

# PLANES Documentation

Olivier DAZEL  
LAUM UMR CNRS 6613  
olivier.dazel@univ-lemans.fr

August 22, 2016



# Contents

<b>1</b>	<b>General overview</b>	<b>5</b>
1.1	Presentation . . . . .	5
<b>2</b>	<b>Tables</b>	<b>7</b>
<b>3</b>	<b>Figures</b>	<b>9</b>
<b>4</b>	<b>Weak forms</b>	<b>11</b>
4.1	Finite-Elements . . . . .	11
4.1.1	Fluid . . . . .	11
4.1.2	Elastic solid . . . . .	11
4.2	Discontinuous Galerkin . . . . .	11
4.2.1	Abstract weak forms . . . . .	11
4.2.2	Characteristics . . . . .	13
4.2.3	Boundary term . . . . .	13
4.2.4	Interfaces terms for DGM/DGM . . . . .	14
4.2.5	Interfaces terms for FEM/FEM . . . . .	15
4.2.6	Interfaces terms for FEM/DGM . . . . .	16
<b>5</b>	<b>Plane waves</b>	<b>19</b>
5.1	Fluid . . . . .	19
<b>6</b>	<b>Finite-Element Matrices</b>	<b>21</b>
6.1	Global system . . . . .	21
6.1.1	Unknown vector . . . . .	21
6.2	Number of coefficient in matrices . . . . .	22
6.3	DtN operators . . . . .	22
6.3.1	Fields properties in incident and transmission media . . . . .	22
6.3.2	Formalisation of the fields . . . . .	22
6.4	Case of a interface with fluid/Biot 1998 . . . . .	23

6.5	Case of a interface with an elastic medium . . . . .	23
6.5.1	Reflexion side . . . . .	23
6.5.2	Transmission side . . . . .	24
6.6	Case of integration of a plate medium . . . . .	24
6.6.1	Reflexion side . . . . .	24
6.7	Elementary matrix . . . . .	25
6.7.1	<b>I</b> . . . . .	25
<b>7</b>	<b>Benchmarks</b>	<b>27</b>
7.1	Kundt Tube . . . . .	27
<b>8</b>	<b>Hermite Finite-Elements</b>	<b>29</b>
<b>9</b>	<b>Miscellaneous</b>	<b>31</b>
9.1	Integration of an exponential on one element . . . . .	31
<b>10</b>	<b>Plane Waves 3D</b>	<b>33</b>
10.1	Fluid medium . . . . .	33
10.1.1	Compressional wave . . . . .	33
10.2	Elastic Medium . . . . .	33
10.2.1	Compressional wave . . . . .	33
10.2.2	Shear wave 1 . . . . .	34
10.2.3	Shear wave 2 . . . . .	35
<b>11</b>	<b>Rotations</b>	<b>37</b>

# Chapter 1

## General overview

### 1.1 Presentation

PLANES (Porous LAum NumErical Simulator) project is a collection of Matlab/Fortran scripts to simulate the vibroacoustics response of coupled systems including acoustic, elastic, porous materials, PML...



## Chapter 2

# Tables

- $\text{interface}(n_1, n_2, e_1, e_2, n_{\text{middle}})$
- $\text{boundary}(n_1, n_2, \text{typ}(e), \text{typ}(\text{boundary}), 0, n_{\text{middle}})$





# Chapter 3

## Figures

The number of the figure is xxxyyy. xxx is associated to the the type of display and yyy corresponds to the type of field

Type of display xxx

- 1yyy: Modulus of the projection on  $x$  axis
- 2yyy: Modulus of the projection on  $y$  axis
- 3yyy: Angle of the projection on  $x$  axis
- 4yyy: Angle of the projection on  $y$  axis
- 5000: Shape of the displacement
- 10yyy: Map of the modulus for FEM
- 11yyy: Map of the angle for FEM
- 20yyy: Map of the modulus for DGM
- 21yyy: Map of the angle for DGM
- 30yyy: Map of the modulus in case of a FEM/DGM model
- 31yyy: Map of the angle in case of a FEM/DGM model

Type of fields xxx

- xxx1:  $v_x$  of the fluid or  $v_x^{eq}$  for an equivalent fluid or  $u_x^t$  for a Biot PEM.
- xxx2:  $v_y$  of the fluid or  $v_y^{eq}$  for an equivalent fluid or  $v_y^t$  for a Biot PEM
- xxx3:  $|v|$  of the fluid or  $|v^{eq}|$  for an equivalent fluid or  $u_y^t$  for a Biot PEM

- xxx4:  $v_x^s$  for a Biot PEM.
- xxx5:  $v_y^s$  for a Biot PEM
- xx10:  $p$

# Chapter 4

## Weak forms

### 4.1 Finite-Elements

#### 4.1.1 Fluid

Unknown: the pressure field  $p$ .  $\rho$  is the density

$$\forall q, \quad \int_{\Omega} \frac{\nabla p \cdot \nabla q}{\rho \omega^2} - \frac{p q}{\rho c^2} d\Omega = \oint_{\partial\Omega} \frac{1}{\rho \omega^2} \frac{\partial p}{\partial n} d\Gamma \quad (4.1)$$

#### 4.1.2 Elastic solid

Unknown: the solid displacement  $\mathbf{u}$ .

$$\forall \mathbf{v}, \quad \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - \omega^2 \rho_s \mathbf{u} \cdot \mathbf{v} d\Omega = \oint_{\partial\Omega} [\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n}] \cdot \mathbf{v} d\Gamma \quad (4.2)$$

### 4.2 Discontinuous Galerkin

#### 4.2.1 Abstract weak forms

$$\left( j\omega[\mathbf{M}] + [\mathbf{B}_x] \frac{\partial}{\partial x} + [\mathbf{B}_y] \frac{\partial}{\partial y} \right) \mathbf{S} = \mathbf{0} \text{ on } \Omega. \quad (4.3)$$

$\mathbf{S}$  is the State Vector of the medium. In a conservative form:

$$\left( j\omega + [\mathbf{A}_x] \frac{\partial}{\partial x} + [\mathbf{A}_y] \frac{\partial}{\partial y} \right) \mathbf{S} = \mathbf{0} \text{ on } \Omega. \quad (4.4)$$

$$[\mathbf{A}_x] = [\mathbf{M}]^{-1}[\mathbf{B}_x], \quad [\mathbf{A}_y] = [\mathbf{M}]^{-1}[\mathbf{B}_y]. \quad (4.5)$$

$\Omega$  is partitioned in  $n$  elements  $\Omega_e$  with  $e = 1, \dots, n$ .

$$\sum_{e=1}^n \int_{\Omega_e} \mathbf{T}_e^T \left( j\omega[\mathbf{M}]\mathbf{S}_e + [\mathbf{B}_x] \frac{\partial \mathbf{S}_e}{\partial x} + [\mathbf{B}_y] \frac{\partial \mathbf{S}_e}{\partial y} \right) d\Omega = 0. \quad (4.6)$$

