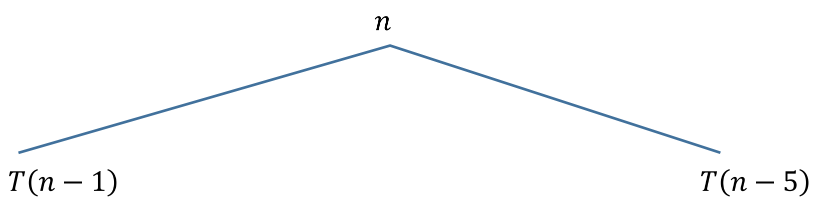
## Raskime rekurentinės lygties $T(n) = T(n-1) + T(n-5) + n$ sprendinį.

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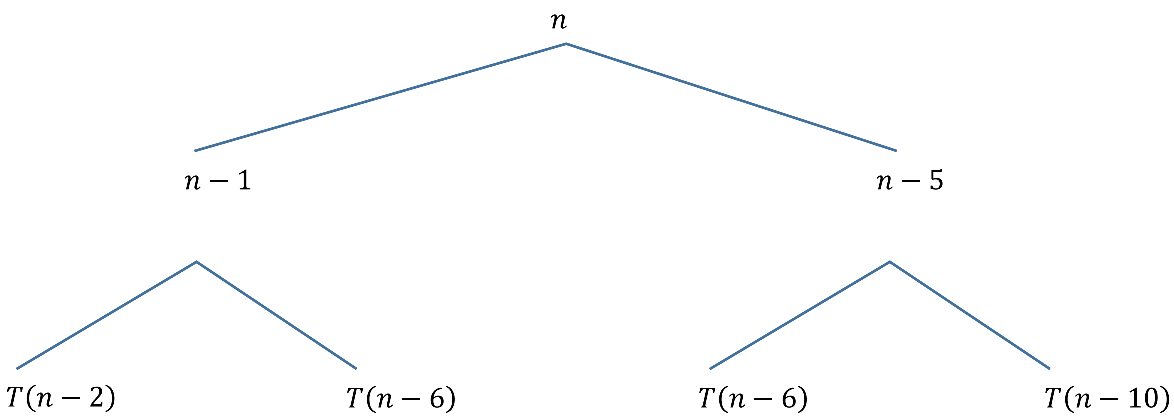
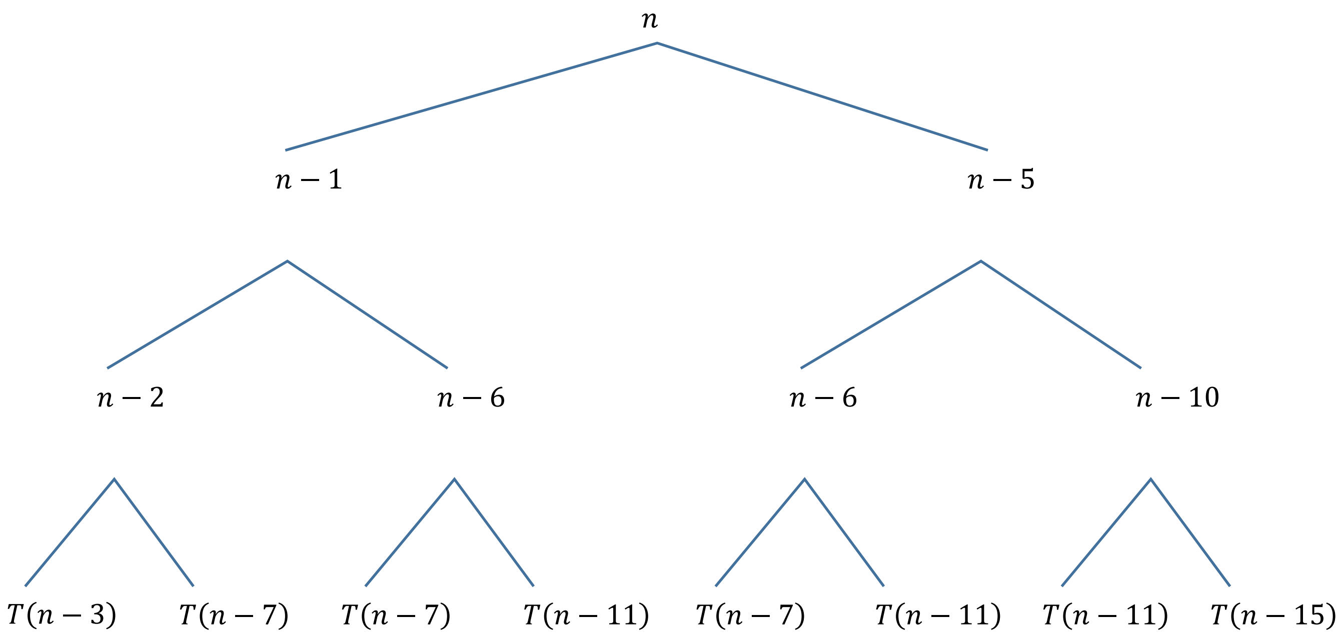
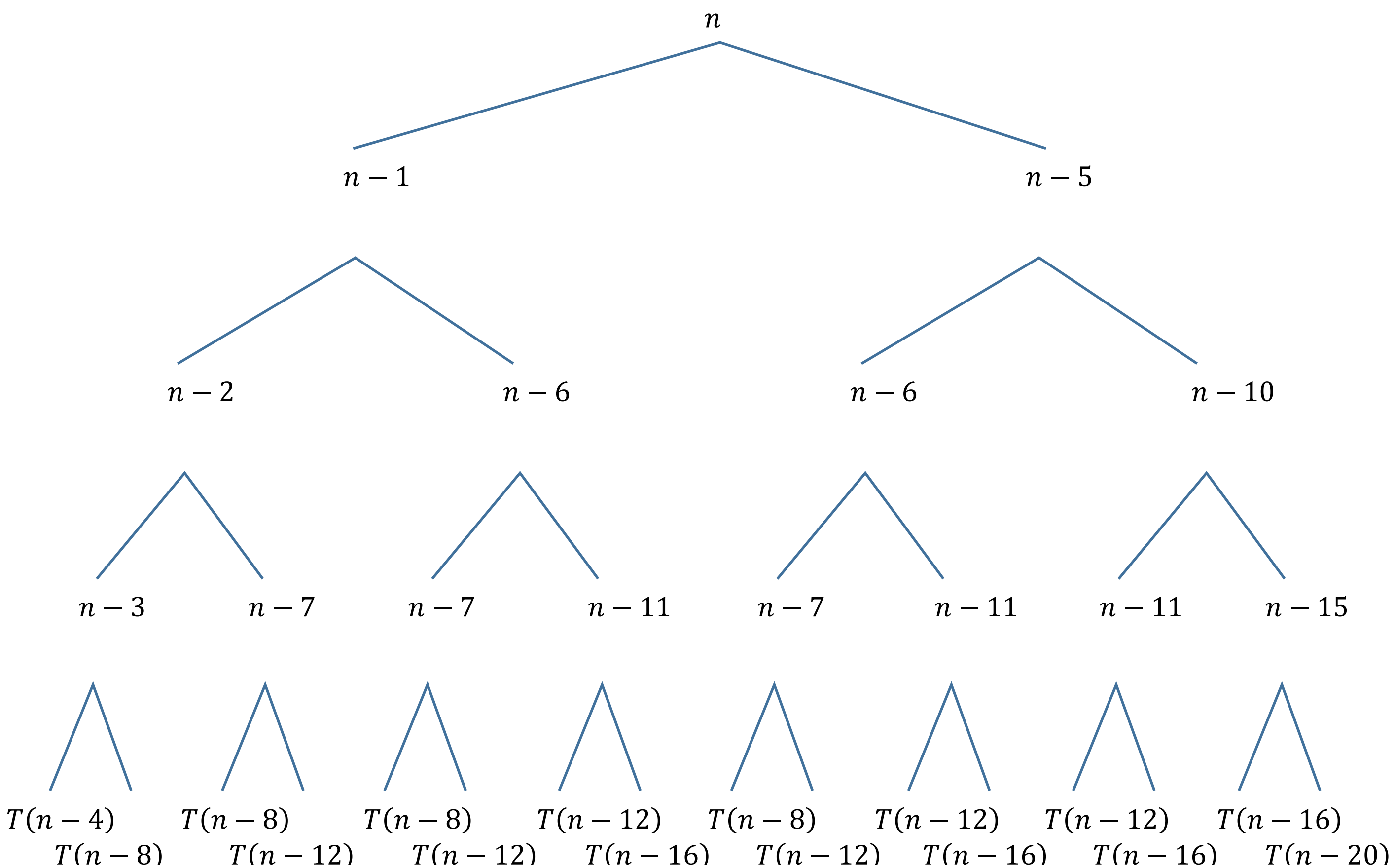
*1 pav.*



