Chapitre 1

Électromagnétisme

Note : Certaines parties de ce chapitre sur l'électromagnétisme résultent de notes de cours antérieures rédigées en anglais ; elles ne sont pas traduites en français pour l'instant. Il y aura donc un mélange d'anglais et de français pour un bout.

Le champ électromagnétique (EM) représente l'état d'excitation qui existe dans l'espace dû à la présence de charges électriques. Le champ EM est généralement décrit comme étant composé de deux champs, soit les champs électrique et magnétique, le champ électrique étant causé par des charges stationnaires et le champ magnétique par des charges en mouvement (aussi appelées courants). Pour cette raison, les charges stationnaires et en mouvement sont souvent considérées comme étant les sources du champ EM. Le champ EM peut être envisagé comme un système auto-suffisant dans la mesure où les champs électrique et magnétique sont produits et modifiés par les charges électriques et qu'en même temps ces champs exercent des forces sur les charges électriques, les mettant en mouvement, ce qui conséquemment altère le champ EM. Ces interactions avec les charges, et plus généralement avec la matière, et les variations dans l'espace et le temps du champ EM sont décrites par les équations de Maxwell et la force de Lorentz.

Pour terminer cette introduction, il est à noter que de la perspective de la théorie de la relativité restreinte, les champs électrique et magnétique sont un tout. Un exemple frappant de cela est celui d'une charge électrique se déplaçant à vitesse constante dans un repère donné. Si un observateur se déplace avec la charge à la même vitesse que celle-ci, ce qui en d'autres mots signifie que cet observateur est dans un second repère avec une vitesse relative par rapport au premier, alors dans ce second repère il n'y a pas de champ magnétique, mais plutôt seulement un champ électrostatique. La théorie de la relativité indique comment le champ magnétique dans le premier repère se transforme en un champ électrique dans le second par l'entremise d'une transformation de Lorentz. Pour cette raison il est plus approprié de considérer le champ électromagnétique comme un tout, plutôt que de l'envisager comme étant composé de deux entités différentes. Une poursuite de la présente discussion sur la théorie de la relativité en lien avec l'électromagnétisme dépasse le cadre du présent ouvrage, pour cela le lecteur est référé à la littérature, v. p. ex. [JACKSON, CLASSICAL ELECTRODYNAMICS, 3RD ED] qui donne une excellente description.[1]

[1] Pour une introduction plus de base et intuitive, mais néanmoins complète à la relativité restreinte, une excellente référence est le livre de French [FRENCH, SPECIAL RELATIVITY].

1.1 Équations de Maxwell

Les équations de Maxwell sont un des grands aboutissements de la physique du XIXe siècle avec la mécanique statistique. Elles forment un ensemble de quatre équations aux dérivées partielles couplées qui relient les dérivées spatiales et temporelles des champs électriques et magnétiques aux charges et courants électriques.[2] Les équations de Maxwell constituent une synthèse de tout ce qui est connu sur les phénomènes électromagnétiques classiques, dont elles permettent une description complète. Il a fallu plus de 50 ans pour arriver à ces équations. Elles font suite à de nombreuses découvertes expérimentales. On ne décrira pas ici comment elles ont été déduites historiquement à partir de ces expériences étant donné que cela est généralement couvert dans des livres sur l'électromagnétisme et l'électrodynamique classique ; [changement à l'anglais] see for instance *Introduction to Electrodynamics* (3rd Edition) by D. J. Griffiths. Most notably, the experiments that have allowed at arriving at a complete description of electromagnetic phenomena are those of

- Coulomb and Cavendish on electrostatics ;
- Ohm on currents and electrical resistance ;
- Oersted who was the first to establish a connection between electricity and magnetism by observing that an electrical current deflected the needle of a compass ;
- Biot and Savart who further elaborated on Oersted's experiment ; and
- Faraday, who, fascinated by Oersted's discovery, set out to determine if magnetic fields, in turn, allowed the onset of electrical currents, thereby arriving at his celebrated law of induction.

