PLANES Documentation

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Contents

Chapter 1

General overview

1.1 Presentation

PLANES (Porous LAum NumErical Simulator) project is a collection of Matlab/Fortran scripts to simulate the vibroacoustics response of coupled systems including acoustic, elastic, porous materials, PML...

Chapter 2

Tables

- interface(n_1,n_2,e_1,e_2,n_{middle})
- boundary(n_1,n_2,typ(e),typ(boundary),0,n_{middle})

Chapter 3

Figures

The number of the figure is xxxyyy. xxx is associated to the the type of display and yyy corresponds to the type of field

Type of display xxx

- 1yyy: Modulus of the projection on x axis
- 2yyy: Modulus of the projection on y axis
- 3yyy: Angle of the projection on x axis
- 4yyy: Angle of the projection on y axis
- 5000: Shape of the displacement
- 10yyy: Map of the modulus for FEM
- 11yyy: Map of the angle for FEM
- 20yyy: Map of the modulus for DGM
- 21yyy: Map of the angle for DGM
- 30yyy: Map of the modulus in case of a FEM/DGM model
- 31yyy: Map of the angle in case of a FEM/DGM model

Type of fields xxx

- xxx1: v_x of the fluid or v_x^{eq} for an equivalent fluid or u_x^t for a Biot PEM.
- xxx2: v_y of the fluid or v_y^{eq} for an equivalent fluid or v_y^t for a Biot PEM
- xxx3: $|v|$ of the fluid or $|v^{eq}|$ for an equivalent fluid or u_y^t for a Biot PEM

- xxx4: v_x^s for a Biot PEM.
- xxx5: v_y^s for a Biot PEM
- xx10: p

Chapter 4

Weak forms

4.1 Finite-Elements

4.1.1 Fluid

Unknown: the pressure field p. ρ is the density

$$\forall\, q, \quad \int_{\Omega} \frac{\boldsymbol{\nabla} p\; \boldsymbol{\nabla} q}{\rho\, \omega^2} - \frac{p\, q}{\rho\, c^2}\, \mathrm{d}\Omega = \oint_{\partial\Omega} \frac{1}{\rho\, \omega^2} \frac{\partial p}{\partial n}\, \mathrm{d}\Gamma \tag{4.1}$$

4.1.2 Elastic solid

Unknown: the solid displacement $\mathbf{u}$.

$$\forall\, \mathbf{v}, \quad \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \omega^2 \rho_s \mathbf{u}.\mathbf{v}\, \mathrm{d}\Omega = \oint_{\partial\Omega} [\boldsymbol{\sigma}(\mathbf{u}).\mathbf{n}].\mathbf{v}\, \mathrm{d}\Gamma \tag{4.2}$$

4.2 Discontinuous Galerkin

4.2.1 Abstract weak forms

$$\left(j\omega[\mathbf{M}] + [\mathbf{B}_x]\frac{\partial}{\partial x} + [\mathbf{B}_y]\frac{\partial}{\partial y} \right) \mathbf{S} = \mathbf{0} \text{ on } \Omega. \tag{4.3}$$

$\mathbf{S}$ is the State Vector of the medium. In a conservative form:

$$\left(j\omega + [\mathbf{A}_x]\frac{\partial}{\partial x} + [\mathbf{A}_y]\frac{\partial}{\partial y} \right) \mathbf{S} = \mathbf{0} \text{ on } \Omega. \tag{4.4}$$

$$[\mathbf{A}_x] = [\mathbf{M}]^{-1}[\mathbf{B}_x], \quad [\mathbf{A}_y] = [\mathbf{M}]^{-1}[\mathbf{B}_y]. \tag{4.5}$$

Ω is partitioned in n elements Ω_e with $e = 1, ..., n$.

$$\sum_{e=1}^{n} \int_{\Omega_e} \mathbf{T}_e^T \left(j\omega[\mathbf{M}]\mathbf{S}_e + [\mathbf{B}_x]\frac{\partial \mathbf{S}_e}{\partial x} + [\mathbf{B}_y]\frac{\partial \mathbf{S}_e}{\partial y} \right) \, \mathrm{d}\Omega = 0. \tag{4.6}$$

$\mathbf{T}_e$ is the test field on each element.

$$-\sum_{e=1}^{n} \int_{\Omega_e} \left(j\omega[\mathbf{M}]^T\mathbf{T}_e + [\mathbf{B}_x]^T\frac{\partial \mathbf{T}_e}{\partial x} + [\mathbf{B}_y]^T\frac{\partial \mathbf{T}_e}{\partial y} \right)^T \mathbf{S}_e \, \mathrm{d}\Omega + \sum_{e=1}^{n} \int_{\partial\Omega_e} \mathbf{T}_e^T[\mathbf{G}_e]\mathbf{S}_e \, \mathrm{d}\Gamma = 0.$$

where we have introduced the matrix $[\mathbf{G}_e] = [\mathbf{B}_x]n_x + [\mathbf{B}_y]n_y$ which represents the normal fluxes across the boundary of the element Ω_e. The unit normal $\mathbf{n} = (n_x, n_y)$ on the element boundary $\partial\Omega_e$ points out of element e. A key aspect of the wave-based DGM is to use test functions $\mathbf{T}$ whose restrictions $\mathbf{T}_e$ on each elements are solutions of the adjoint problem defined on each element:

$$\left(j\omega[\mathbf{M}]^T\mathbf{T}_e + [\mathbf{B}_x]^T\frac{\partial \mathbf{T}_e}{\partial x} + [\mathbf{B}_y]^T\frac{\partial \mathbf{T}_e}{\partial y} \right) = \mathbf{0} = 0. \tag{4.7}$$

With this choice of test functions the integral over each element Ω_e vanishes and one is left with integrals on the interfaces between elements and on the boundary of the domain.

$$\sum_{e=1}^{n} \int_{\partial\Omega_e} \mathbf{T}_e^T[\mathbf{G}_e]\mathbf{S}_e \, \mathrm{d}\Gamma = 0. \tag{4.8}$$

$$[\mathbf{F}_e] = [\mathbf{A}_x]n_x + [\mathbf{A}_y]n_y \implies [\mathbf{G}_e] = [\mathbf{M}_e][\mathbf{F}_e] \tag{4.9}$$

$$\sum_{e=1}^{n} \int_{\partial\Omega_e} \mathbf{T}_e^T[\mathbf{M}_e][\mathbf{F}_e]\mathbf{S}_e \, \mathrm{d}\Gamma = 0. \tag{4.10}$$

$$\sum_{e=1}^{n} \int_{\partial\Omega_e} \mathbf{V}_e[\mathbf{F}_e]\mathbf{S}_e \, \mathrm{d}\Gamma = 0, \quad \mathbf{V}_e = [\mathbf{M}_e]\mathbf{T}_e. \tag{4.11}$$

