

PHYS465: Statistical Data Analysis in Physics

Week 2: Hypothesis Testing, Common distributions, Model testing

Dr. Mathew Smith

mat.smith@lancaster.ac.uk

Physics Building; C46

Module Structure

‘model testing’: does our data agree with our model?

Week 11: Fitting a model to data; estimating parameters

Week 12: Hypothesis testing; the likelihood; estimating uncertainties

Week 13: Posterior sampling; Bayesian statistics

‘data driven’: what does our data tell us?

Week 14: Clustering and Classification algorithms

Week 15: Machine learning techniques

Week 12: Previous session

Last week we learnt:

- Our measurements drawn from an underlying continuous distribution
 - The shape of this distribution is dependent on our experiment
 - The probability of our data can be described through a PDF
- Given a model we can find the best-fit parameters using least-squares fitting
- This is generalised to χ^2 -fitting in the presence of gaussian uncertainties
- A good model will have $\chi_{red}^2 \sim 1$
 - Where $\chi_{red}^2 = \chi^2/n_{dof}$
 - $\chi_{red}^2 > 1$ suggests the need for more complicated models; $\chi_{red}^2 < 1$ is the reverse
- Our knowledge of model parameters is expressed through confidence intervals
- Question: how do we measure a confidence interval or parameter uncertainty?

Week 12: Learning aims

Today we will introduce

- Probabilities and how they relate to measurements
- The Null Hypothesis
 - And how to test it
- How our experiment will change how the data is distributed
- How to test which distribution is correct
- How do we estimate confidence intervals?
- How to select which model is most likely

Practical examples on Friday!

Revision: Probabilities

Given an experiment:

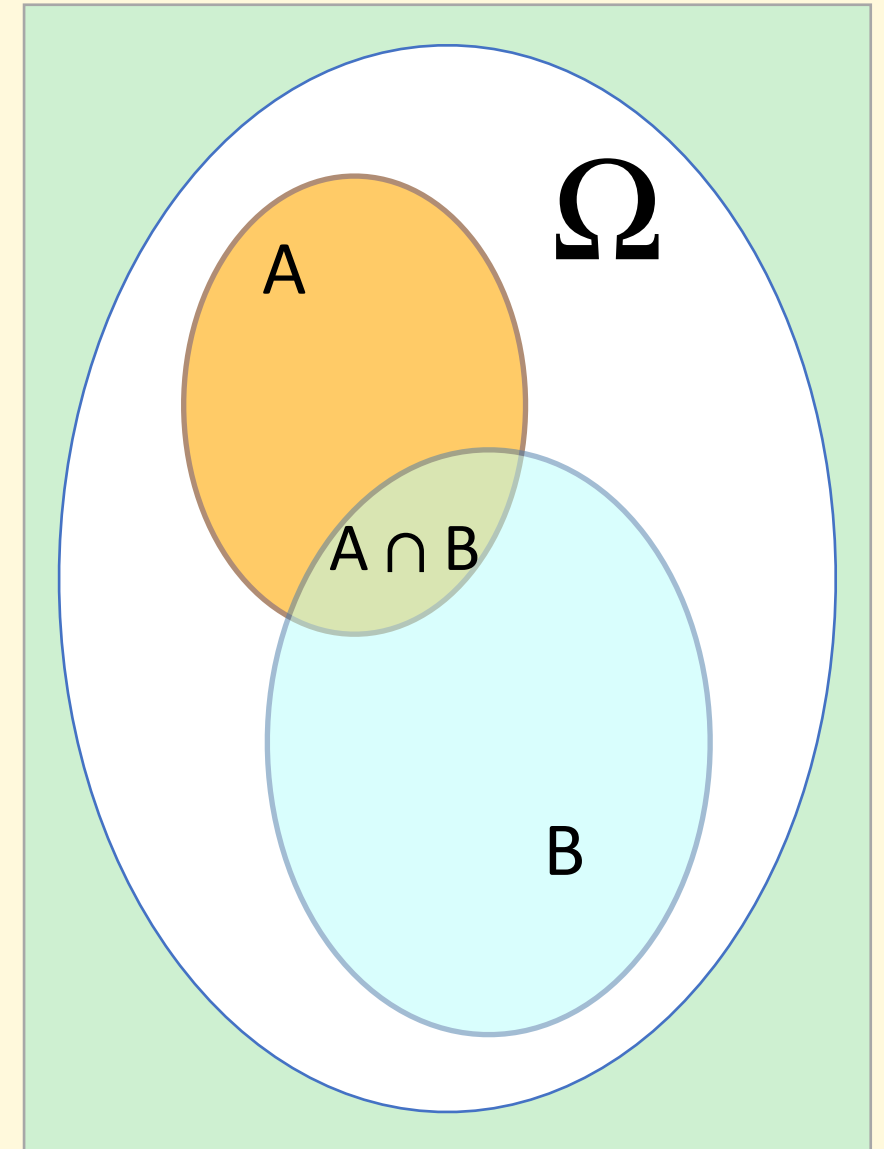
- Ω is the list of all possible outcomes

Axioms:

- $0 \leq P(A) \leq 1$
- $P(\Omega) = 1$
- $P(A_1 \cup A_2 \cup A_3 \dots) = P(A_1) + P(A_2) + P(A_3) + \dots$

Consequences:

- $P(A) + P(A^c) = 1$
- $P(A \cup B) = P(A) + P(B) + P(A \cap B)$
- $P(A \cap B) = P(A) \times P(B)$ if A, B are independent



