応用数理解析

No.

Date

氏名：東田悠希　ID：1116191012

演習問題1

(1) $f(t) = e^{-\alpha|t|} \quad (\alpha > 0)$

$$f(t) = \begin{cases} e^{-\alpha t} & (t \geq 0) \\ e^{\alpha t} & (t < 0) \end{cases}$$

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{0} e^{\alpha t} e^{-i\omega t} dt + \int_0^\infty e^{-\alpha t} e^{-i\omega t} dt \\
&= \int_{-\infty}^{0} e^{t(\alpha - i\omega)} dt + \int_0^\infty e^{t(-\alpha - i\omega)} dt \\
&= \left[\frac{1}{\alpha - i\omega} e^{t(\alpha - i\omega)}\right]_{-\infty}^{0} + \left[\frac{1}{-\alpha - i\omega} e^{t(-\alpha - i\omega)}\right]_0^\infty \\
&= \frac{1}{\alpha - i\omega} - 0 + 0 - \left(\frac{1}{-\alpha - i\omega}\right) \\
&= \frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}
\end{aligned}$$

(2) $f(t) = \begin{cases} 1 - \frac{|t|}{T} & (|t| \leq T) \\ 0 & (|t| > T) \end{cases}$

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\
&= \int_{-\infty}^{-T} 0 \times e^{-i\omega t} dt + \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) e^{-i\omega t} dt + \int_T^\infty 0 \times e^{-i\omega t} dt \\
&= \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) e^{-i\omega t} dt = \int_{-T}^{0} \left(1 + \frac{t}{T}\right) e^{-i\omega t} dt + \int_0^T \left(1 - \frac{t}{T}\right) e^{-i\omega t} dt \\
&= \int_{-T}^{0} \left(e^{-i\omega t} + \frac{t}{T} e^{-i\omega t}\right) dt + \int_0^T \left(e^{-i\omega t} - \frac{t}{T} e^{-i\omega t}\right) dt \\
&= \int_{-T}^{0} e^{-i\omega t} dt + \frac{1}{T} \int_{-T}^{0} t e^{-i\omega t} dt + \int_0^T e^{-i\omega t} dt - \frac{1}{T} \int_0^T t e^{-i\omega t} dt \, dt
\end{aligned}$$

ここで $\int t e^{-i\omega t} dt = -\frac{1}{i\omega} t e^{-i\omega t} + \frac{1}{i\omega} \int e^{-i\omega t} dt$

$= -\frac{t e^{-i\omega t}}{i\omega} + \frac{1}{\omega^2} \cdot e^{-i\omega t} + C$ （Cは積分定数）

であるから

$$\begin{aligned}
F(\omega) &= \left[\frac{1}{-i\omega} e^{-i\omega t}\right]_{-T}^{0} + \frac{1}{T}\left[\frac{t e^{-i\omega t}}{-i\omega} + \frac{1}{\omega^2} e^{-i\omega t}\right]_{-T}^{0} + \left[\frac{1}{-i\omega} e^{-i\omega t}\right]_0^T \\
&\quad - \frac{1}{T}\left[\frac{t e^{-i\omega t}}{-i\omega} + \frac{1}{\omega^2} e^{-i\omega t}\right]_0^T \\
&= \frac{1}{-i\omega}\left(1 - e^{i\omega T}\right) + \frac{1}{T}\left(\frac{1}{\omega^2} - \frac{T e^{i\omega T}}{i\omega} - \frac{1}{\omega^2} e^{i\omega T}\right) - \frac{1}{i\omega}\left(e^{-i\omega T} - 1\right) \\
&\quad - \frac{1}{T}\left(\frac{T e^{-i\omega T}}{-i\omega} + \frac{1}{\omega^2} e^{-i\omega T} - \frac{1}{\omega^2}\right) \\
&= \frac{1}{i\omega}\left(e^{i\omega T} - e^{-i\omega T}\right) + \frac{1}{T}\left\{\frac{1}{\omega^2}\left(2 - e^{i\omega T} - e^{-i\omega T}\right) - \frac{T e^{i\omega T}}{i\omega} + \frac{T e^{-i\omega T}}{i\omega}\right\} \\
&= \frac{2 \sin \omega T}{\omega} + \frac{1}{T}\left\{\frac{1}{\omega^2}(2 - 2\cos \omega T) - \frac{T}{\omega} 2 \sin \omega T\right\} \\
&= \frac{2}{\omega^2 T}(1 - \cos \omega T)
\end{aligned}$$

No.

Date

(3) $f(t) = \begin{cases} \sin t & (|t| \leq \pi) \\ 0 & (|t| > \pi) \end{cases}$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$
$$= \int_{-\infty}^{-\pi} 0 \times e^{-i\omega t} dt + \int_{-\pi}^{\pi} \sin t\, e^{-i\omega t} dt + \int_{\pi}^{\infty} 0 \times e^{-i\omega t} dt$$
$$= \int_{-\pi}^{\pi} \sin t \times e^{-i\omega t} dt$$

ここで $\int \sin t \times e^{-i\omega t} dt = I$ とおくと

$$I = \int \sin t\, e^{-i\omega t} dt = -\frac{1}{i\omega} e^{-i\omega t} \sin t + \frac{1}{i\omega} \int e^{-i\omega t} \cos t\, dt + C$$
$$= -\frac{1}{i\omega} e^{-i\omega t} \sin t + \frac{1}{i\omega} \left\{ -\frac{1}{i\omega} e^{-i\omega t} \cdot \cos t - \frac{1}{i\omega} \int e^{-i\omega t} \sin t\, dt \right\} + C$$
$$= -\frac{1}{i\omega} e^{-i\omega t} \sin t + \frac{1}{\omega^2} e^{-i\omega t} \cos t + \frac{1}{\omega^2} \int e^{-i\omega t} \sin t\, dt + C$$

$$\left(1 - \frac{1}{\omega^2}\right) I = \frac{e^{-i\omega t}}{\omega} \left(\frac{\cos t}{\omega} + i \sin t \right)$$ (C: 積分定数)

$$I = \frac{1}{\omega^2 - 1} (\cos t + i\omega \sin t)\, e^{-i\omega t}$$

よって

$$F(\omega) = \int_{-\pi}^{\pi} \sin t \times e^{-i\omega t} dt$$
$$= \frac{1}{\omega^2 - 1} \left[(\cos t + i\omega \sin t) e^{-i\omega t} \right]_{-\pi}^{\pi}$$
$$= \frac{1}{\omega^2 - 1} \left\{ (-1) e^{-i\omega\pi} - (-1) e^{i\omega\pi} \right\}$$
$$= \frac{1}{\omega^2 - 1} \left(-e^{-i\omega\pi} + e^{i\omega\pi} \right)$$
$$= \frac{1}{\omega^2 - 1} \left(e^{i\omega\pi} - e^{-i\omega\pi} \right)$$
$$= \frac{1}{\omega^2 - 1} \left\{ \cos \omega\pi + i \sin \omega\pi - \cos(-\omega\pi) - i \sin(-\omega\pi) \right\}$$
$$= \frac{1}{\omega^2 - 1} \times 2i \sin \omega\pi$$
$$= \frac{2i}{\omega^2 - 1} \sin \omega\pi$$ //