PHYS465: Statistical Data Analysis in Physics

Week 2: Hypothesis Testing, Common distributions, Model testing

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Module Structure

'*model testing*': does our data agree with our model?

Week 11: Fitting a model to data; estimating parameters

Week 12: Hypothesis testing; the likelihood; estimating uncertainties

Week 13: Posterior sampling; Bayesian statistics

'*data driven*': what does our data tell us?

Week 14: Clustering and Classification algorithms

Week 15: Machine learning techniques

Week 12: Previous session

Last week we learnt:

- Our measurements drawn from an underlying continuous distribution
  - The shape of this distribution is dependent on our experiment
  - The probability of our data can be described through a PDF
- Given a model we can find the best-fit parameters using least-squares fitting
- This is generalised to χ^2-fitting in the presence of gaussian uncertainties
- A good model will have $\chi^2_{red} \sim 1$
  - Where $\chi^2_{red} = \chi^2 / n_{dof}$
  - $\chi^2_{red} > 1$ suggests the need for more complicated models; $\chi^2_{red} < 1$ is the reverse
- Our knowledge of model parameters is expressed through confidence intervals
- <u>Question: how do we measure a confidence interval or parameter uncertainty?</u>

Week 12: Learning aims

Today we will introduce

- Probabilities and how they relate to measurements
- <u>The Null Hypothesis</u>
  - And how to test it
- <u>How our experiment will change how the data is distributed</u>
- How to test which distribution is correct
- How do we estimate confidence intervals?
- <u>How to select which model is most likely</u>

Practical examples on Friday!

Revision: Probabilities

Given an experiment:

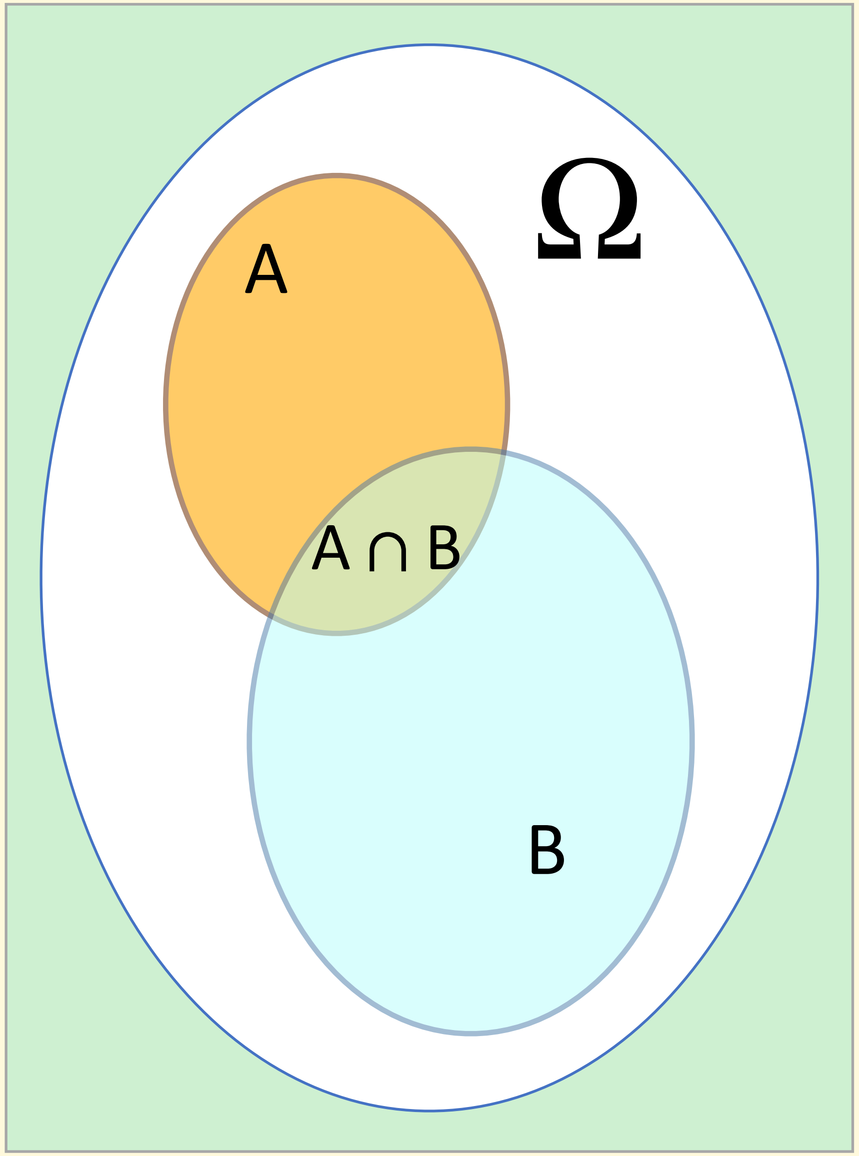
- Ω is the list of all possible outcomes

Axioms:

- $0 \leq P(A) \leq 1$
- $P(\Omega) = 1$
- $P(A_1 \cup A_2 \cup A_3 \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$

Consequences:

- $P(A) + P(A^c) = 1$
- $P(A \cup B) = P(A) + P(B) + P(A \cap B)$
- $P(A \cap B) = P(A) \times P(B)$ if A, B are independent