4.2.2 Characteristics

$$[\mathbf{F}_e][\mathbf{P}_e] = [\mathbf{P}_e][\mathbf{\Lambda}_e], \quad [\mathbf{F}_e] = [\mathbf{P}_e][\mathbf{\Lambda}_e][\mathbf{Q}_e], \quad [\mathbf{Q}_e] = [\mathbf{P}_e]^{-1} \tag{4.12}$$

$$[\mathbf{\Lambda}_e] = \mathrm{diag}([\mathbf{\Lambda}_e^{in}], [\mathbf{\Lambda}_e^{out}], [\mathbf{0}]). \tag{4.13}$$

$$[\mathbf{F}_e] = [\mathbf{P}_e^{in}][\mathbf{\Lambda}_e^{in}][\mathbf{Q}_e^{in}] + [\mathbf{P}_e^{out}][\mathbf{\Lambda}_e^{out}][\mathbf{Q}_e^{out}] \tag{4.14}$$

4.2.3 Boundary term

$$\mathbf{S}_e = [\mathbf{P}_e^{in}]\mathbf{S}_e^{in} + [\mathbf{P}_e^{out}]\mathbf{S}_e^{out} \tag{4.15}$$

$$[\mathbf{C}_e]\mathbf{S}_e = \mathbf{E}_e. \tag{4.16}$$

$$[\mathbf{C}_e][\mathbf{P}^{out}]\mathbf{S}_e^{out} = \mathbf{E}_e - [\mathbf{C}_e][\mathbf{P}^{in}]\mathbf{S}_e^{in} \tag{4.17}$$

$$\mathbf{S}_e^{out} = [\widetilde{\mathbf{R}}_e]\mathbf{S}_e^{in} + \widetilde{\mathbf{E}}_e, \tag{4.18}$$

$$[\widetilde{\mathbf{R}}_e] = -\left([\mathbf{C}_e][\mathbf{P}^{in}]\right)^{-1}[\mathbf{C}_e][\mathbf{P}^{in}], \quad \widetilde{\mathbf{E}}_e = \left([\mathbf{C}_e][\mathbf{P}^{in}]\right)^{-1}\mathbf{E}_e. \tag{4.19}$$

$$\mathbf{S}_e = \Big([\mathbf{P}^{in}] + [\mathbf{P}^{out}][\widetilde{\mathbf{R}}_e]\Big)\mathbf{S}_e^{in} + [\mathbf{P}^{out}]\widetilde{\mathbf{E}}_e \tag{4.20}$$

$$\mathbf{S}_e = \underbrace{\Big([\mathbf{P}^{in}] + [\mathbf{P}^{out}][\widetilde{\mathbf{R}}_e]\Big)[\mathbf{Q}_e^{in}]}_{[\widetilde{\mathbf{P}}_e]}\mathbf{S}_e + [\mathbf{P}^{out}]\widetilde{\mathbf{E}}_e \tag{4.21}$$

$$\int_{\partial\Omega_e} \mathbf{T}_e^T[\mathbf{M}_e][\mathbf{F}_e]\mathbf{S}_e \,\mathrm{d}\Gamma = \int_{\partial\Omega_e} \mathbf{T}_e^T[\mathbf{M}_e][\mathbf{F}_e]\Big[[\widetilde{\mathbf{P}}_e]\mathbf{S}_e + [\mathbf{P}^{out}]\widetilde{\mathbf{E}}_e\Big]\,\mathrm{d}\Gamma \tag{4.22}$$

$$= \int_{\partial\Omega_e} \mathbf{T}_e^T[\widetilde{\mathbf{F}}_e]\mathbf{S}_e \,\mathrm{d}\Gamma + \int_{\partial\Omega_e} \mathbf{T}_e^T\widetilde{\mathbf{S}}_e \,\mathrm{d}\Gamma \tag{4.23}$$

$$[\widetilde{\mathbf{F}}_e] = [\mathbf{M}_e][\mathbf{F}_e][\widetilde{\mathbf{P}}_e], \quad \widetilde{\mathbf{S}}_e = [\mathbf{M}_e][\mathbf{F}_e][\mathbf{P}^{out}]\widetilde{\mathbf{E}}_e \tag{4.24}$$

4.2.4 Interfaces terms for DGM/DGM

$$I_{12} = \int_{\Gamma_{12}} \mathbf{T}_1[\mathbf{M}_1][\mathbf{F}_1]\mathbf{S}_1 \,\mathrm{d}\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2[\mathbf{M}_2][\mathbf{F}_2]\mathbf{S}_2 \,\mathrm{d}\Gamma \tag{4.25}$$

$$\mathbf{S}_1 = [\mathbf{P}_1^{in}]\mathbf{S}_1^{in} + [\mathbf{P}_1^{out}]\mathbf{S}_1^{out} \tag{4.26}$$

$$\mathbf{S}_2 = [\mathbf{P}_2^{in}]\mathbf{S}_2^{in} + [\mathbf{P}_2^{out}]\mathbf{S}_2^{out}. \tag{4.27}$$

$$[\mathbf{C}_1]\mathbf{S}_1 = [\mathbf{C}_2]\mathbf{S}_2. \tag{4.28}$$

$$[\mathbf{C}_1][\mathbf{P}_1^{out}]\mathbf{S}_1^{out} - [\mathbf{C}_2][\mathbf{P}_2^{out}]\mathbf{S}_2^{out} = [\mathbf{C}_2][\mathbf{P}_2^{in}]\mathbf{S}_2^{in} - [\mathbf{C}_1][\mathbf{P}_1^{in}]\mathbf{S}_1^{in} \tag{4.29}$$

$$\left\{ \begin{array}{c} \mathbf{S}_1^{out} \\ \mathbf{S}_2^{out} \end{array} \right\} = [\widetilde{\mathbf{R}}] \left\{ \begin{array}{c} \mathbf{S}_1^{in} \\ \mathbf{S}_2^{in} \end{array} \right\}. \tag{4.30}$$

$$[\widetilde{\mathbf{R}}] = \left[[\mathbf{C}_1][\mathbf{P}_1^{out}] \,|\, -[\mathbf{C}_2][\mathbf{P}_2^{out}] \right]^{-1} \left[-[\mathbf{C}_1][\mathbf{P}_1^{in}] \,|[\mathbf{C}_2][\mathbf{P}_2^{in}] \right] \tag{4.31}$$

