 + a_1(x_1 - x_0) = y_0 + a_1 h$$

$$a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

substituting x_2 in eq. (1) we get.

$$y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$= y_0 + \frac{y_1 - y_0}{h}(2h) + a_2(2h)h$$

$$\Rightarrow a_2 = \frac{y_2 - y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2h^2}$$

similarly,

$$a_3 = \frac{\Delta^3 y_0}{3!h^3} \qquad a_5 = \frac{\Delta^5 y_0}{5!h^5}$$

$$a_4 = \frac{\Delta^4 y_0}{4!h^4} \qquad a_n = \frac{\Delta^n y_0}{n!h^n}$$

substituting in eq(1) the values $a_0; a_1, \dots a_n$ we have,

$$\phi(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \dots + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\dots(x - x_{n-1}) \quad \text{—(2)}$$

This is Newton's formula for forward interpolation written in term of

This formula can be simplified by a change of variable.

we can rewrite eq(2) in the following equivalent form,

$$\phi(x) = y_0 + \Delta y_0\left(\frac{x - x_0}{1!h}\right) + \frac{\Delta^2 y_0}{2!}\left(\frac{x - x_0}{h}\right)\left(\frac{x - x_1}{h}\right) + \dots + \frac{\Delta^n y_0}{n!}\left(\frac{x - x_0}{h}\right)\left(\frac{x - x_1}{h}\right)\dots\left(\frac{x - x_{n-1}}{h}\right) \quad \text{—(3)}$$

Now, put the following in eq.(3)

$$\frac{x - x_0}{h} = u \quad \text{or,} \quad x = x_0 + hu$$

Then, since $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, etc. we have

$$\frac{x - x_1}{h} = \frac{x - (x_0 + h)}{h} = \frac{x - x_0}{h} - \frac{h}{h} = u - 1$$

Similarly,

$$\frac{x - x_2}{h} = \frac{x - (x_0 + 2h)}{h} = \frac{x - x_0}{h} - \frac{2h}{h} = u - 2,$$

. . . .

$$\frac{x - x_{n-1}}{h} = \frac{x - [x_0 + (n-1)h]}{h} = \frac{x - x_0}{h} - \frac{(n-1)h}{h} = u - n + 1$$

Substituting the values of $(x - x_0)/h$, $(x - x_1)/h$ etc. in eq.(3),

$$\phi(x) = \phi(x_0 + hu) = g(u)$$

$$= y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \cdots \frac{u(u-1)(u-2)(u-3)\cdots(u-n+1)}{n!} \Delta^n y_0 \quad \text{——(4)}$$

This is the form in which Newton's formula for forward interpolation is usually written.

forward → from y_0 onward to the right (forward from y_0) and none to the left of this value

For this, this formula is used mainly for interpolating the values of y near the begining of a set of tabular values.

The population of a town is given below for a range of years. Estimate the population for the year 1895.

Year : x	1891	1901	1911	1921	1931
Population: y (in thou)	46	66	81	93	101

Ans. 54.85 K

Newton's formula for Backward Interpolation :

let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2 \dots y_n$ for the equidistant values $x_0, x_1, x_2 \dots x_n$ of the independent variable x.

It is required to find $\phi(x)$, a polynomial of the n-th degree such that y and $\phi(x)$ agree at the tabulated points (i.e. they have the same values).

let $\phi(x)$ denote a polynomial of the n-th degree.

This polynomial can be written in the form

$$\begin{cases} \phi(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + a_3(x-x_n)(x-x_{n-1})(x-x_{n-2}) \\ \qquad + \dots + a_n(x-x_n) \dots (x-x_1) \text{———} (1) \end{cases}$$

we shall now determine the coefficients $a_0, a_1, a_2 \dots a_n$ so that we can get,

$\phi(x_n) = y_n$, $\phi(x_{n-1}) = y_{n-1}$, $\phi(x_{n-2}) = y_{n-2}$, $\dots$ $\phi(x_0) = y_0$.

we know that,

$$\phi(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + a_3(x-x_n)(x-x_{n-1})(x-x_{n-2})$$
$$+ \dots + a_n(x-x_n) \dots (x-x_1) \text{———} (1)$$

we can substitute the given successive values $x_n, x_{n-1}, x_{n-2}, \dots x_0$ in eq.(1)

At the same time we can put $\phi(x_n) = y_n$, $\phi(x_{n-1}) = y_{n-1}$, $\phi(x_{n-2}) = y_{n-2}$, $\dots$ $\phi(x_0) = y_0$.

And let $x_{n-1} - x_n = -h$, Then, $x_{n-2} - x_n = -2h$, etc. (Since the values of x are equidistance).

In eq.(1) we have,

$$\phi(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + a_3(x-x_n)(x-x_{n-1})(x-x_{n-2})$$
$$+ \dots + a_n(x-x_n)(x-x_{n-1}) \dots (x-x_1) \text{———} (1)$$

That is,

at $x = x_n$ (substituting x with x_n in eq (1)) we have

$$\phi(x_n) = a_0 + a_1(x_n - x_n) + a_2(x_n - x_n)(x_n - x_{n-1}) + \cdots$$

$$+ a_n(x_n - x_n)(x_n - x_{n-1}) \cdots (x_n - x_1)$$

$$\text{or, } \phi(x_n) = a_0 = y_n$$

$\therefore \; a_0 = y_n$

Similarly,

$$y_{n-1} = a_0 + a_1(x_{n-1} - x_n) = y_n - a_1 h$$

$$\text{or, } a_1 = \frac{y_n - y_{n-1}}{h} = \frac{\nabla y_n}{h}$$

substituting x_2 in eq.(1) we get,

$$y_{n-2} = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$= y_n + \frac{y_n - y_{n-1}}{h}(-2h) + a_2(-2h)(-h)$$

$$\Rightarrow a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2h^2}$$

substituting x_3 in eq.(1) we get,

$$y_{n-3} = a_0 + a_1(x_{n-3} - x_n) + a_2(x_{n-3} - x_n)(x_{n-3} - x_{n-1}) +$$

$$a_3(x_{n-3} - x_n)(x_{n-3} - x_{n-1})(x_{n-3} - x_{n-2})$$

$$a_3 = \frac{\nabla^3 y_n}{3!\,h^3}$$

Similarly,

$$a_4 = \frac{\nabla^4 y_n}{4!\,h^4} \quad \cdots \quad a_n = \frac{\nabla^n y_n}{n!\,h^n}$$

substituting in eq.(1) the values $a_0, a_1, a_2 \dots a_n$ we have,

$$\phi(x) = y_n + \frac{\nabla y_n}{h}(x-x_n) + \frac{\nabla^2 y_n}{2h^2}(x-x_n)(x-x_{n-1}) + \frac{\nabla^3 y_n}{3!\,h^3}(x-x_n)(x-x_{n-1})(x-x_{n-2}) + \dots + \frac{\nabla^n y_n}{n!\,h^n}(x-x_n)(x-x_{n-1})\dots(x-x_1) \quad \text{---(2)}$$

This is Newton's formula for backward interpolation, written in term of x.