Conditional Probabilities

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

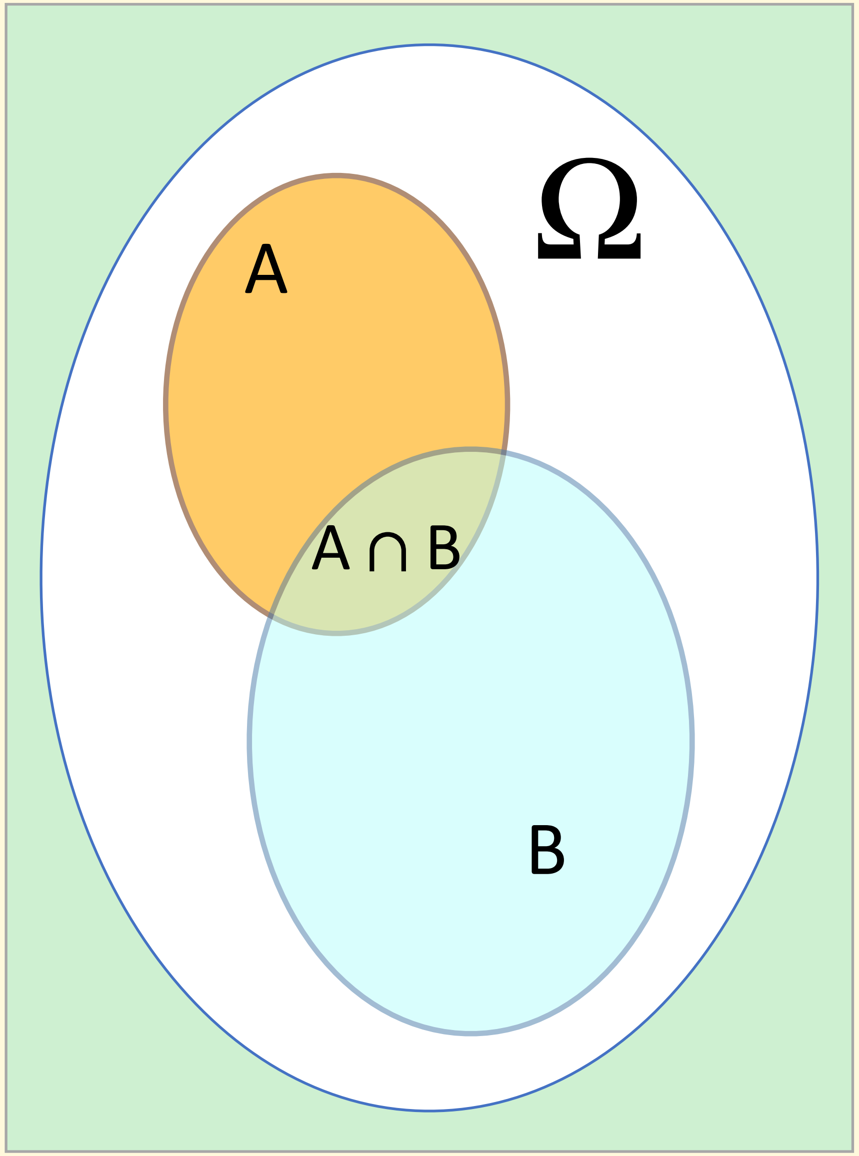
The probability of A given B (an event or condition)

when θ is a model, or parameter

$P(A \mid \theta)$ is called the ***likelihood:*** $\mathscr{L}(\theta \mid A)$

"the ***likelihood*** of obtaining the data 'A' given the model"

Plenty more on this with Bayesian statistics next week



Hypothesis testing

How good is the model?

Fundamental question: Is the model that we are testing a good fit to our measured data?

Is there a more likely alternative?

Null Hypothesis: Our starting assumption

Can be either parameter values or choice of model

In practice: Compares our assumption (or prediction) with experimental measurements
e.g. does the luminosity distribution of galaxies match a Schechter function?

Outcome: Make a statement on the probability of obtaining our result (see confidence level slides).

Null Hypothesis testing

e.g. Given a model $f(\theta)$, with parameter θ, is the value of θ in a set of possible values Θ_0?

H_0 : **The Null Hypothesis:** $\theta \in \Theta_0$

H_1 : An Alternative Hypothesis:

- $\theta \in \Theta_1$ where $\Theta_0 \cap \Theta_1 = 0$

α : threshold of rejection (e.g. 0.05)

With new observations X, such that $p(X, \theta)$

$p(X) < \alpha$ reject H_0 and accept H_1

(N.B. This is does not mean accept another model)

$p(X) > \alpha$ no evidence to reject H_0

(N.B. This is not the same as accepting H_0)

		Outcome	
		H_0 not rejected	H_0 rejected
Truth	H_0 is true	Probability of this: $1 - \alpha$	Type I error Will happen α % of the time
	H_1 is true	Type II error Happens $1 - \beta$ % of the time	$P_{H_1}(H_0$ is rejected$)$ $= \beta$

A *good* experiment is defined around these requirements

Confidence Intervals

Given a probability threshold, what is the allowed range of the parameter:

$$\hat{p} - \delta p \leq P_0 \leq \hat{p} + \delta p$$

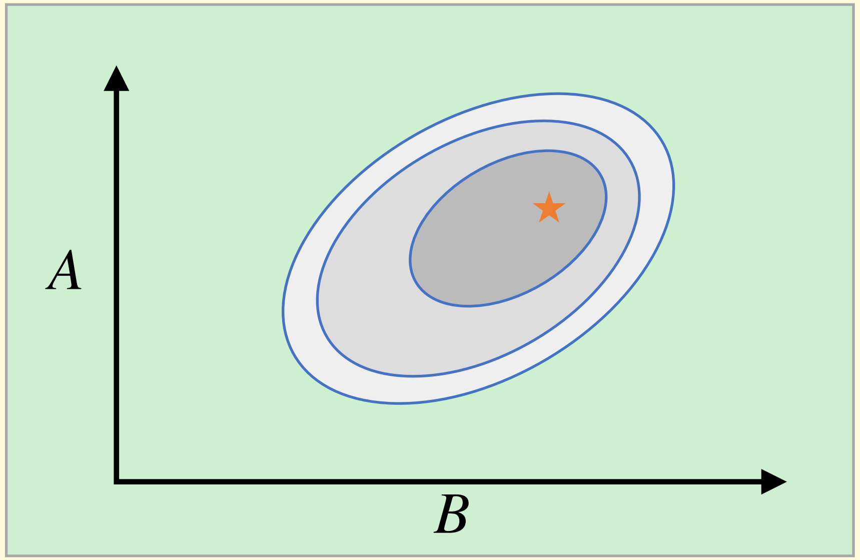
Interpretation: if we repeat our experiment, how often will P_0 be within our quoted range

For multiple variables (A, B):

An error ellipse containing δp

of the probability

Typical values are 68%, 90%, 95%.



Key Assumption:

The probability, p, needed for **Hypothesis testing** and **confidence intervals** depend on how the data and model are distributed

Are they both drawn from a Gaussian distribution?

