Reinforcement Learning

Lecture 3: Markov Decision Processes

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Lecture Overview

Lecture covers Chapter 3 in Sutton & Barto [3] and uses David Silver's examples [2]

1. **Markov Chains**
   - markov property
   - state transition matrix
   - definition and example
2. **Markov Reward Process**
   - definition and example
   - the return
   - state value function
   - the Bellman equation
3. **Markov Decision Process**
   - definition and example
   - policies
   - state and action value functions
   - the Bellman equation
   - optimal state and action value functions
   - the Bellman optimality equations

Markov Chain markov property recap

With the **Markov property** , we can throw away the history and just use the agents state:

Definition: Markov property

A state S_t is **Markov** if and only if

$$P(S_{t+1} \mid S_t) = P(S_{t+1} \mid S_1, S_2, ..., S_t)$$

- For example, a chess board
  - We don't need to know how the game was played up to this point
- The state fully characterises the distribution over future events:

$$H_{1:t} \rightarrow S_t \rightarrow H_{t+1:\infty}$$

Markov Chain state transition matrix

The probability of transitioning from state s to s' for a Markov state is:

$$\mathcal{P}_{ss'} = P(S_{t+1} = s' \mid S_t = s),$$

where the **state transition probability** for all states to all successor states can be expressed as a large matrix:

$$\mathcal{P} = \overbrace{\begin{bmatrix} \mathcal{P}_{11} & \cdots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \cdots & \mathcal{P}_{nn} \end{bmatrix}}^{\text{to}},$$

and each row sums to 1.

Click to try a demo [1]

Markov Chain definition

A **Markov chain** (also called Markov Process) is a set of states and a state-transition matrix

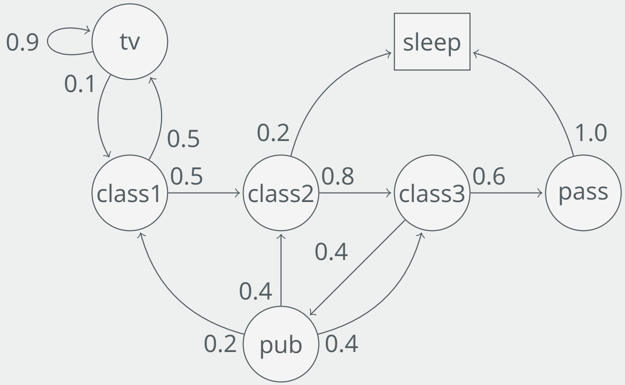
Definition: Markov chain

A **Markov chain** is a tuple $\langle \mathcal{S}, \mathcal{P} \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{P}$ is the state-transition matrix where $\mathcal{P}_{ss'} = P(S_{t+1} = s' \mid S_t = s)$

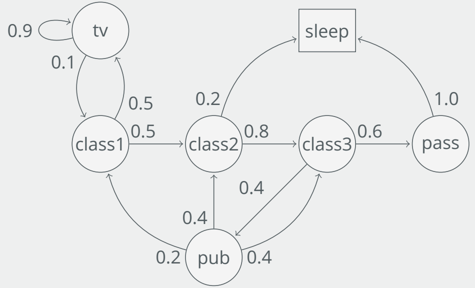
Markov Chain example

Example: Markov Chain



Example from [2]

Example: Markov Chain

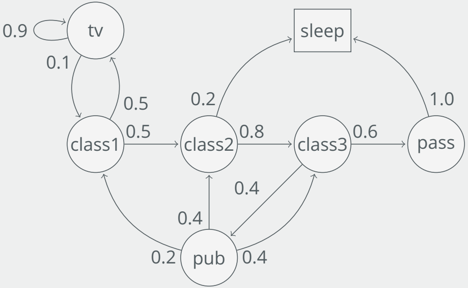


State Transition Matrix

$$
\mathcal{P} = \begin{array}{cc}
 & \begin{array}{ccccccc} \text{c1} & \text{c2} & \text{c3} & \text{pass} & \text{pub} & \text{tv} & \text{sleep} \end{array} \\
\begin{array}{c} \text{c1} \\ \text{c2} \\ \text{c3} \\ \text{pass} \\ \text{pub} \\ \text{tv} \\ \text{sleep} \end{array} &
\left[\begin{array}{ccccccc}
 & 0.5 & & & & 0.5 & \\
 & & 0.8 & & & & 0.2 \\
 & & & 0.6 & 0.4 & & \\
 & & & & & & 1.0 \\
0.2 & 0.4 & 0.4 & & & & \\
0.1 & & & & & 0.9 & \\
 & & & & & & 1.0
\end{array}\right]
\end{array}
$$

Example from [2]

Example: Markov Chain



Episode

An episode is a varying-length sample of a Markov chain:

$$S_1, S_2, ..., S_T,$$

for example starting from $S_1 =$ class1:

