
Relational Reasoning (Relationel ræsonnement)

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Bachelor Report (15 ECTS) in Computer Science

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October 2021



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Abstract

►in English...◄

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Chapter 1

Introduction

►motivate and explain the problem to be addressed◄

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Chapter 2

Definition of Language

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To talk about contextual equivalence of programs, we first need to define a language, which captures what programs can be. Consequently, the choice of language influences the definition of contextual equivalence. We will specifically how this manifests when we reach chapter 3. In this chapter, we will focus on formally defining the language, and showing some facts about it, which will be useful in later chapters.

2.1 The language

We define the language in three parts: the syntax, the typing rules, and finally the dynamics (semantics). We note here that we will be working with a Curry-style language, as opposed to a Church-style language. This essentially just means that types are not an intrinsic part of the semantics. Instead, types become a property that programs can have. Thus, we may attempt to evaluate a program even though it has type-errors.

2.1.1 Syntax

The syntax captures exactly all programs that can be written in our language. We define it in Backus-Naur form as follows:

$e ::= ()$	(unit value)
x	(variables)
$\bar{n} \mid e + e \mid e - e \mid e \leq e \mid e < e \mid e = e$	(integers)
$\text{true} \mid \text{false} \mid \text{if } e \text{ then } e \text{ else } e$	(booleans)
$(e, e) \mid \text{fst } e \mid \text{snd } e$	(products)
$\text{inj}_1 e \mid \text{inj}_2 e \mid \text{match } e \text{ with } \text{inj}_1 x \Rightarrow e \mid \text{inj}_2 x \Rightarrow e \text{ end}$	(sums)
$\text{rec } f(x) := e \mid e e$	(recursive functions)
$\Lambda e \mid e _$	(polymorphism)

$v ::= () \mid \bar{n} \mid \text{true} \mid \text{false} \mid (v, v) \mid \text{inj}_1 v \mid \text{inj}_2 v \mid \text{rec } f(x) := e \mid \Lambda e$ (values)

$\tau ::= \text{Unit} \mid \mathbb{Z} \mid \mathbb{B} \mid \tau \times \tau \mid \tau + \tau \mid \tau \rightarrow \tau \mid \forall X. \tau$ (types)

$K ::= [] \mid K + e \mid v + K \mid K - e \mid v - K \mid K \leq e \mid v \leq K \mid K < e \mid v < K \mid$ (evaluation context)
 $K = e \mid v = K \mid \text{if } K \text{ then } e \text{ else } e \mid (K, e) \mid (v, K) \mid \text{fst } K \mid \text{snd } K \mid$
 $\text{inj}_1 K \mid \text{inj}_2 K \mid \text{match } K \text{ with } \text{inj}_1 x \Rightarrow e \mid \text{inj}_2 x \Rightarrow e \text{ end} \mid K e \mid v K \mid K _$

Thus, all programs in our language can be derived using the rules above. Evaluation contexts will be used when we get to defining the dynamics. Note that programs in our language may not necessarily make "sense". For example, using the rules above, we can construct the program $\text{true} + 1$. To avert this problem, we introduce the typing rules, which will allow us to only talk about well-typed programs.

2.1.2 Typing rules

The typing rules gives us a way to derive the type of a program in our language. They are defined using inference rules as follows:

$$\begin{array}{c}
\begin{array}{c} \text{T-VAR} \\ \frac{(x : \tau) \in \Gamma}{\Xi \mid \Gamma \vdash x : \tau} \end{array} \qquad \begin{array}{c} \text{T-UNIT} \\ \frac{}{\Xi \mid \Gamma \vdash () : \text{Unit}} \end{array} \qquad \begin{array}{c} \text{T-INT} \\ \frac{}{\Xi \mid \Gamma \vdash \bar{n} : \mathbb{Z}} \end{array} \\[10pt]
\begin{array}{c} \text{T-ADD} \\ \frac{\Xi \mid \Gamma \vdash e_1 : \mathbb{Z} \quad \Xi \mid \Gamma \vdash e_2 : \mathbb{Z}}{\Xi \mid \Gamma \vdash e_1 + e_2 : \mathbb{Z}} \end{array} \qquad \begin{array}{c} \text{T-SUB} \\ \frac{\Xi \mid \Gamma \vdash e_1 : \mathbb{Z} \quad \Xi \mid \Gamma \vdash e_2 : \mathbb{Z}}{\Xi \mid \Gamma \vdash e_1 - e_2 : \mathbb{Z}} \end{array} \\[10pt]
\begin{array}{c} \text{T-LE} \\ \frac{\Xi \mid \Gamma \vdash e_1 : \mathbb{Z} \quad \Xi \mid \Gamma \vdash e_2 : \mathbb{Z}}{\Xi \mid \Gamma \vdash e_1 \leq e_2 : \mathbb{B}} \end{array} \qquad \begin{array}{c} \text{T-LT} \\ \frac{\Xi \mid \Gamma \vdash e_1 : \mathbb{Z} \quad \Xi \mid \Gamma \vdash e_2 : \mathbb{Z}}{\Xi \mid \Gamma \vdash e_1 < e_2 : \mathbb{B}} \end{array} \\[10pt]
\begin{array}{c} \text{T-EQ} \\ \frac{\Xi \mid \Gamma \vdash e_1 : \mathbb{Z} \quad \Xi \mid \Gamma \vdash e_2 : \mathbb{Z}}{\Xi \mid \Gamma \vdash e_1 = e_2 : \mathbb{B}} \end{array} \qquad \begin{array}{c} \text{T-TRUE} \\ \frac{}{\Xi \mid \Gamma \vdash \text{true} : \mathbb{B}} \end{array} \qquad \begin{array}{c} \text{T-FALSE} \\ \frac{}{\Xi \mid \Gamma \vdash \text{false} : \mathbb{B}} \end{array} \\[10pt]
\begin{array}{c} \text{T-IF} \\ \frac{\Xi \mid \Gamma \vdash e_1 : \mathbb{B} \quad \Xi \mid \Gamma \vdash e_2 : \tau \quad \Xi \mid \Gamma \vdash e_3 : \tau}{\Xi \mid \Gamma \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 : \tau} \end{array} \\[10pt]
\begin{array}{c} \text{T-PAIR} \\ \frac{\Xi \mid \Gamma \vdash e_1 : \tau_1 \quad \Xi \mid \Gamma \vdash e_2 : \tau_2}{\Xi \mid \Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2} \end{array} \qquad \begin{array}{c} \text{T-FST} \\ \frac{\Xi \mid \Gamma \vdash e : \tau_1 \times \tau_2}{\Xi \mid \Gamma \vdash \text{fst } e : \tau_1} \end{array} \\[10pt]
\begin{array}{c} \text{T-SND} \\ \frac{\Xi \mid \Gamma \vdash e : \tau_1 \times \tau_2}{\Xi \mid \Gamma \vdash \text{snd } e : \tau_2} \end{array} \qquad \begin{array}{c} \text{T-INJ1} \\ \frac{\Xi \mid \Gamma \vdash e : \tau_1}{\Xi \mid \Gamma \vdash \text{inj}_1 e : \tau_1 + \tau_2} \end{array} \qquad \begin{array}{c} \text{T-INJ2} \\ \frac{\Xi \mid \Gamma \vdash e : \tau_2}{\Xi \mid \Gamma \vdash \text{inj}_2 e : \tau_1 + \tau_2} \end{array} \\[10pt]
\begin{array}{c} \text{T-MATCH} \\ \frac{\Xi \mid \Gamma \vdash e : \tau_1 + \tau_2 \quad \Xi \mid \Gamma, x : \tau_1 \vdash e_2 : \tau \quad \Xi \mid \Gamma, x : \tau_2 \vdash e_3 : \tau}{\Xi \mid \Gamma \vdash \text{match } e_1 \text{ with } \text{inj}_1 x \Rightarrow e_2 \mid \text{inj}_2 x \Rightarrow e_3 \text{ end} : \tau} \end{array} \\[10pt]
\begin{array}{c} \text{T-REC} \\ \frac{\Xi \mid \Gamma, f : \tau_1 \rightarrow \tau_2, x : \tau_1 \vdash e : \tau_2}{\Xi \mid \Gamma \vdash \text{rec } f(x) := e : \tau_1 \rightarrow \tau_2} \end{array} \qquad \begin{array}{c} \text{T-APP} \\ \frac{\Xi \mid \Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \Xi \mid \Gamma \vdash e_2 : \tau_1}{\Xi \mid \Gamma \vdash e_1 e_2 : \tau_2} \end{array} \\[10pt]
\begin{array}{c} \text{T-TLAM} \\ \frac{\Xi, X \mid \Gamma \vdash e : \tau}{\Xi \mid \Gamma \vdash \Lambda e : \forall X. \tau} \end{array} \qquad \begin{array}{c} \text{T-TAPP} \\ \frac{\Xi \mid \Gamma \vdash e : \forall X. \tau}{\Xi \mid \Gamma \vdash e _ : \tau[\tau'/X]} \end{array}
\end{array}$$

2.1.3 Dynamics

Finally, we define the semantics of the language. We do this using a single "step" rule, and several "head-step" rules. The head-step rules govern all possible

reductions, and the "head-step-step" rule tells us where in our program we can apply these reductions.

$$\frac{\text{HEAD-STEP-STEP} \quad e \rightarrow_h e'}{K[e] \rightarrow K[e']}$$

