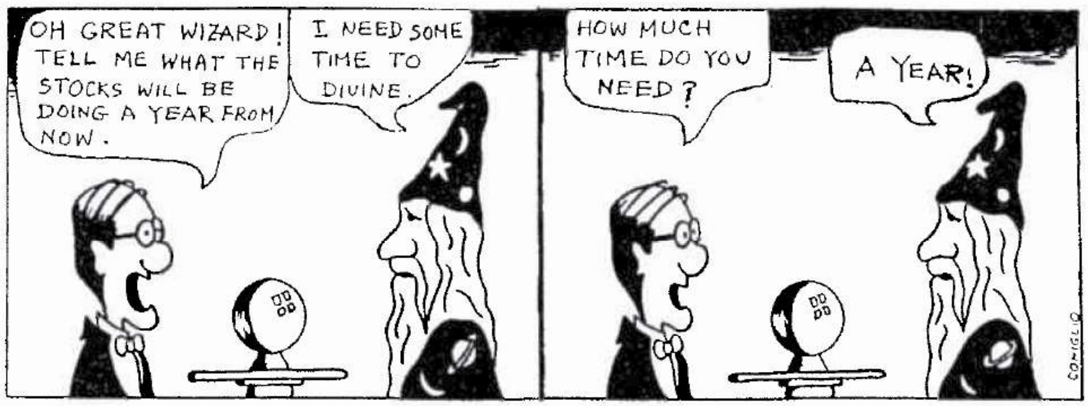
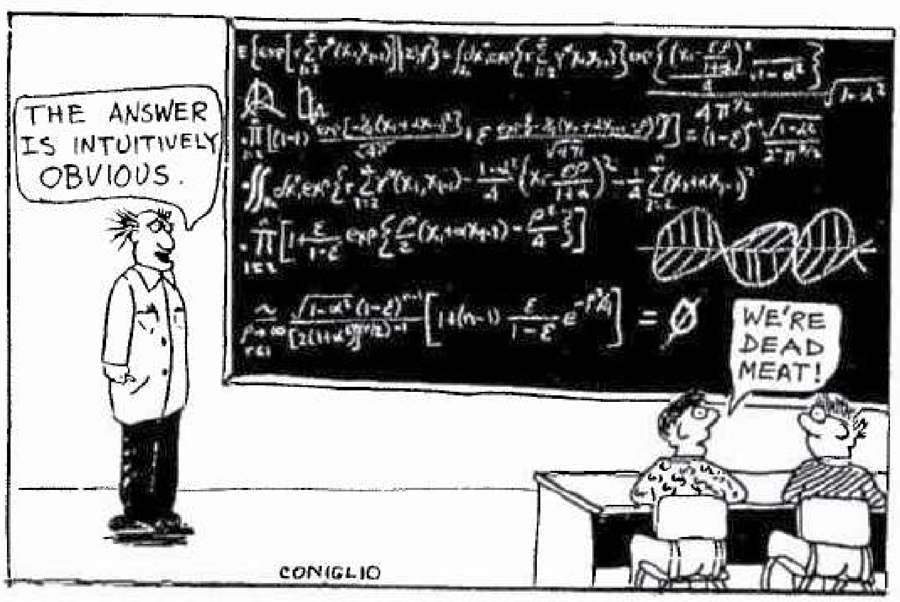
Annexes



Noncausal systems are realizable with time delay!



Intuition can cut the math jungle instantly!

Annexe A

Nombres complexes

B

Background

The topics discussed in this chapter are not totally new to students taking this course. You have already studied many of these topics in earlier courses or are expected to know them from your previous training. Even so, this background material deserves a review because it is so pervasive in the area of signals and systems. Investing a little time in such a review will pay big dividends later. Furthermore, this material is useful not only for this course but also for several courses that follow. It will also be helpful as reference material in your future professional career.

B.1 COMPLEX NUMBERS

Complex numbers are an extension of ordinary numbers and are an integral part of the modern number system. Complex numbers, particularly **imaginary numbers**, sometimes seem mysterious and unreal. This feeling of unreality comes from their unfamiliarity and novelty rather than their supposed nonexistence! It was a blunder in mathematics to call these numbers "imaginary" because the term immediately prejudices perception. Had these numbers been called by some other name, they would have become demystified long ago, just as irrational numbers or negative numbers were. Many futile attempts were made to ascribe some physical meaning to imaginary numbers. This effort was needless. In mathematics we assign symbols and operations any meaning as long as there is an internal consistency. A healthier approach would have been to define a symbol i (with any name but "imaginary"), which has a property $i^2 = -1$. The history of mathematics is full of entities which were unfamiliar and held in abhorrence until familiarity made them acceptable. This fact will become clear from the following historical note.

B.1-1 A Historical Note

Among early people the number system consisted only of natural numbers (positive integers) that were needed to count the number of children, cattle, and quivers of arrows. They had no use for fractions. Whoever heard of two-and-one-half children or three-and-one-fourth cows!

Page 206

With the advent of agriculture, people needed to measure continuously varying quantities, such as the length of a field, the weight of a quantity of butter, and so on. The number system therefore was extended to include fractions. The ancient Egyptians and Babylonians knew how to handle fractions, but **Pythagoras** discovered that some numbers (like the diagonal of a unit square) could not be expressed as a whole number or a fraction. Pythagoras, a number mystic who regarded numbers as the essence and principle of all things in the universe, was so appalled at his discovery that he swore his followers to secrecy and provided a death penalty for divulging this secret.[1] These numbers, however, were included in the number system by the time of Descartes, and they are now known as **irrational numbers**.

Until recently **negative numbers** were not a part of the number system. The idea of negative numbers must have appeared absurd to early man. However, the medieval Hindus had had a clear understanding of the significance of positive and negative numbers.[2,3] They were also the first to recognize the existence of absolute negative quantities.[4] The works of **Bhaskar** (1114-1185) on arithmetic (*Līlāvatī*) and algebra (*Bījaganit*) not only use the decimal system but also give rules for dealing with negative quantities. Bhaskar recognized that positive numbers have two square roots.[5] Much later in Europe, the banking system that arose in Florence and Venice during the late Renaissance (fifteenth century) is credited with developing a crude form of negative numbers. The seemingly absurd subtraction of 7 from 5 seemed reasonable when bankers began to allow their clients to draw seven gold ducats while their deposit stood at five. All that was necessary for this purpose was to write the difference, 2, on the debit side of a ledger.[6]

Thus the number system was once again broadened (generalized) to include negative numbers. The acceptance of negative numbers made it possible to solve equations such as $x+5=0$, which had no solution before. Yet for equations such as $x^2+1=0$, leading to $x^2=-1$, the solution could not be found in the real number system. It was therefore necessary to define a completely new kind of number with its square equal to -1. During the time of Descartes and Newton, imaginary (or complex) numbers came to be accepted as part of the number system, but they were still regarded as algebraic fiction. The Swiss mathematician **Leonhard Euler** introduced the notation i (for **imaginary**) around 1777 to represent $\sqrt{-1}$. (Electrical engineers use the notation j instead of i to avoid confusion with the notation i often used for electrical current.) Thus

$$j^2 = -1$$

and

$$\sqrt{-1} = \pm j$$

This allows us to determine the square root of any negative number. For example,

$$\sqrt{-4} = \sqrt{4} \times \sqrt{-1} = \pm 2j$$

When we include imaginary numbers in the number system, the resulting numbers are called **complex numbers**.

Origins of Complex Numbers

Ironically (and contrary to popular belief), it was not the solution of a quadratic equation, such as $x^2+1=0$, but a cubic equation with real roots that made

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Gerolamo Cardano (left) and Karl Friedrich Gauss (right).

imaginary numbers plausible and acceptable to early mathematicians. They could dismiss $\sqrt{-1}$ as pure nonsense when it appeared as a solution to $x^2 + 1 = 0$ because this equation has no real solution. But in 1545, **Gerolamo Cardano** of Milan published *Ars Magna* (*The Great Art*), the most important algebraic work of the Renaissance. In this book he gave a method of solving a general cubic equation in which a root of a negative number appeared in an intermediate step. According to his method, the solution to a third-order equation†

$$x^3 + ax + b = 0$$

is given by

$$x = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

For example, to find a solution of $x^3 + 6x - 20 = 0$, we substitute $a = 6$, $b = -20$ in the above equation to obtain

$$x = \sqrt[3]{10 + \sqrt{108}} + \sqrt[3]{10 - \sqrt{108}} = \sqrt[3]{20.392} - \sqrt[3]{0.392} = 2$$

We can readily verify that 2 is indeed a solution of $x^3 + 6x - 20 = 0$. But when Cardano tried to solve the equation $x^3 - 15x - 4 = 0$ by this formula, his solution

†This equation is known as the *depressed cubic* equation. A general cubic equation

$$y^3 + py^2 + qy + r = 0$$

can always be reduced to a depressed cubic form by substituting $y = x - \frac{p}{3}$. Therefore any general cubic equation can be solved if we know the solution to the depressed cubic. The depressed cubic was independently solved, first by **Scipione del Ferro** (1465-1526) and then by **Niccolo Fontana** (1499-1557). The latter is better known in the history of mathematics as **Tartaglia** ("Stammerer"). Cardano learned the secret of the depressed cubic solution from Tartaglia. He then showed that by using the substitution $y = x - \frac{p}{3}$, a general cubic is reduced to a depressed cubic.

Page 208

was

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

What was Cardano to make of this equation in the year 1545? In those days negative numbers were themselves suspect, and a square root of a negative number was doubly preposterous! Today we know that

$$(2 \pm j)^3 = 2 \pm j11 = 2 \pm \sqrt{-121}$$

Therefore Cardano's formula gives

$$x = (2 + j) + (2 - j) = 4$$

We can readily verify that $x = 4$ is indeed a solution of $x^3 - 15x - 4 = 0$. Cardano tried to explain halfheartedly the presence of $\sqrt{-121}$ but ultimately dismissed the whole enterprise as being "as subtle as it is useless." A generation later, however, **Raphael Bombelli** (1526-1573), after examining Cardano's results, proposed acceptance of imaginary numbers as a necessary vehicle that would transport the mathematician from the *real* cubic equation to its *real* solution. In other words, while we begin and end with real numbers, we seem compelled to move into an unfamiliar world of imaginaries to complete our journey. To mathematicians of the day, this seemed incredibly strange.[7] Yet they could not dismiss the idea of imaginary numbers so easily because it yielded the real solution of an equation. It took two more centuries for the full importance of complex numbers to become evident in the works of Euler, Gauss, and Cauchy. Still, Bombelli deserves credit for recognizing that such numbers have a role to play in algebra.[7]

In 1799, the German mathematician **Karl Friedrich Gauss**, at the ripe age of 22, proved the fundamental theorem of algebra, namely that every algebraic equation in one unknown has a root in the form of a complex number. He showed that every equation of the nth order has exactly n solutions (roots), no more and no less. Gauss was also one of the first to give a coherent account of complex numbers and to interpret them as points in a complex plane. It is he who introduced the term *complex numbers* and paved the way for general and systematic use of complex numbers. The number system was once again broadened or generalized to include imaginary numbers. Ordinary (or real) numbers became a special case of generalized (or complex) numbers.

The utility of complex numbers can be understood readily by an analogy with two neighboring countries X and Y as shown in Fig. B.1. If we want to travel from City a to City b (both in Country X), the shortest route is through Country Y, although the journey begins and ends in Country X. We may, if we desire, perform this journey by an alternate route that lies exclusively in X, but this alternate route is longer. In mathematics we have a similar situation with real numbers (Country X) and complex numbers (Country Y). All real-world problems must start with real numbers, and all the final results must also be in real numbers. But the derivation of results is considerably simplified by using complex numbers as an intermediary. It is also possible to solve all real-world problems by an alternate method, using real numbers exclusively, but this would increase the work needlessly.

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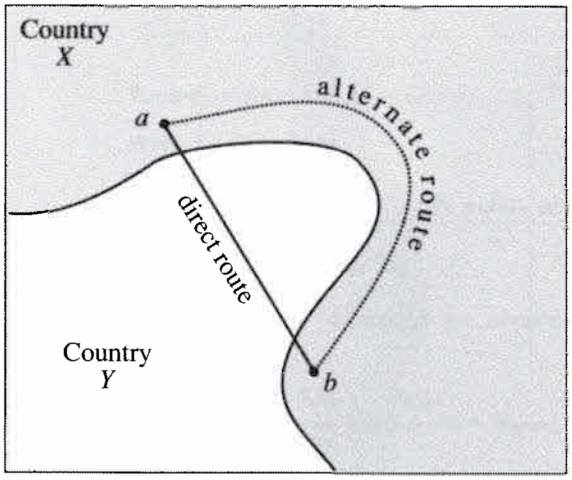

Fig. B.1 Use of complex numbers can reduce the work.

B.1-2 Algebra of Complex Numbers

A complex number (a, b) or $a + jb$ can be represented graphically by a point whose Cartesian coordinates are (a, b) in a complex plane (Fig. B.2). Let us denote this complex number by z so that

$$z = a + jb \tag{B.1}$$

The numbers a and b (the abscissa and the ordinate) of z are the **real part** and the **imaginary part**, respectively, of z. They are also expressed as

$$\text{Re}\, z = a$$

$$\text{Im}\, z = b$$

Note that in this plane all real numbers lie on the horizontal axis, and all imaginary numbers lie on the vertical axis.

Complex numbers may also be expressed in terms of polar coordinates. If (r, θ) are the polar coordinates of a point $z = a + jb$ (see Fig. B.2), then

$$a = r\cos\theta$$

$$b = r\sin\theta$$

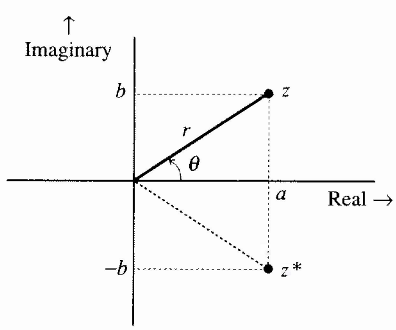

Fig. B.2 Representation of a number in the complex plane.

Page 210

and

$$
\begin{aligned}
z = a + jb &= r\cos\theta + jr\sin\theta \\
&= r(\cos\theta + j\sin\theta) \qquad \text{(B.2)}
\end{aligned}
$$

The **Euler formula** states that

$$e^{j\theta} = \cos\theta + j\sin\theta$$

To prove Euler's formula, we expand $e^{j\theta}$, $\cos\theta$, and $\sin\theta$ using a Maclaurin series:

$$
\begin{aligned}
e^{j\theta} &= 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \frac{(j\theta)^6}{6!} + \cdots \\
&= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \cdots \\
\cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!}\cdots \\
\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots
\end{aligned}
$$

From these results, we conclude that

$$e^{j\theta} = \cos\theta + j\sin\theta \qquad \text{(B.3)}$$

Substituting (B.3) in (B.2) yields

$$
\begin{aligned}
z &= a + jb \\
&= re^{j\theta} \qquad \text{(B.4)}
\end{aligned}
$$

Thus a complex number can be expressed in Cartesian form $a + jb$ or polar form $re^{j\theta}$ with

$$a = r\cos\theta, \qquad b = r\sin\theta \qquad \text{(B.5)}$$

and

$$r = \sqrt{a^2 + b^2}, \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right) \qquad \text{(B.6)}$$

Observe that r is the distance of the point z from the origin. For this reason, r is also called the **magnitude** (or **absolute value**) of z and is denoted by $|z|$. Similarly θ is called the angle of z and is denoted by $\angle z$. Therefore

$$|z| = r, \qquad \angle z = \theta$$

and

$$z = |z|e^{j\angle z} \qquad \text{(B.7)}$$

Also

$$\frac{1}{z} = \frac{1}{re^{j\theta}} = \frac{1}{r}e^{-j\theta} = \frac{1}{|z|}e^{-j\angle z} \qquad \text{(B.8)}$$

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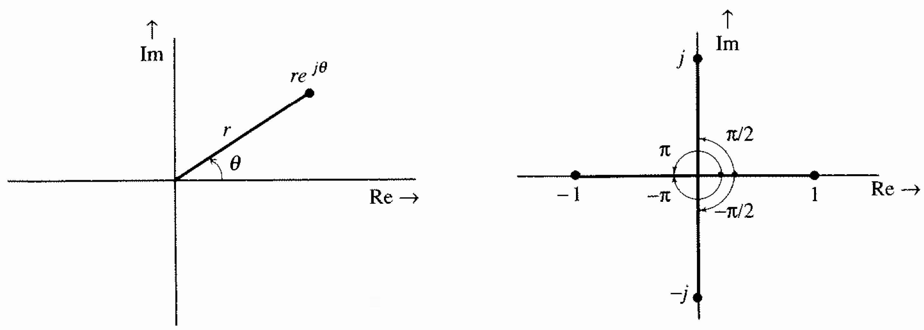

Fig. B.3 Understanding some useful identities in terms of $re^{j\theta}$.

Conjugate of a Complex Number

We define z^*, the **conjugate** of $z = a + jb$, as

$$z^* = a - jb = re^{-j\theta} \tag{B.9a}$$

$$= |z|e^{-j\angle z} \tag{B.9b}$$

The graphical representation of a number z and its conjugate z^* is shown in Fig. B.2. Observe that z^* is a mirror image of z about the horizontal axis. **To find the conjugate of any number, we need only to replace j by $-j$ in that number** (which is the same as changing the sign of its angle).