Episode samples
c1,c2,c3,pass,sleep
c1,tv,tv,tv,c1,c2,c3,pub,c2,sleep

Example from [2]

Markov Reward Process definition

A Markov **reward** process is a Markov Chain with a **reward** function

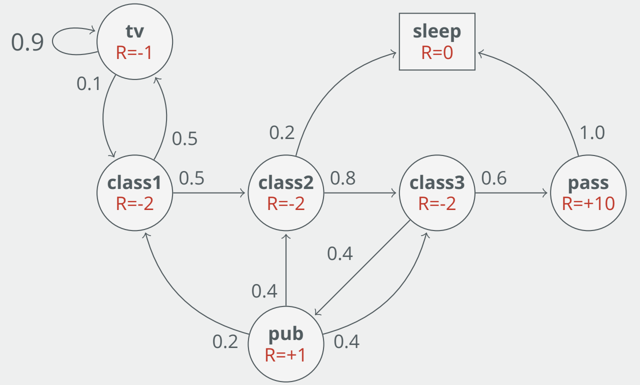
Definition: Markov reward process

A **Markov reward process** is a tuple $\langle \mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{P}$ is the state-transition matrix where $\mathcal{P}_{ss'} = P(S_{t+1} = s' \mid S_t = s)$
- $\mathcal{R}$ is a **reward** function where $\mathcal{R}_s = \mathbb{E}[R_{t+1} \mid S_t = s]$
- γ is the **discount** rate $\gamma \in [0, 1]$

Markov Reward Process example

Example: Markov Reward Process



Example from [2]

Markov Reward Process the return

The **return** G_t, in the simplest case, is the total future reward:

$$G_t = R_{t+1} + R_{t+2} + R_{t+3} + \ldots + R_T$$

In practice, we discount rewards into the future by the *discount rate* $\gamma \in [0, 1]$.

Definition: The return

The return G_t is the discounted total future reward:

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

Definition: The state value function

The **state value function** $v(s)$ in an MRP is the long-term value of a state:

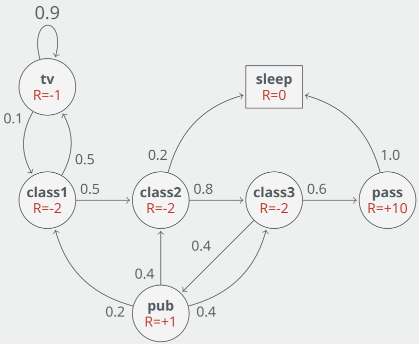
$$v(s) = \mathbb{E}[G_t \mid S_t = s],$$

for example calculated by sampling episodes...

Sample episodes
c1,c2,c3,pass,sleep
c1,tv,tv,tv,c1,c2,c3,pub,c2,sleep
c1,c2,sleep
...

Example: Puppy

Example: MRP



Example: The state value function

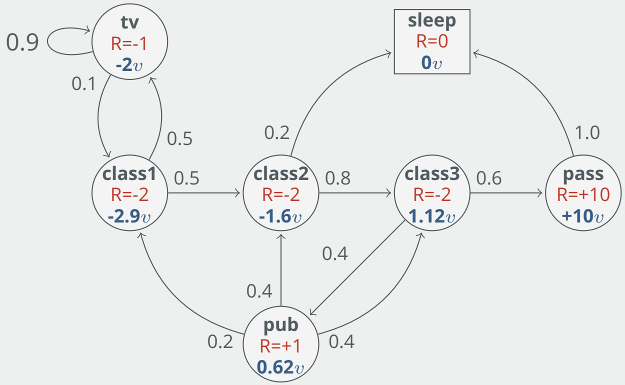
This is an example $v(s)$ with $s =$ 'class1' and $\gamma = \frac{1}{2}$:

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots \\ &= R_{t+1} + \frac{1}{2} R_{t+2} + \frac{1}{4} R_{t+3} + \ldots \end{aligned}$$

Episode samples	Value function
c1,c2,c3,pass,sleep	$v_1 = -2 - \frac{1}{2} \cdot 2 - \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 10 = -2.25$
c1,tv,tv,c1,c2,c3,pub,c2,sleep	$v_1 = -2 - \frac{1}{2} \cdot 1 - \frac{1}{4} \cdot 1 + \frac{1}{8} \cdot \ldots = -3.125$
c1,c2,sleep	$v_1 = -2 - \frac{1}{2} \cdot 2 - \frac{1}{4} \cdot 0 + \frac{1}{8} \quad = -3$
...	$\ldots = -2.9$

Markov Reward Process value function example

Example: Markov Reward Process for $\gamma = 0.5$



Example from [2]

Markov Reward Process the Bellman equation

Through a series of identities, we can decompose the value function into the **immediate reward** R_{t+1} and the discounted **value of the next state** $\gamma v(S_{t+1})$.