$\mathbf{T}_e$  is the test field on each element.

$$-\sum_{e=1}^n \int_{\Omega_e} \left( j\omega[\mathbf{M}]^T \mathbf{T}_e + [\mathbf{B}_x]^T \frac{\partial \mathbf{T}_e}{\partial x} + [\mathbf{B}_y]^T \frac{\partial \mathbf{T}_e}{\partial y} \right)^T \mathbf{S}_e d\Omega + \sum_{e=1}^n \int_{\partial\Omega_e} \mathbf{T}_e^T [\mathbf{G}_e] \mathbf{S}_e d\Gamma = 0.$$

where we have introduced the matrix  $[\mathbf{G}_e] = [\mathbf{B}_x]n_x + [\mathbf{B}_y]n_y$  which represents the normal fluxes across the boundary of the element  $\Omega_e$ . The unit normal  $\mathbf{n} = (n_x, n_y)$  on the element boundary  $\partial\Omega_e$  points out of element  $e$ . A key aspect of the wave-based DGM is to use test functions  $\mathbf{T}$  whose restrictions  $\mathbf{T}_e$  on each elements are solutions of the adjoint problem defined on each element:

$$\left( j\omega[\mathbf{M}]^T \mathbf{T}_e + [\mathbf{B}_x]^T \frac{\partial \mathbf{T}_e}{\partial x} + [\mathbf{B}_y]^T \frac{\partial \mathbf{T}_e}{\partial y} \right) = \mathbf{0} = 0. \quad (4.7)$$

With this choice of test functions the integral over each element  $\Omega_e$  vanishes and one is left with integrals on the interfaces between elements and on the boundary of the domain.

$$\sum_{e=1}^n \int_{\partial\Omega_e} \mathbf{T}_e^T [\mathbf{G}_e] \mathbf{S}_e d\Gamma = 0. \quad (4.8)$$

$$[\mathbf{F}_e] = [\mathbf{A}_x]n_x + [\mathbf{A}_y]n_y \implies [\mathbf{G}_e] = [\mathbf{M}_e][\mathbf{F}_e] \quad (4.9)$$

$$\sum_{e=1}^n \int_{\partial\Omega_e} \mathbf{T}_e^T [\mathbf{M}_e][\mathbf{F}_e] \mathbf{S}_e d\Gamma = 0. \quad (4.10)$$

$$\sum_{e=1}^n \int_{\partial\Omega_e} \mathbf{V}_e [\mathbf{F}_e] \mathbf{S}_e d\Gamma = 0, \quad \mathbf{V}_e = [\mathbf{M}_e] \mathbf{T}_e. \quad (4.11)$$

## 4.2.2 Characteristics

$$[\mathbf{F}_e][\mathbf{P}_e] = [\mathbf{P}_e][\boldsymbol{\Lambda}_e], \quad [\mathbf{F}_e] = [\mathbf{P}_e][\boldsymbol{\Lambda}_e][\mathbf{Q}_e], \quad [\mathbf{Q}_e] = [\mathbf{P}_e]^{-1} \quad (4.12)$$

$$[\boldsymbol{\Lambda}_e] = \text{diag}([\boldsymbol{\Lambda}_e^{in}], [\boldsymbol{\Lambda}_e^{out}], [\mathbf{0}]). \quad (4.13)$$

$$[\mathbf{F}_e] = [\mathbf{P}_e^{in}][\boldsymbol{\Lambda}_e^{in}][\mathbf{Q}_e^{in}] + [\mathbf{P}_e^{out}][\boldsymbol{\Lambda}_e^{out}][\mathbf{Q}_e^{out}] \quad (4.14)$$

## 4.2.3 Boundary term

$$\mathbf{S}_e = [\mathbf{P}_e^{in}]\mathbf{S}_e^{in} + [\mathbf{P}_e^{out}]\mathbf{S}_e^{out} \quad (4.15)$$

$$[\mathbf{C}_e]\mathbf{S}_e = \mathbf{E}_e. \quad (4.16)$$

$$[\mathbf{C}_e][\mathbf{P}_e^{out}]\mathbf{S}_e^{out} = \mathbf{E}_e - [\mathbf{C}_e][\mathbf{P}_e^{in}]\mathbf{S}_e^{in} \quad (4.17)$$

$$\mathbf{S}_e^{out} = [\tilde{\mathbf{R}}_e]\mathbf{S}_e^{in} + \tilde{\mathbf{E}}_e, \quad (4.18)$$

$$[\tilde{\mathbf{R}}_e] = -([\mathbf{C}_e][\mathbf{P}_e^{in}])^{-1}[\mathbf{C}_e][\mathbf{P}_e^{in}], \quad \tilde{\mathbf{E}}_e = ([\mathbf{C}_e][\mathbf{P}_e^{in}])^{-1}\mathbf{E}_e. \quad (4.19)$$

$$\mathbf{S}_e = \left([\mathbf{P}_e^{in}] + [\mathbf{P}_e^{out}][\tilde{\mathbf{R}}_e]\right)\mathbf{S}_e^{in} + [\mathbf{P}_e^{out}]\tilde{\mathbf{E}}_e \quad (4.20)$$

$$\mathbf{S}_e = \underbrace{\left([\mathbf{P}_e^{in}] + [\mathbf{P}_e^{out}][\tilde{\mathbf{R}}_e]\right)[\mathbf{Q}_e^{in}]}_{[\tilde{\mathbf{P}}_e]}\mathbf{S}_e + [\mathbf{P}_e^{out}]\tilde{\mathbf{E}}_e \quad (4.21)$$

$$\int_{\partial\Omega_e} \mathbf{T}_e^T[\mathbf{M}_e][\mathbf{F}_e]\mathbf{S}_e \, d\Gamma = \int_{\partial\Omega_e} \mathbf{T}_e^T[\mathbf{M}_e][\mathbf{F}_e] \left([\tilde{\mathbf{P}}_e]\mathbf{S}_e + [\mathbf{P}_e^{out}]\tilde{\mathbf{E}}_e\right) \, d\Gamma \quad (4.22)$$

$$= \int_{\partial\Omega_e} \mathbf{T}_e^T[\tilde{\mathbf{F}}_e]\mathbf{S}_e \, d\Gamma + \int_{\partial\Omega_e} \mathbf{T}_e^T\tilde{\mathbf{S}}_e \, d\Gamma \quad (4.23)$$

$$[\tilde{\mathbf{F}}_e] = [\mathbf{M}_e][\mathbf{F}_e][\tilde{\mathbf{P}}_e], \quad \tilde{\mathbf{S}}_e = [\mathbf{M}_e][\mathbf{F}_e][\mathbf{P}_e^{out}]\tilde{\mathbf{E}}_e \quad (4.24)$$

