Exercices

1.1 Vector spaces

Exercice 1.1. Show that the set $\mathbb{R}_n[X]$ of polynomials of degree lower than $n \in \mathbb{N}$ is a vector space.

Solution 1.1. Let P and Q be two polynomials of degree lower than n. Hence there exists two sequences $\{p_k\}_{k=0..n}$ and $\{q_k\}_{k=0..n}$ so that

$$P(X) = \sum_{k=0}^{n} p_k X^K, \quad Q(X) = \sum_{k=0}^{n} q_k X^K. \tag{1.1}$$

Some of the elements of these sequences can be equal to zero as the degree of these polynomials is not necessarily n. The sum S of P and Q is

$$S(X) = \sum_{k=0}^{n} (p_k + q_k) X^K \tag{1.2}$$

This polynomial is trivially of degree lower than n.
Let now consider a real numbre λ, the polynomial λP defined by

$$\lambda P(X) = \sum_{k=0}^{n} \lambda p_k X^K \tag{1.3}$$

is of degree lower than n.

Exercice 1.2. Show that the set of functions y solution of

$$y''(x) + 5y'(x) + 3y(x) = 0 \tag{1.4}$$

is a vector space.

Solution 1.2. Let y_1 and y_2 be two solutions of (1.4):

$$y_1''(x) + 5y_1'(x) + 3y_1(x) = 0, \quad y_2''(x) + 5y_2'(x) + 3y_2(x) = 0 \tag{1.5}$$

By adding these two equations

$$y_1''(x) + y_2''(x) + 5(y_1'(x) + y_2'(x)) + 3(y_1(x) + y_2(x)) = 0. \tag{1.6}$$

By linearity of the derivation

$$(y_1 + y_2)''(x) + 5(y_1 + y_2)'(x) + 3(y_1(x) + y_2(x)) = 0. \tag{1.7}$$

Hence $y_1 + y_2$ is solution of (1.4). Let λ be a real number

$$\lambda(y_1''(x) + 5y_1'(x) + 3y_1(x)) = 0. \tag{1.8}$$

Consequently

$$(\lambda y)_1''(x) + 5(\lambda y)_1'(x) + 3(\lambda y_1)(x)) = 0. \tag{1.9}$$

Hence λy_1 is solution of (1.4).

Exercice 1.3. Let $[\mathbf{M}]$ be a $n \times n$ matrix. Show that the solution of

$$[\mathbf{M}]\mathbf{X} = \mathbf{0} \tag{1.10}$$

is a vector space.

Solution 1.3. Let X_1 and X_2 be two solutions of (1.10):

$$[\mathbf{M}]\mathbf{X}_1 = \mathbf{0}, \quad [\mathbf{M}]\mathbf{X}_2 = \mathbf{0}. \tag{1.11}$$

By adding these two equations

$$[\mathbf{M}](\mathbf{X}_1 + \mathbf{X}_2) = \mathbf{0}. \tag{1.12}$$

Let λ be a real number

$$[\mathbf{M}](\lambda \mathbf{X}_1) = \mathbf{0}. \tag{1.13}$$

Hence $\lambda \mathbf{X}_1$ is solution of (1.10).

Exercice 1.4. What is the dimension of the nullspace[a] of matrix

$$[\mathbf{M}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{1.14}$$

[a] the set of vector $\mathbf{x}$ so that $[\mathbf{M}]\mathbf{x} = \mathbf{0}$

Solution 1.4. Let $\mathbf{x} = \{x, y, z\}$ be one element of the nullspace with x, y and z in $/mbR$. The relation between components are

$$x - y = 0, \tag{1.15}$$
$$-x + 2y - z = 0, \tag{1.16}$$
$$-y + z \qquad\qquad = 0. \tag{1.17}$$

Hence

$$x = y = z \tag{1.18}$$

The nullspace is then of dimension 1.