Definition: Bellman equation for MRP

The Bellman equation is:

$$\begin{aligned} v(s) &= \mathbb{E}[G_t \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \ldots \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma(R_{t+2} + \gamma R_{t+3} + \gamma^2 R_{t+4} + \ldots) \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma G_{t+1} \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s], \end{aligned}$$

which is equivalent to:

$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$$

Markov Reward Process solving the Bellman equation

The Bellman equation can be expressed with matrices:

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \cdots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \cdots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix},$$

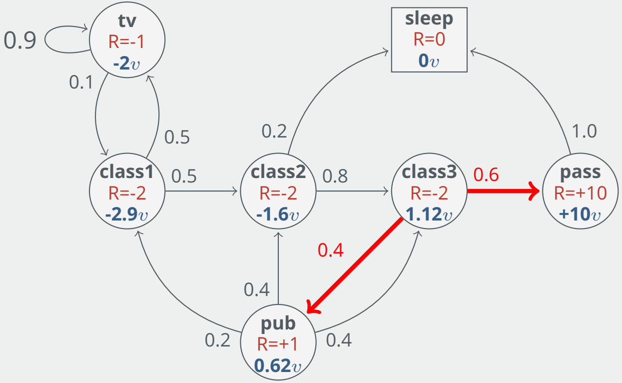
which is a linear equation that can be solved:

$$\begin{aligned} v &= \mathcal{R} + \gamma \mathcal{P} v \\ (\mathrm{I} - \gamma \mathcal{P}) v &= \mathcal{R} \\ v &= (\mathrm{I} - \gamma \mathcal{P})^{-1} \mathcal{R}, \end{aligned}$$

where I is the identity matrix. Unfortunately this matrix inversion is too slow, except for small MDPs, so we use iterative methods for larger MDP (MC evaluation and TD learning).

Verification: MRP for $\gamma = 0.5$

$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s') = -2 + 0.5 * (0.6 * \mathbf{10} + 0.4 * \mathbf{0.62}) = \mathbf{1.12}$$



Markov Decision Process definition

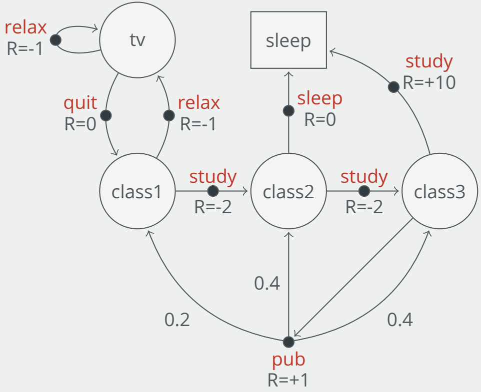
A Markov **decision** process adds 'actions' so the transition probability matrix now depends on which action the agent takes.

Definition: Markov decision process

A **Markov decision process** is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{P}$ is the state-transition matrix where $\mathcal{P}^a_{ss'} = P(S_{t+1} = s' \mid S_t = s, A_t = a)$
- $\mathcal{R}$ is a **reward** function where $\mathcal{R}^a_s = \mathbb{E}[R_{t+1} \mid S_t = s, A_t = a]$
- γ is the **discount** rate $\gamma \in [0, 1]$

Example: Markov Decision Process



Example from [2]

Markov Decision Process policies

A policy is a distribution over actions which determines how agents should behave in the environment.

- A lazy agent will sample relaxing actions more than frequently than studying
- A high-performing agent will study at all classes, then study more at home!

Definition: Policy

A policy π is a distribution over actions given a state:

$$\pi(a|s) = P(A_t = a \mid S_t = s)$$

Definition: The state-value function

The **state-value function** $v_\pi(s)$ is the same, but its the return when following a given policy π:

$$v_\pi(s) = \mathbb{E}_\pi[G_t \mid S_t = s]$$

Definition: The action-value function

The **action-value function** is the long term-value of a state when choosing an action with policy π:

$$q_\pi(s, a) = \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a]$$

Example: Arizona trail

Markov Decision Process the Bellman equation

Similarly to MRPs, the state-value function can be decomposed into the immediate reward and the discounted value of the next state:

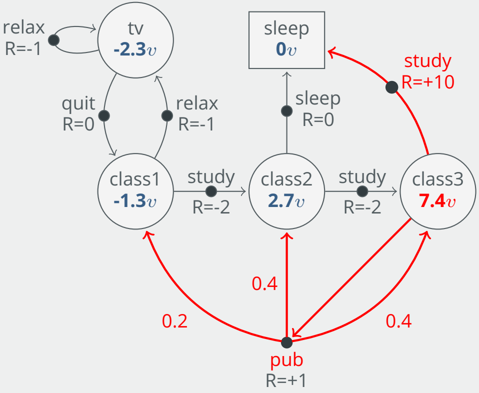
$$\begin{aligned} v_\pi(s) &= \mathbb{E}_\pi[G_t \mid S_t = s] \\ &= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\ &= \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s, a), \end{aligned}$$

which is also the case for the action-value function, where:

$$\begin{aligned} q_\pi(s, a) &= \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a] \\ &= \mathbb{E}_\pi[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a] \\ &= \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s'). \end{aligned}$$