This formula can be simplified by a change of variable.

Now, we can rewrite eq(2) in the following equivalent form

$$\phi(x) = y_n + \nabla y_n\left(\frac{x-x_n}{h}\right) + \frac{\nabla^2 y_n}{2}\left(\frac{x-x_n}{h}\right)\left(\frac{x-x_{n-1}}{h}\right) + \frac{\nabla^3 y_n}{3!}\left(\frac{x-x_n}{h}\right)\cdot\left(\frac{x-x_{n-1}}{h}\right)\left(\frac{x-x_{n-2}}{h}\right) + \dots + \frac{\nabla^n y_n}{n!}\left(\frac{x-x_n}{h}\right)\dots\left(\frac{x-x_1}{h}\right) \quad \text{---(3)}$$

Now, put the following in eq.(3),

$$\frac{x-x_n}{h} = u \quad \text{or.} \quad x = x_n + hu$$

Then, since $x_{n-1} = x_n - h$, $x_{n-2} = x_n - 2h$. etc. we have

$$\frac{x-x_{n-1}}{h} = \frac{x-(x_n-h)}{h} = \frac{x-x_n}{h} + \frac{h}{u} = u+1$$

similarly,

$$\frac{x-x_{n-2}}{h} = \frac{x-(x_n-2h)}{h} = \frac{x-x_n}{h} + \frac{2h}{h} = u+2,$$

$$\frac{x-x_1}{h} = \frac{x-[x_n-(n-1)h]}{h} = \frac{x-x_n}{h} + \frac{(n-1)h}{h} = u+n-1$$

substituting the value of $(x-x_n)/h$, $(x-x_{n-1})/h$ etc. in eq.(3)

$$\phi(x) = \phi(x_n + hu) = g(u)$$

$$= y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \dots + \frac{u(u+1)(u+2)(u+3)\dots(u+n-1)}{n!}\nabla^n y_n \quad \text{---(2)}$$

This is the form in which Newton's formula for backward interpolation is usually written.

The reason for the name 'backward' interpolation formula since the formula contains values of the tabulated function from y_n onward to the left (backward from y_n) and none to the right of this value.

Because of this fact this formula is used mainly for interpolating the values of y near the end of a set of tabular values.

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Class work:

$\tan(x)$ for $0.10 \leq x \leq 0.30$ Find $\tan(0.26)$

Ans: 0.265952

x	0.10	0.15	0.20	0.25	0.30
$y=\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

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population (1925) Ans. 98.837 K

Extrapolation:

If the nth differences of a tabulated function are constant when the values of the independent variable are taken in arithematic progression, the function is a polynomial of degree n.

The process of finding the value of y for some value of x outside the given range is called extrapolation.

If a tabulated value is a polynomial, then interpolation and exterpolation would give exact values.

Newton's forward difference formula is used to extrapolate values to the right of y_n.

Newton's Backward difference formula is used to extrapolate value to the left of y_0.

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Class work :

The table below gives the values of $\tan x$ for $0.10 \leq x \leq 0.30$,

Find $\tan(0.05)$ and $\tan(0.50)$

$\tan(0.5) = 0.05004 8$

$\tan(5.0) = 0.549836$

x	0.10	0.15	0.20	0.25	0.30
$Y = \tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Solution :

Example:

The table below gives the values of y are consecutive terms of a series of which the number 21.6 is the 6^{th} term.

Find the first and tenth terms of the series

x	3	4	5	6	7	8	9
y	2.7	6.4	12.5	21.6	34.3	51.2	72.9

solution:

x	y	Δ	Δ^2	Δ^3	Δ^4
3	2.7				
		3.7			
4	6.4		2.4		
		6.1		0.6	
5	12.5		3.0		0
		9.1		0.6	
6	21.6		3.6		0
		12.7		0.6	
7	34.3		4.2		0
		16.9		0.6	
8	51.2		4.8		
		21.7			
9	72.9				

third differences are constant

hence the tabulated function represents a polynomial of the third degree.

solution,

$y(1) = 0.1$

$y(10) = 100$

Interpolation With Unequal Intervals of Argument

→ The interpolation formulas derived before are applicable only when the values of the functions are given at equidistant intervals.

→ It is sometimes inconvenient or even impossible to obtain values of a function at equidistant values of its argument.

→ Two formulas can be applicable on a functional values with unequal intervals of arguments.

1. Newton's formula — Divided diff. form
2. Lagrange's formula

Divided difference :

The differences used in the Newton's formula are called divided diff.

Let $y = f(x)$ denote a function which takes the values $y_0, y_1, y_2 \ldots y_n$ for the values $x_0, x_1, x_2, \ldots x_n$ of the independent variable x.

First order Difference is,

$$\delta(x_1, x_0) = \frac{y_1 - y_0}{x_1 - x_0}$$

$$\delta(x_2, x_1) = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\delta(x_3, x_2) = \frac{y_3 - y_2}{x_3 - x_2} \quad \text{etc.}$$

Second order diff. is,

$$\delta(x_2, x_1, x_0) = \frac{\delta(x_2, x_1) - \delta(x_1, x_0)}{x_2 - x_0}$$

$$\delta(x_3, x_2, x_1) = \frac{\delta(x_3, x_2) - \delta(x_2, x_1)}{x_3 - x_1}$$

Third order diff.

$$\delta(x_3, x_2, x_1, x_0) = \frac{\delta(x_3, x_2, x_1) - \delta(x_2, x_1, x_0)}{x_3 - x_0}$$

Note that the order of any divided difference is one less than the no. of the arguments in it.

$(x_0, y_0), (x_1, y_1), (x_2, y_2) \rightarrow$ points on a curve

1st order diff : slope of the line through any 2 pts

x	y	δ	δ^2	δ^3
x_0	y_0			
		$\delta(x_1, x_0)$		
x_1	y_1		$\delta(x_2, x_1, x_0)$	
		$\delta(x_2, x_1)$		$\delta(x_3, x_2, x_1, x_0)$
x_2	y_2		$\delta(x_3, x_2, x_1)$	
		$\delta(x_3, x_2)$		
x_3	y_3			

Symmetry of Divided diff.