Exercice 1.5. Is the family of three vectors a basis of $\mathbb{R}^3$? If yes, is this basis orthonormal ?

$$[\mathbf{\Phi}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{1.19}$$

Solution 1.5. This family is a basis and this can be justified in several ways

- The three vectors are independent: Let suppose that we have a combination of the three vectors $\mathbf{\Phi}_i$ associated to the columns of the matrix so that

$$\lambda_1 \mathbf{\Phi}_1 + \lambda_2 \mathbf{\Phi}_2 + \lambda_3 \mathbf{\Phi}_3 = \mathbf{0} \tag{1.20}$$

The first line leads to $\lambda_1 = \lambda_2$ and the last one to $\lambda_2 = 2\lambda_3$. This allows to eliminate λ_1 and λ_3 in the second equation which reads:

Exercice 1.6. Let V be the subspace of $\mathbb{R}^3$ spanned by the two vectors $\mathbf{e}_1 = \begin{Bmatrix} \sqrt{1/2} \\ \sqrt{1/2} \\ 0 \end{Bmatrix}$ and $\mathbf{e}_2 = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}$

Find an orthonormal basis of V including $\mathbf{e}_1$

Solution 1.6.

Exercice 1.7. Show that the following applications define inner products on the given vector space E.

- $E = \mathbb{R}^n$, $< \mathbf{x}, \mathbf{y} >= \sum_{i=1}^{n} x_i y_i$.
- $E = \mathbb{R}^3$ $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^t [\mathbf{A}] \mathbf{y}$ with

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- $E = \mathbb{R}^2$ $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^t [\mathbf{A}] \mathbf{y}$ with

$$[\mathbf{A}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Solution 1.7.

Exercice 1.8. Show that the following applications define inner products on the given vector space E.

- $E = \mathbb{R}^n$, $< \mathbf{x}, \mathbf{y} >= \sum_{i=1}^{n} x_i y_i$.
- $E = \mathbb{R}^3$ $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^t [\mathbf{A}] \mathbf{y}$ with

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- $E = \mathbb{R}^2$ $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^t [\mathbf{A}] \mathbf{y}$ with

$$[\mathbf{A}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Solution 1.8.

Exercice 1.9. Let $u = \{u_n\}_{n \in \mathbb{N}}$ be a numerical sequence on $\mathbb{N}$. u is said square summable (or $u \in l^2(\mathbb{N})$) if:

$$u \in l^2(\mathbb{N}) \Leftrightarrow \sum_{n \in \mathbb{N}} u_n^2 < +\infty$$

- Show that $l^2(\mathbb{N})$ is a vector space.
- Show that $< u, v >= \sum_{n \in \mathbb{N}} u_n v_n$ is an inner product on $l^2(\mathbb{N})$.

- Show that $\mathcal{N}(u) = \sqrt{\sum_{n \in \mathbb{N}} u_n^2}$ is a norm on $l^2(\mathbb{N})$.

Solution 1.9.

Exercice 1.10. Let $\mathbb{P}_3$ be the vector space of polynomials with degree lower than 3. We consider the application:

$$< P, Q >= \int_0^1 P(x)Q(x)\,dx$$

Show that this application is an inner product.
Let $\{P_0 = 1, P_1 = x, P_2 = x^2, P_3 = x^3\}$ be a basis of this space.
Is this basis orthonormal?

Solution 1.10.

1.2 Approximation by polynoms and Least Mean Squares

Exercice 1.11.

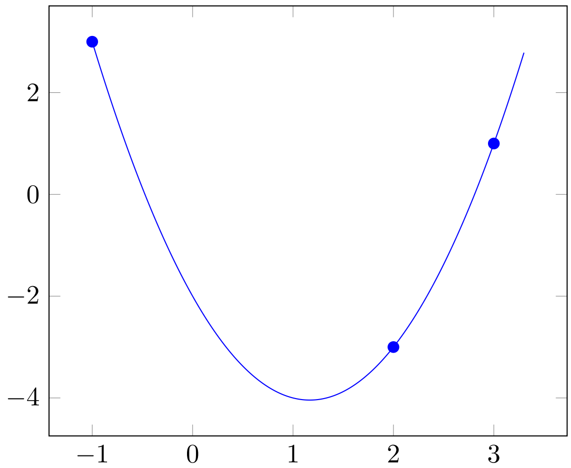
- What is the Taylor expansion at order 2 of the cos function in $x = 0$?
- What is the Taylor expansion at order 2 of the cos function in $x = \frac{\pi}{4}$?

