Refresher courses in Mathematics

Matrices

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Chapter 1

Matrices

1.1 Algebraic Calculus

Exercice 1.1. Let consider the following matrices

$$[\mathbf{\Phi}(\xi)] = \begin{bmatrix} (1-\xi)/2 \\ \xi/2 \end{bmatrix} \tag{1.1}$$

Compute (if it exists)

$$\int_0^1 [\mathbf{\Phi}(\xi)][\mathbf{\Phi}(\xi)]^T \, \mathrm{d}\xi, \tag{1.2}$$

where ^{T} is the transposition.

Exercice 1.2. The Forward Euler's Method is defined by the recurence relation:

$$\mathbf{S}_{k+1} = ([\mathbf{I}_2] + \Delta t[\boldsymbol{\alpha}])\, \mathbf{S}_k. \tag{1.3}$$

Compute the first three steps $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ of Forward Euler's Method for

$$[\boldsymbol{\alpha}] = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{S}_0 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{1.4}$$

Exercice 1.3. Compute $[\mathbf{M}]^n \mathbf{X}$ for

$$[\mathbf{M}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \tag{1.5}$$

Exercice 1.4. What are the conditions all the possible matrix $[\mathbf{M}] \in \mathbb{R}^{2\times 2}$ which are their own inverse (i.e. $[\mathbf{M}]^{-1} = [\mathbf{M}]$)

$$[\mathbf{M}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{1.6}$$

Exercice 1.5. Let consider the following matrix

$$[\mathbf{M}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.7}$$

- What is the value of $|[\mathbf{M}]|$?
- What is the value of $[\mathbf{M}]^5$?
- What is the value of $|[\mathbf{M}]|$?

Exercice 1.6. The rotation matrices $[\mathbf{R}_x(\theta_x)]$, $[\mathbf{R}_y(\theta_y)]$ and $[\mathbf{R}_z(\theta_z)]$ are respectively defined by:

$$[\mathbf{R}_x(\theta_x)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x) & \sin(\theta_x) \\ 0 & -\sin(\theta_x) & \cos(\theta_x) \end{bmatrix}, \quad [\mathbf{R}_y(\theta_y)] = \begin{bmatrix} \cos(\theta_y) & 0 & \sin(\theta_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y) & 0 & \cos(\theta_y) \end{bmatrix}, \tag{1.8}$$

$$[\mathbf{R}_z(\theta_z)] = \begin{bmatrix} \cos(\theta_z) & \sin(\theta_z) & 0 \\ -\sin(\theta_z) & \cos(\theta_z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.9}$$

- What is the determinant of $[\mathbf{R}_x(\theta_x)][\mathbf{R}_y(\theta_y)][\mathbf{R}_z(\theta_z)]$?
- What is the value of $[\mathbf{R}_x(\pi/2)][\mathbf{R}_y(\pi/2)][\mathbf{R}_z(\pi/2)]$ and the value of $[\mathbf{R}_z(\pi/2)][\mathbf{R}_y(\pi/2)][\mathbf{R}_x(\pi/2)]$? Comment this result.

Compute the following transformations:

Exercice 1.7. The Hooke's matrix $[\mathbf{C}]$ for a material relates the vector of strain $\boldsymbol{\varepsilon}$ to the vector of stresses $\boldsymbol{\sigma}$.

$$\boldsymbol{\sigma} = [\mathbf{C}]\,\boldsymbol{\varepsilon}. \tag{1.10}$$

For an isotropic material, the Hooke's matrix depends on only two coefficients called the Lamé coefficients λ and μ and

$$[\mathbf{C}] = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \tag{1.11}$$

The Lamé coefficients λ and μ can be expressed from the young's modulus E and Poisson coeficient ν by

$$\lambda = \frac{1+\nu}{1-2\nu}, \quad \mu = \frac{E}{2(1+\nu)} \tag{1.12}$$

The compliance matrix $[\mathbf{S}]$ is defined as the inverse of $[\mathbf{C}]$.

What is the value of the compliance matrix as a function of E and ν?

