**departamento de matemática** **universidade de aveiro**

1. Em cada uma das alíneas que se seguem, averigúe se a aplicação considerada é uma aplicação linear:
   (a) $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ tal que $\varphi(x, y, z) = (2x - y - z, x + y, x)$, $\forall (x, y, z) \in \mathbb{R}^3$;
   (b) $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ tal que $\varphi(x, y, z) = (y, x, 0)$, $\forall (x, y, z) \in \mathbb{R}^3$;
   (c) $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ tal que $\varphi(x, y, z) = xyz$, $\forall (x, y, z) \in \mathbb{R}^3$;
   (d) $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ tal que $\varphi(x, y) = (x^2 + y, x - 2y, x)$, $\forall (x, y) \in \mathbb{R}^2$;
   (e) $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ tal que $\varphi(x, y) = |x - y|$, $\forall (x, y) \in \mathbb{R}^2$;
   (f) $\varphi : P_2[x] \longrightarrow P_1[x]$ tal que $\varphi(ax^2 + bx + c) = b + 2cx$, $\forall ax^2 + bx + c \in P_2[x]$;
   (g) $\varphi : P_2[x] \longrightarrow \mathbb{R}$ tal que $\varphi(ax^2 + bx + c) = c - 2b - a$, $\forall ax^2 + bx + c \in P_2[x]$;
   (h) $\varphi : P_2[x] \longrightarrow P_2[x]$ tal que $\varphi(ax^2 + bx + c) = a(x + 1)^2 + b(x + 1) + c$, $\forall ax^2 + bx + c \in P_2[x]$;
   (i) $\varphi : \mathbb{R}^2 \longrightarrow M_{2\times 1}(\mathbb{R})$ tal que $\varphi(x, y) = \begin{bmatrix} 2x - 5y \\ 2x \end{bmatrix}$, $\forall (x, y) \in \mathbb{R}^2$;
   (j) $\varphi : M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R})$ tal que $\varphi(A) = AB + BA$, com $B \in M_{n\times n}(\mathbb{R})$, $\forall A \in M_{n\times n}(\mathbb{R})$;
   (k) $\varphi : M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$ tal que $\varphi(A) = \det(A)$, $\forall A \in M_{n\times n}(\mathbb{R})$;
   (l) $\varphi : M_{n\times m}(\mathbb{R}) \longrightarrow M_{m\times n}(\mathbb{R})$ tal que $\varphi(A) = A^T$, $\forall A \in M_{n\times m}(\mathbb{R})$.

2. (a) Seja $E$ um espaço vectorial real e seja $\varphi : E \longrightarrow \mathbb{R}$ uma aplicação linear e sejam $v_1, v_2 \in E$ tais que
   $$\varphi(v_1) = 1 \quad \text{e} \quad \varphi(v_2) = -1.$$
   Determine $\varphi(3v_1 - 5v_2)$.
   (b) Seja $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ uma aplicação linear tal que
   $$\psi(1, 3) = (1, 1) \quad \text{e} \quad \psi(1, 1) = (0, 1).$$
   Determine $\psi(-1, 3)$.
   (c) Seja $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ uma aplicação linear tal que
   $$\phi(3, -1, 2) = 5, \quad \phi(1, 0, 1) = 2 \quad \text{e} \quad \phi(0, 0, 1) = -1.$$
   Determine $\phi(-1, 1, 0)$.
   (d) Seja $\theta : P_2[x] \longrightarrow P_1[x]$ uma aplicação linear tal que
   $$\theta(x + 1) = x, \quad \theta(x - 1) = 1 \quad \text{e} \quad \theta(x^2) = 0.$$
   Determine $\theta(2 - x + 3x^2)$.

3. Para cada caso, verifique se existe uma aplicação linear que satisfaça as condições indicadas. Em caso afirmativo, determine a expressão geral de tal aplicação linear.

   (a) $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ tal que $\varphi(1,1) = (1,-2)$ e $\varphi(1,0) = (-4,1)$;

   (b) $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ tal que $\varphi(1,0) = (1,2,-1)$ e $\varphi(0,1) = (0,1,0)$;

   (c) $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ tal que $\varphi(1,2) = (3,-1,5)$ e $\varphi(0,1) = (2,1,-1)$;

   (d) $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ tal que $\varphi(1,1,1) = 3$, $\varphi(0,1,-2) = 1$ e $\varphi(0,0,1) = -2$;

   (e) $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ tal que
   $$\varphi(1,1,1) = (1,2,3), \quad \varphi(1,2,3) = (1,4,9) \quad \text{e} \quad \varphi(2,3,4) = (1,8,27);$$

   (f) $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ tal que $\varphi(1,0) = (0,0,0)$, $\varphi(1,1) = (1,0,1)$ e $\varphi(3,2) = (2,0,2)$;

   (g) $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ tal que
   $$\varphi(1,2) = (4,5,0), \quad \varphi(0,1) = (1,-1,2) \quad \text{e} \quad \varphi(1,1) = (1,3,-2);$$

   (h) $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ tal que $\varphi(1,2) + \varphi(2,4) = -\varphi(1,2)$, $\varphi(0,1) = (1,1,-1)$;