The sum of a complex number and its conjugate is a real number equal to twice the real part of the number:

$$z + z^* = (a + jb) + (a - jb) = 2a = 2\,\mathrm{Re}\,z \tag{B.10a}$$

The product of a complex number z and its conjugate is a real number $|z|^2$, the square of the magnitude of the number:

$$zz^* = (a + jb)(a - jb) = a^2 + b^2 = |z|^2 \tag{B.10b}$$

Understanding Some Useful Identities

In a complex plane, $re^{j\theta}$ represents a point at a distance r from the origin and at an angle θ with the horizontal axis as shown in Fig. B.3a. For example, the number -1 is at a unit distance from the origin and has an angle π or $-\pi$ (in fact, any odd multiple of $\pm\pi$), as seen from Fig. B.3b. Therefore,

$$1e^{\pm j\pi} = -1$$

In fact,

$$e^{\pm jn\pi} = -1 \qquad n \text{ odd integer} \tag{B.11}$$

The number 1, on the other hand, is also at a unit distance from the origin, but has an angle 2π (in fact, $\pm 2n\pi$ for any integral value of n). Therefore,

$$e^{\pm j2n\pi} = 1 \qquad n \text{ integer} \tag{B.12}$$

Page 212

The number j is at unit distance from the origin and its angle is $\pi/2$ (see Fig. B.3b). Therefore

$$e^{j\pi/2} = j$$

Similarly,

$$e^{-j\pi/2} = -j$$

Thus

$$e^{\pm j\pi/2} = \pm j \tag{B.13a}$$

In fact,

$$e^{\pm jn\pi/2} = \pm j \qquad n = 1,\ 5,\ 9,\ 13, \cdots \tag{B.13b}$$

and

$$e^{\pm jn\pi/2} = \mp j \qquad n = 3,\ 7,\ 11,\ 15, \cdots \tag{B.13c}$$

These results are summarized in Table B.1.

TABLE B.1

r	θ	$re^{j\theta}$	
1	0	$e^{j0} = 1$	
1	$\pm\pi$	$e^{\pm j\pi} = -1$	
1	$\pm n\pi$	$e^{\pm jn\pi} = -1$	n odd integer
1	$\pm 2\pi$	$e^{\pm j2\pi} = 1$	
1	$\pm 2n\pi$	$e^{\pm j2n\pi} = 1$	n integer
1	$\pm\pi/2$	$e^{\pm j\pi/2} = \pm j$	
1	$\pm n\pi/2$	$e^{\pm jn\pi/2} = \pm j$	$n = 1, 5, 9, 13, \ldots$
1	$\pm n\pi/2$	$e^{\pm jn\pi/2} = \mp j$	$n = 3, 7, 11, 15, \ldots$

This discussion shows the usefulness of the graphic picture of $re^{j\theta}$. This picture is also helpful in several other applications. For example, to determine the limit of $e^{(\alpha+j\omega)t}$ as $t \to \infty$, we note that

$$e^{(\alpha+j\omega)t} = e^{\alpha t}e^{j\omega t}$$

Now the magnitude of $e^{j\omega t}$ is unity regardless of the value of ω or t because $e^{j\omega t} = re^{j\theta}$ with $r = 1$. Therefore, $e^{\alpha t}$ determines the behavior of $e^{(\alpha+j\omega)t}$ as $t \to \infty$ and

$$\lim_{t\to\infty} e^{(\alpha+j\omega)t} = \lim_{t\to\infty} e^{\alpha t}e^{j\omega t} = \begin{cases} 0 & \alpha < 0 \\ \infty & \alpha > 0 \end{cases} \tag{B.14}$$

In future discussions you will find it very useful to remember $re^{j\theta}$ as a number at a distance r from the origin and at an angle θ with the horizontal axis of the complex plane.

A Warning about Using Electronic Calculators in Computing Angles

From Cartesian form $a + jb$ we can readily compute polar form $re^{j\theta}$ [see Eq. (B.6)]. Electronic calculators provide ready conversion of rectangular into polar and vice versa. However, if a calculator is used to compute an angle of a complex number using an inverse trigonometric function $\theta = \tan^{-1}(b/a)$, proper attention must be paid to the quadrant in which the number is located. For instance, θ corresponding to the number $-2 - j3$ is $\tan^{-1}(\frac{-3}{-2})$. This is not the same as $\tan^{-1}(\frac{3}{2})$. The former is $-123.7°$, whereas the latter is $56.3°$. An electronic calculator cannot make this distinction and can give a correct answer only for angles in the first and fourth quadrant. It will read $\tan^{-1}(\frac{-3}{-2})$ as $\tan^{-1}(\frac{3}{2})$, which is clearly wrong. In computing inverse trigonometric functions if the angle is in the second or third quadrant, the answer of the calculator is off by 180°. The correct answer is obtained by adding or subtracting 180° to the value found with the calculator (either adding or subtracting yields the correct answer). For this reason it is advisable to draw the point in the complex plane and determine the quadrant in which the point lies. This issue will be clarified by the following examples.

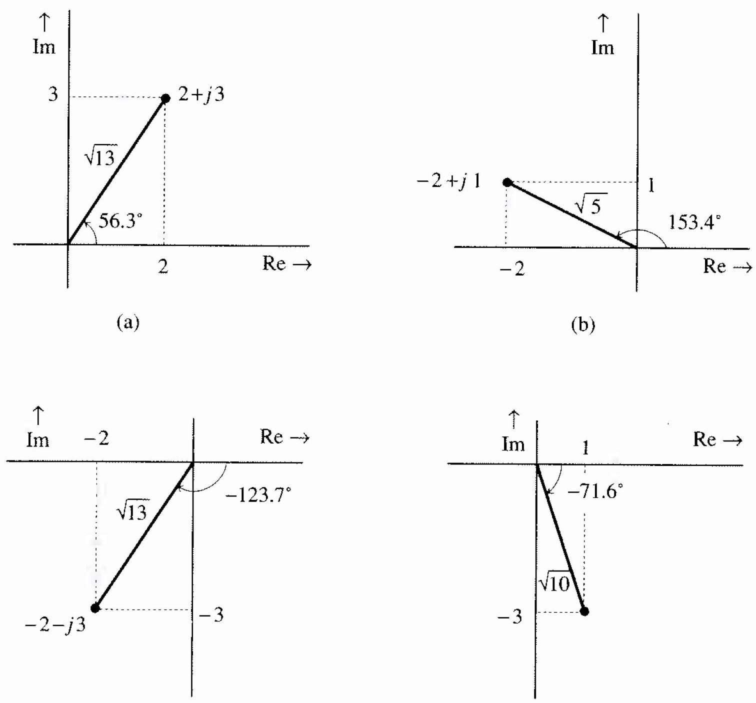

Fig. B.4 From Cartesian to polar form.

■ **Example B.1**

Express the following numbers in polar form:

(a) $2 + j3$ **(b)** $-2 + j1$ **(c)** $-2 - j3$ **(d)** $1 - j3$

(a)

$$|z| = \sqrt{2^2 + 3^2} = \sqrt{13} \qquad \angle z = \tan^{-1}\left(\tfrac{3}{2}\right) = 56.3°$$

In this case the number is in the first quadrant, and a calculator will give the correct value of 56.3°. Therefore (see Fig. B.4a)

$$2 + j3 = \sqrt{13}\, e^{j56.3°}$$

(b)

$$|z| = \sqrt{(-2)^2 + 1^2} = \sqrt{5} \qquad \angle z = \tan^{-1}\left(\tfrac{1}{-2}\right) = 153.4^\circ$$

In this case the angle is in the second quadrant (see Fig. B.4b), and therefore the answer found with the calculator ($\tan^{-1}(\frac{1}{-2}) = -26.6^\circ$) is off by 180°. The correct answer is $(-26.6 \pm 180)^\circ = 153.4^\circ$ or -206.6°. Both values are correct because they represent the same angle. As a matter of convenience we choose an angle whose numerical value is less than 180°, which in this case is 153.4°. Therefore

$$-2 + j1 = \sqrt{5}e^{j153.4^\circ}$$

(c)

$$|z| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13} \qquad \angle z = \tan^{-1}(\tfrac{-3}{-2}) = -123.7^\circ$$

In this case the angle is in the third quadrant (see Fig. B.4c), and therefore the answer found with the calculator ($\tan^{-1}(\frac{-3}{-2}) = 56.3^\circ$) is off by 180°. The correct answer is $(56.3 \pm 180)^\circ = 236.3^\circ$ or -123.7°. As a matter of convenience we choose the latter and (see Fig. B.4c)

$$-2 - j3 = \sqrt{13}e^{-j123.7^\circ}$$

(d)

$$|z| = \sqrt{1^2 + (-3)^2} = \sqrt{10} \qquad \angle z = \tan^{-1}(\tfrac{-3}{1}) = -71.6^\circ$$

In this case the angle is in the fourth quadrant (see Fig. B.4d), and therefore the answer found with the calculator ($\tan^{-1}(\frac{-3}{1}) = -71.6^\circ$) is correct (see Fig. B.4d).

$$1 - j3 = \sqrt{10}e^{-j71.6^\circ} \quad \blacksquare$$

⊙ Computer Example CB.1

Express the following numbers in polar form: **(a)** $2 + j3$ **(b)** $1 - j3$

(a)

```
z=2+i*3
magz=abs(z)                      % magz = |z|
argz_in_radian=angle(z)          % ∠z (in radians)
argz_in_deg=angle(z)*(180/pi)    % ∠z (in degrees)
disp('strike any key for part (b)')
pause
clc
```

(b)

```
z=1-3*i
magz=abs(z)
argz_in_radian=angle(z)
argz_in_deg=angle(z)*(180/pi)
```
⊙

■ Example B.2

Represent the following numbers in the complex plane and express them in Cartesian form: **(a)** $2e^{j\pi/3}$ **(b)** $4e^{-j3\pi/4}$ **(c)** $2e^{j\pi/2}$ **(d)** $3e^{-j3\pi}$ **(e)** $2e^{j4\pi}$ **(f)** $2e^{-j4\pi}$.

(a) $2e^{j\pi/3} = 2\left(\cos\frac{\pi}{3} + j\sin\frac{\pi}{3}\right) = 1 + j\sqrt{3}$ (see Fig. B.5a)

(b) $4e^{-j3\pi/4} = 4\left(\cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}\right) = -2\sqrt{2} - j2\sqrt{2}$ (see Fig. B.5b)

(c) $2e^{j\pi/2} = 2\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right) = 2(0 + j1) = j2$ (see Fig. B.5c)

(d) $3e^{-j3\pi} = 3(\cos 3\pi - j\sin 3\pi) = 3(-1 + j0) = -3$ (see Fig. B.5d)

(e) $2e^{j4\pi} = 2(\cos 4\pi + j\sin 4\pi) = 2(1 + j0) = 2$ (see Fig. B.5e)

(f) $2e^{-j4\pi} = 2(\cos 4\pi - j\sin 4\pi) = 2(1 - j0) = 2$ (see Fig. B.5f) ■

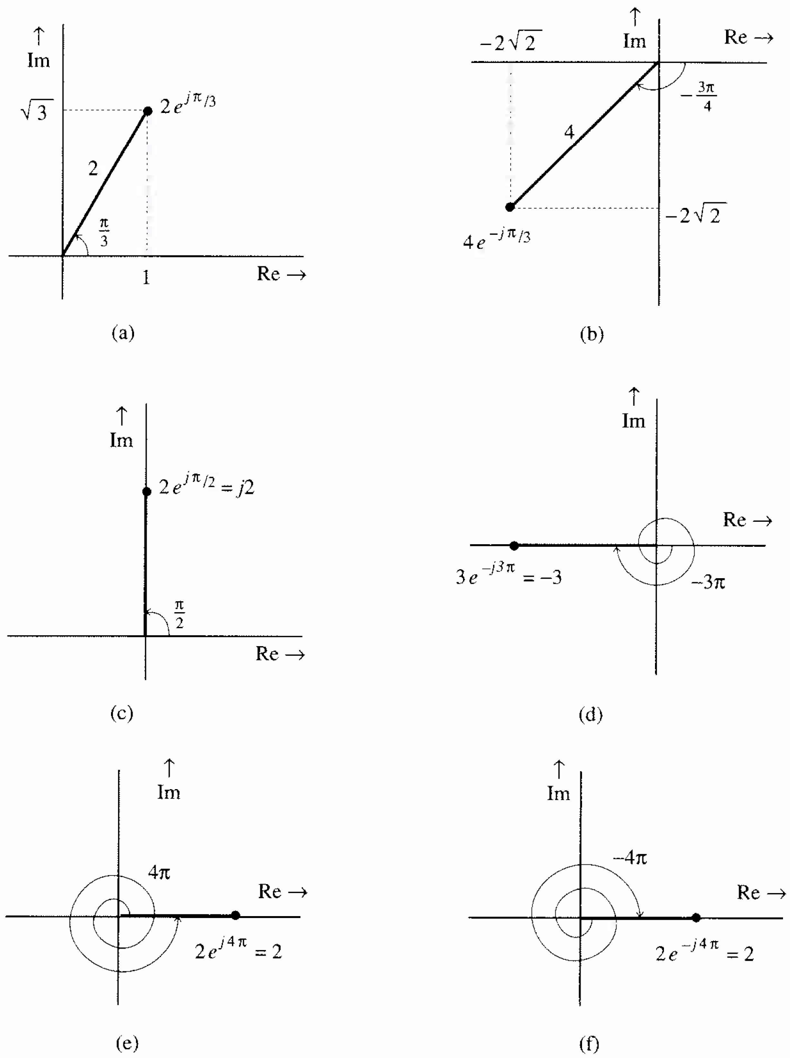

Fig. B.5 From polar to Cartesian form.

⊙ **Computer Example CB.2**

Represent the following numbers in the complex plane and express them in Cartesian form: **(a)** $2e^{-j\frac{\pi}{3}}$ **(b)** $4e^{-j\frac{3\pi}{4}}$

```
(a)
    z=2*exp(-i*pi/3)
(b)
    z=4*exp(-i*3*pi/4)
```
⊙

Arithmetical Operations, Powers, and Roots of Complex Numbers

To perform addition and subtraction, complex numbers should be expressed in Cartesian form. Thus, if

$$z_1 = 3 + j4 = 5e^{j53.1^\circ}$$

$$z_2 = 2 + j3 = \sqrt{13}e^{j56.3^\circ}$$

then

$$z_1 + z_2 = (3 + j4) + (2 + j3) = 5 + j7$$

If z_1 and z_2 are given in polar form, we would need to convert them into Cartesian form for the purpose of adding (or subtracting). Multiplication and division, however, can be carried out in either Cartesian or polar form, although the latter proves to be much more convenient. This is because if z_1 and z_2 are expressed in polar form as

$$z_1 = r_1 e^{j\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{j\theta_2}$$

then

$$z_1 z_2 = \left(r_1 e^{j\theta_1}\right)\left(r_2 e^{j\theta_2}\right) = r_1 r_2 e^{j(\theta_1+\theta_2)} \tag{B.15a}$$

and

$$\frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)} \tag{B.15b}$$

Moreover,

$$z^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta} \tag{B.15c}$$

and

$$z^{1/n} = \left(re^{j\theta}\right)^{1/n} = r^{1/n} e^{j\theta/n} \tag{B.15d}$$

This shows that the operations of multiplication, division, powers, and roots can be carried out with remarkable ease when the numbers are in polar form.

■ Example B.3

Determine $z_1 z_2$ and z_1/z_2 for the numbers

$$z_1 = 3 + j4 = 5e^{j53.1^\circ}$$

$$z_2 = 2 + j3 = \sqrt{13}e^{j56.3^\circ}$$

We shall solve this problem in both polar and Cartesian forms.

Multiplication: Cartesian Form

$$z_1 z_2 = (3 + j4)(2 + j3) = (6 - 12) + j(8 + 9) = -6 + j17$$

Multiplication: Polar Form

$$z_1 z_2 = \left(5e^{j53.1^\circ}\right)\left(\sqrt{13}e^{j56.3^\circ}\right) = 5\sqrt{13}e^{j109.4^\circ}$$

Division: Cartesian Form

$$\frac{z_1}{z_2} = \frac{3 + j4}{2 + j3}$$

In order to eliminate the complex number in the denominator, we multiply both the numerator and the denominator of the right-hand side by $2 - j3$, the denominator's conjugate. This yields

$$\frac{z_1}{z_2} = \frac{(3+j4)(2-j3)}{(2+j3)(2-j3)} = \frac{18-j1}{2^2+3^2} = \frac{18-j1}{13} = \frac{18}{13} - j\frac{1}{13}$$

Division: Polar Form

$$\frac{z_1}{z_2} = \frac{5e^{j53.1^\circ}}{\sqrt{13}e^{j56.3^\circ}} = \frac{5}{\sqrt{13}}e^{j(53.1^\circ - 56.3^\circ)} = \frac{5}{\sqrt{13}}e^{-j3.2^\circ} \quad \blacksquare$$

It is clear from this example that multiplication and division are easier to accomplish in polar form than in Cartesian form.