#### 4.2.4 Interfaces terms for DGM/DGM

$$I_{12} = \int_{\Gamma_{12}} \mathbf{T}_1[\mathbf{M}_1][\mathbf{F}_1]\mathbf{S}_1 \, d\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2[\mathbf{M}_2][\mathbf{F}_2]\mathbf{S}_2 \, d\Gamma \quad (4.25)$$

$$\mathbf{S}_1 = [\mathbf{P}_1^{in}]\mathbf{S}_1^{in} + [\mathbf{P}_1^{out}]\mathbf{S}_1^{out} \quad (4.26)$$

$$\mathbf{S}_2 = [\mathbf{P}_2^{in}]\mathbf{S}_2^{in} + [\mathbf{P}_2^{out}]\mathbf{S}_2^{out}. \quad (4.27)$$

$$[\mathbf{C}_1]\mathbf{S}_1 = [\mathbf{C}_2]\mathbf{S}_2. \quad (4.28)$$

$$[\mathbf{C}_1][\mathbf{P}_1^{out}]\mathbf{S}_1^{out} - [\mathbf{C}_2][\mathbf{P}_2^{out}]\mathbf{S}_2^{out} = [\mathbf{C}_2][\mathbf{P}_2^{in}]\mathbf{S}_2^{in} - [\mathbf{C}_1][\mathbf{P}_1^{in}]\mathbf{S}_1^{in} \quad (4.29)$$

$$\begin{Bmatrix} \mathbf{S}_1^{out} \\ \mathbf{S}_2^{out} \end{Bmatrix} = [\tilde{\mathbf{R}}] \begin{Bmatrix} \mathbf{S}_1^{in} \\ \mathbf{S}_2^{in} \end{Bmatrix}. \quad (4.30)$$

$$[\tilde{\mathbf{R}}] = [[\mathbf{C}_1][\mathbf{P}_1^{out}] | -[\mathbf{C}_2][\mathbf{P}_2^{out}]]^{-1} [-[\mathbf{C}_1][\mathbf{P}_1^{in}] | [\mathbf{C}_2][\mathbf{P}_2^{in}]] \quad (4.31)$$

$$= \begin{bmatrix} [\tilde{\mathbf{R}}_{11}] & [\tilde{\mathbf{R}}_{12}] \\ [\tilde{\mathbf{R}}_{21}] & [\tilde{\mathbf{R}}_{22}] \end{bmatrix}. \quad (4.32)$$

$$\mathbf{S}_1 = \underbrace{\left([\mathbf{P}_1^{in}] + [\mathbf{P}_1^{out}][\tilde{\mathbf{R}}_{11}]\right)}_{[\tilde{\mathbf{P}}_{11}]} [\mathbf{Q}_1^{in}] \mathbf{S}_1 + \underbrace{[\mathbf{P}_1^{out}][\tilde{\mathbf{R}}_{12}][\mathbf{Q}_2^{in}]}_{[\tilde{\mathbf{P}}_{12}]} \mathbf{S}_2 \quad (4.33)$$

$$\mathbf{S}_2 = \underbrace{[\mathbf{P}_2^{out}][\tilde{\mathbf{R}}_{21}][\mathbf{Q}_1^{in}]}_{[\tilde{\mathbf{P}}_{21}]} \mathbf{S}_1 + \underbrace{\left([\mathbf{P}_2^{in}] + [\mathbf{P}_2^{out}][\tilde{\mathbf{R}}_{22}]\right) [\mathbf{Q}_2^{in}]}_{[\tilde{\mathbf{P}}_{22}]} \mathbf{S}_2 \quad (4.34)$$

$$I_{12} = \int_{\Gamma_{12}} \mathbf{T}_1[\tilde{\mathbf{F}}_{11}]\mathbf{S}_1 \, d\Gamma + \int_{\Gamma_{12}} \mathbf{T}_1[\tilde{\mathbf{F}}_{12}]\mathbf{S}_2 \, d\Gamma \quad (4.35)$$

$$+ \int_{\Gamma_{12}} \mathbf{T}_2[\tilde{\mathbf{F}}_{21}]\mathbf{S}_1 \, d\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2[\tilde{\mathbf{F}}_{22}]\mathbf{S}_2 \, d\Gamma \quad (4.36)$$

$$[\tilde{\mathbf{F}}_{11}] = [\mathbf{M}_1][\mathbf{F}_1][\tilde{\mathbf{P}}_{11}], \quad [\tilde{\mathbf{F}}_{12}] = [\mathbf{M}_1][\mathbf{F}_1][\tilde{\mathbf{P}}_{12}] \quad (4.37)$$

$$[\tilde{\mathbf{F}}_{21}] = [\mathbf{M}_2][\mathbf{F}_2][\tilde{\mathbf{P}}_{21}], \quad [\tilde{\mathbf{F}}_{22}] = [\mathbf{M}_2][\mathbf{F}_2][\tilde{\mathbf{P}}_{22}] \quad (4.38)$$

## 4.2.5 Interfaces terms for FEM/FEM

$$I_{12} = - \int_{\Gamma_{12}} q_1 \frac{\mathbf{v}_1 \cdot \mathbf{n}_1}{j\omega} d\Gamma - \int_{\Gamma_{12}} q_2 \frac{\mathbf{v}_2 \cdot \mathbf{n}_2}{j\omega} d\Gamma \quad (4.39)$$

– comes from the transposition of the right hand side integral.

Like in the previous section

$$\hat{\mathbf{S}}_1 = [\tilde{\mathbf{P}}_{11}] \mathbf{S}_1 + [\tilde{\mathbf{P}}_{12}] \hat{\mathbf{S}}_2 \quad (4.40)$$

$$\hat{\mathbf{S}}_2 = [\tilde{\mathbf{P}}_{21}] \mathbf{S}_1 + [\tilde{\mathbf{P}}_{22}] \mathbf{S}_2 \quad (4.41)$$

$$-\frac{\mathbf{v}_1 \cdot \mathbf{n}_1}{j\omega} = -\frac{1}{j\omega} \left[ \hat{\mathbf{S}}_1(1) \mathbf{n}_1(1) + \hat{\mathbf{S}}_1(2) \mathbf{n}_1(2) \right] \quad (4.42)$$

$$= [\tilde{\mathbf{F}}_{11}] \mathbf{S}_1 + [\tilde{\mathbf{F}}_{12}] \mathbf{S}_2 \quad (4.43)$$

with

$$[\tilde{\mathbf{F}}_{11}] = -\frac{1}{j\omega} \left[ [\tilde{\mathbf{P}}_{11}(1, :)] \mathbf{n}_1(1) + [\tilde{\mathbf{P}}_{11}(2, :)] \mathbf{n}_1(2) \right] \quad (4.44)$$

$$[\tilde{\mathbf{F}}_{12}] = -\frac{1}{j\omega} \left[ [\tilde{\mathbf{P}}_{12}(1, :)] \mathbf{n}_1(1) + [\tilde{\mathbf{P}}_{12}(2, :)] \mathbf{n}_1(2) \right] \quad (4.45)$$

