Mathematics for Acousticians

Exercices

1.1 Vector spaces

Exercice 1.1. Show that the set $\mathbb{R}_n[X]$ of polynomials of degree lower than $n \in \mathbb{N}$ is a vector space.

Solution 1.1. Let P and Q be two polynomials of degree lower than n. Hence there exists two sequences $\{p_k\}_{k=0..n}$ and $\{q_k\}_{k=0..n}$ so that

$$P(X) = \sum_{k=0}^{n} p_k X^K, \quad Q(X) = \sum_{k=0}^{n} q_k X^K. \tag{1.1}$$

Some of the elements of these sequences can be equal to zero as the degree of these polynomials is not necessarily n. The sum S of P and Q is

$$S(X) = \sum_{k=0}^{n} (p_k + q_k) X^K \tag{1.2}$$

This polynomial is trivially of degree lower than n.
Let now consider a real numbre λ, the polynomial λP defined by

$$\lambda P(X) = \sum_{k=0}^{n} \lambda p_k X^K \tag{1.3}$$

is of degree lower than n.

Exercice 1.2. Show that the set of functions y solution of

$$y''(x) + 5y'(x) + 3y(x) = 0 \tag{1.4}$$

is a vector space.

Solution 1.2. Let y_1 and y_2 be two solutions of (1.4):

$$y_1''(x) + 5y_1'(x) + 3y_1(x) = 0, \quad y_2''(x) + 5y_2'(x) + 3y_2(x) = 0 \tag{1.5}$$

By adding these two equations

$$y_1''(x) + y_2''(x) + 5(y_1'(x) + y_2'(x)) + 3(y_1(x) + y_2(x)) = 0. \tag{1.6}$$

By linearity of the derivation

$$(y_1 + y_2)''(x) + 5(y_1 + y_2)'(x) + 3(y_1(x) + y_2(x)) = 0. \tag{1.7}$$

Hence $y_1 + y_2$ is solution of (1.4). Let λ be a real number

$$\lambda(y_1''(x) + 5y_1'(x) + 3y_1(x)) = 0. \tag{1.8}$$

Consequently

$$(\lambda y)_1''(x) + 5(\lambda y)_1'(x) + 3(\lambda y_1)(x)) = 0. \tag{1.9}$$

Hence λy_1 is solution of (1.4).

Exercice 1.3. Let $[\mathbf{M}]$ be a $n \times n$ matrix. Show that the solution of

$$[\mathbf{M}]\mathbf{X} = \mathbf{0} \tag{1.10}$$

is a vector space.

Solution 1.3. Let X_1 and X_2 be two solutions of (1.10):

$$[\mathbf{M}]\mathbf{X}_1 = \mathbf{0}, \quad [\mathbf{M}]\mathbf{X}_2 = \mathbf{0}. \tag{1.11}$$

By adding these two equations

$$[\mathbf{M}](\mathbf{X}_1 + \mathbf{X}_2) = \mathbf{0}. \tag{1.12}$$

Let λ be a real number

$$[\mathbf{M}](\lambda \mathbf{X}_1) = \mathbf{0}. \tag{1.13}$$

Hence $\lambda \mathbf{X}_1$ is solution of (1.10).

Exercice 1.4. What is the dimension of the nullspace[a] of matrix

$$[\mathbf{M}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{1.14}$$

[a] the set of vector $\mathbf{x}$ so that $[\mathbf{M}]\mathbf{x} = \mathbf{0}$

Solution 1.4. Let $\mathbf{x} = \{x, y, z\}$ be one element of the nullspace with x, y and z in $/mbR$. The relation between components are

$$x - y = 0, \tag{1.15}$$
$$-x + 2y - z = 0, \tag{1.16}$$
$$-y + z = 0. \tag{1.17}$$

Hence

$$x = y = z \tag{1.18}$$

The nullspace is then of dimension 1.