One should not overlook that what enabled all these developments was the critically important invention of the electrochemical battery by Volta. This was a technology which allowed, for the first time, the generation of electrical currents almost at will and in a controlled way, thus making feasible experiments with coil electromagnets.[3]

Gauss and Ampère also made important theoretical contributions in mathematically stating three of the four basic laws of electromagnetism. The genius of Maxwell was to put all these equations together, rewriting in passing Faraday's law in more mathematical form, and, most importantly, finding out that a correction was needed in Ampère's law in order that the set of equations be coherent with the necessary conservation of electric charge (continuity equation). This correction led Maxwell to introduce the displacement current which is a time derivative of the electric field that accounts for the possibility of generating magnetic fields via time-varying electric fields. This is analogous and somewhat reciprocal to Faraday's law which says that a time-varying magnetic field induces an electric field. This reciprocity between the electric and magnetic fields allows for the physical possibility of self-sustaining *electromagnetic waves* that can travel through empty space. Maxwell's correction indeed allowed him to mathematically derive wave equations for the electric and magnetic fields, thereby predicting the existence of electromagnetic waves (1864). Astonishingly, he discovered that these waves had to propagate at a speed equal to the speed of light, which had already been measured at that time with relatively good accuracy, but

[2] Pour revenir à la notion de champ EM en lien avec la relativité restreinte, on notera qu'il est possible de mettre les équations de Maxwell sous une forme manifestement invariante sous les transformations de Lorentz à l'aide du formalisme mathématique de l'analyse tensorielle. Dans ce formalisme, le tenseur, appelé tenseur de Faraday, regroupe les champs électrique et magnétique et représente ainsi le champ EM comme un tout.

[3] Up to this invention, the only means to produce electricity was through electrostatic discharges, which were highly unpredictable and uncontrollable.

which could now be connected with fundamental electrical and magnetic constants. This further led to the unification of two seemingly independent phenomena : electromagnetism and optics. This was a tremendous contribution to science building on the then very recent unification of electricity and magnetism. Moreover, it provided a solid theory on which to found optics, allowing to naturally describe such phenomena as the interference and polarization of light based on first principles.

Maxwell's prediction of EM waves remained theoretical and not well accepted by his contemporaries until Hertz demonstrated their existence in a series of clever experiments (1887). Hertz then convincingly showed that he could generate, transmit and detect such waves with electrical apparatus, and that these waves had the same properties as light, such as being polarized and being refracted by materials. Soon after, Marconi exploited the existence of EM waves for telecommunications. He carried out the first wireless transmission of messages over the Atlantic Ocean. Finally, as mentioned before, modern theoretical developments in classical electromagnetism, including relativity theory, owe much to H. A. Lorentz. For more extensive descriptions on the fascinating history of electromagnetism, the reader should consult the two-volume treatise of Whittaker *A History of the Theories of Aether and Electricity*, along with the historical introductions in the books of Jackson *Classical Electrodynamics* and Born & Wolf *Principles of Optics*.

Given this brief historical introduction, Maxwell's equations in SI units can be stated as

$$\nabla \cdot \vec{D} = \rho_f, \qquad (\textit{Gauss's law}) \tag{1.1}$$

$$\nabla \cdot \vec{B} = 0, \qquad (\textit{Gauss's law for magnetism}) \tag{1.2}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = \vec{0}, \qquad (\textit{Faraday's law of induction (Maxwell-Faraday equation)}) \tag{1.3}$$

$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}_f. \qquad (\textit{Ampère's circuital law - with Maxwell's correction}) \tag{1.4}$$