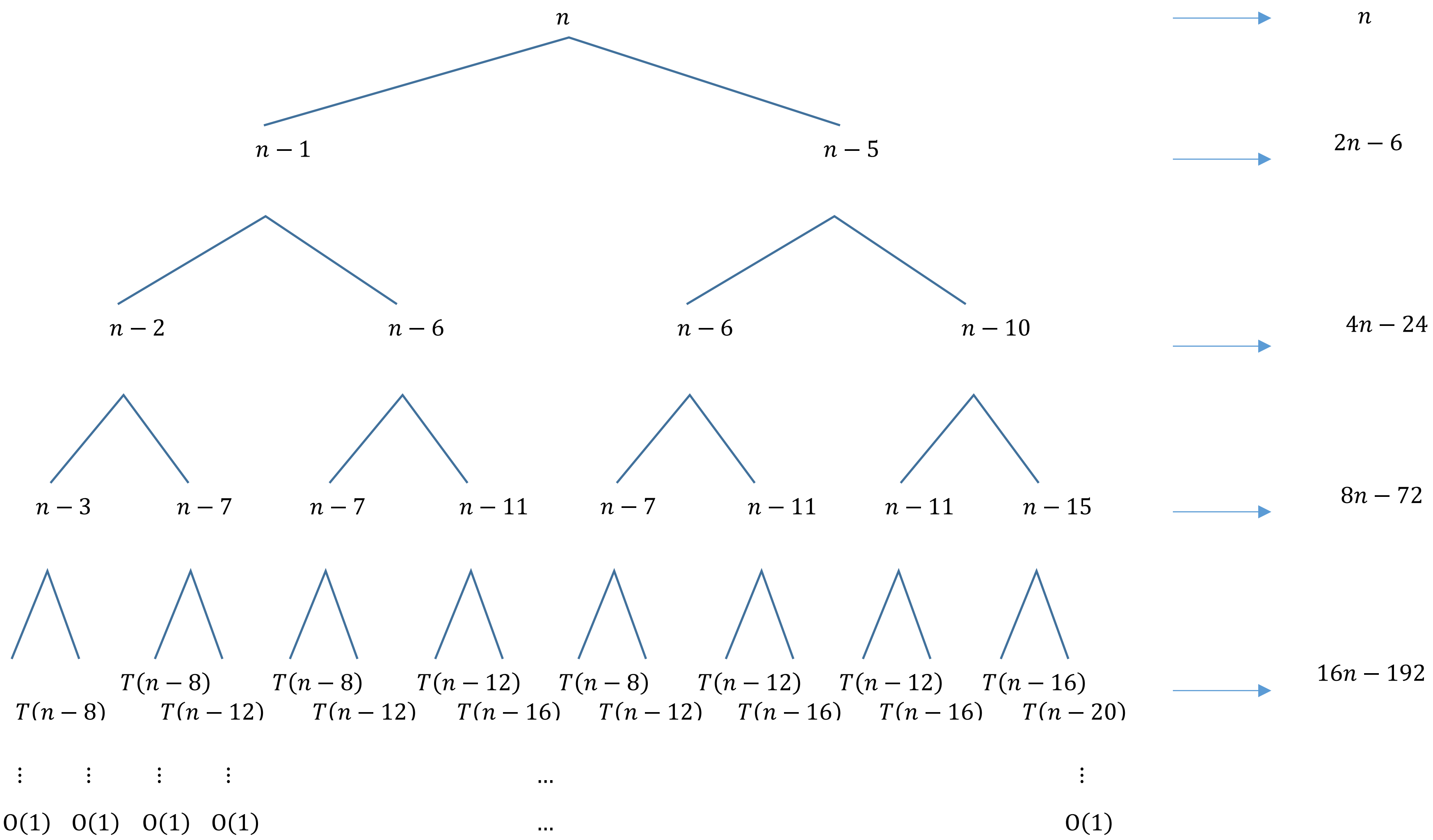

*2 pav.*



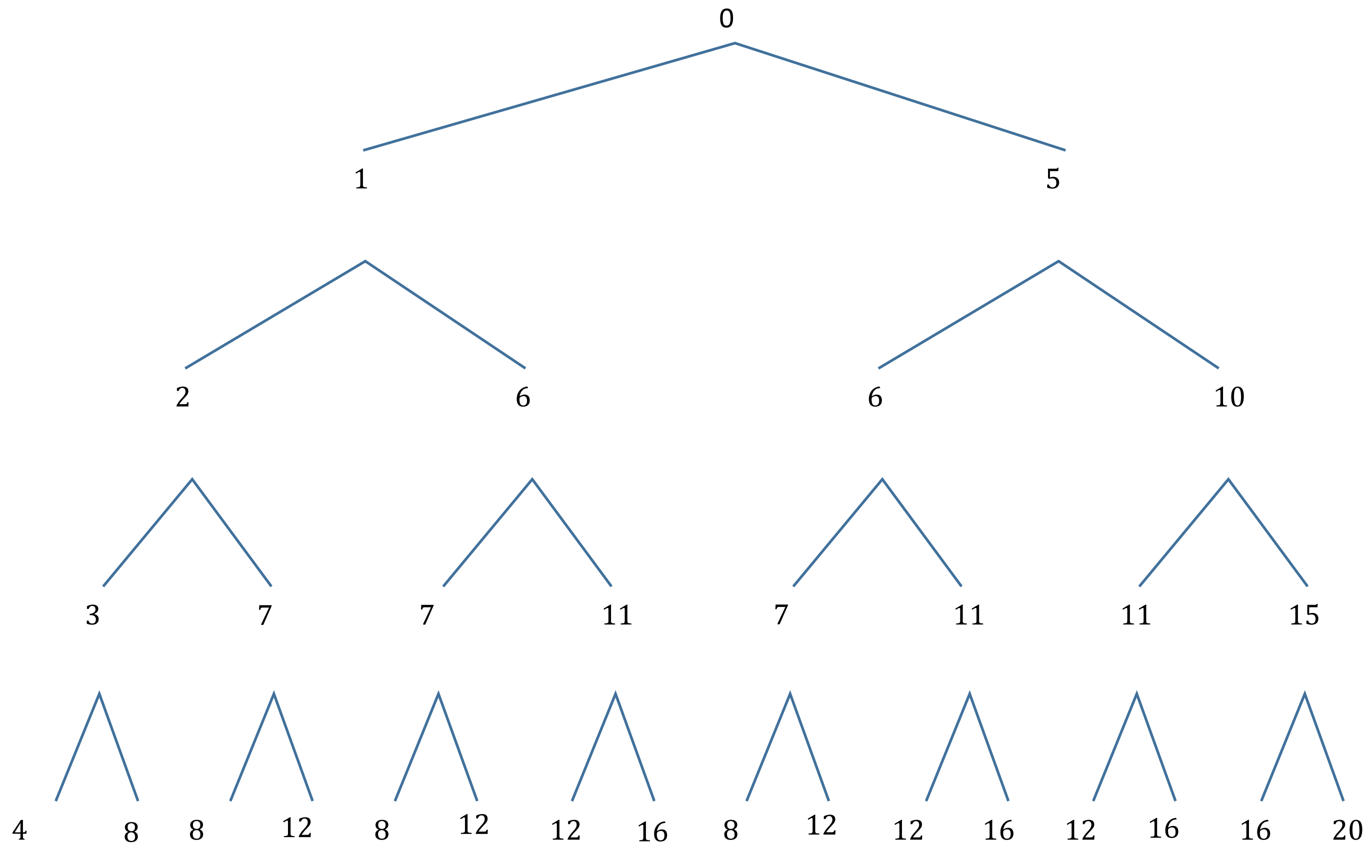

*3 pav.*


$n$
$n-1$
$n-5$
$n-2$
$n-6$
$n-6$
$n-10$
$n-3$
$n-7$
$n-7$
$n-11$
$n-7$
$n-11$
$n-11$
$n-15$
$T(n-4)$
$T(n-8)$
$T(n-8)$
$T(n-12)$
$T(n-8)$
$T(n-12)$
$T(n-12)$
$T(n-16)$
$T(n-8)$
$T(n-12)$
$T(n-12)$
$T(n-16)$
$T(n-12)$
$T(n-16)$
$T(n-16)$
$T(n-20)$

*4 pav.*


$n$
$n$
$n-1$
$n-5$
$2n-6$
$n-2$
$n-6$
$n-6$
$n-10$
$4n-24$
$n-3$
$n-7$
$n-7$
$n-11$
$n-7$
$n-11$
$n-11$
$n-15$
$8n-72$
$T(n-8)$
$T(n-8)$
$T(n-12)$
$T(n-8)$
$T(n-12)$
$T(n-12)$
$T(n-12)$
$T(n-16)$
$T(n-12)$
$T(n-16)$
$T(n-8)$
$T(n-12)$
$T(n-16)$
$T(n-16)$
$T(n-16)$
$T(n-20)$
$16n-192$
$\vdots$
...
$O(1)$
$O(1)$
$O(1)$
$O(1)$
...
$O(1)$