The evaluation context, K , tells us where in our program we may apply the reduction. If we have an expression e , and we have $e = K[f]$, and $f \rightarrow_h f'$, then we may conclude that $e \rightarrow K[f']$. Now, the head-steps are as follows:

$$\begin{array}{c} \text{E-ADD} \quad \frac{}{\overline{n_1} + \overline{n_2} \rightarrow_h \overline{n_1 + n_2}} \quad \text{E-SUB} \quad \frac{}{\overline{n_1} - \overline{n_2} \rightarrow_h \overline{n_1 - n_2}} \quad \text{E-EQ} \quad \frac{n_1 = n_2}{\overline{n_1} = \overline{n_2} \rightarrow_h \text{true}} \\ \\ \text{E-NOT-EQ} \quad \frac{n_1 \neq n_2}{\overline{n_1} = \overline{n_2} \rightarrow_h \text{false}} \quad \text{E-LE} \quad \frac{n_1 \leq n_2}{\overline{n_1} \leq \overline{n_2} \rightarrow_h \text{true}} \quad \text{E-NOT-LE} \quad \frac{n_1 \not\leq n_2}{\overline{n_1} \leq \overline{n_2} \rightarrow_h \text{false}} \\ \\ \text{E-LT} \quad \frac{n_1 < n_2}{\overline{n_1} < \overline{n_2} \rightarrow_h \text{true}} \quad \text{E-NOT-LT} \quad \frac{n_1 \not< n_2}{\overline{n_1} < \overline{n_2} \rightarrow_h \text{false}} \quad \text{E-IF-TRUE} \quad \frac{}{\text{if true then } e_2 \text{ else } e_3 \rightarrow_h e_2} \\ \\ \text{E-IF-FALSE} \quad \frac{}{\text{if false then } e_2 \text{ else } e_3 \rightarrow_h e_3} \quad \text{E-FST} \quad \frac{}{\text{fst}(v_1, v_2) \rightarrow_h v_1} \quad \text{E-SND} \quad \frac{}{\text{snd}(v_1, v_2) \rightarrow_h v_2} \\ \\ \text{E-MATCH-INJ1} \quad \frac{}{\text{match } (\text{inj}_1 v) \text{ with } \text{inj}_1 x \Rightarrow e_2 \mid \text{inj}_2 x \Rightarrow e_3 \text{ end} \rightarrow_h e_2[v/x]} \\ \\ \text{E-MATCH-INJ2} \quad \frac{}{\text{match } (\text{inj}_2 v) \text{ with } \text{inj}_1 x \Rightarrow e_2 \mid \text{inj}_2 x \Rightarrow e_3 \text{ end} \rightarrow_h e_3[v/x]} \\ \\ \text{E-REC-APP} \quad \frac{}{(\text{rec } f(x) := e)v \rightarrow_h e[\text{rec } f(x) := e/f][v/x]} \quad \text{E-TAPP-TLAM} \quad \frac{}{(\Lambda e) _ \rightarrow_h e} \end{array}$$

At this point, our language has been formally defined, so next we will note down some facts and results about it.

2.2 Facts about the language

The first point we note about the language is that it is non-terminating. This essentially means that we can have well-typed programs that never terminate. Take for example the following program, which we will denote ω .

$$\omega = (\text{rec } f(x) := fx) 0$$

We can make this program have any type τ_2 we like. By the T-rec rule and T-app rule, we may conclude $(\text{rec } f(x) := fx)$ has type $\mathbb{Z} \rightarrow \tau_2$. And by T-app rule ω has type τ_2 . This well-typed program never terminates; this becomes

evident when we take one step: $(\text{rec } f(x) := fx) 0 \rightarrow (\text{rec } f(x) := fx) 0$.

Next we note down some results about the evaluation context.

Lemma 1. $K[e] \rightarrow_h e' \wedge \neg(K = []) \implies \text{Val}(e)$

Proof. This can be shown by doing case distinction on the head-step $K[e] \rightarrow_h e'$. We show it here only for case E-ADD, as all the other cases are similar.

So assume $K[e] = \overline{n_1} + \overline{n_2}$, and $e' = \overline{n_1 + n_2}$. Then there are three cases: $K = []$ and $e = \overline{n_1} + \overline{n_2}$, $K = [] + \overline{n_2}$ and $e = \overline{n_1}$, or $K = \overline{n_1} + []$ and $e = \overline{n_2}$. The first case raises a contradiction as we have assumed $\neg(K = [])$. In the remaining two cases, we may conclude $\text{Val}(e)$, as wanted. \square

Lemma 2 (Evaluation under Context). $K[e] \rightarrow^* e' \implies \exists e''. (e \rightarrow^* e'') \wedge ((\text{Val}(e'') \wedge K[e''] \rightarrow^* e') \vee (\neg \text{Val}(e'') \wedge K[e''] = e'))$

Proof. So assuming $K[e] \rightarrow^* e'$, we must show

$$\exists e''. (e \rightarrow^* e'') \wedge ((\text{Val}(e'') \wedge K[e''] \rightarrow^* e') \vee (\neg \text{Val}(e'') \wedge K[e''] = e')) \quad (2.1)$$

We proceed by induction on the number of steps in the evaluation $K[e] \rightarrow^* e'$. Let n denote the number of steps taken, so that $K[e] \rightarrow^n e'$.

- Base Case $n = 0$. In this case, we have that $K[e] \rightarrow^0 e'$, which means that $K[e] = e'$. Now use e for e'' in 2.1. We must show

$$(e \rightarrow^* e) \wedge ((\text{Val}(e) \wedge K[e] \rightarrow^* e') \vee (\neg \text{Val}(e) \wedge K[e] = e'))$$

Trivially, $e \rightarrow^* e$. For the second part, we proceed by case distinction on $\text{Val}(e)$.

- $\text{Val}(e)$. We have that $K[e] \rightarrow^0 e'$, so $K[e] \rightarrow^* e'$. Thus, we have $\text{Val}(e) \wedge K[e] \rightarrow^* e'$, which matches the left part of the " \vee ".
- $\neg \text{Val}(e)$. We know that $K[e] = e'$, so we have $\neg \text{Val}(e) \wedge K[e] = e'$, which matches the right part of the " \vee ".

- Inductive Step $n = m + 1$. Now we have $K[e] \rightarrow^{m+1} e'$. By the Induction Hypothesis, we have:

$$\forall F, f, f'. F[f] \rightarrow^m f' \implies \exists f''. (f \rightarrow^* f'') \wedge ((\text{Val}(f'') \wedge F[f''] \rightarrow^* f') \vee (\neg \text{Val}(f'') \wedge K[f''] = f')) \quad (2.2)$$

Split the evaluation, $K[e] \rightarrow^{m+1} e'$, up, so that $K[e] \rightarrow g \wedge g \rightarrow^m e'$. Looking at our dynamics, we must have that $K[e] = H[h]$, and $g = H[h']$, for some evaluation context H , and expressions h, h' , and $h \rightarrow_h h'$. There are now three possible cases. Either $e = h$, e is a superexpression of h , or e is a subexpression of h . We will consider each in turn.

- $K = H$ and $e = h$.

Then $g = H[h'] = K[h']$, and $e \rightarrow_h h'$. Furthermore, $K[h'] \rightarrow^m e'$.
 Instantiate I.H. with this to get

$$\begin{aligned} & \exists f''. (h' \rightarrow^* f'') \wedge \\ & ((\text{Val}(f'') \wedge K[f''] \rightarrow^* e') \vee (\neg \text{Val}(f'') \wedge K[f''] = e')) \end{aligned} \quad (2.3)$$

Call this quantified expression for f'' , and use it for e'' in 2.1. We must then show

$$(e \rightarrow^* f'') \wedge ((\text{Val}(f'') \wedge K[f''] \rightarrow^* e') \vee (\neg \text{Val}(f'') \wedge K[f''] = e'))$$

We know that $e \rightarrow h'$, as $e \rightarrow_h h'$, and by 2.3, we know that $h' \rightarrow^* f''$, so that $e \rightarrow^* f''$. The second part follows directly from 2.3.

- $K[E[]] = H$ and $e = E[h]$.

Then $g = H[h'] = K[E[]][h'] = K[E[h']]$, thus $K[E[h']] \rightarrow^m e'$.
 Instantiate I.H. with this to get

$$\begin{aligned} & \exists f''. (E[h'] \rightarrow^* f'') \wedge \\ & ((\text{Val}(f'') \wedge K[f''] \rightarrow^* e') \vee (\neg \text{Val}(f'') \wedge K[f''] = e')) \end{aligned} \quad (2.4)$$

Call this quantified expression for f'' , and use it for e'' in 2.1. We must then show

$$(e \rightarrow^* f'') \wedge ((\text{Val}(f'') \wedge K[f''] \rightarrow^* e') \vee (\neg \text{Val}(f'') \wedge K[f''] = e'))$$

We know that $E[h] \rightarrow E[h']$, as $h \rightarrow_h h'$, and since $e = E[h]$, then $e \rightarrow E[h']$. By 2.4, we know that $E[h'] \rightarrow^* f''$, so that $e \rightarrow^* f''$. The second part follows directly from 2.4.

- $K = H[E[]]$ and $E[e] = h$. Here we have $h = E[e]$, so $E[e] \rightarrow_h h'$. Note that E is not the empty evaluation context, as otherwise, we would be in case 1. So by lemma 1, we know that $\text{Val}(e)$. Now, pick e for e'' in 2.1. We must show

$$(e \rightarrow^* e) \wedge ((\text{Val}(e) \wedge K[e] \rightarrow^* e') \vee (\neg \text{Val}(e) \wedge K[e] = e'))$$

Trivially, $e \rightarrow^* e$. We also have that $\text{Val}(e)$, and since $K[e] \rightarrow^{m+1} e'$, then $K[e] \rightarrow^* e'$.

□

Corollary 2.1. $K[e] \rightarrow^* v \implies \exists e''. (e \rightarrow^* e'') \wedge (\text{Val}(e'') \wedge K[e''] \rightarrow^* v)$

Proof. Assuming $K[e] \rightarrow^* v$, we must show $\exists e''. (e \rightarrow^* e'') \wedge (\text{Val}(e'') \wedge K[e''] \rightarrow^* v)$. By lemma 2, we know $\exists e''. (e \rightarrow^* e'') \wedge ((\text{Val}(e'') \wedge K[e''] \rightarrow^* v) \vee (\neg \text{Val}(e'') \wedge K[e''] = v))$. Now, if $\neg \text{Val}(e'')$ then $\neg \text{Val}(K[e''])$, which one may see simply by inspecting the possible evaluation contexts. Therefore, $K[e''] \neq v$, so $\neg(\neg \text{Val}(e'') \wedge K[e''] = v)$. Thus, we conclude $\exists e''. (e \rightarrow^* e'') \wedge (\text{Val}(e'') \wedge K[e''] \rightarrow^* v)$. \square

2.3 substitution

We define a substitution $\gamma = \{x_1 \mapsto v_1, x_2 \mapsto v_2, \dots\}$ inductively as follows:

$$\begin{aligned}\emptyset(e) &= e \\ \gamma[x \mapsto v](e) &= \gamma(e[v/x])\end{aligned}$$

Lemma 3 (Substitution). $\gamma[x \mapsto v](e) = \gamma(e)[v/x]$

Proof. Lemma 10 in notes by Lau. \square

Chapter 3

Contextual Equivalence

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Imagine you are writing a larger program, and as part of that program, you need to use a stack. So as part of your program, you implement a stack. However, your implementation is naive, and not efficient. Thus you implement a new stack, that is more complex, but also more efficient. You would of course want to use the more efficient stack implementation in your larger program, but since it is more complex, you are not really sure whether you can justify the refactoring – the implementations might show differing behaviour.

