① (30 pts) Sea $\mathbb{R}[t]_3$ el $\mathbb{R}$-espacio vectorial de polinomios de grado mayor o igual que 3, y que sea $T: \mathbb{R}[t]_3 \longrightarrow \mathbb{R}^3$ la tl dada por $T(p(x)) = (p(0), p(1), p(2))$

ⓐ Calcular el Nucleo y Imagen de T

Base de $\mathbb{R}[t]_3 = \{1, t, t^2, t^3\}$

$Nu(T) = \{p(x) \in \mathbb{R}[t]_3 \mid p(0)=0,\ p(1)=0,\ p(2)=0\} \therefore\ x(x-1)(x-2)\cdot c$ con $c \in \mathbb{R}$

$T(1) = (1,1,1)$
$T(t) = (0,1,2)$
$T(t^2) = (0,1,4)$
$T(t^3) = (0,1,8)$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 1 & 8 \end{pmatrix} \xrightarrow[\substack{f_3 - f_2 \\ f_4 - f_2}]{f_1 - f_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{pmatrix} \xrightarrow{f_3 \cdot \frac{1}{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{pmatrix} \xrightarrow[\substack{f_2 - 2f_3 \\ f_4 - 4f_3}]{f_1 + f_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

ⓑ Dadas las bases $\beta_2 = \{(1,1,0), (0,1,1), (0,0,-1)\}$ de $\mathbb{R}^3$ y $\beta_1 = \{1+t,\ t^2+t+1,\ t^2+1,\ t^3+1\}$ de $\mathbb{R}[t]_3$ calcular $[T]_{\beta_1\beta_2}$

$T(1+t) = (1,2,3) \Rightarrow a(1,1,0) + b(0,1,1) + c(0,0,-1) \Rightarrow [T(1+t)]_{\beta_2} = (1,1,-2)$
$T(t^2+t+1) = (1,3,7) \quad a(1,1,0) + b(0,1,1) + c(0,0,-1) \Rightarrow [T(t^2+t+1)]_{\beta_2} = (1,2,-5)$
$T(t^2+1) = (1,2,5) \quad a(1,1,0) + b(0,1,1) + c(0,0,-1) \Rightarrow [T(t^2+1)]_{\beta_2} = (1,1,-4)$
$T(t^3+1) = (1,2,9) \quad a(1,1,0) + b(0,1,1) + c(0,0,-1) \Rightarrow [T(t^3+1)]_{\beta_1} = (1,1,-8)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow \begin{cases} a = 1 \\ a + b = 2 \\ b - c = 3 \end{cases} \quad b=1,\ c=-2 \qquad \begin{cases} a = 1 \\ a + b = 2 \\ b - c = 5 \end{cases} \quad b=1 \quad c=-4$$

$$\begin{cases} a = 1 \\ a + b = 3 \\ b - c = 7 \end{cases} \quad b=2,\ c=-5 \qquad \begin{cases} a = 1 \\ a + b = 2 \\ b - c = 9 \end{cases} \quad b=1 \quad c=-8$$

$$[T]_{\beta_1\beta_2} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ -2 & -5 & -4 & -8 \end{pmatrix}$$