1

a)
$$\begin{pmatrix} 4 & 7 \\ 5 & 3 \end{pmatrix} = (4\cdot 3) - 5\cdot 7 = 12 - 35 = -23$$

b)
$$\begin{pmatrix} -3 & 2 & 4 \\ 1 & -1 & 2 \\ -1 & 4 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} -3 & 2 & 4 \\ 1 & -1 & 2 \\ -1 & 4 & 0 \\ -3 & 2 & 4 \\ 1 & -1 & 2 \end{array}$$

$$((-3)\cdot(-1)\cdot 0) + (1\cdot 4\cdot 4) + ((-1)\cdot 2\cdot 2) - (4\cdot(-1)\cdot(-1)) - (2\cdot 4\cdot(-3)) - (0\cdot 2\cdot 1)$$
$$16 - 4 - 4 + 24 = 32$$

c)
$$\begin{pmatrix} 2 & 3 & 1 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 5 & 1 & 1 \\ 1 & 1 & 2 & 5 \end{pmatrix} \quad A(1|1) = \begin{pmatrix} 2 & -1 & 3 \\ 5 & 1 & 1 \\ 1 & 2 & 5 \end{pmatrix} \quad A(1|2) = \begin{pmatrix} 0 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 5 \end{pmatrix} \quad A(1|3) = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 5 & 1 \\ 1 & 1 & 5 \end{pmatrix} \quad A(1|4) = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 5 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$\begin{array}{ccc} 2 & -1 & 3 \\ 5 & 1 & 1 \\ 1 & 2 & 5 \\ 2 & -1 & 3 \\ 5 & 1 & 1 \end{array}$$

$$\begin{aligned} \det(A(1|1)) &= (2\cdot 1\cdot 5) + (5\cdot 2\cdot 3) + (1\cdot(-1)\cdot 1) - (3\cdot 1\cdot 1) - (1\cdot 2\cdot 2) - (5\cdot(-1)\cdot 5) \\ &= 10 + 30 - 1 - 3 - 4 + 25 \\ &= 65 - 8 \\ &= 57 \end{aligned}$$

$$\begin{array}{ccc} 0 & -1 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 5 \\ 0 & -1 & 3 \\ 0 & 1 & 1 \end{array}$$

$$\begin{aligned} \det(A(1|2)) &= (0\cdot 1\cdot 5) + (0\cdot 2\cdot 3) + (1\cdot(-1)\cdot 1) - (3\cdot 1\cdot 1) - (1\cdot 2\cdot 0) - (5\cdot(-1)\cdot 0) \\ &= 0 + 0 - 1 - 3 - 0 - 0 \\ &= -4 \end{aligned}$$

$$\begin{array}{ccc} 0 & 2 & 3 \\ 0 & 5 & 1 \\ 1 & 1 & 5 \\ 0 & 2 & 3 \\ 0 & 5 & 1 \end{array}$$

$$\begin{aligned} \det(A(1|3)) &= (0\cdot 5\cdot 5) + (0\cdot 1\cdot 3) + (1\cdot 2\cdot 1) - (3\cdot 5\cdot 1) - (1\cdot 1\cdot 0) - (5\cdot 2\cdot 0) \\ &= 0 + 0 + 2 - 15 - 0 - 0 \\ &= -13 \end{aligned}$$

$$\begin{array}{ccc} 0 & 2 & -1 \\ 0 & 5 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & 5 & 1 \end{array}$$

$$\begin{aligned} \det(A(1|4)) &= (0\cdot 5\cdot 2) + (0\cdot 1\cdot(-1)) + (1\cdot 2\cdot 1) - ((-1)\cdot 5\cdot 1) - (1\cdot 1\cdot 0) - (2\cdot 2\cdot 0) \\ &= 0 + 0 + 2 + 5 - 0 - 0 \\ &= 7 \end{aligned}$$

$$\begin{aligned} \det(A) &= (-1)^2\cdot 2 \det(A(1|1)) + (-1)^3\cdot 3\cdot \det(A(1|2)) + (-1)^4\cdot \det(A(1|3)) + (-1)^5\cdot \det(A(1|4)) \\ &= 2\cdot \det(A(1|1)) - 3\cdot \det(A(1|2)) + \det(A(1|3)) - \det(A(1|4)) \\ &= 2\cdot 57 - (3\cdot(-4)) + (-24) - 7 \\ &= 114 + 12 - 13 - 7 \\ &= 106 \end{aligned}$$