Key Distributions: Gaussian

Aka a normal

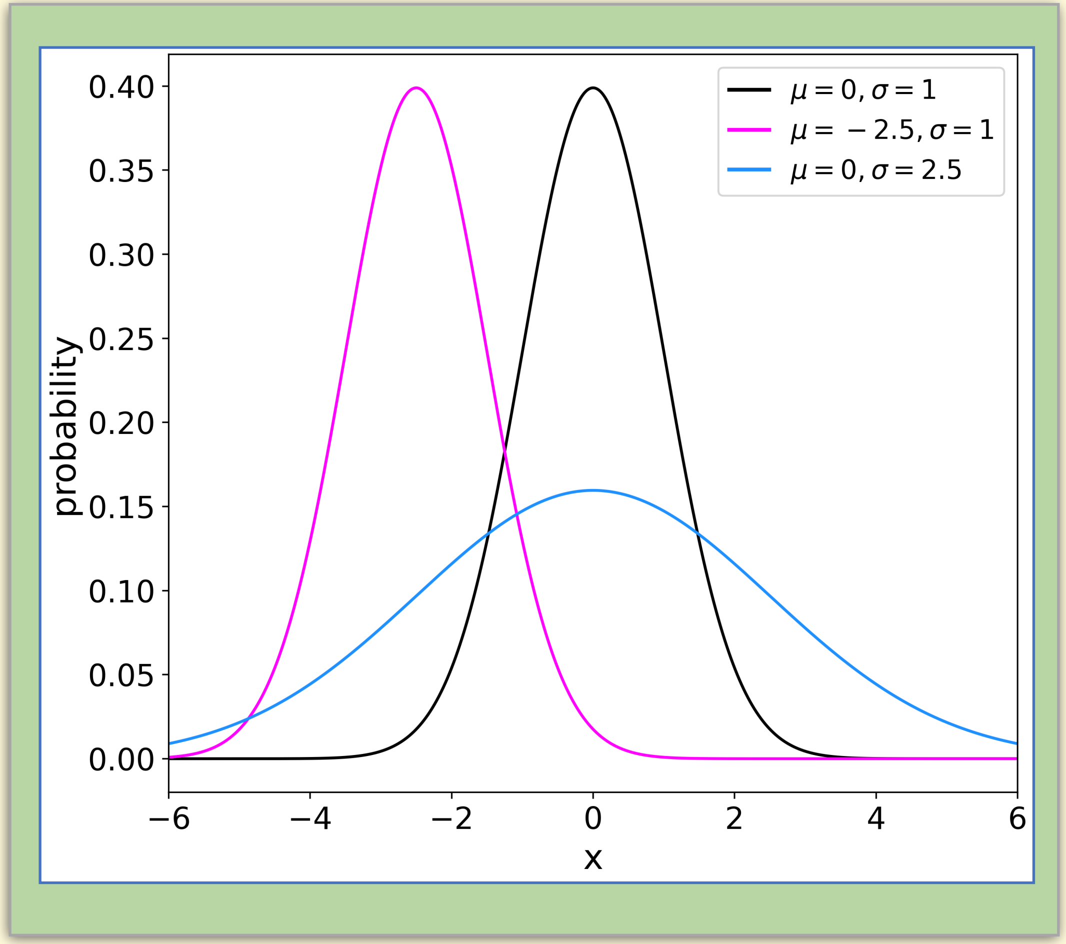
$$P(x) = \frac{1}{\sigma\sqrt{(2\pi)}} \exp\{-(x-\mu)^2/2\sigma^2\}$$

The limiting case ($\rightarrow \infty$) of most other distributions

Expectation value: $\text{mean} = \mu$

Variance: $\sigma^2 = \sigma^2$

$$P(-t\sigma < x_0 < t\sigma) = \frac{1}{\sqrt{2}} \int_{-t}^{t} e^{-z^2/2} dz$$



Common assumption across physics

Key Distributions: Binomial

Fixed number of outcomes

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

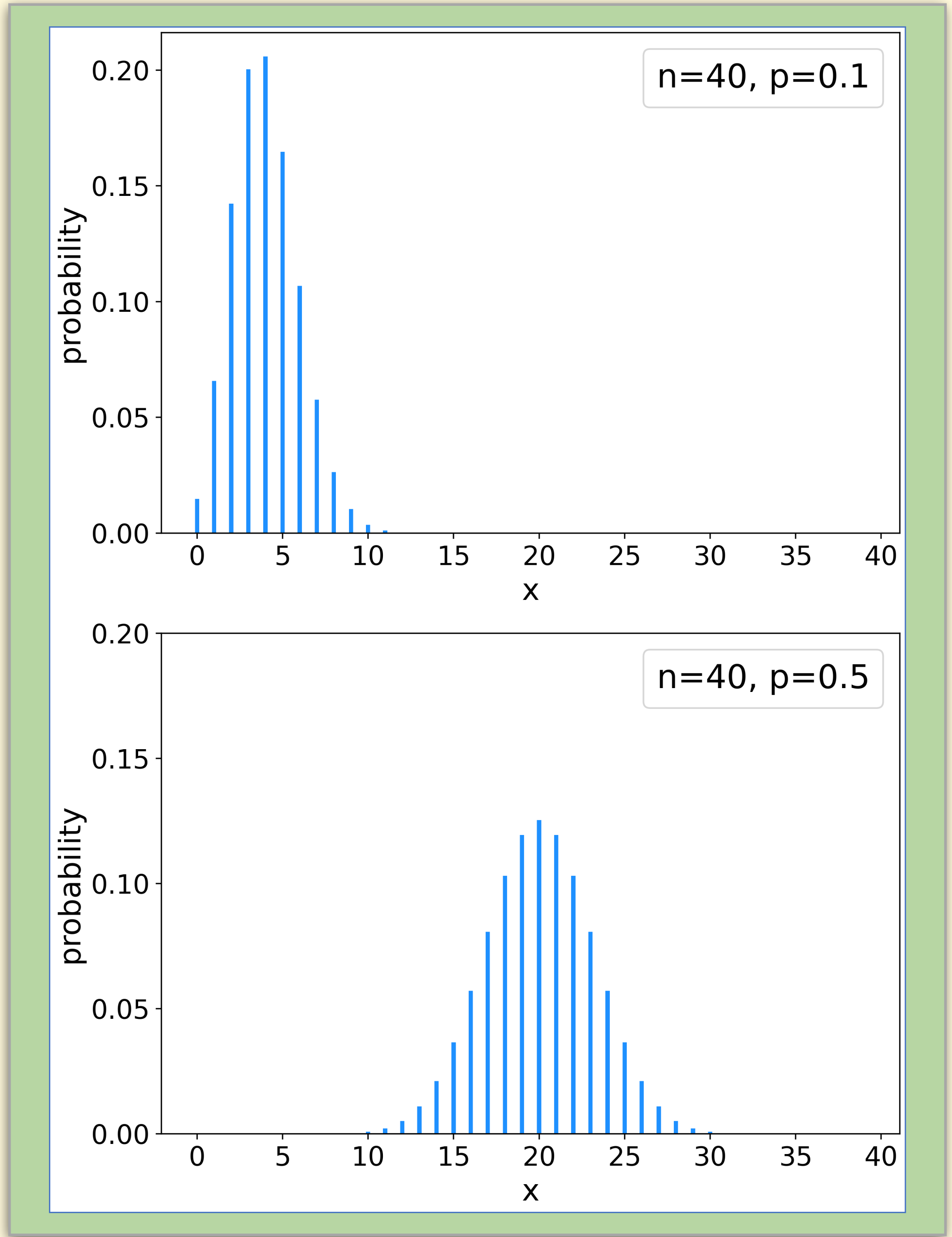
draw n events from k possibilities each with probability p