Conditional Probabilities

$$P(A | B) = \frac{P(A \cap B)}{P(B)}$$

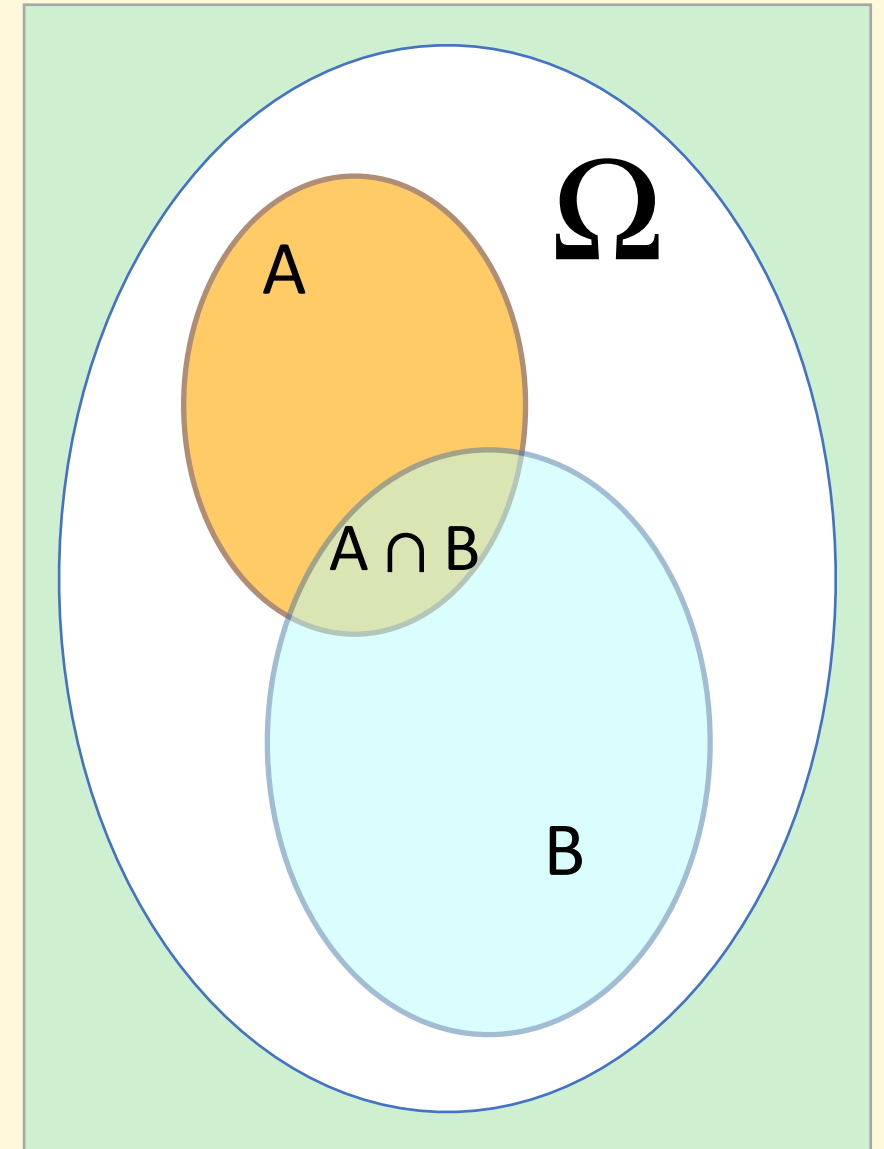
The probability of A given B (an event or condition)

when θ is a model, or parameter

$P(A | \theta)$ is called the **likelihood**: $\mathcal{L}(\theta | A)$

“the **likelihood** of obtaining the data ‘A’ given the model”

Plenty more on this with Bayesian statistics next week



Hypothesis testing

How good is the model?

Fundamental question: Is the model that we are testing a good fit to our measured data?
Is there a more likely alternative?

Null Hypothesis: Our starting assumption
Can be either parameter values or choice of model

In practice: Compares our assumption (or prediction) with experimental measurements
e.g. does the luminosity distribution of galaxies match a Schechter function?

Outcome: Make a statement on the probability of obtaining our result
(see confidence level slides).

Null Hypothesis testing

e.g. Given a model $f(\theta)$, with parameter θ , is the value of θ in a set of possible values Θ_0 ?

H_0 : **The Null Hypothesis:** $\theta \in \Theta_0$

H_1 : An Alternative Hypothesis:

- $\theta \in \Theta_1$ where $\Theta_0 \cap \Theta_1 = \emptyset$

α : threshold of rejection (e.g. 0.05)

With new observations X , such that $p(X, \theta)$

$p(X) < \alpha$ reject H_0 and accept H_1
(N.B. This does not mean accept another model)

$p(X) > \alpha$ no evidence to reject H_0
(N.B. This is not the same as accepting H_0)

		Outcome	
		H_0 not rejected	H_0 rejected
Truth	H_0 is true	Probability of this: $1 - \alpha$	Type I error Will happen α % of the time
	H_1 is true	Type II error Happens $1 - \beta$ % of the time	$P_{H_1}(H_0 \text{ is rejected}) = \beta$

A good experiment is defined around these requirements 8

Confidence Intervals

Given a probability threshold, what is the allowed range of the parameter:

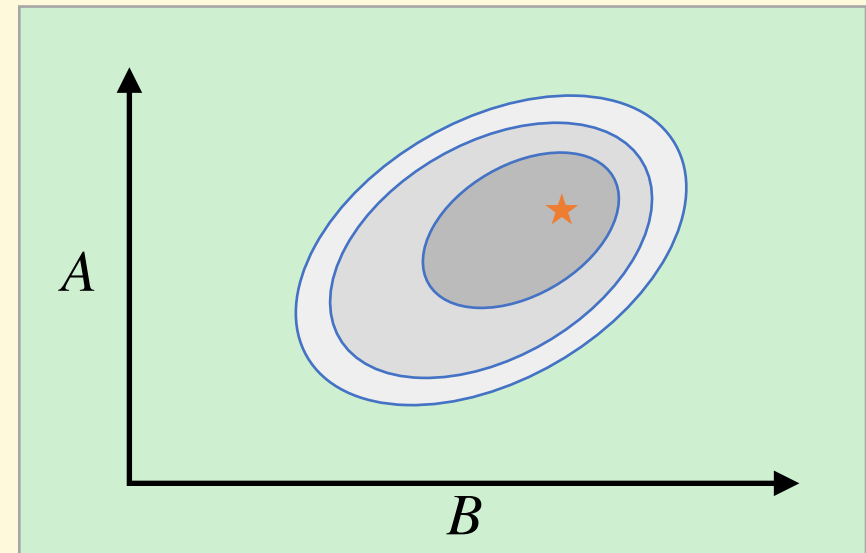
$$\hat{p} - \delta p \leq P_0 \leq \hat{p} + \delta p$$

Interpretation: if we repeat our experiment, how often will P_0 be within our quoted range

For multiple variables (A, B) :

An error ellipse containing δp
of the probability

Typical values are 68%, 90%, 95%.



Key Assumption:

The probability, p , needed for **Hypothesis testing** and **confidence intervals** depend on how the data and model are distributed

Are they both drawn from a Gaussian distribution?