② $\det(A) = -1, \quad \det(B) = 2, \quad \det(C) = 3$

ⓘ $\det(A^2 B \cdot C^T B^{-1}) = \det(A^2) \cdot \det(B) \cdot \det(C^T) \cdot \det(B^{-1})$

$= (-1)^2 \cdot 2 \cdot 3 \cdot \frac{1}{2}$

$= 3$

ⓘⓘ $\det(B^2 C^{-1} A B^{-1} C^T)$

$= \det(B^2) \cdot \det(C^{-1}) \cdot \det(A) \cdot \det(B^{-1}) \cdot \det(C^T)$

$= 2^2 \cdot \frac{1}{3} \cdot (-1) \cdot \frac{1}{2} \cdot 3$

$= -2$

③ ⓐ $\begin{pmatrix} -2 & 3 & 2 & -6 \\ 0 & 4 & 4 & -5 \\ 5 & -6 & -3 & 2 \\ -3 & 7 & 0 & 0 \end{pmatrix}$ $\quad A(1|1) = \begin{pmatrix} 4 & 4 & -5 \\ -6 & -3 & 2 \\ 7 & 0 & 0 \end{pmatrix}$ $\quad A(1|2) = \begin{pmatrix} 0 & 4 & -5 \\ 5 & -3 & 2 \\ -3 & 0 & 0 \end{pmatrix}$ $\quad A(1|3) = \begin{pmatrix} 0 & 4 & -5 \\ 5 & -6 & 2 \\ -3 & 4 & 0 \end{pmatrix}$ $\quad A(1|4) = \begin{pmatrix} 0 & 4 & 4 \\ 5 & -6 & -3 \\ -3 & 7 & 0 \end{pmatrix}$

$\begin{matrix} 4 & 4 & -5 \\ -6 & -3 & 2 \\ 7 & 0 & 0 \\ 4 & 4 & -5 \\ -6 & -3 & 2 \end{matrix}$

$\det(A(1|1)) = (4 \cdot (-3) \cdot 0) + ((-6) \cdot 0 \cdot (-5)) + (7 \cdot 4 \cdot 2) - ((-5) \cdot (-3) \cdot 7) - (2 \cdot 0 \cdot 4) - (0 \cdot 4 \cdot (-6))$

$= 0 + 0 + 56 - 105 - 0 - 0$

$= -49$

$\begin{matrix} 0 & 4 & -5 \\ 5 & -3 & 2 \\ -3 & 0 & 0 \\ 0 & 4 & -5 \\ 5 & -3 & 2 \end{matrix}$

$\det(A(1|2)) = (0 \cdot (-3) \cdot 0) + (5 \cdot 0 \cdot (-5)) + ((-3) \cdot 4 \cdot 2) - ((-5) \cdot (-3) \cdot (-3)) - (2 \cdot 0 \cdot 0) - (0 \cdot 4 \cdot 5)$

$= 0 + 0 - 24 + 45 - 0 - 0$

$= 21$

$\begin{matrix} 0 & 4 & -5 \\ 5 & -6 & 2 \\ -3 & 4 & 0 \\ 0 & 4 & -5 \\ 5 & -6 & 2 \end{matrix}$

$\det(A(1|3)) = (0 \cdot (-6) \cdot 0) + (5 \cdot 7 \cdot (-5)) + ((-3) \cdot 4 \cdot 2) - ((-5) \cdot (-6) \cdot (-3)) - (2 \cdot 7 \cdot 0) - (0 \cdot 4 \cdot 5)$

$= 0 - 175 - 24 + 90 - 0 - 0$

$= -109$

$\begin{matrix} 0 & 4 & 4 \\ 5 & -6 & -3 \\ -3 & 7 & 0 \\ 0 & 4 & 4 \\ 5 & -6 & -3 \end{matrix}$

$\det(A(1|4)) = (0 \cdot (-6) \cdot 0) + (5 \cdot 7 \cdot 4) + ((-3) \cdot 4 \cdot (-3)) - (4 \cdot (-6) \cdot (-3)) - ((-3) \cdot 7 \cdot 0) - (0 \cdot 4 \cdot 5)$