Expectation value: $\text{mean} = np$

Variance: $np(1-p)$

=> Gaussian distribution as $N \to \infty$

Example: coin tossing



NB: Other common distributions include 'uniform', 'gamma', 'log-normal'…

Key Distributions: Poisson

The counting distribution

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

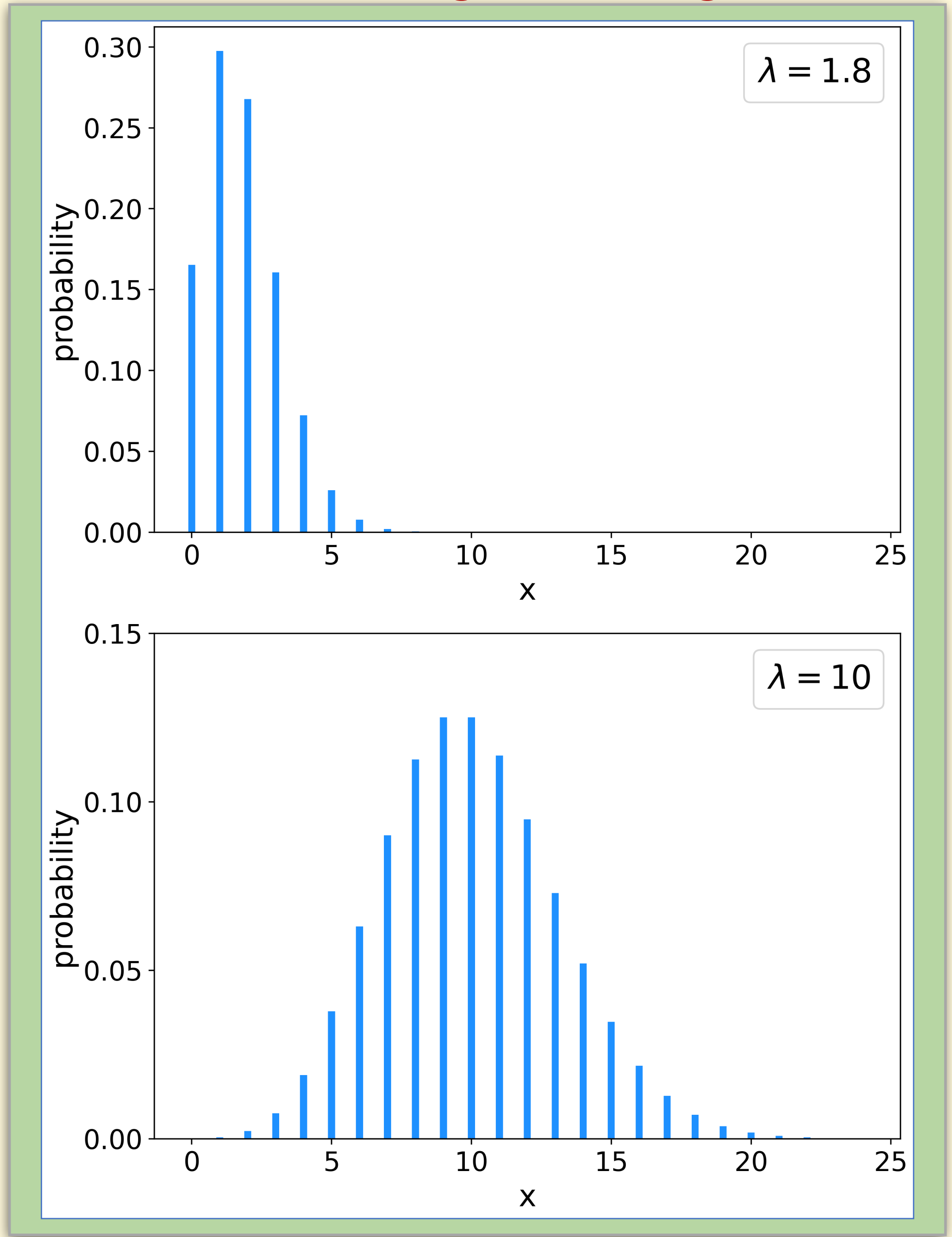
observe k events drawn from λ possibilities