$$= \left[\begin{array}{cc} [\widetilde{\mathbf{R}}_{11}] & [\widetilde{\mathbf{R}}_{12}] \\ [\widetilde{\mathbf{R}}_{21}] & [\widetilde{\mathbf{R}}_{22}] \end{array} \right]. \tag{4.32}$$

$$\mathbf{S}_1 = \underbrace{\left([\mathbf{P}_1^{in}] + [\mathbf{P}_1^{out}][\widetilde{\mathbf{R}}_{11}]\right)[\mathbf{Q}_1^{in}]}_{[\widetilde{\mathbf{P}}_{11}]} \mathbf{S}_1 + \underbrace{[\mathbf{P}_1^{out}][\widetilde{\mathbf{R}}_{12}][\mathbf{Q}_2^{in}]}_{[\widetilde{\mathbf{P}}_{12}]} \mathbf{S}_2 \tag{4.33}$$

$$\mathbf{S}_2 = \underbrace{[\mathbf{P}_2^{out}][\widetilde{\mathbf{R}}_{21}][\mathbf{Q}_1^{in}]}_{[\widetilde{\mathbf{P}}_{21}]} \mathbf{S}_1 + \underbrace{\left([\mathbf{P}_2^{in}] + [\mathbf{P}_2^{out}][\widetilde{\mathbf{R}}_{22}]\right)[\mathbf{Q}_2^{in}]}_{[\widetilde{\mathbf{P}}_{22}]} \mathbf{S}_2 \tag{4.34}$$

$$I_{12} = \int_{\Gamma_{12}} \mathbf{T}_1[\widetilde{\mathbf{F}}_{11}]\mathbf{S}_1 \,\mathrm{d}\Gamma + \int_{\Gamma_{12}} \mathbf{T}_1[\widetilde{\mathbf{F}}_{12}]\mathbf{S}_2 \,\mathrm{d}\Gamma \tag{4.35}$$

$$+ \int_{\Gamma_{12}} \mathbf{T}_2[\widetilde{\mathbf{F}}_{21}]\mathbf{S}_1 \,\mathrm{d}\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2[\widetilde{\mathbf{F}}_{22}]\mathbf{S}_2 \,\mathrm{d}\Gamma \tag{4.36}$$

$$[\widetilde{\mathbf{F}}_{11}] = [\mathbf{M}_1][\mathbf{F}_1][\widetilde{\mathbf{P}}_{11}], \quad [\widetilde{\mathbf{F}}_{12}] = [\mathbf{M}_1][\mathbf{F}_1][\widetilde{\mathbf{P}}_{12}] \tag{4.37}$$

$$[\widetilde{\mathbf{F}}_{21}] = [\mathbf{M}_2][\mathbf{F}_2][\widetilde{\mathbf{P}}_{21}], \quad [\widetilde{\mathbf{F}}_{22}] = [\mathbf{M}_2][\mathbf{F}_2][\widetilde{\mathbf{P}}_{22}] \tag{4.38}$$

4.2.5 Interfaces terms for FEM/FEM

$$I_{12} = -\int_{\Gamma_{12}} q_1 \frac{\mathbf{v}_1.\mathbf{n}_1}{j\omega}\,\mathrm{d}\Gamma - \int_{\Gamma_{12}} q_2 \frac{\mathbf{v}_2.\mathbf{n}_2}{j\omega}\,\mathrm{d}\Gamma \tag{4.39}$$

$-$ comes from the transposition of the right hand side integral.

Like in the previous section

$$\hat{\mathbf{S}}_1 = [\widetilde{\mathbf{P}}_{11}]\mathbf{S}_1 + [\widetilde{\mathbf{P}}_{12}]\hat{\mathbf{S}}_2 \tag{4.40}$$

$$\hat{\mathbf{S}}_2 = [\widetilde{\mathbf{P}}_{21}]\mathbf{S}_1 + [\widetilde{\mathbf{P}}_{22}]\mathbf{S}_2 \tag{4.41}$$

$$-\frac{\mathbf{v}_1.\mathbf{n}_1}{j\omega} = -\frac{1}{j\omega}\left[\hat{\mathbf{S}}_1(1)\mathbf{n}_1(1) + \hat{\mathbf{S}}_1(2)\mathbf{n}_1(2)\right] \tag{4.42}$$

$$= [\widetilde{\mathbf{F}}_{11}]\mathbf{S}_1 + [\widetilde{\mathbf{F}}_{12}]\mathbf{S}_2 \tag{4.43}$$

with

$$[\widetilde{\mathbf{F}}_{11}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{11}(1,:)]\mathbf{n}_1(1) + [\widetilde{\mathbf{P}}_{11}(2,:)]\mathbf{n}_1(2)\right] \tag{4.44}$$

$$[\widetilde{\mathbf{F}}_{12}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{12}(1,:)]\mathbf{n}_1(1) + [\widetilde{\mathbf{P}}_{12}(2,:)]\mathbf{n}_1(2)\right] \tag{4.45}$$

Concerning the second term

$$-\frac{\mathbf{v}_2.\mathbf{n}_2}{j\omega} = \frac{1}{j\omega}\left[\hat{\mathbf{S}}_2(1)\mathbf{n}_2(1) + \hat{\mathbf{S}}_2(2)\mathbf{n}_2(2)\right] \tag{4.46}$$

$$= [\widetilde{\mathbf{F}}_{21}]\mathbf{S}_1 + [\widetilde{\mathbf{F}}_{22}]\mathbf{S}_2 \tag{4.47}$$

with

$$[\widetilde{\mathbf{F}}_{21}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{21}(1,:)]\mathbf{n}_2(1) + [\widetilde{\mathbf{P}}_{21}(2,:)]\mathbf{n}_2(2)\right] \tag{4.48}$$

$$[\widetilde{\mathbf{F}}_{22}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{22}(1,:)]\mathbf{n}_2(1) + [\widetilde{\mathbf{P}}_{22}(2,:)]\mathbf{n}_2(2)\right] \tag{4.49}$$

$$p_1(x,y) \approx [\mathbf{N}_1(x,y)]\mathbf{p}_1, \quad \mathbf{p}_2(x,y) \approx [\mathbf{N}_2(x,y)]\mathbf{p}_2 \tag{4.50}$$

$$\mathbf{S}_i \approx \begin{bmatrix} [\mathbf{V}_i^x(x,y)] \\ [\mathbf{V}_i^y(x,y)] \\ [\mathbf{N}_i(x,y)] \end{bmatrix} \mathbf{p}_i \tag{4.51}$$

$$[\mathbf{V}_i^x(x,y)] = -\frac{1}{j\omega\rho}\left[\frac{\partial \mathbf{N}_i(x,y)}{\partial x}\right], \quad [\mathbf{V}_i^y(x,y)] = -\frac{1}{j\omega\rho}\left[\frac{\partial \mathbf{N}_i(x,y)}{\partial y}\right] \tag{4.52}$$