Concerning the second term

$$-\frac{\mathbf{v}_2 \cdot \mathbf{n}_2}{j\omega} = \frac{1}{j\omega} \left[ \hat{\mathbf{S}}_2(1) \mathbf{n}_2(1) + \hat{\mathbf{S}}_2(2) \mathbf{n}_2(2) \right] \quad (4.46)$$

$$= [\tilde{\mathbf{F}}_{21}] \mathbf{S}_1 + [\tilde{\mathbf{F}}_{22}] \mathbf{S}_2 \quad (4.47)$$

with

$$[\tilde{\mathbf{F}}_{21}] = -\frac{1}{j\omega} \left[ [\tilde{\mathbf{P}}_{21}(1, :)] \mathbf{n}_2(1) + [\tilde{\mathbf{P}}_{21}(2, :)] \mathbf{n}_2(2) \right] \quad (4.48)$$

$$[\tilde{\mathbf{F}}_{22}] = -\frac{1}{j\omega} \left[ [\tilde{\mathbf{P}}_{22}(1, :)] \mathbf{n}_2(1) + [\tilde{\mathbf{P}}_{22}(2, :)] \mathbf{n}_2(2) \right] \quad (4.49)$$

$$p_1(x, y) \approx [\mathbf{N}_1(x, y)] \mathbf{p}_1, \quad \mathbf{p}_2(x, y) \approx [\mathbf{N}_2(x, y)] \mathbf{p}_2 \quad (4.50)$$

$$\mathbf{S}_i \approx \begin{bmatrix} [\mathbf{V}_i^x(x, y)] \\ [\mathbf{V}_i^y(x, y)] \\ [\mathbf{N}_i(x, y)] \end{bmatrix} \mathbf{p}_i \quad (4.51)$$

$$[\mathbf{V}_i^x(x, y)] = -\frac{1}{j\omega\rho} \left[ \frac{\partial \mathbf{N}_i(x, y)}{\partial x} \right], \quad [\mathbf{V}_i^y(x, y)] = -\frac{1}{j\omega\rho} \left[ \frac{\partial \mathbf{N}_i(x, y)}{\partial y} \right] \quad (4.52)$$

The boundary terms

$$\int_{\Gamma_{12}} q_i [\tilde{\mathbf{F}}_{ij}] \mathbf{S}_j \, d\Gamma \approx \int_{\Gamma_{12}} \mathbf{q}_i^T [\mathbf{N}_i(x, y)]^T [\tilde{\mathbf{I}}_{ij}] \mathbf{p}_j \, d\Gamma \quad (4.53)$$

with

$$[\tilde{\mathbf{I}}_{ij}] = \tilde{\mathbf{F}}_{ij}(1) [\mathbf{V}_j^x(x, y)] + \tilde{\mathbf{F}}_{ij}(2) [\mathbf{V}_j^y(x, y)] + \tilde{\mathbf{F}}_{ij}(3) [\mathbf{N}_j(x, y)] \quad (4.54)$$

#### 4.2.6 Interfaces terms for FEM/DGM

$$I_{12} = -\int_{\Gamma_{12}} q_1 \frac{\mathbf{v}_1 \cdot \mathbf{n}_1}{j\omega} \, d\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2 [\mathbf{M}_2] [\mathbf{F}_2] \mathbf{S}_2 \, d\Gamma \quad (4.55)$$

– for the first comes from the transposition of the right hand side integral.

Like in the previous section

$$\hat{\mathbf{S}}_1 = [\tilde{\mathbf{P}}_{11}] \mathbf{S}_1 + [\tilde{\mathbf{P}}_{12}] \hat{\mathbf{S}}_2 \quad (4.56)$$

$$\hat{\mathbf{S}}_2 = [\tilde{\mathbf{P}}_{21}] \mathbf{S}_1 + [\tilde{\mathbf{P}}_{22}] \mathbf{S}_2 \quad (4.57)$$

$$-\frac{\mathbf{v}_1 \cdot \mathbf{n}_1}{j\omega} = \frac{1}{j\omega} \left[ \hat{\mathbf{S}}_1(1) \mathbf{n}_1(1) + \hat{\mathbf{S}}_1(2) \mathbf{n}_1(2) \right] \quad (4.58)$$

$$= [\tilde{\mathbf{F}}_{11}] \mathbf{S}_1 + [\tilde{\mathbf{F}}_{12}] \mathbf{S}_2 \quad (4.59)$$

with

$$[\tilde{\mathbf{F}}_{11}] = -\frac{1}{j\omega} \left[ [\tilde{\mathbf{P}}_{11}(1, :)] \mathbf{n}_1(1) + [\tilde{\mathbf{P}}_{11}(2, :)] \mathbf{n}_1(2) \right] \quad (4.60)$$

$$[\tilde{\mathbf{F}}_{12}] = -\frac{1}{j\omega} \left[ [\tilde{\mathbf{P}}_{12}(1, :)] \mathbf{n}_1(1) + [\tilde{\mathbf{P}}_{12}(2, :)] \mathbf{n}_1(2) \right] \quad (4.61)$$



$$[\tilde{\mathbf{F}}_{21}] = [\mathbf{M}_2][\mathbf{F}_2][\tilde{\mathbf{P}}_{21}], \quad [\tilde{\mathbf{F}}_{22}] = [\mathbf{M}_2][\mathbf{F}_2][\tilde{\mathbf{P}}_{22}] \quad (4.62)$$

$$I_{12} = \int_{\Gamma_{12}} q_1 [\tilde{\mathbf{F}}_{11}] \mathbf{S}_1 \, d\Gamma + \int_{\Gamma_{12}} q_1 [\tilde{\mathbf{F}}_{12}] \mathbf{S}_2 \, d\Gamma \quad (4.63)$$

$$+ \int_{\Gamma_{12}} \mathbf{T}_2^T [\tilde{\mathbf{F}}_{21}] \mathbf{S}_1 \, d\Gamma + \int_{\Gamma_{12}} \mathbf{T}_2^T [\tilde{\mathbf{F}}_{22}] \mathbf{S}_2 \, d\Gamma \quad (4.64)$$

$$p_1(x, y) \approx [\mathbf{N}_1(x, y)] \mathbf{p}_1, \quad \mathbf{S}_2(x, y) \approx [\mathbf{N}_2(x, y)] \mathbf{X}_2 \quad (4.65)$$