Expectation value: $\text{mean} = \lambda$

Variance: $\sigma^2 = \lambda$

=> Gaussian distribution as $k \to \infty$

Example: histogram (or photon) counting



Key Distributions: χ^2

Related to the Gamma distribution

$$P \propto (\chi^2)^{\frac{\nu-2}{2}} \exp(-\chi^2/2)$$

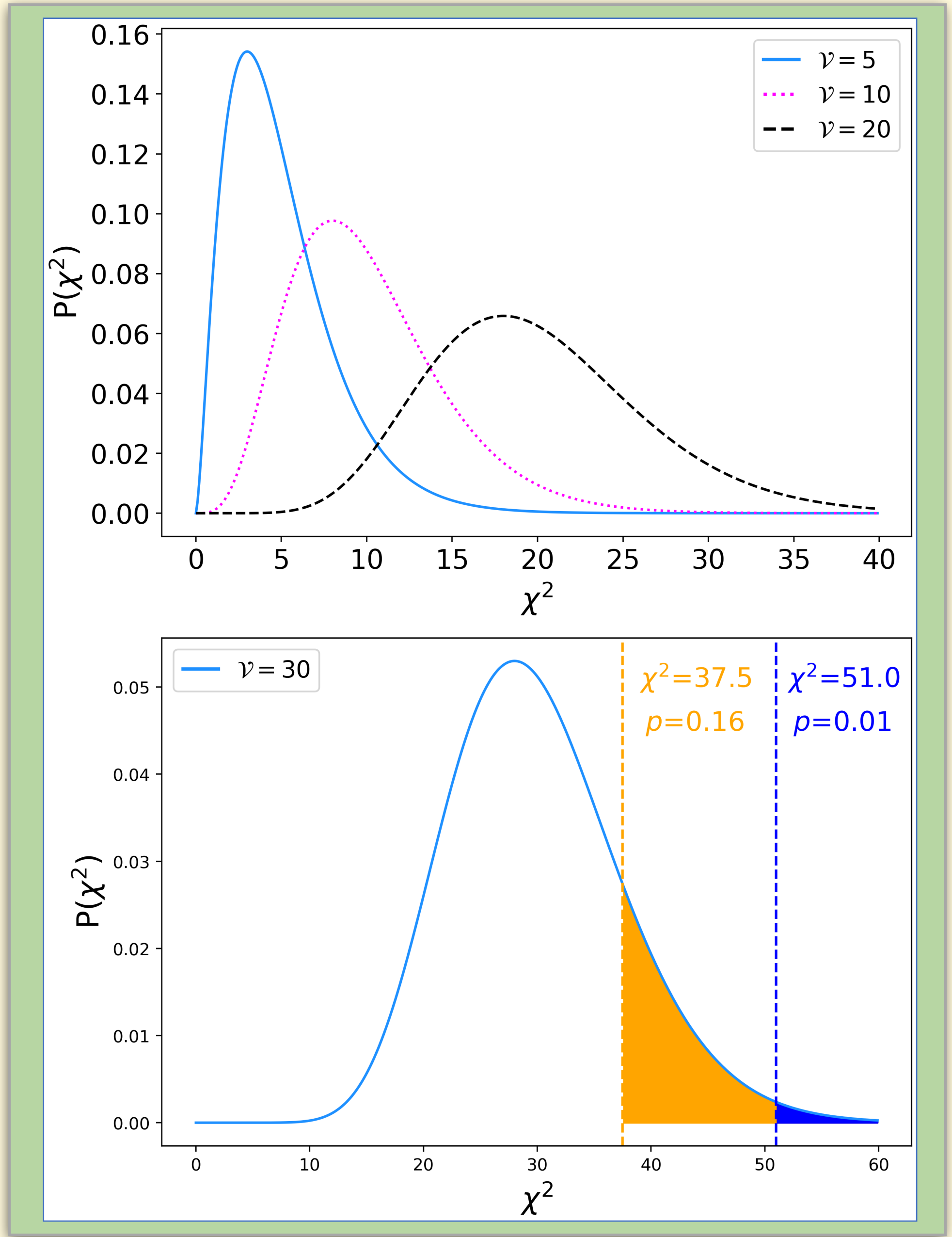
ν is the number of degrees of freedom

Expectation value: $\quad \text{mean} = \nu = N - p$

Variance: $\quad \sigma^2 = 2\nu$

i.e. if the model is correct then we expect $\chi^2 \sim \nu \pm \sqrt{(2\nu)}$

$\chi^2/\nu \sim 1$



Gaussian Distribution: probabilities

Recall $P(x) = \frac{1}{\sigma\sqrt{(2\pi)}} \exp\{-(x-\mu)^2/2\sigma^2\}$

$$P(-t < x_0 < t) = \frac{1}{\sqrt{2}} \int_{-t}^{t} e^{-z^2/2} dz$$

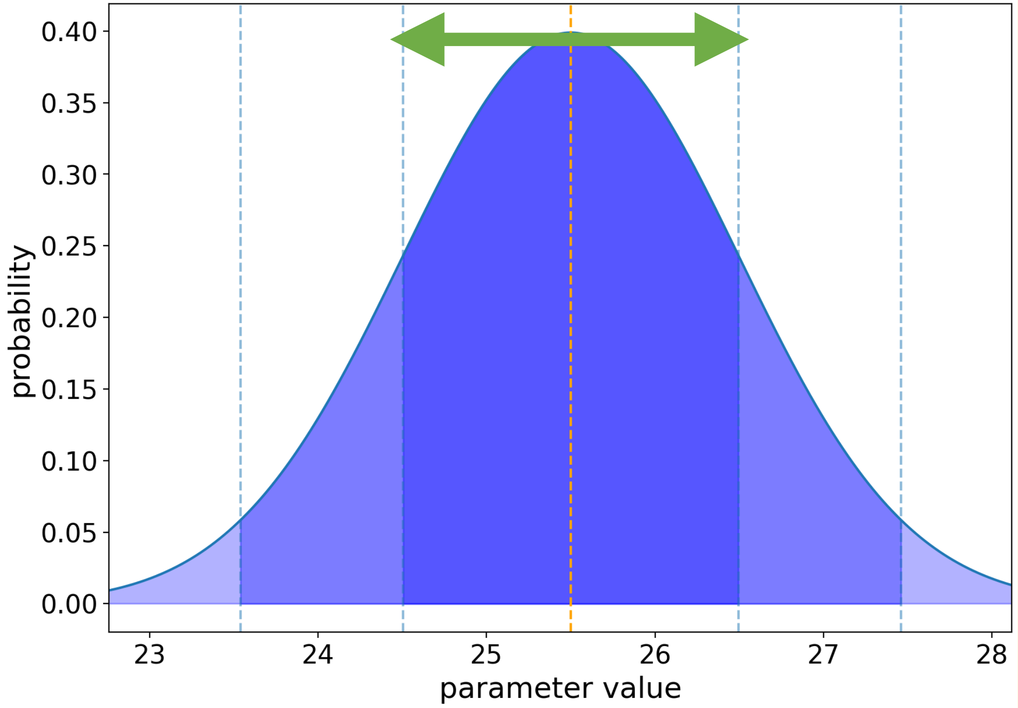
1σ : contains **68.3%** of the probability

2σ : contains **95.5%** of the probability

3σ : contains **99.7%** of the probability

5σ : contains **99.99994%** of the probability

area containing 68% of the probability



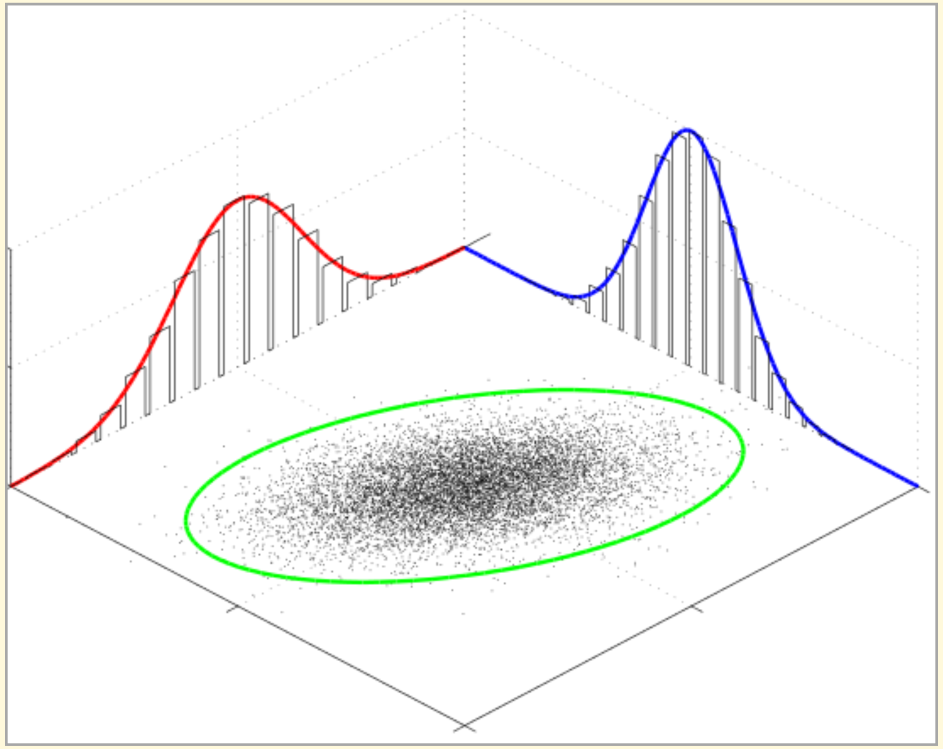
WARNING: This process knows nothing about physics: beware of unphysical regions

