Reinforcement Learning

Lecture 2: Markov Decision Processes

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Lecture Overview

Lecture covers Chapter 3 in Sutton & Barto [3] and uses David Silver's examples [2]

1. **Markov Chains**
   - markov property
   - state transition matrix
   - definition and example
2. **Markov Reward Process**
   - definition and example
   - the return
   - state value function
   - the Bellman equation
3. **Markov Decision Process**
   - definition and example
   - policies
   - state and action value functions
   - the Bellman equation
   - optimal state and action value functions
   - the Bellman optimality equations

Markov Chain markov property recap

With the **Markov property** , we can throw away the history and just use the agents state:

Definition: Markov property

A state S_t is **Markov** if and only if

$$P(S_{t+1} \mid S_t) = P(S_{t+1} \mid S_1, S_2, \ldots, S_t)$$

- For example, a chess board
  - We don't need to know how the game was played up to this point
- The state fully characterises the distribution over future events:

$$H_{1:t} \to S_t \to H_{t+1:\infty}$$

Markov Chain state transition matrix

The probability of transitioning from state s to s' for a Markov state is:

$$\mathcal{P}_{ss'} = P(S_{t+1} = s' \mid S_t = s),$$

where the **state transition probability** for all states to all successor states can be expressed as a large matrix:

$$\mathcal{P} = \overbrace{\begin{bmatrix} \mathcal{P}_{11} & \cdots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \cdots & \mathcal{P}_{nn} \end{bmatrix}}^{\text{to}},$$

and each row sums to 1.

Click to try a demo [1]

Markov Chain definition

A **Markov chain** (also called Markov Process) is a set of states and a state-transition matrix

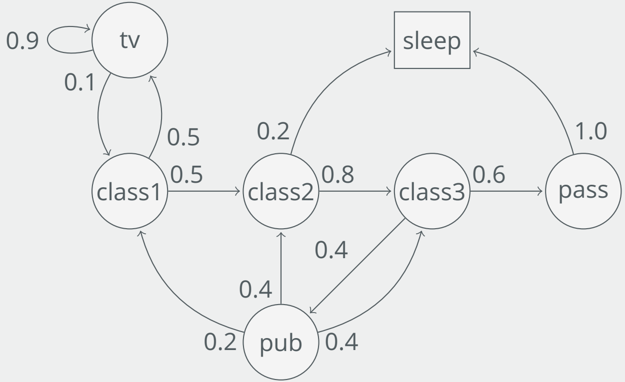
Definition: Markov chain

A **Markov chain** is a tuple $\langle \mathcal{S}, \mathcal{P} \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{P}$ is the state-transition matrix where $\mathcal{P}_{ss'} = P(S_{t+1} = s' \mid S_t = s)$

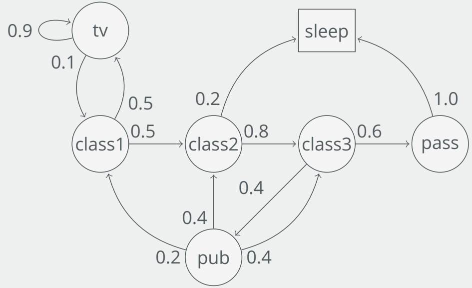
Markov Chain example

Example: Markov Chain



Example from [2]

Example: Markov Chain

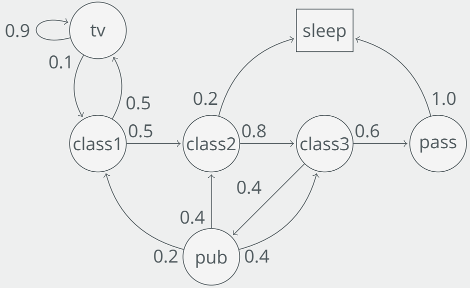


State Transition Matrix

$$
\mathcal{P} = \begin{array}{c} \\ c1 \\ c2 \\ c3 \\ pass \\ pub \\ tv \\ sleep \end{array}
\begin{array}{c}
\begin{array}{ccccccc} c1 & c2 & c3 & pass & pub & tv & sleep \end{array} \\
\left[\begin{array}{ccccccc}
 & 0.5 & & & & 0.5 & \\
 & & 0.8 & & & & 0.2 \\
 & & & 0.6 & 0.4 & & \\
 & & & & & & 1.0 \\
0.2 & 0.4 & 0.4 & & & & \\
0.1 & & & & & 0.9 & \\
 & & & & & & 1.0
\end{array}\right]
\end{array}
$$