$$\mathbf{S}_1 \approx \begin{bmatrix} [\mathbf{V}_1^x(x, y)] \\ [\mathbf{V}_1^y(x, y)] \\ [\mathbf{N}_1(x, y)] \end{bmatrix} \mathbf{p}_1 \quad (4.66)$$

$$[\mathbf{V}_1^x(x, y)] = -\frac{1}{j\omega\rho} \left[ \frac{\partial \mathbf{N}_1(x, y)}{\partial x} \right], \quad [\mathbf{V}_1^y(x, y)] = -\frac{1}{j\omega\rho} \left[ \frac{\partial \mathbf{N}_1(x, y)}{\partial y} \right] \quad (4.67)$$

$$\int_{\Gamma_{12}} q_1 [\tilde{\mathbf{F}}_{11}] \mathbf{S}_1 \, d\Gamma \approx \int_{\Gamma_{12}} \mathbf{q}_1^T [\mathbf{N}_1(x, y)]^T [\tilde{\mathbf{I}}_{11}] \mathbf{p}_1 \, d\Gamma \quad (4.68)$$

with

$$[\tilde{\mathbf{I}}_{11}] = \tilde{\mathbf{F}}_{11}(1) [\mathbf{V}_1^x(x, y)] + \tilde{\mathbf{F}}_{11}(2) [\mathbf{V}_1^y(x, y)] + \tilde{\mathbf{F}}_{11}(3) [\mathbf{N}_1(x, y)] \quad (4.69)$$

$$\int_{\Gamma_{12}} q_1 [\tilde{\mathbf{F}}_{12}] \mathbf{S}_2 \, d\Gamma \approx \int_{\Gamma_{12}} \mathbf{q}_1^T [\tilde{\mathbf{I}}_{12}] \mathbf{X}_2 \, d\Gamma \quad (4.70)$$

with

$$[\tilde{\mathbf{I}}_{12}] = [\mathbf{N}_1(x, y)]^T \tilde{\mathbf{F}}_{12} [\mathbf{N}_2(x, y)] \quad (4.71)$$

$$\int_{\Gamma_{12}} \mathbf{T}_2^T [\tilde{\mathbf{F}}_{21}] \mathbf{S}_1 \, d\Gamma \approx \int_{\Gamma_{12}} \mathbf{X}_2^T [\mathbf{N}_2(x, y)]^T [\tilde{\mathbf{I}}_{21}] \mathbf{p}_1 \, d\Gamma \quad (4.72)$$

with

$$[\tilde{\mathbf{I}}_{21}] = \tilde{\mathbf{F}}_{11}(:, 1) [\mathbf{V}_1^x(x, y)] + \tilde{\mathbf{F}}_{11}(:, 2) [\mathbf{V}_1^y(x, y)] + \tilde{\mathbf{F}}_{11}(:, 3) [\mathbf{N}_1(x, y)] \quad (4.73)$$

$$\int_{\Gamma_{12}} \mathbf{T}_2^T [\tilde{\mathbf{F}}_{22}] \mathbf{S}_2 \, d\Gamma \approx \int_{\Gamma_{12}} \mathbf{X}_2^T [\tilde{\mathbf{I}}_{22}] \mathbf{X}_2 \, d\Gamma \quad (4.74)$$

with

$$[\tilde{\mathbf{I}}_{22}] = [\mathbf{N}_2(x, y)]^T [\tilde{\mathbf{F}}_{22}] [\mathbf{N}_2(x, y)] \quad (4.75)$$

## Chapter 5

# Plane waves

### 5.1 Fluid

$$j\omega\rho v_x = -\frac{\partial p}{\partial x} \quad (5.1)$$

$$j\omega\rho v_y = -\frac{\partial p}{\partial y} \quad (5.2)$$

$$j\omega p = -\rho c^2 \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right). \quad (5.3)$$

$$\mathbf{S} = \begin{Bmatrix} v_x \\ v_y \\ p \end{Bmatrix}. \quad (5.4)$$

$[\mathbf{M}]$ ,  $[\mathbf{A}_x]$  and  $[\mathbf{A}_y]$  are square matrices of size  $m$  defined as

$$[\mathbf{M}] = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \frac{1}{\rho c^2} \end{bmatrix}, \quad (5.5)$$

and

$$[\mathbf{B}_x] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad [\mathbf{B}_y] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \quad (5.6)$$



## Chapter 6

# Finite-Element Matrices

### 6.1 Global system

#### 6.1.1 Unknown vector

1.  $\mathbf{X}_{TR6}$
2.  $\mathbf{X}_{H12}$
3.  $\mathbf{X}_{DGM}$
4.  $\mathbf{X}_{Bloch}$

$$\left[ \begin{array}{cccc|cc}
 [\mathbf{K}_e] - \omega^2[\mathbf{M}_e] & & & & \mathbf{I}_R^e & \mathbf{I}_T^e \\
 & [\mathbf{K}] - \omega^2\tilde{\rho}[\mathbf{M}] & -\tilde{\gamma}[\mathbf{C}] & & \mathbf{I}_R^s & \mathbf{I}_T^s \\
 & -\tilde{\gamma}[\mathbf{C}]^t & \frac{[\mathbf{H}]}{\omega^2\tilde{\rho}_{eq}} - \frac{[\mathbf{Q}]}{\tilde{K}_{eq}} & & \mathbf{I}_R^p & \mathbf{I}_T^p \\
 & & & \frac{[\mathbf{H}]}{\omega^2\rho_0} - \frac{[\mathbf{Q}]}{K_0} & \mathbf{I}_R^a & \mathbf{I}_T^a \\
 \hline
 \mathbf{J}_R^e & \mathbf{J}_R^s & \mathbf{J}_R^p & \mathbf{J}_R^a & & \\
 \mathbf{J}_T^e & \mathbf{J}_T^s & \mathbf{J}_T^p & \mathbf{J}_T^a & & 
 \end{array} \right] \left\{ \begin{array}{c} \mathbf{u}^e \\ \mathbf{u}^s \\ \mathbf{p} \\ \mathbf{p}^a \\ \mathbf{R} \\ \mathbf{T} \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{f}^e \\ \mathbf{f}^s \\ \mathbf{u} \\ \mathbf{u}^a \\ \mathbf{R}' \\ \mathbf{T}' \end{array} \right\} \quad (6.1)$$