The boundary terms

$$\int_{\Gamma_{12}} q_i [\widetilde{\mathbf{F}}_{ij}]\mathbf{S}_j \,\mathrm{d}\Gamma \approx \int_{\Gamma_{12}} \mathbf{q}_i^T \,[\mathbf{N}_i(x,y)]^T\, [\widetilde{\mathbf{I}}_{ij}]\mathbf{p}_j \,\mathrm{d}\Gamma \tag{4.53}$$

with

$$[\widetilde{\mathbf{I}}_{ij}] = \widetilde{\mathbf{F}}_{ij}(1)\left[\mathbf{V}_j^x(x,y)\right] + \widetilde{\mathbf{F}}_{ij}(2)\left[\mathbf{V}_j^y(x,y)\right] + \widetilde{\mathbf{F}}_{ij}(3)\left[\mathbf{N}_j(x,y)\right] \tag{4.54}$$

4.2.6 Interfaces terms for FEM/DGM

$$I_{12} = -\int_{\Gamma_{12}} q_1 \frac{\mathbf{v}_1.\mathbf{n}_1}{j\omega}\,\mathrm{d}\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2[\mathbf{M}_2][\mathbf{F}_2]\mathbf{S}_2\,\mathrm{d}\Gamma \tag{4.55}$$

$-$ for the first comes from the transposition of the right hand side integral.

Like in the previous section

$$\hat{\mathbf{S}}_1 = [\widetilde{\mathbf{P}}_{11}]\mathbf{S}_1 + [\widetilde{\mathbf{P}}_{12}]\hat{\mathbf{S}}_2 \tag{4.56}$$

$$\hat{\mathbf{S}}_2 = [\widetilde{\mathbf{P}}_{21}]\mathbf{S}_1 + [\widetilde{\mathbf{P}}_{22}]\mathbf{S}_2 \tag{4.57}$$

$$-\frac{\mathbf{v}_1.\mathbf{n}_1}{j\omega} = \frac{1}{j\omega}\left[\hat{\mathbf{S}}_1(1)\mathbf{n}_1(1) + \hat{\mathbf{S}}_1(2)\mathbf{n}_1(2)\right] \tag{4.58}$$

$$= [\widetilde{\mathbf{F}}_{11}]\mathbf{S}_1 + [\widetilde{\mathbf{F}}_{12}]\mathbf{S}_2 \tag{4.59}$$

with

$$[\widetilde{\mathbf{F}}_{11}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{11}(1,:)]\mathbf{n}_1(1) + [\widetilde{\mathbf{P}}_{11}(2,:)]\mathbf{n}_1(2)\right] \tag{4.60}$$

$$[\widetilde{\mathbf{F}}_{12}] = -\frac{1}{j\omega}\left[[\widetilde{\mathbf{P}}_{12}(1,:)]\mathbf{n}_1(1) + [\widetilde{\mathbf{P}}_{12}(2,:)]\mathbf{n}_1(2)\right] \tag{4.61}$$

$$[\widetilde{\mathbf{F}}_{21}] = [\mathbf{M}_2][\mathbf{F}_2][\widetilde{\mathbf{P}}_{21}], \quad [\widetilde{\mathbf{F}}_{22}] = [\mathbf{M}_2][\mathbf{F}_2][\widetilde{\mathbf{P}}_{22}] \tag{4.62}$$

$$I_{12} = \int_{\Gamma_{12}} q_1[\widetilde{\mathbf{F}}_{11}]\mathbf{S}_1 \,\mathrm{d}\Gamma + \int_{\Gamma_{12}} q_1[\widetilde{\mathbf{F}}_{12}]\mathbf{S}_2 \,\mathrm{d}\Gamma \tag{4.63}$$

$$+ \int_{\Gamma_{12}} \mathbf{T}_2^T[\widetilde{\mathbf{F}}_{21}]\mathbf{S}_1 \,\mathrm{d}\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2^T[\widetilde{\mathbf{F}}_{22}]\mathbf{S}_2 \,\mathrm{d}\Gamma \tag{4.64}$$

$$p_1(x,y) \approx [\mathbf{N}_1(x,y)]\mathbf{p}_1, \quad \mathbf{S}_2(x,y) \approx [\mathbf{N}_2(x,y)]\mathbf{X}_2 \tag{4.65}$$

$$\mathbf{S}_1 \approx \begin{bmatrix} [\mathbf{V}_1^x(x,y)] \\ [\mathbf{V}_1^y(x,y)] \\ [\mathbf{N}_1(x,y)] \end{bmatrix} \mathbf{p}_1 \tag{4.66}$$

$$[\mathbf{V}_1^x(x,y)] = -\frac{1}{j\omega\rho}\left[\frac{\partial \mathbf{N}_1(x,y)}{\partial x}\right], \quad [\mathbf{V}_1^y(x,y)] = -\frac{1}{j\omega\rho}\left[\frac{\partial \mathbf{N}_1(x,y)}{\partial y}\right] \tag{4.67}$$

$$\int_{\Gamma_{12}} q_1[\widetilde{\mathbf{F}}_{11}]\mathbf{S}_1 \,\mathrm{d}\Gamma \approx \int_{\Gamma_{12}} \mathbf{q}_1^T \,[\mathbf{N}_1(x,y)]^T\, [\widetilde{\mathbf{I}}_{11}]\mathbf{p}_1 \,\mathrm{d}\Gamma \tag{4.68}$$

with

$$[\widetilde{\mathbf{I}}_{11}] = \widetilde{\mathbf{F}}_{11}(1)\,[\mathbf{V}_1^x(x,y)] + \widetilde{\mathbf{F}}_{11}(2)\,[\mathbf{V}_1^y(x,y)] + \widetilde{\mathbf{F}}_{11}(3)\,[\mathbf{N}_1(x,y)] \tag{4.69}$$

$$\int_{\Gamma_{12}} q_1[\widetilde{\mathbf{F}}_{12}]\mathbf{S}_2 \,\mathrm{d}\Gamma \approx \int_{\Gamma_{12}} \mathbf{q}_1^T[\widetilde{\mathbf{I}}_{12}]\mathbf{X}_2 \,\mathrm{d}\Gamma \tag{4.70}$$

with

$$[\widetilde{\mathbf{I}}_{12}] = [\mathbf{N}_1(x,y)]^T\, \widetilde{\mathbf{F}}_{12}\, [\mathbf{N}_2(x,y)] \tag{4.71}$$