*5 pav.*




*6 pav.*

Sumuojant anksčiau pateiktą medį nesunku pastebėti, kad sekančio lygį sudaro dvigubai daugiau šakų. Pusė šakų prideda 1, kita pusė 5 prie buvusio ankstesnio lygio sumos. Pažymėkime, kad $S_i$ – $i$-tojo lygio suma, kurią galima išreikšti rekurentiškai:

$$\begin{cases} S_0 = 0 \\ S_n = 2S_{n-1} + 2^{n-1} + 5 \times 2^{n-1} = 2S_{n-1} + 2^{n-1} + 5 \times 2^{n-1} \end{cases}$$

Matematinės indukcijos būdu nesunku pasitikrinti, kad rekurentinės lygties sprendinys

$$S_n = 6n2^{n-1}$$

Bazė akivaizdžiai tenkinama, o

$$S_{n+1} = 2 \times 6n2^{n-1} + 6 \times 2^n = 6(n+1)2^n$$

Randame sprendinį

$$T(n) = \sum_{i=0}^{h} (2^i n - 6i2^{i-1}) = n \sum_{i=0}^{h} 2^i - 6 \sum_{i=0}^{h} i2^{i-1} = n(2^{h+1} - 1) - 6(h2^h - 2^h + 1)$$

Žinoma, kad $\sum_{i=0}^{n} i2^i = 2(n2^n - 2^n + 1)$. Žiūrėti https://www.wolframalpha.com/input/?i=sum+%28i*2%5Ei%29+from+i%3D0+to+n .

Apatinį įvertinimą galima gauti imant medį, kurio visos šakos neilgesnės kaip $h = \lfloor n/5 \rfloor$ ...

Atsakymas

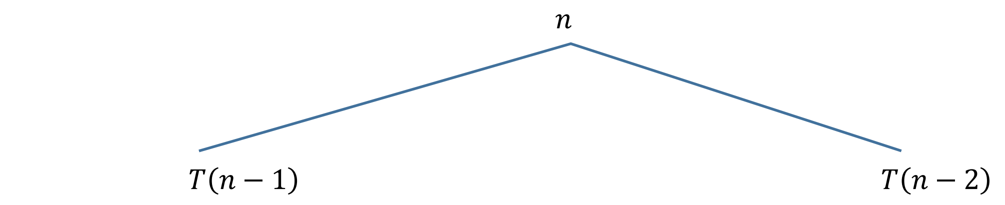
$$T(n) = \Omega\left(n2^{\frac{n}{5}}\right).$$

Su viršutiniu rėžiu kiek blogiau, jei imtume $h = n$, į medį bus įtraukta daug neigiamų reikšmių visose šakose išskyrus keliose ilgiausiose. Atmeskime neigiamą dalį

$$T(n) = O(n2^n).$$

Šiuo atveju, neradome asimptotiškai tikslaus įvertinimo.