Computer Example CB.3

Determine $z_1 z_2$ and z_1/z_2 if $z_1 = 3 + j4$ and $z_1 = 2 + j3$

Multiplication and Division: Cartesian Form

```
z1=3+4*i;
z2=2+3*i;
disp('multiplication and division using cartesian form of z1 and z2')
z1z2=z1*z2
z1divz2=z1/z2
disp('multiplication and division using polar Form')
z1z2 = (abs(z1)*exp(i*angle(z1))*abs(z2)*exp(i*angle(z2)))
z1divz2 = (abs(z1)*exp(i*angle(z1)))/(abs(z2)*exp(i*angle(z2)))
```

■ Example B.4

For $z_1 = 2e^{j\pi/4}$ and $z_2 = 8e^{j\pi/3}$, find **(a)** $2z_1 - z_2$ **(b)** $\frac{1}{z_1}$ **(c)** $\frac{z_1}{z_2^2}$ **(d)** $\sqrt[3]{z_2}$

(a) Since subtraction cannot be performed directly in polar form, we convert z_1 and z_2 to Cartesian form:

$$z_1 = 2e^{j\pi/4} = 2\left(\cos\tfrac{\pi}{4} + j\sin\tfrac{\pi}{4}\right) = \sqrt{2} + j\sqrt{2}$$
$$z_2 = 8e^{j\pi/3} = 8\left(\cos\tfrac{\pi}{3} + j\sin\tfrac{\pi}{3}\right) = 4 + j4\sqrt{3}$$

Therefore

$$\begin{aligned} 2z_1 - z_2 &= 2(\sqrt{2}+j\sqrt{2}) - (4+j4\sqrt{3}) \\ &= (2\sqrt{2}-4) + j(2\sqrt{2}-4\sqrt{3}) \\ &= -1.17 - j4.1 \end{aligned}$$

(b)

$$\frac{1}{z_1} = \frac{1}{2e^{j\pi/4}} = \frac{1}{2}e^{-j\pi/4}$$

(c)

$$\frac{z_1}{z_2^2} = \frac{2e^{j\pi/4}}{(8e^{j\pi/3})^2} = \frac{2e^{j\pi/4}}{64e^{j2\pi/3}} = \frac{1}{32}e^{j(\frac{\pi}{4}-\frac{2\pi}{3})} = \frac{1}{32}e^{-j\frac{5\pi}{12}}$$

(d)

$$\sqrt[3]{z_2} = z_2^{1/3} = \left(8e^{j\pi/3}\right)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(e^{j\pi/3}\right)^{1/3} = 2e^{j\pi/9} \quad ■$$

■ **Example B.5**

Consider $F(\omega)$, a complex function of a real variable ω:

$$F(\omega) = \frac{2+j\omega}{3+j4\omega} \tag{B.16a}$$

(a) Express $F(\omega)$ in Cartesian form, and find its real and imaginary parts. **(b)** Express $F(\omega)$ in polar form, and find its magnitude $|F(\omega)|$ and angle $\angle F(\omega)$.

(a) To obtain the real and imaginary parts of $F(\omega)$, we must eliminate imaginary terms in the denominator of $F(\omega)$. This is readily done by multiplying both the numerator and denominator of $F(\omega)$ by $3 - j4\omega$, the conjugate of the denominator $3 + j4\omega$ so that

$$F(\omega) = \frac{(2+j\omega)(3-j4\omega)}{(3+j4\omega)(3-j4\omega)} = \frac{(6+4\omega^2)-j5\omega}{9+16\omega^2} = \frac{6+4\omega^2}{9+16\omega^2} - j\frac{5\omega}{9+\omega^2} \tag{B.16b}$$

This is the Cartesian form of $F(\omega)$. Clearly the real and imaginary parts $F_r(\omega)$ and $F_i(\omega)$ are given by

$$F_r(\omega) = \frac{6+4\omega^2}{9+16\omega^2}, \qquad F_i(\omega) = \frac{-5\omega}{9+16\omega^2}$$

(b)

$$F(\omega) = \frac{2+j\omega}{3+j4\omega} = \frac{\sqrt{4+\omega^2}\, e^{j\tan^{-1}\left(\frac{\omega}{2}\right)}}{\sqrt{9+16\omega^2}\, e^{j\tan^{-1}\left(\frac{4\omega}{3}\right)}}$$

$$= \sqrt{\frac{4+\omega^2}{9+16\omega^2}}\, e^{j\left[\tan^{-1}\left(\frac{\omega}{2}\right)-\tan^{-1}\left(\frac{4\omega}{3}\right)\right]} \tag{B.16c}$$

This is the polar representation of $F(\omega)$. Observe that

$$|F(\omega)| = \sqrt{\frac{4+\omega^2}{9+16\omega^2}}, \qquad \angle F(\omega) = \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{4\omega}{3}\right) \tag{B.17}$$

■

B.2 SINUSOIDS

Consider the sinusoid

$$f(t) = C\cos(2\pi\mathcal{F}_0 t + \theta) \tag{B.18}$$

We know that

$$\cos\varphi = \cos(\varphi + 2n\pi) \qquad n = 0, \pm1, \pm2, \pm3, \cdots$$

Therefore, $\cos\varphi$ repeats itself for every change of 2π in the angle $\angle\varphi$. For the sinusoid in Eq. (B.18), the angle $2\pi\mathcal{F}_0 t + \theta$ changes by 2π when t changes by

$1/\mathcal{F}_0$. Clearly
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Annexe B

Tables de mathématiques

B.10 MISCELLANEOUS

B.10-1 L'Hôpital's Rule

If $\lim f(x)/g(x)$ results in the indeterministic form 0/0 or ∞/∞, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{\dot{f}(x)}{\dot{g}(x)}$$

B.10-2 The Taylor and Maclaurin Series

$$f(x) = f(a) + \frac{(x-a)}{1!}\dot{f}(a) + \frac{(x-a)^2}{2!}\ddot{f}(a) + \cdots$$

$$f(x) = f(0) + \frac{x}{1!}\dot{f}(0) + \frac{x^2}{2!}\ddot{f}(0) + \cdots$$

B.10-3 Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots \qquad x^2 < \pi^2/4$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \binom{n}{k}x^k + \cdots + x^n$$

$$\approx 1 + nx \qquad |x| << 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

B.10-4 Sums

$$\sum_{m=0}^{k} r^m = \frac{r^{k+1}-1}{r-1} \qquad r \neq 1$$

$$\sum_{m=M}^{N} r^m = \frac{r^{N+1}-r^M}{r-1} \qquad r \neq 1$$

$$\sum_{m=0}^{k} \left(\frac{a}{b}\right)^m = \frac{a^{k+1}-b^{k+1}}{b^k(a-b)} \qquad a \neq b$$

B.10 Misc

B.10-5 C

B.10-6 T

B.10-5 Complex Numbers

$$e^{\pm j\pi/2} = \pm j$$

$$e^{\pm jn\pi} = \begin{cases} 1 & n\,\text{even} \\ -1 & n\,\text{odd} \end{cases}$$

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$a + jb = re^{j\theta} \qquad r = \sqrt{a^2 + b^2}, \qquad \theta = \tan^{-1}\left(\tfrac{b}{a}\right)$$

$$(re^{j\theta})^k = r^k e^{jk\theta}$$

$$(r_1 e^{j\theta_1})(r_2 e^{j\theta_2}) = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

B.10-6 Trigonometric Identities

$$e^{\pm jx} = \cos x \pm j\sin x$$

$$\cos x = \tfrac{1}{2}[e^{jx} + e^{-jx}]$$

$$\sin x = \tfrac{1}{2j}[e^{jx} - e^{-jx}]$$

$$\cos(x \pm \tfrac{\pi}{2}) = \mp \sin x$$

$$\sin(x \pm \tfrac{\pi}{2}) = \pm \cos x$$

$$2\sin x \cos x = \sin 2x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cos^2 x - \sin^2 x = \cos 2x$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

$$\cos^3 x = \tfrac{1}{4}(3\cos x + \cos 3x)$$

$$\sin^3 x = \tfrac{1}{4}(3\sin x - \sin 3x)$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$

$$\cos x \cos y = \tfrac{1}{2}[\cos(x - y) + \cos(x + y)]$$

$$\sin x \cos y = \tfrac{1}{2}[\sin(x - y) + \sin(x + y)]$$

$$a\cos x + b\sin x = C\cos(x + \theta)$$

in which $C = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left(\dfrac{-b}{a}\right)$

Page 223

B.10-7 Indefinite Integrals

$$\int u\,dv = uv - \int v\,du$$

$$\int f(x)\dot{g}(x)\,dx = f(x)g(x) - \int \dot{f}(x)g(x)\,dx$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \qquad \int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int x\sin ax\,dx = \frac{1}{a^2}(\sin ax - ax\cos ax)$$

$$\int x\cos ax\,dx = \frac{1}{a^2}(\cos ax + ax\sin ax)$$

$$\int x^2\sin ax\,dx = \frac{1}{a^3}(2ax\sin ax + 2\cos ax - a^2x^2\cos ax)$$

$$\int x^2\cos ax\,dx = \frac{1}{a^3}(2ax\cos ax - 2\sin ax + a^2x^2\sin ax)$$

$$\int \sin ax\sin bx\,dx = \frac{\sin(a-b)x}{2(a-b)} - \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int \sin ax\cos bx\,dx = -\left[\frac{\cos(a-b)x}{2(a-b)} + \frac{\cos(a+b)x}{2(a+b)}\right] \qquad a^2 \neq b^2$$

$$\int \cos ax\cos bx\,dx = \frac{\sin(a-b)x}{2(a-b)} + \frac{\sin(a+b)x}{2(a+b)} \qquad a^2 \neq b^2$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$\int x^2e^{ax}\,dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{x}{x^2+a^2}\,dx = \frac{1}{2}\ln(x^2+a^2)$$

B.10-8 Differentiation Table

$\frac{d}{dx}f(u) = \frac{d}{du}f(u)\frac{du}{dx}$	$\frac{d}{dx}a^{bx} = b(\ln a)a^{bx}$
$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$	$\frac{d}{dx}\sin ax = a\cos ax$
$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	$\frac{d}{dx}\cos ax = -a\sin ax$
$\frac{dx^n}{dx} = nx^{n-1}$	$\frac{d}{dx}\tan ax = \frac{a}{\cos^2 ax}$
$\frac{d}{dx}\ln(ax) = \frac{1}{x}$	$\frac{d}{dx}(\sin^{-1} ax) = \frac{a}{\sqrt{1-a^2x^2}}$
$\frac{d}{dx}\log(ax) = \frac{\log e}{x}$	$\frac{d}{dx}(\cos^{-1} ax) = \frac{-a}{\sqrt{1-a^2x^2}}$
$\frac{d}{dx}e^{bx} = be^{bx}$	$\frac{d}{dx}(\tan^{-1} ax) = \frac{a}{1+a^2x^2}$

B.10-9 Some Useful Constants

$\pi \approx 3.1415926535$

$e \approx 2.7182818284$

$\frac{1}{e} \approx 0.3678794411$

$\log_{10} 2 = 0.30103$

$\log_{10} 3 = 0.47712$

B.10-10 Solution of Quadratic and Cubic Equations

Any **quadratic** equation can be reduced to the form

$$ax^2 + bx + c = 0$$

The solution of this equation is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Table 6.2

Fourier Series Representation of a Periodic Signal of Period T_0 ($\omega_0 = 2\pi/T_0$)

Series form	Coefficient computation	Conversion formulas
Trigonometric	$a_0 = \frac{1}{T_0}\int_{T_0} f(t)\,dt$	$a_0 = C_0 = D_0$
$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$	$a_n = \frac{2}{T_0}\int_{T_0} f(t)\cos n\omega_0 t\,dt$	$a_n - jb_n = C_n e^{j\theta_n} = 2D_n$
	$b_n = \frac{2}{T_0}\int_{T_0} f(t)\sin n\omega_0 t\,dt$	$a_n + jb_n = C_n e^{-j\theta_n} = 2D_{-n}$
Compact Trigonometric	$C_0 = a_0$	$C_0 = D_0$
$f(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$	$C_n = \sqrt{{a_n}^2 + {b_n}^2}$	$C_n = 2\lvert D_n\rvert \quad n \geq 1$
	$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$	$\theta_n = \angle D_n$
Exponential		
$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$	$D_n = \frac{1}{T_0}\int_{T_0} f(t)e^{-jn\omega_0 t}\,dt$	

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($|D_n|$ vs. ω) is
function of ω w

■ **Example 6.**
Find the ex
In this case

where

From the time-differentiation property (7.47)

$$\frac{d^2 f}{dt^2} \Longleftrightarrow (j\omega)^2 F(\omega) = -\omega^2 F(\omega) \tag{7.50a}$$

Also, from the time-shifting property (7.37)

$$\delta(t - t_0) \Longleftrightarrow e^{-j\omega t_0} \tag{7.50b}$$

Taking Fourier transform of Eq. (7.49) and using the results in Eqs. (7.50), we obtain

$$-\omega^2 F(\omega) = \tfrac{2}{\tau}[e^{j\frac{\omega\tau}{2}} - 2 + e^{-j\frac{\omega\tau}{2}}] = \tfrac{4}{\tau}\left(\cos \tfrac{\omega\tau}{2} - 1\right) = -\tfrac{8}{\tau}\sin^2\left(\tfrac{\omega\tau}{4}\right)$$

and

$$F(\omega) = \frac{8}{\omega^2 \tau}\sin^2\left(\frac{\omega\tau}{4}\right) = \frac{\tau}{2}\left[\frac{\sin\left(\frac{\omega\tau}{4}\right)}{\frac{\omega\tau}{4}}\right]^2 = \frac{\tau}{2}\,\text{sinc}^2\left(\frac{\omega\tau}{4}\right) \tag{7.51}$$

The spectrum $F(\omega)$ is shown in Fig. 7.24d. This procedure of finding the Fourier transform can be applied to any function made up of straight line segments. The second derivative of such a signal yields a sequence of impulses whose Fourier transform can be found by inspection. This example suggests a numerical method of finding the Fourier transform of an arbitrary signal $f(t)$ by approximating the signal by straight-line segments. ■

△ **Exercise E7.10**

Find the Fourier transform of rect $\left(\frac{t}{\tau}\right)$, using the time-differentiation property. ▽

Table 7.2

Fourier Transform Operations

Operation	$f(t)$	$F(\omega)$
Addition	$f_1(t) + f_2(t)$	$F_1(\omega) + F_2(\omega)$
Scalar multiplication	$kf(t)$	$kF(\omega)$
Duality	$F(t)$	$2\pi f(-\omega)$
Scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega - \omega_0)$
Time convolution	$f_1(t) * f_2(t)$	$F_1(\omega)F_2(\omega)$
Frequency convolution	$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega) * F_2(\omega)$
Time differentiation	$\frac{d^n f}{dt^n}$	$(j\omega)^n F(\omega)$
Time integration	$\int_{-\infty}^{t} f(x)\,dx$	$\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$

Page 227

Table 4.2

Laplace Transform Operations

Operation	$f(t)$	$F(s)$
Addition	$f_1(t) + f_2(t)$	$F_1(s) + F_2(s)$
Scalar multiplication	$kf(t)$	$kF(s)$
Time differentiation	$\dfrac{df}{dt}$	$sF(s) - f(0^-)$
	$\dfrac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - \dot{f}(0^-)$
	$\dfrac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - s\dot{f}(0^-) - \ddot{f}(0^-)$
Time integration	$\displaystyle\int_{0^-}^{t} f(t)\,dt$	$\dfrac{1}{s}F(s)$
	$\displaystyle\int_{-\infty}^{t} f(t)\,dt$	$\dfrac{1}{s}F(s) + \dfrac{1}{s}\displaystyle\int_{-\infty}^{0^-} f(t)\,dt$
Time shift	$f(t-t_0)u(t-t_0)$	$F(s)e^{-st_0} \quad t_0 \geq 0$
Frequency shift	$f(t)e^{s_0t}$	$F(s-s_0)$
Frequency differentiation	$-tf(t)$	$\dfrac{dF(s)}{ds}$
Frequency integration	$\dfrac{f(t)}{t}$	$\displaystyle\int_{s}^{\infty} F(s)\,ds$
Scaling	$f(at),\ a \geq 0$	$\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$
Time convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$
Frequency convolution	$f_1(t)f_2(t)$	$\dfrac{1}{2\pi j}F_1(s) * F_2(s)$
Initial value	$f(0^+)$	$\lim\limits_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim\limits_{s\to 0} sF(s)$ (poles of $sF(s)$ in LHP)

Page 228

Table 7.1

A Short Table of Fourier Transforms

	$f(t)$	$F(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\frac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\displaystyle\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\displaystyle\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

Table 4.1

A Short Table of Laplace Transforms

	$f(t)$	$F(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
8a	$\cos bt\, u(t)$	$\frac{s}{s^2+b^2}$
8b	$\sin bt\, u(t)$	$\frac{b}{s^2+b^2}$
9a	$e^{-at}\cos bt\, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
9b	$e^{-at}\sin bt\, u(t)$	$\frac{b}{(s+a)^2+b^2}$
10a	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{(r\cos\theta)s+(ar\cos\theta-br\sin\theta)}{s^2+2as+(a^2+b^2)}$
10b	$re^{-at}\cos(bt+\theta)\, u(t)$	$\frac{0.5re^{j\theta}}{s+a-jb}+\frac{0.5re^{-j\theta}}{s+a+jb}$
10c	$re^{-at}\cos(bt+\theta)\, u(t)$ $r=\sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}}$, $\theta=\tan^{-1}\frac{Aa-B}{A\sqrt{c-a^2}}$ $b=\sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$
10d	$e^{-at}\left[A\cos bt+\frac{B-Aa}{b}\sin bt\right]u(t)$ $b=\sqrt{c-a^2}$	$\frac{As+B}{s^2+2as+c}$

