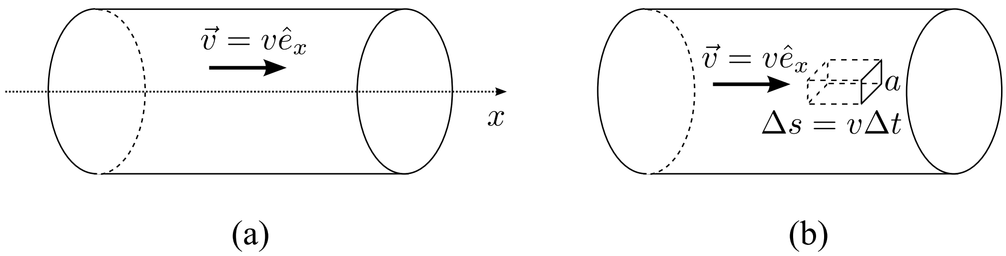


FIGURE 5.4

## 5.5 Flux d'un champ de vecteurs à travers une surface

Cette section est en anglais.

A situation we shall often have to consider is the flux of a vector field through a surface in space (this surface can be real or fictitious). To explain this notion, let us consider the flow of a fluid inside a cylindrical pipe with axis along the $x$ direction as illustrated in Fig. 5.4 (a). Suppose for simplicity that the fluid has a uniform mass per unit volume given by $\rho$ (which is called the density of the fluid; the larger the value of $\rho$ the more mass there is per unit volume, that is the denser the fluid is; hence the name). Now, assume that the fluid travels parallel to the axis of the pipe at speed $v$ everywhere (hence its velocity, which is a vector, is given by $\vec{v} = v\hat{e}_x$, where $\hat{e}_x$ indicates the unit vector along $x$. Consider a small element of surface with area $a$ inside the pipe and perpendicular to its axis (*i.e.* perpendicular to the velocity of the fluid inside the pipe). Let us now calculate the total mass that will cross this element of surface in a small time interval $\Delta t$. During this time interval, since the speed of the fluid is $v$, the fluid contained in a length $\Delta s = v\Delta t$ behind the surface will have time to cross the surface element (see Fig. 5.4 (b)). The volume contained in this length is $\Delta V = \Delta s a$. Thus the total mass that will cross the element of surface will be given by

$$\Delta m = \rho\Delta V = \rho v a\Delta t. \tag{5.6}$$

Hence, in the limiting case where $\Delta t$ becomes very small, the instantaneous mass per unit time that crosses the surface element of area $a$ is

$$\frac{\Delta m}{\Delta t} = \rho v a. \tag{5.7}$$

This mass per unit time is called the flux (more precisely the flux of mass in this case) and is usually denoted by the Greek letter $\Phi$. In this example, the vector field considered is the mass-velocity field given by $\vec{w} = \rho\vec{v}$. It has units of mass per unit time per unit area (kg s$^{-1}$ m$^{-2}$). At each point $\vec{r} = (x, y, z)$ in the fluid, we can assign a vector $\vec{w} = \rho\vec{v}$ (with magnitude $w = \rho v$), and we can think of this vector as a directed rate of mass per unit area flowing at that point (such rates per unit area are also generally called current densities). Ultimately, we know that a fluid is composed of individual particles, and if we divide $\rho$ by the mass $m_{particle}$ of one particle to obtain $n = \rho/m_{particle}$, then $n$ will be the density of particles, and we can then view the fluid as a stream of particles traveling inside the pipe, with each particle having a velocity vector attached to it. In this case, the vector field would be more naturally described in terms of the velocity field $\vec{v}$ of the particles. The trajectory (or path) followed by each particle is called a *field line*. In the present case, the field lines are straight lines along the $x$ direction, but one can easily imagine more complicated situations where the field lines would not be straight, such as for instance if the pipe