Key Distributions: Gaussian

Aka a normal

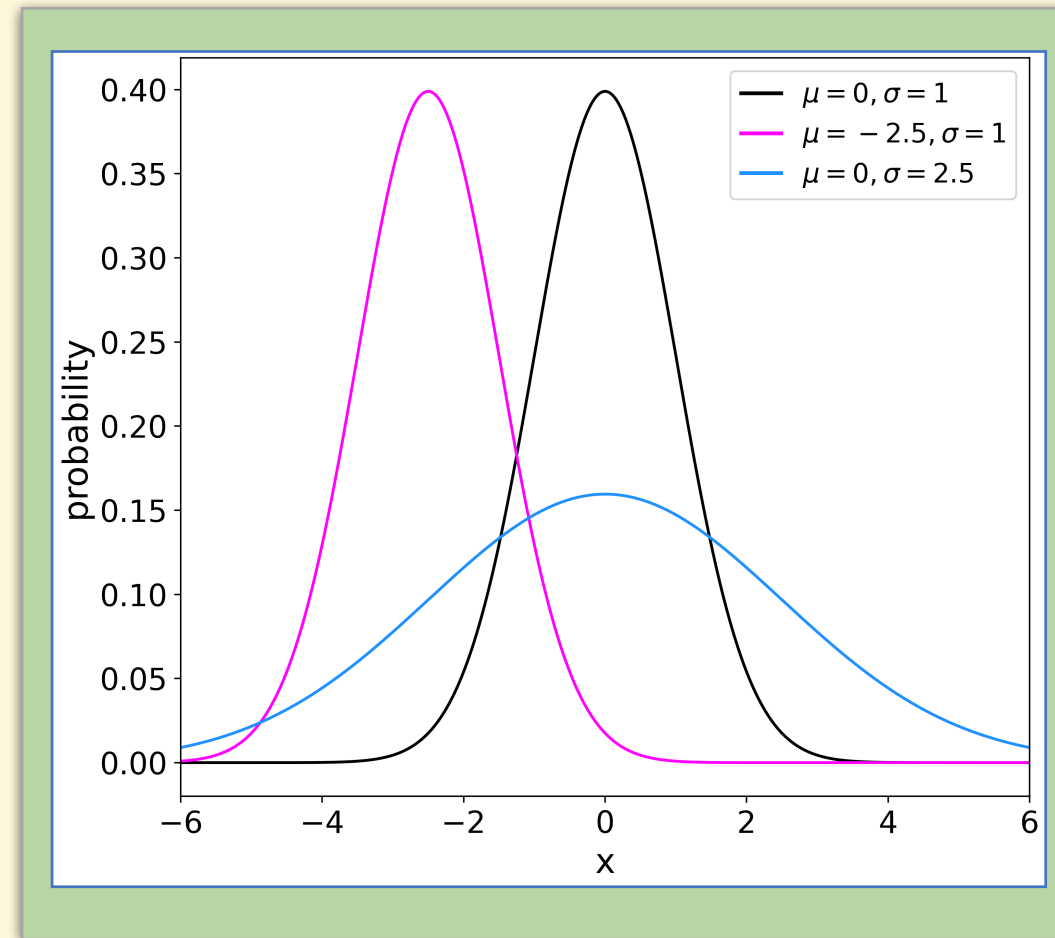
$$P(x) = \frac{1}{\sigma\sqrt{(2\pi)}} \exp\{ - (x - \mu)^2 / 2\sigma^2 \}$$

The limiting case ($\rightarrow \infty$) of most other distributions

Expectation value: mean = μ

Variance: $\sigma^2 = \sigma^2$

$$P(-t\sigma < x_0 < t\sigma) = \frac{1}{\sqrt{2}} \int_{-t}^t e^{-z^2/2} dz$$



Common assumption across physics

Key Distributions: Binomial

Fixed number of outcomes

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

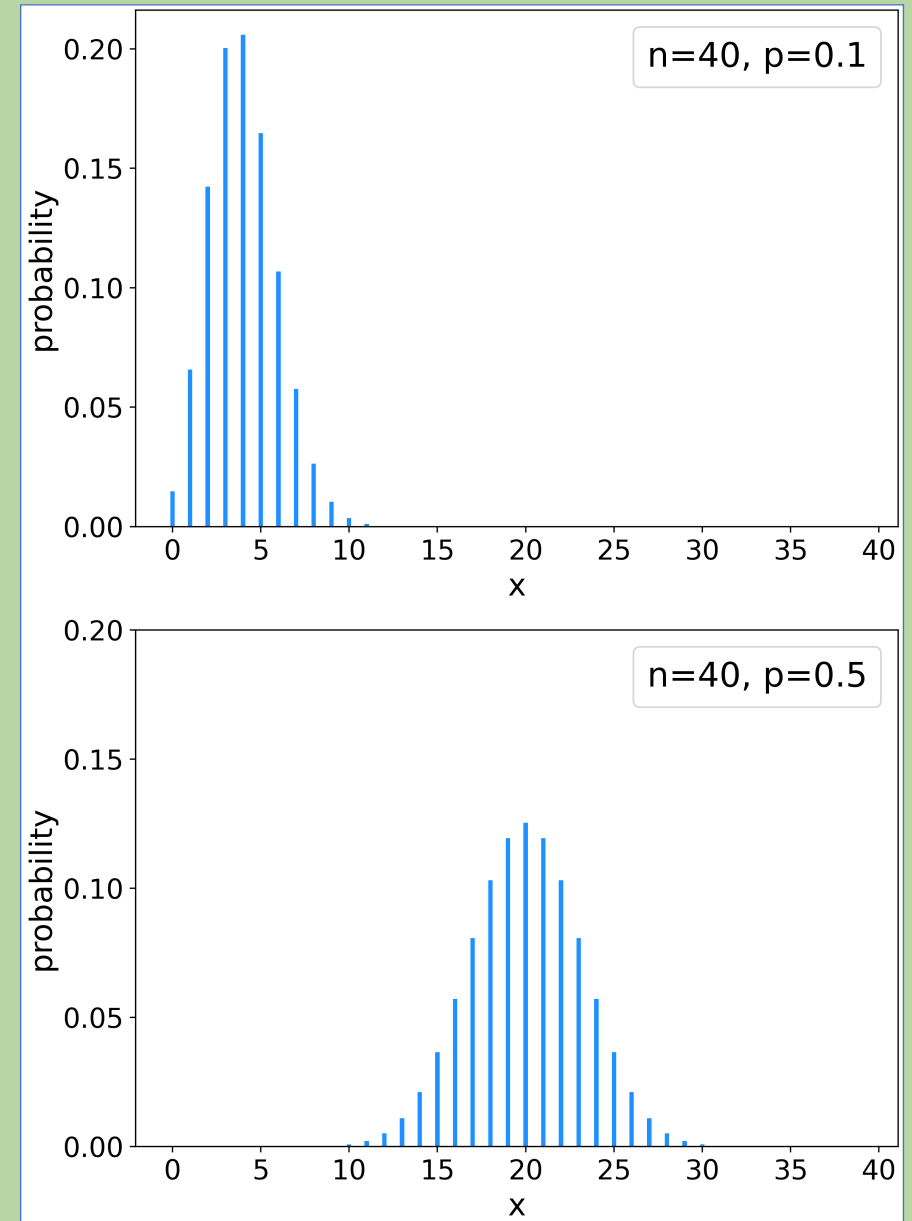
draw n events from k possibilities each with probability p