What you want to show, then, is that your two stack implementation *behaves* the same. In other words, no matter which *context* you use your two stack implementations in, they will always behave the same as part of the context. This is intuitively what we want to define with *Contextual Equivalence*.

In this section we give two equivalent definitions of what it means for programs to be contextually equivalent. The first definition will be more intuitive and instructive, while the second will give a deeper insight into the theoretical underpinnings of contextual equivalence.

3.1 Definition of Contextual Equivalence

Context

$$\begin{aligned} C ::= & [\cdot] \mid C + e \mid e + C \mid C - e \mid e - C \mid C \leq e \mid e \leq C \mid C < e \mid e < C \mid C = e \mid e = C \mid \\ & \text{if } C \text{ then } e \text{ else } e \mid \text{if } e \text{ then } C \text{ else } e \mid \text{if } e \text{ then } e \text{ else } C \mid (C, e) \mid (e, C) \mid \\ & \text{fst } C \mid \text{snd } C \mid \text{inj}_1 C \mid \text{inj}_2 C \mid \text{match } C \text{ with } \text{inj}_1 x \Rightarrow e \mid \text{inj}_2 x \Rightarrow e \text{ end} \mid \\ & \text{match } e \text{ with } \text{inj}_1 x \Rightarrow C \mid \text{inj}_2 x \Rightarrow e \text{ end} \mid \\ & \text{match } e \text{ with } \text{inj}_1 x \Rightarrow e \mid \text{inj}_2 x \Rightarrow C \text{ end} \mid \text{rec } f(x) := C \mid C e \mid e C \mid \Lambda C \mid C _ \end{aligned}$$

Context Typing

$$\frac{\text{T-CTX} \quad \Xi \mid \Gamma \vdash e : \tau \quad \Xi' \mid \Gamma' \vdash C[e] : \tau'}{C : (\Xi \mid \Gamma \vdash \tau) \Rightarrow (\Xi' \mid \Gamma' \vdash \tau')}$$

Read as: the context takes an expression of type τ under Ξ, Γ , and outputs an expression of type τ' under Ξ', Γ' .

Definition 3.1.1 (Contextual Equivalence). We define contextual equivalence of two expressions e_1 and e_2 at type τ under Ξ and Γ as

$$\begin{aligned} & \Xi \mid \Gamma \vdash e_1 \approx^{ctx} e_2 : \tau \\ & \iff \\ & \Xi \mid \Gamma \vdash e_1 : \tau \quad \wedge \quad \Xi \mid \Gamma \vdash e_2 : \tau \quad \wedge \\ & \forall C : (\Xi \mid \Gamma \vdash \tau) \Rightarrow (\bullet \mid \bullet \vdash \text{Unit}). (C[e_1] \Downarrow \iff C[e_2] \Downarrow) \end{aligned}$$

We define CE so that two programs are CE when under all Contexts, one program terminates iff the other does. One may ponder whether this definition is really strong enough. If both terminate, we really want the result of the two programs to have values that are behaviourally the same, otherwise, we can certainly put them in a context, where they have different behaviour. For integer values, for example, this means that the two values must be the same number, and for product types, this means that the first value in each pair should be behaviourally the same, and likewise with the second value in each pair. To give an intuition as to why this definition is strong enough, consider two contextually equivalent programs e_1 and e_2 , where $\bullet \mid \bullet \vdash e_1 : \tau$ and $\bullet \mid \bullet \vdash e_2 : \tau$. Note first that if both e_1 and e_2 don't terminate, then they will be behaviourally the same; if we plug them into any context, and the context makes us evaluate the expressions, both will not terminate. And if the context doesn't make us evaluate the expressions, then they have no influence in the run of the programs, so they will behave the same.

Now consider what happens if e_1 terminates with some value v_1 . Can we then guarantee that e_2 also terminates with some value v_2 , and v_1 and v_2 behave the same? Since e_1 and e_2 are contextually equivalent, then we can put them into any context, C , and one will terminate if and only if the other one does. Let's assume that $\tau = \mathbb{Z}$. Then consider when C has the form **if** $[\cdot] = v_1$ **then** $()$ **else** ω . Here $C[e_1] \Downarrow ()$. But what about $C[e_2]$? If $v_2 \neq v_1$, then our evaluation rules tell us that we will take the else branch, and hence not terminate. However, since e_1 and e_2 are contextually equivalent, and $C[e_1]$ terminates, then we know that $C[e_2]$ must also terminate. Hence it is not the case that $v_2 \neq v_1$, thus $v_2 = v_1$. So if our two programs of type integer are contextually equivalent, and they both don't run forever, then they must both evaluate to the same number.

Now if τ was \mathbb{B} instead, the context **if** $[\cdot]$ **then** $()$ **else** ω would suffice in showing that $v_2 = v_1$.

It becomes a little more difficult to reason about when τ is not a base-type, like \mathbb{Z} or \mathbb{B} . Take function type, for instance. If e_1 and e_2 both evaluate down to functions, how do we know that those functions are behaviourally equivalent (in the sense of functional extensionality)? To see this, consider the following "congruence" rule:

$$\frac{\text{CNG-CTX-APP} \quad \Xi \mid \Gamma \vdash f \approx^{ctx} f' : \tau \rightarrow \tau' \quad \Xi \mid \Gamma \vdash t \approx^{ctx} t' : \tau}{\Xi \mid \Gamma \vdash f t \approx^{ctx} f' t' : \tau'}$$

This essentially gives us what we want: if we have two contextually equivalent functions, then, as long as we give them contextually equivalent inputs, the output will also be contextually equivalent. That output may be another function, but then this rule applies again to that output, and so on. In other words, we can keep on applying this rule until we at some point get a base type, like \mathbb{Z} , and at that point, we know that the two numbers will be the same, by the argument made above.

Of course, it may happen that the output is of another non-base type, such as type abstraction. But we may do something similar for all other non-base types. However proving all of them is quite tedious, so we will here only prove the congruence rule for function types.

Proof. By the two hypotheses of the rule, we know that $\Xi \mid \Gamma \vdash f : \tau \rightarrow \tau'$ and $\Xi \mid \Gamma \vdash t : \tau$. So by the T-APP rule, we may conclude $\Xi \mid \Gamma \vdash f t : \tau'$. Likewise for the prime variants. So all that remains to be shown is that $\forall C : (\Xi \mid \Gamma \vdash \tau') \Rightarrow (\bullet \mid \bullet \vdash \text{Unit}). (C[f t] \Downarrow \iff C[f' t'] \Downarrow)$. So assume some context, C , of the right type. Consider now the context $C[[\cdot] t]$. We know that f is contextually equivalent to f' , so $C[f t] \Downarrow \iff C[f' t] \Downarrow$. Now consider the context $C[f' [\cdot]]$. Since t is contextually equivalent to t' , then we know $C[f' t] \Downarrow \iff C[f' t'] \Downarrow$. In other words, we have that $C[f t] \Downarrow \iff C[f' t] \Downarrow \iff C[f' t'] \Downarrow$, which was what we wanted. \square

The way we have defined contextual equivalence is quite instructive and intuitive. However, one may define contextual equivalence in another, equivalent way, which gives much insight into contextual equivalence, and will make working with the logical relation later more intuitive. We will explore this alternative definition in the next section.

3.2 Alternative Definition of Contextual Equivalence

Before giving the alternative definition, we first introduce a few concepts. In the following, we shall let R denote any relation over $\Xi \times \Gamma \times e \times e \times \tau$. I.e. $R \subseteq \Xi \times \Gamma \times e \times e \times \tau$. We also introduce the following notation: $R(\Xi, \Gamma, e, e', \tau) \triangleq \Xi \mid \Gamma \vdash e \approx e' : \tau$, and use them interchangeably.

Now we define two properties that a relation R may have.

Definition 3.2.1 (Adequacy). We say R is an adequate relation when it holds for R that $\bullet \mid \bullet \vdash e \approx e' : \text{Unit} \implies e \Downarrow \iff e' \Downarrow$.

Definition 3.2.2 (Congruency). We say R is a congruence relation (with respect to the typing rules) when it satisfies all the congruence rules that arises from the typing rules.

For instance, the congruence rules for T-unit and T-app would be:

$$\begin{array}{c} \text{CNG-UNIT} \\ \hline \Xi \mid \Gamma \vdash () \approx () : \text{Unit} \end{array} \qquad \begin{array}{c} \text{CNG-TAPP} \\ \Xi \mid \Gamma \vdash e \approx e' : \forall X. \tau \\ \hline \Xi \mid \Gamma \vdash e _ \approx e' _ : \tau[\tau'/X] \end{array}$$

The remaining congruence rules can be found in the appendix (A.1).
Now, we may define contextual equivalence in the following way

Definition 3.2.3. Contextual Equivalence is a relation, $CE \subseteq \Xi \times \Gamma \times e \times e \times \tau$, such that

- all expressions in CE are well-typed
- CE is a congruence relation
- CE is an adequate relation

Finally, it is the coarsest such relation.

CE being the coarsest such relation means that if given a relation R satisfying the three properties, then $\forall \Xi, \Gamma, e, e', \tau. R(\Xi, \Gamma, e, e', \tau) \implies CE(\Xi, \Gamma, e, e', \tau)$. We of course have to show that this definition of Contextual Equivalence is actually equivalent to the one given in definition 3.1.1:

Theorem 4. $\Xi \mid \Gamma \vdash e \approx^{ctx} e' : \tau \iff CE(\Xi, \Gamma, e, e', \tau)$

We will show this in two steps, where each step essentially corresponds to one way of the double implication. Each step will be phrased as a theorem.

Theorem 5. $\Xi \mid \Gamma \vdash e \approx^{ctx} e' : \tau \subseteq \Xi \times \Gamma \times e \times e \times \tau$ is a congruence relation, an adequate relation, and all expressions in the relation are well-typed.

Proof. First, the well-typedness follows directly from the definition of contextual equivalence.

Second, let's show that it is an adequate relation. So assuming $\bullet \mid \bullet \vdash e \approx^{ctx} e' : \text{Unit}$, we must show $e \Downarrow \iff e' \Downarrow$. We know that $\forall C : (\bullet \mid \bullet \vdash \text{Unit}) \implies (\bullet \mid \bullet \vdash \text{Unit}).(C[e] \Downarrow \iff C[e'] \Downarrow)$. So specifically for C being the empty context, we get $e \Downarrow \iff e' \Downarrow$, which was what we wanted.