Multiple parameters: *Marginalisation*

Distributions are often NOT the function of 1 parameter

$$p(x, y)$$

NB: Coursework 1: is 'happiness' a function solely of GDP?



The probability of x is calculated by integrating over y

$$p(x) = \int p(x, y)\, dy$$

NB: We wanted to measure $p(\textit{happiness} \,|\, \textit{GDP})$, but we measured $p(\textit{happiness}, \textit{country} \,|\, \textit{GDP})$

$$p(\textit{happiness} \,|\, \textit{GDP}) = p(\textit{happiness}, \textit{UK} \,|\, \textit{GDP}) + p(\textit{happiness}, \textit{France} \,|\, \textit{GDP}) + p(\textit{happiness}, \textit{Spain} \,|\, \textit{GDP}) + \ldots$$

An aside: *Covariance*

When two measurements are not independent, we call them covariant

For two connected variables ($z = x + y$): $\sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2}$

if x is related to y

$$\sigma_z = \sqrt{\left(\frac{\partial z}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial z}{\partial y}\sigma_y\right)^2 + 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}cov(x, y)}$$

$$cov(x, y) = \frac{1}{N-1}\sum (x - \hat{x})(y - \hat{y})$$

NB: Parameter transformations (e.g. $y = \log(x)$) change the functional form of the distribution

An aside: *Covariance*

Often expressed in matrix form:

$$\begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}$$

Variance in the diagonal terms; covariance in the off-diagonals

Finding the Distribution

Kolmogorov-Smirnov (KS) testing

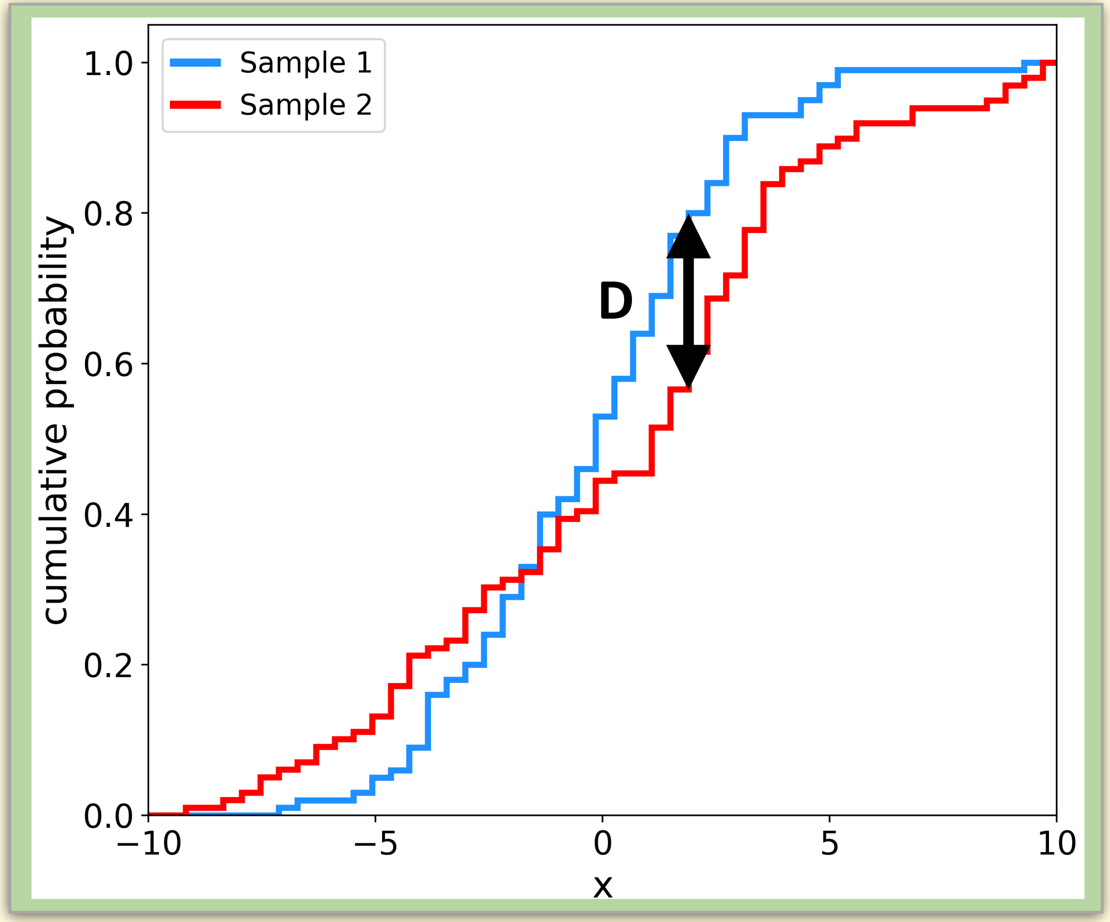
Null hypothesis: the **two distributions are the same**

The maximal distance between

Cumulative Density Functions (CDF)

$$D = \max_{-\infty < x < \infty} |S_N(x) - F(x)|$$

Recall: a CDF is a rank-ordered PDF.



No assumptions about the underlying distribution

NB: See also Anderson-Darling test

Testing for Correlations

<u>Spearman's Rank (SR)</u>

Testing whether **two variables are related**

Compares the ordered rankings (R) of two datasets

$$p_s = 1 - \frac{\sum_{i=1}^{N} [R(x_i) - R(y_i)]^2}{N(N^2 - 1)}$$

Output will be a value between -1 and 1.