TABLE 2.1: Convolution Table

No	$f_1(t)$	$f_2(t)$	$f_1(t) * f_2(t) = f_2(t) * f_1(t)$
1	$f(t)$	$\delta(t-T)$	$f(t-T)$
2	$e^{\lambda t}u(t)$	$u(t)$	$\frac{-1}{\lambda}(1-e^{\lambda t})u(t)$
3	$u(t)$	$u(t)$	$tu(t)$
4	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{1}{\lambda_1-\lambda_2}[e^{\lambda_1 t}-e^{\lambda_2 t}]u(t) \quad \lambda_1 \neq \lambda_2$
5	$e^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$te^{\lambda t}u(t)$
6	$te^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$\frac{1}{2}t^2e^{\lambda t}u(t)$
7	$t^n u(t)$	$e^{\lambda t}u(t)$	$\frac{n!}{\lambda^{n+1}}e^{\lambda t}u(t)-\sum_{j=o}^{n}\frac{n!}{\lambda^{j+1}(n-j)!}t^{n-j}u(t)$
8	$t^m u(t)$	$t^n u(t)$	$\frac{m!n!}{(m+n+1)!}t^{m+n+1}u(t)$
9	$te^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{1}{(\lambda_1-\lambda_2)^2}[e^{\lambda_2 t}-e^{\lambda_1 t}+(\lambda_1-\lambda_2)te^{\lambda_1 t}]u(t)$
10	$t^m e^{\lambda t}u(t)$	$t^n e^{\lambda t}u(t)$	$\frac{m!\,n!}{(n+m+1)!}t^{m+n+1}e^{\lambda t}u(t)$
11	$t^m e^{\lambda_1 t}u(t)$	$t^n e^{\lambda_2 t}u(t)$	$\sum_{j=0}^{m}\frac{(-1)^j m!(n+j)!}{j!(m-j)!(\lambda_1-\lambda_2)^{n+j+1}}t^{m-j}e^{\lambda_1 t}u(t)$
	$\lambda_1 \neq \lambda_2$		$+\sum_{k=0}^{n}\frac{(-1)^k n!(m+k)!}{k!(n-k)!(\lambda_2-\lambda_1)^{m+k+1}}t^{n-k}e^{\lambda_2 t}u(t)$
12	$e^{-\alpha t}\cos(\beta t+\theta)u(t)$	$e^{\lambda t}u(t)$	$\frac{\cos(\theta-\phi)e^{\lambda t}-e^{-\alpha t}\cos(\beta t+\theta-\phi)}{\sqrt{(\alpha+\lambda)^2+\beta^2}}u(t)$
			$\phi=\tan^{-1}[-\beta/(\alpha+\lambda)]$
13	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(-t)$	$\frac{1}{\lambda_2-\lambda_1}[e^{\lambda_1 t}u(t)+e^{\lambda_2 t}u(-t)] \quad \mathrm{Re}\,\lambda_2 > \mathrm{Re}\,\lambda_1$
14	$e^{\lambda_1 t}u(-t)$	$e^{\lambda_2 t}u(-t)$	$\frac{1}{\lambda_2-\lambda_1}(e^{\lambda_1 t}-e^{\lambda_2 t})u(-t)$

Annexe C

Les fractions partielles

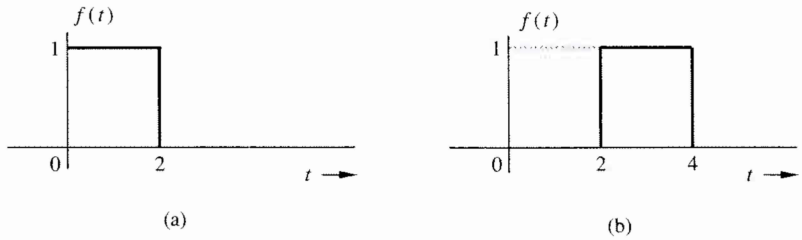

Fig. 4.2 Signals for Exercise E4.1.

$$
\begin{aligned}
\mathcal{L}[\cos \omega_0 t\, u(t)] &= \frac{1}{2}\left[\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0}\right] \qquad \operatorname{Re}(s \pm j\omega) = \operatorname{Re} s > 0 \\
&= \frac{s}{s^2 + {\omega_0}^2} \qquad \operatorname{Re} s > 0
\end{aligned}
\tag{4.14}
$$

■

For the unilateral Laplace transform, there is a unique inverse transform of $F(s)$; consequently there is no need to specify the region of convergence explicitly. For this reason we shall generally ignore any mention of the region of convergence for unilateral transforms.

In unilateral Laplace transform (discussed so far) it is understood that every signal $f(t)$ is zero for $t < 0$; the practice of multiplying $f(t)$ by $u(t)$ (as done here) is redundant and unnecessary. However, such a precaution may save us from some potential pitfalls, as pointed out later.

△ **Exercise E4.1**

By direct integration, find the Laplace transform $F(s)$ and the region of convergence of $F(s)$ for signals shown in Fig. 4.2.

Answer: **(a)** $\frac{1}{s}(1 - e^{-2s})$ for all s. **(b)** $\frac{1}{s}(e^{-2s} - e^{-4s})$ for all s. ▽

4.1-1 Finding the Inverse Transform

Finding the inverse Laplace transform by using the definition (4.2) requires integration in the complex plane. This is beyond the scope of this book.[2] For our purpose, we can find the inverse transforms from the transform table. All we need to do is to express $F(s)$ as a sum of simpler functions of the form listed in the table. Most of the transforms $F(s)$ of practical interest are **rational functions**, that is, ratios of polynomials in s. Such functions can be expressed as a sum of simpler functions by using partial fraction expansion (see Sec. B.8). Values of s for which $F(s) = 0$ are called the zeros of $F(s)$; the values of s for which $F(s) \to \infty$ and are called the **poles** of $F(s)$. If $F(s)$ is a rational function of the form $P(s)/Q(s)$, the roots of $P(s)$ are the zeros and the roots of $Q(s)$ are the poles of $F(s)$.

■ **Example 4.3**

Find the inverse Laplace transforms of:

(a) $\dfrac{7s - 6}{s^2 - s - 6}$ **(b)** $\dfrac{2s^2 + 5}{s^2 + 3s + 2}$ **(c)** $\dfrac{6(s + 34)}{s(s^2 + 10s + 34)}$ **(d)** $\dfrac{8s + 10}{(s + 1)(s + 2)^3}$

The inverse transform of none of the above functions is directly available in Table 4.1. We need to expand these functions into partial fractions. Readers should familiarize themselves thoroughly with the techniques of partial fraction expansion discussed in Sec. B.8.

Page 234

(a)

$$F(s) = \frac{7s-6}{(s+2)(s-3)}$$
$$= \frac{k_1}{s+2} + \frac{k_2}{s-3}$$

To determine k_1, corresponding to the term $(s+2)$, we cover the term $(s+2)$ in $F(s)$ and substitute $s=-2$ (the value of s that makes $s+2=0$) in the remaining expression (see Sec. B.8-2):

$$k_1 = \left.\frac{7s-6}{\boxed{(s+2)}(s-3)}\right|_{s=-2} = \frac{-14-6}{-2-3} = 4$$

Similarly, to determine k_2 corresponding to the term $(s-3)$, we cover up the term $(s-3)$ in $F(s)$ and substitute $s=3$ in the remaining expression

$$k_2 = \left.\frac{7s-6}{(s+2)\boxed{(s-3)}}\right|_{s=3} = \frac{21-6}{3+2} = 3$$

Therefore

$$F(s) = \frac{7s-6}{(s+2)(s-3)} = \frac{4}{s+2} + \frac{3}{s-3} \tag{4.15a}$$

Checking the answer

It is easy to make a mistake in partial fraction computations. Fortunately it is simple to check the answer by recognizing that $F(s)$ and its partial fractions must be equal for every value of s if the partial fractions are correct. Let us verify this in Eq. (4.15a) for some convenient value, say $s=1$. Substitution of $s=1$ in Eq. (4.15a) yields†

$$-\frac{1}{6} = \frac{4}{3} - \frac{3}{2} = -\frac{1}{6}$$

We can now be sure of our answer with a high margin of confidence. Using Pair 5 (Table 4.1) in Eq. (4.15a), we obtain

$$f(t) = \mathcal{L}^{-1}\left(\frac{4}{s+2} + \frac{3}{s-3}\right) = \left(4e^{-2t} + 3e^{3t}\right)u(t) \tag{4.15b}$$

(b)

$$F(s) = \frac{2s^2+5}{s^2+3s+2} = \frac{2s^2+5}{(s+1)(s+2)}$$

Observe that $F(s)$ is an improper function with $m=n$. In such a case we can express $F(s)$ as a sum of the coefficient b_n (the coefficient of the highest power in the numerator) plus partial fractions corresponding to the poles of $F(s)$ (see Sec.B.8-4). In the present case $b_n=2$. Therefore

$$F(s) = 2 + \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

where

$$k_1 = \left.\frac{2s^2+5}{\boxed{(s+1)}(s+2)}\right|_{s=-1} = \frac{2+5}{-1+2} = 7$$

†Because $F(s)=\infty$ at its poles, we should avoid the pole values (-2 and 3 in the present case) for checking. It is possible that the answers may check even if partial fractions are wrong. This can happen when two or more errors cancel their effects. But the chances of this happening for a randomly selected s are extremely small.

and

$$k_2 = \left.\frac{2s^2+5}{(s+1)(s+2)}\right|_{s=-2} = \frac{8+5}{-2+1} = -13$$

Therefore

$$F(s) = 2 + \frac{7}{s+1} - \frac{13}{s+2}$$

From Table 4.1, Pairs 1 and 5, we obtain

$$f(t) = 2\delta(t) + \left(7e^{-t} - 13e^{-2t}\right)u(t) \tag{4.16}$$

(c)

$$\begin{aligned} F(s) &= \frac{6(s+34)}{s(s^2+10s+34)} \\ &= \frac{6(s+34)}{s(s+5-j3)(s+5+j3)} \\ &= \frac{k_1}{s} + \frac{k_2}{s+5-j3} + \frac{{k_2}^*}{s+5+j3} \end{aligned}$$

Note that the coefficients (k_2 and ${k_2}^*$) of the conjugate terms must also be conjugate (see Sec. B.8). Now

$$k_1 = \left.\frac{6(s+34)}{s(s^2+10s+34)}\right|_{s=0} = \frac{6\times 34}{34} = 6$$

$$k_2 = \left.\frac{6(s+34)}{s(s+5-j3)(s+5+j3)}\right|_{s=-5+j3} = \frac{29+j3}{-3-j5} = -3+j4$$

Therefore

$${k_2}^* = -3-j4$$

In order to be able to use Pair 10b (Table 4.1), we need to express k_2 and ${k_2}^*$ in polar form

$$-3+j4 = \left(\sqrt{3^2+4^2}\right)e^{j\tan^{-1}(4/-3)} = 5e^{j\tan^{-1}(4/-3)}$$

Observe that $\tan^{-1}(\frac{4}{-3}) \neq \tan^{-1}(-\frac{4}{3})$. This is evident from Fig. 4.3. Remember that electronic calculators can give answers only for the angles in the first and the fourth quadrant. For this reason it is important to plot the point (e.g. $-3+j4$) in the complex plane, as shown in Fig. 4.3, and determine the angle. For further discussion of this topic, see Example B.1.

From Fig. 4.3, we observe that

$$k_2 = -3+j4 = 5e^{j126.9^\circ}$$

so that

$${k_2}^* = 5e^{-j126.9^\circ}$$

Therefore

$$F(s) = \frac{6}{s} + \frac{5e^{j126.9^\circ}}{s+5-j3} + \frac{5e^{-j126.9^\circ}}{s+5+j3}$$

From Table 4.1 (Pairs 2 and 10b), we obtain

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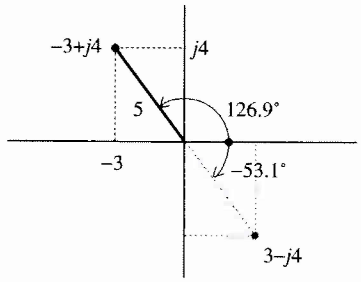


Fig. 4.3 $\tan^{-1}\left(\frac{-4}{3}\right) \neq \tan^{-1}\left(\frac{4}{-3}\right)$.

$$f(t) = \left[6 + 10e^{-5t}\cos(3t + 126.9^\circ)\right]u(t) \tag{4.17}$$

Alternative Method Using Quadratic Factors

The above procedure involves considerable manipulation of complex numbers. As indicated by Pair 10c (Table 4.1), the inverse transform of quadratic terms (with complex conjugate poles) can be found directly without having to find first-order partial fraction. We now show the alternative method (also discussed in Sec. B.8-2), which exploits this observation. For this purpose we shall express $F(s)$ as

$$F(s) = \frac{6(s+34)}{s(s^2+10s+34)} = \frac{k_1}{s} + \frac{As+B}{s^2+10s+34}$$

We have already found $k_1 = 6$ by the (Heaviside) "cover-up" method. Therefore

$$\frac{6(s+34)}{s(s^2+10s+34)} = \frac{6}{s} + \frac{As+B}{s^2+10s+34}$$

Clearing the fractions by multiplying both sides with $s(s^2+10s+34)$ yields

$$\begin{aligned} 6(s+34) &= 6\left(s^2+10s+34\right) + s(As+B) \\ &= (6+A)s^2 + (60+B)s + 204 \end{aligned}$$

Now equating the coefficients of s^2 and s on both sides yields

$$0 = (6+A) \implies A = -6$$
$$6 = 60 + B \implies B = -54$$

and

$$F(s) = \frac{6}{s} + \frac{-6s-54}{s^2+10s+34}$$

We now use Pairs 2 and 10c to find the inverse Laplace transform. The parameters for Pair 10c are $A = -6$, $B = -54$, $a = 5$, $c = 34$, and $b = \sqrt{c-a^2} = 3$, and

$$r = \sqrt{\frac{A^2c+B^2-2ABa}{c-a^2}} = 10, \quad \theta = \tan^{-1}\frac{Aa-B}{A\sqrt{c-a^2}} = 126.9^\circ$$

Therefore

$$f(t) = [6 + 10e^{-5t}\cos(3t + 126.9^\circ)]u(t)$$

which agrees with previous result.

Short-Cuts

The partial fractions with quadratic terms also can be found by using short cuts. We have

$$F(s) = \frac{6(s+34)}{s(s^2+10s+34)} = \frac{6}{s} + \frac{As+B}{s^2+10s+34}$$

We can determine A by eliminating B on the right-hand side. This can be done by multiplying both sides of the above equation by s and then letting $s \to \infty$. This yields

$$0 = 6 + A \implies A = -6$$

Therefore

$$\frac{6(s+34)}{s(s^2+10s+34)} = \frac{6}{s} + \frac{-6s+B}{s^2+10s+34}$$

To find B, we let s take on any convenient value, say $s = 1$, in this equation to yield

$$\frac{210}{45} = 6 + \frac{B-6}{45}$$

Multiplying both sides of this equation by 45 yields

$$210 = 270 + B - 6 \implies B = -54$$

which agrees with the results we found earlier.

(d)

$$\begin{aligned} F(s) &= \frac{8s+10}{(s+1)(s+2)^3} \\ &= \frac{k_1}{s+1} + \frac{a_0}{(s+2)^3} + \frac{a_1}{(s+2)^2} + \frac{a_2}{s+2} \end{aligned}$$

where (see Sec. B.8-3)

$$\begin{aligned} k_1 &= \left. \frac{8s+10}{(s+1)(s+2)^3} \right|_{s=-1} = 2 \\ a_0 &= \left. \frac{8s+10}{(s+1)(s+2)^3} \right|_{s=-2} = 6 \\ a_1 &= \left\{ \frac{d}{ds} \left[\frac{8s+10}{(s+1)(s+2)^3} \right] \right\}_{s=-2} = -2 \\ a_2 &= \frac{1}{2} \left\{ \frac{d^2}{ds^2} \left[\frac{8s+10}{(s+1)(s+2)^3} \right] \right\}_{s=-2} = -2 \end{aligned}$$

Therefore

$$F(s) = \frac{2}{s+1} + \frac{6}{(s+2)^3} - \frac{2}{(s+2)^2} - \frac{2}{s+2}$$

and

$$f(t) = \left[2e^{-t} + (3t^2 - 2t - 2)e^{-2t}\right] u(t) \tag{4.18}$$

Alternative Method: A Hybrid of Heaviside and Clearing Fractions

In this method the simpler coefficients k_1 and a_0 are determined by the Heaviside "cover-up" procedure, as discussed earlier. To determine the remaining coefficients, we use the clearing-fraction method (see Sec. B.8-3). Using the values $k_1 = 2$ and $a_0 = 6$ found earlier by the Heaviside "cover-up" method, we have

$$\frac{8s+10}{(s+1)(s+2)^3} = \frac{2}{s+1} + \frac{6}{(s+2)^3} + \frac{a_1}{(s+2)^2} + \frac{a_2}{s+2}$$

We now clear fractions by multiplying both sides of the equation by $(s+1)(s+2)^3$. This yields†

$$\begin{aligned} 8s+10 &= 2(s+2)^3 + 6(s+1) + a_1(s+1)(s+2) + a_2(s+1)(s+2)^2 \\ &= (2+a_2)s^3 + (12+a_1+5a_2)s^2 + (30+3a_1+8a_2)s + (22+2a_1+4a_2) \end{aligned}$$

Equating coefficients of s^3 and s^2 on both sides, we obtain

$$0 = (2+a_2) \implies a_2 = -2$$

$$0 = 12 + a_1 + 5a_2 = 2 + a_1 \implies a_1 = -2$$

We can stop here if we wish, since the two desired coefficients a_1 and a_2 are already found. However, equating the coefficients of s^1 and s^0 serves as a check on our answers. This yields

$$8 = 30 + 3a_1 + 8a_2$$

$$10 = 22 + 2a_1 + 4a_2$$

Substitution of $a_1 = a_2 = -2$, found earlier, satisfies these equations. This assures the correctness of our answers.

Alternative Method: A Hybrid of Heaviside and Short-Cuts

In this method the simpler coefficients k_1 and a_0 are determined by the Heaviside "cover-up" procedure, as discussed earlier. The usual short-cuts then are used to determine the remaining coefficients. Using the values $k_1 = 2$ and $a_0 = 6$, found earlier by the Heaviside method, we have

$$\frac{8s+10}{(s+1)(s+2)^3} = \frac{2}{s+1} + \frac{6}{(s+2)^3} + \frac{a_1}{(s+2)^2} + \frac{a_2}{s+2}$$

There are two unknowns, a_1 and a_2. If we multiply both sides by s and then let $s \to \infty$, we eliminate a_1. This yields

$$0 = 2 + a_2 \implies a_2 = -2$$

Therefore

$$\frac{8s+10}{(s+1)(s+2)^3} = \frac{2}{s+1} + \frac{6}{(s+2)^3} + \frac{a_1}{(s+2)^2} - \frac{2}{s+2}$$

†We could have cleared fractions without finding k_1 and a_0. This , however, proves more laborious because it increases the number of unknowns to 4. By predetermining k_1 and a_0, we reduce the unknowns to 2. Moreover, this method provides a convenient check on the solution. This hybrid procedure achieves the best of both methods.