Solution 1.11.

- $\cos(x) \approx 1 - \frac{x^2}{2} + \mathrm{o}(x^2)$
- $\cos(x) \approx \frac{\sqrt{2}}{2}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{\left(x - \frac{\pi}{4}\right)^2}{2}\right] + \mathrm{o}\left(\left(x - \frac{\pi}{4}\right)^2\right)$

Exercice 1.12. What is the Lagrange interpolation polynomial that at each point x_j assumes the corresponding value y_j with

$$\begin{aligned} x_1 &= -1 \quad & y_1 &= 3 \\ x_2 &= 2 \quad & y_1 &= -3 \\ x_3 &= 3 \quad & y_1 &= 1 \end{aligned}$$



Solution 1.12. We are looking for an interpolation polynomial associated to a set of 3 points. Its degree is then 2 and this polynomial is the sum of three elementary polynomials P_i. For each one of them the value of P_i in x_i is y_i and equal to zero at the other points x_j with $j \neq i$. The expression of P_1 is:

$$\begin{aligned} P_1 &= y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \\ &= 3\frac{(x-2)(x-3)}{(-1-2)(-1-3)} \\ &= \frac{1}{4}\left(x^2-5x+6\right) \end{aligned}$$

Similarily

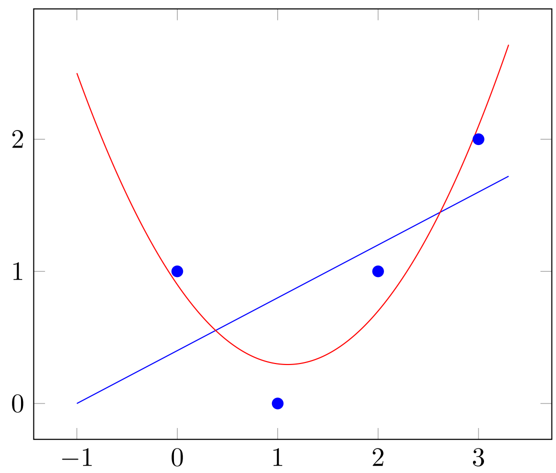
$$P_2 = (-3)\frac{(x+1)(x-3)}{(3)(-1)} = x^2 - 2x - 3$$
$$P_3 = (1)\frac{(x+1)(x-2)}{(4)(1)} = \frac{1}{4}\left(x^2 - x - 2\right)$$

Hence, the interpolation polynomial is:

$$P(x) = \frac{3}{2}x^2 - \frac{7}{2}x - 2 \tag{1.21}$$

Exercice 1.13. What is the best approximation in a least mean square sense by a polynomial of degree 1 of the following cloud of points $\{x_j, y_j\}$ with

$$\begin{aligned} x_1 &= 0 \quad y_1 = 1 \\ x_2 &= 1 \quad y_1 = 0 \\ x_3 &= 2 \quad y_1 = 1 \\ x_4 &= 3 \quad y_1 = 2 \end{aligned}$$



Solution 1.13. concerning the polynomial of degree 1, an approximation $P(x) = \alpha x + \beta$ is sought which is associated to the minimisation of

$$\mathcal{C}(\alpha, \beta) = \sum_{i=1}^{4} (y_i - (\alpha x_i + \beta))^2 \tag{1.22}$$

This can be rewritten in a matrical form:

$$\mathcal{C}(\alpha,\beta) = \left\| \mathbf{y} - [\mathbf{M}] \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \right\|^2 \tag{1.23}$$

with

$$[\mathbf{M}] = \left[\begin{array}{c|c} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{array}\right], \quad \mathbf{y} = \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{Bmatrix}. \tag{1.24}$$

The unknowns are then solution of

$$[\mathbf{M}]^T[\mathbf{M}] \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = [\mathbf{M}]^T \mathbf{y} \tag{1.25}$$

Hence the system is

$$\begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \tag{1.26}$$

whose solution is $\alpha = \beta = 0.4$.