$$\int_{\Gamma_{12}} \mathbf{T}_2^T[\widetilde{\mathbf{F}}_{21}]\mathbf{S}_1 \,\mathrm{d}\Gamma \approx \int_{\Gamma_{12}} \mathbf{X}_2^T\, [\mathbf{N}_2(x,y)]^T\, [\widetilde{\mathbf{I}}_{21}]\mathbf{p}_1 \,\mathrm{d}\Gamma \tag{4.72}$$

with

$$[\widetilde{\mathbf{I}}_{21}] = \widetilde{\mathbf{F}}_{11}(:,1)\,[\mathbf{V}_1^x(x,y)] + \widetilde{\mathbf{F}}_{11}(:,2)\,[\mathbf{V}_1^y(x,y)] + \widetilde{\mathbf{F}}_{11}(:,3)\,[\mathbf{N}_1(x,y)] \tag{4.73}$$

$$\int_{\Gamma_{12}} \mathbf{T}_2^T [\widetilde{\mathbf{F}}_{22}] \mathbf{S}_2 \,\mathrm{d}\Gamma \approx \int_{\Gamma_{12}} \mathbf{X}_2^T [\widetilde{\mathbf{I}}_{22}] \mathbf{X}_2 \,\mathrm{d}\Gamma \tag{4.74}$$

with

$$[\widetilde{\mathbf{I}}_{22}] = [\mathbf{N}_2(x,y)]^T \, [\widetilde{\mathbf{F}}_{22}] \, [\mathbf{N}_2(x,y)] \tag{4.75}$$

Chapter 5

Plane waves

5.1 Fluid

$$
\begin{aligned}
j\omega\rho v_x &= -\frac{\partial p}{\partial x} && (5.1)\\
j\omega\rho v_y &= -\frac{\partial p}{\partial y} && (5.2)\\
j\omega p &= -\rho c^2 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right). && (5.3)
\end{aligned}
$$

$$
\mathbf{S} = \left\{ \begin{array}{c} v_x \\ v_y \\ p \end{array} \right\}. \tag{5.4}
$$

$[\mathbf{M}]$, $[\mathbf{A}_x]$ and $[\mathbf{A}_y]$ are square matrices of size m defined as

$$
[\mathbf{M}] = \left[\begin{array}{ccc} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \dfrac{1}{\rho c^2} \end{array} \right], \tag{5.5}
$$

and

$$
[\mathbf{B}_x] = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \right], \quad [\mathbf{B}_y] = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right]. \tag{5.6}
$$

Chapter 6

Finite-Element Matrices

6.1 Global system

6.1.1 Unknown vector

1. $\mathbf{X}_{TR6}$
2. $\mathbf{X}_{H12}$
3. $\mathbf{X}_{DGM}$
4. $\mathbf{X}_{Bloch}$

$$\left[\begin{array}{cccc|cc}
[\mathbf{K}_e]-\omega^2[\mathbf{M}_e] & & & & \mathbf{I}_R^e & \mathbf{I}_T^e \\
 & [\mathbf{K}]-\omega^2\widetilde{\rho}[\mathbf{M}] & -\widetilde{\gamma}[\mathbf{C}] & & \mathbf{I}_R^s & \mathbf{I}_T^s \\
 & -\widetilde{\gamma}[\mathbf{C}]^t & \dfrac{[\mathbf{H}]}{\omega^2\widetilde{\rho}_{eq}}-\dfrac{[\mathbf{Q}]}{\widetilde{K}_{eq}} & & \mathbf{I}_R^p & \mathbf{I}_T^p \\
 & & & \dfrac{[\mathbf{H}]}{\omega^2\rho_0}-\dfrac{[\mathbf{Q}]}{K_0} & \mathbf{I}_R^a & \mathbf{I}_T^a \\
\hline
\mathbf{J}_R^e & \mathbf{J}_R^s & \mathbf{J}_R^p & \mathbf{J}_R^a & & \\
\mathbf{J}_T^e & \mathbf{J}_T^s & \mathbf{J}_T^p & \mathbf{J}_T^a & &
\end{array}\right]
\left\{\begin{array}{c}\mathbf{u}^e\\ \mathbf{u}^s\\ \mathbf{p}\\ \mathbf{p}^a\\ \hline \mathbf{R}\\ \mathbf{T}\end{array}\right\}
=
\left\{\begin{array}{c}\mathbf{f}^e\\ \mathbf{f}^s\\ \mathbf{u}\\ \mathbf{u}^a\\ \hline \mathbf{R}'\\ \mathbf{T}'\end{array}\right\}
\tag{6.1}$$

6.2 Number of coefficient in matrices

6.3 DtN operators

6.3.1 Fields properties in incident and transmission media

$$p^{inc}(x,y,t) = 1\, e^{j(\omega t - k_x x - k_y y)} + \sum_{l \in \mathbb{Z}} e^{j(\omega t - k_x x + k_y(l) y)} R_l \tag{6.2}$$

$$k_x = k_0\ sin(\theta), \quad k_y = k_0\ \cos(\theta), \quad \theta \in \left[0; \frac{\pi}{2}\right[, \quad k_x(l) = k_x^i + \frac{2\pi l}{D} \tag{6.3}$$

$$u^{inc}(x,y,t) = \frac{\boldsymbol{\nabla} p^{\,inc}}{\rho_0 \omega^2} = \left| \begin{array}{c} -\dfrac{jk_x}{\rho\omega^2} \\ -\dfrac{jk_z}{\rho\omega^2} \end{array} \right. e^{j(\omega t - k_x x - k_y y)} + \sum_{l \in \mathbb{Z}} \left| \begin{array}{c} -\dfrac{jk_x}{\rho\omega^2} \\ \dfrac{jk_y(l)}{\rho\omega^2} \end{array} \right. e^{j(\omega t - k_x x + k_y(l) y)} R_l \tag{6.4}$$

$$p^{tr}(x,y,t) = \sum_{l \in \mathbb{Z}} e^{j(\omega t - k_x x - k_y(l) y)} T_l \tag{6.5}$$

$$u^{tr}(x,y,t) = \frac{\boldsymbol{\nabla} p^{\,tr}}{\rho_0 \omega^2} = \sum_{l \in \mathbb{Z}} \left| \begin{array}{c} -\dfrac{jk_x}{\rho\omega^2} \\ -\dfrac{jk_y(l)}{\rho\omega^2} \end{array} \right. e^{j(\omega t - k_x x - k_y(l) y)} T_l \tag{6.6}$$

6.3.2 Formalisation of the fields

Let $\mathbf{X}_l$ be an information vector associated to Bloch wave l.

$$\mathbf{Y} = \left\{ \begin{array}{c} 1 \\ \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_n \end{array} \right\} \tag{6.7}$$