## 6.2 Number of coefficient in matrices

## 6.3 DtN operators

### 6.3.1 Fields properties in incident and transmission media

$$p^{inc}(x, y, t) = 1 e^{j(\omega t - k_x x - k_y y)} + \sum_{l \in \mathbb{Z}} e^{j(\omega t - k_x x + k_y(l)y)} R_l \quad (6.2)$$

$$k_x = k_0 \sin(\theta), \quad k_y = k_0 \cos(\theta), \quad \theta \in \left[0; \frac{\pi}{2}\right], \quad k_x(l) = k_x^i + \frac{2\pi l}{D} \quad (6.3)$$

$$u^{inc}(x, y, t) = \frac{\nabla p^{inc}}{\rho_0 \omega^2} = \left[ \begin{array}{c} -\frac{jk_x}{\rho \omega^2} \\ -\frac{jk_z}{\rho \omega^2} \end{array} \right] e^{j(\omega t - k_x x - k_y y)} + \sum_{l \in \mathbb{Z}} \left[ \begin{array}{c} -\frac{jk_x}{\rho \omega^2} \\ \frac{jk_y(l)}{\rho \omega^2} \end{array} \right] e^{j(\omega t - k_x x + k_y(l)y)} R_l \quad (6.4)$$

$$p^{tr}(x, y, t) = \sum_{l \in \mathbb{Z}} e^{j(\omega t - k_x x - k_y(l)y)} T_l \quad (6.5)$$

$$u^{tr}(x, y, t) = \frac{\nabla p^{tr}}{\rho_0 \omega^2} = \sum_{l \in \mathbb{Z}} \left[ \begin{array}{c} -\frac{jk_x}{\rho \omega^2} \\ -\frac{jk_y(l)}{\rho \omega^2} \end{array} \right] e^{j(\omega t - k_x x - k_y(l)y)} T_l \quad (6.6)$$

### 6.3.2 Formalisation of the fields

Let  $\mathbf{X}_l$  be an information vector associated to Bloch wave  $l$ .

$$\mathbf{Y} = \left\{ \begin{array}{c} 1 \\ \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_n \end{array} \right\} \quad (6.7)$$

On each interface with the FEM model a physical field  $\varphi$  can be written

$$\varphi(x, y) = [\mathbf{\Omega}_F^\varphi] e^{-jk_x} + \sum_l [\mathbf{\Omega}_l^\varphi] \mathbf{X}_l e^{-jk_x(l)}. \quad (6.8)$$

## 6.4 Case of a interface with fluid/Biot 1998

$$I_{FEM} = \int_{|\Gamma_R} \frac{1}{\rho\omega^2} \frac{\partial p^{inc}}{\partial n} q \, d\Gamma = \int_{|\Gamma_R} -u_y(x)q(x) \, d\Gamma \quad (6.9a)$$

$$= \int_{|\Gamma_R} [\mathbf{\Omega}_F^{u_y}] e^{-jk_x} q \, d\Gamma - \sum_l \int_{|\Gamma_R} [\mathbf{\Omega}_l^{u_y}] \mathbf{X}_l e^{-jk_x(l)} q \, d\Gamma \quad (6.9b)$$

- is highlighted as this term will end in the matrix system.

The information vector is scalar  $\mathbf{X}_l = R_l$  and correspond to the reflexion coefficient.

$$[\mathbf{\Omega}_F^{u_y}] = \frac{jk_z}{\rho_0\omega^2}, \quad [\mathbf{\Omega}_l^{u_y}] = \frac{jk_z(l)}{\rho_0\omega^2} \quad (6.9c)$$

The additional equation is the projection of the pressure

$$\int_{\Gamma_R} p(x) e^{jk_x(n)x} \, d\Gamma = \underbrace{\int_{\Gamma_R} e^{j\frac{2\pi nx}{D}} \, d\Gamma}_{D\delta_{n0}} + \underbrace{\int_{\Gamma_R} \sum_{l \in \mathbb{Z}} e^{-j\frac{2\pi(l-n)x}{D}} \, d\Gamma}_{D\delta_{nl}} R_l \quad (6.10)$$

## 6.5 Case of a interface with an elastic medium

### 6.5.1 Reflexion side

$$I_{FEM} = \int_{|\Gamma_R} \sigma_{xy}v_x - \sigma_{yy}v_y \, d\Gamma = \int_{|\Gamma_R} pv_y \, d\Gamma \quad (6.11a)$$

$$= \int_{|\Gamma_R} [\mathbf{\Omega}_F^p] e^{-jk_x} q \, d\Gamma - \sum_l \int_{|\Gamma_R} [\mathbf{\Omega}_l^p] \mathbf{X}_l e^{-jk_x(l)} v_y \, d\Gamma \quad (6.11b)$$

The information vector is scalar  $\mathbf{X}_l = R_l$  and correspond to the reflexion coefficient.

$$[\mathbf{\Omega}_F^p] = 1, \quad [\mathbf{\Omega}_l^p] = -1 \quad (6.11c)$$

The additional equation is the projection of the pressure

$$\int_{\Gamma_R} u_y(x) e^{jk_x(n)x} \, d\Gamma = \frac{-jk_y}{\rho_0\omega^2} D\delta_{n0} + \frac{jk_y(l)}{\rho_0\omega^2} D\delta_{nl} R_l \quad (6.12)$$

### 6.5.2 Transmission side

$$I_{FEM} = \int_{|\Gamma_R} \sigma_{xy} v_x + \sigma_{yy} v_y \, d\Gamma = \int_{|\Gamma_R} p v_y \, d\Gamma \quad (6.13a)$$

$$= \int_{|\Gamma_R} [\mathbf{\Omega}_F^p] e^{-jk_x} q \, d\Gamma - \sum_l \int_{|\Gamma_R} [\mathbf{\Omega}_l^p] \mathbf{X}_l e^{-jk_x(l)} v_y \, d\Gamma \quad (6.13b)$$

The information vector is scalar  $\mathbf{X}_l = R_l$  and correspond to the reflexion coefficient.

$$[\mathbf{\Omega}_F^p] = -1, \quad [\mathbf{\Omega}_l^p] = 1 \quad (6.13c)$$

The additional equation is the projection of the pressure

$$\int_{\Gamma_T} u_y(x) e^{jk_x(n)x} \, d\Gamma = -\frac{jk_y(l)}{\rho_0 \omega^2} D \delta_{nl} R_l \quad (6.14)$$