1.2 Determinant of matrices

Exercice 1.8. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}? \tag{1.13}$$

Exercice 1.9. What is the link between determinants of matrice $[\mathbf{M}]$ and $[\mathbf{N}]$ defined by

$$[\mathbf{M}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}, \quad [\mathbf{N}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 1 & 0 & 1 & 5 \\ 2 & 3 & 1 & 5 \end{bmatrix} \tag{1.14}$$

Exercice 1.10. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}? \tag{1.15}$$

Exercice 1.11. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}? \tag{1.16}$$

Exercice 1.12. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}? \tag{1.17}$$

Exercice 1.13. What is the determinant of

$$[\mathbf{M}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}? \tag{1.18}$$

Exercice 1.14. Given

$$[\mathbf{A}] = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 3 & 4 & 1 & 3 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 3 \end{bmatrix}, \quad [\mathbf{P}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{1.19}$$

evaluate:

1. $[\mathbf{P}]\,[\mathbf{A}]$
2. $\det([\mathbf{A}])$
3. $\det([\mathbf{P}]\,[\mathbf{A}])$

What is the effect of $[\mathbf{P}]$ on $[\mathbf{A}]$?

Exercice 1.15. Given

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix} \quad , \quad [\mathbf{P}]_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad , \quad [\mathbf{P}]_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{1.20}$$

evaluate:

1. $\det([\mathbf{A}])$
2. $\det([\mathbf{P}]_1 [\mathbf{A}])$
3. $\det([\mathbf{P}]_2 [\mathbf{A}])$

Can you deduce a more general rule for the determinants of permutated matrices?

Exercice 1.16. Given

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 7 & 9 \\ 0 & 2 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix} \quad , \quad D = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix} \tag{1.21}$$

and $\mathbf{0}_n$ a $n \times n$ matrix of zeros;
evaluate

$$\begin{vmatrix} [\mathbf{A}] & \mathbf{0}_2 \\ \mathbf{0}_2 & [\mathbf{D}] \end{vmatrix} \quad , \quad \begin{vmatrix} [\mathbf{A}] & [\mathbf{B}] \\ \mathbf{0}_2 & [\mathbf{D}] \end{vmatrix} \quad , \quad \begin{vmatrix} [\mathbf{A}] & \mathbf{0}_2 \\ [\mathbf{C}] & [\mathbf{D}] \end{vmatrix} \tag{1.22}$$

What rule(s) can you deduce from the previous calculations?

1.3 Operation on Matrices

1.4 Change of basis

Exercice 1.17. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ in the $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ basis

Exercice 1.18. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ in the $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$ basis

Exercice 1.19. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ in the $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ basis

Exercice 1.20. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$ in the $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ basis

Exercice 1.21. What is the image $[\mathbf{M}']$ of $[\mathbf{M}] = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$ in the $\begin{bmatrix} 2 & 10 \\ 3 & 5 \end{bmatrix}$ basis

1.5 Eigenvalues and Eigenvectors

Exercice 1.22. Let $\xi \neq 1$. What are the eigenvalues p and the associated eigenvectors of

$$\begin{bmatrix} -2\omega_0\xi & -\omega_0^2 \\ 1 & 0 \end{bmatrix} \tag{1.23}$$

<u>Exercice</u> 1.23. Compute the eigenvalues δ and the associated eigenvectors $\boldsymbol{\Phi}$ of problem

$$[\mathbf{K}]\,\boldsymbol{\Phi} = \delta^2\,[\mathbf{M}]\,\boldsymbol{\Phi}, \tag{1.24}$$

with

$$[\mathbf{K}] = \begin{bmatrix} \hat{P} & 0 \\ 0 & \widetilde{K}_{eq} \end{bmatrix} [\mathbf{M}] = \begin{bmatrix} \widetilde{\rho}_s & \widetilde{\gamma}\widetilde{\rho}_{eq} \\ \widetilde{\gamma}\widetilde{\rho}_{eq} & \widetilde{\rho}_{eq} \end{bmatrix} \tag{1.25}$$

1.6 Resolution of linear systems

<u>Exercice</u> 1.24. What is the solution of

$$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 5 \\ 3 \end{Bmatrix}$$

<u>Exercice</u> 1.25. What is the solution of

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{X} = \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix}$$

Exercice 1.26. What is the solution of

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{bmatrix} \mathbf{X} = \left\{ \begin{matrix} 4 \\ 12 \\ 38 \end{matrix} \right\}$$

Exercice 1.27. What is the solution of

$$\begin{bmatrix} 3 & 2 & 7 \\ 5 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \mathbf{X} = \left\{ \begin{matrix} 9 \\ 3 \\ 2 \end{matrix} \right\}$$

Exercice 1.28. What is the solution of

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix} \mathbf{X} = \left\{ \begin{matrix} 18 \\ 6 \\ 14 \end{matrix} \right\}$$

<u>Exercice</u> 1.29. What is the solution of

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 5 & 4 & 3 & 2 \\ 0 & 0 & 2 & -1 \end{bmatrix} \mathbf{X} = \left\{ \begin{matrix} 4 \\ 9 \\ 24 \\ 5 \end{matrix} \right\}$$