$\delta(1, 5, 9) = \delta(5, 9, 1)$
$= \delta(1, 9, 5)$
$= \delta(5, 1, 9) \ldots$

Example :

distance unequal

x	y	δ	δ^2	δ^3	δ^4	δ^5
-2	729					
		418.5				
0	108		-95.7			
		-60		17.1		
3	-72		24		-2.9	
		60		9		0.49
5	48		-39		2	
		-96		7		
7	-144		-4			
		-108				
8	-252					

$\delta^2(x_3, x_2, x_1)$

$$= \frac{\delta(x_3, x_2) - \delta(x_2, x_1)}{x_3 - x_1}$$

$$= \frac{60 - (-60)}{5 - 0} = 24$$

$\delta^5(x_5, x_4, x_3, x_2, x_1, x_0)$

$$= \frac{\delta^4(x_5, x_4, x_3, x_2, x_1) - \delta^4(x_4, x_3, x_2, x_1, x_0)}{x_5 - x_0}$$

$$= \frac{2 - (-2.9)}{8 - (-2)} = 0.49$$

Lecture - 5

Date: 26.2.22

Newton's Divided Difference Formula :

let $y = f(x)$ be a function which takes the value $(x_0, y_0), (x_1, y_1) \ldots (x_n, y_n)$. Since there are $n+1$ pairs of values of x and y, so $f(x)$ can be represented as a polynomial in x of degree n.

$$y = f(x) = y_0 + (x - x_0)\underbrace{[x_0, x_1]}_{\text{1st order difference}} + (x - x_0)(x - x_1)\underbrace{[x_0, x_1, x_2]}_{\text{2nd order}}$$

$$+ \ldots (x - x_0)(x - x_1) + \ldots + [x_0, x_1, x_2 \ldots x_n]$$

x	$f(x) = y$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x_0	y_0			
		$[x_0, x_1]$		
x_1	y_1		$[x_0, x_1, x_2]$	
		$[x_1, x_2]$		$[x_0, x_1, x_2, x_3]$
x_2	y_2		$[x_1, x_2, x_3]$	
		$[x_2, x_3]$		
x_3	y_3			
$\vdots$		x		
x_n	y_n			

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

$$\boxed{y = f(x) = f(x_0) + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0) + \ldots}$$

Derivation of formula :

let $y = f(x)$ be a function which takes the value $(x_0, y_0), (x_1, y_1) \ldots (x_n, y_n)$

$$[x, x_0, x_1] = \frac{[x, x_1] - [x_0, x_1]}{x - x_1}$$

$$[x - x_0] = \frac{y_0 - y}{x_0 - x} = \frac{y - y_0}{x - x_0}$$

$$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

$$y = y_0 + (x - x_0)[x, x_0]$$

$$\boxed{y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1]}$$

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Derivation of formula : let $y = f(x)$ be a function which takes the value $(x_0, y_0), (x_1, y_1) \dots (x_n, y_n)$

$$[x, x_0] = \frac{y_0 - y}{x_0 - y} = \frac{y - y_0}{x - x_0}$$

$$y = y_0 + (x - x_0)[x, x_0]$$

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[\cancel{x_0}\ x, x_0, x_1]$$

$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$$

$\dots$

General formula

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) \cdot [x_0, x_1, x_2, x_3] + (x - x_0)(x - x_1)(x - x_2)(x - x_3)[x, x_0, x_1, x_2, x_3]$$

$$y = f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1) \cdot (x - x_2)[x_0, x_1, x_2, x_3] + \dots + (x - x_0)(x - x_1) \dots + (x - x_{n-1})[x_0, x_1, \dots x_n]$$

This is known as Newton's Divided difference interpolation formula.

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_1}$$

Example:

Lagrange's Interpolation formula :

This formula can be applied on non-equidistant variables.

It is based on the theorem that the divided differences of a polynomial of the nth degree are constant.

Hence the (n+1)th divided differences of a polynomial of the nth degree is zero.

Let f(x) denote a polynomial of the nth degree which takes the values $y_0, y_1, y_2 \dots y_n$ when x has the values $x_0, x_1, x_2 \dots x_n$ respectively.

Then the (n+1)th differences of this polynomial are zero.

This formula is used to find the value of the independent variable corresponding to a given value of the function.

Therefore,

$$\delta(x, x_0, x_1, x_2 \dots x_{n-1}, x_n) = 0$$

$$\Rightarrow \frac{y}{(x-x_0)(x-x_1)\cdots(x-x_n)} + \frac{y_0}{(x_0-x)(x_0-x_1)\cdots(x_0-x_n)} +$$

$$\frac{y_1}{(x_1-x)(x_1-x_0)\cdots(x_1-x_n)} + \dots + \frac{y_n}{(x_n-x)(x_n-x_0)\cdots(x_n-x_{n-1})} = 0$$

Thus,

$$y = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} \times y_1$$

$$+ \dots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} \cdot y_n$$

- This is the Lagrange's formula for interpolation.
- In this formula, $y = y_0, y_1, y_2, \dots y_n$ when $x = x_0, x_1, x_2, \dots x_n$ respectively.
- The values of the independent variable may or may not be equidistant.

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Class work:

The following table gives certain corresponding values of x and $\log_{10} x$. Compute the value of $\log_{10}(325.5)$ using Lagrange's interpolation formula.

	x_0	x_1	x_2	x_3
x	321.0	322.8	324.2	325.0
$\log_{10} x$	2.50651	2.50893	2.51081	2.51188

$x = 323.5$

Ans: 2.50987

using lagrange's interpolation formula, find the form of the function $y(x)$ from the following table.

x	0	1	3	4
y	-12	0	12	24

Ans. $y = (x-1)(x^2-5x+12)$
$= x^3 - 6x^2 + 17x - 12$

Find from lagrange's interpolating polynomial of degree 2 approximating the function $y = \ln(x)$ defined by the following table of values.

Hence, determine the value of $\ln(2.7)$

x	2	2.5	3.0
$y=\ln(x)$	0.69315	0.91629	1.09861

Ans. $y = -0.08169x^2 + 0.81366x - 0.60761$
$\ln 2.7 = 0.9941164$

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Newton's General Interpolation Formula :

From $\frac{y-y_0}{x-x_0} = \delta(x, x_0)$ we can derive

$$y = y_0 + (x - x_0)\,\delta(x, x_0)$$

From $\frac{\delta(x, x_0) - \delta(x_0, x_1)}{x - x_1} = \delta(x, x_0, x_1)$ we can derive

$$\delta(x, x_0) = \delta(x_0, x_1) + (x - x_1)\,\delta(x, x_0, x_1)$$

Example :

The following table gives certain corresponding values of x and $\log_{10} x$. Find $\log_{10}(323.5)$ by Newton's General formula.

x	$\log_{10} x$	δ'	
322.8	2.50893		
		0.00134286	
324.2	2.51081		-0.00000244
		0.00133750	
325.0	2.51188		

$y = \log_{10}(323.5)$

$= 2.50893 + (323.5 - 322.8)(0.00134286)$

$+ (323.5 - 322.8)(323.5 - 324.2)(-0.00000244)$

$= 2.50893 + 0.000940 + 0.0000012$

$= 2.50987$

Find $\log_{10}(301)$ using Newton's formula

x	$\log_{10} x$	δ^1	δ^2	δ^3
300	2.4771			
		0.00144		
304	2.4829		-2.3653E-06	
		0.00143		5.15322E-09
305	2.4843		-2.3292E-06	
		0.00142		
307	2.4871			

$\log_{10}(301) = 2.4771 + (1)\,0.00144 + (1)(-3)(-2.3653E-06) + (1)(-3)(-4)(5.15322E-09)$

$= 2.478567$

Find the polynomial representation of $f(x)$ using the following table:

x	$f(x)$
-1	3
0	-6
3	39
6	822
7	1611

x	$f(x)$	δ	δ^2	δ^3	δ^4
-1	3				
		-9			
0	-6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

$$f(x) = 3 + (x+1)(-9) + x(x+1)(6) + x(x+1)(x-3)(5) + x(x+1)(x-3)(x-6)$$
$$= x^4 - 3x^3 + 5x^2 - 6$$

using Newton's general interpolation formula, find from the following table the value of y for $x = 5.60275$

x	5.600	5.602	5.605	5.607	5.608
y	0.77556588	0.77682616	0.77871250	0.77996571	0.78059114

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Errors in Polynomial Interpolation :

$$y(x) - \phi_n(x) = \frac{\Pi_{n+1}(x)}{(n+1)!} \, y^{(n+1)}(\varepsilon)$$

which is the required expression for the error.