Example from [2]

Example: Markov Chain



Episode

An episode is a varying-length sample of a Markov chain:

$$S_1, S_2, \ldots, S_T,$$

for example starting from $S_1 = \text{class1}$:

Episode samples
c1,c2,c3,pass,sleep
c1,tv,tv,tv,c1,c2,c3,pub,c2,sleep

Example from [2]

Markov Reward Process definition

A Markov **reward** process is a Markov Chain with a **reward** function

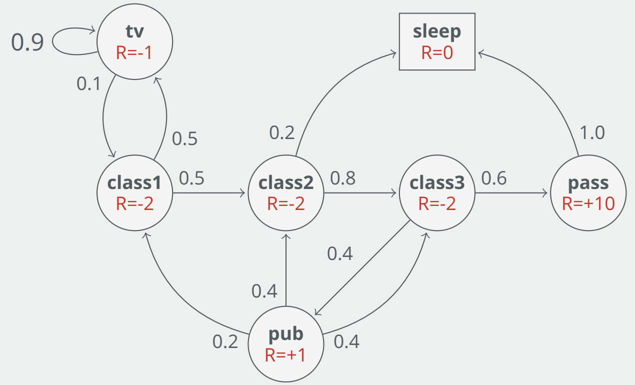
Definition: Markov reward process

A **Markov reward process** is a tuple $\langle \mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{P}$ is the state-transition matrix where $\mathcal{P}_{ss'} = P(S_{t+1} = s' \mid S_t = s)$
- $\mathcal{R}$ is a **reward** function where $\mathcal{R}_s = \mathbb{E}[R_{t+1} \mid S_t = s]$
- γ is the **discount** rate $\gamma \in [0, 1]$

Markov Reward Process example

Example: Markov Reward Process



Example from [2]

The **return** G_t, in the simplest case, is the total future reward:

$$G_t = R_{t+1} + R_{t+2} + R_{t+3} + \ldots + R_T$$

In practice, we discount rewards into the future by the *discount rate* $\gamma \in [0, 1]$.

Definition: The return

The return G_t is the discounted total future reward:

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1}$$

Markov Reward Process state value function

Definition: The state value function

The **state value function** $v(s)$ in an MRP is the long-term value of a state:

$$v(s) = \mathbb{E}[G_t \mid S_t = s],$$

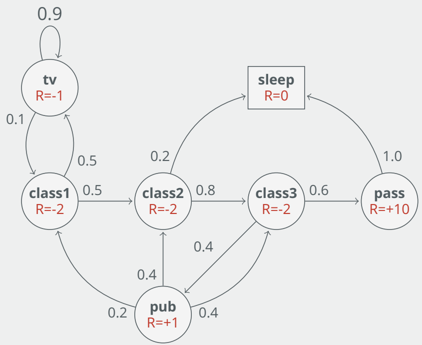
for example calculated by sampling episodes...

Sample episodes
c1,c2,c3,pass,sleep
c1,tv,tv,tv,c1,c2,c3,pub,c2,sleep
c1,c2,sleep
...