The Maxwell equations just given relate to macroscopic media, and are for this reason called the ***macroscopic Maxwell equations***. Here, $\vec{E}$ (in volt per meter - V/m, or equivalently in newton per coulomb - N/C) is called the *electric field*, $\vec{B}$ (in tesla - T, or equivalently in volt-second per square meter - V·s/m^2) is the *magnetic induction* (also called the magnetic flux density, or often simply the magnetic field when there is no possible confusion), $\vec{D}$ (in Coulomb per square meter - C/m^2 or equivalently in newton per volt-meter - N/(V·m)) is the (Maxwell) *electric displacement*, and finally $\vec{H}$ (in ampere per meter - A/m) is the *magnetizing field* (also often called the magnetic field, or sometimes the auxiliary magnetic field). The term $\frac{\partial \vec{D}}{\partial t}$ appearing in the left hand side of Eq. (1.4) is Maxwell's displacement current (correction to Ampère's law). The quantity ρ_f (in Coulomb per cubic meter - C/m^3) is the density of free electric charges (usually simply called the free electric charge density), and $\vec{J}_f$ (in ampere per square meter - A/m^2) is the free electric current density. Here, free charges have the meaning of electric charges that are free to move in a material, in opposition to bound charges, which relate to charges bound to the medium and on which we shall come back briefly later. Having $\vec{J}_f$, the free electrical current $\mathrm{d}I_f$ flowing through an infinitesimal element of surface $\mathrm{d}\vec{a}$ is given by

$$\mathrm{d}I_f = \vec{J}_f \cdot \mathrm{d}\vec{a}. \tag{1.5}$$

For a finite surface S, the total free current is obtained through integration over the entire surface S :

$$I_f = \int_S \vec{J}_f \cdot \mathrm{d}\vec{a}. \tag{1.6}$$

An immediate consequence Maxwell's equations is the conservation of the electric charge in the neighborhood of any point. This follows from Ampère's and Gauss's laws. Taking the divergence of Eq. (1.4) leads to (since div rot ≡ 0)

$$\nabla\cdot\vec{J}_f = -\nabla\cdot\left(\frac{\partial\vec{D}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla\cdot\vec{D}\right).$$

Differentiating Eq. (1.1) with respect to time gives

$$\frac{\partial}{\partial t}\left(\nabla\cdot\vec{D}\right) = \frac{\partial\rho_f}{\partial t}.$$

Combining the last two equations gives

$$\frac{\partial\rho_f}{\partial t} + \nabla\cdot\vec{J}_f = 0. \tag{1.7}$$

This is the so-called *equation of continuity* for the charge (this is a form analogous to a relation encountered for the conservation of mass in hydrodynamics for a non-compressible fluid).

Équations de continuité Les équations de continuité étaient bien connues à l'époque de Maxwell, car on en rencontre en hydrodynamique, qui était une théorie alors assez bien développée, notamment suite aux travaux d'Euler. Une des équations de continuité les plus simples est celle pour exprimer la loi de conservation de la masse pour un fluide dans un volume donné. Soit $\rho(\vec{r},t)$ (kg/m^3) la masse volumique (ou densité[4]) d'un fluide qui s'écoule en suivant un champ de vitesses $\vec{v}(\vec{r})$ dans une région de l'espace. Si on considère un volume V dans cette région, alors le taux de variation de la masse dans ce volume doit être égal à la masse nette qui sort de ce volume. Cette masse sortante nette correspond au flux de masse sortant de la surface S qui borne le volume V. Pour évaluer ce flux, il faut connaître la masse par unité de surface qui se déplace dans le fluide. Or, en un point $\vec{r}$ donné du volume, la matière se déplace dans la direction de la vitesse en ce point, soit $\vec{v}(\vec{r})$. Ainsi, la masse qui se déplace par unité de surface dans la direction de $\vec{v}$ est donnée par $\rho\vec{v}$ (on se convainc aisément de cela à l'aide d'une analyse dimensionnelle qui montre que $\rho\vec{v}$ a des unités de masse par unité de surface (kg/m^2). Avec cela, le flux de masse est donné par l'intégrale de flux suivante :

$$\Phi = \oint_S (\rho\vec{v})\cdot\hat{n}\,\mathrm{d}a, \tag{1.8}$$

où $\hat{n}\,\mathrm{d}a \equiv \mathrm{d}\vec{a}$ est un élément d'aire de la surface bornant le volume V et $\hat{n}$ le vecteur normal unitaire à cet élément d'aire. Comme la masse dans le volume à un instant t est donnée par

$$M(t) = \int_V \rho\,\mathrm{d}^3\vec{r}, \tag{1.9}$$

on a alors

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\Phi, \tag{1.10}$$

ce qui est équivalent à

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathrm{d}^3\vec{r} = -\oint_S (\rho\vec{v})\cdot\hat{n}\,\mathrm{d}a. \tag{1.11}$$