Interpolation?
→ Extrapolation (newton forward/back)
→ newton forward/back →
→ divided differ.
general for
degree

Lecture - 6
Curve Fitting

Least-square curve fitting procedure

Let the set of data points be (x_i, y_i), $i = 1, 2 \ldots m$ and let the curve given by $Y = f(x)$ be fitted to this data. At $x = x_i$, the experimental (or observed) value of the ordinate is y_i and the corresponding value on the fitting curve is $f(x_i)$. If e_i is the error of approximation at $x = x_i$, then we have

x_i → original
$f(x_i)$ → measured

$$e_i = y_i - f(x_i)$$

If we write

$$S = [y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \cdots + [y_m - f(x_m)]^2$$

$$= e_1^2 + e_2^2 + \cdots + e_m^2 \quad \text{—— (2)}$$

→ **Fitting a straight line**

Let $Y = a_0 + a_1 x$ be the straight line to be fitted to the given data. Then corresponding to eq. (2) we have

$$S = [y_1 - (a_0 + a_1 x_1)]^2 + [y_2 - (a_0 + a_1 x_2)]^2 + \cdots + [y_m - (a_0 + a_1 x_m)]^2 . \quad \text{—— (3)}$$

For S to be minimum, we have

$$\frac{\partial S}{\partial a_0} = 0 = -2\,[y_1 - (a_0 + a_1 x_1)] - 2\,[y_2 - (a_0 + a_1 x_2)] - \cdots - 2\,[y_m - (a_0 + a_1 x_m)] \quad \text{—— (4)}$$

and,

$$\frac{\partial S}{\partial a_1} = 0 = -2\,x_1\,[y_1 - (a_0 + a_1 x_1)] - 2\,x_2\,[y_2 - (a_0 + a_1 x_2)] - \cdots - 2\,x_m\,[y_m - (a_0 + a_1 x_m)] \quad \text{—— (5)}$$

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The above equations simplify to

$$ma_0 + a_1(x_1 + x_2 + \cdots + x_m) = y_1 + y_2 + \cdots + y_m \quad \text{———} (5)(a)$$

and

$$a_0(x_1 + x_2 + \cdots + x_m) + a_1(x_1^2 + x_2^2 + \cdots + x_m^2) = x_1 y_1 + x_2 y_2 + \cdots + x_m y_m \quad \text{———} (5)b$$

or, more compactly to

$$ma_0 + a_1 \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i \quad \text{———} (5)a$$

and,

$$a_0 + \sum_{i=1}^{m} x_i + a_1 \sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i y_i \quad \text{———} (5b)$$

Since the x_i and y_i are known quantities, called the normal equations, can be solved for the two unknown a_0 and a_1.

Differentiating eqs. 4(a) and 4(b) with respect to a_0 and a_1 respectively, we find that $\partial^2 S/\partial a_0^2$ and $\partial^2 S/\partial a_1^2$ will both be positive at the points a_0 and a_1. Hence these values provide a minimum of S.

Example :

The table below gives the temperatures T (in °c) and lengths l (in nm) of a heated rod. If $l = a_0 + a_1 T$, find the best values for a_0 and a_1.

T (in °c)	L (in nm)
20	800·3
30	800·4
40	800·6
50	800·7
60	800·9
70	801·0

Solution :

To use formula, we require ΣT, ΣL, ΣT^2 and ΣTL and these are computed below:

T (in °c)	L(in nm)	T^2	TL
20	800·3	400	16006
30	800·4	900	24012
40	800·6	1600	32024
50	800·7	2500	40035
60	800·9	3600	48054
70	801·0	4900	56070

Using formula, we obtain

$6a_0 + 270a_1 = 4803 \cdot 9$

$270a_0 + 13900a_1 = 216201$

$a_0 = 800$ $\quad a_1 = 0 \cdot 0146$

Linear Curve Fitting Method:

If $y = a_0 + a_1 x$ find a_0, a_1

x	0	2	5	7
y	-1	5	12	20

Answer.

$4a_0 + 14a_1 = 36$

$14a_0 + 78a_1 = 210$

$a_0 = -1 \cdot 1381$

$a_1 = 2 \cdot 8966$

Non-Linear Curve Fitting Method

- Taking a straight line as an approximation for a curve is not sufficient for some curves.

- The following non-linear curve fitting methods can be used in such cases:
  - → Polynomial of nth degree
  - → Power function
  - → Exponential function

⇒ Polynomial of nth degree

Let the polynomial of the nth degree, that is,

$$Y = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n \quad \text{---} (1)$$

be fitted to the data points (x_i, y_i), $i = 1, 2, \ldots, m$. We then have

$$S = [y_1 - (a_0 + a_1 x_1 + \cdots + a_n x_1^n)]^2 + [y_2 - (a_0 + a_1 x_2 + \cdots + a_n x_2^n)]^2 + \cdots + [y_m - (a_0 + a_1 x_m + \cdots + a_n x_m^n)]^2$$

Equating, as before, the first partial derivatives to zero and simplifying, we get the following normal equations

$$m a_0 + a_1 \sum_{i=1}^{m} x_i + a_2 \sum_{i=1}^{m} x_i^2 + \cdots + a_n \sum_{i=1}^{m} x_i^n = \sum_{i=1}^{m} y_i$$

$$a_0 \sum_{i=1}^{m} x_i + a_1 \sum_{i=1}^{m} x_i^2 + \cdots + a_n \sum_{i=1}^{m} x_i^{n+1} = \sum_{i=1}^{m} x_i y_i$$

$$\vdots$$

$$a_0 \sum_{i=1}^{m} x_i^n + a_1 \sum_{i=1}^{m} x_i^{n+1} + \cdots + a_n \sum_{i=1}^{m} x_i^{2n} = \sum_{i=1}^{m} x_i^n y_i$$

These are $(n+1)$ equations in $(n+1)$ unknowns and hence can be solved for $a_0, a_1, \ldots a_n$. eq. (1) then gives the required polynomial of the nth degree

Example :

Fit a polynomial of second degree to the data points given in the following table .