$= 0 + 140 + 36 - 72 - 0 - 0$

$= 104$

$\begin{matrix} -2 & 3 & 2 & -6 \end{matrix}$

$\det(A) = (-2) \cdot (-49) - 3 \cdot 21 + 2 \cdot (-109) - (-6) \cdot 104$

$= 98 - 63 - 218 + 624$

$= 441$

ⓑ $B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ $\quad B(1|1) = \begin{pmatrix} 0 & 3 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ $\quad B(1|2) = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ $\quad B(1|3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ $\quad B(1|4) = \begin{pmatrix} 0 & 0 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\begin{matrix} 0 & 3 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 0 \\ -1 & 0 & 0 \end{matrix}$

$\det(B(1|1)) = (0 \cdot 0 \cdot 4) + ((-1) \cdot 0 \cdot 0) + (0 \cdot 3 \cdot 0) - 0 - 0 - (4 \cdot 3 \cdot (-1))$

$= 0 + 0 + 0 - 0 - 0 + 12$

$= 12$

$\det(B) = 2 \cdot 12 = 24$

c

$C = \begin{pmatrix} 1 & -1 & 2 & 0 & 0 \\ 3 & 1 & 4 & 0 & 0 \\ 2 & -1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 & 4 \end{pmatrix}$

$\overset{A}{\uparrow}$ $C(1|1) = \begin{pmatrix} 1 & 4 & 0 & 0 \\ -1 & 5 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$ $\overset{B}{\uparrow}$ $C(1|2) = \begin{pmatrix} 3 & 4 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$ $\overset{C}{\uparrow}$ $C(1|3) = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$

$\overset{D}{\uparrow}$ $C(1|4) = \begin{pmatrix} 3 & 1 & 4 & 0 \\ 2 & -1 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ $\overset{E}{\uparrow}$ $C(1|5) = \begin{pmatrix} 3 & 1 & 4 & 0 \\ 2 & -1 & 5 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$A(1|1) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix}$ $A(1|2) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix}$

$\begin{array}{ccc} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \\ 5 & 0 & 0 \\ 0 & 2 & 1 \end{array}$

$$\det(A(1|1)) = (5\cdot 2\cdot 4) + (0\cdot(-1)\cdot 0) + (0\cdot 0\cdot 1) - (0\cdot 2\cdot 0) - (1\cdot(-1)\cdot 5) - (4\cdot 0\cdot 0)$$
$$= 40 + 0 + 0 - 0 + 5 - 0$$
$$= 45$$

$\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \\ -1 & 0 & 0 \\ 0 & 2 & 1 \end{array}$

$$\det(A(1|2)) = ((-1)\cdot 2\cdot 4) + (0\cdot(-1)\cdot 0) + (0\cdot 0\cdot 1) - (0\cdot 2\cdot 0) - (1\cdot(-1)\cdot(-1)) - (4\cdot 0\cdot 0)$$
$$= -8 + 0 + 0 - 0 - 1 - 0$$
$$= -9$$

$$\det(A) = 45 + 4\cdot 9 = 81 = \det(C(1|1))$$

$B(1|1) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix}$ $B(1|2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix}$

$\begin{array}{ccc} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \\ 5 & 0 & 0 \\ 0 & 2 & 1 \end{array}$

$$\det(B(1|1)) = (5\cdot 2\cdot 4) + (0\cdot(-1)\cdot 0) + (0\cdot 0\cdot 1) - (0\cdot 2\cdot 0) - (1\cdot(-1)\cdot 5) - (4\cdot 0\cdot 0)$$
$$= 40 + 0 + 0 - 0 + 5 - 0$$
$$= 45$$

$\begin{array}{ccc} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \\ 2 & 0 & 0 \\ 0 & 2 & 1 \end{array}$

$$\det(B(1|2)) = (2\cdot 2\cdot 4) + (0\cdot(-1)\cdot 0) + (0\cdot 0\cdot 1) - (0\cdot 2\cdot 0) - (1\cdot(-1)\cdot 2) - (4\cdot 0\cdot 0)$$
$$= 16 + 0 + 0 - 0 + 2 - 0$$
$$= 18$$

$$\det(B) = 3\cdot 45 - 4\cdot 18 = 135 - 72 = 63$$

$C(1|1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix}$ $C(1|2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \end{pmatrix}$