## Raskime rekurentinės lygties $T(n) = T(n-1) + T(n-2) + n^2$ sprendinį.

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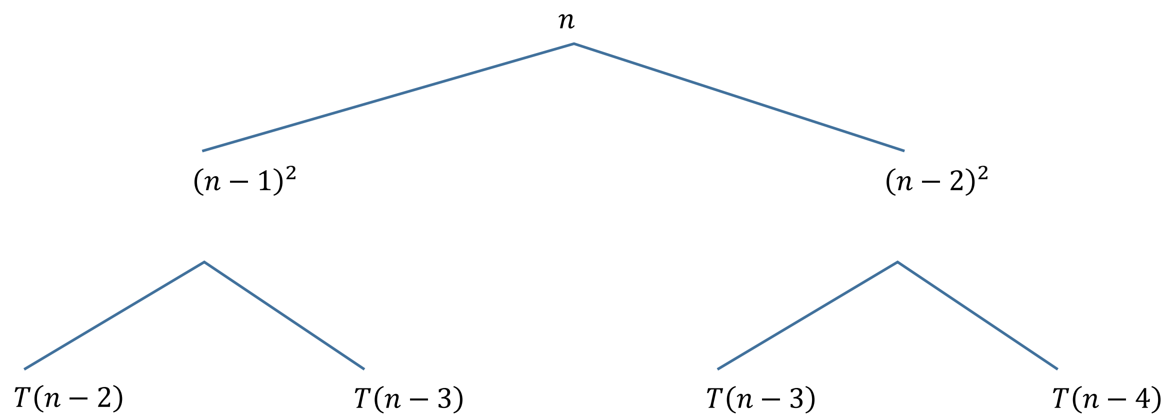
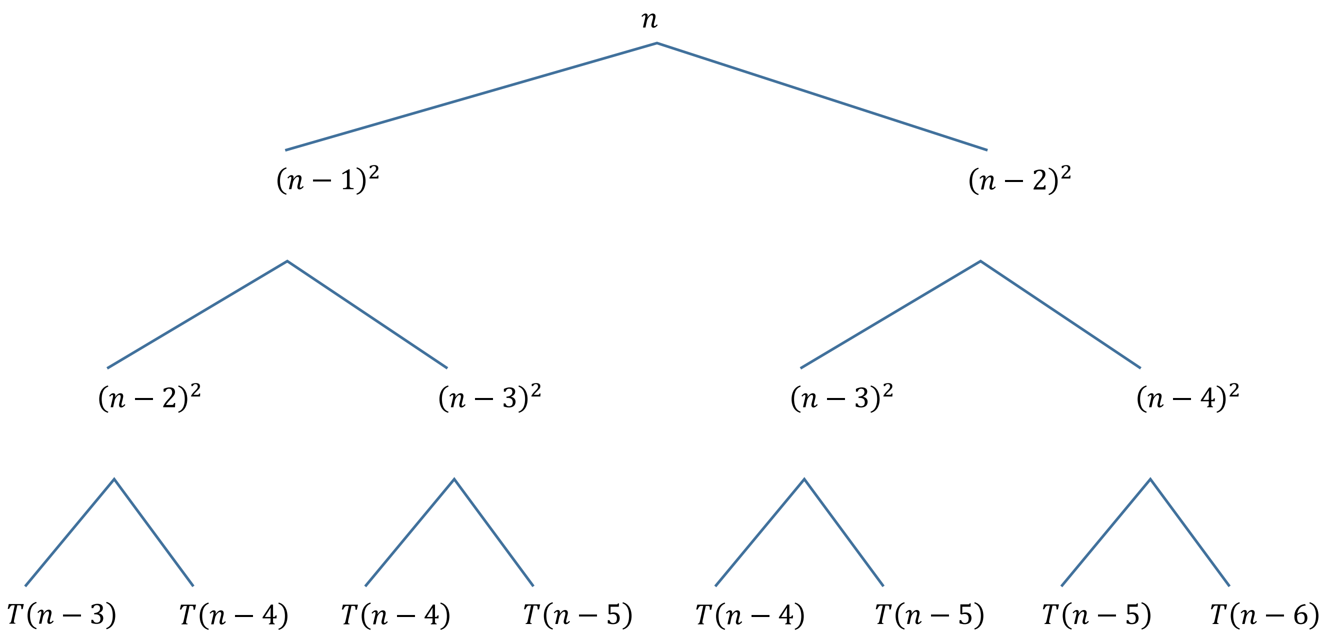
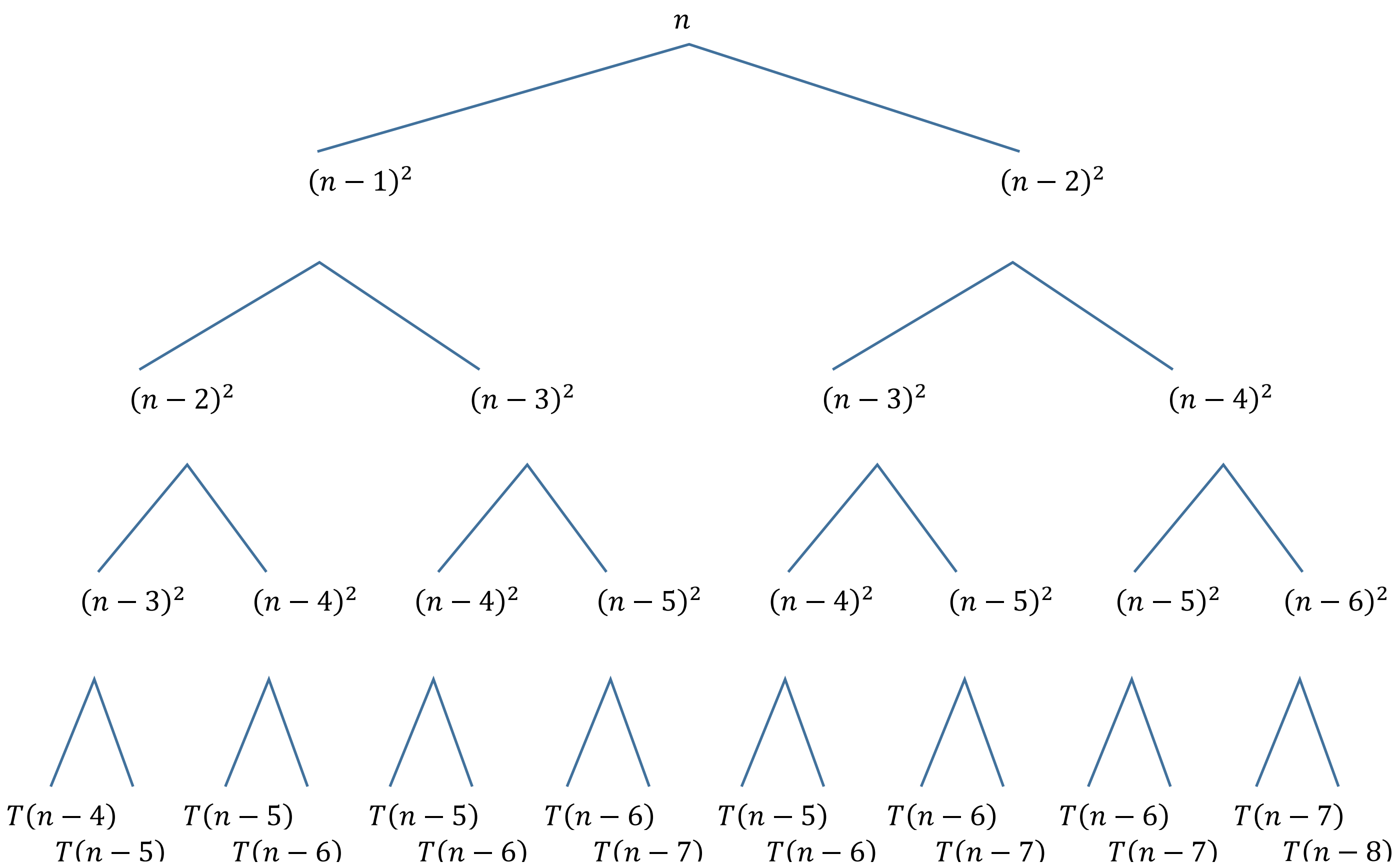
*7 pav.*