To show that it is a congruence relation, we must go through all the congruence rules, and show that they hold. We did the one for function application above, Cng-ctx-app. The rest will not be shown here. \square

To prove the next theorem, we will need to show two lemmas.

Lemma 6 (Reflexivity). Let R be a congruence relation. Then $\Xi \mid \Gamma \vdash e : \tau \implies R(\Xi, \Gamma, e, e, \tau)$.

Proof. By induction on the typing derivation of e . It follows immediately from applying the induction hypothesis and applying congruency rules. We show one case here.

case T-tlam.

Assuming $\Xi \mid \Gamma \vdash \Lambda e : \forall X. \tau$ and $\Xi, X \mid \Gamma \vdash e : \tau$, we must show $R(\Xi, \Gamma, \Lambda e, \Lambda e, \forall X. \tau)$. By our induction hypothesis, we get $\Xi, X \mid \Gamma \vdash e \approx e : \tau$. By the congruency rule *Cng – tlam*, we then get $\Xi \mid \Gamma \vdash \Lambda e \approx \Lambda e : \forall X. \tau$, which was what we wanted.

□

Lemma 7. *Let R be a congruence relation, then $R(\Xi, \Gamma, e, e', \tau) \wedge C : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma' \vdash \tau') \implies R(\Xi', \Gamma', C[e], C[e'], \tau')$*

Proof. By induction on C . We show here only a few interesting cases. For each non-base-case we show, the induction hypothesis will be $\forall \Xi, \Gamma, \Xi', \Gamma', e, e', \tau, \tau'. R(\Xi, \Gamma, e, e', \tau) \wedge C' : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma' \vdash \tau') \implies R(\Xi', \Gamma', C'[e], C'[e'], \tau')$, where C' is structurally smaller than C .

case $C = [\cdot]$.

So assuming $R(\Xi, \Gamma, e, e', \tau)$ and $C : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma' \vdash \tau')$, we must show $R(\Xi', \Gamma', C[e], C[e'], \tau')$. We may conclude that $\Xi = \Xi'$, $\Gamma = \Gamma'$, and $\tau = \tau'$, given that $[\cdot] : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi \mid \Gamma \vdash \tau)$. So since $e = [\cdot][e] = C[e]$, and $e' = [\cdot][e'] = C[e']$, we get $R(\Xi', \Gamma', C[e], C[e'], \tau')$, which was what we wanted.

case $C = C' + e''$

So assuming $R(\Xi, \Gamma, e, e', \tau)$ and $C : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma' \vdash \tau')$, we must show $R(\Xi', \Gamma', C[e], C[e'], \tau')$. In this case, $C : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma' \vdash \mathbb{Z})$ and $C' : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi \mid \Gamma \vdash \mathbb{Z})$, meaning $\tau' = \mathbb{Z}$. So we have $R(\Xi, \Gamma, e, e', \mathbb{Z})$, and our goal then is $R(\Xi', \Gamma', C[e], C[e'], \mathbb{Z})$. From the context typing and from rule T-add, we may conclude $\Xi' \mid \Gamma' \vdash e'' : \mathbb{Z}$. So by lemma 6, we know that $\Xi' \mid \Gamma' \vdash e'' \approx e'' : \mathbb{Z}$. And by I.H. we get $R(\Xi', \Gamma', C'[e], C'[e'], \mathbb{Z}) \equiv \Xi' \mid \Gamma' \vdash C'[e] \approx C'[e'] : \mathbb{Z}$. Finally, from the congruence rule *cng-add*, we get $\Xi' \mid \Gamma' \vdash C'[e] + e'' \approx C'[e'] + e'' : \mathbb{Z} \equiv \Xi' \mid \Gamma' \vdash C[e] \approx C[e'] : \mathbb{Z}$, which was what we wanted.

case $C = \text{fst } C'$

Then $C : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma' \vdash \tau_1)$ and $C' : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma' \vdash \tau_1 \times \tau_2)$, so our goal is $R(\Xi', \Gamma', C[e], C[e'], \tau_1)$. By I.H. we get $\Xi' \mid \Gamma' \vdash C'[e] \approx C'[e'] : \tau_1 \times \tau_2$, and by *cng-fst* with this, we get $\Xi' \mid \Gamma' \vdash \text{fst } C'[e] \approx \text{fst } C'[e'] : \tau_1 \equiv \Xi' \mid \Gamma' \vdash C[e] \approx C[e'] : \tau_1$.

case $C = \text{rec } f(x) := C'$

Then $C : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma' \vdash \tau_1 \rightarrow \tau_2)$ and $C' : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma', f : \tau_1 \rightarrow \tau_2, x : \tau_1 \vdash \tau_2)$, so our goal is $R(\Xi', \Gamma', C[e], C[e'], \tau_1 \rightarrow \tau_2)$. By I.H. we get $\Xi' \mid \Gamma', f : \tau_1 \rightarrow \tau_2, x : \tau_1 \vdash C'[e] \approx C'[e'] : \tau_2$, and by *cng-rec* with

this, we get $\Xi' \mid \Gamma' \vdash \text{rec } f(x) := C'[e] \approx \text{rec } f(x) := C'[e'] : \tau_1 \rightarrow \tau_2 \equiv \Xi' \mid \Gamma' \vdash C[e] \approx C[e'] : \tau_1 \rightarrow \tau_2$.

case $C = C' e''$

Then $C : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma' \vdash \tau_2)$ and $C' : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma' \vdash \tau_1 \rightarrow \tau_2)$, so our goal is $R(\Xi', \Gamma', C[e], C[e'], \tau_2)$. From the context-typing and T-app, we have $\Xi' \mid \Gamma' \vdash e'' : \tau_1$. By lemma 6, we get $\Xi' \mid \Gamma' \vdash e'' \approx e'' : \tau_1$. Now, by the induction hypothesis, we get $\Xi' \mid \Gamma' \vdash C'[e] \approx C'[e'] : \tau_1 \rightarrow \tau_2$. And by cng-app with this, we get $\Xi' \mid \Gamma' \vdash C'[e] e'' \approx C'[e'] e'' : \tau_2 \equiv \Xi' \mid \Gamma' \vdash C[e] \approx C[e'] : \tau_2$.

case $C = \Lambda C'$

Then $C : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi' \mid \Gamma' \vdash \forall X. \tau')$ and $C' : (\Xi \mid \Gamma \vdash \tau \Rightarrow \Xi', X \mid \Gamma' \vdash \tau')$, so our goal is $R(\Xi', \Gamma', C[e], C[e'], \forall X. \tau')$. By I.H. we get $\Xi', X \mid \Gamma' \vdash C'[e] \approx C'[e'] : \tau'$, and by cng-tlam with this, we get $\Xi' \mid \Gamma' \vdash \Lambda C'[e] \approx \Lambda C'[e'] : \forall X. \tau' \equiv \Xi' \mid \Gamma' \vdash C[e] \approx C[e'] : \forall X. \tau'$.

□

With this lemma proved, we are ready to prove the next theorem.

Theorem 8. *Let R be a relation satisfying the three properties of definition 3.2.3, then $R(\Xi, \Gamma, e, e', \tau) \implies \Xi \mid \Gamma \vdash e \approx^{ctx} e' : \tau$.*

Proof. So assuming $R(\Xi, \Gamma, e, e', \tau)$, we must show $\Xi \mid \Gamma \vdash e : \tau \wedge \Xi \mid \Gamma \vdash e' : \tau \wedge \forall C : (\Xi \mid \Gamma \vdash \tau) \Rightarrow (\bullet \mid \bullet \vdash \text{Unit}).(C[e] \Downarrow \iff C[e'] \Downarrow)$. Again, the well-typedness follows from the fact that all expressions in R are well-typed. So what remains to be shown is that $\forall C : (\Xi \mid \Gamma \vdash \tau) \Rightarrow (\bullet \mid \bullet \vdash \text{Unit}).(C[e] \Downarrow \iff C[e'] \Downarrow)$. So assume some context C of the right type. By lemma 7, we have $R(\bullet, \bullet, C[e], C[e'], \text{Unit})$. By adequacy of R , we have that $C[e] \Downarrow \iff C[e'] \Downarrow$, which was what we wanted. □

Now, the proof of the two definitions of contextual equivalence being equivalent follows easily:

Proof of theorem 4. By theorem 5 and from the fact that CE is the coarsest relation satisfying the three properties, it follows that $\Xi \mid \Gamma \vdash e \approx^{ctx} e' : \tau \implies CE(\Xi, \Gamma, e, e', \tau)$.

By theorem 8 and from the fact that CE is one such relation R , it follows that $CE(\Xi, \Gamma, e, e', \tau) \implies \Xi \mid \Gamma \vdash e \approx^{ctx} e' : \tau$. □

Chapter 4

Logical Relations Model for Contextual Equivalence

►draft◄

Logical relations model

Value interpretation

$$\begin{aligned}
\mathcal{V}[\![\tau]\!]_{\rho}(v_1, v_2) &\triangleq \bullet \mid \bullet \vdash v_1 : \rho_1(\tau) \wedge \bullet \mid \bullet \vdash v_2 : \rho_2(\tau) \wedge \dots \\
\mathcal{V}[\![\text{Unit}]\!]_{\rho}(v_1, v_2) &\triangleq (v_1, v_2) \in \{((), ())\} \\
\mathcal{V}[\![\mathbb{Z}]\!]_{\rho}(v_1, v_2) &\triangleq (v_1, v_2) \in \{(\bar{n}, \bar{n}) \mid \bar{n} \in \mathbb{Z}\} \\
\mathcal{V}[\![\mathbb{B}]\!]_{\rho}(v_1, v_2) &\triangleq (v_1, v_2) \in \{(\text{true}, \text{true}), (\text{false}, \text{false})\} \\
\mathcal{V}[\![\tau_1 \times \tau_2]\!]_{\rho}(v_1, v_2) &\triangleq (v_1, v_2) \in \{((w_{11}, w_{12}), (w_{21}, w_{22})) \mid \mathcal{V}[\![\tau_1]\!]_{\rho}(w_{11}, w_{21}) \wedge \mathcal{V}[\![\tau_2]\!]_{\rho}(w_{12}, w_{22})\} \\
\mathcal{V}[\![\tau_1 + \tau_2]\!]_{\rho}(v_1, v_2) &\triangleq (v_1, v_2) \in \{(\text{inj}_1 w_1, \text{inj}_1 w_2) \mid \mathcal{V}[\![\tau_1]\!]_{\rho}(w_1, w_2)\} \vee \\
&\quad (v_1, v_2) \in \{(\text{inj}_2 w_1, \text{inj}_2 w_2) \mid \mathcal{V}[\![\tau_2]\!]_{\rho}(w_1, w_2)\} \\
\mathcal{V}[\![\tau_1 \rightarrow \tau_2]\!]_{\rho}(v_1, v_2) &\triangleq (v_1, v_2) \in \{(\text{rec } f(x) := e_1, \text{rec } f(x) := e_2) \mid \forall w_1, w_2. \mathcal{V}[\![\tau_1]\!]_{\rho}(w_1, w_2) \implies \\
&\quad \mathcal{E}[\![\tau_2]\!]_{\rho}(e_1[\text{rec } f(x) := e_1/f][w_1/x], e_2[\text{rec } f(x) := e_2/f][w_2/x])\} \\
\mathcal{V}[\![\forall X. \tau]\!]_{\rho}(v_1, v_2) &\triangleq (v_1, v_2) \in \{(\Lambda e_1, \Lambda e_2) \mid \forall \tau_1, \tau_2, R \in \text{Rel}[\tau_1, \tau_2]. \\
&\quad \mathcal{E}[\![\tau]\!]_{\rho[X \mapsto (\tau_1, \tau_2, R)]}(e_1[\tau_1/X], e_2[\tau_2/X])\} \\
\mathcal{V}[\![X]\!]_{\rho}(v_1, v_2) &\triangleq (v_1, v_2) \in \rho_R[X]
\end{aligned}$$