On each interface with the FEM model a physical field φ can be written

$$\varphi(x,y) = [\boldsymbol{\Omega}_F^{\varphi}] e^{-jk_x} + \sum_{l} [\boldsymbol{\Omega}_l^{\varphi}] \mathbf{X}_l e^{-jk_x(l)}. \tag{6.8}$$

6.4 Case of a interface with fluid/Biot 1998

$$
\begin{aligned}
I_{FEM} &= \int_{|\Gamma_R} \frac{1}{\rho\omega^2}\frac{\partial p^{inc}}{\partial n} q \,\mathrm{d}\Gamma = \int_{|\Gamma_R} - u_y(x)q(x)\,\mathrm{d}\Gamma && (6.9a)\\
&= \int_{|\Gamma_R} [\boldsymbol{\Omega}_F^{u_y}]e^{-jk_x} q \,\mathrm{d}\Gamma \;\text{-}\; \sum_l \int_{|\Gamma_R} [\boldsymbol{\Omega}_l^{u_y}]\mathbf{X}_l e^{-jk_x(l)} q\,\mathrm{d}\Gamma && (6.9b)
\end{aligned}
$$

- is highlighted as this term will end in the matrix system.

The information vector is scalar $\mathbf{X}_l = R_l$ and correspond to the reflexion coefficient.

$$
[\boldsymbol{\Omega}_F^{u_y}] = \frac{jk_z}{\rho_0\omega^2}, \quad [\boldsymbol{\Omega}_l^{u_y}] = \frac{jk_z(l)}{\rho_0\omega^2} \tag{6.9c}
$$

The additional equation is the projection of the pressure

$$
\int_{\Gamma_R} p(x)e^{jk_x(n)x}\,\mathrm{d}\Gamma = \underbrace{\int_{\Gamma_R} e^{j\frac{2\pi n x}{D}}\,\mathrm{d}\Gamma}_{D\delta_{n0}} + \underbrace{\int_{\Gamma_R}\sum_{l\in\mathbb{Z}} e^{-\frac{j2\pi(l-n)x}{D}}\,\mathrm{d}\Gamma}_{D\delta_{nl}} R_l \tag{6.10}
$$

6.5 Case of a interface with an elastic medium

6.5.1 Reflexion side

$$
\begin{aligned}
I_{FEM} &= \int_{|\Gamma_R} \sigma_{xy}v_x - \sigma_{yy}v_y \,\mathrm{d}\Gamma = \int_{|\Gamma_R} p v_y\,\mathrm{d}\Gamma && (6.11a)\\
&= \int_{|\Gamma_R} [\boldsymbol{\Omega}_F^{p}]e^{-jk_x} q \,\mathrm{d}\Gamma - \sum_l \int_{|\Gamma_R} [\boldsymbol{\Omega}_l^{p}]\mathbf{X}_l e^{-jk_x(l)} v_y\,\mathrm{d}\Gamma && (6.11b)
\end{aligned}
$$

The information vector is scalar $\mathbf{X}_l = R_l$ and correspond to the reflexion coefficient.

$$
[\boldsymbol{\Omega}_F^{p}] = 1, \quad [\boldsymbol{\Omega}_l^{p}] = -1 \tag{6.11c}
$$

The additional equation is the projection of the pressure

$$
\int_{\Gamma_R} u_y(x)e^{jk_x(n)x}\,\mathrm{d}\Gamma = \frac{-jk_y}{\rho_0\omega^2}D\delta_{n0} + \frac{jk_y(l)}{\rho_0\omega^2}D\delta_{nl}R_l \tag{6.12}
$$

6.5.2 Transmission side

$$I_{FEM} = \int_{|\Gamma_R} \sigma_{xy} v_x + \sigma_{yy} v_y \ \mathrm{d}\Gamma = \int_{|\Gamma_R} p v_y \ \mathrm{d}\Gamma \tag{6.13a}$$

$$= \int_{|\Gamma_R} [\boldsymbol{\Omega}_F^p] e^{-jk_x} q \ \mathrm{d}\Gamma - \sum_l \int_{|\Gamma_R} [\boldsymbol{\Omega}_l^p] \mathbf{X}_l e^{-jk_x(l)} v_y \ \mathrm{d}\Gamma \tag{6.13b}$$

The information vector is scalar $\mathbf{X}_l = R_l$ and correspond to the reflexion coefficient.

$$[\boldsymbol{\Omega}_F^p] = -1, \quad [\boldsymbol{\Omega}_l^p] = 1 \tag{6.13c}$$

The additional equation is the projection of the pressure

$$\int_{\Gamma_T} u_y(x) e^{jk_x(n)x} \ \mathrm{d}\Gamma = -\frac{jk_y(l)}{\rho_0 \omega^2} D \delta_{nl} R_l \tag{6.14}$$

6.6 Case of integration of a plate medium

6.6.1 Reflexion side

$$I_{FEM} = \int_{|\Gamma_R} - \sigma_{xz} v_x - \sigma_{zz} v_z \ \mathrm{d}\Gamma \tag{6.15}$$

$$\left\{ \begin{array}{c} \sigma_{xz} \\ u_z \\ \sigma_{zz} \\ u_x \end{array} \right\} = \begin{bmatrix} 0 & 0 & 0 \\ -\dfrac{jk_z}{\rho\omega^2} & \dfrac{jk_z}{\rho\omega^2} & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} 1 \\ R \\ u_x \end{array} \right\} \tag{6.16}$$

$$\left\{ \begin{array}{c} \sigma_{xz} \\ u_z \\ \sigma_{zz} \\ u_x \end{array} \right\}_{\mathbf{F}} = [\mathbf{T}] \left\{ \begin{array}{c} 0 \\ -\dfrac{jk_z}{\rho\omega^2} \\ -1 \\ 0 \end{array} \right\}, \quad \left\{ \begin{array}{c} \sigma_{xz} \\ u_z \\ \sigma_{zz} \\ u_x \end{array} \right\}_{\mathbf{R}} = [\mathbf{T}] \begin{bmatrix} 0 & 0 \\ \dfrac{jk_z}{\rho\omega^2} & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} R \\ u_x \end{array} \right\} \tag{6.17}$$

$$p_{|\Gamma_R}(x,z) = e^{-j(k_x^i x + k_z^i z)} + \sum_{l\in\mathbb{Z}} e^{-j(k_x(l)x - k_z(l)z)} R_l \tag{6.18}$$

$$\frac{\partial p}{\partial z}_{|\Gamma_R}(x,z) = -jk_z^i e^{-j(k_x^i x + k_z^i z)} + \sum_{l\in\mathbb{Z}} jk_y(l) e^{-j(k_x(l)x - k_z(l)z)} R_l \tag{6.19}$$

$$p_{|\Gamma_R}(x,z) = \sum_{l\in\mathbb{Z}} e^{-j(k_x(l)x + k_z(l)z)} T_l \tag{6.20}$$

$$\frac{\partial p}{\partial z}_{|\Gamma_T}(x,z) = \sum_{l\in\mathbb{Z}} - jk_z(l) e^{-j(k_x(l)x + k_z(l)z)} T_l \tag{6.21}$$