Expectation value: mean = np

Variance: $np(1 - p)$

=> Gaussian distribution as $N \rightarrow \infty$

Example: coin tossing



Key Distributions: Poisson

The counting distribution

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

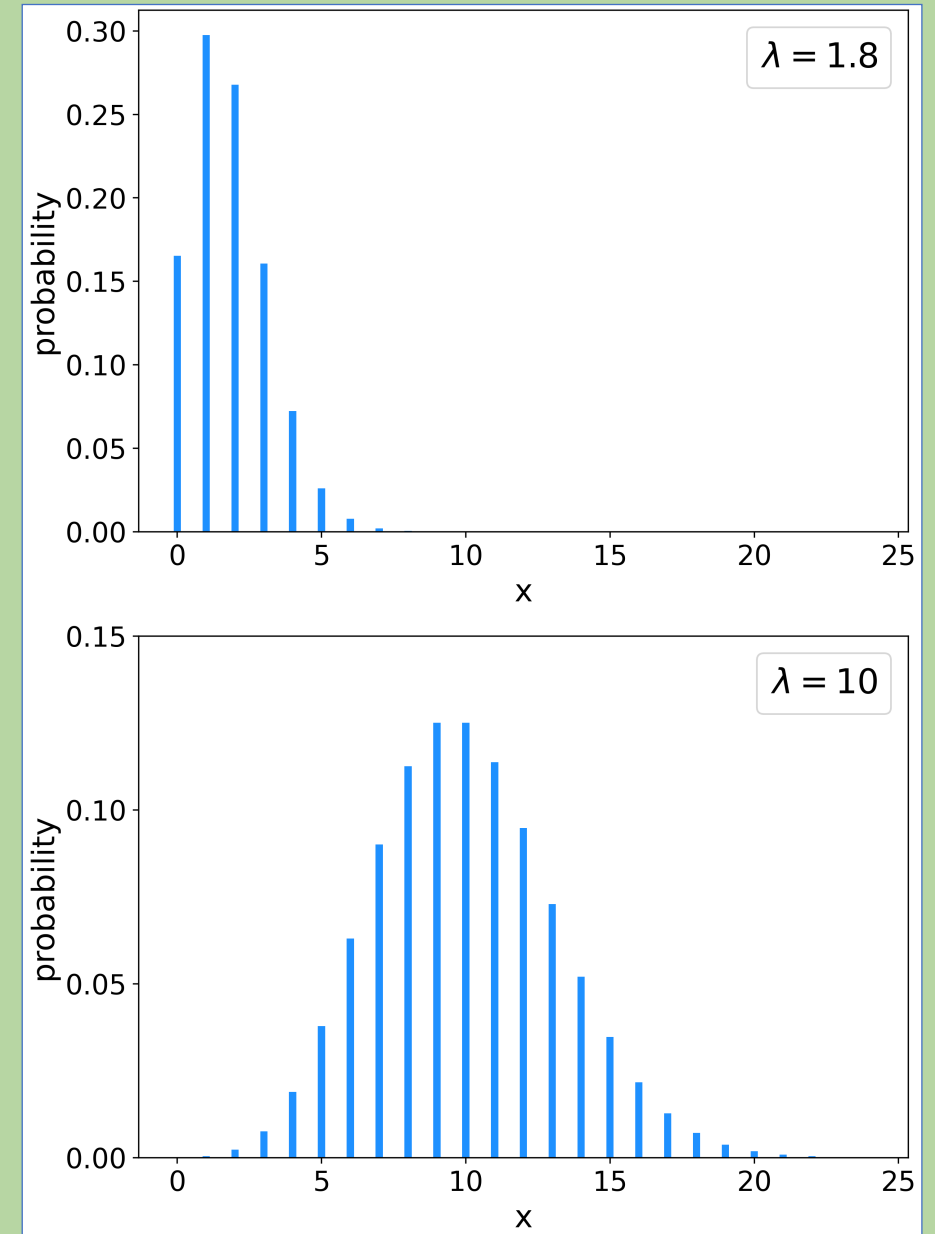
observe k events drawn from λ possibilities