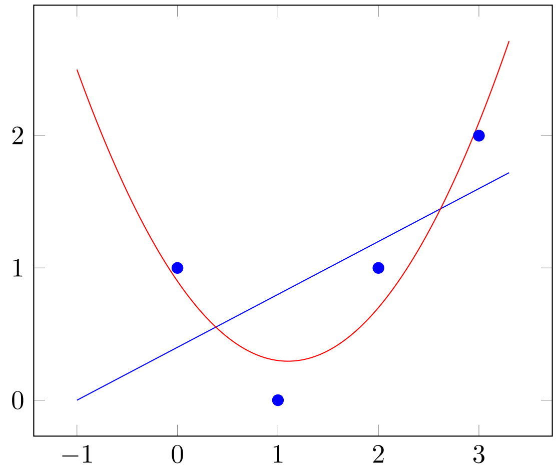
For the polynomial of degree 2 ($P(x) = \alpha^2 x + \beta x + \gamma$), the matrix $[\mathbf{M}]$ and the linear system read:

$$[\mathbf{M}] = \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{array}\right], \quad \begin{bmatrix} 98 & 36 & 14 \\ 36 & 14 & 6 \\ 14 & 6 & 4 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \\ \gamma \end{Bmatrix} = \begin{bmatrix} 22 \\ 8 \\ 4 \end{bmatrix}. \tag{1.27}$$

The solution is $\alpha = 0.5$, $\beta = -1.1$ and $\gamma = 0.9$

Exercice 1.14. What is the best approximation in a least mean square sense by a polynomial of degree 1 of the following cloud of points $\{x_j, y_j\}$ with

$$\begin{aligned} x_1 &= 0 \quad y_1 = 1 \\ x_2 &= 1 \quad y_1 = 0 \\ x_3 &= 2 \quad y_1 = 1 \\ x_4 &= 3 \quad y_1 = 2 \end{aligned}$$



Solution 1.14. concerning the polynomial of degree 1, an approximation $P(x) = \alpha x + \beta$ is sought which is associated to the minimisation of

$$\mathcal{C}(\alpha, \beta) = \sum_{i=1}^{4} (y_i - (\alpha x_i + \beta))^2 \tag{1.28}$$

This can be rewritten in a matrical form:

$$\mathcal{C}(\alpha, \beta) = \left\| \mathbf{y} - [\mathbf{M}] \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \right\|^2 \tag{1.29}$$

with

$$[\mathbf{M}] = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}, \quad \mathbf{y} = \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{Bmatrix}. \tag{1.30}$$

The unknowns are then solution of

$$[\mathbf{M}]^T[\mathbf{M}] \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = [\mathbf{M}]^T \mathbf{y} \tag{1.31}$$

Hence the system is

$$\begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \tag{1.32}$$

whose solution is $\alpha = \beta = 0.4$.

For the polynomial of degree 2 ($P(x) = \alpha^2 x + \beta x + \gamma$), the matrix $[\mathbf{M}]$ and the linear system read:

$$[\mathbf{M}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 98 & 36 & 14 \\ 36 & 14 & 6 \\ 14 & 6 & 4 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \\ \gamma \end{Bmatrix} = \begin{bmatrix} 22 \\ 8 \\ 4 \end{bmatrix}. \tag{1.33}$$

The solution is $\alpha = 0.5$, $\beta = -1.1$ and $\gamma = 0.9$

1.3 Singular Value Decomposition

Exercice 1.15. What is the SVD and the pseudo-inverse of the following matrix:

$$[\mathbf{M}] = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{1.34}$$

Solution 1.15.

$$[\mathbf{M}] = [\mathbf{U}][\mathbf{\Sigma}][\mathbf{V}]^t \tag{1.35}$$

with

$$[\mathbf{U}] = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \tag{1.36}$$

$$[\mathbf{\Sigma}] = \begin{bmatrix} \sqrt{2} & 0 \\ & \\ 0 & 0 \end{bmatrix} \tag{1.37}$$

$$[\mathbf{V}] = \begin{bmatrix} 0 & -1 \\ & \\ 1 & 0 \end{bmatrix} \tag{1.38}$$

1.4 Bessel Function

Exercice 1.16. Acoustic propagation in circular coordinates leads to the Bessel Equation:

$$r^2 p''(r) + rp'(r) + (r^2 - n^2)p(r) = 0, \quad n \in \mathbb{N}.$$

Show that the $J_n(r)$ function defined as a parametric integral is solution of this equation

$$J_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - r\sin(\theta))}\, d\theta$$