Example: Puppy

Markov Reward Process state value function

Example: MRP



Example: The state value function

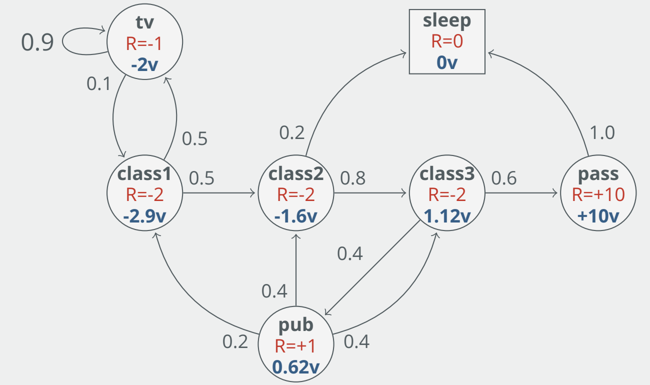
This is an example $v(s)$ with $s =$ 'class1' and $\gamma = \frac{1}{2}$:

$$
\begin{aligned}
G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots \\
&= R_{t+1} + \frac{1}{2} R_{t+2} + \frac{1}{4} R_{t+3} + \ldots
\end{aligned}
$$

Episode samples	Value function
c1,c2,c3,pass,sleep	$v_1 = -2 - \frac{1}{2} \cdot 2 - \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 10 = -2.25$
c1,tv,tv,c1,c2,c3,pub,c2,sleep	$v_1 = -2 - \frac{1}{2} \cdot 1 - \frac{1}{4} \cdot 1 + \frac{1}{8} \cdot \ldots = -3.125$
c1,c2,sleep	$v_1 = -2 - \frac{1}{2} \cdot 2 - \frac{1}{4} \cdot 0 + \frac{1}{8} \quad = -3$
...	$\ldots = -2.9$

Markov Reward Process value function example

Example: Markov Reward Process for $\gamma = 0.5$



Example from [2]

Through a series of identities, we can decompose the value function into the **immediate reward** R_{t+1} and the discounted **value of the next state** $\gamma v(S_{t+1})$.

Definition: Bellman equation for MRP

The Bellman equation is:

$$\begin{aligned} v(s) &= \mathbb{E}[G_t \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \ldots \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma (R_{t+2} + \gamma R_{t+3} + \gamma^2 R_{t+4} + \ldots) \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma G_{t+1} \mid S_t = s] \\ &= \mathbb{E}[R_{t+1} + \gamma v(S_{t+1}) \mid S_t = s], \end{aligned}$$

which is equivalent to:

$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s')$$

The Bellman equation can be expressed with matrices:

$$\begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 \\ \vdots \\ \mathcal{R}_n \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \cdots & \mathcal{P}_{1n} \\ \vdots & & \\ \mathcal{P}_{n1} & \cdots & \mathcal{P}_{nn} \end{bmatrix} \begin{bmatrix} v(1) \\ \vdots \\ v(n) \end{bmatrix},$$

which is a linear equation that can be solved:

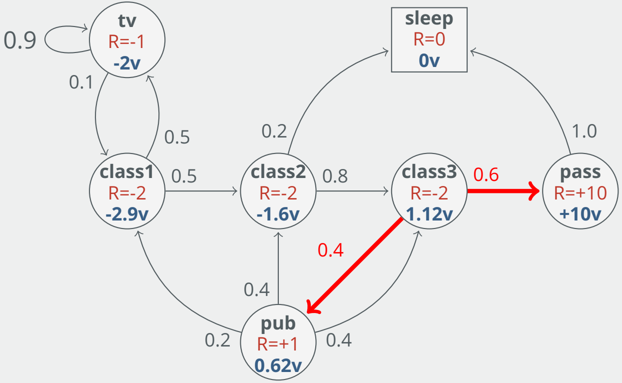
$$\begin{aligned} v &= \mathcal{R} + \gamma \mathcal{P} v \\ (\mathrm{I} - \gamma \mathcal{P}) v &= \mathcal{R} \\ v &= (\mathrm{I} - \gamma \mathcal{P})^{-1} \mathcal{R}, \end{aligned}$$

where I is the identity matrix. Unfortunately this matrix inversion is too slow, except for small MDPs, so we use iterative methods for larger MDP (MC evaluation and TD learning).

Markov Reward Process example verifying the Bellman equation

Verification: MRP for $\gamma = 0.5$

$$v(s) = \mathcal{R}_s + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'} v(s') = -2 + 0.5 * (0.6 * \mathbf{10} + 0.4 * \mathbf{0.62}) = \mathbf{1.12}$$



Markov Decision Process definition

A Markov **decision** process adds 'actions' so the transition probability matrix now depends on which action the agent takes.

