Refresher courses in Mathematics

Olivier DAZEL & Mathieu GABORIT

September 13, 2016

Chapter 1

Matrices

1.1 Algebraic Calculus

Exercice 1.1. Let consider the following matrices

$$[\mathbf{\Phi}(\xi)] = \begin{bmatrix} (1-\xi)/2 \\ \xi/2 \end{bmatrix} \tag{1.1}$$

Compute (if it exists)

$$\int_0^1 [\mathbf{\Phi}(\xi)][\mathbf{\Phi}(\xi)]^T \, \mathrm{d}\xi, \tag{1.2}$$

where ^{T} is the transposition.

Solution 1.1. The product of the matrices lead to:

$$[\mathbf{\Phi}(\xi)][\mathbf{\Phi}(\xi)]^T = \frac{1}{4} \begin{bmatrix} (1-\xi)^2 & (1-\xi)\xi \\ (1-\xi)\xi & \xi^2 \end{bmatrix} \tag{1.3}$$

The integration can be done term by term and

$$\int_0^1 [\mathbf{\Phi}(\xi)][\mathbf{\Phi}(\xi)]^T \, \mathrm{d}\xi = \frac{1}{4} \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \tag{1.4}$$

Exercice 1.2. The Forward Euler's Method is defined by the recurence relation:

$$\mathbf{S}_{k+1} = ([\mathbf{I}_2] + \Delta t[\boldsymbol{\alpha}])\,\mathbf{S}_k. \tag{1.5}$$

Compute the first three steps $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ of Forward Euler's Method for

$$[\boldsymbol{\alpha}] = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{S}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{1.6}$$

Solution 1.2.

$$\mathbf{S}_1 = \begin{Bmatrix} 1 \\ \Delta t \end{Bmatrix}, \quad \mathbf{S}_2 = \begin{Bmatrix} 1 - \omega_0^2 \Delta t^2 \\ 2\Delta t \end{Bmatrix}, \quad \mathbf{S}_2 = \begin{Bmatrix} 1 - 3\omega_0^2 \Delta t^2 \\ \Delta t\left(3 - \omega_0^2 \Delta t^2\right) \end{Bmatrix}. \tag{1.7}$$

Exercice 1.3. Compute $[\mathbf{M}]^n\,\mathbf{X}$ for

$$[\mathbf{M}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{1.8}$$

Solution 1.3.

$$[\mathbf{M}]\,\mathbf{X} = \begin{Bmatrix} 3 \\ -3 \end{Bmatrix} = 3\mathbf{X}. \tag{1.9}$$

$\mathbf{X}$ is then an eigenvector associated to eigenvalue $\lambda = 3$.

$$[\mathbf{M}]^2 \mathbf{X} = 3^2 \mathbf{X} \implies [\mathbf{M}]^n \mathbf{X} = 3^n \mathbf{X} = \begin{Bmatrix} 3^n \\ -3^n \end{Bmatrix}. \tag{1.10}$$

Exercice 1.4. What are the conditions all the possible matrix $[\mathbf{M}] \in \mathbb{R}^{2\times 2}$ which are their own inverse (i.e. $[\mathbf{M}]^{-1} = [\mathbf{M}]$)

$$[\mathbf{M}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{1.11}$$

Solution 1.4.

$$[\mathbf{M}]\,[\mathbf{M}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix} \tag{1.12}$$

Hence

$$\begin{cases} a^2 + bc = 1 \\ bc + d^2 = 1 \\ b(a+d) = 0 \\ c(a+d) = 0 \end{cases} \tag{1.13}$$

The combination of the first two equations provides $a^2 = d^2$. Then we have several cases

If $a = d = 0$. In this case, the condition is $bc = 1$.

If $a = d \neq 0$. In this case, b and c are zero due to the last two equations. Then, $a^2 = d^2 = 1$ so a and d are simultaneously 1 or -1

If $a = -d$ then the last two equations are satisfied and the condition is $a^2 + bc = 1$. If

$a^2 = 1$ then $bc = 0$ so either b or c is zero. The matrices are then

$$\begin{bmatrix} 0 & \beta \\ 1/\beta & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \pm \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}, \quad \pm \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix}, \quad \pm \begin{bmatrix} \alpha & \beta \\ (1-\alpha^2)/\beta & -\alpha \end{bmatrix} \tag{1.14}$$

with $\beta \neq 0$ and $\alpha^2 \neq \pm 1$

Exercice 1.5. Let consider the following matrix

$$[\mathbf{M}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.15}$$

- What is the value of $|[\mathbf{M}]|$?
- What is the value of $[\mathbf{M}]^5$?
- What is the value of $|[\mathbf{M}]|$?

Solution 1.5.