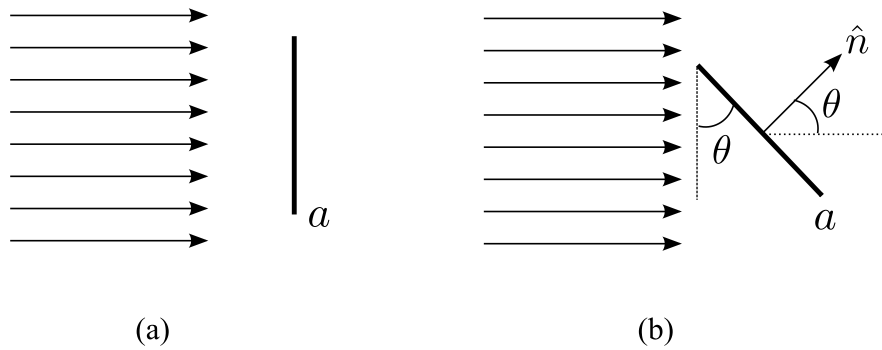


FIGURE 5.5

were curved, rather than being straight (or if the pipe were constricted), then the field lines would also be curved. Using field lines, another interpretation of the flux is that it is a measure of the number of field lines crossing a surface.

The previous discussion considered a surface perpendicular to the flow of the fluid. This is schematically depicted in Fig. 5.5 (a) which shows a side view of the situation along with a few vector field lines with their direction (arrows). Now, consider the case where the element of area is tilted by an angle $\theta$ with respect to the vector field lines as shown in Fig. 5.5 (b). To analyze this situation, we need to assign an orientation to the surface given by the normal vector to it. The normal vector is usually normalized so to have a length of 1 (unit length vector called a unit vector). It will be denoted by $\hat{n}$, where the hat over a vector denotes a unit vector. Clearly, as Fig. 5.5 (b) shows, the flux through the element of area when it is tilted is not the same as when it is not, even though the area itself has not changed. In fact, what matters for the flux is the projected area perpendicular to the field lines. This projected area is given by $a\cos\theta$ in Fig. 5.5 (b). Hence the flux through the tilted surface is given by

$$\Phi = wa\cos\theta. \tag{5.8}$$

Recalling the definition of the scalar product between two vectors, this is equivalent to

$$\Phi = \vec{w}\cdot\hat{n}a. \tag{5.9}$$

The flux is therefore the component of the vector normal to the surface, that is $\vec{w}\cdot\hat{n}$ times the area.

As the reader may have noticed, the unit vector could have been chosen on either side of the surface element. There is indeed an arbitrariness in choosing the direction of the normal vector to a surface. We could have chosen the normal vector to point in the opposite direction, then we would have had as normal vector $\hat{n}' = -\hat{n}$. Often in practice, the surfaces through which fluxes are considered are closed surfaces, such as the surface of a sphere. In such cases, by convention, the normal vector will be chosen to point outwards (*i.e.* towards the exterior of the volume enclosed by the surface). Also, often the notation $\vec{a} = \hat{n}a$ is used to denote an oriented element of surface, *i.e.* the element of surface is considered to be a vector. This just means that its unit normal has been incorporated into the vector $\vec{a}$.

Thus far, a flat element of surface has been considered, and moreover, it was of finite spatial extent. More generally, fluxes will be calculated through curved surfaces. In such cases, elemental fluxes through infinitesimal elements of surfaces will be considered and these elemental fluxes will be added up using integration to obtain the total flux through the entire surface. We thus arrive at the following definitions. Given a vector field $\vec{B}$, then the flux $\mathrm{d}\Phi$ of $\vec{B}$ through an infinitesimal element of surface $\mathrm{d}\vec{a} = \hat{n}\mathrm{d}a$ is given by

$$\mathrm{d}\Phi = \vec{B} \cdot \mathrm{d}\vec{a}. \tag{5.10}$$

Again, the interpretation of this is that the flux of $\vec{B}$ is component of $\vec{B}$ normal to the surface multiplied by the element of area $\mathrm{d}a$. For a surface $S$ of finite area, the flux through it is given by

$$\Phi = \int_S \vec{B} \cdot \mathrm{d}\vec{a}. \tag{5.11}$$

**Exercice de lecture 5.4.** Calculate the flux of the vector field given by $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{e}_r$ (the Coulomb field of a charge $q$) through the surface of a sphere of radius $R$. Does the result depend on the radius? Explain why.

**Exercice de lecture 5.5.** Consider a fictitious spherical surface placed anywhere within the pipe considered in this section. Explain why, without any calculations, the mass flux through this surface is zero.