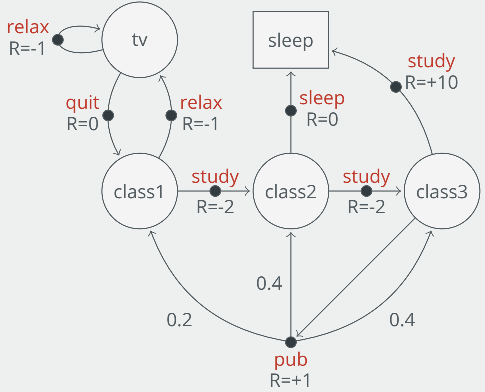
Definition: Markov decision process

A **Markov decision process** is a tuple $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$

- $\mathcal{S}$ is a finite set of states
- $\mathcal{A}$ is a finite set of actions
- $\mathcal{P}$ is the state-transition matrix where $\mathcal{P}^a_{ss'} = P(S_{t+1} = s' \mid S_t = s, A_t = a)$
- $\mathcal{R}$ is a **reward** function where $\mathcal{R}^a_s = \mathbb{E}[R_{t+1} \mid S_t = s, A_t = a]$
- γ is the **discount** rate $\gamma \in [0, 1]$

Markov Decision Process example

Example: Markov Decision Process



Example from [2]

Markov Decision Process policies

A policy is a distribution over actions which determines how agents should behave in the environment.

- A lazy agent will sample relaxing actions more than frequently than studying
- A high-performing agent will study at all classes, then study more at home!

Definition: Policy

A policy π is a distribution over actions given a state:

$$\pi(a|s) = P(A_t = a \mid S_t = s)$$

Markov Decision Process state and action value functions

Definition: The state-value function

The **state-value function** $v_\pi(s)$ is the same, but its the return when following a given policy π:

$$v_\pi(s) = \mathbb{E}_\pi[G_t \mid S_t = s]$$

Definition: The action-value function

The **action-value function** is the long term-value of a state when choosing an action with policy π:

$$q_\pi(s, a) = \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a]$$

Example: Arizona trail

Markov Decision Process the Bellman equation

Similarly to MRPs, the state-value function can be decomposed into the immediate reward and the discounted value of the next state:

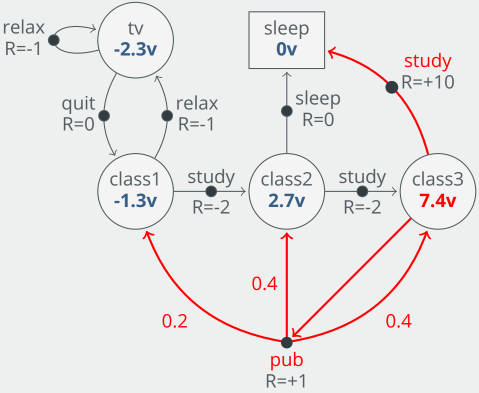
$$
\begin{aligned}
v_\pi(s) &= \mathbb{E}_\pi[G_t \mid S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\
&= \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s, a),
\end{aligned}
$$

which is also the case for the action-value function, where:

$$
\begin{aligned}
q_\pi(s, a) &= \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma q_\pi(S_{t+1}, A_{t+1}) \mid S_t = s, A_t = a] \\
&= \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s').
\end{aligned}
$$

Markov Decision Process example verifying the Bellman equation

Verification: MDP with average policy



Verification

Under the policy π where we do everything {study,pub} with 50% probability and $\gamma = 1$:

$$
\begin{aligned}
v_\pi(s) &= \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s,a) \\
&= \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_\pi(s') \right) \\
&= \frac{1}{2} * 10 \\
&+ \frac{1}{2} (1 + 0.2(-1.3v) + 0.4(2.7v) + 0.4(7.4v)) \\
&= \color{red}{7.4v}
\end{aligned}
$$