There is now only one unknown, a_1. This can be found readily by setting s equal to any convenient value, say $s = 0$. This yields

$$\frac{10}{8} = 2 + \frac{3}{4} + \frac{a_1}{4} - 1 \implies a_1 = -2 \quad \blacksquare$$

⊙ Computer Example C4.1

Find the coefficients of the inverse Laplace transform for the following quadratic function using pair 10c in table 4.1:

$$F(s) = \frac{7s+6}{s^2+s+7}$$

```
num = [7 6];     % Enter the numerator polynomial.
den = [1 1 7];  % Enter the denominator polynomial.
b =sqrt(den(3) - (den(2)/2)^2) % Solve for b
% Solve for theta.
theta = atan((num(1)*den(2)/2-num(2))/(num(1)*b))
r = sqrt((num(1)^2*den(3)+num(2)^2-num(1)*num(2)*den(2))/b^2)
% Calculates the coefficient r.
a = den(2)/2;    % Calculate the coefficient a.
t = 0:.1:10;     % Create a time vector.
f=r*exp(-a*t).*cos(b*t+theta);
disp('Strike any key to see the plot of f(t)')
pause
plot(t,f),grid       % You should see a dampened sinusoid.
xlabel('t'),ylabel('f(t)'),title('plot of f(t)')
```
⊙

△ Exercise E4.2

(i) Show that the Laplace transform of $10e^{-3t}\cos(4t + 53.13°)$ is $\dfrac{6s-14}{s^2+6s+25}$. Use Pair 10a from Table 4.1.

(ii) Find the inverse Laplace transform of: **(a)** $\dfrac{s+17}{s^2+4s-5}$

(b) $\dfrac{3s-5}{(s+1)(s^2+2s+5)}$ **(c)** $\dfrac{16s+43}{(s-2)(s+3)^2}$

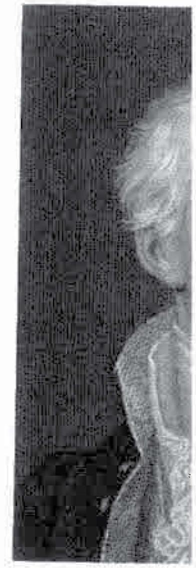
Answers: **(a)** $\left(3e^{t} - 2e^{-5t}\right)u(t)$ **(b)** $\left[-2e^{-t} + \frac{5}{2}e^{-t}\cos(2t - 36.87°)\right]u(t)$

(c) $\left[3e^{2t} + (t-3)e^{-3t}\right]u(t)$ ▽

A Historical Note: Marquis Pierre-Simon De Laplace (1749-1827)

The Laplace transform is named after the great French mathematician and astronomer Laplace, who first presented the transform and its applications to differential equations in a paper published in 1779.

Laplace developed the foundations of potential theory and made important contributions to special functions, probability theory, astronomy, and celestial mechanics. In his *Exposition du systeme du Monde* (1796), he formulated a nebular hypothesis of cosmic origin and tried to explain the universe as a pure mechanism.



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P.S. de Laplace (left) and Oliver Heaviside (right).

In his *traite de Mechanique Celeste (celestial mechanics)*, which completed the work of Newton, he used mathematics and physics to subject the solar system and all heavenly bodies to the laws of motion and the principle of gravitation. Newton was unable to explain the irregularities of some heavenly bodies; in desperation, he concluded that God himself must intervene now and then to prevent some catastrophes, such as Jupiter eventually falling into the sun (and the moon in the earth) as predicted by Newton's calculations. Laplace proposed to show that these irregularities would correct themselves periodically, and that a little patience—in Jupiter's case, 929 years—would see everything returning automatically to order. He concluded that there was no reason why the solar and the stellar systems could not continue to operate by the laws of Newton and Laplace to the end of time.[3]

Laplace presented a copy of *Mechanique Celeste* to Napoleon, who, after reading the book, took Laplace to task for not including God in his scheme: "You have written this huge book on the system of the world without once mentioning the author of the universe." "Sire," Laplace retorted, "I had no need of that hypothesis." Napoleon was not amused, and when he reported this reply to Lagrange, the latter remarked, "Ah, but that is a fine hypothesis. It explains so many things."[4]

Napoleon, following his policy of honoring and promoting scientists, made Laplace the minister of the interior. To Napoleon's dismay, he found the great mathematician-astronomer bringing "the spirit of infinitesimals" into administration, and so had Laplace transferred hastily to the senate.

Oliver Heaviside (1850-1925)

Although Laplace published his transform method to solve differential equations in 1779, it did not catch on until a century later. It was rediscovered independently in a rather awkward form by an eccentric British engineer, Oliver Heaviside (1850-1925), one of the tragic figures in the history of electrical engineering. Despite his prolific contributions to electrical engineering, he was severely criticized

Page 241

Annexe D

Exercices sur la convolution

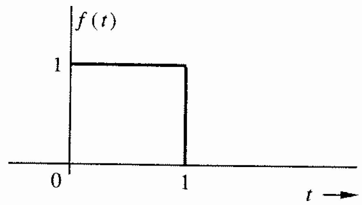


Fig. P2.4-8

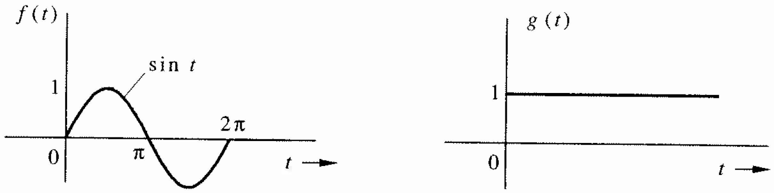


Fig. P2.4-11

2.4-3 Using direct integration, find $\sin t\, u(t) * u(t)$ and $\cos t\, u(t) * u(t)$.

2.4-4 The unit impulse response of an LTIC system is $h(t) = e^{-t}u(t)$. Find this system's (zero-state) response $y(t)$ if the input $f(t)$ is:

(a) $u(t)$ **(b)** $e^{-t}u(t)$ **(c)** $e^{-2t}u(t)$ **(d)** $\sin 3t\, u(t)$.

Use the convolution table to find your answers.

2.4-5 Repeat Prob. 2.4-4 if

$$h(t) = \left[2e^{-3t} - e^{-2t}\right] u(t)$$

and if the input $f(t)$ is: **(a)** $u(t)$ **(b)** $e^{-t}u(t)$ **(c)** $e^{-2t}u(t)$.

2.4-6 Repeat Prob. 2.4-4 if

$$h(t) = (1 - 2t)e^{-2t}u(t)$$

and if the input $f(t) = u(t)$.

2.4-7 Repeat Prob. 2.4-4 if $h(t) = 4e^{-2t}\cos 3t\, u(t)$ and if the input $f(t)$ is: **(a)** $u(t)$ **(b)** $e^{-t}u(t)$.

2.4-8 Repeat Prob. 2.4-4 if

$$h(t) = e^{-t}u(t)$$

and if the input $f(t)$ is: **(a)** $e^{-2t}u(t)$ **(b)** $e^{-2(t-3)}u(t)$ **(c)** $e^{-2t}u(t-3)$ **(d)** the gate pulse shown in Fig. P2.4-8. For **(d)**, sketch $y(t)$.

Hint: $e^{-2(t-3)} = e^{6}e^{-2t}$, and $e^{-2t}u(t-3) = e^{-6}e^{-2(t-3)}u(t-3)$. Also, the input in **(d)** can be expressed as $u(t) - u(t-1)$. For parts **(c)** and **(d)**, use the shift property (2.63) of convolution. (Alternatively, you may want to invoke the system's time-invariance and superposition properties).

2.4-9 A first-order allpass filter impulse response is given by

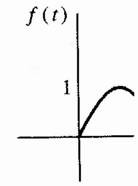
$$h(t) = -\delta(t) + 2e^{-t}u(t)$$

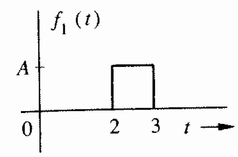
(a) Find the zero-state response of this filter for the input $e^{t}u(-t)$.

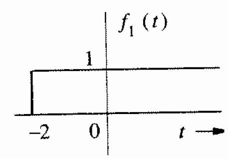
(b) Sketch the input and the corresponding zero-state response.

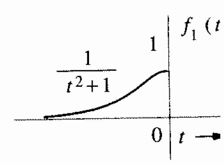
2.4-10 Sketch the functions $f(t) = \frac{1}{t^2+1}$ and $u(t)$. Now find $f(t) * u(t)$ and sketch the result.

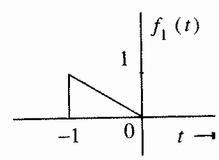
2.4-11 Figure P2.4-11 shows $f(t)$ and $g(t)$. Find and sketch $c(t) = f(t) * g(t)$.











2.4-12 Find and ske

2.4-13 Find and ske

2.4-14 A line charge

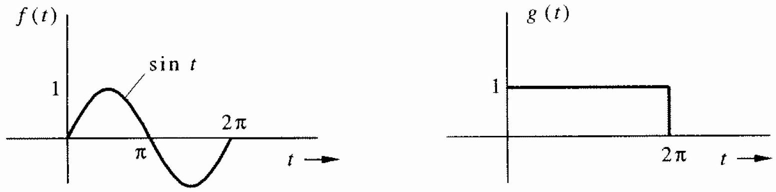


Fig. P2.4-12

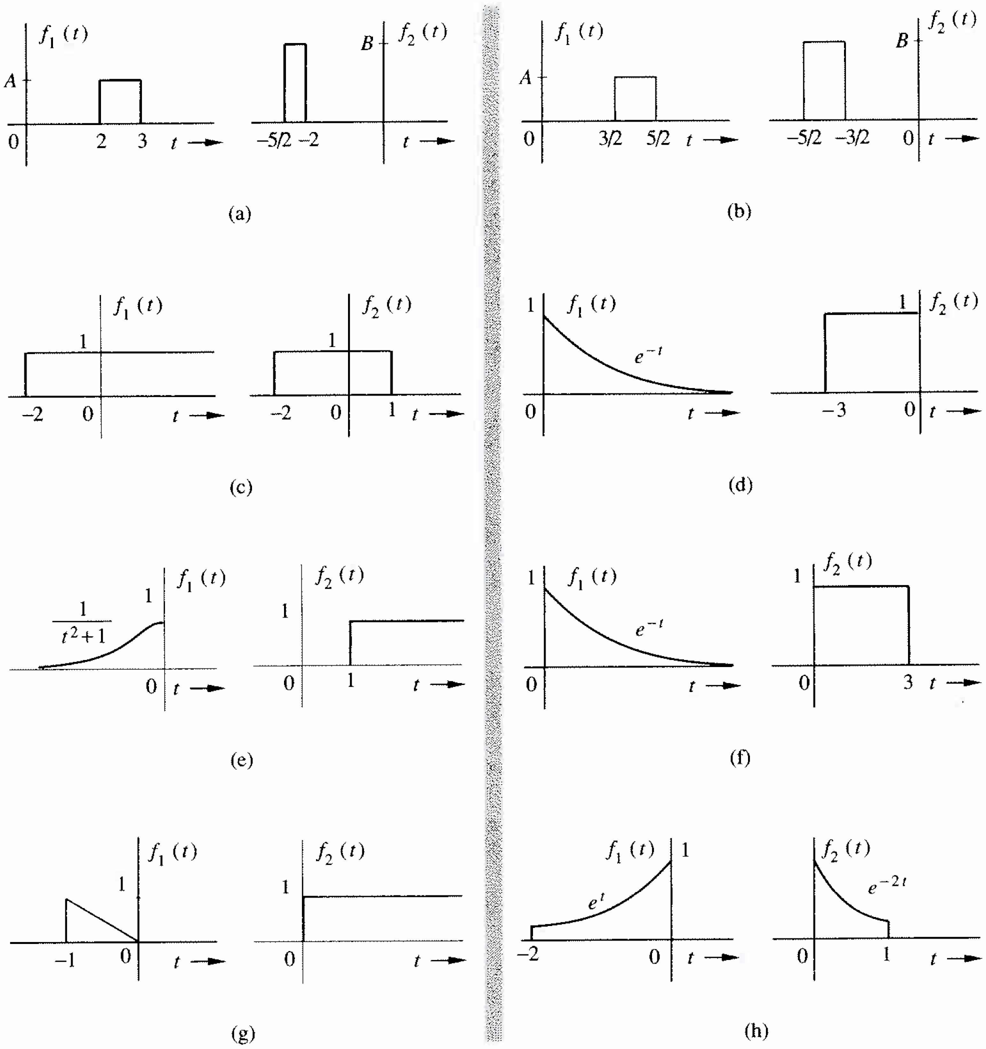


Fig. P2.4-13

2.4-12 Find and sketch $c(t) = f(t) * g(t)$ for the functions shown in Fig. P2.4-12.

2.4-13 Find and sketch $c(t) = f_1(t) * f_2(t)$ for the pairs of functions shown in Fig. P2.4-13.

2.4-14 A line charge is located along the x axis with a charge density $f(x)$. Show that the

Page 245

Annexe E

Exercices sur les séries de Fourier

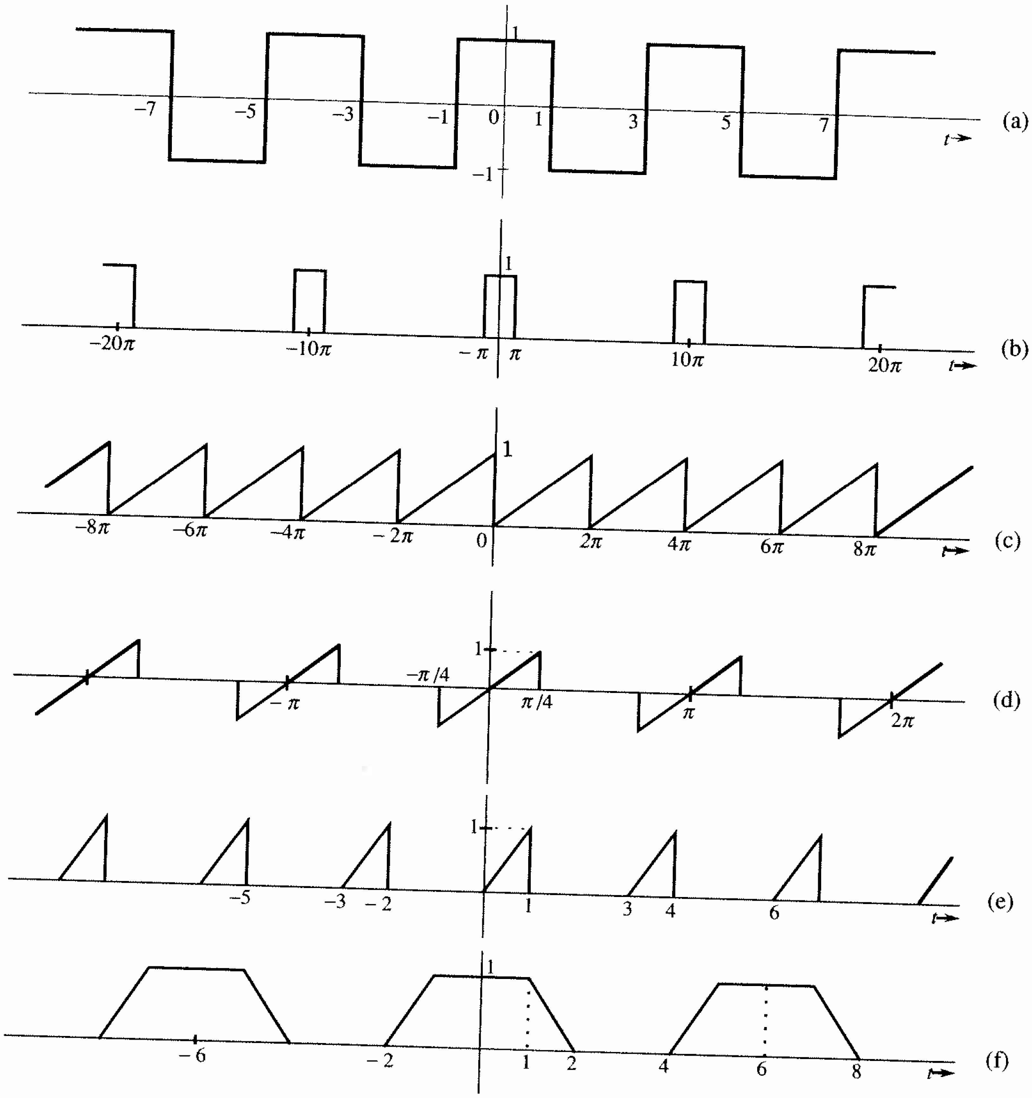


Fig. P6.1-1.

In this case, show that all the even-numbered harmonics vanish, and that the odd-numbered harmonic coefficients are given by

$$a_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\cos n\omega_0 t\,dt \quad \text{and} \quad b_n = \frac{4}{T_0}\int_0^{T_0/2} f(t)\sin n\omega_0 t\,dt$$

Using these results, find the Fourier series for the periodic signals in Fig. P6.1-2.

6.1-3 State with reasons whether the following signals are periodic or nonperiodic. For periodic signals, find the period and state which of the harmonics are present in the series.

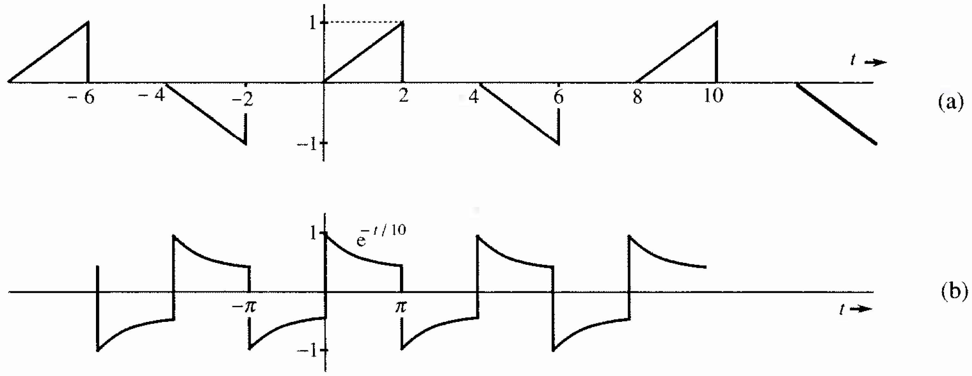


Fig. P6.1-2.