Cette équation traduit en termes mathématiques la loi de conservation de la masse. On peut convertir l'intégrale de surface du membre de droit de cette équation en une intégrale de volume par le théorème de la divergence et en entrant la dérivée temporelle de la masse dans l'intégrale du membre de gauche, on obtient

$$\oint_V \frac{\partial\rho}{\partial t}\,\mathrm{d}^3\vec{r} = -\int_V \nabla\cdot(\rho\vec{v})\,\mathrm{d}^3\vec{r}. \tag{1.12}$$

[4] Qu'on pourrait appeler « densité massique ».

Le volume V étant arbitraire, on obtient

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{1.13}$$

qui est l'*équation de continuité correspondant à la loi de conservation de la masse.*

On peut interpréter $\rho\vec{v}$ comme une « densité de courant massique » $\vec{J}_m$ qu'on définira par

$$\vec{J}_m = \rho\vec{v}. \tag{1.14}$$

$\vec{J}_m$ formalise en fait mathématiquement la notion de courant au sens usuel, p. ex. le courant dans une rivière, car plus la vitesse de l'eau dans une rivière est grande, plus on dira que le courant est fort. Également, plus la densité du fluide est grande, plus grande sera la quantité de mouvement ou l'énergie cinétique et plus on dira alors que le courant est fort. Ceci traduit la notion intuitive de transport de masse.

Pour terminer, on note que des équations de continuité semblables à celle obtenue ici sont omniprésentes en physique (dynamique des fluides, transfert de chaleur, propagation de la lumière en milieux diffusants, etc...). Ici, on en rencontrera une autre en électromagnétisme en lien avec l'énergie du champ électromagnétiques ; une équation de continuité sera aussi obtenue pour le courant de probabilité en mécanique quantique.

C'est précisément pour arriver à l'équation de continuité de la charge qu'il croyait vraie, que Maxwell a ajouté sa correction à l'équation d'Ampère. That this equation expresses the fact that the charge is conserved in the neighborhood of any point can be seen by integrating it over any region V of space ; this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho_f \,\mathrm{d}^3\vec{r} \;+\; \int_V \nabla\cdot\vec{J}_f \,\mathrm{d}^3\vec{r} = 0.$$

Applying the divergence theorem to the last integral leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho_f \,\mathrm{d}^3\vec{r} \;+\; \oint_S \vec{J}_f\cdot\hat{n}\,\mathrm{d}a = 0, \tag{1.15}$$

where $\hat{n}\,\mathrm{d}a \equiv \mathrm{d}\vec{a}$ is the element of area of the surface bounding the region, and $\hat{n}$ is the unit outward normal vector to that surface. The integral

$$j_f = \oint_S \vec{J}_f\cdot\hat{n}\,\mathrm{d}a \tag{1.16}$$

accounts for the current flowing out of the region, whereas the total charge contained within that region is given by

$$Q_f = \int_V \rho_f \,\mathrm{d}^3\vec{r}. \tag{1.17}$$

Using these definitions of charge and current, Eq. (1.15) can be rearranged as

$$\frac{\mathrm{d}Q_f}{\mathrm{d}t} = -j_f. \tag{1.18}$$

Considering that $-j_f$ represents the current flowing into the region, this last equation implies that the total charge within the region can only increase due to the inward flow of electric current. This is a statement on the local conservation of charge. Some terminology : In situations where all field quantities appearing in Maxwell's equations are independent of time, and if additionally there are no currents (*i.e.* $\vec{J}_f = \vec{0}$), the EM field is then called a *static field*. If all field quantities are independent of time, but currents are present ($\vec{J}_f \neq \vec{0}$), then the EM field is said to be a *stationary field*.