Expectation value: mean = λ

Variance: $\sigma^2 = \lambda$

=> **Gaussian distribution as $k \rightarrow \infty$**

Example: histogram (or photon) counting



Key Distributions: χ^2

Related to the Gamma distribution

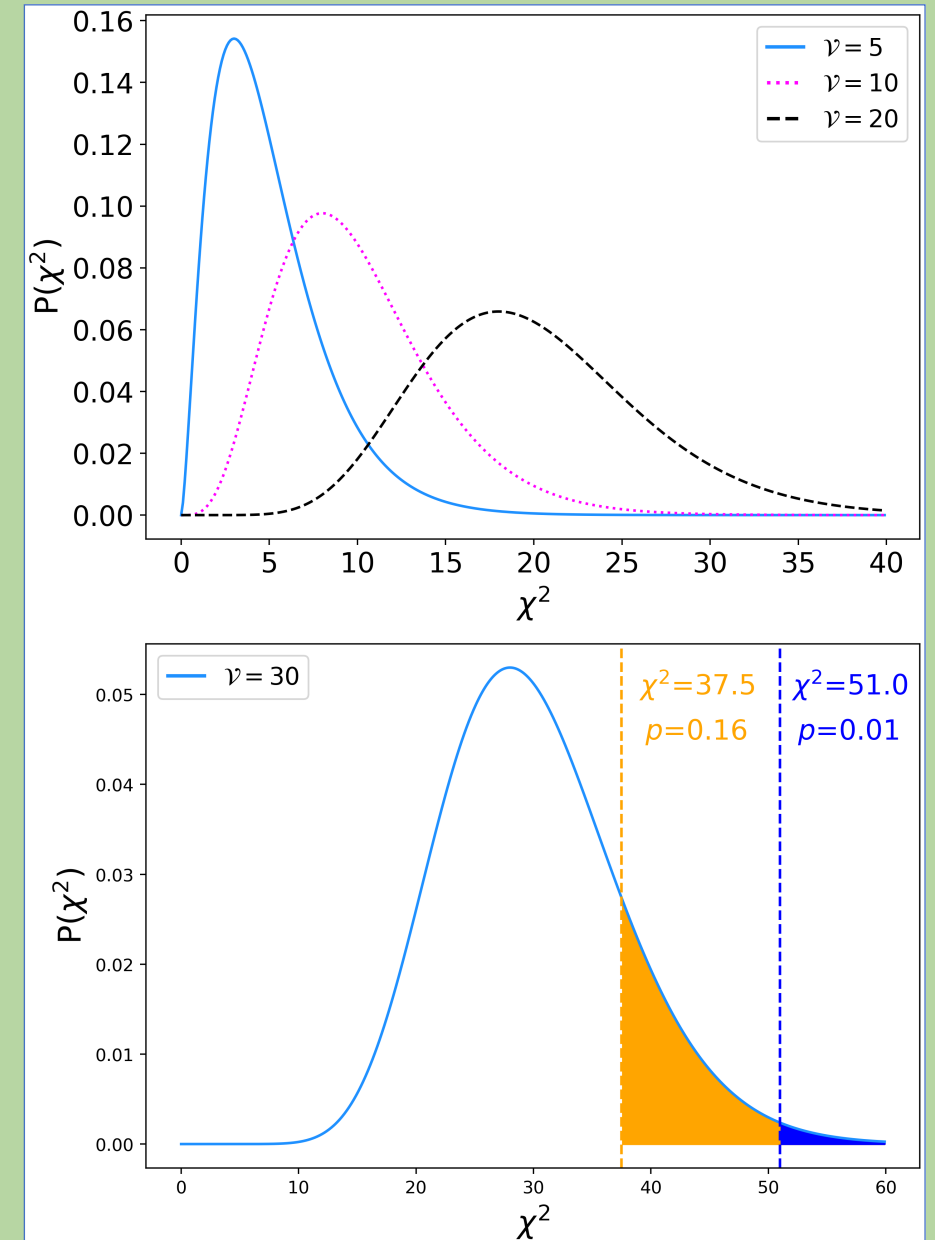
$$P \propto (\chi^2)^{\frac{\nu-2}{2}} \exp(-\chi^2/2)$$

ν is the number of degrees of freedom

Expectation value: $\text{mean} = \nu = N - p$

Variance: $\sigma^2 = 2\nu$

*i.e. if the model is correct then we expect $\chi^2 \sim \nu \pm \sqrt{(2\nu)}$
 $\chi^2/\nu \sim 1$*



Finding the Distribution

Kolmogorov-Smirnov (KS) testing

Null hypothesis: the **two distributions are the same**

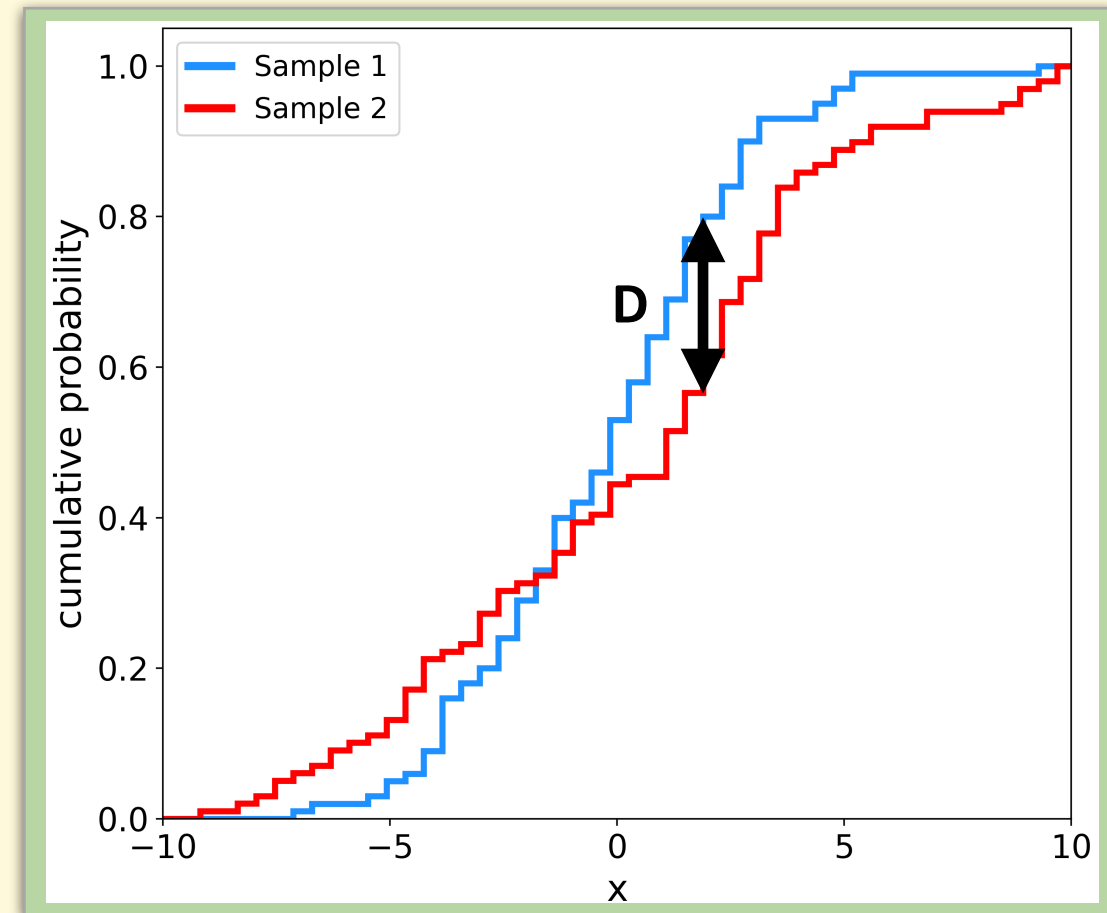
The maximal distance between
Cumulative Density Functions (CDF)

$$D = \max_{-\infty < x < \infty} |S_N(x) - F(x)|$$

Recall: a CDF is a rank-ordered PDF.

No assumptions about the underlying distribution

NB: See also Anderson-Darling test



Testing for Correlations

Spearman's Rank (SR)

Testing whether **two variables** are related

Compares the ordered rankings (R) of two datasets

$$p_s = 1 - \frac{\sum_{i=1}^N [R(x_i) - R(y_i)]^2}{N(N^2 - 1)}$$

Output will be a value between -1 and 1.