$n=2$

$n+1$ term

x	0	1	2
y	1	6	17

solution :

To find the 2nd degree polynomial, we need to find Σx_i, Σy_i, Σx_i^2 + Σx_i^3 + Σx_i^4, $\Sigma x_i y_i$ and $\Sigma x_i^2 y_i$

x	y	x^2	x^3	x^4	xy	x^2y
0	1	0	0	0	0	0
1	6	1	1	1	6	6
2	17	4	8	16	34	68
3	24	5	9	17	40	74

we know that,

$$ma_0 + a_1 \Sigma x_i + a_2 \Sigma x_i^2 = \Sigma \cancel{xy}\, y_i$$

$$a_0 \Sigma x_i + a_1 \Sigma x_i^2 + a_2 \Sigma x_i^3 = \Sigma x_i y_i$$

$$a_0 \Sigma x_i^2 + a_1 \Sigma x_i^3 + a_2 \Sigma x_i^4 = \Sigma x_i^2 y_i$$

Therefore,

$$3a_0 + 3a_1 + 5a_2 = 24$$

$$3a_0 + 5a_1 + 9a_2 = 40$$

$$5a_0 + 9a_1 + 17a_2 = 74$$

$a_0 = 1$, $a_1 = 2$ $a_3 = 3$

$\therefore$ the polynomial is, $1 + 2x + 3x^2$.

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Power function :

→ In power function method we approximate the actual curve by substituting y_i by a power function of x.

→ Then, the approximation Y becomes a power function of x.

→ Let, $Y = f(x) = ax^c$ (power function of x)

→ Taking logarithm of both sides,

$$\log y = \log a + c \log x$$

→ This eq. is in the form $Y = a_0 + a_1 x$ where

$$Y = \log y,\ a_0 = \log a,\ a_1 = c,\ x = \log x.$$

st line
polynomial
power fun
exponen

→ Now we can use the least square method to solve this eq.

Exponential function :

→ In exponential function method we approximate the actual curve by substituting y_i by an exponential function of x.

→ Then, the approximation Y becomes a exponential function of x.

→ Let $Y = f(x) = a_0 e^{a_1 x}$ (i.e. a exponential function of x)

→ Taking logarithm of both sides,

$$\log Y = \log a_0 + a_1 x$$

$$Y = \log y,\ a_0 = \log a_0$$

use least square method

Exponential

Example :

Determine the constants a and b by the method of least square such that $y = ae^{bx}$ fits the following data

x	0	1	2
y	1	6	17

Solution:

Given, $y = ae^{bx}$

Taking logarithm on both sides,

$$\ln y = \ln a + bx$$

Setting, $\ln y = Y$, $\ln a = a_0$ and $b = a_1$ we get $Y = a_0 + a_1 x$.

Example:

Using the least square method,

$$5a_0 + 30a_1 = 17.025$$

$$30a_0 + 220a_1 = 122.150$$

$X = x$	y	$Y = \ln y$	x^2	xY
2	4.077	1.405	4	2.811
4	11.084	2.406	16	9.622
6	30.128	3.405	36	20.433
8	81.897	4.405	64	35.244
10	222.60	5.405	100	54.055
30	349.806	17.0272	220	122.164

So,

$$a_0 = 0.405$$

$$a_1 = 0.5$$

Hence,

$$a = e^{a_0} = e^{0.405} = 1.499$$

$$b = a_1 = 0.5$$

Weighted least sq. Approximation:

$$S = w_1 [y_1 - f(x_1)]^2 + w_2 [y_2 - f(x_2)]^2 + \ldots w_m [y_m - f(x_m)]^2$$

$$S = w_1 e_1^2 + w_2 e_2^2 + \ldots + w_m e_m^2$$

w_i are prescribed positive numbers and are called weights.

Matrix Equation

Matrix notation for following linear system of equation is as follows.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$A \cdot X = B$

Elementary transformation OR operation on Matrix

$\left.\begin{matrix} R_{ij} \text{ or} \\ R_i \leftrightarrow R_j \end{matrix}\right\} \rightarrow$ Interchange of ith and jth rows

$K \cdot R_i \} \rightarrow$ Multiplication of all elements of ith row by K (nonzero)

$\left.\begin{matrix} R_{ij}(K) \text{ or} \\ R_j + K \cdot R_i \end{matrix}\right\} \rightarrow$ Mul ith row by K and added to jth row.

Row echelon form of Matrix:

Steps:

(1) Every zero row of the matrix occurs below the non zero rows.

(2) Arrange all the rows in strictly decreasing order.

(3) Make all the entries zero below the leading (first non zero entry of the row) element of 1st row.

(4) Repeat steps-3 for each row.

Reduced Row echelon form of Matrix:

To convert the matrix : steps –

1. convert given matrix into row echelon form
2. Make all leading elements 1 (one)
3. Make all the entries zero above the leading element 1 of each row.

Numerical methods for solution of linear Eq.

1. Direct method
2. Iterative method

Direct Method :

This method produce the exact solution after a finite number of steps but are subject to errors due to round-off and other factors.

- Gauss Elimination method
- Gauss - Jordan method

Indirect Method :

Here, an approximation to the true solution is assumed initially to start method. By applying the method repeatedly, better and better approximations are obtained. For large systems, iterative methods are faster than direct methods and round-off error are also smaller:

1. Gauss seidel method
2. Gauss jacobi method

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Gauss Elimation method :

step :

1. start with augmented matrix [A:B]
2. convert matrix A into row echelon form (leading 1)
3. Apply back substitution for getting eqs.
4. solve the eqs and find the unknown variables (solution)

Example :

Solve by Gauss ~~Elimatia~~ Elimination method :

$$x+y+z = 9$$
$$2x-3y+4z = 13$$
$$3x+4y+5z = 40$$

By Augmented matrix,

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{array}\right]$$

$R_{12}(-2)$
$R_{13}(-3)$
(multiply row 1 by -3 and add to row 3)

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & \frac{12}{5} & 12 \end{array}\right]$$

$R_{23}(1/5)$

Now, solving eq. by back substitution

$\therefore \frac{12}{5}z = 12$

$\therefore z = 5$

$\therefore -5y + 2z = -5$

$\therefore -5y+10 = -5$

$\therefore y = 3$

$x+y+z = 9$

$\therefore x+3+5 = 9$

$\therefore x = 1$

Ans. $(x, y, z) = (1, 3, 5)$

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Lecture 7

Date: 4.3.22

Example: Solve by Gauss Elimination method :

$$x + 2y + z = 3$$
$$2x + 3y + 3z = 10$$
$$x - y + 2z = 13$$

Solution :

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 1 & -1 & 2 & 13 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & 1 & 4 \end{array}\right] \quad \begin{array}{l} R_{12}(-2) \\ R_{13}(-3) \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & 12/5 & 12 \end{array}\right] \quad R_{23}(1/5)$$

Now solving equations by back-substitution

$\therefore \frac{12}{5} z = 12$

$\therefore z = 5$

$\therefore -5y + 2z = -5$

$\therefore -5y + 10 = -5$

$\therefore y = 3$

$\therefore x + y + z = 9$

$\therefore x = 1$

Ans. $(x, y, z) = (1, 3, 5)$

Gauss Elimination Method with Partial Pivoting

1. Find largest absolute value (pivot element) in first column.
2. Make the pivot element row to first row.
3. Eliminate x_1 below the pivot element.
4. Again find pivot element in 2^{nd} and 3^{rd} row.
5. Make the pivot element row to second row.