(a) $3\sin t + 2\sin 3t$
(b) $2 + 5\sin 4t + 4\cos 7t$
(c) $2\sin 3t + 7\cos \pi t$
(d) $7\cos \pi t + 5\sin 2\pi t$
(e) $3\cos \sqrt{2}t + 5\cos 2t$
(f) $\sin \frac{5t}{2} + 3\cos \frac{6t}{5} + 3\sin\left(\frac{t}{7} + 30^\circ\right)$
(g) $\sin 3t + \cos \frac{15}{4}t$
(h) $(3\sin 2t + \sin 5t)^2$
(i) $(5\sin 2t)^3$

Hint for parts **h** and **i**: If a signal is periodic then any power of that signal is also periodic. To find the period use identities in Sec. B.10-6.

6.2-1 Derive Eq. (6.32) directly from Eq. (6.30), as explained in the footnote on P.000.

6.2-2 For each of the periodic signals in Fig. P6.1-1, find exponential Fourier series and sketch the corresponding spectra.

6.2-3 A periodic signal $f(t)$ is expressed by the following Fourier series:

$$f(t) = 3\cos t + \sin\left(5t - \frac{\pi}{6}\right) - 2\cos\left(8t - \frac{\pi}{3}\right)$$

(a) Sketch the amplitude and phase spectra for the trigonometric series.
(b) By inspection of spectra in part **a**, sketch the exponential Fourier series spectra.
(c) By inspection of spectra in part **b**, write the exponential Fourier series for $f(t)$.

6.2-4 The trigonometric Fourier series of a certain periodic signal is given by

$$f(t) = 3 + \sqrt{3}\cos 2t + \sin 2t + \sin 3t - \frac{1}{2}\cos\left(5t + \frac{\pi}{3}\right)$$

(a) Sketch the trigonometric Fourier spectra.
(b) By inspection of spectra in part **a**, sketch the exponential Fourier series spectra.
(c) By inspection of spectra in part **b**, write the exponential Fourier series for $f(t)$.

6.2-5 The exponential Fourier series of a certain function is given as

$$f(t) = (2 + j2)e^{-j3t} + j2e^{-jt} + 3 - j2e^{jt} + (2 - j2)e^{j3t}$$

(a) Sketch the exponential Fourier spectra.
(b) By inspection of the spectra in part **a**, sketch the trigonometric Fourier spectra for $f(t)$. Find the compact trigonometric Fourier series from these spectra.
(c) Find the signal bandwidth.

6.2-6 Figure P6.2
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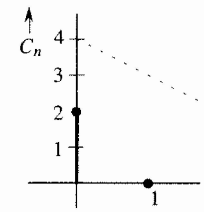
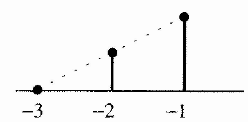
6.2-7 Figure P6.2





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P6.2-8c, wh

6.2-9 If a periodic

(a) Show th

6.2-6 Figure P6.2-6 shows the trigonometric Fourier spectra of a periodic signal $f(t)$.
(a) By inspection of Fig. P6.2-6, find the trigonometric Fourier series representing $f(t)$.
(b) By inspection of Fig. P6.2-6, sketch the exponential Fourier spectra of $f(t)$.
(c) By inspection of the exponential Fourier spectra obtained in part **b**, find the exponential Fourier series for $f(t)$.
(d) Show that the series found in parts **a** and **c** are equivalent.

6.2-7 Figure P6.2-7 shows the exponential Fourier spectra of a periodic signal $f(t)$.

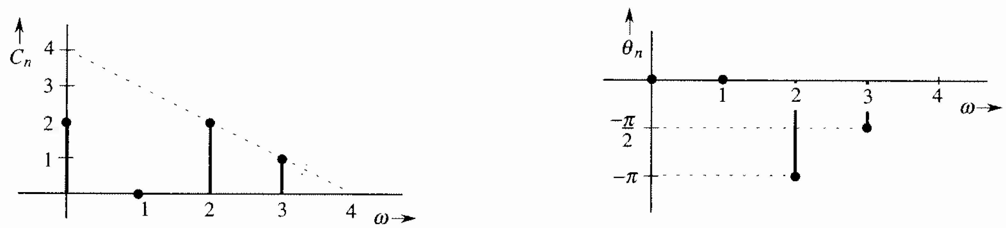

Fig. P6.2-6

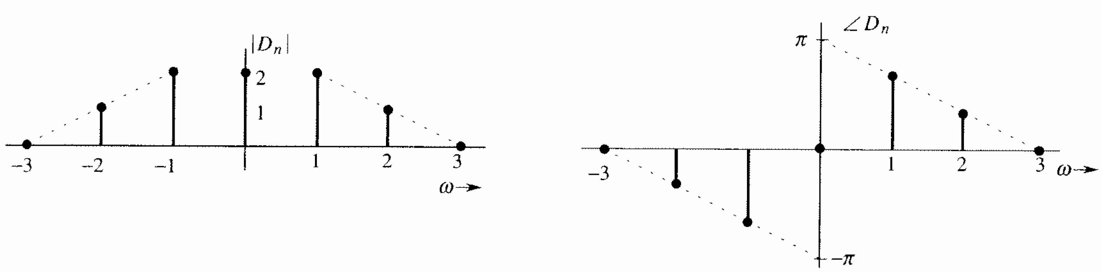

Fig. P6.2-7.

(a) By inspection of Fig. P6.2-7, find the exponential Fourier series representing $f(t)$.
(b) By inspection of Fig. P6.2-7, sketch the trigonometric Fourier spectra for $f(t)$.
(c) By inspection of the trigonometric Fourier spectra found in part **b**, find the trigonometric Fourier series for $f(t)$.
(d) Show that the series found in parts **a** and **c** are equivalent.

6.2-8 **(a)** Find the exponential Fourier series for the signal in Fig. P6.2-8a.
(b) Using the results in part **(a)**, find the Fourier series for the signal $\hat{f}(t)$ in Fig. P6.2-8b, which is a time-shifted version of the signal $f(t)$.
(c) Using the results in part **(a)**, find the Fourier series for the signal $\tilde{f}(t)$ in Fig. P6.2-8c, which is a time-scaled version of the signal $f(t)$.

6.2-9 If a periodic signal $f(t)$ is expressed as an exponential Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

(a) Show that the exponential Fourier series for $\hat{f}(t) = f(t - T)$ is given by

$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} \hat{D}_n e^{jn\omega_0 t}$$

Page 250

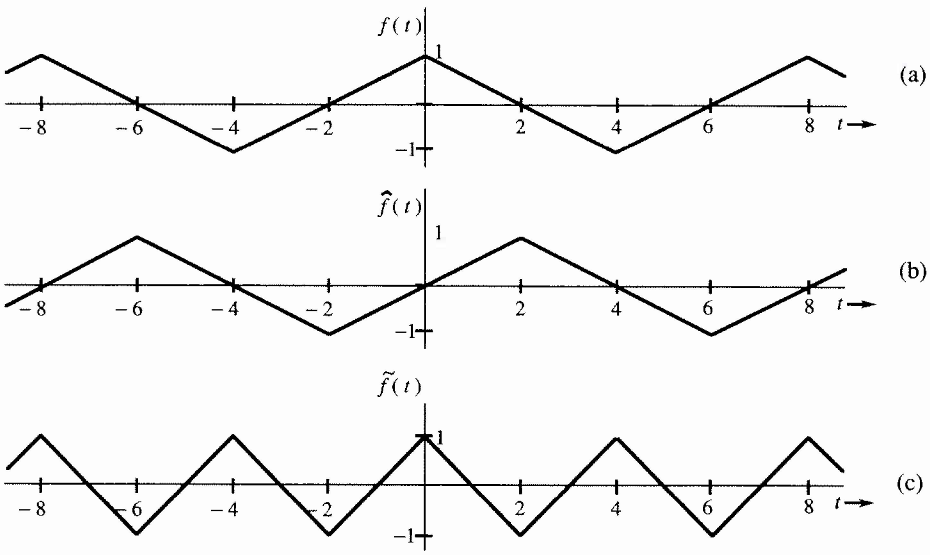

Fig. P6.2-8.

in which

$$|\hat{D}_n| = |D_n| \quad \text{and} \quad \angle\hat{D}_n = \angle D_n - n\omega_0 T$$

(b) Verify this result by finding the exponential Fourier series for the signals in Figs. P6.2-8a and P6.2-8b.

6.3-1 Let $x_1(t)$ and $x_2(t)$ be two signals orthogonal over an interval $[t_1, t_2]$. Consider a signal $f(t)$ where

$$f(t) = c_1 x_1(t) + c_2 x_2(t) \qquad t_1 \le t \le t_2$$

Represent this signal by a two-dimensional vector $\mathbf{F}(c_1, c_2)$.
(a) Determine the vector representation of the following six signals in the two-dimensional vector space

(i) $f_1(t) = 2x_1(t) - x_2(t)$	**(iv)** $f_4(t) = x_1(t) + 2x_2(t)$
(ii) $f_2(t) = -x_1(t) + 2x_2(t)$	**(v)** $f_5(t) = 2x_1(t) + x_2(t)$
(iii) $f_3(t) = -x_2(t)$	**(vi)** $f_6(t) = 3x_1(t)$

(b) Point out pairs of mutually orthogonal vectors among these six vectors. Verify that the pairs of signal corresponding to these orthogonal vectors are also orthogonal.

6.3-2 A signal $f(t)$ is approximated in terms of a signal $x(t)$ over an interval $[t_1, t_2]$:

$$f(t) \simeq c x(t) \qquad t_1 \le t \le t_2$$

where c is chosen to minimize the mean-squared error.
(a) Show that $x(t)$ and the error $e(t) = f(t) - cx(t)$ are orthogonal over the interval $[t_1, t_2]$.
(b) Can you explain the result in terms of analogy with vectors?

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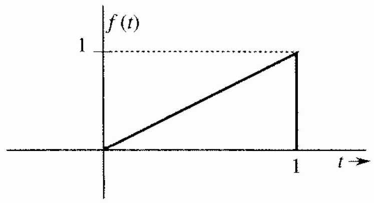

Fig. P6.3-3.

(c) Verify this result for the square signal $f(t)$ in Fig. 6.17 and its approximation in terms of signal sin t.

6.3-3 Represent the signal $f(t)$ shown in Fig. P6.3-3 over the interval [0, 1] by a trigonometric Fourier series of fundamental frequency $\omega_0 = 2\pi$. Compute the mean-squared error in the representation of $f(t)$ by only the first k terms of this series for $k = 1$, 2, 3, and 4.

6.3-4 Represent $f(t) = t$ over the interval [0, 1] by a trigonometric Fourier series which has
(a) $\omega_0 = 2\pi$ and only sine terms.
(b) $\omega_0 = \pi$ and only sine terms.
(c) $\omega_0 = \pi$ and only cosine terms.
You may use a dc term in these series if necessary.

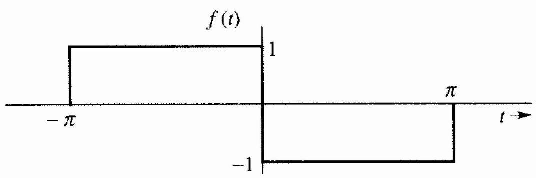

Fig. P6.3-5.

6.3-5 In Example 6.10, we represented the function in Fig. 6.21 by Legendre polynomials.
(a) Using the results in this example, represent the signal $g(t)$ in Fig. P6.3-5 by Legendre polynomials. Hint: Use appropriate time-scaling.
(b) Compute the mean-squared error for the one- and two-(nonzero)term approximations.

COMPUTER PROBLEMS

C6-1 Find the compact trigonometric Fourier series for the periodic signal $f(t)$ shown in Fig. 6.4a. Use a computer to plot the amplitude and phase spectra of $f(t)$ for 5, 10, 50 100 points.

C6-2 For the periodic signal in Fig. 6.3a, use a computer to observe how a sum of harmonics [the Fourier series in Eq. (6.19)] approaches $f(t)$ as the number of harmonics are increased.

Annexe F

Exercices sur la transformée de Fourier

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PROBLEMS

7.1-1 Show that if $f(t)$ is an even function of t, then

$$F(\omega) = 2\int_0^\infty f(t)\cos \omega t\, dt$$

and if $f(t)$ is an odd function of t, then

$$F(\omega) = -2j\int_0^\infty f(t)\sin \omega t\, dt$$

Hence, prove that if $f(t)$ is a real and even function of t, then $F(\omega)$ is a real and even function of ω. In addition if $f(t)$ is a real and odd function of t, then $F(\omega)$ is an imaginary and odd function of ω.

7.1-2 Show that for a real $f(t)$, Eq. (7.7) can be expressed as

$$f(t) = \frac{1}{\pi}\int_0^\infty |F(\omega)| \cos[\omega t + \angle F(\omega)]\, d\omega$$

This is the trigonometric form of the Fourier integral. Compare this with the compact trigonometric Fourier series.

7.1-3 A signal $f(t)$ can be expressed as the sum of even and odd components (see Sec. B.6-2):

$$f(t) = f_e(t) + f_o(t)$$

(a) If $f(t) \Longleftrightarrow F(\omega)$, show that for real $f(t)$,

$$f_e(t) \Longleftrightarrow \text{Re}[F(\omega)] \qquad \text{and} \qquad f_o(t) \Longleftrightarrow j\ \text{Im}[F(\omega)]$$

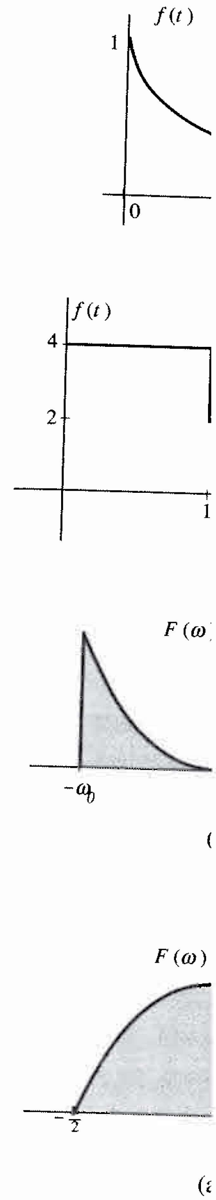

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Fig. P7.1-6

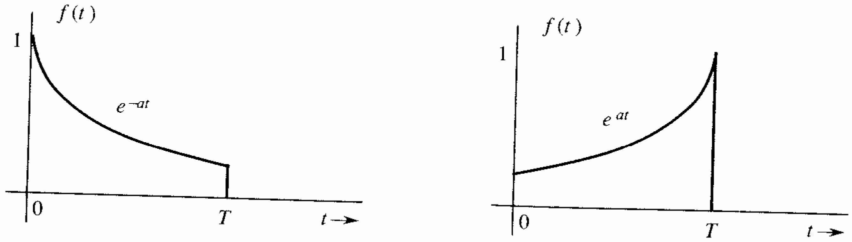

Fig. P7.1-4

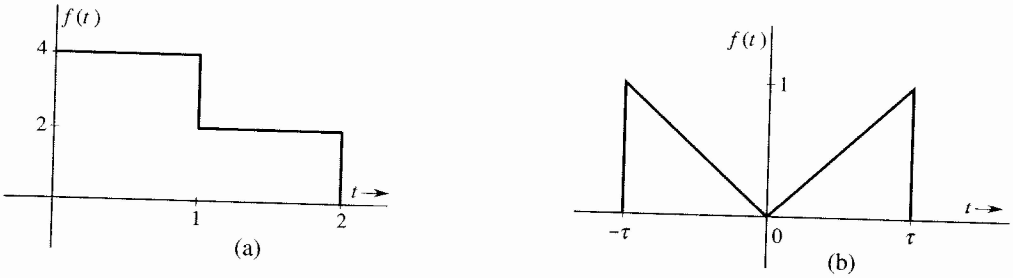

Fig. P7.1-5

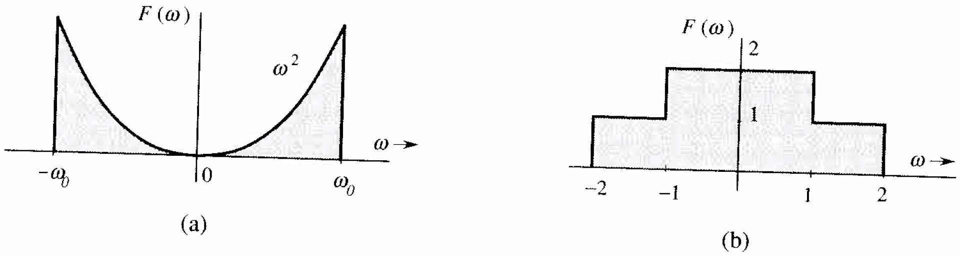

Fig. P7.1-6

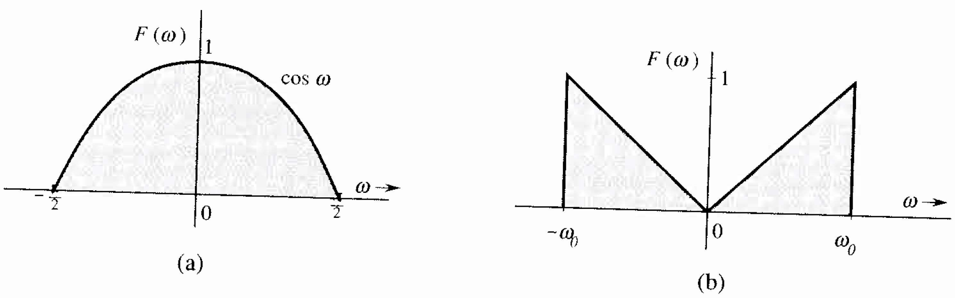

Fig. P7.1-7

(b) Verify these results by finding the Fourier transforms of the even and odd components of the following signals: **(i)** $u(t)$ **(ii)** $e^{-at}u(t)$.

