

Vector definitions, identities, and theorems

Definitions

Rectangular coordinates

1. $\nabla f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}$
2. $\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
3. $\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{z}$
4. $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
5. $\nabla^2 \mathbf{A} = \nabla^2 A_x \hat{x} + \nabla^2 A_y \hat{y} + \nabla^2 A_z \hat{z} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$

Cylindrical coordinates

6. $\nabla f = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\phi} + \frac{\partial f}{\partial z} \hat{z}$
7. $\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$
8. $\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\phi} + \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right] \hat{z}$
9. $\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$
10. $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$ (Sec. 1.11.6).

Spherical coordinates

11. $\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{\partial f}{r \partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi}$
12. $\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$
13. $\nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\phi}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial (r A_\phi)}{\partial r} \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \hat{\phi}$
14. $\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$
15. $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$ (Sec. 1.11.6).

Identities

1. $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$
2. $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ identité "BAC - CAB"
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla(a/b) = (1/b)\nabla a - (a/b^2)\nabla b$
5. $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times (\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla \times \mathbf{B})$
6. $\nabla \cdot (f\mathbf{A}) = (\nabla f) \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A})$
7. $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
8. $\nabla \cdot \nabla \mathbf{A} = \nabla^2 \mathbf{A}$
9. $\nabla \times (\nabla f) = 0$
10. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
11. $\nabla \times (f\mathbf{A}) = (\nabla f) \times \mathbf{A} + f(\nabla \times \mathbf{A})$
12. $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B}$
13. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ (Sec. 1.11.6)
14. $(\mathbf{A} \cdot \nabla)\mathbf{B} = \left[A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} \right] \hat{x} + \left[A_x \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z} \right] \hat{y} + \left[A_x \frac{\partial B_z}{\partial x} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z} \right] \hat{z}$
15. $\nabla'(1/r) = \hat{r}/r^2$. This is the gradient calculated at (x', y', z') , and \mathbf{r} is the vector \mathbf{r} pointing from (x', y', z') to (x, y, z) .
16. $\nabla(1/r) = -\hat{r}/r^2$. This is the gradient calculated at (x, y, z) with the same vector \mathbf{r} .
17. $\mathcal{A} = \frac{1}{2} \oint_C \mathbf{r} \times d\mathbf{l}$, where the surface of area \mathcal{A} is plane. The vector \mathbf{r} extends from an arbitrary origin to a point on the curve C that bounds \mathcal{A} .
18. $\int_v \nabla f dv = \int_{\mathcal{A}} f d\mathcal{A}$
19. $\int_v (\nabla \times \mathbf{A}) dv = -\int_{\mathcal{A}} \mathbf{A} \times d\mathcal{A}$, where \mathcal{A} is the area of the closed surface that bounds the volume v .
20. $\oint_C f d\mathbf{l} = -\int_{\mathcal{A}} \nabla f \times d\mathcal{A}$ where C is the closed curve that bounds the open surface of area \mathcal{A} .

Theorems

1. The divergence theorem. $\int_{\mathcal{A}} \mathbf{A} \cdot d\mathcal{A} = \int_v \nabla \cdot \mathbf{A} dv$ where \mathcal{A} is the area of the closed surface that bounds the volume v .
2. Stokes's theorem: $\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_{\mathcal{A}} (\nabla \times \mathbf{A}) \cdot d\mathcal{A}$.