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6. Eliminate x_2 below the pivot element

7. Apply back substitution for getting equations.

8. Solve the equations and find the unknown variables (i.e. solution)

Example: solve by Gauss Elimination method with partial pivoting

$$8x_2 + 2x_3 = -7$$
$$3x_1 + 5x_2 + 2x_3 = 8$$
$$6x_1 + 2x_2 + 8x_3 = 26$$

solution: By Augmented matrix,

$$\left[\begin{array}{ccc:c} 0 & 8 & 2 & -7 \\ 3 & 5 & 2 & 8 \\ 6 & 2 & 8 & 26 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 6 & 2 & 8 & 26 \\ 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc:c} 6 & 2 & 8 & 26 \\ 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{array}\right] \quad R_{12}(-1/2)$$

largest abs value in second column

$$\sim \left[\begin{array}{ccc:c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 4 & -2 & -5 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc:c} 6 & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -3/2 \end{array}\right] \quad R_{23}(-1/2)$$

By back substitution:

$\therefore -3x_3 = -3/2 \quad \therefore x_3 = 1/2$

$\therefore 8x_2 + 2x_3 = -7$

$\Rightarrow 8x_2 + 1 = -7$

$\therefore x_2 = -1$

$\therefore 6x_1 + 2x_2 + 8x_3 = 26$

$\Rightarrow 6x_1 - 2 + 4 = 26$

$\therefore x_1 = 4$

$\therefore (x_1, x_2, x_3) = (4, -1, 1/2)$

Gauss - Jordan Method

This method is modification of the gauss elimation method. This method solves a given system of equation by transforming the coefficient matrix ~~matx~~ into unit matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Steps :

(1) Write the matrix form of the system of equations.

(2) Write the augmented matrix

(3) Reduce the coefficient matrix to unit matrix by applying elementary row transformations to the augmented matrix.

(4) Write the corresponding linear system of equations to obtain the sol^n.

Example :

$$10x_1 + x_2 + x_3 = 12$$
$$x_1 + 10x_2 - x_3 = 10$$
$$x_1 - 2x_2 + 10x_3 = 9$$

Solution :

By Augmented matrix,

$$\sim \left[\begin{array}{ccc:c} 10 & 1 & 1 & 12 \\ 1 & 10 & -1 & 10 \\ 1 & -2 & 10 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 10 & -1 & 10 \\ 10 & 1 & 1 & 12 \\ 1 & -2 & 10 & 9 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 10 & -1 & 10 \\ 0 & -99 & 11 & -88 \\ 0 & -12 & 11 & -1 \end{array}\right] \quad \begin{array}{l} R_{12}(-10) \\ R_{13}(-1) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 10 & -1 & 10 \\ 0 & 1 & -1/9 & 8/9 \\ 0 & -12 & 11 & -1 \end{array}\right] \quad R_2(-1/99)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1/9 & 10/9 \\ 0 & 1 & -1/9 & 8/9 \\ 0 & 0 & \frac{87}{9} & \frac{87}{9} \end{array}\right] \quad \begin{array}{l} R_{21}(-10) \\ R_{23}(12) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1/9 & 10/9 \\ 0 & 1 & -1/9 & 8/9 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_3\left(\frac{9}{87}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_{31}(-1/9) \\ R_{32}(1/9) \end{array}$$

LU Decomposition of a matrix
L lower, upper
chapter 7 sastry 257 page

non singular $|A| \neq 0$ .L .U

By back substitution,

$$(x, y, z) = (1, 1, 1)$$

$$\boxed{A^{-1} = (LU)^{-1} = U^{-1} L^{-1}}$$

LU Decomposition
also can be solved to gauss elimination
Dolittle method

Example: Factorize the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ into the LU form.

Let,

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Then,

$u_{11} = 2 \quad u_{12} = 3 \quad u_{13} = 1$

$l_{21} = 1/2 \quad l_{31} = 3/2$

$u_{22} = 1/2 \quad u_{23} = 5/2$

$l_{32} = -7 \quad u_{33} = 18$

$$l_{21} = \frac{a_{21}}{a_{11}} \quad l_{31} = \frac{a_{31}}{a_{11}}$$

$$u_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

$$u_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13}$$

$$l_{32} = \frac{a_{32} - \frac{a_{31}}{a_{11}} a_{12}}{u_{22}}$$

$u_{11} = a_{11}$

$u_{12} = a_{12}$

$u_{13} = a_{13}$

$u_{33} = a_{33} - l_{31}u_{13} + l_{32}u_{23}$

It follows that.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

~~and hence the given~~ $$A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \quad \text{— (3)}$$

and hence the given system of equations can be written as.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \quad \text{— (4)}$$

or, as,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} \quad \text{— (5)}$$

where,

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \text{— (6)}$$

Solving the system in (v) by forward substitution we get,

$y_1 = 9$

$y_2 = 3/2$

$y_3 = 5$

$y_1 = 9$

$\frac{1}{2}y_1 + y_2 = 6$

$\frac{9}{2} + y_2 = 6$

$y_2 = 6 - \frac{9}{2}$

$\frac{12-9}{2} = 3/2$

$\frac{3}{2} \cdot 9 - 7 \cdot \frac{3}{2} + y_3 = 8$

$y_3 = 8 - \frac{27}{2} + \frac{21}{2}$

$= \frac{16 - 27 + 21}{2}$

$= \frac{37-27}{2} = \frac{10}{2} = 5$

with these values of y_1, y_2, y_3 in eq. (6) can now be solved by the back substitution process and we obtain,

$x = 35/18$

$y = 29/18$

$z = 5/18$

$18z = 5$

$\therefore z = 5/18$

$\frac{1}{2}y + 5/2 z = 3/2$

$\therefore y = 29/18$

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Finding the [U] matrix :

Using the forward ~~Elimat~~ Elimination Procedure of Gauss Elimination

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

step 1 : $\frac{64}{25} = 2.56$; $R2 - R1(2.56) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix}$

$\frac{144}{25} = 5.76$; $R3 - R1(5.76) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix}$

step 2 : $\frac{-16.8}{-4.8}$; $R3 - R2(3.5) = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$

upper matrix $[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$

Finding the [L] matrix :

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \qquad \begin{matrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{matrix}$$

From the first step of forward elimination :

$$l_{21} = \frac{a_{21}}{a_{11}} = \frac{64}{25} = 2.56$$

$$l_{31} = \frac{a_{31}}{a_{11}} = \frac{144}{25} = 5.76$$

$$l_{32} = \frac{a_{32}}{a_{22}} = \frac{-16.8}{-4.8} = 3.5$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 0 & 0 \\ 5.76 & 3.5 & 0 \end{bmatrix}$$

$LU = A$ //

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Iterative Method :

Gauss-Jacobi Method :

This method is applicable to the system of equations in which leading diagonal elements of the coefficient matrix are dominant (large in magnitude) in their respective rows.

consider the system of equations.