Markov Decision Process example verifying the Bellman equation

Verification: MDP with average policy



Verification

Under the policy π where we do everything {study,pub} with 50% probability and $\gamma = 1$:

$$
\begin{aligned}
v_\pi(s) &= \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s,a) \\
&= \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right) \\
&= \frac{1}{2} * 10 \\
&+ \frac{1}{2} (1 + 0.2(-1.3v) + 0.4(2.7v) + 0.4(7.4v)) \\
&= 7.4v
\end{aligned}
$$

Definition: The optimal state-value function

The **optimal state-value function** $v_*(s)$ is the maximum value function over all policies:

$$v_*(s) = \max_{\pi} v_\pi(s)$$

Definition: The optimal action-value function

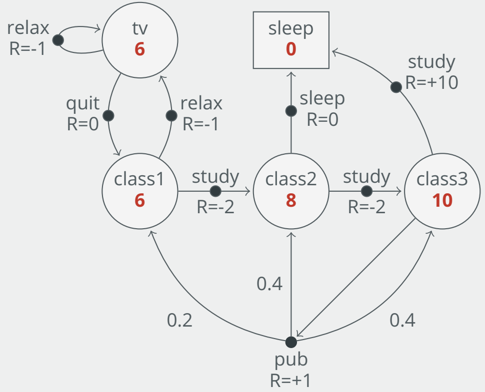
The **optimal action-value function** is the maximum action value function over all policies:

$$q_*(s, a) = \max_{\pi} q_\pi(s, a)$$

Example: Mo Farah

Markov Decision Process optimal state-value function example

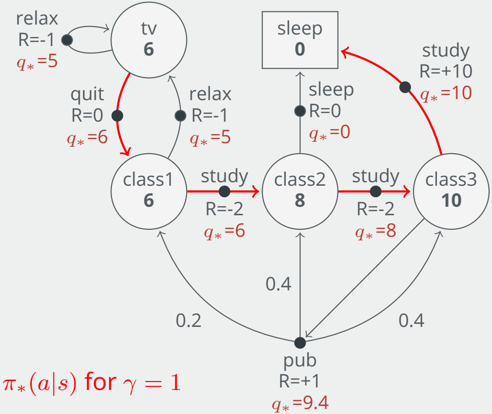
Example: $v_*(s)$ for $\gamma = 1$



Example from [2]

Markov Decision Process optimal action-value and optimal policy

Example: $q_*(s,a)$ for $\gamma = 1$



$\pi_*(a|s)$ for $\gamma = 1$

Example from [2]

The optimal value functions are similarly recursively related by the Bellman optimality equations, where:

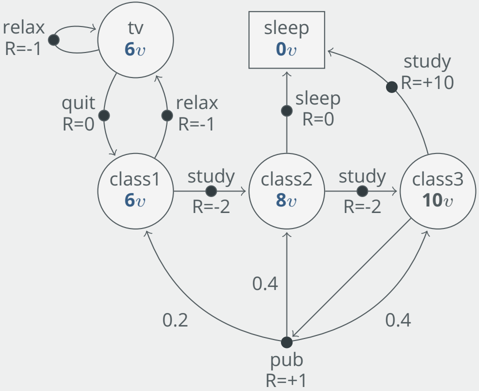
$$\begin{aligned} v_*(s) &= \max_\pi v_\pi(s) \\ &= \max_a q_*(s,a), \end{aligned}$$

and the optimal action-value function:

$$\begin{aligned} q_*(s,a) &= \max_\pi q_\pi(s,a) \\ &= \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s'). \end{aligned}$$

Markov Decision Process verifying the Bellman optimality equation

Verification: MDP with average policy



Verification

The optimal state-value for class3 following $\gamma = 1$ requires q_* for the pub action:

$$
\begin{aligned}
v_*(s) &= \max_a q_*(s,a) \\
&= \max_a \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s') \\
&= \max \Big\{ 10 + 1 * (0v), \\
&\quad (1 + 0.2(6v) + 0.4(8v) + 0.4(10v)) \Big\} \\
&= \max\{q_* = 10, q_* = 9.4\} \\
&= \mathbf{10}v
\end{aligned}
$$

References I

[1] V. Powell.
Markov chains: A visual explanation.
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[2] D. Silver.
Reinforcement learning lectures.
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[3] R. S. Sutton and A. G. Barto.
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