Exercice de lecture 1.1. Show that if Maxwell's correction is not included in Ampère's law, then Maxwell's equations are not consistent with the equation of conservation of charge. What would the equations predict in this case ? Give an equation.

The fields $\vec{E}$ and $\vec{D}$, and $\vec{B}$ and $\vec{H}$ appearing in the Maxwell macroscopic equations are not independent ; they are connected by so-called *constitutive relations* (also called *material equations*) that describe how matter interacts with the EM field. In their simplest form, and the ones we shall use in the present work, these constitutive relations are

$$\vec{D} = \varepsilon \vec{E}, \tag{1.19}$$

where ε is the *permittivity* (also called *dielectric constant*) of the material , and

$$\vec{B} = \mu \vec{H}, \tag{1.20}$$

where μ is the *permeability* of the material. For *homogeneous materials* (materials everywhere the same), ε and μ are constant throughout the material, whereas for inhomogeneous materials they depend on the position $\vec{r}$ within the material. For *isotropic materials* (materials in which there is no dependence of the properties with direction), ε and μ are scalars (numbers) that do not depend on the directions of the applied fields,[5] whereas in anisotropic media ε and μ are tensors (*i.e.* matrices). Unless otherwise stated, the media considered here will be homogeneous and isotropic.

In cases where the fields are very strong, the right hand sides of the constitutive relations may have to include terms containing components of the fields in powers higher than the first. In such cases, materials respond non-linearly to the fields. An example where this occurs is in non-linear optics where very intense laser light fields are used.

There are equivalent ways of specifying constitutive relations by resorting to the *electrical polarization* $\vec{P}$ and *magnetization* $\vec{M}$ of a material. In this case, the first constitutive relation is replaced by

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}, \tag{1.21}$$

where ε_0 is the permittivity of empty space ($\varepsilon_0 \approx 8.854 \times 10^{-12}$ F/m - farad per meter), and $\vec{P}$ represents the electric dipole moment per unit volume in the medium.[6] The electrical constant ε_0 relates the units for electric charge to mechanical quantities such as length and force. For instance, it appears in Coulomb's law for the force between two charges Q_1 and Q_2 in vacuum separated by a distance r : $||\vec{F}_e|| = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}$. The second constitutive relation is given by

$$\vec{B} = \mu_0(\vec{H} + \vec{M}), \tag{1.22}$$

where μ_0 is the permeability of empty space ($\mu_0 = 4\pi \times 10^{-7}$ V·s/(A·m)), and $\vec{M}$ represents the magnetic dipole moment per unit volume in the medium. The magnetic constant μ_0 appears in Ampère's force

[5] This does not mean, however, that ε and μ do not depend on position in this case, the terms homogeneous and isotropic should not be confused !

[6] In so-called linear media, $\vec{P}$ is proportional to $\vec{E}$, which is written as $\vec{P} = \varepsilon_0 \chi_e \vec{E}$, where χ_e is called the electric susceptibility. Again, in a homogeneous medium, χ_e is constant ; it is a scalar in an isotropic material, and in general it is a tensor. In nonlinear media the relationship between $\vec{P}$ and $\vec{E}$ involves powers of the components of $\vec{E}$.

law for the force exerted on each other by two parallel wires each carrying a current I and separated by a distance r : $||\vec{F}_m|| = \frac{\mu_0}{2\pi}\frac{I^2}{r}$.