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

where co-efficient matrix A must be diagonally dominant,

$$|a_1| \geq |b_1| + |c_1|$$

$$|b_2| \geq |a_2| + |c_2|$$

$$|c_3| \geq |a_3| + |b_3| \quad \text{———} (1)$$

And the inequality is strictly greater than for at least one row. Solving the system (1) for x, y, z respectively, we obtain

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y) \quad \text{———} (2)$$

we start with $x_0 = 0$, $y_0 = 0$ and $z_0 = 0$ in eq.(2)

$$\therefore x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2x_0 - c_2z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3x_0 - b_3y_0)$$

Again substituting these value x_1, y_1, z_1 in eq(2) the next approximation is obtained.

This process is continued till the values of x, y, z are obtained to desired degree of accuracy.

Example: solve by Gauss Jacobi method up to three iteration.

$$20x + y - 2z = 17$$
$$2x - 3y + 20z = 25$$
$$3x + 20y - z = -18$$

Solution:

$|20| > |1| + |-2|$

$|-3| \not> |2| + |20|$

so, It is not diagonally dominant.

we need to rearrange the equations.

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

$|20| > |1| + |2|$

$|20| > |3| + |-1|$

$|20| > |2| + |-3|$

so, All equations are diagonally dominant.

(Make subject x, y, z from diagonally dominant equations.)

Here,

$$x = \frac{1}{20}(17 - y + 2z)$$
$$y = \frac{1}{20}(-18 - 3x + z)$$
$$z = \frac{1}{20}(25 - 3x + 3y)$$

let the initial values are $x = y = z = 0$.

1st iteration.

$$x^1 = \frac{1}{20}(17 - 0 + 0) = 0.85$$
$$y^1 = \frac{1}{20}(-18 - 0 + 0) = 0.9$$
$$z^1 = \frac{1}{20}(25 - 0 + 0) = 1.25$$

Iteration	x	y	z
1	0.85	1.00	1.25
2	1.02	-0.97	1.03
3	1.00 0.02	-1.00 0.03	1.00 0.03

Ans. $(x, y, z) = (1, -1, 1)$

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III Gauss – seidal Method

This is a modification of Gauss - Jacobi method. In this method, we replace the approximation by the corresponding new ones as soon as they are calculated.

consider the system of equations:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

where .... same as Jacobi method upto eq.(2)

we start with $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ in eq.(2).

$$\therefore x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)$$

Now subtituting $x = x_1$, and $z = z_0$ in the second eq. of (2)

$$\therefore y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)$$

Now substituting $x = x_1$ and $y = y_1$ in the second eq. of (2)

$$\therefore z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)$$

This process is continued till the values of x, y, z are obtained to desired degree of accuracy.

Let the initial values are $x_1 = x_2 = x_3 = 0$.

$$x = \frac{1}{20}(30 - 2y - z) = \frac{1}{20}(30 - 0 - 0) = 1.5$$

$$y = \frac{1}{-40}(-75 - x - 3z) = \frac{1}{-40}(-75 - 1.5 - 0) = 1.91$$

$$z = \frac{1}{10}(30 - 2x + y) = \frac{1}{10}(30 - 2(1.5) + 1.91) = 2.89$$

Iteration	x	y	z
1	1.5	1.91	2.89
2	1.16	2.12	2.98
3	1.14	2.13	2.99
4	1.14	2.13	2.99

In 3rd and 4th iteration all values are almost same.

so $(x, y, z) = (1.14, 2.13, 2.99)$

Introduction

The general problem of numerical integration may be stated as follows:

→ Given a set of data points $(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$ of a function $y = f(x)$, where $f(x)$ is not known explicitly.

→ It is required to compute the value of the definite integral

$$I = \int_a^b y \, dx$$

→ In this case we have to replace $f(x)$ by an interpolating polynomial $\phi(x)$ and obtain an approximate value of the definite integral by integrating $\phi(x)$.

→ Thus, different integration formulae can be obtained depending upon the type of the interpolation formula used.

Numerical differentiation: is the process of calculating the derivatives of a function from a set of given values of that function.

↳ by Interpolation formula

equi dis	non-equi dis
• forward	• logra
backward	

Errors in numerical diff

Example:

From the following table of values of x and y, obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $x = 1.2$

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y	2.7183	3.3201	4.0552	4.953	6.0496	7.3891	9.025

Solution:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183						
		0.6081					
x_0 1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.953		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.3891		0.2964				
		1.6359					
2.2	9.025						

Here,

$x_0 = 1.2 \quad y_0 = 3.3201$ and $h = 0.2$

$$\left[\frac{dy}{dx}\right]_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014)\right]$$

$$= 3.3205 \quad (3.3201)\%$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = \frac{1}{0.04}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0086) - \frac{5}{6}(0.0014)\right]$$

$$= 3.318$$

when $x = 1.2$ then we get $\frac{dy}{dx} = 3.3205$ and $\frac{d^2y}{dx^2} = 3.318$

But, here $y = e^x$, therefore, $\frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x$ and $\frac{d^2y}{dx^2} = e^x$

Therefore, here we can see with each differentiation, some error occurs in the derivatives. The error increases with higher derivatives.

This is because, in interpolation the new polynomial would agree at the set of points.

III Maximum value of a Tabulated Function

→ It is known that the maximum values of a function can be found by equating the first derivative to zero and solving for the variable.

→ The same procedure can be applied to determine the maxima of a tabulated function.

Example:

From the following table, find x, correct to two decimal places, for which y the function has the maximum value and find the value of y.

x	1.2	1.3	1.4	1.5	1.6
y	0.9320	0.9636	0.9855	0.9975	0.9996

Solution:

x	y	Δy	$\Delta^2 y$
1.2	0.9320		
		0.0316	
1.3	0.9636		−0.0097
		0.0219	
1.4	0.9855		−0.0099
		0.0120	
1.5	0.9975		−0.0099
		0.0021	
1.6	0.9996		

Let $x_0 = 1.2$ and we can terminate the formula after the second difference (since the difference is very negligible).

Now we have,

$$0.0316 + (2u-1)(-0.0097)/2 = 0$$

Therefore, $u = 3.8$ and $x = x_0 + uh = 1.2 + (3.8)(0.1) = 1.58$

For $x = 1.58$, we have the maximum value of y.

Using Newton's backward difference formula at $x_n = 1.6$ gives,

$y(1.58) = 1.0$

This is the maximum value of y in the function.

Numerical Integration:

Let, the interval $[a, b]$ be divided into n equal subintervals such that $a = x_0 < x_1 < \cdots < x_n = b$.