Exercice 1.6. The rotation matrices $[\mathbf{R}_x(\theta_x)]$, $[\mathbf{R}_y(\theta_y)]$ and $[\mathbf{R}_z(\theta_z)]$ are respectively defined by:

$$[\mathbf{R}_x(\theta_x)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x) & \sin(\theta_x) \\ 0 & -\sin(\theta_x) & \cos(\theta_x) \end{bmatrix}, \quad [\mathbf{R}_y(\theta_y)] = \begin{bmatrix} \cos(\theta_y) & 0 & \sin(\theta_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y) & 0 & \cos(\theta_y) \end{bmatrix}, \tag{1.16}$$

$$[\mathbf{R}_z(\theta_z)] = \begin{bmatrix} \cos(\theta_z) & \sin(\theta_z) & 0 \\ -\sin(\theta_z) & \cos(\theta_z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.17}$$

- What is the determinant of $[\mathbf{R}_x(\theta_x)][\mathbf{R}_y(\theta_y)][\mathbf{R}_z(\theta_z)]$?
- What is the value of
$$[\mathbf{R}_x(\pi/2)][\mathbf{R}_y(\pi/2)][\mathbf{R}_z(\pi/2)]$$
and the value of
$$[\mathbf{R}_z(\pi/2)][\mathbf{R}_y(\pi/2)][\mathbf{R}_x(\pi/2)]$$
? Comment this result.

Solution 1.6.

Exercice 1.7. The Hooke's matrix $[\mathbf{C}]$ for a material relates the vector of strain $\boldsymbol{\varepsilon}$ to the vector of stresses $\boldsymbol{\sigma}$.

$$\boldsymbol{\sigma} = [\mathbf{C}]\,\boldsymbol{\varepsilon}. \tag{1.18}$$

For an isotropic material, the Hooke's matrix depends on only two coefficients called the Lamé coefficients λ and μ and

$$[\mathbf{C}] = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \tag{1.19}$$

The Lamé coefficients λ and μ can be expressed from the young's modulus E and Poisson coeficient ν by

$$\lambda = \frac{1+\nu}{1-2\nu}, \quad \mu = \frac{E}{2(1+\nu)} \tag{1.20}$$

The compliance matrix $[\mathbf{S}]$ is defined as the inverse of $[\mathbf{C}]$.

What is the value of the compliance matrix as a function of E and ν?

Solution 1.7.

$$[\mathbf{S}] = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \tag{1.21}$$

1.2 Determinant of matrices

Exercice 1.8. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}? \tag{1.22}$$

Solution 1.8. The expansion along the first row leads to

$$|[\mathbf{M}]| = 1 \times \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix} = -3 \tag{1.23}$$

Exercice 1.9. What is the link between determinants of matrice [**M**] and [**N**] defined by

$$[\mathbf{M}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}, \quad [\mathbf{N}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 1 & 0 & 1 & 5 \\ 2 & 3 & 1 & 5 \end{bmatrix} \tag{1.24}$$

Solution 1.9. The two matrices are identical except in the last column. The column of [**N**] is 5 times the column of [**M**] then the determinant of [**N**] is 5 times the determinant of [**M**]

Exercice 1.10. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} ? \tag{1.25}$$

Solution 1.10.

$$|[\mathbf{M}]| = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (-3) \times (-2) = 6 \tag{1.26}$$

Exercice 1.11. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}? \tag{1.27}$$

Solution 1.11.

$$|[\mathbf{M}]| = (2-1)(3-1)(3-2) = 2 \tag{1.28}$$

Exercice 1.12. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}? \tag{1.29}$$

Solution 1.12.

$$|[\mathbf{M}]| = 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 6 - 2 = 4 \tag{1.30}$$

Exercice 1.13. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}? \tag{1.31}$$

Solution 1.13. The sum of the three lines is zero. They are then linearly dependent. The determinant is zero

Exercice 1.14. Given

$$[\mathbf{A}] = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 3 & 4 & 1 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 3 \end{bmatrix} \quad , \quad [\mathbf{P}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{1.32}$$

evaluate:

1. $[\mathbf{P}]\,[\mathbf{A}]$
2. $\det([\mathbf{A}])$
3. $\det([\mathbf{P}]\,[\mathbf{A}])$

What is the effect of $[\mathbf{P}]$ on $[\mathbf{A}]$?

Solution 1.14.

1.

$$[\mathbf{P}]\,[\mathbf{A}] = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 1 & 0 & 0 & 3 \\ 1 & 1 & 0 & 1 \\ 3 & 4 & 1 & 3 \end{bmatrix}$$

2. 2
3. -2

$[\mathbf{P}]$ *permutes* lines 2 and 4 of $[\mathbf{A}]$.

Exercice 1.15. Given

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix} \quad , \quad [\mathbf{P}_1] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad , \quad [\mathbf{P}_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{1.33}$$

evaluate:

1. $\det([\mathbf{A}])$
2. $\det([\mathbf{P}_1]\,[\mathbf{A}])$
3. $\det([\mathbf{P}_2]\,[\mathbf{A}])$

What is the property of determinants for permutated matrices?