Relational substitution

$$\text{Rel}[\tau_1, \tau_2] \triangleq \left\{ R \mid \begin{array}{l} R \in \mathcal{P}(\text{Val} \times \text{Val}) \wedge \bullet \mid \bullet \vdash \tau_1 \wedge \bullet \mid \bullet \vdash \tau_1 \wedge \\ \forall (v_1, v_2) \in R. \bullet \mid \bullet \vdash v_1 : \tau_1 \wedge \bullet \mid \bullet \vdash v_2 : \tau_2 \end{array} \right\}$$

Expression interpretation

$$\begin{aligned}
\mathcal{E}[\![\tau]\!]_{\rho}(e_1, e_2) &\triangleq \bullet \mid \bullet \vdash e_1 : \rho_1(\tau) \wedge \bullet \mid \bullet \vdash e_2 : \rho_2(\tau) \wedge e_1 \Downarrow \iff e_2 \Downarrow \wedge \\
&\quad (\forall v_1, v_2. e_1 \Downarrow v_1 \wedge e_2 \Downarrow v_2 \implies \mathcal{V}[\![\tau]\!]_{\rho}(v_1, v_2))
\end{aligned}$$

Variable environment interpretation

$$\begin{aligned}\mathcal{G}[\bullet]_\rho &\triangleq \{\emptyset\} \\ \mathcal{G}[\Gamma, x : \tau]_\rho &\triangleq \{\gamma[x \mapsto (v_1, v_2)] \mid \gamma \in \mathcal{G}[\Gamma]_\rho \wedge \mathcal{V}[\tau]_\rho(v_1, v_2)\}\end{aligned}$$

Type environment interpretation

$$\begin{aligned}\mathcal{D}[\bullet] &\triangleq \{\emptyset\} \\ \mathcal{D}[\Xi, X] &\triangleq \{\rho[X \mapsto (\tau_1, \tau_2, R)] \mid \rho \in \mathcal{D}[\Xi] \wedge R \in \text{Rel}[\tau_1, \tau_2]\}\end{aligned}$$

Definition 4.0.1 (Logical relation).

$$\begin{aligned}\Xi \mid \Gamma \vdash e \approx^{LR} e' : \tau \\ \iff \\ \Xi \mid \Gamma \vdash e : \tau \wedge \Xi \mid \Gamma \vdash e' : \tau \wedge \\ \forall \rho \in \mathcal{D}[\Xi], \gamma \in \mathcal{G}[\Gamma]_\rho. \mathcal{E}[\tau]_\rho(\rho_1(\gamma_1(e)), \rho_2(\gamma_2(e')))\end{aligned}$$

Compatibility lemmas Essentially just congruency rules as stated in appendix (but with LR being the relation). We have to prove these (and adequacy) to show that $LR \implies CE$, meaning we can state theorems in terms of LR instead of CE, and then get CE as a consequence.

Lemma 9 (Cmpt-unit). $\frac{}{\Xi \mid \Gamma \vdash () \approx^{LR} () : \text{Unit}}$

Proof. By definition 4.0.1, we must show $\Xi \mid \Gamma \vdash () : \text{Unit} \wedge \Xi \mid \Gamma \vdash () : \text{Unit} \wedge \forall \rho \in \mathcal{D}[\Xi], \gamma \in \mathcal{G}[\Gamma]_\rho. \mathcal{E}[\text{Unit}]_\rho(\rho_1(\gamma_1(())), \rho_2(\gamma_2(())))$. The well-typedness simply follows by the typing rule T-unit.

So assume some $\rho \in \mathcal{D}[\Xi]$ and $\gamma \in \mathcal{G}[\Gamma]_\rho$, we must show $\mathcal{E}[\text{Unit}]_\rho(\rho_1(\gamma_1(())), \rho_2(\gamma_2(()))) \equiv \mathcal{E}[\text{Unit}]_\rho((), ())$. Thus, we must show four things

1. $\bullet \mid \bullet \vdash () : \rho_1(\text{Unit})$
2. $\bullet \mid \bullet \vdash () : \rho_2(\text{Unit})$
3. $() \Downarrow \iff () \Downarrow$
4. $\forall v_1, v_2. () \Downarrow v_1 \wedge () \Downarrow v_2 \implies \mathcal{V}[\text{Unit}]_\rho(v_1, v_2)$

Given that $\rho_1(\text{Unit}) = \rho_2(\text{Unit}) = \text{Unit}$, the well-typedness once again simply follows from typing rule T-unit. (3) holds trivially. So let's show (4). Assume some v_1, v_2 , and $() \Downarrow v_1$ and $() \Downarrow v_2$, then we must show $\mathcal{V}[\text{Unit}]_\rho(v_1, v_2)$. Since $()$ is already a value, we have $v_1 = v_2 = ()$, meaning we must show $\mathcal{V}[\text{Unit}]_\rho((), ())$. Once again, the well-typedness holds by T-unit, and we have that $((), ()) \in \{(() , (()))\}$, so we are done. \square

$$\textbf{Lemma 10 (Cmpt-add). } \frac{\Xi \mid \Gamma \vdash e_1 \approx^{LR} e'_1 : \mathbb{Z} \quad \Xi \mid \Gamma \vdash e_2 \approx^{LR} e'_2 : \mathbb{Z}}{\Xi \mid \Gamma \vdash e_1 + e_2 \approx^{LR} e'_1 + e'_2 : \mathbb{Z}}$$

Proof. So assuming

$$\Xi \mid \Gamma \vdash e_1 \approx^{LR} e'_1 : \mathbb{Z} \tag{4.1}$$

$$\Xi \mid \Gamma \vdash e_2 \approx^{LR} e'_2 : \mathbb{Z} \tag{4.2}$$

we must show $\Xi \mid \Gamma \vdash e_1 + e_2 : \mathbb{Z} \wedge \Xi \mid \Gamma \vdash e'_1 + e'_2 : \mathbb{Z} \wedge$
 $\forall \rho \in \mathcal{D}[\Xi], \gamma \in \mathcal{G}[\Gamma]_\rho. \mathcal{E}[\mathbb{Z}]_\rho(\rho_1(\gamma_1(e_1 + e_2)), \rho_2(\gamma_2(e'_1 + e'_2)))$. From 4.1, we have
 $\Xi \mid \Gamma \vdash e_1 : \mathbb{Z}$, and from 4.2 we have $\Xi \mid \Gamma \vdash e_2 : \mathbb{Z}$. So by the T-add rule, we
get $\Xi \mid \Gamma \vdash e_1 + e_2 : \mathbb{Z}$. The well-typedness of $e'_1 + e'_2$ follows in the same way.
So let's assume some $\rho \in \mathcal{D}[\Xi]$ and $\gamma \in \mathcal{G}[\Gamma]_\rho$. We must show $\mathcal{E}[\mathbb{Z}]_\rho(\rho_1(\gamma_1(e_1 +$
 $e_2)), \rho_2(\gamma_2(e'_1 + e'_2)))$, which is equivalent to $\mathcal{E}[\mathbb{Z}]_\rho(\rho_1(\gamma_1(e_1)) + \rho_1(\gamma_1(e_2)), \rho_2(\gamma_2(e'_1)) +$
 $\rho_2(\gamma_2(e'_2)))$. Unfolding the expression interpretation, what we need to show is
the following four points

1. $\bullet \mid \bullet \vdash \rho_1(\gamma_1(e_1)) + \rho_1(\gamma_1(e_2)) : \rho_1(\mathbb{Z})$
2. $\bullet \mid \bullet \vdash \rho_2(\gamma_2(e'_1)) + \rho_2(\gamma_2(e'_2)) : \rho_2(\mathbb{Z})$
3. $\rho_1(\gamma_1(e_1)) + \rho_1(\gamma_1(e_2)) \Downarrow \iff \rho_2(\gamma_2(e'_1)) + \rho_2(\gamma_2(e'_2)) \Downarrow$
4. $\forall v_1, v_2. \rho_1(\gamma_1(e_1)) + \rho_1(\gamma_1(e_2)) \Downarrow v_1 \wedge \rho_2(\gamma_2(e'_1)) + \rho_2(\gamma_2(e'_2)) \Downarrow v_2 \implies$
 $\mathcal{V}[\mathbb{Z}]_\rho(v_1, v_2)$

To prove these points, we will need to use some more information. From the last
part of 4.1 instantiated with ρ and γ , we get

$$\bullet \mid \bullet \vdash \rho_1(\gamma_1(e_1)) : \rho_1(\mathbb{Z}) \tag{4.3}$$

$$\bullet \mid \bullet \vdash \rho_2(\gamma_2(e'_1)) : \rho_2(\mathbb{Z}) \tag{4.4}$$

$$\rho_1(\gamma_1(e_1)) \Downarrow \iff \rho_2(\gamma_2(e'_1)) \Downarrow \tag{4.5}$$

$$\forall v_1, v_2. \rho_1(\gamma_1(e_1)) \Downarrow v_1 \wedge \rho_2(\gamma_2(e'_1)) \Downarrow v_2 \implies \mathcal{V}[\mathbb{Z}]_\rho(v_1, v_2) \tag{4.6}$$

Likewise, from the last part of 4.2 instantiated with ρ and γ , we get

$$\bullet \mid \bullet \vdash \rho_1(\gamma_1(e_2)) : \rho_1(\mathbb{Z}) \tag{4.7}$$

$$\bullet \mid \bullet \vdash \rho_2(\gamma_2(e'_2)) : \rho_2(\mathbb{Z}) \tag{4.8}$$

$$\rho_1(\gamma_1(e_2)) \Downarrow \iff \rho_2(\gamma_2(e'_2)) \Downarrow \tag{4.9}$$

$$\forall v_1, v_2. \rho_1(\gamma_1(e_2)) \Downarrow v_1 \wedge \rho_2(\gamma_2(e'_2)) \Downarrow v_2 \implies \mathcal{V}[\mathbb{Z}]_\rho(v_1, v_2) \tag{4.10}$$

Now to prove the four points above. Given that \mathbb{Z} is a closed type, $\rho(\mathbb{Z}) = \mathbb{Z}$.
Thus, point 1 follows from 4.3, 4.7 and the T-add rule. Similarly, point 2 follows
from 4.4, 4.8 and the T-add rule.

