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On Ostrowski inequality for quantum calculus

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ABSTRACT

We disprove a version of Ostrowski inequality for quantum calculus appearing in the literature. We derive a correct statement and prove that our new inequality is sharp. We also derive a midpoint inequality.

Quantum calculus is a calculus based on finite difference quotients, without the concept of limits. Over time, two different approaches to quantum calculus have been developed: Jackson's q-calculus [1,2] (see also [3,4]), and finite difference calculus (or h-calculus) with its roots going back to Taylor and Stirling [5]. Both approaches have been unified in Hahn's calculus which was introduced in [6]. Each of these variants has, in recent years, yielded interesting results both in pure and applied mathematics [7–19].

In this paper, we obtain a sharp version of Ostrowski inequality for Hahn calculus in $0 < q < 1$ regime (Corollary 17). We frame our results in terms of the shifted quantum calculus introduced in [20], which is equivalent to Hahn calculus, when $0 < q < 1$ (see Remark 1).

Recall that the well known **Ostrowski inequality** gives an estimate of the difference of function values and its integral mean on a segment [21]:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\left\| f' \right\|_\infty . \tag{1}$$

It holds for every $x \in [a, b]$, whenever $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), with derivative $f' : (a, b) \to \mathbb{R}$ bounded on (a, b), i.e.

$$\left\| f' \right\|_\infty := \sup_{t \in (a,b)} \left| f'(t) \right| < +\infty.$$

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Further, by means of a counterexample (Remark 10), we disprove a recently published result in [22, Theorem 3.5], which claims that for $0<q<1$, and for any q-differentiable function $f:[a,b]\to\mathbb{R}$, with $D_q^a f$ continuous on $[a,b]$, one has

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)d_q^a t\right|\leqslant\left[\frac{2q}{1+q}\left(\frac{x-\frac{(3q-1)a+(1+q)b}{4q}}{b-a}\right)^2+\frac{-q^2+6q-1}{8q(1+q)}\right](b-a)\left\|D_q^a f\right\|_\infty. \tag{2}$$

As we discuss later, this inequality does hold for some, but not all $x\in[a,b]$, as is claimed in [22]. The proof given in [22] uses the standard Lagrange's mean value theorem, which, as we show in Section 2, does not hold for quantum calculus. In Section 3 we give a counterexample to (2), and in Section 4 we derive a correct version of this inequality that holds for all $x\in[a,b]$. In particular, we prove the following result.

Corollary 17 (Full q-Ostrowski inequality) Let $f:[a,b]\to\mathbb{R}$ be q-integrable over $[a,b]$, and assume that f is continuous at $x=a$. Then, for all $x\in[a,b]$, the following sharp inequality holds

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,d_q^a t\right|\leqslant M(x)\|D_q^a f\|_\infty^{(a,b]},$$

where $M(x)$ is a discontinuous function:

$$M(x)=\begin{cases}(b-a)\left(\frac{1+2q\left(\frac{x-a}{b-a}\right)^2}{1+q}-\frac{x-a}{b-a}\right), & \text{if } x=a+q^m(b-a)\text{ for } m\in\mathbb{N}\cup\{0\},\\ (b-a)\left(\frac{x-a}{b-a}+\frac{1}{1+q}\right), & \text{otherwise.}\end{cases}$$

The rest of the paper is organized as follows: in Section 1 we give preliminaries for Hahn calculus; in Section 2 we obtain Lagrange's mean value theorem for quantum calculus, and use it in Section 3 to obtain a sharp bound for q-Ostrowski inequality for $x=a+q^m(b-a)$, $m\in\mathbb{N}\cup\{0\}$; finally, in Section 4 we obtain Ostrowski inequality for all possible values $x\in[a,b]$, and show that our bound is optimal.