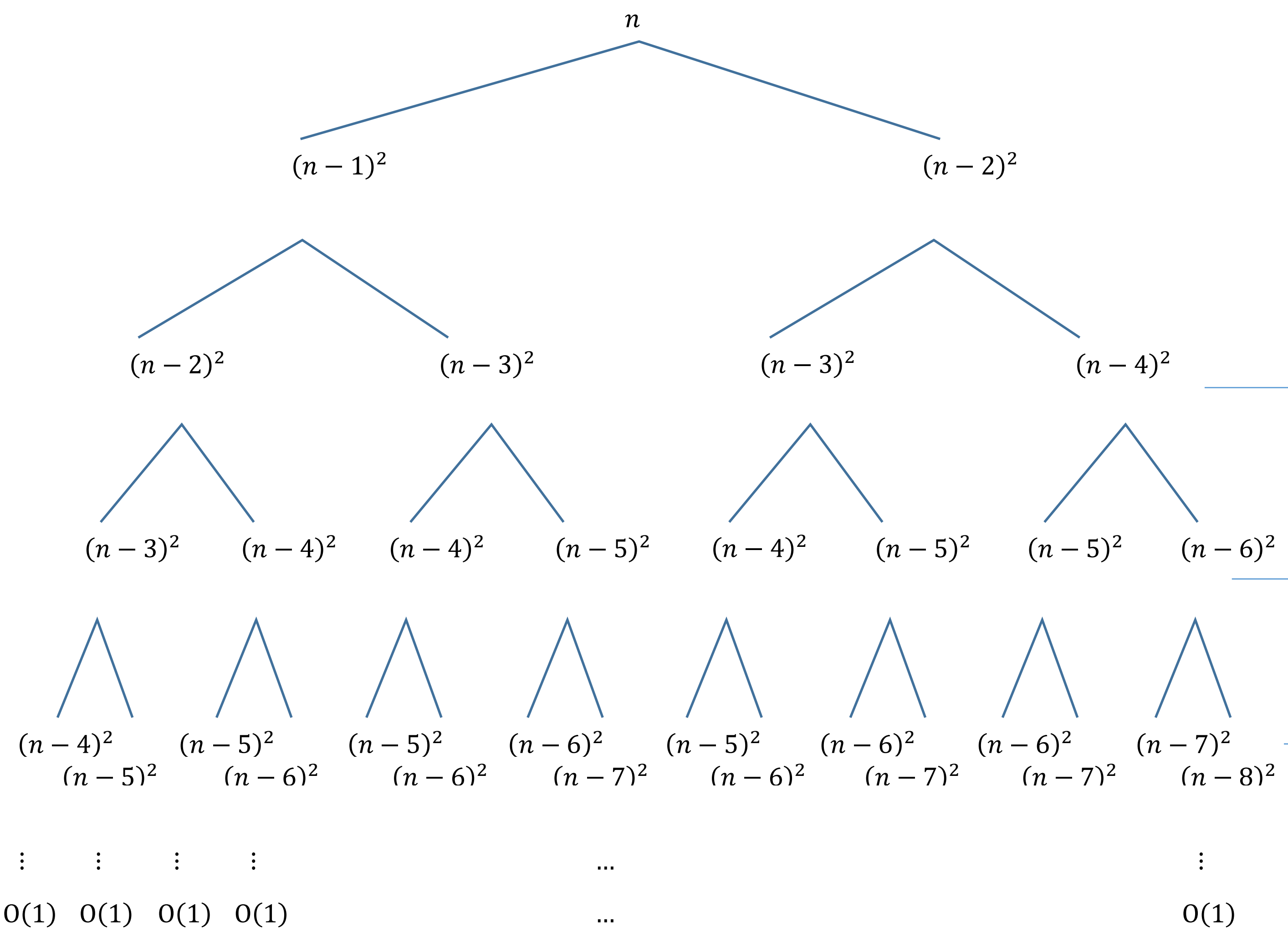

*8 pav.*




*9 pav.*


$n$
$(n-1)^2$
$(n-2)^2$
$(n-2)^2$
$(n-3)^2$
$(n-3)^2$
$(n-4)^2$
$(n-3)^2$
$(n-4)^2$
$(n-4)^2$
$(n-5)^2$
$(n-4)^2$
$(n-5)^2$
$(n-5)^2$
$(n-6)^2$
$T(n-4)$
$T(n-5)$
$T(n-5)$
$T(n-6)$
$T(n-5)$
$T(n-6)$
$T(n-6)$
$T(n-7)$
$T(n-5)$
$T(n-6)$
$T(n-6)$
$T(n-7)$
$T(n-6)$
$T(n-7)$
$T(n-7)$
$T(n-8)$

*10 pav.*




$$n$$

$$(n-1)^2 + (n-2)^2$$

$$\binom{2}{0}(n-2)^2 + \binom{2}{1}(n-3)^2 + \binom{2}{2}(n-4)^2$$

$$\binom{3}{0}(n-3)^2 + \binom{3}{1}(n-4)^2 + \binom{3}{2}(n-5)^2 + \binom{3}{3}(n-6)^2$$

$$\binom{4}{0}(n-4)^2 + \binom{4}{1}(n-5)^2 + \binom{4}{2}(n-6)^2 + \binom{4}{3}(n-7)^2 + \binom{4}{4}(n-8)^2$$

*11 pav.*

*12 pav.*

$i$-tojo medžio lygio suma yra

$$S_i = \sum_{j=0}^{n} \binom{n}{j} (n-i-j)^2 = (n-i)^2 \sum_{j=0}^{n} \binom{n}{j} - 2(n-i) \sum_{j=0}^{n} \binom{n}{j} j + \sum_{j=0}^{n} \binom{n}{j} j^2$$