Exercice 1.5. Is the family of three vectors a basis of $\mathbb{R}^3$? If yes, is this basis orthonormal ?

$$[\mathbf{\Phi}] = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{1.19}$$

Solution 1.5. This family is a basis and this can be justified in several ways

- The three vectors are independent: Let suppose that we have a combination of the three vectors $\mathbf{\Phi}_i$ associated to the columns of the matrix so that

$$\lambda_1 \mathbf{\Phi}_1 + \lambda_2 \mathbf{\Phi}_2 + \lambda_3 \mathbf{\Phi}_3 = \mathbf{0} \tag{1.20}$$

The first line leads to $\lambda_1 = \lambda_2$ and the last one to $\lambda_2 = 2\lambda_3$. This allows to eliminate λ_1 and λ_3 in the second equation which reads:

Exercice 1.6. Let V be the subspace of $\mathbb{R}^3$ spanned by the two vectors $\mathbf{e}_1 = \begin{Bmatrix} \sqrt{1/2} \\ \sqrt{1/2} \\ 0 \end{Bmatrix}$ and $\mathbf{e}_2 = \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix}$

Find an orthonormal basis of V including $\mathbf{e}_1$

Solution 1.6.

Exercice 1.7. Show that the following applications define inner products on the given vector space E.

- $E = \mathbb{R}^n$, $< \mathbf{x}, \mathbf{y} >= \sum_{i=1}^{n} x_i y_i$.
- $E = \mathbb{R}^3$ $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^t [\mathbf{A}] \mathbf{y}$ with

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- $E = \mathbb{R}^2$ $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^t [\mathbf{A}] \mathbf{y}$ with

$$[\mathbf{A}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Solution 1.7.

Exercice 1.8. Show that the following applications define inner products on the given vector space E.

- $E = \mathbb{R}^n$, $< \mathbf{x}, \mathbf{y} >= \sum_{i=1}^{n} x_i y_i$.
- $E = \mathbb{R}^3$ $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^t[\mathbf{A}]\mathbf{y}$ with

$$[\mathbf{A}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- $E = \mathbb{R}^2$ $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^t[\mathbf{A}]\mathbf{y}$ with

$$[\mathbf{A}] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Solution 1.8.

Exercice 1.9. Let $u = \{u_n\}_{n \in \mathbb{N}}$ be a numerical sequence on $\mathbb{N}$. u is said square summable (or $u \in l^2(\mathbb{N})$) if:

$$u \in l^2(\mathbb{N}) \Leftrightarrow \sum_{n \in \mathbb{N}} u_n^2 < +\infty$$

- Show that $l^2(\mathbb{N})$ is a vector space.
- Show that $< u, v >= \sum_{n \in \mathbb{N}} u_n v_n$ is an inner product on $l^2(\mathbb{N})$.

- Show that $\mathcal{N}(u) = \sqrt{\sum_{n \in \mathbb{N}} u_n^2}$ is a norm on $l^2(\mathbb{N})$.

Solution 1.9.

Exercice 1.10. Let $\mathbb{P}_3$ be the vector space of polynomials with degree lower than 3. We consider the application:

$$< P, Q > = \int_0^1 P(x)Q(x)\,dx$$

Show that this application is an inner product.
Let $\{P_0 = 1, P_1 = x, P_2 = x^2, P_3 = x^3\}$ be a basis of this space.
Is this basis orthonormal?

Solution 1.10.

1.2 Approximation by polynoms and Least Mean Squares

Exercice 1.11.

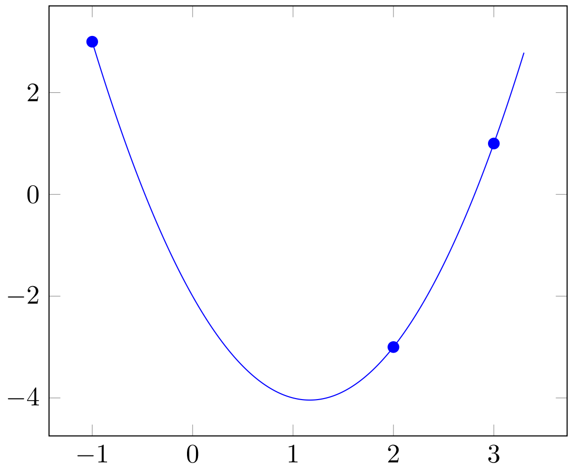
- What is the Taylor expansion at order 2 of the cos function in $x = 0$?
- What is the Taylor expansion at order 2 of the cos function in $x = \frac{\pi}{4}$?

Solution 1.11.