$\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \\ -1 & 0 & 0 \\ 0 & 2 & 1 \end{array}$

$$\det(C(1|1)) = (-1\cdot 2\cdot 4) + (0\cdot(-1)\cdot 0) + (0\cdot 0\cdot 1) - (0\cdot 2\cdot 0) - (1\cdot(-1)\cdot(-1)) - (4\cdot 0\cdot 0)$$
$$= -8 + 0 + 0 - 0 - 1 - 0$$
$$= -9$$

$$\begin{matrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \\ 2 & 0 & 0 \\ 0 & 2 & 1 \end{matrix}$$

$\det(C(1|2)) = (2\cdot 2\cdot 4) + (0\cdot(-1)\cdot 0) + (0\cdot 0\cdot 1) - (0\cdot 2\cdot 0) - (1\cdot(-1)\cdot 2) - (4\cdot 0\cdot 0)$

$= 16 + 0 + 0 - 0 + 2 - 0$

$= 18$

$\det(C) = 3\cdot(-9) - 1\cdot 18 = -27 - 18 = -45$

$\det(C) = 1\cdot 81 + 1\cdot 63 + 2\cdot(-45)$

$= 54$

④ Probar por inducción que si $a_0, \dots, a_{n-1} \in k$ entonces $\det \begin{pmatrix} t & 0 & \cdots & 0 & a_0 \\ -1 & t & 0 & \cdots & a_1 \\ 0 & -1 & t & \cdots & a_2 \\ \vdots & \ddots & -1 & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & t+a_{n-1} \end{pmatrix} = t^n + a_{n-1}t^{n-1} + \dots + a_1 t + a_0$

Para $n=2$

$$\begin{pmatrix} t & a_0 \\ -1 & t+a_n \end{pmatrix}$$

Para $n+1$

$$\begin{pmatrix} t & 0 & 0 & 0 & a_0 \\ -1 & t & 0 & 0 & a_1 \\ 0 & -1 & t & 0 & \vdots \\ 0 & 0 & -1 & \ddots & a_{n-1} \\ 0 & 0 & 0 & -1 & t+a_n \end{pmatrix} = t^{n+1} + a_n t^n + \dots + a_1 t + a_0$$

Matriz Roja $=$ H.I $\Rightarrow$ $t^n + a_n t^{n-1} + \dots + a_2 t + a_1$

Matriz Violeta es diagonal $\Rightarrow$ $\det = (-1)^n$

$\Rightarrow t(t^n + a_n t^{n-1} + \dots + a_2 t + a_1) + a_0\cdot(-1)^{n+2}\cdot(-1)^n$

$t^{n+1} + a_n t^n + \dots + a_2 t^2 + a_1 t + a_0 (-1)^{2n+2}$

$t^{n+1} + a_n t^n + \dots + a_2 t^2 + a_1 t + a_0$

⑤ Determinar para que valores de $c \in \mathbb{R}$ las siguientes matrices son invertibles

ⓐ

$$A = \begin{pmatrix} 0 & c & -c \\ -1 & 2 & -1 \\ c & -c & c \end{pmatrix} \Rightarrow 0 \cdot \det\begin{pmatrix} 2 & -1 \\ -c & c \end{pmatrix} - c \cdot \det\begin{pmatrix} -1 & -1 \\ c & c \end{pmatrix} - c \cdot \det\begin{pmatrix} -1 & 2 \\ c & -c \end{pmatrix} = 0$$

$-c\,((-1 \cdot c) - (-1 \cdot c)) = 0$

$-c = 0$

$-c \cdot ((-1)\cdot(-c) - 2 \cdot c) = 0$

$-c \cdot (c - 2c) = 0$

$(-c)\cdot(-c) = 0$

$c^2 = 0$

$\therefore$ A es invertible $\forall c \neq 0,\ c \in \mathbb{R}$

ⓑ

$$\begin{pmatrix} 4 & c & 3 \\ c & 2 & c \\ 5 & c & 4 \end{pmatrix} = 4 \cdot \det\begin{pmatrix} 2 & c \\ c & 4 \end{pmatrix} - c \cdot \det\begin{pmatrix} c & c \\ 5 & 4 \end{pmatrix} + 3 \cdot \det\begin{pmatrix} c & 2 \\ 5 & c \end{pmatrix} = 0$$