Markov Decision Process optimal state and action value functions

Definition: The optimal state-value function

The **optimal state-value function** $v_*(s)$ is the maximum value function over all policies:

$$v_*(s) = \max_\pi v_\pi(s)$$

Definition: The optimal action-value function

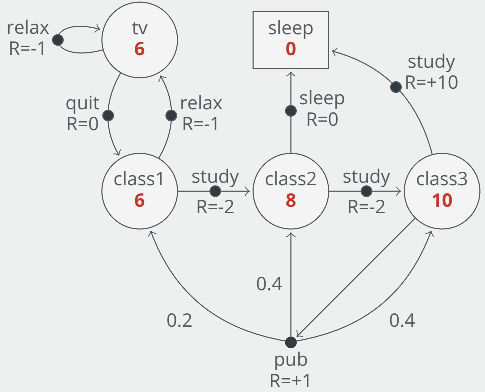
The **optimal action-value function** is the maximum action value function over all policies:

$$q_*(s, a) = \max_\pi q_\pi(s, a)$$

Example: Mo Farah

Markov Decision Process optimal state-value function example

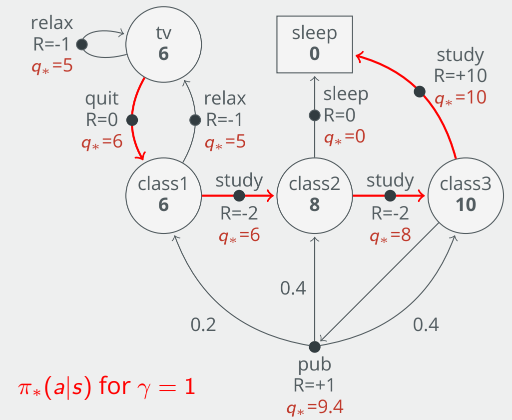
Example: $v_*(s)$ for $\gamma = 1$



Example from [2]

Markov Decision Process optimal action-value and optimal policy

Example: $q_*(s,a)$ for $\gamma = 1$



Example from [2]

The optimal value functions are similarly recursively related by the Bellman optimality equations, where:

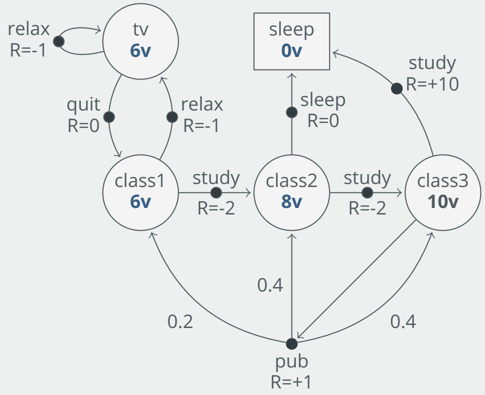
$$\begin{aligned} v_*(s) &= \max_\pi v_\pi(s) \\ &= \max_a q_*(s, a), \end{aligned}$$

and the optimal action-value function:

$$\begin{aligned} q_*(s, a) &= \max_\pi q_\pi(s, a) \\ &= \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s'). \end{aligned}$$

Markov Decision Process verifying the Bellman optimality equation

Verification: MDP with average policy



Verification

The optimal state-value for class3 following $\gamma = 1$ requires q_* for the pub action:

$$
\begin{aligned}
v_*(s) &= \max_a q_*(s, a) \\
&= \max_a \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s') \\
&= \max \Big\{ 10 + 1 * (0v), \\
&\quad (1 + 0.2(6v) + 0.4(8v) + 0.4(10v)) \Big\} \\
&= \max\{ q_* = 10, q_* = 9.4 \} \\
&= \mathbf{10v}
\end{aligned}
$$

References I

[1] V. Powell and L. Lehe.
Markov chains explained visually.
https://setosa.io/markov/index.html, 2014.

[2] D. Silver.
Reinforcement learning lectures.
https://www.davidsilver.uk/teaching/, 2015.

[3] R. S. Sutton and A. G. Barto.
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Available online, MIT press, 2018.