1. q-Calculus preliminaries

Jackson [1] in 1908 has defined what is now known as **Euler-Jackson$q$-difference operator** (q-derivative of the function) by

$$D_q f(x)=\frac{f(x)-f(qx)}{(1-q)x},\ \ x\in(0,b],\ \ q\in(0,1)$$

for an arbitrary function $f:[0,b]\to\mathbb{R}$, where $b>0$. Note that every such function is q-differentiable for every $x\in(0,b]$. When $\lim\limits_{x\to 0}D_q f(x)$ exists, it is said that f is q-differentiable on $[0,b]$ and

$$D_q f(0)=\lim_{x\to 0}D_q f(x).$$

The q-derivative is a discretization of ordinary derivative and if f is differentiable function then

$$\lim_{q\to 1}D_q f(x)=f'(x).$$

Jackson [2] in 1910 has also defined q-**integral** (or Jackson integral) by

$$\int_0^x f(t)d_q t=(1-q)x\sum_{k=0}^{\infty}q^k f\left(q^k x\right),\quad x\in(0,b].$$

If the series on the right hand side is convergent, then q-integral $\int_0^x f(t)d_q t$ exists. If f is continuous on $[0,b]$ as $q\to 1$, then the series $(1-q)x\sum\limits_{k=0}^{\infty}q^k f\left(q^k x\right)$ tends to the Riemann integral ([3,4]):

$$\lim_{q\to 1}\int_0^x f(t)d_q t=\int_0^x f(t)dt.$$

Previous definitions and results for $f:[0,b]\to\mathbb{R}$ can easily be generalized for $f:[a,b]\to\mathbb{R}$ (see [20]). If we have a function $f:[a,b]\to\mathbb{R}$, then the "shifted" q-derivative for $q\in(0,1)$ can be defined as:

$$D_q^a f(x)=\frac{f(x)-f(a+q(x-a))}{(1-q)(x-a)},\qquad \text{if } x\in(a,b].$$

If $\lim_{x\to a} D_q^a f(x)$ exists, then $f:[a,b]\to\mathbb{R}$ is said to be q-differentiable on $[a,b]$ and

$$D_q^a f(a) = \lim_{x\to a} D_q^a f(x).$$

"Shifted" q-integral is defined by

$$\int_a^x f(t)d_q^a t = (1-q)(x-a)\sum_{k=0}^{\infty} q^k f\big(a+q^k(x-a)\big), \quad x\in[a,b].$$

If the series on the right hand side is convergent, then q-integral $\int_a^x f(t)d_q^a t$ exists and $f:[a,b]\to\mathbb{R}$ is q-integrable on $[a,x]$. If $c\in(a,x)$, then q-integral over $[c,x]$ is defined by

$$\int_c^x f(t)d_q^a t = \int_a^x f(t)d_q^a t - \int_a^c f(t)d_q^a t.$$

An important difference between the definite q-integral and Riemann integral is that even if integrate a function on an interval $[c,b]$, $a<c<b$, we have to take into account its behaviour at $t=a$, as well as its values on $[a,c]$. Beside the improper use of Lagrange's mean value theorem, this is the other reason for mistakes made in [22].

Remark 1 (Hahn's calculus)**.** Let $0<q<1$ and $\omega\in\mathbb{R}$ be fixed, and set $\omega_0=\frac{\omega}{1-q}$. Some authors require $\omega>0$ but, for our purposes, this restriction is unnecessary. Hahn's difference operator (see [23]), or Hahn's derivative of f, is defined as:

$$\begin{cases} D_{q,\omega}f(x) = \dfrac{f(x)-f(qx+\omega)}{x-(qx+\omega)}, & \text{if } x\neq\omega_0,\\ D_{q,\omega}f(\omega_0) = f'(\omega_0), & \text{if } f'(\omega_0) \text{ exists.}\end{cases}$$

Note that when $f'(\omega_0)$ exists, it is also true that $D_{q,\omega}f(\omega_0)=\lim_{x\to\omega_0} D_{q,\omega}f(x)$. The converse does not hold, but it makes sense to define $D_{q,\omega}f(\omega_0)$ as $\lim_{x\to\omega_0} D_{q,\omega}f(x)$ whenever this limit exists, even when $f'(\omega_0)$ does not.