To prove point 3, we need to show both ways of the implication. However, since they are similar, we just show one way here. So assume $\rho_1(\gamma_1(e_1)) + \rho_1(\gamma_1(e_2))$, we must show $\rho_2(\gamma_2(e'_1)) + \rho_2(\gamma_2(e'_2)) \Downarrow$. By definition of " \Downarrow ", we know that $\exists v_f. \rho_1(\gamma_1(e_1)) + \rho_1(\gamma_1(e_2)) \rightarrow^* v_f$. From this and two applications of corollary 2.1, we may conclude $\exists v_{f1}. \rho_1(\gamma_1(e_1)) \rightarrow^* v_{f1}$ and $\exists v_{f2}. \rho_1(\gamma_1(e_2)) \rightarrow^* v_{f2}$. But then $\rho_1(\gamma_1(e_1)) \Downarrow$ and $\rho_1(\gamma_1(e_2)) \Downarrow$. From this and 4.5 and 4.9, we get $\rho_2(\gamma_2(e'_1)) \Downarrow$ and $\rho_2(\gamma_2(e'_2)) \Downarrow$. Thus $\exists v_{f1'}. \rho_2(\gamma_2(e'_1)) \rightarrow^* v_{f1'}$ and $\exists v_{f2'}. \rho_2(\gamma_2(e'_2)) \rightarrow^* v_{f2'}$. By 4.10, we know that $v_{f1'}$ and $v_{f2'}$ are of type integer: $\overline{v_{f1'}}, \overline{v_{f2'}}$. So finally, we may conclude $\rho_2(\gamma_2(e'_1)) + \rho_2(\gamma_2(e'_2)) \rightarrow^* \overline{v_{f1'}} + \rho_2(\gamma_2(e'_2)) \rightarrow^* \overline{v_{f1'}} + \overline{v_{f2'}} \rightarrow^* \overline{v_{f1'}} + \overline{v_{f2'}} \rightarrow \overline{v_{f1'} + v_{f2'}}$, which was what we wanted to show.

The last part of the proof is point 4. So assume some v_1 and v_2 , and $\rho_1(\gamma_1(e_1)) + \rho_1(\gamma_1(e_2)) \Downarrow v_1$ and $\rho_2(\gamma_2(e'_1)) + \rho_2(\gamma_2(e'_2)) \Downarrow v_2$, we must show $\mathcal{V}[\![\mathbb{Z}]\!]_\rho(v_1, v_2)$. By two applications of corollary 2.1 on $\rho_1(\gamma_1(e_1)) + \rho_1(\gamma_1(e_2)) \rightarrow^* v_1$, we get $\rho_1(\gamma_1(e_1)) \Downarrow v_{11}$, $\rho_1(\gamma_1(e_2)) \Downarrow v_{12}$ and $v_{11} + v_{12} \rightarrow^* v_1$, for some v_{11} and v_{12} . Likewise, we get $\rho_2(\gamma_2(e'_1)) \Downarrow v_{21}$, $\rho_2(\gamma_2(e'_2)) \Downarrow v_{22}$ and $v_{21} + v_{22} \rightarrow^* v_2$. From $\rho_1(\gamma_1(e_1)) \Downarrow v_{11}$, $\rho_2(\gamma_2(e'_1)) \Downarrow v_{21}$, and 4.6 we get $\mathcal{V}[\![\mathbb{Z}]\!]_\rho(v_{11}, v_{21})$. Similarly, from $\rho_1(\gamma_1(e_2)) \Downarrow v_{12}$, $\rho_2(\gamma_2(e'_2)) \Downarrow v_{22}$, and 4.10, we get $\mathcal{V}[\![\mathbb{Z}]\!]_\rho(v_{12}, v_{22})$. We thusly know all our values are well-typed, and are in fact integers: $\overline{v_{11}}, \overline{v_{12}}, \overline{v_{21}}, \overline{v_{22}}$, and that $\overline{v_{11}} = \overline{v_{21}}$ and $\overline{v_{12}} = \overline{v_{22}}$. By T-add rule, we have that v_1 and v_2 are also well-typed, and are integers. We may then conclude $\overline{v_{11}} + \overline{v_{21}} \rightarrow \overline{v_{11} + v_{21}}$, and $\overline{v_{12}} + \overline{v_{22}} \rightarrow \overline{v_{12} + v_{22}}$, meaning $\overline{v_{11}} + \overline{v_{21}} = \overline{v_1}$ and $\overline{v_{12}} + \overline{v_{22}} = \overline{v_2}$. Finally, from this we get $\overline{v_1} = \overline{v_{11}} + \overline{v_{21}} = \overline{v_{21}} + \overline{v_{22}} = \overline{v_2}$, meaning $(\overline{v_1}, \overline{v_2}) \in \{(\overline{n}, \overline{n}) \mid \overline{n} \in \mathbb{Z}\}$, which was what we needed to show. \square

Lemma 11 (Cmpt-fst).
$$\frac{\Xi \mid \Gamma \vdash e \approx^{LR} e' : \tau_1 \times \tau_2}{\Xi \mid \Gamma \vdash \text{fst } e \approx^{LR} \text{fst } e' : \tau_1}$$

Proof. So assuming $\Xi \mid \Gamma \vdash e \approx^{LR} e' : \tau_1 \times \tau_2$, we must show (ignoring well-typedness): $\forall \rho \in \mathcal{D}[\![\Xi]\!], \gamma \in \mathcal{G}[\![\Gamma]\!]_\rho. \mathcal{E}[\![\tau_1]\!]_\rho(\rho_1(\gamma_1(\text{fst } e)), \rho_2(\gamma_2(\text{fst } e')))$. So assume some $\rho \in \mathcal{D}[\![\Xi]\!]$, and $\gamma \in \mathcal{G}[\![\Gamma]\!]_\rho$, we must show $\mathcal{E}[\![\tau_1]\!]_\rho(\rho_1(\gamma_1(\text{fst } e)), \rho_2(\gamma_2(\text{fst } e'))) \equiv \mathcal{E}[\![\tau_1]\!]_\rho(\text{fst } \rho_1(\gamma_1(e)), \text{fst } \rho_2(\gamma_2(e')))$. By definition of our expression interpretation, we must show two things

1. $\text{fst } \rho_1(\gamma_1(e)) \Downarrow \iff \text{fst } \rho_2(\gamma_2(e')) \Downarrow$
2. $\forall v_1, v_2. \text{fst } \rho_1(\gamma_1(e)) \Downarrow v_1 \wedge \text{fst } \rho_2(\gamma_2(e')) \Downarrow v_2 \implies \mathcal{V}[\![\tau_1]\!]_\rho(v_1, v_2)$

Note first that if we instantiate our initial assumption with ρ and γ , we get

$$\rho_1(\gamma_1(e)) \Downarrow \iff \rho_2(\gamma_2(e')) \Downarrow \quad (4.11)$$

$$\forall v_1, v_2. \rho_1(\gamma_1(e)) \Downarrow v_1 \wedge \rho_2(\gamma_2(e')) \Downarrow v_2 \implies \mathcal{V}[\![\tau_1 \times \tau_2]\!]_\rho(v_1, v_2) \quad (4.12)$$

Now to prove the two goals. For the first part, we show only the " \implies " direction. So assuming $\text{fst } \rho_1(\gamma_1(e)) \Downarrow$, we must show $\text{fst } \rho_2(\gamma_2(e')) \Downarrow$. By definition of

\Downarrow , we know $\exists v. \text{fst } \rho_1(\gamma_1(e)) \Downarrow v$. By corollary 2.1 on our assumption, we know $\exists v_f. \rho_1(\gamma_1(e)) \rightarrow^* v_f \wedge \text{fst } v_f \rightarrow^* v$. By 4.11 we then get $\exists v_{f'}. \rho_2(\gamma_2(e')) \Downarrow v_{f'}$. From this and by 4.12 we get $\mathcal{V}[\tau_1 \times \tau_2]_\rho(v_f, v_{f'})$. Thus, we know $v_{f'} = (y_{21}, y_{22})$, for some values y_{21}, y_{22} , and we may conclude $\text{fst } \rho_2(\gamma_2(e')) \rightarrow^* \text{fst } v_{f'} = \text{fst } (w_{21}, w_{22}) \rightarrow w_{21}$, where the last step is justified by the E-fst rule. Thus, $\text{fst } \rho_2(\gamma_2(e')) \Downarrow$.

