# Refresher courses in Mathematics

Matrices

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# Chapter 1

# Matrices

## 1.1 Algebraic Calculus

**Exercice 1.1.** Let consider the following matrices

$$[\mathbf{\Phi}(\xi)] = \begin{bmatrix} (1-\xi)/2 \\ \xi/2 \end{bmatrix} \tag{1.1}$$

Compute (if it exists)

$$\int_0^1 [\mathbf{\Phi}(\xi)][\mathbf{\Phi}(\xi)]^T \, \mathrm{d}\xi, \tag{1.2}$$

where $^T$ is the transposition.

**Exercice 1.2.** The Forward Euler's Method is defined by the recurence relation:

$$\mathbf{S}_{k+1} = ([\mathbf{I}_2] + \Delta t[\boldsymbol{\alpha}])\,\mathbf{S}_k. \tag{1.3}$$

Compute the first three steps $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ of Forward Euler's Method for

$$[\boldsymbol{\alpha}] = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{S}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{1.4}$$

**Exercice 1.3.** Compute $[\mathbf{M}]^n\,\mathbf{X}$ for

$$[\mathbf{M}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{1.5}$$

**Exercice 1.4.** What are the conditions on $a$, $b$, $c$ and $d$ for matrix $[\mathbf{M}]$ to be its own inverse with

$$[\mathbf{M}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{1.6}$$

## 1.2 Determinant of matrices

**Exercice 1.5.** What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}? \tag{1.7}$$

**Exercice 1.6.** What is the link between determinants of matrice $[\mathbf{M}]$ and $[\mathbf{N}]$ defined by

$$[\mathbf{M}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}, \quad [\mathbf{N}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 1 & 0 & 1 & 5 \\ 2 & 3 & 1 & 5 \end{bmatrix} \tag{1.8}$$

**Exercice 1.7.** What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}? \tag{1.9}$$

**Exercice 1.8.** What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}? \tag{1.10}$$

**Exercice 1.9.** What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} ? \tag{1.11}$$

**Exercice 1.10.** What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} ? \tag{1.12}$$

## 1.3 Operation on Matrices

## 1.4 Change of basis

**Exercice 1.11.** What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ in the $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ basis

**Exercice 1.12.** What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ in the $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ basis

**Exercice 1.13.** What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ in the $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ basis

**Exercice 1.14.** What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$ in the $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ basis

**Exercice 1.15.** What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$ in the $\begin{bmatrix} 2 & 10 \\ 3 & 5 \end{bmatrix}$ basis

## 1.5 Eigenvalues and Eigenvectors

**Exercice 1.16.** What are the eigenvalues of

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.17.** What are the eigenvalues of

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.18.** What are the eigenvalues of

$$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.19.** What are the eigenvalues of

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.20.** What are the eigenvalues of

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.21.** What are the eigenvalues of

$$\begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.22.** What are the eigenvalues of

$$\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.23.** What are the eigenvalues of

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.24.** What are the eigenvalues of

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.25.** What are the eigenvalues of

$$\begin{bmatrix} 3 & 1 \\ 3 & 3 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.26.** What are the eigenvalues of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.27.** What are the eigenvalues of

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.28.** What are the eigenvalues of

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**<u>Exercice</u> 1.29.** What are the eigenvalues of

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**<u>Exercice</u> 1.30.** What are the eigenvalues of

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 4 & 2 \\ -1 & 0 & 3 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**<u>Exercice</u> 1.31.** What are the eigenvalues of

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.32.** What are the eigenvalues of

$$\begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.33.** What are the eigenvalues of

$$\begin{bmatrix} 3 & 0 & 0 \\ -5 & 2 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.34.** What are the eigenvalues of

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 2 \\ -1 & -2 & 0 \end{bmatrix}$$

For each eigenvalue $\lambda$, propose one eigenvector $\mathbf{X}$

**Exercice 1.35.** Let $\xi \neq 1$. What are the eigenvalues $p$ and the associated eigenvectors of

$$\begin{bmatrix} -2\omega_0\xi & -\omega_0^2 \\ 1 & 0 \end{bmatrix} \tag{1.13}$$

## 1.6 Resolution of linear systems

**Exercice 1.36.** What is the solution of

$$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 5 \\ 3 \end{Bmatrix}$$

**Exercice 1.37.** What is the solution of

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix}$$

**Exercice 1.38.** What is the solution of

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 4 \\ 12 \\ 38 \end{Bmatrix}$$

**<u>Exercice</u> 1.39.** What is the solution of

$$\begin{bmatrix} 3 & 2 & 7 \\ 5 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \mathbf{X} = \left\{ \begin{matrix} 9 \\ 3 \\ 2 \end{matrix} \right\}$$

**<u>Exercice</u> 1.40.** What is the solution of

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix} \mathbf{X} = \left\{ \begin{matrix} 18 \\ 6 \\ 14 \end{matrix} \right\}$$

**<u>Exercice</u> 1.41.** What is the solution of

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 5 & 4 & 3 & 2 \\ 0 & 0 & 2 & -1 \end{bmatrix} \mathbf{X} = \left\{ \begin{matrix} 4 \\ 9 \\ 24 \\ 5 \end{matrix} \right\}$$