Termes de surface

$$\frac{1}{\rho_a \omega^2} \int_{\Gamma_R} \frac{\partial p}{\partial n} q \,\mathrm{d}\Gamma = ... \tag{6.22}$$

Equation supplmentaires

$$\int_{\Gamma_R} p(x) e^{jk_x(n)x} \,\mathrm{d}\Gamma = \underbrace{\int_{\Gamma_R} e^{j\dfrac{2\pi n x}{D}} \,\mathrm{d}\Gamma}_{D\delta_{n0}} + \underbrace{\int_{\Gamma_R} \sum_{l\in\mathbb{Z}} e^{-\dfrac{j2\pi(l-n)x}{D}} \,\mathrm{d}\Gamma\, R_l}_{D\delta_{nl}} \tag{6.23}$$

6.7 Elementary matrix

6.7.1 I

$$\hat{I} = \int_a^b (AX^2 + BX + C) e^{-jk_x X} \,\mathrm{d}X \tag{6.24}$$

$x = X - a$

$$\hat{I} = e^{-jk_x a} \int_0^h \Phi(x) e^{-jk_x x} \,\mathrm{d}x \tag{6.25}$$

$\xi = -1 + \dfrac{2x}{h}, \quad x = \dfrac{h}{2}(\xi + 1)$

$$\hat{I} = e^{-jk_x a}\frac{h}{2}\int_{-1}^{1}\Phi(\xi)e^{-jk_x\frac{h(\xi+1)}{2}}\,\mathrm{d}\xi = e^{-jk_x\left(a+\frac{h}{2}\right)}\frac{h}{2}\int_{-1}^{1}\Phi(\xi)e^{-jk_x\frac{h\xi}{2}}\,\mathrm{d}\xi \tag{6.26}$$

Chapter 7

Benchmarks

7.1 Kundt Tube

Sous projets

- 0: TR6
- 1: H12
- 2: TR6/H12
- 3: DGM on TR
- 4: DGM on H
- 5: DGM on TR / DGM on H
- 6: H12 / DGM on H
- 7: TR6 / DGM on H

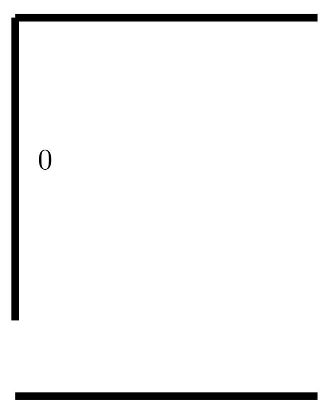

Figure 7.1: caption

Chapter 8

Hermite Finite-Elements

$$\Psi_1(x) = 1 - 3\left(\frac{x}{l_x}\right)^2 + 2\left(\frac{x}{l_x}\right)^3 \tag{8.1}$$

$$\Psi_2(x) = \left(\frac{x}{l_x}\right) - 2\left(\frac{x}{l_x}\right)^2 + \left(\frac{x}{l_x}\right)^3 \tag{8.2}$$

$$\Psi_3(x) = -\left(\frac{x}{l_x}\right)^2 + \left(\frac{x}{l_x}\right)^3 \tag{8.3}$$

$$\Psi_4(x) = 3\left(\frac{x}{l_x}\right)^2 - 2\left(\frac{x}{l_x}\right)^3 \tag{8.4}$$

$$\Psi_{ij}(x,y) = \Psi_i(x)\Psi_j(y) \tag{8.5}$$

Valeurs

	Ψ_1	Ψ_2	Ψ_3	Ψ_4
$f(0)$	1	0	0	0
$f(l_x)$	0	0	0	1
$f'(0)$	0	$1/l$	0	0
$f'(1_x)$	0	0	$1/l$	0

Ordre

	p	$\dfrac{\partial p}{\partial x}$	$\dfrac{\partial p}{\partial y}$	$\emptyset$
$(0,0)$	Ψ_{11}	Ψ_{21}	Ψ_{12}	Ψ_{22}
	$1//1$	$2//2$	$3//3$	$4//\emptyset$
$(1,0)$	Ψ_{41}	Ψ_{31}	Ψ_{42}	Ψ_{23}
	$5//4$	$6//5$	$7//6$	$8//\emptyset$
$(1,1)$	Ψ_{44}	Ψ_{34}	Ψ_{43}	Ψ_{33}
	$9//7$	$10//8$	$11//9$	$12//\emptyset$
$(0,1)$	Ψ_{14}	Ψ_{24}	Ψ_{13}	Ψ_{32}
	$13//10$	$14//11$	$15//12$	$16//\emptyset$

Chapter 9

Miscellaneous

9.1 Integration of an exponential on one element

$$\int_{\Omega} e^{j\mathbf{k}.(\mathbf{x}-\mathbf{x}_c)}\,\mathrm{d}\Omega = \oint \nabla. \left(- \left| \begin{array}{c} \dfrac{jk_x}{k_x^2+k_y^2} \\ \dfrac{jk_y}{k_x^2+k_y^2} \end{array} \right. e^{j\mathbf{k}.(\mathbf{x}-\mathbf{x}_c)} \right) \mathrm{d}\Omega \tag{9.1}$$

$$= - \oint_{\partial\Omega} \left(\frac{jk_x n_x}{k_x^2+k_y^2} + \frac{jk_y n_y}{k_x^2+k_y^2} \right) e^{j\mathbf{k}.(\mathbf{x}-\mathbf{x}_c)}\,\mathrm{d}\Omega \tag{9.2}$$

Chapter 10

Plane Waves 3D

10.1 Fluid medium

10.1.1 Compressional wave

$$\delta_P^2 = k_x^2 + k_y^2 + k_z^2 \tag{10.1}$$

$$u^\varepsilon(x,y,z,t) = \left| \begin{array}{c} k_x \\ k_y \\ \varepsilon k_z \end{array} \right. e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \tag{10.2}$$

$$\begin{aligned} p &= (-K_0)(-j)(k_x^2 + k_y^2 + \varepsilon^2 k_z^2) \qquad (10.3) \\ &= jK_0\delta_P^2 \qquad (10.4) \end{aligned}$$