Null Hypothesis: **the two variables are uncorrelated**

NB: A non-parametric version of the Pearson test

Likelihoods

Given a set of measurements $x = (x_1, x_2, \ldots)$

the ***likelihood:***

$$\mathscr{L}(\theta \,|\, x) = \prod_{i=0}^{N} P(x_i \,|\, \theta)$$

Best-fit value:	The value that maximises $\mathscr{L}$
Plausible values:	Determined by the width of $\mathscr{L}$
Uncertainty:	For a Gaussian distribution 1σ

Log Likelihoods

Given a set of measurements $x = (x_1, x_2, \ldots)$

the ***log-likelihood:***

$$\ell(\theta) = \ln \mathscr{L}(\theta) = \sum_{i=0}^{N} P(x_i \,|\, \theta)$$

Much easier to maximise

Best-fit value:	The value that maximises ℓ
Plausible values:	Determined by the width of ℓ
Uncertainty:	$\ell = \ell_{\text{max}} - 0.5$

NB: Assumes Gaussianity

Assumes that our measurements are uncorrelated

Log Likelihoods: the Gaussian case

Given a set of measurements $x = (x_1, x_2, \ldots)$

$$p(x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(x_i - \mu_i(\theta))^2}{2\sigma_i^2}\right)$$

$$\mathscr{L}(\theta \,|\, x) = p(x_1 \,|\, \theta) * p(x_2 \,|\, \theta) * \ldots$$

$$\mathscr{L}(\theta \,|\, x) = e^{-\chi^2/2} * \prod_{i=1}^{N} \frac{1}{\sigma_i} \times (2\pi)^{(-N/2)}$$

$$\boxed{-2 \ln \mathscr{L}(\theta \,|\, x) = \chi^2 + 2 \sum_{i=1}^{N} \ln \sigma_i + N \ln(2\pi)}$$

Maximising $\mathscr{L}(\theta \,|\, x)$ is equivalent to minimising χ^2

Confidence levels & χ^2

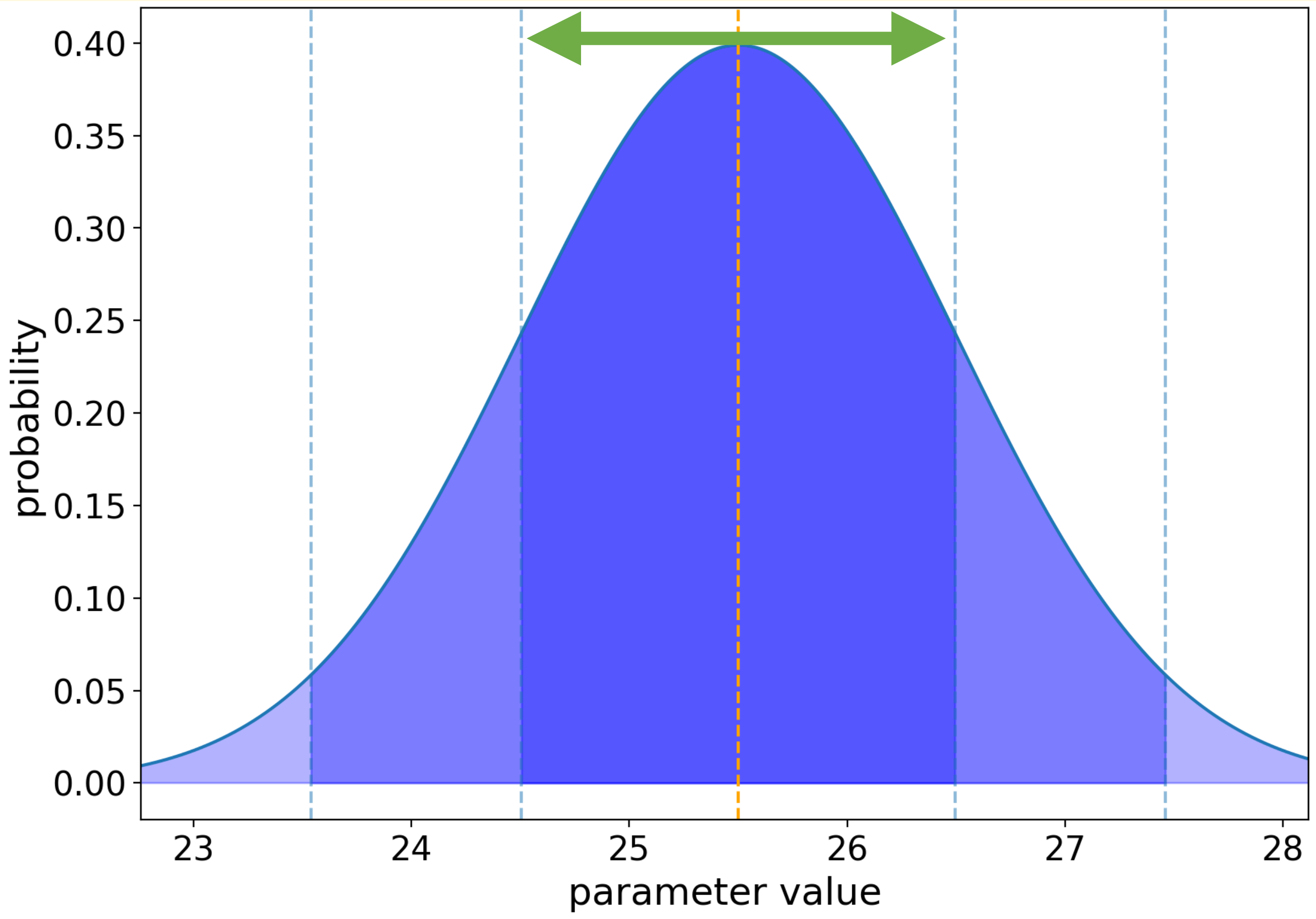
If our best-fit value has $\chi^2 = \chi^2_{min}$:

Contours of equal probability are defined by

$$\chi^2 = \chi^2_{min} + \Delta$$

CI		N_{params}: 1	2	3
68%	1σ	1.0	2.30	3.53
95.4%	2σ	4.0	6.17	8.02
99.7%	3σ	9.0	11.8	14.2