Null Hypothesis: **the two variables are uncorrelated**

NB: A non-parametric version of the Pearson test

Gaussian Distribution: probabilities

Recall
$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{ - (x - \mu)^2 / 2\sigma^2 \right\}$$

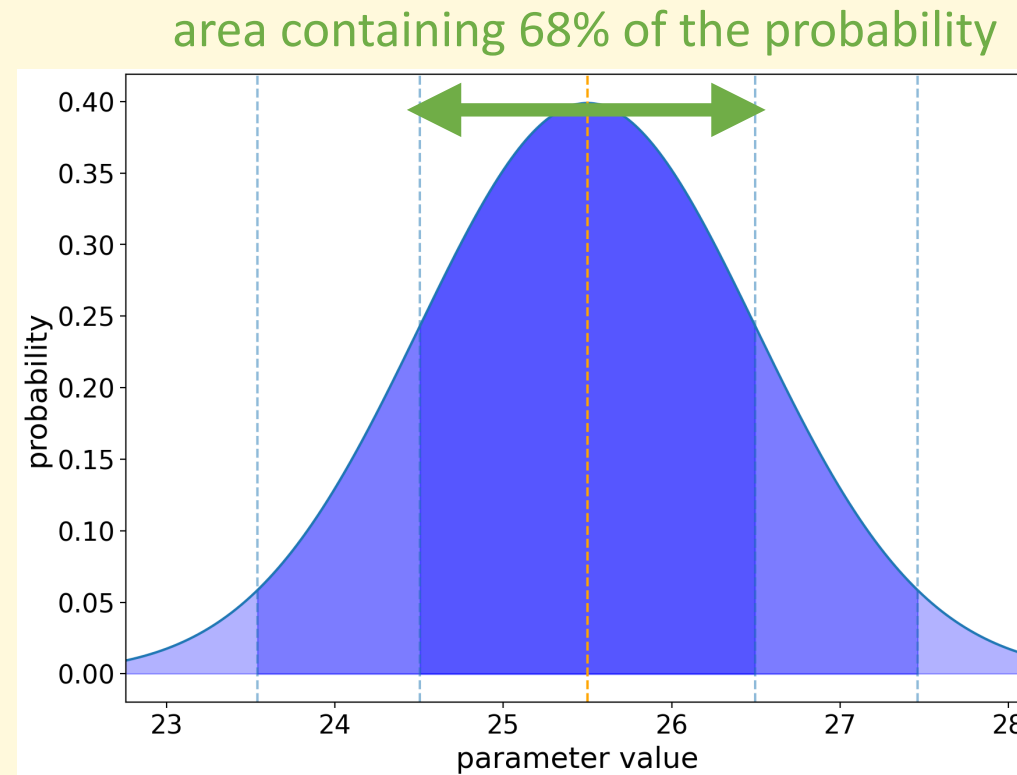
$$P(-t < x_0 < t) = \frac{1}{\sqrt{2}} \int_{-t}^t e^{-z^2/2} dz$$

1σ : contains **68.3%** of the probability

2σ : contains **95.5%** of the probability

3σ : contains **99.7%** of the probability

5σ : contains **99.99994%** of the probability



WARNING: This process knows nothing about physics: beware of unphysical regions

Multiple parameters: *Marginalisation*

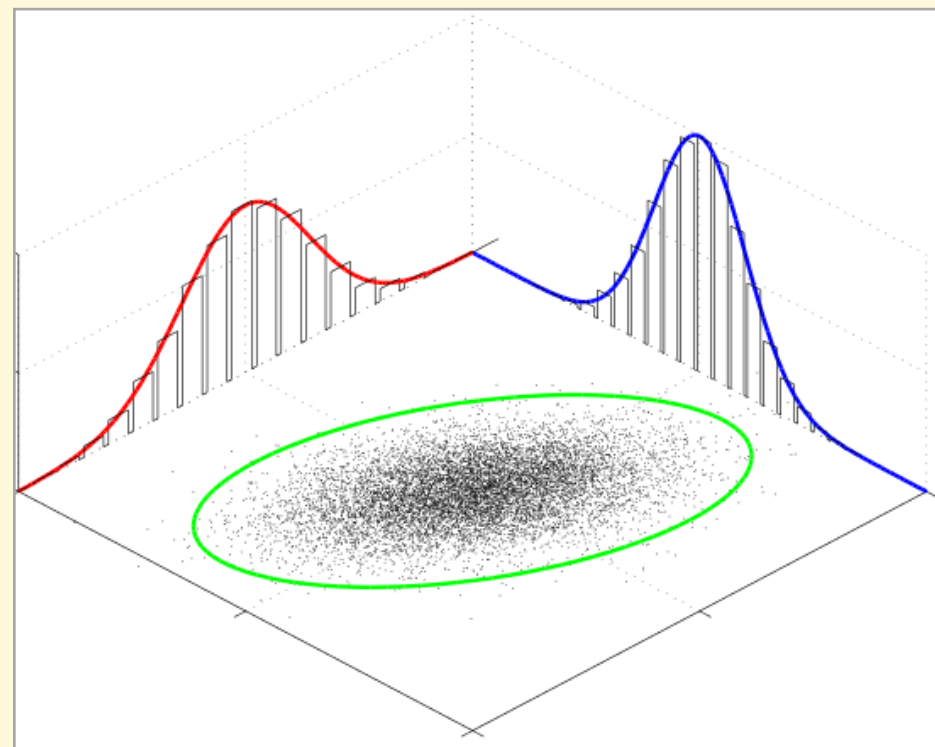
Distributions are often NOT the function of 1 parameter

$$p(x, y)$$

NB: Coursework 1: is 'happiness' a function solely of GDP?

The probability of x is calculated by integrating over y

$$p(x) = \int p(x, y) dy$$



NB: We wanted to measure $p(\text{happiness} \mid \text{GDP})$, but we measured $p(\text{happiness, country} \mid \text{GDP})$

$p(\text{happiness} \mid \text{GDP}) = p(\text{happiness, UK} \mid \text{GDP}) + p(\text{happiness, France} \mid \text{GDP}) + p(\text{happiness, Spain} \mid \text{GDP}) + \dots$

An aside: Covariance

When two measurements are not independent, we call them covariant

For two connected variables ($z = x + y$): $\sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2}$

if x is related to y

$$\sigma_z = \sqrt{\left(\frac{\partial z}{\partial x} \sigma_x\right)^2 + \left(\frac{\partial z}{\partial y} \sigma_y\right)^2 + 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \text{cov}(x, y)}$$

$$\text{cov}(x, y) = \frac{1}{N-1} \sum (x - \hat{x})(y - \hat{y})$$

NB: Parameter transformations (e.g. $y = \log(x)$) change the functional form of the distribution