Randame sumas

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n$$

$$\sum_{j=0}^{n} \binom{n}{j} j = n\, 2^{n-1}$$

$$\sum_{j=0}^{n} \binom{n}{j} j^2 = n(n+1)\, 2^{n-2}$$

https://www.wolframalpha.com/input?i=sum+i%3D0+to+n+of+%28+n+choose+i+%29

https://www.wolframalpha.com/input?i=sum+i%3D0+to+n+of+%28i++*+%28++n+choose+i+%29+%29

https://www.wolframalpha.com/input?i=sum+i%3D0+to+n+of+%28i+%5E+2+*+%28++n+choose+i+%29+%29

$$S_i = (n-i)^2 2^{n-i} - 2(n-i) n\, 2^{n-1} + n(n+1)\, 2^{n-2}$$

Tokiu atveju

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{h} S_i = \sum_{i=0}^{h}\left((n-i)^2 2^{n-i} - 2(n-i)n\,2^{n-1} + n(n+1)\,2^{n-2}\right) = \sum_{i=0}^{h}(n-i)^2 2^{n-i} - n\,2^{n-1}\sum_{i=0}^{h}(n-i) + n(n+1)2^{n-2}\sum_{i=0}^{h}1 \\
&= n^2\sum_{i=0}^{h}2^{n-i} - 2n\sum_{i=0}^{h}i2^{n-i} + \sum_{i=0}^{h}i^2 2^{n-i} \; - n\,2^{n-1}\left(n(h+1) - \frac{1}{2}h(h+1)\right) + n(n+1)2^{n-2}(h+1) \\
&= n^2\left(2^{n+1} - 2^{n-h}\right) - 2n\left(2^{n-h}\left(-h + 2^{h+1} - 2\right)\right) + 2^{n-h}\left(-h^2 - 4h + 6\left(2^h - 1\right)\right) \; - n\,2^{n-1}\left(n(h+1) - \frac{1}{2}h(h+1)\right) \\
&+ n(n+1)2^{n-2}(h+1)
\end{aligned}
$$

nes

$$
\sum_{i=0}^{h} 2^{n-i} = 2^n\sum_{i=0}^{h}\left(\frac{1}{2}\right)^i = 2^{n+1} - 2^{n-h}
$$

$$
\sum_{i=0}^{h} i2^{n-i} = 2^n\sum_{i=0}^{h}i\left(\frac{1}{2}\right)^i = 2^{n-h}\left(-h + 2^{h+1} - 2\right)
$$

$$
\sum_{i=0}^{h} i^2 2^{n-i} = 2^n\sum_{i=0}^{h}i^2\left(\frac{1}{2}\right)^i = 2^{n-h}\left(-h^2 - 4h + 6\left(2^h - 1\right)\right)
$$

o medžio aukštis $h$

$$
\frac{n}{2} \leq h \leq n
$$

Kai $n = h$

$$
\begin{aligned}
T(n) &= n^2(2^{n+1} - 2^{n-n}) - 2n\left(2^{n-n}\left(-n + 2^{h+1} - 2\right)\right) + 2^{n-n}\left(-n^2 - 4n + 6(2^n - 1)\right) \; - n\,2^{n-1}\left(n(n+1) - \frac{1}{2}n(n+1)\right) + n(n+1)2^{n-2}(n+1) \\
&= n^2\left(2^{n+1} - 2^{n-n}\right) - 2n\left(2^{n-n}\left(-n + 2^{n+1} - 2\right)\right) + 2^{n-n}\left(-n^2 - 4n + 6(2^n - 1)\right) \; - n^2(n+1)\,2^{n-2} + n(n+1)^2 2^{n-2} \\
&= n^2\left(2^{n+1} - 1\right) - 2n(-n + 2^{n+1} - 2) + \left(-n^2 - 4n + 6(2^n - 1)\right) \; + n(n+1)2^{n-2} \\
&= n^2 2^{n+1} - n^2 + 2n^2 - n2^{n+2} + 4n - n^2 - 4n + 6(2^n - 1) + n(n+1)2^{n-2} = n^2 2^{n+1} - n2^{n+2} + 6(2^n - 1) + n(n+1)2^{n-2} \\
&= O(n^2 2^n)
\end{aligned}
$$

Analogiškai, kai $h = \frac{n}{2}$

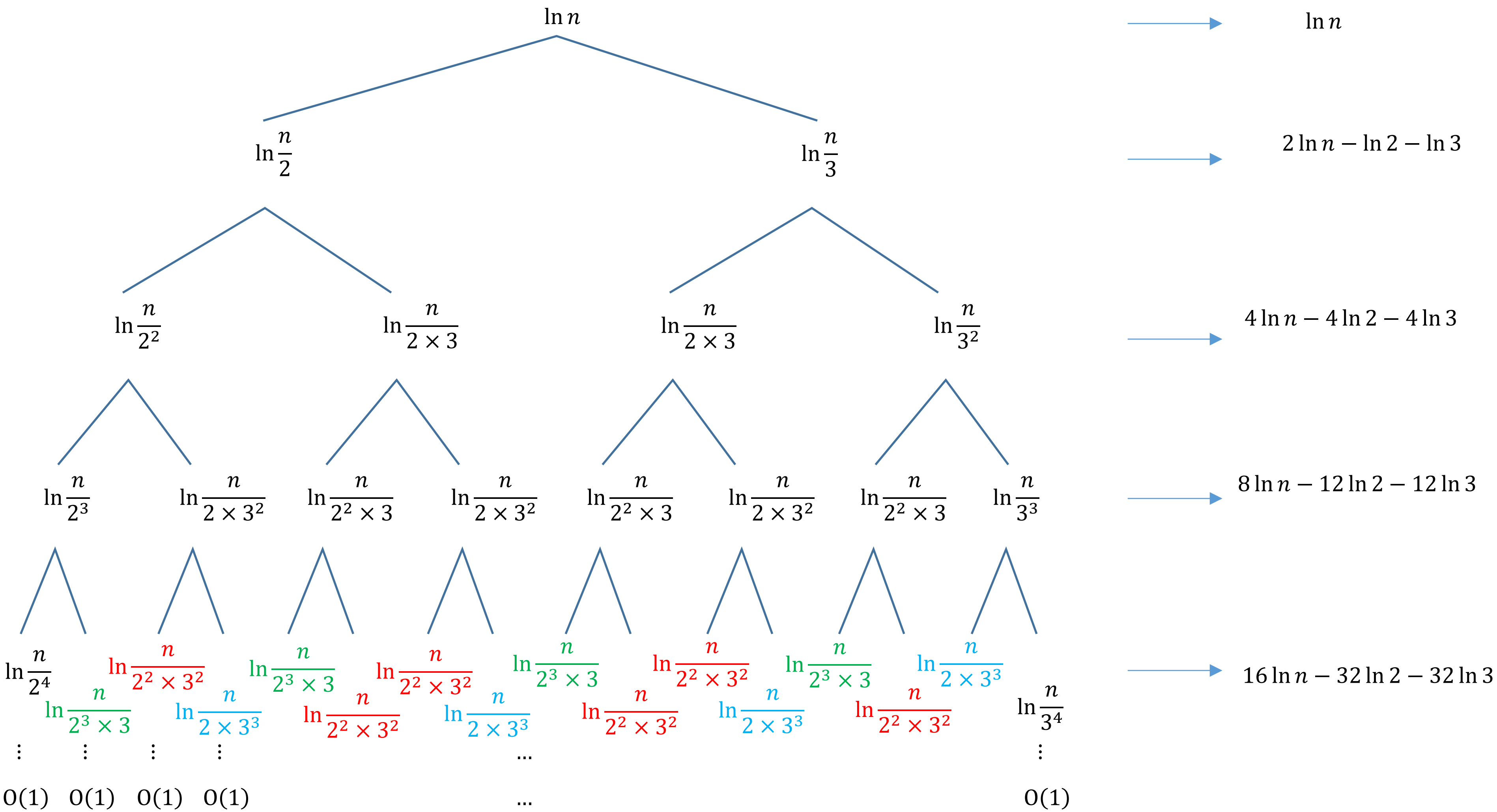
...