Solution 1.16. The first step consists in expressing the first and second derivative of $J_n(r)$. This is done by a partial derivation with respect to r of the function of the integral:

$$J_n'(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [i\sin(\theta)] e^{-i(n\theta - r\sin(\theta))}\, d\theta \tag{1.39}$$

$$J_n''(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [-\sin^2(\theta)] e^{-i(n\theta - r\sin(\theta))}\, d\theta \tag{1.40}$$

$$\tag{1.41}$$

Hence

$$\nu(r) = 2\pi[rJ_n''(r) + rJ_n'(r) + (r^2 - n^2)J_n(r)] \tag{1.42}$$
$$= \int_{-\pi}^{\pi} [-r^2\sin^2(\theta) + ri\sin(\theta) + r^2 - n^2] e^{-i(n\theta - r\sin(\theta))}\, d\theta \tag{1.43}$$
$$= \int_{-\pi}^{\pi} [r^2\cos^2(\theta) - n^2] e^{-i(n\theta - r\sin(\theta))}\, d\theta + \int_{-\pi}^{\pi} ri\sin(\theta) e^{-i(n\theta - r\sin(\theta))}\, d\theta \tag{1.44}$$

The first integral can be rewritten

$$\int_{-\pi}^{\pi} [r^2\cos^2(\theta) - n^2] e^{-i(n\theta - r\sin(\theta))}\, d\theta = \int_{-\pi}^{\pi} [r\cos(\theta) + n] \underbrace{[r\cos(\theta) - n] e^{-i(n\theta - r\sin(\theta))}}_{\frac{1}{i}\frac{\partial}{\partial\theta}\left[e^{-i(n\theta - r\sin(\theta))}\right]}\, d\theta \tag{1.45}$$

This can be integrated by part

$$\int_{-\pi}^{\pi} [r^2\cos^2(\theta) - n^2] e^{-i(n\theta - r\sin(\theta))}\, d\theta = \tag{1.46}$$
$$\underbrace{\left[(r\cos(\theta) + n)\frac{e^{-i(n\theta - r\sin(\theta))}}{i}\right]_{-\pi}^{\pi}}_{0} - \int_{-\pi}^{\pi} ri\sin(\theta) e^{-i(n\theta - r\sin(\theta))}\, d\theta \tag{1.47}$$

Hence this first part of $\nu(r)$ is the oppposite of the remaining one. $\nu(r)$ is then zero and the Bessel function J_n verifies the Bessel equation.

1.5 Chebishev polynomials

Exercice 1.17. Chebishev polynomials are defined by

$$T_n(x) = \cos(n \arccos(x))$$

- What are the expression of T_0, T_1 and T_2?
- Show that $T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$
- Show that the Chebishev polynoms are orthogonal with respect to:

$$< P, Q >= \int_{-1}^{1} \frac{P(x)Q(x)}{\sqrt{1 - x^2}}\, \mathrm{d}x$$

Solution 1.17.

$$T_0(x) = 1 \tag{1.48}$$
$$T_1(x) = x \tag{1.49}$$
$$T_2(x) = 2x^2 - 1 \tag{1.50}$$

1.6 Legendre Polynomials

Exercice 1.18. Let $\mathbb{P}[X]$ be the vector space of polynomials. We consider the application:

$$< P, Q >= \int_{-1}^{1} P(x)Q(x)\ \mathrm{d}x$$

- Show that this application is an inner product.

Let $\{P_0 = 1, P_1 = x, P_2 = \frac{3x^2-1}{2}\}$ be a family of 3 polynomials.

- Is this family orthonormal with respect to the inner product?
- If not,orthonormalize this family

Legendre equation L_n of order n is defined by:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + n(n+1)P = 0.$$

- Show that P_0, P_1 and P_2 are respectively solution of L_0, L_1 and L_2

Legendre polynomials P_n can be determined with a recurrence relation.

- Show that the recurrence relation is

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

- Show that:

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right].$$

Solution 1.18. the first step consists in