- $\cos(x) \approx 1 - \frac{x^2}{2} + \mathrm{o}(x^2)$
- $\cos(x) \approx \frac{\sqrt{2}}{2}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{\left(x - \frac{\pi}{4}\right)^2}{2}\right] + \mathrm{o}\left(\left(x - \frac{\pi}{4}\right)^2\right)$

Exercice 1.12. What is the Lagrange interpolation polynomial that at each point x_j assumes the corresponding value y_j with

$$
\begin{aligned}
x_1 = -1 \quad & y_1 = 3 \\
x_2 = 2 \quad & y_1 = -3 \\
x_3 = 3 \quad & y_1 = 1
\end{aligned}
$$



Solution 1.12. We are looking for an interpolation polynomial associated to a set of 3 points. Its degree is then 2 and this polynomial is the sum of three elementary polynomials P_i. For each one of them the value of P_i in x_i is y_i and equal to zero at the other points x_j with $j \neq i$. The expression of P_1 is:

$$
\begin{aligned}
P_1 &= y_1 \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} \\
&= 3\frac{(x-2)(x-3)}{(-1-2)(-1-3)} \\
&= \frac{1}{4}\left(x^2-5x+6\right)
\end{aligned}
$$

Similarily

$$P_2 = (-3)\frac{(x+1)(x-3)}{(3)(-1)} = x^2 - 2x - 3$$

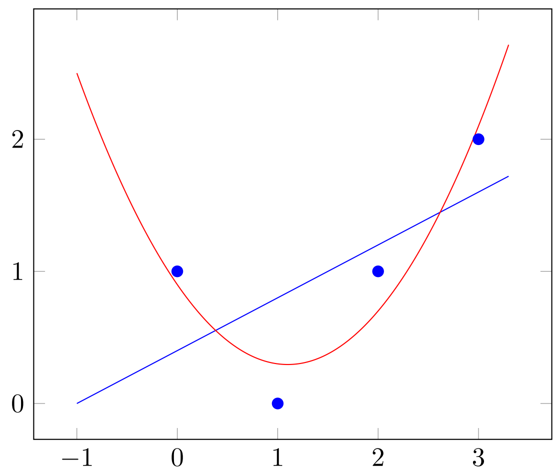
$$P_3 = (1)\frac{(x+1)(x-2)}{(4)(1)} = \frac{1}{4}\left(x^2 - x - 2\right)$$

Hence, the interpolation polynomial is:

$$P(x) = \frac{3}{2}x^2 - \frac{7}{2}x - 2 \tag{1.21}$$

Exercice 1.13. What is the best approximation in a least mean square sense by a polynomial of degree 1 of the following cloud of points $\{x_j, y_j\}$ with

$$\begin{aligned} x_1 &= 0 \quad y_1 = 1 \\ x_2 &= 1 \quad y_1 = 0 \\ x_3 &= 2 \quad y_1 = 1 \\ x_4 &= 3 \quad y_1 = 2 \end{aligned}$$



Solution 1.13. concerning the polynomial of degree 1, an approximation $P(x) = \alpha x + \beta$ is sought which is associated to the minimisation of

$$\mathcal{C}(\alpha, \beta) = \sum_{i=1}^{4} (y_i - (\alpha x_i + \beta))^2 \tag{1.22}$$

This can be rewritten in a matrical form:

$$\mathcal{C}(\alpha,\beta) = \left\| \mathbf{y} - [\mathbf{M}] \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \right\|^2 \tag{1.23}$$

with

$$[\mathbf{M}] = \left[\begin{array}{c|c} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{array}\right], \quad \mathbf{y} = \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{Bmatrix}. \tag{1.24}$$

The unknowns are then solution of

$$[\mathbf{M}]^T[\mathbf{M}] \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = [\mathbf{M}]^T\mathbf{y} \tag{1.25}$$

Hence the system is

$$\begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \tag{1.26}$$

whose solution is $\alpha = \beta = 0.4$.

For the polynomial of degree 2 ($P(x) = \alpha^2 x + \beta x + \gamma$), the matrix $[\mathbf{M}]$ and the linear system read:

$$[\mathbf{M}] = \left[\begin{array}{c|c|c} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{array}\right], \quad \begin{bmatrix} 98 & 36 & 14 \\ 36 & 14 & 6 \\ 14 & 6 & 4 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \\ \gamma \end{Bmatrix} = \begin{bmatrix} 22 \\ 8 \\ 4 \end{bmatrix}. \tag{1.27}$$

The solution is $\alpha = 0.5$, $\beta = -1.1$ and $\gamma = 0.9$