## 6.6 Case of integration of a plate medium

### 6.6.1 Reflexion side

$$I_{FEM} = \int_{|\Gamma_R} -\sigma_{xz} v_x - \sigma_{zz} v_z \, d\Gamma \quad (6.15)$$

$$\begin{Bmatrix} \sigma_{xz} \\ u_z \\ \sigma_{zz} \\ u_x \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{jk_z}{\rho\omega^2} & \frac{jk_z}{\rho\omega^2} & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ R \\ u_x \end{Bmatrix} \quad (6.16)$$

$$\begin{Bmatrix} \sigma_{xz} \\ u_z \\ \sigma_{zz} \\ u_x \end{Bmatrix}_{\mathbf{F}} = [\mathbf{T}] \begin{Bmatrix} 0 \\ -\frac{jk_z}{\rho\omega^2} \\ -1 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_{xz} \\ u_z \\ \sigma_{zz} \\ u_x \end{Bmatrix}_{\mathbf{R}} = [\mathbf{T}] \begin{bmatrix} 0 & 0 \\ \frac{jk_z}{\rho\omega^2} & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} R \\ u_x \end{Bmatrix} \quad (6.17)$$



$$p|_{\Gamma_R}(x, z) = e^{-j(k_x^i x + k_z^i z)} + \sum_{l \in \mathbb{Z}} e^{-j(k_x(l)x - k_z(l)z)} R_l \quad (6.18)$$

$$\frac{\partial p}{\partial z}|_{\Gamma_R}(x, z) = -jk_z^i e^{-j(k_x^i x + k_z^i z)} + \sum_{l \in \mathbb{Z}} jk_y(l) e^{-j(k_x(l)x - k_z(l)z)} R_l \quad (6.19)$$

$$p|_{\Gamma_R}(x, z) = \sum_{l \in \mathbb{Z}} e^{-j(k_x(l)x + k_z(l)z)} T_l \quad (6.20)$$

$$\frac{\partial p}{\partial z}|_{\Gamma_T}(x, z) = \sum_{l \in \mathbb{Z}} -jk_z(l) e^{-j(k_x(l)x + k_z(l)z)} T_l \quad (6.21)$$

Termes de surface

$$\frac{1}{\rho_a \omega^2} \int_{\Gamma_R} \frac{\partial p}{\partial n} q \, d\Gamma = \dots \quad (6.22)$$

Equation supplementaires

$$\int_{\Gamma_R} p(x) e^{jk_x(n)x} \, d\Gamma = \underbrace{\int_{\Gamma_R} e^{j \frac{2\pi n x}{D}} \, d\Gamma}_{D\delta_{n0}} + \underbrace{\int_{\Gamma_R} \sum_{l \in \mathbb{Z}} e^{-j \frac{2\pi(l-n)x}{D}} \, d\Gamma}_{D\delta_{nl}} R_l \quad (6.23)$$

## 6.7 Elementary matrix

### 6.7.1 I

$$\hat{I} = \int_a^b (AX^2 + BX + C) e^{-jk_x X} \, dX \quad (6.24)$$

$$x = X - a$$

$$\hat{I} = e^{-jk_x a} \int_0^h \Phi(x) e^{-jk_x x} \, dx \quad (6.25)$$

$$\xi = -1 + \frac{2x}{h}, \quad x = \frac{h}{2} (\xi + 1)$$

$$\hat{I} = e^{-jk_x a} \frac{h}{2} \int_{-1}^1 \Phi(\xi) e^{-jk_x \frac{h(\xi+1)}{2}} d\xi = e^{-jk_x (a+\frac{h}{2})} \frac{h}{2} \int_{-1}^1 \Phi(\xi) e^{-jk_x \frac{h\xi}{2}} d\xi \quad (6.26)$$

# Chapter 7

## Benchmarks

### 7.1 Kundt Tube

Sous projets

- 0: TR6
- 1: H12
- 2: TR6/H12
- 3: DGM on TR
- 4: DGM on H
- 5: DGM on TR / DGM on H
- 6: H12 / DGM on H
- 7: TR6 / DGM on H

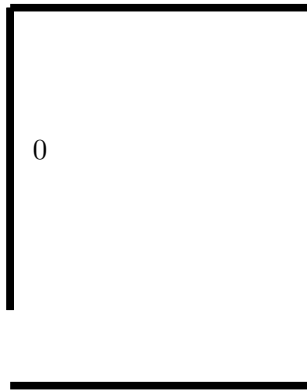


Figure 7.1: caption

## Chapter 8

# Hermite Finite-Elements

$$\Psi_1(x) = 1 - 3\left(\frac{x}{l_x}\right)^2 + 2\left(\frac{x}{l_x}\right)^3 \quad (8.1)$$

$$\Psi_2(x) = \left(\frac{x}{l_x}\right) - 2\left(\frac{x}{l_x}\right)^2 + \left(\frac{x}{l_x}\right)^3 \quad (8.2)$$

$$\Psi_3(x) = -\left(\frac{x}{l_x}\right)^2 + \left(\frac{x}{l_x}\right)^3 \quad (8.3)$$

$$\Psi_4(x) = 3\left(\frac{x}{l_x}\right)^2 - 2\left(\frac{x}{l_x}\right)^3 \quad (8.4)$$

$$\Psi_{ij}(x, y) = \Psi_i(x)\Psi_j(y) \quad (8.5)$$

Valeurs

	$\Psi_1$	$\Psi_2$	$\Psi_3$	$\Psi_4$
$f(0)$	1	0	0	0
$f(l_x)$	0	0	0	1
$f'(0)$	0	$1/l$	0	0
$f'(1_x)$	0	0	$1/l$	0

Ordre

	$p$	$\frac{\partial p}{\partial x}$	$\frac{\partial p}{\partial y}$	$\emptyset$
(0, 0)	$\Psi_{11}$	$\Psi_{21}$	$\Psi_{12}$	$\Psi_{22}$
	1//1	2//2	3//3	4// $\emptyset$
(1, 0)	$\Psi_{41}$	$\Psi_{31}$	$\Psi_{42}$	$\Psi_{23}$
	5//4	6//5	7//6	8// $\emptyset$
(1, 1)	$\Psi_{44}$	$\Psi_{34}$	$\Psi_{43}$	$\Psi_{33}$
	9//7	10//8	11//9	12// $\emptyset$
(0, 1)	$\Psi_{14}$	$\Psi_{24}$	$\Psi_{13}$	$\Psi_{32}$
	13//10	14//11	15//12	16// $\emptyset$

## Chapter 9

# Miscellaneous

### 9.1 Integration of an exponential on one element

$$\int_{\Omega} e^{j\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_c)} d\Omega = \oint \nabla \cdot \left( - \begin{vmatrix} \frac{jk_x}{k_x^2 + k_y^2} \\ \frac{jk_y}{k_x^2 + k_y^2} \end{vmatrix} e^{j\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_c)} \right) d\Omega \quad (9.1)$$

$$= - \oint_{\partial\Omega} \left( \frac{jk_x n_x}{k_x^2 + k_y^2} + \frac{jk_y n_y}{k_x^2 + k_y^2} \right) e^{j\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}_c)} d\Omega \quad (9.2)$$





