Vector definitions, identities, and theorems

Definitions

Rectangular coordinates

1. $\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$

2. $\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

3. $\nabla \times A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{dA_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$

4. $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

5. $\nabla^2 A = \nabla^2 A_x \hat{x} + \nabla^2 A_y \hat{y} + \nabla^2 A_z \hat{z} = \nabla(\nabla \cdot A) - \nabla \times (\nabla \times A)$

Cylindrical coordinates

6. $\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$

7. $\nabla \cdot A = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$

8. $\nabla \times A = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right]\hat{z}$

9. $\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$

10. $\nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times (\nabla \times A)$ (Sec. 1.11.6).

Spherical coordinates

11. $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{\partial f}{r\,\partial \theta}\hat{\theta} + \frac{1}{r \sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$

12. $\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$

13. $\nabla \times A = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi}$

14. $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$

15. $\nabla^2 A = \nabla(\nabla \cdot A) - \nabla \times \nabla \times A$ (Sec. 1.11.6).

Identities

1. $(A \times B) \cdot C = A \cdot (B \times C)$
2. $A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$ identité "BAC - CAB"
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla(a/b) = (1/b)\nabla a - (a/b^2)\nabla b$

identité de Jacobi pour le produit vectoriel:
$A \times (B \times C) + C \times (A \times B) + B \times (C \times A) = 0$

5. $\nabla(A \cdot B) = (B \cdot \nabla)A + (A \cdot \nabla)B + B \times (\nabla \times A) + A \times (\nabla \times B)$
6. $\nabla \cdot (fA) = (\nabla f) \cdot A + f(\nabla \cdot A)$
7. $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$
8. $\nabla \cdot \nabla A = \nabla^2 A$
9. $\nabla \times (\nabla f) = 0$
10. $\nabla \cdot (\nabla \times A) = 0$
11. $\nabla \times (fA) = (\nabla f) \times A + f(\nabla \times A)$
12. $\nabla \times (A \times B) = (B \cdot \nabla)A - (A \cdot \nabla)B + (\nabla \cdot B)A - (\nabla \cdot A)B$.
13. $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$ (Sec. 1.11.6)
14. $(A \cdot \nabla)B = \left[A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right]\hat{x} + \left[A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right]\hat{y} + \left[A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right]\hat{z}$
15. $\nabla'(1/r) = \hat{r}/r^2$. This is the gradient calculated at (x', y', z'), and r is the vector r pointing from (x', y', z') to (x, y, z).
16. $\nabla(1/r) = -\hat{r}/r^2$. This is the gradient calculated at (x, y, z) with the same vector r.
17. $\mathcal{A} = \frac{1}{2}\oint_C r \times dl$, where the surface of area $\mathcal{A}$ is plane. The vector r extends from an arbitrary origin to a point on the curve C that bounds $\mathcal{A}$.
18. $\int_v \nabla f\, dv = \int_{\mathcal{A}} f\, d\mathcal{A}$
19. $\int_v (\nabla \times A)\, dv = -\int_{\mathcal{A}} A \times d\mathcal{A}$, where $\mathcal{A}$ is the area of the closed surface that bounds the volume v.
20. $\oint_C f\, dl = -\int_{\mathcal{A}} \nabla \times d\mathcal{A}$ where C is the closed curve that bounds the open surface of area $\mathcal{A}$.

Theorems

1. The divergence theorem. $\int_{\mathcal{A}} A \cdot d\mathcal{A} = \int_v \nabla \cdot A\, dv$ where $\mathcal{A}$ is the area of the closed surface that bounds the volume v.
2. Stokes's theorem: $\oint_C A \cdot dl = \int_{\mathcal{A}} (\nabla \times A) \cdot d\mathcal{A}$.