Next we prove the second part. So assume some v_1, v_2 , and $\text{fst } \rho_1(\gamma_1(e)) \Downarrow v_1$ and $\text{fst } \rho_2(\gamma_2(e')) \Downarrow v_2$. We must show $\mathcal{V}[\tau_1]_\rho(v_1, v_2)$. By corollary 2.1, we get $\exists v_f. \rho_1(\gamma_1(e)) \rightarrow^* v_f \wedge \text{fst } v_f \rightarrow^* v_1$, and $\exists v_{f'}. \rho_1(\gamma_1(e')) \rightarrow^* v_{f'} \wedge \text{fst } v_{f'} \rightarrow^* v_2$. By 4.12 with this, we know $\mathcal{V}[\tau_1 + \tau_2]_\rho(v_f, v_{f'})$. Thus $v_f = (w_{11}, w_{12})$ and $v_{f'} = (w_{21}, w_{22})$, and $\mathcal{V}[\tau_1]_\rho(w_{11}, w_{21})$. So since $\text{fst } \rho_1(\gamma_1(e)) \rightarrow^* \text{fst } v_f = \text{fst } (w_{11}, w_{12}) \rightarrow w_{11}$, then $w_{11} = v_1$. Likewise $w_{21} = v_2$, then $\mathcal{V}[\tau_1]_\rho(v_1, v_2)$. \square

Lemma 12 (Cmpt-match).

$$\frac{\Xi \mid \Gamma \vdash e_1 \approx^{LR} e'_1 : \tau_1 + \tau_2 \quad \Xi \mid \Gamma, x : \tau_1 \vdash e_2 \approx^{LR} e'_2 : \tau \quad \Xi \mid \Gamma, x : \tau_2 \vdash e_3 \approx^{LR} e'_3 : \tau}{\Xi \mid \Gamma \vdash \text{match } e_1 \text{ with } \text{inj}_1 x \Rightarrow e_2 \mid \text{inj}_2 x \Rightarrow e_3 \text{ end} \approx^{LR} \text{match } e'_1 \text{ with } \text{inj}_1 x \Rightarrow e'_2 \mid \text{inj}_2 x \Rightarrow e'_3 \text{ end} : \tau}$$

Proof. First, we introduce the following notation: $e = \text{match } e_1 \text{ with } \text{inj}_1 x \Rightarrow e_2 \mid \text{inj}_2 x \Rightarrow e_3 \text{ end}$, and $e' = \text{match } e'_1 \text{ with } \text{inj}_1 x \Rightarrow e'_2 \mid \text{inj}_2 x \Rightarrow e'_3 \text{ end}$. So we assume

$$\Xi \mid \Gamma \vdash e_1 \approx^{LR} e'_1 : \tau_1 + \tau_2 \quad (4.13)$$

$$\Xi \mid \Gamma, x : \tau_1 \vdash e_2 \approx^{LR} e'_2 : \tau \quad (4.14)$$

$$\Xi \mid \Gamma, x : \tau_2 \vdash e_3 \approx^{LR} e'_3 : \tau \quad (4.15)$$

and must show (ignoring well-typedness) $\forall \rho \in \mathcal{D}[\Xi], \gamma \in \mathcal{G}[\Gamma]_\rho. \mathcal{E}[\tau]_\rho(e, e')$. So assume some $\rho \in \mathcal{D}[\Xi]$, and $\gamma \in \mathcal{G}[\Gamma]_\rho$, then we must show $\mathcal{E}[\tau]_\rho(\rho_1(\gamma_1(e)), \rho_1(\gamma_1(e')))$.

We again introduce the following notation $f = \text{match } \rho_1(\gamma_1(e_1)) \text{ with } \text{inj}_1 x \Rightarrow \rho_1(\gamma_1(e_2)) \mid \text{inj}_2 x \Rightarrow \rho_1(\gamma_1(e_3)) \text{ end}$, and $f' = \text{match } \rho_2(\gamma_2(e'_1)) \text{ with } \text{inj}_1 x \Rightarrow \rho_2(\gamma_2(e'_2)) \mid \text{inj}_2 x \Rightarrow \rho_2(\gamma_2(e'_3)) \text{ end}$, and note that $\rho_1(\gamma_1(e)) = f$ and $\rho_2(\gamma_2(e')) = f'$, so that we must show. $\mathcal{E}[\tau]_\rho(f, f')$. This corresponds to showing

1. $f \Downarrow \iff f' \Downarrow$
2. $\forall v_1, v_2. f \Downarrow v_1 \wedge f' \Downarrow v_2 \implies \mathcal{V}[\tau]_\rho(v_1, v_2)$

By 4.13 instantiated with ρ, γ , we get

$$\rho_1(\gamma_1(e_1)) \Downarrow \iff \rho_2(\gamma_2(e'_1)) \Downarrow \quad (4.16)$$

$$\forall v_1, v_2. \rho_1(\gamma_1(e_1)) \Downarrow v_1 \wedge \rho_2(\gamma_2(e'_1)) \Downarrow v_2 \implies \mathcal{V}[\tau_1 + \tau_2]_\rho(v_1, v_2) \quad (4.17)$$

Now let's show the first part. We show only the \implies direction. So assume $f \Downarrow \equiv \exists v_{f_1}. f \Downarrow v_{f_1}$, we must show $f' \Downarrow$. By corollary 2.1, we get $\exists v_{1_f}. \rho_1(\gamma_1(e_1)) \rightarrow^* v_{1_f} \wedge \text{match } v_{1_f} \text{ with } \text{inj}_1 x \Rightarrow \rho_1(\gamma_1(e_2)) \mid \text{inj}_2 x \Rightarrow \rho_1(\gamma_1(e_3)) \text{ end} \rightarrow^*$

v_{f_1} . Now, from 4.16 we get $\rho_2(\gamma_2(e'_1)) \Downarrow$. So $\exists v_{2_f}. \rho_2(\gamma_2(e'_1)) \rightarrow^* v_{2_f}$. By 4.17 we know $\mathcal{V}[\tau_1 + \tau_2]_\rho(v_{1_f}, v_{2_f})$. Thus, $(v_{1_f}, v_{2_f}) \in \{(\text{inj}_1 w_1, \text{inj}_1 w_2) \mid \mathcal{V}[\tau_1]_\rho(w_1, w_2)\} \vee (v_{1_f}, v_{2_f}) \in \{(\text{inj}_2 w_1, \text{inj}_2 w_2) \mid \mathcal{V}[\tau_2]_\rho(w_1, w_2)\}$. We do case distinction on the " \vee ", but show only the first case as they are similar. So assume $(v_{1_f}, v_{2_f}) \in \{(\text{inj}_1 w_1, \text{inj}_1 w_2) \mid \mathcal{V}[\tau_1]_\rho(w_1, w_2)\}$. Then $v_{1_f} = \text{inj}_1 w_1$, $v_{2_f} = \text{inj}_1 w_2$, and $\mathcal{V}[\tau_1]_\rho(w_1, w_2)$. So we have
 $f \rightarrow^* \text{match } \text{inj}_1 w_1 \text{ with } \text{inj}_1 x \Rightarrow \rho_1(\gamma_1(e_2)) \mid \text{inj}_2 x \Rightarrow \rho_1(\gamma_1(e_3)) \text{ end} \rightarrow \rho_1(\gamma_1(e_2))[w_1/x] \rightarrow^* v_{f_1}$, and
 $f' \rightarrow^* \text{match } \text{inj}_1 w_2 \text{ with } \text{inj}_1 x \Rightarrow \rho_2(\gamma_2(e'_2)) \mid \text{inj}_2 x \Rightarrow \rho_2(\gamma_2(e'_3)) \text{ end} \rightarrow \rho_2(\gamma_2(e'_2))[w_2/x]$. By **do substitution lemma** $\rho_1(\gamma_1(e_2))[w_1/x] = \rho_1(\gamma_1[x \mapsto (w_1, w_2)](e_2))$. Now, from 4.14 instantiated with $\rho, \gamma' = \gamma[x \mapsto (w_1, w_2)]$ we get

$$\rho_1(\gamma'_1(e_2)) \Downarrow \iff \rho_2(\gamma'_2(e'_2)) \Downarrow \quad (4.18)$$

$$\forall v_1, v_2. \rho_1(\gamma'_1(e_2)) \Downarrow v_1 \wedge \rho_2(\gamma'_2(e'_2)) \Downarrow v_2 \implies \mathcal{V}[\tau]_\rho(v_1, v_2) \quad (4.19)$$

So by 4.18, we get $\rho_2(\gamma_2[x \mapsto (w_1, w_2)](e'_2)) \Downarrow$, which again, by **do substitution lemma** gives us $\rho_2(\gamma_2(e'_2))[w_2/x] \Downarrow$. Thus $f' \Downarrow$, which was what we wanted.

Next we show the second part. So assume v, v' , and $f \Downarrow v_1$ and $f' \Downarrow v_2$, we must show $\mathcal{V}[\tau]_\rho(v_1, v_2)$. By corollary 2.1, we get $\exists v_1. \rho_1(\gamma_1(e_1)) \rightarrow^* v_1 \wedge \text{match } v_1 \text{ with } \text{inj}_1 x \Rightarrow \rho_1(\gamma_1(e_2)) \mid \text{inj}_2 x \Rightarrow \rho_1(\gamma_1(e_3)) \text{ end} \rightarrow^* v$. Likewise $\exists v_2. \rho_2(\gamma_2(e'_1)) \rightarrow^* v_2 \wedge \text{match } v_2 \text{ with } \text{inj}_1 x \Rightarrow \rho_2(\gamma_2(e'_2)) \mid \text{inj}_2 x \Rightarrow \rho_2(\gamma_2(e'_3)) \text{ end} \rightarrow^* v'$. By 4.17 we know $\mathcal{V}[\tau_1 + \tau_2]_\rho(v_1, v_2)$. Thus, $(v_1, v_2) \in \{(\text{inj}_1 w_1, \text{inj}_1 w_2) \mid \mathcal{V}[\tau_1]_\rho(w_1, w_2)\} \vee (v_1, v_2) \in \{(\text{inj}_2 w_1, \text{inj}_2 w_2) \mid \mathcal{V}[\tau_2]_\rho(w_1, w_2)\}$. We do case distinction on the " \vee ", but show only the second case as they are similar. So assume $(v_1, v_2) \in \{(\text{inj}_2 w_1, \text{inj}_2 w_2) \mid \mathcal{V}[\tau_2]_\rho(w_1, w_2)\}$. Then $v_1 = \text{inj}_2 w_1$, $v_2 = \text{inj}_2 w_2$, and $\mathcal{V}[\tau_2]_\rho(w_1, w_2)$. So we have
 $f \rightarrow^* \text{match } \text{inj}_2 w_1 \text{ with } \text{inj}_1 x \Rightarrow \rho_1(\gamma_1(e_2)) \mid \text{inj}_2 x \Rightarrow \rho_1(\gamma_1(e_3)) \text{ end} \rightarrow \rho_1(\gamma_1(e_3))[w_1/x] \rightarrow^* v$, and
 $f' \rightarrow^* \text{match } \text{inj}_2 w_2 \text{ with } \text{inj}_1 x \Rightarrow \rho_2(\gamma_2(e'_2)) \mid \text{inj}_2 x \Rightarrow \rho_2(\gamma_2(e'_3)) \text{ end} \rightarrow \rho_2(\gamma_2(e'_3))[w_2/x] \rightarrow^* v'$. By **do substitution lemma** $\rho_1(\gamma_1(e_3))[w_1/x] = \rho_1(\gamma_1[x \mapsto w_1](e_3))$, and $\rho_2(\gamma_2(e'_3))[w_2/x] = \rho_2(\gamma_2[x \mapsto w_2](e'_3))$. Now, from 4.15 instantiated with $\rho, \gamma' = \gamma[x \mapsto (w_1, w_2)]$ we get

$$\rho_1(\gamma'_1(e_3)) \Downarrow \iff \rho_2(\gamma'_2(e'_3)) \Downarrow \quad (4.20)$$

$$\forall v_1, v_2. \rho_1(\gamma'_1(e_3)) \Downarrow v_1 \wedge \rho_2(\gamma'_2(e'_3)) \Downarrow v_2 \implies \mathcal{V}[\tau]_\rho(v_1, v_2) \quad (4.21)$$