Any pair $(\vec{E},\vec{B})$ or $(\vec{E},\vec{H})$ (or equivalently $(\vec{D},\vec{B})$ or $(\vec{D},\vec{H})$) can be considered to represent the EM field. However, $\vec{E}$ and $\vec{B}$ are to be considered as the fundamental components of the EM field ; this being notably supported by considerations linked with relativity theory. The auxiliary fields, $\vec{D}$ and $\vec{H}$, are introduced as a matter of convenience in order to take into account, in an averaged manner, the contributions of atomic charges and currents to ρ and $\vec{J}$. In the present work, the description shall be in terms of $\vec{E}$ and $\vec{B}$, or $\vec{E}$ and $\vec{H}$ as convenient (it is always possible to convert from $\vec{H}$ to $\vec{B}$ and vice-versa using Eq. (1.20)).

Referring back to free versus bound charges, it was mentioned that free charges are those that are free to move in a medium, whereas bound charges, as their name implies, are bound to the medium. An example of bound charges would be the electrons orbiting around atoms or molecules in a solid (bound electrons), whereas free charges would be the free electrons that can move in a metal. Bound charges are generally taken into account by the material properties, or through the relationships for a given medium linking $\vec{D}$ to $\vec{E}$ and $\vec{B}$ to $\vec{H}$ as in Eqs. (1.19) and (1.20) (or Eqs. (1.21) and (1.22)).

The Maxwell equations as presented so far are for macroscopic media. It should be noted, however, that Maxwell's equations based on $\vec{E}$ and $\vec{B}$ as the fundamental fields with $\vec{D}$ and $\vec{H}$ defined by $\vec{D} = \varepsilon_0\vec{E}$ and $\vec{H} = \frac{1}{\mu_0}\vec{B}$ (that is a vacuum except for the presence of charges) go far beyond those for macroscopic media as they are also valid in the realm of the microscopic world described by quantum physics (more precisely quantum electronynamics as regards electromagnetism). The equations thus obtained are referred to as the ***microscopic Maxwell equations***, and are given by

$$\nabla\cdot\vec{E} = \frac{\rho_f}{\varepsilon_0}, \tag{1.23}$$

$$\nabla\cdot\vec{B} = 0, \tag{1.24}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \tag{1.25}$$

$$\nabla\times\vec{B} = \mu_0\vec{J}_f + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}. \tag{1.26}$$

The microscopic Maxwell equations can in principle also be used to describe macroscopic media (rather than use the macroscopic equations), but this would be exceedingly complicated, since taking into account all microscopic charges and currents present in a macroscopic material would represent a formidable task. For this reason, the macroscopic Maxwell equations are preferred in such situations. Les équations de Maxwell microscopiques sont celles qui sont généralement pertinentes en mécanique quantique lorsqu'on étudie des processus fondamentaux au niveau microscopique.

Media containing charges that are free to move, will display the onset of a current when subjected to an electric field. Such current is proportional to the applied electric field and is described by

$$\vec{J}_f = \sigma_c\vec{E}, \tag{1.27}$$

where σ_c is the *conductivity* of the medium.[7] This last equation is *Ohm's law in differential form* (v. exercice **??**). Substances for which σ_c is not negligibly small are called conductors. Metals are very good

[7] Here, the subscript c stands for *conductivity*. It was appended to the letter σ in order not to confuse the σ for the conductivity discussed here with the σ commonly used to denote a surface charge density.

conductors, as are ionic liquid solutions. Conductivity in metals decreases with increasing temperature, whereas in semiconductors, conductivity increases with temperature over a wide range of temperatures. Substances displaying conductivity are called lossy, because conduction leads to the loss of electromagnetic energy in the form of Joule heat. This will be considered further when the energy of the EM field is discussed later. Substances for which σ_c is negligibly small are called *dielectrics* or *insulators* ; ε and μ then completely determine their electrical and magnetic properties.

The magnetic permeability of most substances encountered in nature is equal to that of vacuum, *i.e.* $\mu \approx \mu_0$. Substances for which this is not the case are called *magnetic*. *Paramagnetic substances* (*e.g.* oxygen, platinum) are characterized by $\mu > \mu_0$, whereas *diamagnetic substances* (water, hydrogen, copper) are characterized by $\mu < \mu_0$.

