

Refresher courses in Mathematics
Matrices

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Chapter 1

Matrices

1.1 Algebraic Calculus



Exercise 1.1. Let consider the following matrices

$$[\Phi(\xi)] = \begin{bmatrix} (1 - \xi)/2 \\ \xi/2 \end{bmatrix} \quad (1.1)$$

Compute (if it exists)

$$\int_0^1 [\Phi(\xi)][\Phi(\xi)]^T d\xi, \quad (1.2)$$

where T is the transposition.



Exercise 1.2. The Forward Euler's Method is defined by the recurrence relation:

$$\mathbf{S}_{k+1} = ([\mathbf{I}_2] + \Delta t[\boldsymbol{\alpha}]) \mathbf{S}_k. \quad (1.3)$$

Compute the first three steps \mathbf{S}_1 , \mathbf{S}_2 and \mathbf{S}_3 of Forward Euler's Method for

$$[\boldsymbol{\alpha}] = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{S}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad (1.4)$$



Exercise 1.3. Compute $[\mathbf{M}]^n \mathbf{X}$ for

$$[\mathbf{M}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (1.5)$$



Exercise 1.4. What are the conditions all the possible matrix $[\mathbf{M}] \in \mathbb{R}^{2 \times 2}$ which are their own inverse (i.e. $[\mathbf{M}]^{-1} = [\mathbf{M}]$)

$$[\mathbf{M}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \quad (1.6)$$



Exercise 1.5. Let consider the following matrix

$$[\mathbf{M}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.7)$$

- What is the value of $||[\mathbf{M}]||$?
- What is the value of $[\mathbf{M}]^5$?
- What is the value of $||[\mathbf{M}]||$?



Exercise 1.6. The rotation matrices $[\mathbf{R}_x(\theta_x)]$, $[\mathbf{R}_y(\theta_y)]$ and $[\mathbf{R}_z(\theta_z)]$ are respectively

defined by:

$$[\mathbf{R}_x(\theta_x)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x) & \sin(\theta_x) \\ 0 & -\sin(\theta_x) & \cos(\theta_x) \end{bmatrix}, \quad [\mathbf{R}_y(\theta_y)] = \begin{bmatrix} \cos(\theta_y) & 0 & \sin(\theta_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y) & 0 & \cos(\theta_y) \end{bmatrix}, \quad (1.8)$$

$$[\mathbf{R}_z(\theta_z)] = \begin{bmatrix} \cos(\theta_z) & \sin(\theta_z) & 0 \\ -\sin(\theta_z) & \cos(\theta_z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.9)$$

- What is the determinant of $[\mathbf{R}_x(\theta_x)][\mathbf{R}_y(\theta_y)][\mathbf{R}_z(\theta_z)]$?
- What is the value of $[\mathbf{R}_x(\pi/2)][\mathbf{R}_y(\pi/2)][\mathbf{R}_z(\pi/2)]$ and the value of $[\mathbf{R}_z(\pi/2)][\mathbf{R}_y(\pi/2)][\mathbf{R}_x(\pi/2)]$? Comment this result.

Compute the following transformations:



Exercise 1.7. The Hooke's matrix $[\mathbf{C}]$ for a material relates the vector of strain $\boldsymbol{\varepsilon}$ to the vector of stresses $\boldsymbol{\sigma}$.

$$\boldsymbol{\sigma} = [\mathbf{C}] \boldsymbol{\varepsilon}. \quad (1.10)$$

For an isotropic material, the Hooke's matrix depends on only two coefficients called the Lam coefficients λ and μ and

$$[\mathbf{C}] = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \quad (1.11)$$

The compliance matrix $[\mathbf{S}]$ is defined as the inverse of $[\mathbf{C}]$.

What is the value of the compliance matrix ?

1.2 Determinant of matrices



Exercise 1.8. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix} ? \quad (1.12)$$



Exercise 1.9. What is the link between determinants of matrix $[\mathbf{M}]$ and $[\mathbf{N}]$ defined by

$$[\mathbf{M}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}, \quad [\mathbf{N}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 1 & 0 & 1 & 5 \\ 2 & 3 & 1 & 5 \end{bmatrix} \quad (1.13)$$



Exercise 1.10. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} ? \quad (1.14)$$



Exercise 1.11. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} ? \quad (1.15)$$



Exercise 1.12. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} ? \quad (1.16)$$



Exercise 1.13. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} ? \quad (1.17)$$

1.3 Operation on Matrices

1.4 Change of basis



Exercise 1.14. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ in the $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ basis



Exercise 1.15. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ in the $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ basis



Exercise 1.16. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ in the $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ basis



Exercise 1.17. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$ in the $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ basis



Exercise 1.18. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$ in the $\begin{bmatrix} 2 & 10 \\ 3 & 5 \end{bmatrix}$ basis

1.5 Eigenvalues and Eigenvectors



Exercise 1.19. Let $\xi \neq 1$. What are the eigenvalues p and the associated eigenvectors

of

$$\begin{bmatrix} -2\omega_0\xi & -\omega_0^2 \\ 1 & 0 \end{bmatrix} \quad (1.18)$$



Exercise 1.20. Compute the eigenvalues δ and the associated eigenvectors Φ of problem

$$[\mathbf{K}] \Phi = \delta^2 [\mathbf{M}] \Phi, \quad (1.19)$$

with

$$[\mathbf{K}] = \begin{bmatrix} \hat{P} & 0 \\ 0 & \tilde{K}_{eq} \end{bmatrix} \quad [\mathbf{M}] = \begin{bmatrix} \tilde{\rho}_s & \tilde{\gamma}\tilde{\rho}_{eq} \\ \tilde{\gamma}\tilde{\rho}_{eq} & \tilde{\rho}_{eq} \end{bmatrix} \quad (1.20)$$

1.6 Resolution of linear systems



Exercise 1.21. What is the solution of

$$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 5 \\ 3 \end{Bmatrix}$$



Exercise 1.22. What is the solution of

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix}$$



Exercise 1.23. What is the solution of

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 4 \\ 12 \\ 38 \end{Bmatrix}$$



Exercise 1.24. What is the solution of

$$\begin{bmatrix} 3 & 2 & 7 \\ 5 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 9 \\ 3 \\ 2 \end{Bmatrix}$$



Exercise 1.25. What is the solution of

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 18 \\ 6 \\ 14 \end{Bmatrix}$$



Exercise 1.26. What is the solution of

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 5 & 4 & 3 & 2 \\ 0 & 0 & 2 & -1 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 4 \\ 9 \\ 24 \\ 5 \end{Bmatrix}$$