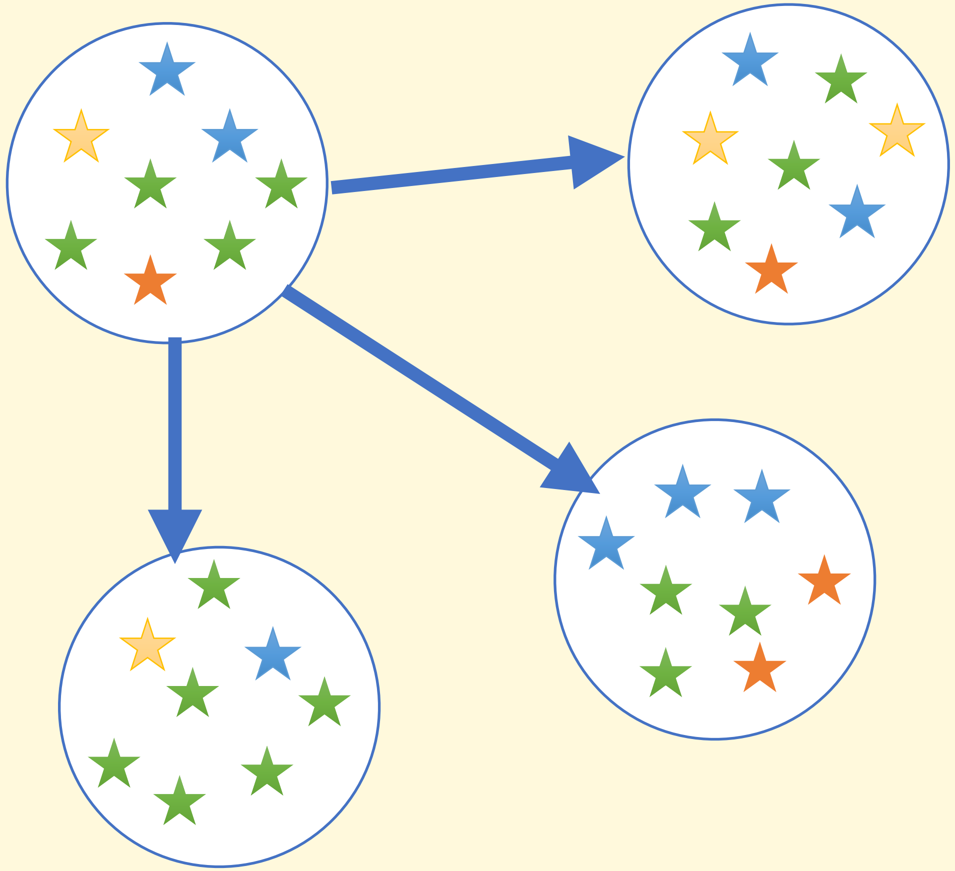
$\chi^2 < \chi^2_{\text{min}} + 1$



WARNING: This process knows nothing about physics: beware of unphysical regions

χ^2 : *estimating uncertainties : bootstrapping*

Resampling the data **with** replacement



(For a dataset with N measurements)

For each of k samples:

> Randomly select N values

> Calculate the χ^2

Calculate the mean, and variance on χ^2

Does not assume a functional form for the distribution

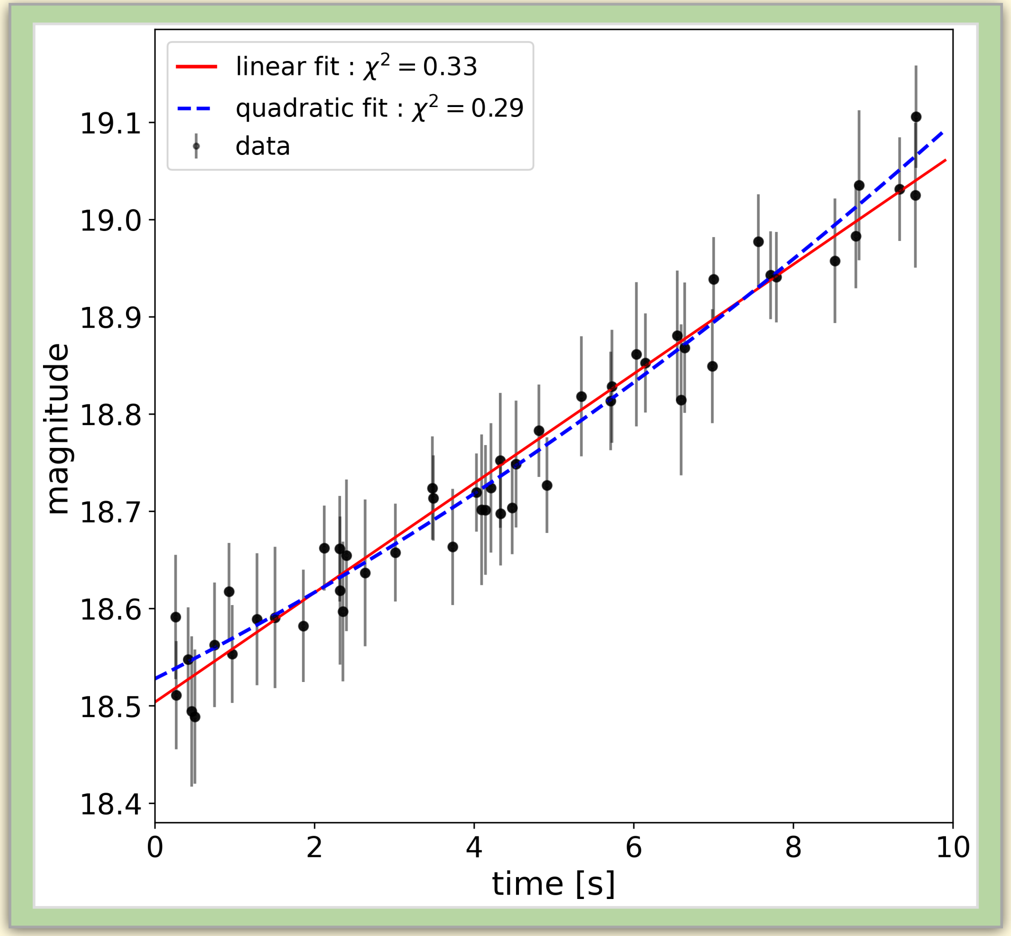
NB: Can also sample from uncertainties : 'MonteCarlo'

See Week 13 for the optimum way to do this

Model Selection

Given two models M_1, M_2 how do we know which one to choose?

Recall that for a good fit, we expect: min$(\chi^2_{red}) \sim 1$



Akaike Information Criteria (AIC)

$$\mathrm{AIC} = -2\ln(\mathscr{L}(\theta)) + 2p$$

Bayes Information Criteria (BIC)

$$\mathrm{BIC} = -2\ln(\mathscr{L}(\theta)) + p\log N$$

$(p = N_{\text{params}}\,;\, n = N_{\text{data}})$

Week 12: Learning outcomes

Today you have learnt

- How to relate probabilities to measurements: the likelihood
- <u>How to define a null hypothesis</u>
  - How to use probabilities to test our hypothesis
- <u>Three key distributions in Physics</u>
- Key tests of whether our hypothesis (assumptions) are correct
  - Distributions: KS test
  - Correlations: Spearman's Rank
- <u>Goodness-of-fit metrics: calculating likelihoods</u>
  - Selecting a model given a likelihood

Practical examples on Friday!