## **Exercices**

## 1.1 Vector spaces

Exercice 1.1. Show that the set  $\mathbb{R}_n[X]$  of polynomials of degree lower than  $n \in \mathbb{N}$  is a vector space.

Exercise 1.2. Show that the set of functions y solution of

$$y''(x) + 5y'(x) + 3y(x) = 0 (1.1)$$

is a vector space.



Exercise 1.3. Let [M] be a  $n \times n$  matrix. Show that the solution of

$$[\mathbf{M}]\mathbf{X} = \mathbf{0} \tag{1.2}$$

is a vector space.



Exercise 1.4. What is the dimension of the nullspace a of matrix

$$[\mathbf{M}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{1.3}$$

Exercice 1.5. Is the family of three vectors a basis of  $\mathbb{R}^3$ ? If yes, is this basis

$$[\mathbf{\Phi}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$
 (1.4)

Exercice 1.6. Let V be the subspace of  $\mathbb{R}^3$  spanned by the two vectors  $\mathbf{e}_1 = \begin{cases} \sqrt{1/2} \\ \sqrt{1/2} \end{cases}$ 

and 
$$\mathbf{e}_2 = \begin{cases} 1 \\ 0 \\ 2 \end{cases}$$

Find an orthonormal basis of V including  $e_1$ 

Exercice 1.7. Show that the following applications define inner products on the given vector space E.

• 
$$E = \mathbb{R}^n, \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i.$$

• 
$$E = \mathbb{R}^3 < \mathbf{x}, \mathbf{y} > = \mathbf{x}^t[\mathbf{A}]\mathbf{y}$$
 with

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

• 
$$E = \mathbb{R}^2 < \mathbf{x}, \mathbf{y} > = \mathbf{x}^t[\mathbf{A}]\mathbf{y}$$
 with

$$[\mathbf{A}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Exercice 1.8. Show that the following applications define inner products on the given

• 
$$E = \mathbb{R}^n, \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i.$$

• 
$$E = \mathbb{R}^3 < \mathbf{x}, \mathbf{y} > = \mathbf{x}^t[\mathbf{A}]\mathbf{y}$$
 with

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

• 
$$E = \mathbb{R}^2 < \mathbf{x}, \mathbf{y} >= \mathbf{x}^t[\mathbf{A}]\mathbf{y}$$
 with

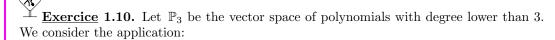
$$[\mathbf{A}] = \left[ \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right]$$

Exercice 1.9. Let  $u = \{u_n\}_{n \in \mathbb{N}}$  be a numerical sequence on  $\mathbb{N}$ . u is said square

summable (or  $u \in l^2(\mathbb{N})$ ) if:

$$u \in l^2(\mathbb{N}) \Leftrightarrow \sum_{n \in \mathbb{N}} u_n^2 < +\infty$$

- Show that  $l^2(\mathbb{N})$  is a vector space.
- Show that  $\langle u, v \rangle = \sum_{n \in \mathbb{N}} u_n v_n$  is an inner product on  $l^2(\mathbb{N})$ .
- Show that  $\mathcal{N}(u) = \sqrt{\sum_{n \in \mathbb{N}} u_n^2}$  is a norm on  $l^2(\mathbb{N})$ .



$$\langle P, Q \rangle = \int_0^1 P(x)Q(x) dx$$

Show that this application is an inner product.

Let  $\{P_0 = 1, P_1 = x, P_2 = x^2, P_3 = x^3\}$  be a basis of this space.

Is this basis orthonormal?