Solution 1.15.

1. -3
2. -3
3. 3

The permutation matrix has a determinant of $(-1)^N$ with N the number of permutations.

Exercice 1.16. Given

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 9 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} \tag{1.34}$$

and $[\mathbf{0}]$ the $n \times n$ matrix of zeros;
evaluate

$$\begin{vmatrix} [\mathbf{A}] & \mathbf{0}_2 \\ \mathbf{0}_2 & [\mathbf{D}] \end{vmatrix}, \quad \begin{vmatrix} [\mathbf{A}] & [\mathbf{B}] \\ \mathbf{0}_2 & [\mathbf{D}] \end{vmatrix}, \quad \begin{vmatrix} [\mathbf{A}] & \mathbf{0}_2 \\ [\mathbf{C}] & [\mathbf{D}] \end{vmatrix} \tag{1.35}$$

What is the property of determinant illustrated by the previous calculations?

Solution 1.16. The three matrices have the same determinant (-16), the extra-diagonal *block* does not come in play. The following equivalence then holds:

$$\begin{vmatrix} [\mathbf{A}] & [\mathbf{B}] \\ \mathbf{0}_2 & [\mathbf{D}] \end{vmatrix} = \begin{vmatrix} [\mathbf{A}] & \mathbf{0}_2 \\ [\mathbf{C}] & [\mathbf{D}] \end{vmatrix} = \det([\mathbf{A}])\det([\mathbf{D}]) \tag{1.36}$$

1.3 Change of basis

Exercice 1.17. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ in the $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ basis

Solution 1.17.

$$[\mathbf{M}'] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercice 1.18. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ in the $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ basis

Solution 1.18.

$$[\mathbf{M}'] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Exercice 1.19. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ in the $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ basis

Solution 1.19.

$$[\mathbf{M}'] = \begin{bmatrix} 0 & -1 \\ 1 & 4 \end{bmatrix}$$

1.4 Eigenvalues and Eigenvectors

Exercice 1.20. What are the eigenvalues of

$$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

For each eigenvalue λ, propose one eigenvector $\mathbf{X}$

Solution 1.20.

$$\lambda = 1, \quad \mathbf{X} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\lambda = 2, \quad \mathbf{X} = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

Exercice 1.21. What are the eigenvalues of

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

For each eigenvalue λ, propose one eigenvector $\mathbf{X}$

Solution 1.21.

$$\lambda = 1, \quad \mathbf{X} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\lambda = 3, \quad \mathbf{X} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Exercice 1.22. Let $\xi \neq 1$. What are the eigenvalues p and the associated eigenvectors of

$$\begin{bmatrix} -2\omega_0\xi & -\omega_0^2 \\ 1 & 0 \end{bmatrix} \tag{1.37}$$

Solution 1.22.

$$p^{\pm} = \omega_0 \left(-\xi \pm \sqrt{\xi^2 - 1}\right). \tag{1.38}$$

One eigenvector is

$$\mathbf{X}^{\pm} = \begin{Bmatrix} p^{\pm} \\ 1 \end{Bmatrix} \tag{1.39}$$

Exercice 1.23. Compute the eigenvalues δ and the associated eigenvectors $\mathbf{\Phi}$ of problem

$$[\mathbf{K}]\, \mathbf{\Phi} = \delta^2\, [\mathbf{M}]\, \mathbf{\Phi}, \tag{1.40}$$

with

$$[\mathbf{K}] = \begin{bmatrix} \hat{P} & 0 \\ 0 & \widetilde{K}_{eq} \end{bmatrix} [\mathbf{M}] = \begin{bmatrix} \widetilde{\rho}_s & \widetilde{\gamma}\widetilde{\rho}_{eq} \\ \widetilde{\gamma}\widetilde{\rho}_{eq} & \widetilde{\rho}_{eq} \end{bmatrix} \tag{1.41}$$

Solution 1.23.

1.5 Resolution of linear systems

Exercice 1.24. What is the solution of

$$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 5 \\ 3 \end{Bmatrix}$$

Solution 1.24.

$$\mathbf{X} = \begin{Bmatrix} \frac{5}{3} \\ \frac{3}{5} \end{Bmatrix}$$

Exercice 1.25. What is the solution of

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix}$$

Solution 1.25.

$$\mathbf{X} = \begin{Bmatrix} -1 \\ -2 \\ -3 \end{Bmatrix}$$

Exercice 1.26. What is the solution of

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 4 \\ 12 \\ 38 \end{Bmatrix}$$

Solution 1.26.

$$\mathbf{X} = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix}$$

Exercice 1.27. What is the solution of

$$\begin{bmatrix} 3 & 2 & 7 \\ 5 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 9 \\ 3 \\ 2 \end{Bmatrix}$$

Solution 1.27.

$$\mathbf{X} = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$