Rappel FEM :

$$\Delta p + k^2 p = 0$$

$$\int \Delta p \, q \, d\Omega + k^2 \int p q \, d\Omega = 0.$$

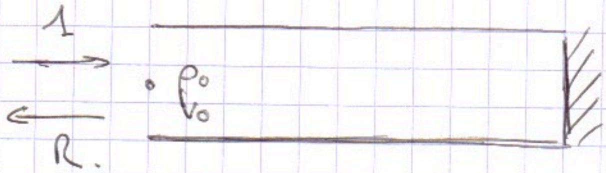
$$\left[\nabla p \, q \right]_\Omega - \int \nabla p \nabla q \, d\Omega + \int_\Omega p q \, d\Omega = 0.$$

$$\int \nabla p \nabla q \, d\Omega - k^2 \int p q \, d\Omega = \int_{\partial\Omega} \frac{\partial p}{\partial n} q \, d\Gamma$$

DGM :

$$\frac{du}{dt} + F \frac{du}{dx} = 0 \qquad u = \begin{Bmatrix} v \\ p \end{Bmatrix}$$

$$\frac{d\tilde{u}}{dt} + \Lambda \frac{d\tilde{u}}{dx} = 0 \qquad \tilde{u} = \begin{Bmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{Bmatrix}$$



(*)

Conditions de la forme $\tilde{u}^\pm = C^\pm$.

$$p \pm \rho_0 c_0 \frac{\frac{\partial p}{\partial n}}{j\omega\rho_0} = C$$

$$\Leftrightarrow jkp - \frac{\partial p}{\partial n} = C \Rightarrow \frac{\partial p}{\partial n} = C + jkp$$

$$\Leftrightarrow \int \frac{\partial p}{\partial n} q \, d\Gamma = \int C q \, d\Gamma + jk \int p q \, d\Gamma$$

Conditions aux limites pour (*)

$$\begin{cases} \rho_0 = 1 + R \\ v_0 = 1 - R. \end{cases} \qquad (**)$$

Diagonalisation de F

$$F = \begin{bmatrix} 0 & 1/\rho_0 \\ \rho_0 c_0^2 & 0 \end{bmatrix}$$

$$|F - \lambda I| = \lambda^2 - c_0^2 \Rightarrow \lambda_{1,2} = \pm c_0$$

- $\lambda_1 = c_0 \Rightarrow F\vec{W}_1 = \lambda_1 \vec{W}_1$

$$\begin{cases} 1/\rho_0 \, W_{12} = c_0 W_{11} \\ \rho_0 c_0^2 W_{11} = c_0 W_{12} \end{cases} \Rightarrow \vec{W}_1 = \begin{Bmatrix} 1 \\ Z_0 \end{Bmatrix}.$$

$$Z_0 = \rho_0 c_0$$

- $\lambda_2 = -c_0 \Rightarrow F\vec{W}_2 = \lambda_2 \vec{W}_2$

$$\begin{cases} \frac{1}{\rho_0} W_{22} = -c_0 W_{21} \\ \rho_0 c_0^2 W_{21} = -c_0 W_{22} \end{cases} \Rightarrow \vec{W}_2 \begin{Bmatrix} 1 \\ -Z_0 \end{Bmatrix}.$$

$$W = \begin{bmatrix} 1 & 1 \\ Z_0 & -Z_0 \end{bmatrix} \Rightarrow W^{-1} = \frac{1}{2Z_0} \begin{bmatrix} Z_0 & 1 \\ -Z_0 & 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} c_0 & 0 \\ 0 & -c_0 \end{bmatrix} \Rightarrow F = W \Lambda W^{-1}$$

From (**).

$$\begin{cases} p_0 = 1 + R \\ v_0 = 1 - R \end{cases} \iff \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{Bmatrix} v \\ p \end{Bmatrix}}_{u} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{C'} \begin{Bmatrix} 1 \\ R \end{Bmatrix}.$$

$$Cu = C' \begin{Bmatrix} 1 \\ R \end{Bmatrix}.$$

$$\iff \quad C W W^{-1} u = C' \begin{Bmatrix} 1 \\ R \end{Bmatrix}.$$

$$\tilde{C} \tilde{u} = C' \begin{Bmatrix} 1 \\ R \end{Bmatrix}.$$

avec : $\tilde{C} = CW$ or $C = I$ donc $\tilde{C} = W$.