An aside: *Covariance*

Often expressed in matrix form:

$$\begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}$$

Variance in the diagonal terms; covariance in the off-diagonals

Likelihoods

Given a set of measurements $x = (x_1, x_2, \dots)$

the *likelihood*:

$$\mathcal{L}(\theta | x) = \prod_{i=0}^N P(x_i | \theta)$$

Best-fit value:

The value that maximises \mathcal{L}

Plausible values:

Determined by the width of \mathcal{L}

Uncertainty:

For a Gaussian distribution 1σ

Log Likelihoods

Given a set of measurements $x = (x_1, x_2, \dots)$

the *log-likelihood*:

$$\ell(\theta) = \ln \mathcal{L}(\theta) = \sum_{i=0}^N P(x_i | \theta)$$

Much easier to maximise

Best-fit value:

The value that maximises ℓ

Plausible values:

Determined by the width of ℓ

Uncertainty:

$$\ell = \ell_{\max} - 0.5$$

Log Likelihoods: the Gaussian case

Given a set of measurements $x = (x_1, x_2, \dots)$

$$p(x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(x_i - \mu_i(\theta))^2}{2\sigma_i^2}\right)$$

$$\mathcal{L}(\theta | x) = p(x_1 | \theta) * p(x_2 | \theta) * \dots$$

$$\mathcal{L}(\theta | x) = e^{-\chi^2/2} * \prod_{i=1}^N \frac{1}{\sigma_i} \times (2\pi)^{(-N/2)}$$

$$-2 \ln \mathcal{L}(\theta | x) = \chi^2 + 2 \sum_{i=1}^N \ln \sigma_i + N \ln(2\pi)$$

Maximising $\mathcal{L}(\theta | x)$ is equivalent to minimising χ^2

Note: χ^2 not χ^2_{red}

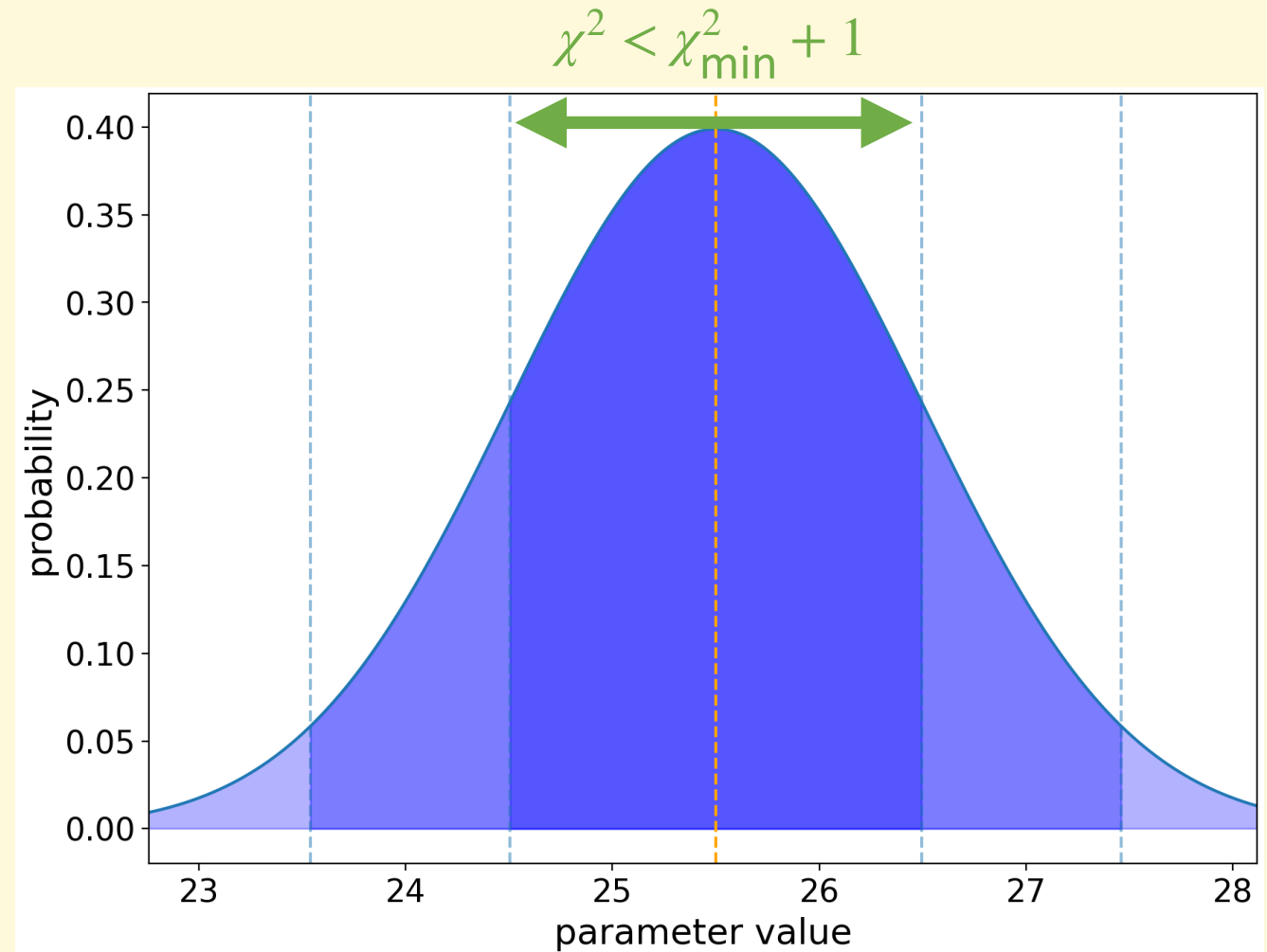
Confidence levels & χ^2

If our best-fit value has $\chi^2 = \chi^2_{min}$:

Contours of equal probability are defined by

$$\chi^2 = \chi^2_{min} + \Delta$$

			N_{params}		
			1	2	3
C	68%	1σ	1.0	2.30	3.53
	95.4%	2σ	4.0	6.17	8.02
	99.7%	3σ	9.0	11.8	14.2



WARNING: This process knows nothing about physics: beware of unphysical regions

Model Selection

Given two models M_1, M_2 how do we know which one to choose?