$4 \cdot (2 \cdot 4 - c^2)$ $\quad$ $-c \cdot (c \cdot 4 - c\,5)$ $\quad$ $3\,(c^2 - 10)$

$4\,(8 - c^2)$ $\quad$ $-c \cdot (-c)$ $\quad$ $3c^2 - 30$

$32 - 4c^2$ $\quad$ c^2

$(32 - 4c^2) + c^2 + (3c^2 - 30) = 2 \neq 0$ $\therefore$ B es invertible $\forall c \in \mathbb{R}$

ⓒ

$$\begin{pmatrix} 1 & c & -1 \\ c & 1 & 1 \\ 0 & 1 & c \end{pmatrix} \quad 1 \cdot \det\begin{pmatrix} 1 & 1 \\ 1 & c \end{pmatrix} - c \cdot \begin{pmatrix} c & 1 \\ 0 & c \end{pmatrix} - 1 \cdot \det\begin{pmatrix} c & 1 \\ 0 & 1 \end{pmatrix}$$

$c - 1 - c\,(c^2) - c$

$-1 - c^3$

$c^3 = -1 \Rightarrow c = -1$

$\therefore$ C es invertible $\forall c \neq -1,\ c \in \mathbb{R}$

⑥ Probar que si $k_1, \dots, k_n \in K$, entonces

$$\det\begin{pmatrix} 1+k_1 & k_2 & k_3 & \dots & k_n \\ k_1 & 1+k_2 & k_3 & \dots & k_n \\ k_1 & k_2 & 1+k_3 & \dots & k_n \\ \vdots & & & & \\ k_1 & k_2 & k_3 & \dots & 1+k_n \end{pmatrix} = 1 + k_1 + \dots + k_n$$

$k_2 \ \dots \ \dots \ k_{n+1}$

Para $n+1$

$$\det\begin{pmatrix} 1+k_1 & k_2 & k_3 & \dots & k_n & k_{n+1} \\ k_1 & 1+k_2 & k_3 & \dots & k_n & k_{n+1} \\ k_1 & k_2 & 1+k_3 & \dots & k_n & k_{n+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ k_1 & k_2 & k_3 & \dots & 1+k_n & k_{n+1} \\ k_1 & k_2 & k_3 & \dots & k_n & 1+k_{n+1} \end{pmatrix} = 1 + k_1 + \dots + k_n + k_{n+1}$$

Mat. verde Por HI $= 1 + k_2 + k_3 + \dots + k_{n+1}$

⑦ Una matriz A compleja $n\times n$ se dice antisimetrica si $A^t = -A$

ⓘ Probar que si n es impar y A es antisimetrica, entonces $\det(A) = 0$

$$\det(A^t) = \det(-A) = \det\begin{pmatrix} -A_1 & \dots & -A_n \end{pmatrix} = (-1)\det\begin{pmatrix} A_1, -A_2, \dots, -A_n \end{pmatrix} = \dots = (-1)^n \det\begin{pmatrix} A_1, \dots, A_n \end{pmatrix} = \begin{cases} -\det(A) & \text{si } n \text{ impar (i)} \\ \det(A) & \text{si } n \text{ es par (ii)} \end{cases}$$

ⓘⓘ Para cada n par encontrar una matriz antisimetrica $n\times n$ tq $\det(A) = 0$

⑧ ⓐ

ⓑ En el EV $\mathbb{R}^{\mathbb{R}}$ consideramos el subconjunto $S_2 = \{e^{\lambda_1 x}, \dots, e^{\lambda_n x}\}$. Probar que S_2 es li $\iff \lambda_j \neq \lambda_i$ para todo $j \neq i$

$$a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + \dots + a_n^{\lambda_n x} = 0 \quad \forall x \in \mathbb{R}$$

$$a_1 \lambda_1 e^{\lambda_1 x} + a_2 \lambda_2 e^{\lambda_2 x} + \dots + a_n \lambda_n e^{\lambda_n x} = 0$$

En particular para $x=0$: $a_1 \lambda_1 + a_2 \lambda_2 + \dots + a_n \lambda_n = 0$

Si seguimos derivando hasta n obtenemos $a_1 \lambda_1^{n-1} + \dots + a_n \lambda_n^{n-1}$

Podemos observar que si armamos un sistema de n ecuaciones obtendremos una matriz de coeficientes y esto seria la matriz Vandermonde