Then, $x_n = x_0 + nh$

Hence, the integral becomes $I = \int_{x_0}^{x_n} y\,dx$

$$I = \int_{x_0}^{x_n} \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \right] dx$$

$$= \int_{x_0}^{x_0+nh} \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \right] dx$$

III Trapezoidal Rule :

→ setting $n=1$ in the general formula (1) and neglecting all differences above the first we obtain for the first interval $[x_0, x_1]$

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}\left[y_0 + y_1\right]$$

⋮

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

Example :

Evaluate $I = \int_0^1 \frac{1}{1+x}\,dx$.

for $h = 0.5$, 0.25 and 0.125 using trapezoidal rule (correct to three decimal pla

2 segment   4 seg   8 seg

Solution :

The value of x and y are tabulated below $h = 0.5$

x	0	0.5	1.0
y	1.0000	0.6667	0.5

Trapezoidal rule gives ,

$$I = \int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_0 + \cdots + y_{n-1}) + y_n\right]$$

$$= \frac{0.5}{2}\left[1.0000 + 2(0.6667) + 0.5\right] = 0.7084$$

when $h = 0.25$

x	0	0.25	0.5	0.75	1.0
y	1	0.8	0.6667	0.5714	0.5

A/C ~~to~~ trapezoidal rule,

$$I = \frac{0.25}{2}\left[1 + 2(0.8 + 0.6667 + 0.5714) + 0.5\right]$$

$$= 0.6970$$

If $h = 0.125$ $\quad I = 0.6941$

Class Work

A solid of revolution is formed by rotating about the x-axis the area between the x-axis, the lines $x = 0$ and $x = 1$, and a curve through the points with the following coordinates

$h = .25$

x	0.00	0.25	0.50	0.75	1.00
y	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using trapezoidal rule, giving the answer to three decimal places.

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curve by 3 points replaced by $\frac{n}{2}$ arcs of second degree

Simpson's 1/3 Rule

$$I = \int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 \cdots\right] \quad (1)$$

setting $n = 2$ and neglecting all differences above the second; we obtain

for the first interval $[x_0, x_2]$

$$\int_{x_0}^{x_2} y\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right] = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Thus,

$$\int_0^{x_n} y\,dx = \frac{h}{3}\left[y_0 + 4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2}) + y_n\right]$$

Example:

Evaluate $I = \int_0^1 \frac{1}{1+x}\,dx$ correct to three decimal places for $h = 0.5, 0.25$ and 0.125 using simpson's 1/3 rule.

Solution:

The values of x and y are tabulated below $h = 0.5$

x	0	0.5	1.0
y	1.0000	0.6667	0.5

Simpson's rule gives.

$$I = \frac{1}{6}\left[1.0000 + 4(0.6667) + 0.5\right] = 0.6945$$

when, $h = 0.25$; $I = 0.6932$

$h = 0.125$; $I = 0.6932$

not accurate as Simpson's rule

Simpson's 3/8 Rule (when $n = 3$)

$$\int_{x_0}^{x_3} y\,dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 + y_0) + \frac{1}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left[y_0 + \cancel{3h}\, 3y_1 + 3y_2 + y_3\right]$$

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + 2y_6 + \dots + 2y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

Weddle's Rule:

7Ph

The rule is obtained by putting $n = 6$ in the general equation i.e. and neglecting all the differences above the sixth we have.

$$\int_{x_0}^{x_6} y\,dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right]$$

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12} + \dots + 2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right]$$

Example :

compute the value of the definite integral for $h = 0.2$ using Weddle's Rule

$$\int_{4}^{5.2} \ln x \, dx$$

Solution :

The values of this function is computed for each point of subdivision.

x	$\ln x$
4.0	1.3863
4.2	1.4351
4.4	1.4816
4.6	1.5261
4.8	1.5686
5.0	1.6094
5.2	1.6487

By weddle's rule,

$$I = 3(0.2)\,[1.3863 + 5(1.4351) + 1.4816 + 6(1.5261) + 1.5686 + 5(1.6094) + 1.6487]/10$$

$$= 1.827858$$

4) Romberg Integration

This method can be used to improve the approximate result obtained by the finite difference methods such as trapezoidal method.

let T_n be the approximation of the integral $I = \int_a^b y\,dx$. using T.R with 2^n subintervals.

- let, $I_{1,1} = T_1$ (here, I is calculated with 2^1 segments)

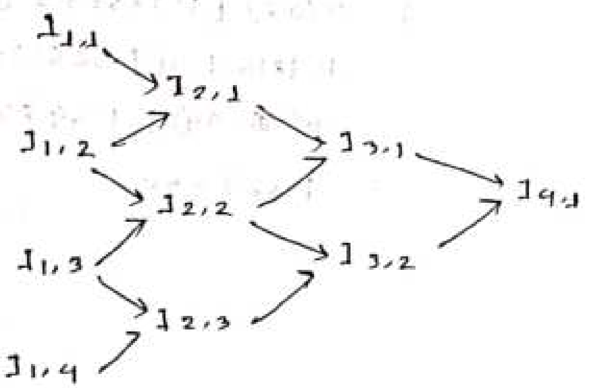
$I_{1,n} \cdot I_{2,n}$

$I_{1,n+1} = T_{n+1}$ 2^2 segment

for $j = 2, 3, \ldots n$

$$I_{j,k} = I_{j-1,k+1} + \frac{I_{j-1,k+1} - I_{j-1,k}}{4^{j-1} - 1} \quad ; \quad j \geq 2$$

- The index j represents the order of ~~interpret~~ interpolation.
- For example, $j=1$ represents the values obtained from the regular Trapezoidal rule.
- The index k represents the more or less accurate estimate of the integral.
- The value of the integral with $k+1$ index is more accurate than with k index.



→ An advantage of this method is that the accuracy of the computed value is known at each step.

For, $j=2$, $k=1$

$$I_{2,1} = I_{12} + \frac{I_{1,2} - I_{1,1}}{4^{2-1} - 1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3}$$

For $j=3$, $k=1$

$$I_{3,1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{4^{3-1} - 1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15}$$

Example :

Use Romberg method to compute the following integral correct to three decimal pla

$$I = \int_0^1 \frac{1}{1+x}\,dx.$$

use 2, 4 and 8 segment Trapezoidal rule result.

0.5 0.25 0.125

0.7084 0.6970 0.6941

Solution :

Here, we have to calculate I = using $2 = 2^1$, $4 = 2^2$, and $8 = 2^3$ intervals.

Therefore,

$I_{1,1} = T_1$, that is calculate I using Trapezoidal rule with $2^1 = 2$ intervals.

$I_{1,2} = T_2$; $2^2 = 4$ intervals

$I_{1,3} = T_3$; $2^3 = 8$ "

$I_{1,1} = 0.7084$ $\quad I_{1,2} = 0.6970$ $\quad I_{1,3} = 0.6941$

Now,

$$I_{j,k} = I_{j-1,k+1} + \frac{I_{j-1,k+1} - I_{j-1,k}}{4^{j-1}-1}, \quad j \geq 2$$

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{4^{2-1}-1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} = 0.6970 + \frac{1}{3}(0.6970 - 0.7084)$$
$$= 0.6932$$

$$I_{2,2} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{4^{2-1}-1} = I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} = 0.6941 + \frac{1}{3}(0.6941 - 0.6970)$$
$$= 0.6931$$

$$I_{31} = I_{22} + \frac{I_{2,2} - I_{2,1}}{4^{3-1}-1} = I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} = 0.6931 + \frac{1}{15}(0.6931 - 0.6932)$$
$$= 0.6931$$

The table of values is therefore

0.7084		
	0.6932	
0.6970		0.6931
	0.6931	
0.6941		

Ans. 0.6931