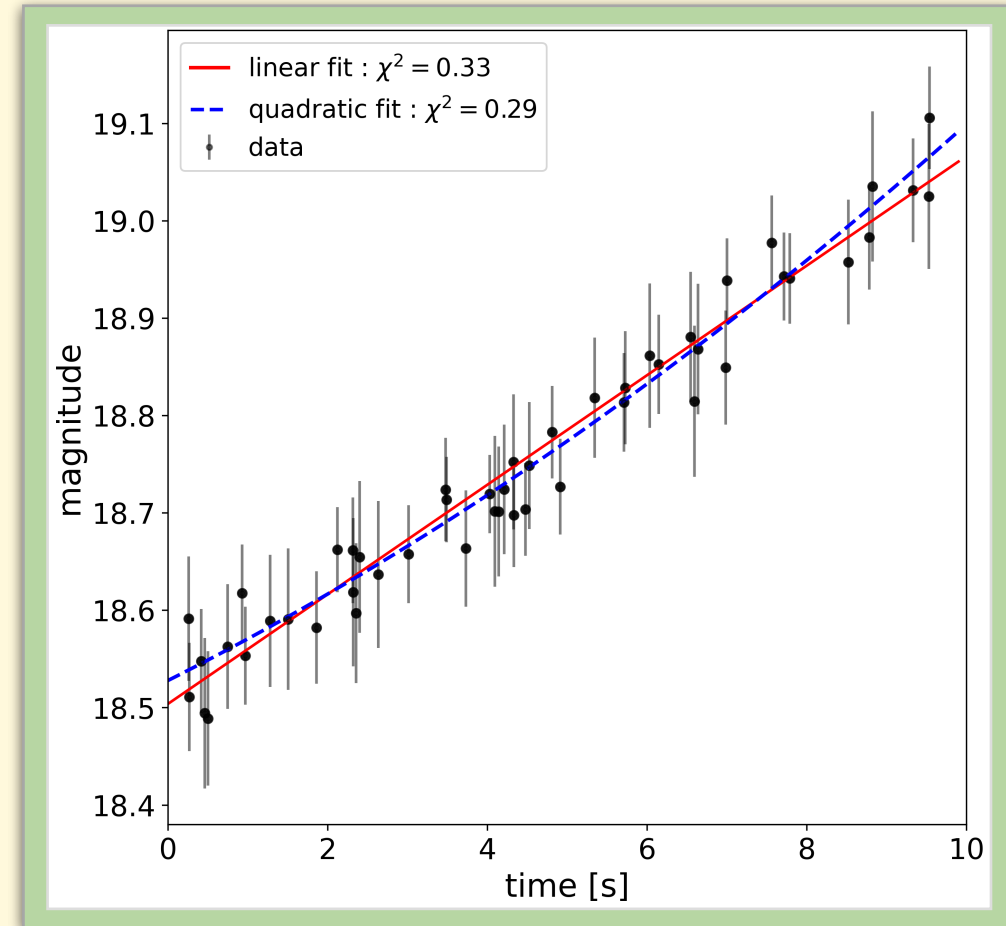
Recall that for a good fit, we expect: $\min(\chi^2_{red}) \sim 1$

Akaike Information Criteria (AIC)

$$AIC = -2 \ln(\mathcal{L}(\theta)) + 2p$$

Bayes Information Criteria (BIC)

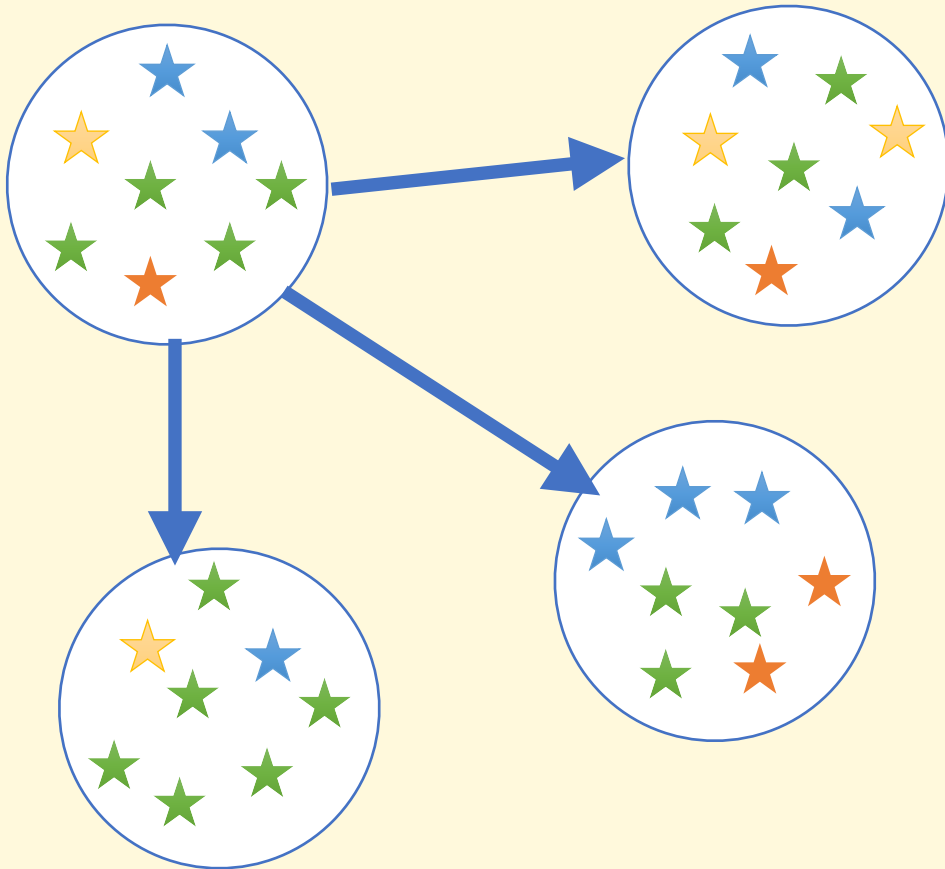
$$BIC = -2 \ln(\mathcal{L}(\theta)) + p \log N$$



$(p = N_{\text{params}} ; n = N_{\text{data}})$ 25

χ^2 : *estimating uncertainties : bootstrapping*

Resampling the data **with** replacement



(For a dataset with N measurements)

For each of k samples:

- > Randomly select N values

- > Calculate the χ^2

Calculate the mean, and variance on χ^2

Does not assume a functional form for the distribution

NB: Can also sample from uncertainties : 'MonteCarlo'

Week 12: Learning outcomes

Today you have learnt

- How to relate probabilities to measurements: the likelihood
- How to define a null hypothesis
 - How to use probabilities to test our hypothesis
- Three key distributions in Physics
- Key tests of whether our hypothesis (assumptions) are correct
 - Distributions: KS test
 - Correlations: Spearman's Rank
- Goodness-of-fit metrics: calculating likelihoods
 - Selecting a model given a likelihood

Practical examples on Friday!