Atsakymas

$$T(n) = \Omega(?).$$

$$T(n) = O(n^2 2^n)$$

## Raskime rekurentinės lygties $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{3}\right) + \ln n$ sprendinį.




*13 pav.*

Medžio $i$-tąją eilutę galima susumuoti kaip

$$S_i = \ln\frac{n}{2^i} + \binom{i}{1}\ln\frac{n}{2^{i-1}3} + \binom{i}{2}\ln\frac{n}{2^{i-2}3^2} + \binom{i}{3}\ln\frac{n}{2^{i-3}3^3} + \cdots + \binom{i}{i-1}\ln\frac{n}{2\times 3^{i-1}} + \ln\frac{n}{3^i} = \sum_{j=0}^{i}\binom{i}{j}\ln\frac{n}{2^{i-j}3^j}$$

$$= \ln n\sum_{j=0}^{i}\binom{i}{j} - \ln 2\sum_{j=0}^{i}(i-j)\binom{i}{j} - \ln 3\sum_{j=0}^{i} j\binom{i}{j} = \ln n\sum_{j=0}^{i}\binom{i}{j} - \ln 2\sum_{j=0}^{i} j\binom{i}{j} - \ln 3\sum_{j=0}^{i} j\binom{i}{j} = \ln n\sum_{j=0}^{i}\binom{i}{j} - (\ln 2 + \ln 3)\sum_{j=0}^{i} j\binom{i}{j}$$

$$= \ln n\sum_{j=0}^{i}\binom{i}{j} - \ln 6\sum_{j=0}^{i} j\binom{i}{j}$$

Kadangi

$$\binom{i}{j} = \binom{i}{i-j}$$

$$C_n = (1+1)^n = \sum_{i=0}^{n}\binom{n}{i} = 2^n$$

$$P_i = \sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1} = 2P_{i-1} + C_{i-1}$$

Antra seka iš Niutono binomo, o trečią galima pasitikrinti matematinės indukcijos būdu...

$$S_i = 2^i \ln n - i2^{i-1}\ln 6$$

$$T(n) = \sum_{i=0}^{h} S_i = \sum_{i=0}^{h}\left(2^i \ln n - i2^{i-1}\ln 6\right) = \ln n\sum_{i=0}^{h} 2^i - \ln 6\sum_{i=0}^{h} i2^{i-1} = \ln n\left(2^{h+1}-1\right) - \ln 6\left(h2^h - 2^h + 1\right) = \ln n\left(2^{h+1}-1\right) - \ln 6\left(h2^h - 2^h + 1\right).$$

Tarkime, kad $h = \lfloor \log_3 n \rfloor \geq \log_3 n - 1$

$$T(n) > \ln n\left(2^{\log_3 n} - 1\right) - \ln 6\left(2^{\log_3 n}\log_3 n - 2^{\log_3 n - 1} + 1\right) = \left(1 - \frac{\ln 6}{\ln 3}\right) n^{\log_3 2}\ln n + 1/2\, n^{\log_3 2} - \ln n - \ln 6 > \left(2 - \frac{\ln 6}{\ln 3}\right) n^{\log_3 2}\ln n$$

Nesunku parodyti, kad $n^{\log_3 2} - \ln n - \ln 6 > 0$, kai $n \geq 9$. Be to, $1 - \frac{\ln 6}{\ln 3} \approx 2 - 1{,}631 > 0$.

Gavome, kad

$$T(n) = \Omega\left(n^{\log_3 2} \ln n\right)$$

Tarkime, kad $h = \lfloor \log_2 n \rfloor$, bet prie greičiausiai augančio dėmens atsiranda neigiamas koeficientas!

$$T(n) < \ln n \left(2^{\log_2 n + 1} - 1\right) - \ln 6 \left(2^{\log_2 n} \log_2 n - 2^{\log_2 n} + 1\right) = \left(2 - \frac{\ln 6}{\ln 2}\right) n \ln n + n - \ln n - \ln 6 < (2 - \log_2 6) n \ln n + n - \ln n - \ln 6$$

Vertinimas blogas!!!!

Tačiau kai $h \geq 1$

$$\ln n \left(2^{h+1} - 1\right) - \ln 6 \left(h 2^h - 2^h + 1\right) < \ln n \left(2^{h+1} - 1\right)$$

Gavome, kad

$$T(n) = O(n \ln n).$$