Often, the material properties ε, μ, and σ_c are independent of the field strengths. There are cases, however, where a material cannot be described in such a simple manner. For instance, in ferromagnetic materials (materials that are highly magnetic such as iron, nickel and cobalt), the value of $\vec{B}$ at a given instant of time is not determined by the instantaneous value of $\vec{H}$ at that instant, but rather by its past history. Such materials are said to exhibit *hysteresis*. A similar behavior depending on history is found in certain dielectric substances for the electric displacement $\vec{D}$.

Finally, in the case of oscillating EM fields, ε, μ, and σ_c will generally depend on the frequency ω of the applied field, *i.e.* $\varepsilon = \varepsilon(\omega)$, $\mu = \mu(\omega)$, and $\sigma_c = \sigma_c(\omega)$. Media for which ε and μ vary with frequency are called *dispersive*. Dispersion accounts, for example in optics, for the separation of light into its constituent colors by prisms, and the appearance of rainbows. Dispersion leads to a relationship between the velocity of light in a medium and its electrical constants (the velocity of light is generally not the same in a medium as in vaccum). Such relationship is called a *dispersion relation*.

In the following, non lossy, homogeneous, isotropic and non-magnetic substances shall be considered and nonlinear effects will not be considered. In such cases, ε and μ are constant (with $\mu \approx \mu_0$) and $\sigma_c = 0$. Moreover, the constitutive relations are simply given by Eqs. (1.19) and (1.20) (linear relationships between $\vec{D}$ and $\vec{E}$, and $\vec{B}$ and $\vec{H}$). Such materials are said to be linear homogeneous isotropic media. To recover results in empty space (*i.e.* resulting from the microscopic Maxwell equations), ε can simply be replaced by ε_0 and μ by μ_0. For reference, the ***Maxwell equations for linear homogeneous isotropic media*** shall be listed here :

$$\nabla \cdot \vec{E} = \frac{\rho_f}{\varepsilon}, \tag{1.28}$$

$$\nabla \cdot \vec{B} = 0, \tag{1.29}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{1.30}$$

$$\nabla \times \vec{B} = \mu \vec{J}_f + \varepsilon\mu \frac{\partial \vec{E}}{\partial t}. \tag{1.31}$$

1.2 Force de Lorentz

From electrostatics, by the very definition of the electric field, the force exerted by an electric field $\vec{E}$ on a particle of charge q is given by

$$\vec{F}_\mathrm{e} = q\vec{E}. \tag{1.32}$$

Recall that this ultimately comes from Coulomb's law. From electromagnetism, a particle with charge q moving at velocity $\vec{v}$ in a magnetic field $\vec{B}$ will undergo a magnetic force given by

$$\vec{F}_{\mathrm{m}} = q\vec{v} \times \vec{B}. \tag{1.33}$$

This last law is based on empirical observations.[8] In the presence of both an electric and a magnetic field, the total force exerted on a charged particle will be given by the combination of the electric and magnetic forces, *i.e.*

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \tag{1.34}$$

This is the so-called *Lorentz force law* that makes the link between fields and charges in the language of mechanics.

1.3 Équations d'onde pour champs électromagnétiques

1.3.1 Dérivation

Les équations de Maxwell sont des équations aux dérivées partielles couplées pour les champs électrique et magnétique. De façon générale, lorsqu'on a des équations couplées, on cherche à les découpler pour obtenir de nouvelles équations qui n'impliquent que chacune des quantitées individuellement. Cela facilite généralement la recherche de solutions. Dans ce qui suit, on va utiliser les équations de Maxwell pour les milieux linéaires homogènes et isotropes (éqs. (1.28) à (1.31)) et on cherchera à partir de celles-ci à obtenir une équation pour $\vec{E}$ en éliminant $\vec{B}$ et inversement obtenir une autre équation pour $\vec{B}$ en éliminant $\vec{E}$. [La suite est en anglais] Taking the rotational of the third Maxwell equation gives

