Assignment 4
Computer simulation course

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Part 1:Generating random variants

1 About Beta Distribution

1.1 Beta Function

- Fixed parameters: $\alpha > 0, \beta > 0$
- We define the beta function

$B(\alpha, \beta) := \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy =$ some constant

- Relationship between the gamma and beta functions:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

- This makes it easy to compute for whole numbers, since $\Gamma(n) = (n-1)!$.

(Plug in values for α and β, and it spits out a number.)
Don't mix up the beta function and the beta distribution!

1.2 Beta Distribution

- We want to define a distribution on $[0, 1]$ using the function

$$y^{\alpha-1}(1-y)^{\beta-1}$$

- Recall that a probability density function must satisfy $\int_0^1 f(y)dy = 1$. To make this work, we divide by $B(\alpha, \beta)$:
- Density function for the beta distribution:

$$f(y) := \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}, 0 \le y \le 1$$

Don't mix up the beta function and the beta distribution!

Key Properties of the Beta Distribution

- Mean:

$$\mu = E(Y) = \frac{\alpha}{\alpha+\beta}$$

- Variance:

$$\sigma^2 = V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

- Standard deviation:

$$\sigma = \sqrt{V(Y)} = \sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

2 Generating Beta variants

2.1 Finding PDF Beta Function

Alpha and beta based on my student ID :$\alpha = 3, \beta = 4$

Calculate $B(3,4)$

$$\begin{aligned} B(\alpha,\beta) &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\ B(3,4) &= \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} \\ &= \frac{2!\cdot 3!}{6!} \\ &= \frac{2\cdot 6}{720} \\ &= \frac{1}{60} \end{aligned}$$

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} = 60 * (x^2) * ((1-x)^3), 0 \le x \le 1$$

2.2 The acceptance-Rejection method

2.2.1 The Acceptance-Rejection Method

Suppose that X and Y are r.v. with density or pmf f and g respectively, and there exists a constant c such that $\frac{f(t)}{g(t)} \leq c$ for all t such that $f(t) > 0$. Then the acceptance-rejection method can be applied to generate the r.v. X.

1. Find a r.v. Y with density g satisfying $f(t)/g(t) \leq c$, for all t such that $f(t) > 0$
2. Generate a random y from the dist. with density g
3. Generate a random u from the $U(0,1)$ dist.
4. If $u < f(y)/(cg(y))$ accept y and deliver $x = y$; otherwise reject y and repeat step 2-4.

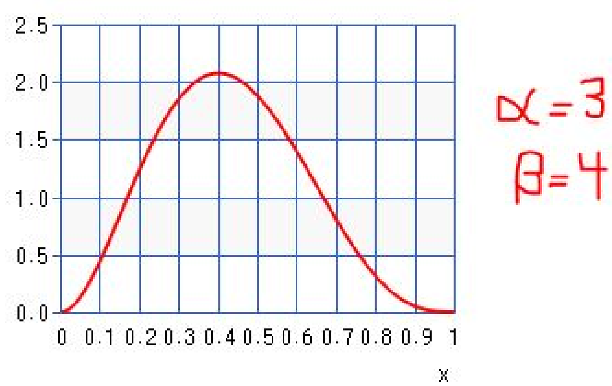
Note that in step 4, $P(\text{ accept } \mid Y) = P\left(U < \frac{f(Y)}{cg(Y)} \mid Y\right) = \frac{f(Y)}{cg(Y)}$.

The total prob. of acceptance for any iteration is

$$\sum_y P(\text{ accept } \mid y)P(Y = y) = \sum_y \frac{f(y)}{cg(y)} g(y) = \frac{1}{c}$$

2.2.2 Finding appropriate C

We have two ways to find a suitable C; the first and best one is calculating derivative of Beta's PDF function (which will tell us the maximum of PDF occurs at $y = 0.4$ and the second method is finding approximate c by the plot of PDF:



So, $c = f(0.4) = 2.07360 \approx 2.1$.

2.2.3 Implementation in R language

```r
n <- 1000; k <- 0 # counter for accepted
y <- numeric(n); j <- 0 # iterations

while (k < n) {u <- runif (1); j <-j+1;
x <- runif (1) # random variate from g
#we accept x if:
if ( (60 * (1/2.1) * (x^2) * ((1-x)^3) ) > u){k <-k +1; y[k] <- x}}

print(j)

## [1] 2118

# compare empirical and theoretical percentiles
p <-seq(.1 ,.9 ,.1); Qhat <- quantile(y,p) # quantiles of sample
Q <- qbeta(p, 3, 4) # theoretical quantiles
se <- sqrt((60 * (p^2) * ((1-p)^3) ) / (n * dbeta(Q, 3, 4)))

# Now we compare
round( rbind (Qhat , Q, se), 3)

##        10%   20%   30%   40%   50%   60%   70%   80%   90%
## Qhat 0.185 0.257 0.322 0.377 0.428 0.477 0.527 0.591 0.675
## Q    0.201 0.269 0.323 0.373 0.421 0.471 0.524 0.585 0.667
## se   0.019 0.027 0.031 0.032 0.030 0.026 0.021 0.014 0.007
```

Part 2: Evaluation of the result

As we can see the result of Q and Qhat, they are approximately the same. Therefore, we can be statistically sure that the generated numbers are of Beta distribution.