7.1-4 From definition (7.8a), find the Fourier transforms of the signals $f(t)$ shown in Fig. P7.1-4.

7.1-5 From definition (7.8a), find the Fourier transforms of signals shown in Fig. P7.1-5.

7.1-6 From definition (7.8b), find the inverse Fourier transforms of the spectra shown in Fig. P7.1-6

7.1-7 From definition (7.8b), find the inverse Fourier transforms of spectra shown in Fig. P7.1-7.

7.3-1 Sketch the following functions:
(a) $\text{rect}\left(\frac{t}{2}\right)$ **(b)** $\Delta\left(\frac{3\omega}{100}\right)$ **(c)** $\text{rect}\left(\frac{t-10}{8}\right)$ **(d)** $\text{sinc}\left(\frac{\pi\omega}{5}\right)$ **(e)** $\text{sinc}\left(\frac{\omega-10\pi}{5}\right)$
(f) $\text{sinc}\left(\frac{t}{5}\right)\text{rect}\left(\frac{t}{10\pi}\right)$.
Hint: $f\left(\frac{x-a}{b}\right)$ is $f\left(\frac{x}{b}\right)$ right-shifted by a.

7.3-2 From definition (7.8a), show that the Fourier transform of $\text{rect}\,(t-5)$ is $\text{sinc}\left(\frac{\omega}{2}\right)e^{-j5\omega}$.

7.3-3 From definition (7.8b), show that the inverse Fourier transform of $\text{rect}\left(\frac{\omega-10}{2\pi}\right)$ is $\text{sinc}\,(\pi t)\,e^{j10t}$.
Hint: $\text{rect}\left(\frac{\omega-10}{2\pi}\right)$ is a gate pulse of width 2π that is centered at $\omega = 10$.

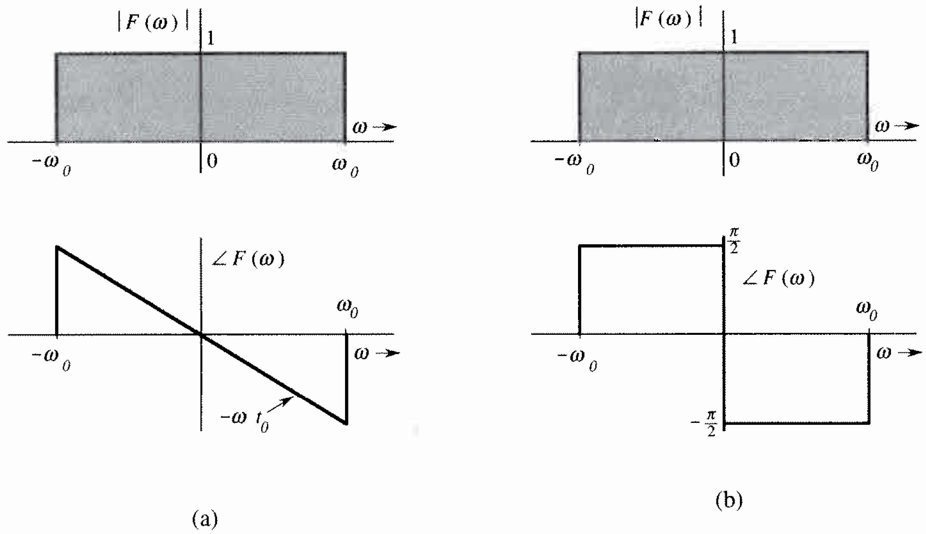

Fig. P7.3-4

7.3-4 Find the inverse Fourier transform of $F(\omega)$ for the spectra shown in Fig. P7.3-4.
Hint: $F(\omega) = |F(\omega)|e^{j\angle F(\omega)}$. For part a, $F(\omega) = 1e^{-j\omega t_0} \quad |\omega| \leq \omega_0$. For part b,

$$F(\omega) = \begin{cases} 1e^{-j\pi/2} = -j & 0 < \omega \leq \omega_0 \\ 1e^{j\pi/2} = j & 0 > \omega \geq -\omega_0 \end{cases}$$

7.4-1 Apply the duality property to the appropriate Pair in Table 7.1 to show that
(a) $\frac{1}{2}[\delta(t) + \frac{j}{\pi t}] \Longleftrightarrow u(\omega)$ **(b)** $\frac{1}{t} \Longleftrightarrow -j\pi\ \text{sgn}(\omega)$
(c) $\delta(t+T) + \delta(t-T) \Longleftrightarrow 2\cos T\omega$ **(d)** $\delta(t+T) - \delta(t-T) \Longleftrightarrow 2j\sin T\omega$.
Hint: $f(-t) \Longleftrightarrow F(-\omega)$ and $\delta(t) = \delta(-t)$.

7.4-2 The Fourier transform of the triangular pulse $f(t)$ in Fig. P7.4-2a is given to be

$$F(\omega) = \frac{1}{\omega^2}(e^{j\omega} - j\omega e^{j\omega} - 1)$$

Using this information, and the time-shifting and time-scaling properties, find the Fourier transforms of signals shown in Figs. P7.4-2b, c, d, e, and f.

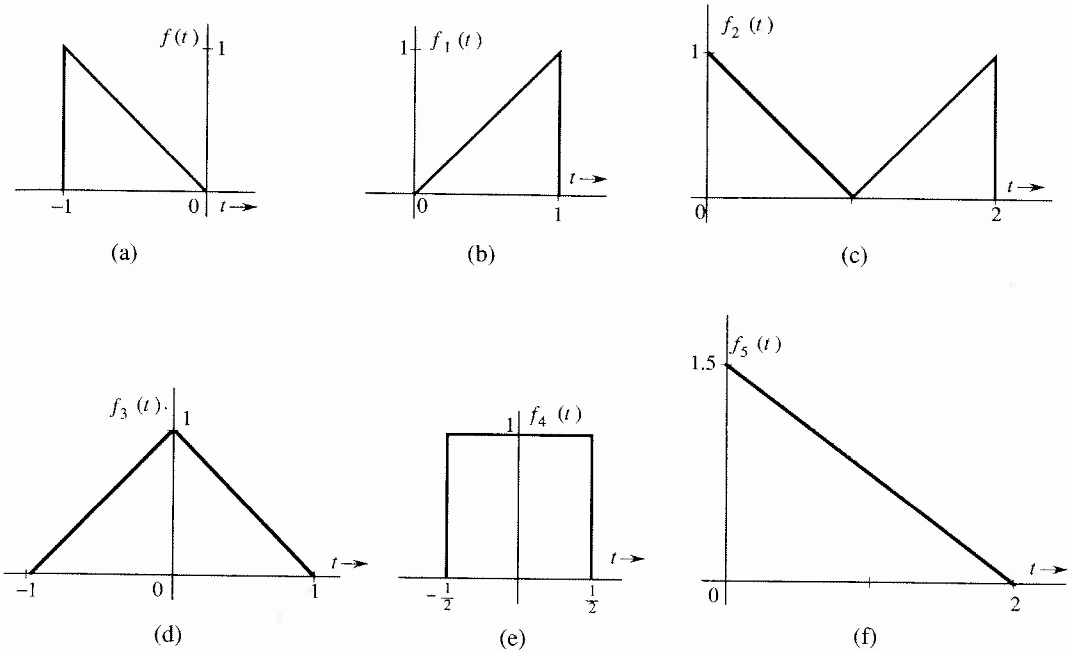

Fig. P7.4-2

Hint: See Sec. B.4 for explanation of various signal operations. Time inversion in $f(t)$ results in the pulse $f_1(t)$ in Fig. b; consequently $f_1(t) = f(-t)$. The pulse in Fig. c can be expressed as $f(t-T) + f_1(t-T)$ (the sum of $f(t)$ and $f_1(t)$ both delayed by T. The pulses in Figs. (d) and (e) both can be expressed as $f(t-T) + f_1(t+T)$ (sum of $f(t)$ delayed by T and $f_1(t)$ advanced by T) for some suitable choice of T. The pulse in Fig. f can be obtained by time-expanding $f(t)$ by a factor of 2 and then delaying the resulting pulse by two seconds (or by first delaying $f(t)$ by one second and then time-expanding by a factor of 2).

7.4-3 Using only the time-shifting property and Table 7.1, find the Fourier transforms of signals shown in Fig. P7.4-3.
Hint: The signal in Fig. a is a sum of two shifted gate pulses. The signal in Fig. b is $\sin t\,[u(t) - u(t-\pi)] = \sin t\,u(t) - \sin t\,u(t-\pi) = \sin t\,u(t) + \sin(t-\pi)\,u(t-\pi)$. The reader should verify that addition of the two sinusoids above indeed results in the pulse in Fig. b. In the same way we can express the signal in figs. c as $\cos tu(t) + \sin(t - \frac{\pi}{2})u(t - \frac{\pi}{2})$ (verify this by sketching these signals). The signal in Fig. d is $e^{-at}[u(t) - u(t-T)] = e^{-at}u(t) - e^{-aT}e^{-a(t-T)}u(t-T)$.

7.4-4 Using the time-shifting property, show that if $f(t) \Longleftrightarrow F(\omega)$, then

$$f(t+T) + f(t-T) \Longleftrightarrow 2F(\omega)\cos T\omega$$

This is the dual of Eq. (7.41). Using this result and Pairs 17 and 19 in Table 7.1, find the Fourier transforms of signals shown in Fig. P7.4-4.

7.4-5 Prove the following results, which are duals of each other:

$$f(t)\sin\omega_0 t \Longleftrightarrow \frac{1}{2j}[F(\omega-\omega_0) - F(\omega+\omega_0)]$$

$$\frac{1}{2j}[f(t+T) - f(t-T)] \Longleftrightarrow F(\omega)\sin T\omega$$

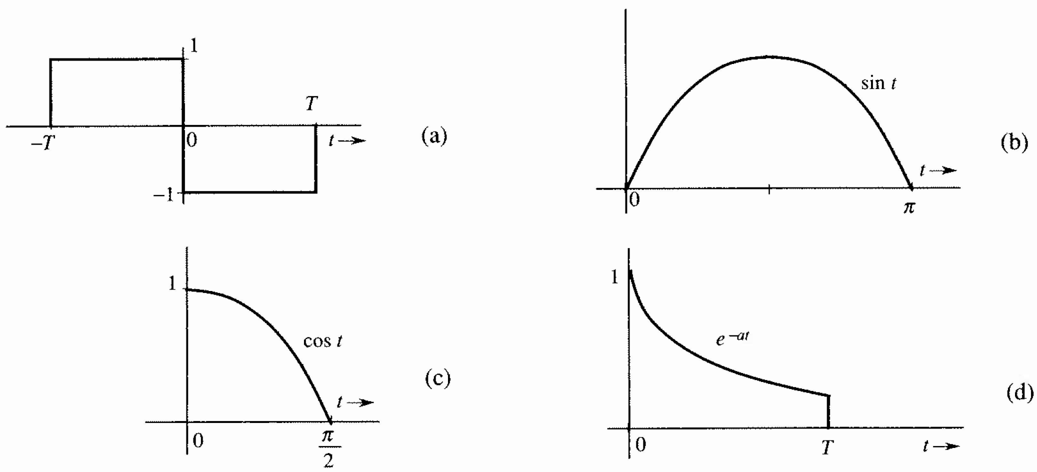

Fig. P7.4-3

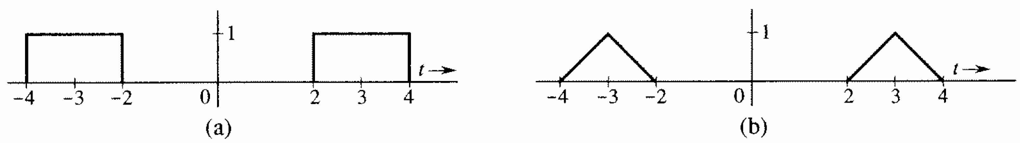

Fig. P7.4-4

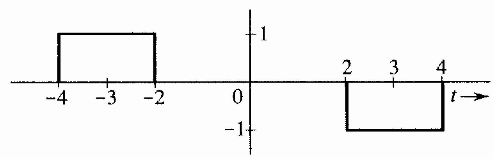

Fig. P7.4-5

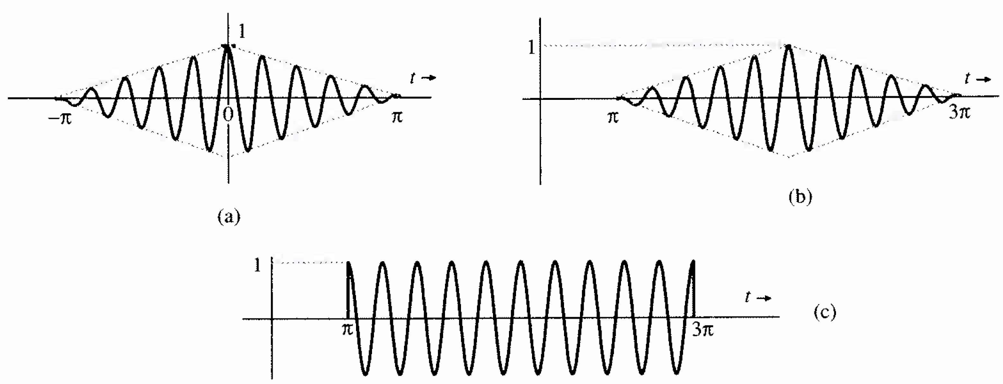

Fig. P7.4-6

Using the latter result and Table 7.1, find the Fourier transform of the signal in Fig. P7.4-5.

7.4-6 The signals in Fig. P7.4-6 are modulated signals with carrier cos $10t$. Find the Fourier transforms of these signals using appropriate properties of the Fourier transform and

Problems

Table 7.1. S
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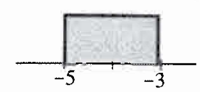

7.4-8 The process
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Hint: 2 cos^2
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7.4-9 Using the tir
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7.4-10 A signal $f(t$
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7.4-11 Find the Fou
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Hint: $1 - \cos$

7.4-12 **(a)** Prove th

(b) Using th
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7.5-1 Find the (zer

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7.5-2 A noncausal

Table 7.1. Sketch the amplitude and phase spectra for parts **a** and **b**.
Hint: These functions can be expressed in the form $g(t)\cos\omega_0 t$.

7.4-7 Using the frequency-shifting property and Table 7.1, find the inverse Fourier transform of spectra shown in Fig. P7.4-7.

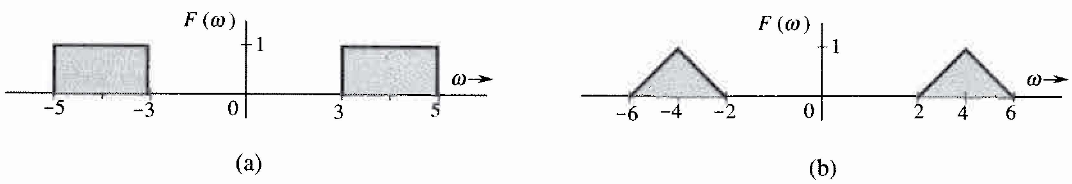

Fig. P7.4-7

7.4-8 The process of recovering a signal $f(t)$ from the modulated signal $f(t)\cos\omega_0 t$ is called **demodulation**. Show that the signal $f(t)\cos\omega_0 t$ can be demodulated by multiplying it with $2\cos\omega_0 t$ and passing the product through a lowpass filter of bandwidth W rps [the bandwidth of $f(t)$]. Assume $W < \omega_0$.
Hint: $2\cos^2\omega_0 t = 1 + \cos 2\omega_0 t$. Recognize that the spectrum of $f(t)\cos 2\omega_0 t$ is centered at $2\omega_0$ and will be suppressed by a lowpass filter of bandwidth W rps.

7.4-9 Using the time-convolution property, prove Pairs 2, 4, 13 and 14 in Table 2.1 (assume $\lambda < 0$ in Pair 2, λ_1 and $\lambda_2 < 0$ in Pair 4, $\lambda_1 < 0$ and $\lambda_2 > 0$ in Pair 13, and λ_1 and $\lambda_2 > 0$ in Pair 14).

7.4-10 A signal $f(t)$ is bandlimited to B Hz. Show that the signal $f^n(t)$ is bandlimited to nB Hz. Hint: $f^2(t) \Longleftrightarrow [F(\omega) * F(\omega)]/2\pi$, and so on. Use the width property of convolution).

7.4-11 Find the Fourier transform of the signal in Fig. 7.4-3a by three different methods:
(a) By direct integration using the definition (7.8a).
(b) Using only Pair 17 Table 7.1 and the time-shifting property.
(c) Using the time-differentiation and the time-shifting properties, along with the fact that $\delta(t) \Longleftrightarrow 1$.
Hint: $1 - \cos 2x = 2\sin^2 x$.

7.4-12 **(a)** Prove the frequency differentiation property (dual of the time differentiation):

$$-jtf(t) \Longleftrightarrow \frac{d}{d\omega}F(\omega)$$

(b) Using this property and Pair 1 (Table 7.1), determine the Fourier transform of $te^{-at}u(t)$.

7.5-1 Find the (zero-state) response of an LTIC system described by the equation

$$(D^2 + 3D + 2)y(t) = (D + 3)f(t)$$

if the input $f(t)$ is **(a)**$e^{-3t}u(t)$ **(b)** $e^{-4t}u(t)$ **(c)** $e^{-2t}u(t)$ **(d)** $e^{t}u(-t)$ **(e)** $u(t)$.