## Chapter 10

# Plane Waves 3D

### 10.1 Fluid medium

#### 10.1.1 Compressional wave

$$\delta_P^2 = k_x^2 + k_y^2 + k_z^2 \quad (10.1)$$

$$u^\varepsilon(x, y, z, t) = \begin{vmatrix} k_x \\ k_y \\ \varepsilon k_z \end{vmatrix} e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \quad (10.2)$$

$$p = (-K_0)(-j)(k_x^2 + k_y^2 + \varepsilon^2 k_z^2) \quad (10.3)$$

$$= jK_0 \delta_P^2 \quad (10.4)$$

### 10.2 Elastic Medium

#### 10.2.1 Compressional wave

$$\delta_P^2 = k_x^2 + k_y^2 + k_z^2 \quad (10.5)$$

$$u^+(x, y, z, t) = \begin{vmatrix} k_x \\ k_y \\ \varepsilon k_z \end{vmatrix} e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \quad (10.6)$$

$$\sigma_{zz} = \lambda [u_{x,x} + u_{y,y} + u_{z,z}] + 2\mu u_{z,z} \quad (10.7)$$

$$= -j\lambda(k_x^2 + k_y^2 + \varepsilon^2 k_z^2) + 2\mu(-j\varepsilon^2 k_z^2) \quad (10.8)$$

$$= -j[\lambda(k_x^2 + k_y^2) + (\lambda + 2\mu)k_z^2] \quad (10.9)$$

$$\sigma_{yz} = \mu [u_{y,z} + u_{z,y}] = -j[\varepsilon k_y k_z + \varepsilon k_z k_y] = -2j\mu \varepsilon k_y k_z \quad (10.10)$$

$$\sigma_{xz} = \mu [u_{x,z} + u_{z,x}] = -j\mu [\varepsilon k_x k_z + \varepsilon k_z k_x] = -2j\mu \varepsilon k_x k_z \quad (10.11)$$

### 10.2.2 Shear wave 1

$$\delta_S^2 = k_x^2 + k_y^2 + k_z^2 \quad (10.12)$$

$$u^\varepsilon(x, y, z, t) = \begin{vmatrix} k_z \\ 0 \\ -\varepsilon k_x \end{vmatrix} e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \quad (10.13)$$

$$\sigma_{zz} = \lambda [u_{x,x} + u_{y,y} + u_{z,z}] + 2\mu u_{z,z} \quad (10.14)$$

$$= 2\mu j k_x k_z \quad (10.15)$$

$$\sigma_{yz} = \mu [u_{y,z} + u_{z,y}] = -j\mu \varepsilon k_x k_y \quad (10.16)$$

$$\sigma_{xz} = \mu [u_{x,z} + u_{z,x}] = -j\mu [\varepsilon k_x k_z - \varepsilon k_z k_x] = -j\mu \varepsilon (k_z^2 - k_x^2) \quad (10.17)$$

**10.2.3 Shear wave 2**

$$\delta_S^2 = k_x^2 + k_y^2 + k_z^2 \quad (10.18)$$

$$u^\varepsilon(x, y, z, t) = \begin{vmatrix} 0 \\ k_z \\ -\varepsilon k_y \end{vmatrix} e^{j(\omega t - k_x x - k_y y - \varepsilon k_z z)} \quad (10.19)$$

$$\sigma_{zz} = \lambda [u_{x,x} + u_{y,y} + u_{z,z}] + 2\mu u_{z,z} \quad (10.20)$$

$$= 2\mu j k_y k_z \quad (10.21)$$

$$\sigma_{yz} = \mu [u_{y,z} + u_{z,y}] = -j\mu\varepsilon(k_z^2 - k_y^2) \quad (10.22)$$

$$\sigma_{xz} = \mu [u_{x,z} + u_{z,x}] = -j\varepsilon k_x k_y \quad (10.23)$$



## Chapter 11

### Rotations

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = [\mathbf{P}] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = [\mathbf{P}] \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad (11.1)$$

$$\frac{\partial f}{\partial x_1} = \frac{\partial F}{\partial X_1} \frac{\partial X_1}{\partial x_1} + \frac{\partial F}{\partial X_2} \frac{\partial X_2}{\partial x_1} + \frac{\partial F}{\partial X_3} \frac{\partial X_3}{\partial x_1} \quad (11.2)$$

$$= \mathbf{P}(1, :) \begin{pmatrix} \frac{\partial}{\partial X_1} \\ \frac{\partial}{\partial X_2} \\ \frac{\partial}{\partial X_3} \end{pmatrix} F(X_1, X_2, X_3) \quad (11.3)$$

$$\begin{pmatrix} \frac{\partial}{\partial X_1} \\ \frac{\partial}{\partial X_2} \\ \frac{\partial}{\partial X_3} \end{pmatrix} F(X_1, X_2, X_3) = [\mathbf{Q}] \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} f(x_1, x_2, x_3) \quad (11.4)$$

$$\frac{\partial U_i}{\partial X_j} = \mathbf{Q}(i, :) \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} \mathbf{Q}(j, :) \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad (11.5)$$

$$= \mathbf{Q}(i, :) \begin{Bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{Bmatrix} (\mathbf{Q}_{j1}u_1 + \mathbf{Q}_{j2}u_2 + \mathbf{Q}_{j3}u_3) \quad (11.6)$$

$$= \mathbf{Q}(i, :) \begin{Bmatrix} \mathbf{Q}_{j1} \frac{\partial u_1}{\partial x_1} + \mathbf{Q}_{j2} \frac{\partial u_2}{\partial x_1} + \mathbf{Q}_{j3} \frac{\partial u_3}{\partial x_1} \\ \mathbf{Q}_{j1} \frac{\partial u_1}{\partial x_2} + \mathbf{Q}_{j2} \frac{\partial u_2}{\partial x_2} + \mathbf{Q}_{j3} \frac{\partial u_3}{\partial x_2} \\ \mathbf{Q}_{j1} \frac{\partial u_1}{\partial x_3} + \mathbf{Q}_{j2} \frac{\partial u_2}{\partial x_3} + \mathbf{Q}_{j3} \frac{\partial u_3}{\partial x_3} \end{Bmatrix} \quad (11.7)$$

$$\frac{\partial U_i}{\partial X_j} = \sum_{k=1}^3 \sum_{k'=1}^3 Q_{ik} Q_{jk'} u_{k',k} \quad (11.8)$$

$$E_{ii} = Q_{i1}^2 \varepsilon_{11} + Q_{i2}^2 \varepsilon_{22} + Q_{i3}^2 \varepsilon_{33} + Q_{i1} Q_{i2} 2\varepsilon_{12} + Q_{i1} Q_{i3} 2\varepsilon_{13} + Q_{i2} Q_{i3} 2\varepsilon_{23} \quad (11.9)$$

$$2E_{ij} = (Q_{i1}Q_{j1} + Q_{j1}Q_{i1}) \varepsilon_{11} + \dots \quad (11.10)$$

$$+ (Q_{i1}Q_{j2} + Q_{j1}Q_{i2}) 2\varepsilon_{12} + \dots \quad (11.11)$$