So by 4.21 we get $\mathcal{V}[\tau]_\rho(v, v')$. □

Lemma 13 (Cmpt-rec).
$$\frac{\Xi \mid \Gamma, f : \tau_1 \rightarrow \tau_2, x : \tau_1 \vdash e \approx^{LR} e' : \tau_2}{\Xi \mid \Gamma \vdash \text{rec } f(x) := e \approx^{LR} \text{rec } f(x) := e' : \tau_1 \rightarrow \tau_2}$$

Proof. **do proof** □

Lemma 14 (Cmpt-app).
$$\frac{\Xi \mid \Gamma \vdash e_1 \approx^{LR} e'_1 : \tau_1 \rightarrow \tau_2 \quad \Xi \mid \Gamma \vdash e_2 \approx^{LR} e'_2 : \tau_1}{\Xi \mid \Gamma \vdash e_1 e_2 \approx^{LR} e'_1 e'_2 : \tau_2}$$

Proof. ►do proof◄ □

Lemma 15 (Cmpt-Tlam).
$$\frac{\Xi, X \mid \Gamma \vdash e \approx^{LR} e' : \tau}{\Xi \mid \Gamma \vdash \Lambda e \approx^{LR} \Lambda e' : \forall X. \tau}$$

Proof. ►write in proof◄ □

Lemma 16 (Cmpt-Tapp).
$$\frac{\Xi \mid \Gamma \vdash e \approx^{LR} e' : \forall X. \tau}{\Xi \mid \Gamma \vdash e _ \approx^{LR} e' _ : \tau[\tau'/X]}$$

Proof. ►write in proof◄ □

Lemma 17 (Adequacy of LR). $\bullet \mid \bullet \vdash e \approx^{LR} e' : \text{Unit} \implies e \Downarrow \iff e' \Downarrow$

Proof. ►do proof◄ □

Theorem 18 (Fundamental Theorem). $\Xi \mid \Gamma \vdash e : \tau \implies \Xi \mid \Gamma \vdash e \approx^{LR} e : \tau$

Proof. Since LR is a congruence relation, it follows directly from lemma 6. □

Theorem 19. $\Xi \mid \Gamma \vdash e \approx^{LR} e' : \tau \implies \Xi \mid \Gamma \vdash e \approx^{ctx} e' : \tau$

Proof. ►write properly◄ LR is a congruence relation, LR is an adequate relation, and all expressions in it are well-formed by definition. It thusly follows from theorem 8. □

Chapter 5

Examples of Application of Contextual Equivalence

►draft◄

Chapter 6

Comparison to Other Work and Ideas for Future Work

►draft◄

Chapter 7

Conclusion

►conclude on the problem statement from the introduction◄

Acknowledgments



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Appendix A

The Technical Details



A.1 Congruence Rules for Alternative Definition of Contextual Equivalence

$$\begin{array}{c}
\text{CNG-VAR} \\
\frac{(x : \tau) \in \Gamma}{\Xi \mid \Gamma \vdash x \approx x : \tau} \\
\\
\text{CNG-UNIT} \\
\frac{}{\Xi \mid \Gamma \vdash () \approx () : \text{Unit}} \\
\\
\text{CNG-INT} \\
\frac{}{\Xi \mid \Gamma \vdash \bar{n} \approx \bar{n} : \mathbb{Z}} \\
\\
\text{CNG-ADD} \\
\frac{\Xi \mid \Gamma \vdash e_1 \approx e'_1 : \mathbb{Z} \quad \Xi \mid \Gamma \vdash e_2 \approx e'_2 : \mathbb{Z}}{\Xi \mid \Gamma \vdash e_1 + e_2 \approx e'_1 + e'_2 : \mathbb{Z}} \\
\\
\text{CNG-SUB} \\
\frac{\Xi \mid \Gamma \vdash e_1 \approx e'_1 : \mathbb{Z} \quad \Xi \mid \Gamma \vdash e_2 \approx e'_2 : \mathbb{Z}}{\Xi \mid \Gamma \vdash e_1 - e_2 \approx e'_1 - e'_2 : \mathbb{Z}} \\
\\
\text{CNG-LE} \\
\frac{\Xi \mid \Gamma \vdash e_1 \approx e'_1 : \mathbb{Z} \quad \Xi \mid \Gamma \vdash e_2 \approx e'_2 : \mathbb{Z}}{\Xi \mid \Gamma \vdash e_1 \leq e_2 \approx e'_1 \leq e'_2 : \mathbb{B}} \\
\\
\text{CNG-LT} \\
\frac{\Xi \mid \Gamma \vdash e_1 \approx e'_1 : \mathbb{Z} \quad \Xi \mid \Gamma \vdash e_2 \approx e'_2 : \mathbb{Z}}{\Xi \mid \Gamma \vdash e_1 < e_2 \approx e'_1 < e'_2 : \mathbb{B}} \\
\\
\text{CNG-EQ} \\
\frac{\Xi \mid \Gamma \vdash e_1 \approx e'_1 : \mathbb{Z} \quad \Xi \mid \Gamma \vdash e_2 \approx e'_2 : \mathbb{Z}}{\Xi \mid \Gamma \vdash e_1 = e_2 \approx e'_1 = e'_2 : \mathbb{B}} \\
\\
\text{CNG-TRUE} \\
\frac{}{\Xi \mid \Gamma \vdash \text{true} \approx \text{true} : \mathbb{B}} \\
\\
\text{CNG-FALSE} \\
\frac{}{\Xi \mid \Gamma \vdash \text{false} \approx \text{false} : \mathbb{B}} \\
\\
\text{CNG-IF} \\
\frac{\Xi \mid \Gamma \vdash e_1 \approx e'_1 : \mathbb{B} \quad \Xi \mid \Gamma \vdash e_2 \approx e'_2 : \tau \quad \Xi \mid \Gamma \vdash e_3 \approx e'_3 : \tau}{\Xi \mid \Gamma \vdash \text{if } e_1 \text{ then } e_2 \text{ else } e_3 \approx \text{if } e'_1 \text{ then } e'_2 \text{ else } e'_3 : \tau} \\
\\
\text{CNG-PAIR} \\
\frac{\Xi \mid \Gamma \vdash e_1 \approx e'_1 : \tau_1 \quad \Xi \mid \Gamma \vdash e_2 \approx e'_2 : \tau_2}{\Xi \mid \Gamma \vdash (e_1, e_2) \approx (e'_1, e'_2) : \tau_1 \times \tau_2} \\
\\
\text{CNG-FST} \\
\frac{\Xi \mid \Gamma \vdash e \approx e' : \tau_1 \times \tau_2}{\Xi \mid \Gamma \vdash \text{fst } e \approx \text{fst } e' : \tau_1} \\
\\
\text{CNG-SND} \\
\frac{\Xi \mid \Gamma \vdash e \approx e' : \tau_1 \times \tau_2}{\Xi \mid \Gamma \vdash \text{snd } e \approx \text{snd } e' : \tau_2} \\
\\
\text{CNG-INJ1} \\
\frac{\Xi \mid \Gamma \vdash e \approx e' : \tau_1}{\Xi \mid \Gamma \vdash \text{inj}_1 e \approx \text{inj}_1 e' : \tau_1 + \tau_2} \\
\\
\text{CNG-INJ2} \\
\frac{\Xi \mid \Gamma \vdash e \approx e' : \tau_2}{\Xi \mid \Gamma \vdash \text{inj}_2 e \approx \text{inj}_2 e' : \tau_1 + \tau_2}
\end{array}$$

$$\begin{array}{c}
\text{CNG-MATCH} \\
\frac{\Xi \mid \Gamma \vdash e_1 \approx e'_1 : \tau_1 + \tau_2 \quad \Xi \mid \Gamma, x : \tau_1 \vdash e_2 \approx e'_2 : \tau \quad \Xi \mid \Gamma, x : \tau_2 \vdash e_3 \approx e'_3 : \tau}{\Xi \mid \Gamma \vdash \text{match } e_1 \text{ with } \text{inj}_1 x \Rightarrow e_2 \mid \text{inj}_2 x \Rightarrow e_3 \text{ end} \approx \text{match } e'_1 \text{ with } \text{inj}_1 x \Rightarrow e'_2 \mid \text{inj}_2 x \Rightarrow e'_3 \text{ end} : \tau} \\
\\
\text{CNG-REC} \\
\frac{\Xi \mid \Gamma, f : \tau_1 \rightarrow \tau_2, x : \tau_1 \vdash e \approx e' : \tau_2}{\Xi \mid \Gamma \vdash \text{rec } f(x) := e \approx \text{rec } f(x) := e' : \tau_1 \rightarrow \tau_2} \\
\\
\text{CNG-APP} \\
\frac{\Xi \mid \Gamma \vdash e_1 \approx e'_1 : \tau_1 \rightarrow \tau_2 \quad \Xi \mid \Gamma \vdash e_2 \approx e'_2 : \tau_1}{\Xi \mid \Gamma \vdash e_1 e_2 \approx e'_1 e'_2 : \tau_2} \\
\\
\begin{array}{cc}
\text{CNG-TLAM} & \text{CNG-TAPP} \\
\frac{\Xi, X \mid \Gamma \vdash e \approx e' : \tau}{\Xi \mid \Gamma \vdash \Lambda e \approx \Lambda e' : \forall X. \tau} & \frac{\Xi \mid \Gamma \vdash e \approx e' : \forall X. \tau}{\Xi \mid \Gamma \vdash e _ \approx e' _ : \tau[\tau'/X]}
\end{array}
\end{array}$$