10.2 Elastic Medium

10.2.1 Compressional wave

$$\delta_P^2 = k_x^2 + k_y^2 + k_z^2 \tag{10.5}$$

$$u^+(x,y,z,t) = \left| \begin{array}{c} k_x \\ k_y \\ \varepsilon k_z \end{array} \right. e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \tag{10.6}$$

$$\begin{aligned}\sigma_{zz} &= \lambda\left[u_{x,x} + u_{y,y} + u_{z,z}\right] + 2\mu u_{z,z} &(10.7)\\ &= -j\lambda(k_x^2 + k_y^2 + \varepsilon^2 k_z^2) + 2\mu(-j\varepsilon^2 k_z^2) &(10.8)\\ &= -j\left[\lambda(k_x^2 + k_y^2) + (\lambda + 2\mu)k_z^2\right] &(10.9)\end{aligned}$$

$$\sigma_{yz} = \mu\left[u_{y,z} + u_{z,y}\right] = -j\left[\varepsilon k_y k_z + \varepsilon k_z k_y\right] = -2j\mu\varepsilon k_y k_z \tag{10.10}$$

$$\sigma_{xz} = \mu\left[u_{x,z} + u_{z,x}\right] = -j\mu\left[\varepsilon k_x k_z + \varepsilon k_z k_x\right] = -2j\mu\varepsilon k_x k_z \tag{10.11}$$

10.2.2 Shear wave 1

$$\delta_S^2 = k_x^2 + k_y^2 + k_z^2 \tag{10.12}$$

$$u^\varepsilon(x,y,z,t) = \left|\begin{array}{c} k_z \\ 0 \\ -\varepsilon k_x \end{array}\right. e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \tag{10.13}$$

$$\begin{aligned}\sigma_{zz} &= \lambda\left[u_{x,x} + u_{y,y} + u_{z,z}\right] + 2\mu u_{z,z} &(10.14)\\ &= 2\mu j k_x k_z &(10.15)\end{aligned}$$

$$\sigma_{yz} = \mu\left[u_{y,z} + u_{z,y}\right] = -j\mu\varepsilon k_x k_y \tag{10.16}$$

$$\sigma_{xz} = \mu\left[u_{x,z} + u_{z,x}\right] = -j\mu\left[\varepsilon k_x k_z - \varepsilon k_z k_x\right] = -j\mu\varepsilon(k_z^2 - k_x^2) \tag{10.17}$$

10.2.3 Shear wave 2

$$\delta_S^2 = k_x^2 + k_y^2 + k_z^2 \tag{10.18}$$

$$u^\varepsilon(x,y,z,t) = \left| \begin{array}{c} 0 \\ k_z \\ -\varepsilon k_y \end{array} \right. e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \tag{10.19}$$

$$\begin{aligned} \sigma_{zz} &= \lambda \left[u_{x,x} + u_{y,y} + u_{z,z}\right] + 2\mu u_{z,z} && (10.20)\\ &= 2\mu j k_y k_z && (10.21) \end{aligned}$$

$$\sigma_{yz} = \mu \left[u_{y,z} + u_{z,y}\right] = -j\mu\varepsilon(k_z^2 - k_y^2) \tag{10.22}$$

$$\sigma_{xz} = \mu \left[u_{x,z} + u_{z,x}\right] = -j\varepsilon k_x k_y \tag{10.23}$$

Chapter 11

Rotations

$$\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = [\mathbf{P}] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} = [\mathbf{P}] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{11.1}$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial F}{\partial X_1}\frac{\partial X_1}{\partial x_1} + \frac{\partial F}{\partial X_1}\frac{\partial X_1}{\partial x_2} + \frac{\partial F}{\partial X_1}\frac{\partial X_1}{\partial x_3} \tag{11.2}$$

$$= \mathbf{P}(1,:) \begin{Bmatrix} \dfrac{\partial}{\partial X_1} \\ \dfrac{\partial}{\partial X_2} \\ \dfrac{\partial}{\partial X_3} \end{Bmatrix} F(X_1, X_2, X_3) \tag{11.3}$$

$$\begin{Bmatrix} \dfrac{\partial}{\partial X_1} \\ \dfrac{\partial}{\partial X_2} \\ \dfrac{\partial}{\partial X_3} \end{Bmatrix} F(X_1, X_2, X_3) = [\mathbf{Q}] \begin{Bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{Bmatrix} f(x_1, x_2, x_3) \tag{11.4}$$

$$\frac{\partial U_i}{\partial X_j} = \mathbf{Q}(i,:) \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \mathbf{Q}(j,:) \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{11.5}$$

$$= \mathbf{Q}(i,:) \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \left(\mathbf{Q}_{j1} u_1 + \mathbf{Q}_{j2} u_2 + \mathbf{Q}_{j3} u_3 \right) \tag{11.6}$$

$$= \mathbf{Q}(i,:) \begin{Bmatrix} \mathbf{Q}_{j1} \frac{\partial u_1}{\partial x_1} + \mathbf{Q}_{j2} \frac{\partial u_2}{\partial x_1} + \mathbf{Q}_{j3} \frac{\partial u_3}{\partial x_1} \\ \mathbf{Q}_{j1} \frac{\partial u_1}{\partial x_2} + \mathbf{Q}_{j2} \frac{\partial u_2}{\partial x_2} + \mathbf{Q}_{j3} \frac{\partial u_3}{\partial x_2} \\ \mathbf{Q}_{j1} \frac{\partial u_1}{\partial x_3} + \mathbf{Q}_{j2} \frac{\partial u_2}{\partial x_3} + \mathbf{Q}_{j3} \frac{\partial u_3}{\partial x_3} \end{Bmatrix} \tag{11.7}$$

$$\frac{\partial U_i}{\partial X_j} = \sum_{k=1}^{3} \sum_{k'=1}^{3} Q_{ik} Q_{jk'} u_{k',k} \tag{11.8}$$

$$E_{ii} = Q_{i1}^2 \varepsilon_{11} + Q_{i2}^2 \varepsilon_{22} + Q_{i3}^2 \varepsilon_{33} + Q_{i1} Q_{i2} 2\varepsilon_{12} + + Q_{i1} Q_{i3} 2\varepsilon_{13} + Q_{i2} Q_{i3} 2\varepsilon_{23} \tag{11.9}$$

$$2E_{ij} = \left(Q_{i1} Q_{j1} + Q_{j1} Q_{i1} \right) \varepsilon_{11} + ... \tag{11.10}$$

$$+ \left(Q_{i1} Q_{j2} + Q_{j1} Q_{i2} \right) 2\varepsilon_{12} + ... \tag{11.11}$$