   (i) $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ tal que $\varphi(0,0) = (1,0,0)$;

   (j) $\varphi : P_2[x] \longrightarrow P_3[x]$ tal que $\varphi(1+x) = x - x^3$, $\varphi(1+x^2) = 1 - x$ e $\varphi(x) = x$;

   (k) $\varphi : P_2[x] \longrightarrow P_2[x]$ tal que $\varphi(1) = 1+x+x^2$, $\varphi(x) = 1+x^2$ e $\varphi(x+2x^2) = 4x^2$;

   (l) $\varphi : M_{2\times 2}(\mathbb{R}) \longrightarrow \mathbb{R}$ tal que
   $$\varphi\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) = 3,\ \varphi\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right) = -1,\ \varphi\left(\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}\right) = 0 \ \text{ e } \ \varphi\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = 2.$$

4. Considere a aplicação $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ definida por $\varphi(x,y,z) = (x-y+z, x+y+2z)$, para todo $(x,y,z) \in \mathbb{R}^3$.

   (a) Verifique que $\varphi$ é aplicação linear.

   (b) Calcule $\varphi(\mathbb{R}^3)$.

   (c) Determine $\varphi^{-1}(\{(1,-2)\})$.

5. Considere a aplicação linear $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ definida por
   $$\varphi(1,2) = (2,3) \quad \text{e} \quad \varphi(0,1) = (1,4).$$
   Seja $S = \{(x,y) \in \mathbb{R}^2 : x + y = 0\}$ um subespaço vectorial de $\mathbb{R}^2$.

   (a) Determine $\varphi(x,y)$, para todo $(x,y) \in \mathbb{R}^2$.

   (b) Calcule $\varphi(S)$.

   (c) Determine $\varphi^{-1}(\{(2,7)\})$.

6. Considere o subespaço vectorial de $P_3[x]$

$$S = \{ax^3 + bx^2 + cx + d \in P_3[x] : d = c + b \ \wedge \ a = -d\}.$$

Seja $\varphi : \mathbb{R}^3 \longrightarrow P_3[x]$ uma aplicação linear definida por

$$\varphi(1,0,0) = x^3 + 2x, \quad \varphi(0,1,1) = x^2 - 2x \quad \text{e} \quad \varphi(0,0,1) = x^3 + x^2.$$

Determine $\varphi^{-1}(S)$ e comprove, usando a definição, que $\varphi^{-1}(S)$ é um subespaço vectorial de $\mathbb{R}^3$.

7. Sejam $E$ e $E'$ espaços vectoriais sobre $\mathbb{K}$ e seja $F$ um subespaço vectorial de $E$. Seja ainda $\varphi$ uma aplicação linear de $E$ para $E'$. Mostre que:

   (a) $\varphi^{-1}(\varphi(F)) \supseteq F$;

   (b) $\varphi^{-1}(\varphi(E)) = \varphi^{-1}(E')$.

1. **<u>Não são</u>** aplicações lineares as alíneas (c) , (d) , (e) e (k) .

2. (a) 8; (b) $(2,-1)$; (c) $-1$ (d) $\frac{1}{2}x-\frac{3}{2}$;.

3. (a) $\varphi(x,y)=(5y-4x,-3y+x)$, $\forall(x,y)\in\mathbb{R}^2$;
   (b) $\varphi(x,y)=(x,2x+y,-x)$, $\forall(x,y)\in\mathbb{R}^2$;
   (c) $\varphi(x,y)=(2y-x,y-3x,7x-y)$, $\forall(x,y)\in\mathbb{R}^2$;
   (d) $\varphi(x,y,z)=8x-3y-2z$, $\forall(x,y,z)\in\mathbb{R}^3$;
   (e) $\varphi$ não é aplicação linear;
   (f) $\varphi(x,y)=(y,0,y)$, $\forall(x,y)\in\mathbb{R}^2$;
   (g) $\varphi$ não é aplicação linear;
   (h) $\varphi(x,y)=(y-2x,y-2x,2x-y)$, $\forall(x,y)\in\mathbb{R}^2$;
   (i) $\varphi$ não é aplicação linear;
   (j) $\varphi(ax^2+bx+c)=(a-c)x^3+bx+a$, $\forall ax^2+bx+c\in P_2[x]$;
   (k) $\varphi(ax^2+bx+c)=\left(c+b+\frac{3}{2}a\right)x^2+cx+c+b-\frac{a}{2}$, $\forall ax^2+bx+c\in P_2[x]$;
   (l) $\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=3a+2b-3c+2d$, $\forall \begin{bmatrix} a & b \\ c & d \end{bmatrix}\in M_{2\times 2}(\mathbb{R})$.

4. (b) $\mathbb{R}^2$; (c) $\{(4+3y,y,-3-2y):y\in\mathbb{R}\}$.

5. (a) $\varphi(x,y)=(y,4y-5x),\forall(x,y)\in\mathbb{R}^2$; (b) $\{(x,9x):x\in\mathbb{R}\}$; (c) $\left\{\left(\frac{1}{5},2\right)\right\}$.

6. $\{(u,u,0):u\in\mathbb{R}\}$.