$$\tilde{u} = W^{-1} u = \begin{Bmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{Bmatrix}.$$

$$\tilde{u} = \tilde{C}^{-1} C' \begin{Bmatrix} 1 \\ R \end{Bmatrix}.$$

$$\tilde{u} = \frac{1}{2Z_0} \begin{bmatrix} Z_0 & 1 \\ -Z_0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ R \end{Bmatrix}.$$

$$\tilde{u} = \frac{1}{2Z_0} \begin{bmatrix} (Z_0 + 1) & (1 - Z_0) \\ (Z_0 - 1) & -(Z_0 + 1) \end{bmatrix} \begin{Bmatrix} 1 \\ R \end{Bmatrix}$$

$$\begin{cases} 2Z_0 \tilde{u}^+ = (Z_0 + 1) + R(1 - Z_0) \\ 2Z_0 \tilde{u}^- = (Z_0 - 1) - R(Z_0 + 1) \end{cases}$$

$$\begin{cases} 2Z_0 \tilde{u}^+ = Z_0 + 1 - \dfrac{2Z_0 \tilde{u}^- - (1 - Z_0)}{Z_0 + 1} (1 - Z_0) \\ R = \dfrac{-2Z_0 \tilde{u}^- + (Z_0 - 1)}{1 + Z_0}. \end{cases}$$

$$\begin{cases} \tilde{u}^+ = \dfrac{(z_0+1)^2 - (1-z_0)^2}{1+z_0} - \dfrac{1-z_0}{1+z_0}\,\tilde{u}^- \\ R = \dfrac{1-z_0}{1+z_0} - \dfrac{2z_0}{1+z_0}\,\tilde{u}^- \end{cases}$$

$$\begin{Bmatrix} \tilde{u}^+ \\ R \end{Bmatrix} = \frac{1}{2z_0} \begin{bmatrix} (1-z_0) & 2 \\ -2z_0 & (1-z_0) \end{bmatrix} \begin{Bmatrix} \tilde{u}^- \\ 1 \end{Bmatrix} \quad (***$$

On the FEM Side

$$\begin{cases} p = \phi_1 q_1 + \phi_2 q_2 \\ v = \dfrac{j}{\rho_0 \omega} \phi_1' q_1 + \dfrac{j}{\rho_0 \omega} \phi_2' q_2 \end{cases}$$

$$\underbrace{\begin{Bmatrix} v \\ p \end{Bmatrix}}_{\equiv u} = \begin{bmatrix} j/\rho_0\omega\, \phi_1' & j/\rho_0\omega\, \phi_2' \\ \phi_1 & \phi_2 \end{bmatrix} \underbrace{\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}}_{\equiv q}$$

$$\tilde{u} = \omega^{-1} u = \frac{1}{2z_0} \begin{bmatrix} z_0 & 1 \\ -z_0 & 1 \end{bmatrix} \begin{bmatrix} \frac{j}{\rho_0\omega} \phi_1' & \frac{j}{\rho_0\omega} \phi_2' \\ \phi_1 & \phi_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{Bmatrix} = \frac{1}{2z_0} \begin{bmatrix} \left(\phi_1 - \dfrac{\phi_1'}{jk}\right) & \left(\phi_2 - \dfrac{\phi_2'}{jk}\right) \\ \left(\dfrac{\phi_1'}{jk} + \phi_1\right) & \left(\dfrac{\phi_2'}{jk} + \phi_2\right) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$\begin{pmatrix} ** \\ ** \end{pmatrix}$$

$\cancel{(***) \Rightarrow}\ R =$

$$\begin{pmatrix} ** \\ ** \end{pmatrix} \Rightarrow \quad \tilde{u}^- = \frac{1}{2Z_0} \underbrace{\left[q_1 \,|\, q_2 \right]}_{q^T} \underbrace{\begin{pmatrix} \dfrac{\Phi_1'}{jk} + \Phi_1 \\[2ex] \dfrac{\Phi_2'}{jk} + \Phi_2 \end{pmatrix}}_{\mathcal{U}}$$

$$(***) \Rightarrow \begin{cases} \tilde{u}^+ = \dfrac{1 - Z_0}{1 + Z_0}\, \tilde{u}^- + \dfrac{2}{1 + Z_0}\, 1 \\[2ex] R = \dfrac{-2Z_0}{1 + Z_0}\, \tilde{u}^- + \dfrac{1 - Z_0}{1 + Z_0}\, 1 \end{cases}$$

On remplace $\tilde{u}^-$ dans $\tilde{u}^+$

$$\tilde{u}^+ = \frac{1}{2Z_0} \frac{1 - Z_0}{1 + Z_0}\, q^T \mathcal{U} + \frac{2}{1 + Z_0}$$

En reprenant : $\tilde{u} = W^{-1} u \Leftrightarrow u = W \tilde{u}$

$$\begin{Bmatrix} v \\ p \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ Z_0 & -Z_0 \end{bmatrix} \begin{Bmatrix} \tilde{u}^+ \\ \tilde{u}^- \end{Bmatrix}.$$

$$\Leftrightarrow \begin{cases} v = \tilde{u}^+ + \tilde{u}^- = -\dfrac{1}{j\omega\rho_0} \dfrac{\partial p}{\partial x} \\[2ex] p = Z_0 \tilde{u}^+ - Z_0 \tilde{u}^- \end{cases}$$

$$\frac{\partial p}{\partial x} = -j\omega\rho_0 \left[\left(1 + \frac{1 - Z_0}{1 + Z_0} \right) \frac{1}{2Z_0}\, q^T \mathcal{U} + \frac{2}{1 + Z_0} \right]$$