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial}{\partial t} \nabla \times \vec{B}, \tag{1.35}$$

where the rotational and the time derivative have been interchanged in the right hand side. By substituting the fourth Maxwell equation in the right hand side, Eq. (1.35) becomes

$$\nabla \times (\nabla \times \vec{E}) = -\varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} - \mu \frac{\partial \vec{J}_f}{\partial t}. \tag{1.36}$$

Using the identity (valid in cartesian coordinates for any vector $\vec{V}$)

$$\nabla \times (\nabla \times \vec{V}) = \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$$

in the previous equation and rearranging the terms leads to

$$\nabla^2 \vec{E} - \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} = \nabla(\nabla \cdot \vec{E}) + \mu \frac{\partial \vec{J}_f}{\partial t}. \tag{1.37}$$

[8] La force magnétique est parfois appelée force de Laplace, mais plus précisément la *force de Laplace* est la force macroscopique résultant de l'addition de forces magnétiques qui s'exercent sur un fil conducteur parcouru par un courant électrique dans un champ magnétique.

Finally, resorting to the first Maxwell equation gives

$$\nabla^2 \vec{E} - \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{\varepsilon}\nabla\rho_f + \mu\frac{\partial \vec{J}_f}{\partial t}. \tag{1.38}$$

This is a vector wave equation for $\vec{E}$, with each cartesian component of $\vec{E}$ satisfying a scalar wave equation. The source term of the vector wave equation is given by the right hand of the equation, that is $\frac{1}{\varepsilon}\nabla\rho_f + \mu\frac{\partial \vec{J}_f}{\partial t}$. By similar reasoning, one arrives at the following equation for $\vec{H}$

$$\nabla^2 \vec{H} - \varepsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2} = -\nabla \times \vec{J}_f, \tag{1.39}$$

which is also a wave equation. Here the source term is $-\nabla \times \vec{J}_f$.

Exercice de lecture 1.2. Faire le développement pour arriver à l'équation d'onde pour le champ magnétique $\vec{H}$ (éq. (1.39)).

Comparing the previous wave equations for $\vec{E}$ and $\vec{H}$ with the well-known standard scalar wave equation

$$\nabla^2 u - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = s(\vec{r}, t), \tag{1.40}$$

where $s(\vec{r}, t)$ is the source term and v represents the speed of the phase of the waves, it is concluded that the speed c of EM waves is given by

$$v = c = \frac{1}{\sqrt{\varepsilon\mu}}. \tag{1.41}$$

When the medium is empty space, that speed is

$$v = c_0 = \frac{1}{\sqrt{\varepsilon_0\mu_0}}, \tag{1.42}$$

whose numerical value corresponds to the speed of light in vacuum. The ratio of the speed of light in vacuum to the speed of light in a medium is called the *refractive index* (or index of refraction) of the medium denoted by n, hence

$$n = \frac{c_0}{c} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}. \tag{1.43}$$

The refractive index of a medium, which was known long before Maxwell's theory was developed, is thus determined by the electrical and magnetic material properties of the medium. It came somewhat as a surprise at first, when this discovery was made, that optics was linked with electricity and magnetism. Furthermore, the polarization of light, which was known since the beginning of the 19th century, could then be traced back to the vectorial nature of the EM field as will be discussed shortly.

Generally the materials used in optics are non-magnetic, hence $\mu = \mu_0$ and the refractive index in such cases is thus solely determined by the permittivity of the material :

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\varepsilon_r}, \tag{1.44}$$

where the ratio $\varepsilon_r = \varepsilon/\varepsilon_0$ is called the *relative permittivity*.

The derivations carried here, similar to those originally published by Maxwell (1864), and which lead to wave equations for the EM field, suggest the existence of electromagnetic waves as Maxwell originally noted. It took time for Maxwell's wave theory to be accepted by his contemporaries. The existence of EM waves was experimentally first demonstrated by Hertz some twenty years later (1887), and that closed the debate on the validity of Maxwell's theory.