7.5-2 A noncausal LTIC system is specified by the transfer function

$$H(\omega) = \frac{-(j\omega + 2)}{(j\omega - 2)(j\omega + 3)}$$

Find the (zero-state) response of this system if the input $f(t)$ is **(a)** $e^{-t}u(t)$ **(b)**$e^{t}u(-t)$ **(c)** $e^{2t}u(-t)$ **(d)**$e^{-3t}u(t)$.

7.5-3 For an LTIC system with transfer function

$$H(s) = \frac{1}{s+1}$$

find the (zero-state) response if the input $f(t)$ is **(a)** $e^{-2t}u(t)$ **(b)** $e^{-t}u(t)$ **(c)** $e^{t}u(-t)$ **(d)** $u(t)$.

7.7-1 Consider a filter with the transfer function

$$H(\omega) = e^{-(k\omega^2 + j\omega t_0)}$$

Show that this filter is physically unrealizable by using the time-domain criterion [noncausal $h(t)$] and frequency-domain (Paley-Wiener) criterion. Can this filter be made approximately realizable by choosing a sufficiently large t_0? Use your own (reasonable) criterion of approximate realizability to determine t_0.
Hint: Use Pair 22 in Table 7.1.

7.7-2 Show that a filter with transfer function

$$H(\omega) = \frac{2(10^5)}{\omega^2 + 10^{10}} e^{-j\omega t_0}$$

is unrealizable. Can this filter be made approximately realizable by choosing a sufficiently large t_0? Use your own (reasonable) criterion of approximate realizability to determine t_0.
Hint: Show that the impulse response is noncausal.

7.9-1 Show that the energy of a Gaussian pulse

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}}$$

is $\frac{1}{2\sigma\sqrt{\pi}}$. Verify this result by deriving the energy E_f from $F(\omega)$ using the Parseval's theorem.
Hint: See Pair 22 in Table 7.1. Use the fact that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

7.9-2 Show that

$$\int_{-\infty}^{\infty} \text{sinc}^2 (kx)\, dx = \frac{\pi}{k}$$

Hint: Recognize that the integral is the energy of $f(t) = \text{sinc}\,(kt)$. Find this energy by using the Parseval's theorem.

7.9-3 Generalize Parseval's theorem to show that for real, Fourier transformable signals $f_1(t)$ and $f_2(t)$

$$\int_{-\infty}^{\infty} f_1(t) f_2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(-\omega)F_2(\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)F_2(-\omega)\, d\omega$$

7.9-4 For the signal

$$f(t) = \frac{2a}{t^2 + a^2}$$

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7.9-5 **(a)** If $f_1(t)$,

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(b) Show th

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Exercices sur la transformée de Laplace

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PROBLEMS

4.1-1 By direct integration [Eq. (4.1)] Find the Laplace transforms and the region of convergence of the following functions:

(a) $u(t) - u(t-1)$

(b) $te^{-t}u(t)$

(c) $t\cos\omega_0 t\, u(t)$

(d) $(e^{2t} - 2e^{-t})u(t)$

(e) $\cos\omega_1 t \cos\omega_2 t\, u(t)$

(f) $\cosh(at)\, u(t)$

(g) $\sinh(at)\, u(t)$

(h) $e^{-2t}\cos(5t+\theta)\, u(t)$

4.1-2 By direct integration find the Laplace transforms of the signals shown in Fig. P4.1-2.

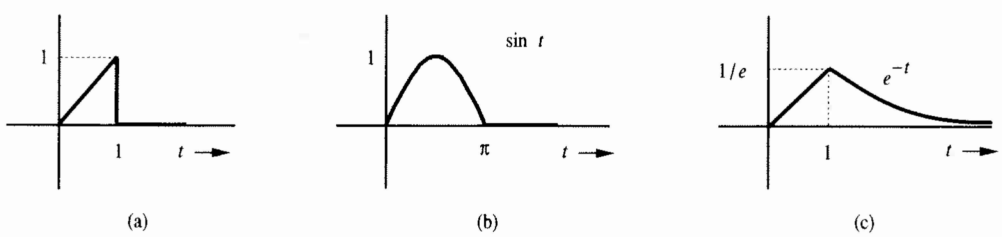

Fig. P4.1-2

4.1-3 Find the inverse (unilateral) Laplace transforms of the following functions:

(a) $\dfrac{2s+5}{s^2+5s+6}$

(b) $\dfrac{3s+5}{s^2+4s+13}$

(c) $\dfrac{(s+1)^2}{s^2-s-6}$

(d) $\dfrac{5}{s^2(s+2)}$

(e) $\dfrac{2s+1}{(s+1)(s^2+2s+2)}$

(f) $\dfrac{s+2}{s(s+1)^2}$

(g) $\dfrac{1}{(s+1)(s+2)^4}$

(h) $\dfrac{s+1}{s(s+2)^2(s^2+4s+5)}$

(i) $\dfrac{s^3}{(s+1)^2(s^2+2s+5)}$

4.2-1 Find the Laplace transforms of the following functions using only Table 4.1 and the time-shifting property (if needed) of the unilateral Laplace transform:

(a) $u(t) - u(t-1)$

(b) $e^{-(t-\tau)}u(t-\tau)$

(c) $e^{-(t-\tau)}u(t)$

(d) $e^{-t}u(t-\tau)$

(e) $te^{-t}u(t-\tau)$

(f) $\sin[\omega_0(t-\tau)]\,u(t-\tau)$

(g) $\sin[\omega_0(t-\tau)]\,u(t)$

(h) $\sin\omega_0 t\,u(t-\tau)$

Hint: $e^{-t} = e^{-\tau}e^{-(t-\tau)}$, and $e^{-(t-\tau)} = e^{\tau}e^{-t}$. For part **(g)** use trigonometric identity in B.10-6 to expand $\sin[\omega_0(t-\tau)] = \sin(\omega_0 t - \omega_0\tau)$, and recognize that $\sin\omega_0\tau$ and $\cos\omega_0\tau$ are constants. For part **(h)**, recognize that $\sin\omega_0 t = \sin[\omega_0(t-\tau)+\omega_0\tau]$. Expand this, using the identity for $\sin(x+y)$.

4.2-2 Using only Table 4.1 and the time-shifting property, determine the Laplace transform of the signals shown in Fig. P4.1-2.
Hint: See Sec. B.5 for discussion of expressing such signals analytically. Recognize that $t = (t-1)+1$ and $e^{-t} = e^{-\tau}e^{-(t-\tau)}$. See also Example 4.4. For **(b)**, show that the signal is $\sin t\,u(t)$ plus the same sine delayed by π.

4.2-3 Find the inverse Laplace transforms of the following functions:

(a) $\dfrac{(2s+5)e^{-2s}}{s^2+5s+6}$

(b) $\dfrac{se^{-3s}+2}{s^2+2s+2}$

(c) $\dfrac{e^{-(s-1)}+3}{s^2-2s+5}$

(d) $\dfrac{e^{-s}+e^{-2s}+1}{s^2+3s+2}$

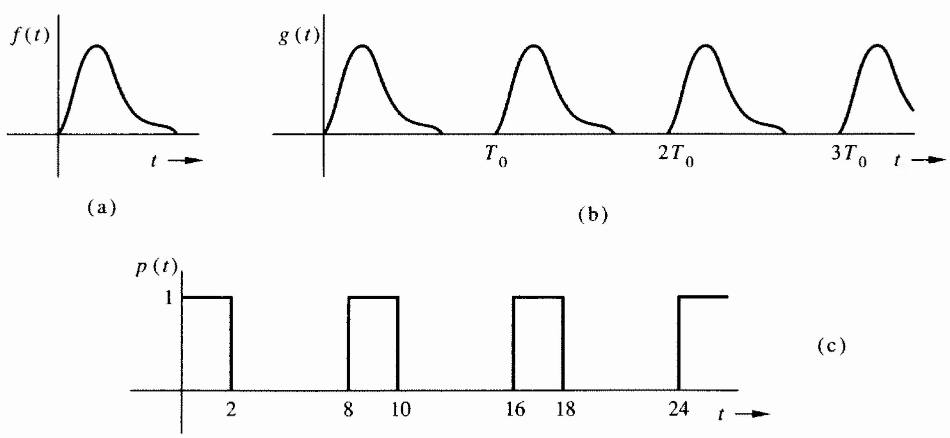

Fig. P4.2-4

4.2-4 The Laplace transform of a causal periodic signal can be found from the knowledge of the Laplace transform of its first cycle (period).

(a) If the Laplace transform of $f(t)$ in Fig. P4.2-4a is $F(s)$, then show that $G(s)$, the Laplace transform of $g(t)$ [Fig. P4.2-4b], is

$$G(s) = \frac{F(s)}{1-e^{-sT_0}} \qquad \operatorname{Re} s > 0$$

(b) Using this result, find the Laplace transform of the signal $p(t)$ shown in Fig. P4.2-4c.
Hint: recognize that $g(t) = f(t)+f(t-T_0)+f(t-2T_0)+\cdots$, and $1+x+x^2+x^3+\cdots = \frac{1}{1-x}$ for $|x| < 1$.

4.2-5 Starting only with the fact that $\delta(t) \Longleftrightarrow 1$, build Pairs 2 through 10b in Table 4.1, using various properties of the Laplace transform. Hint: $u(t)$ is integral of $\delta(t)$, and $tu(t)$ is integral of $u(t)$ (or second integral of $\delta(t)$), and so on.

4.2-6 **(a)** Find the Laplace transform of the pulses in Fig. 4.2 in the text by using only the time-differentiation property, time-shifting property and the fact that $\delta(t) \Longleftrightarrow 1$.
(b) In Example 4.7, the Laplace transform of $f(t)$ is found by finding the Laplace transform of d^2f/dt^2. Find the Laplace transform of $f(t)$ in that example by finding the Laplace transform of df/dt.
Hint for part **(b)**: df/dt can be expressed as a sum of step functions (delayed by various amounts) whose transforms can be determined readily.

4.3-1 Using Laplace transform, solve the following differential equations.

(a) $\left(D^2+3D+2\right)y(t) = Df(t)$ if $y(0^-) = \dot{y}(0^-) = 0$ and $f(t) = u(t)$

(b) $\left(D^2+4D+4\right)y(t) = (D+1)f(t)$ if $y(0^-) = 2$, $\dot{y}(0^-) = 1$ and $f(t) = e^{-t}u(t)$

(c) $\left(D^2+6D+25\right)y(t) = (D+2)f(t)$ if $y(0^-) = \dot{y}(0^-) = 1$ and $f(t) = 25u(t)$

4.3-2 Solve the differential equations in Prob. 4.3-1, using Laplace transform. In each case determine the zero-input and zero-state components of the solution.

4.3-3 Solve the following simultaneous differential equations using Laplace transform, assuming all initial conditions to be zero and the input $f(t) = u(t)$.

$$\textbf{(a)}\ (D+3)y_1(t) - 2y_2(t) = f(t)$$
$$-2y_1(t) + (2D+4)y_2(t) = 0$$
$$\textbf{(b)}\ (D+2)y_1(t) - (D+1)y_2(t) = 0$$
$$-(D+1)y_1(t) + (2D+1)y_2(t) = f(t)$$

Determine the transfer functions relating outputs $y_1(t)$ and $y_2(t)$ to the input $f(t)$.

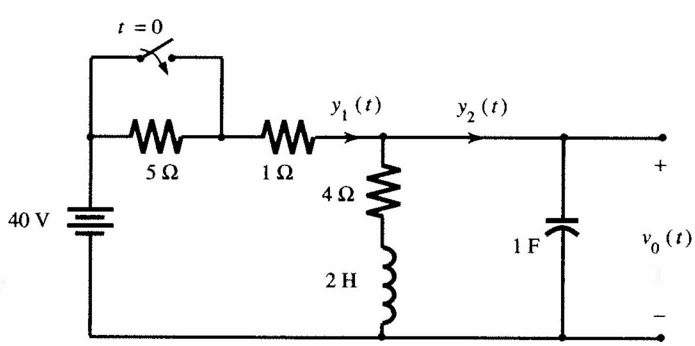

Fig. P4.3-4

4.3-4 For the circuit in Fig. P4.3-4, the switch is in open position for a long time before $t = 0$, when it is closed instantaneously.

(a) Write loop equations (in time domain) for $t \geq 0$.

(b) Solve for $y_1(t)$, and $y_2(t)$ by taking the Laplace transform of loop equations found in part **(a)**.

4.3-5 For each of the systems described by following differential equations, find the system transfer function.

(a) $\dfrac{d^2y}{dt^2} + 11\dfrac{dy}{dt} + 24y(t) = 5\dfrac{df}{dt} + 3f(t)$

(b) $\dfrac{d^3y}{dt^3} + 6\dfrac{d^2y}{dt^2} - 11\dfrac{dy}{dt} + 6y(t) = 3\dfrac{d^2f}{dt^2} + 7\dfrac{df}{dt} + 5f(t)$

(c) $\dfrac{d^4y}{dt^4} + 4\dfrac{dy}{dt} = 3\dfrac{df}{dt} + 2f(t)$

4.3-6 For each of the systems specified by the following transfer functions, find the differential equation relating the output $y(t)$ to the input $f(t)$.

(a) $H(s) = \dfrac{s+5}{s^2+3s+8}$ **(b)** $H(s) = \dfrac{s^2+3s+5}{s^3+8s^2+5s+7}$

(c) $H(s) = \dfrac{5s^2+7s+2}{s^2-2s+5}$

4.3-7 For a system with transfer function

$$H(s) = \frac{s+5}{s^2+5s+6}$$

(a) Find the (zero-state) response if the input $f(t)$ is:

(i) $e^{-3t}u(t)$ **(ii)** $e^{-4t}u(t)$ **(iii)** $e^{-4(t-5)}u(t-5)$ **(iv)** $e^{-4(t-5)}u(t)$ **(v)** $e^{-4t}u(t-5)$.

Hint: $e^{-4(t-5)} = e^{20}e^{-4t}$, and $e^{-4t}u(t-5) = e^{-20}e^{-4(t-5)}u(t-5)$.

(b) For this system write the differential equation relating the output $y(t)$ to the input $f(t)$.

4.3-8 Repeat Prob. 4.3-7 if

$$H(s) = \frac{2s+3}{s^2+2s+5}$$

and the input $f(t)$ is: **(a)** $10u(t)$ **(b)** $u(t-5)$.

4.3-9 Repeat Prob. 4.3-7 if

$$H(s) = \frac{s}{s^2+9}$$

and the input $f(t) = (1-e^{-t})u(t)$

4.3-10 For an LTIC system with zero initial conditions (system initially in zero state), if an input $f(t)$ produces an output $y(t)$, then show that:

(a) the input df/dt produces an output dy/dt, and

(b) the input $\int_0^t f(\tau)\,d\tau$ produces an output $\int_0^t y(\tau)\,d\tau$. Hence show that the unit step response of a system is an integral of the impulse response; that is, $\int_0^t h(\tau)\,d\tau$.

4.4-1 Find the zer
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4.4-3 Find the curr
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4.4-4 Find the loop
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4.4-6 Find the outp
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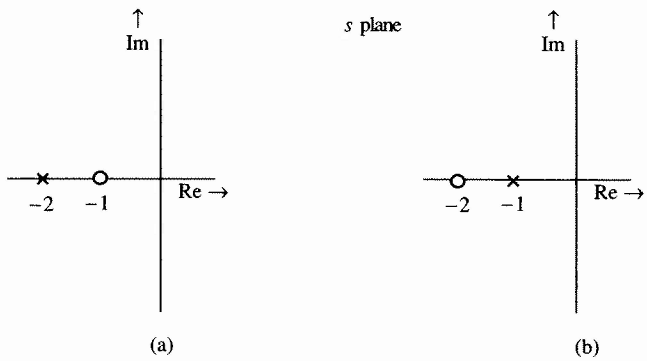

Fig. P4.7-6

4.7-6 Using the graphical method of Sec. 4.7-1, draw a rough sketch of the amplitude and phase response of LTIC systems whose pole-zero plots are shown in Fig. P4.7-6.

4.7-7 Design a second-order bandpass filter with center frequency $\omega = 10$. The gain should be zero at $\omega = 0$ and at $\omega = \infty$. Select poles at $-a \pm j10$. Leave your answer in terms of a. Explain the influence of a on the frequency response.

4.8-1 Find the region of convergence, if it exists, of the (bilateral) Laplace transform of the following signals:

(a) $e^{tu(t)}$

(b) $e^{-tu(t)}$

(c) $\dfrac{1}{1+t^2}$

(d) $\dfrac{1}{1+e^t}$

(e) e^{-kt^2}

4.8-2 Find the (bilateral) Laplace transform and the corresponding region of convergence for the following signals:

(a) $e^{-|t|}$

(b) $e^{-|t|}\cos t$

(c) $e^t u(t) + e^{2t} u(-t)$

(d) $\cos \omega_0 t\, u(t) + e^t\, u(-t)$

(e) $e^{-tu(t)}$

(f) $e^{tu(-t)}$

4.8-3 Find the inverse (bilateral) Laplace transforms of the following functions:

(a) $\dfrac{2s+5}{(s+2)(s+3)} \qquad -3 < \sigma < -2$

(b) $\dfrac{2s-5}{(s-2)(s-3)} \qquad 2 < \sigma < 3$

(c) $\dfrac{2s+3}{(s+1)(s+2)} \qquad \sigma > -1$

(d) $\dfrac{2s+3}{(s+1)(s+2)} \qquad \sigma < -2$

(e) $\dfrac{3s^2-2s-17}{(s+1)(s+3)(s-5)} \qquad -1 < \sigma < 5$

4.8-4 Find

$$\mathcal{L}^{-1}\left[\frac{2s^2-2s-6}{(s+1)(s-1)(s+2)}\right]$$