The integral associated to Hahn's derivative is called Jackson–Nörlund sum [23]. It is first defined for integrals with the lower limit equal to ω_0:

$$\int_{\omega_0}^x f(t)d_{q,\omega}t := (x(1-q)-\omega)\sum_{k=0}^{\infty} q^k f\left(xq^k+\omega\frac{1-q^k}{1-q}\right).$$

The definition can, then, be extended to all other intervals:

$$\int_c^x f(t)d_{q,\omega}t := \int_{\omega_0}^x f(t)d_{q,\omega}t - \int_{\omega_0}^c f(t)d_{q,\omega}t.$$

It turns out that Hahn's calculus is equivalent to the shifted q-calculus in the regime $0<q<1$. This can be seen simply by setting $a=\omega_0=\frac{\omega}{1-q}$. It then follows:

$$D_{q,\omega}f(x) = D_q^a f(x) \text{ and } \int_c^x f(t)d_{q,\omega}t = \int_c^x f(t)d_q^a t.$$

In the rest of this paper, we prefer to maintain the shifted q-calculus perspective, as it, we believe, makes our results and their proofs more accessible to the reader. All our results, if one prefers, can be read as results concerning Hahn's calculus, simply by changing the notation.

Remark 2. In the case $a=0$, when writing $D_q^0 f$ and $\int d_q^0 t$, we will omit superscript zeros. This is consistent with the notation for the original Jackson's derivative and integral.

2. Lagrange's mean value theorem for q-calculus

Here and hereafter the symbol $\|\cdot\|_\infty^{(a,b]}$ denotes the supremum

$$\|f\|_\infty^{(a,b]} = \sup_{t\in(a,b]} |f(t)|.$$

Remark 3. This is not a norm on the space of all functions with domain $[a,b]$. However, this is a norm that coincides with the standard $\|\cdot\|_\infty$ norm, for the class of functions that are continuous at a.

Remark 4. Recall that every function $f:[a,b]\to\mathbb{R}$ has a q-derivative $D_q^a f(t)$ for any $t\in(a,b]$. Further, $D_q^a f(a) = \lim_{x\to a} D_q^a f(x)$, when this limit exists. Thus, q-differentiable functions are, by definition, continuously differentiable at a, and, therefore, the norm of the derivative $\|D_q^a f\|_\infty = \sup_{t\in[a,b]}|f(t)|$ is the same as $\|D_q^a f\|_\infty^{(a,b]}$.

Some of the results that follow do not require q-differentiability of f (at $x=a$) and this notation allows us to state them in full generality.

Let us first see that the standard mean value theorem does not hold in q-calculus. Setting $a=0$ and $b=2$, we consider a function:

$$f(x)=\begin{cases}1, & x\in[1,2],\\ 0, & x\in[0,1).\end{cases}$$

Clearly, $\|D_q f\|_\infty=\frac{1}{1-q}$, but

$$|f(x)-f(y)|\nleqslant \left\|D_q^a f\right\|_\infty |x-y|, \tag{3}$$

for $x=1$ and any $y\in(q,1)$.

Note that this function, although not continuous, is q-differentiable. It is not hard to find examples of continuous functions that also fail the standard mean value theorem. For instance, one could take the same example we give in Remark 10 below.

In the next theorem, we show that inequality (3) does hold when both x and y belong to the same q-*lattice*. This means that $y=a+q^n(x-a)$, for some $n\in\mathbb{Z}$.

Theorem 5 (Mean value inequality for q-calculus, [24] Corollary 3.2)**.** *] Let $f:[a,b]\to\mathbb{R}$ be an arbitrary function. Then for all $x\in(a,b]$, and all $n\in\mathbb{N}\cup\{0\}$ we have*

$$|f(x)-f(a+q^n(x-a))|\leqslant|x-(a+q^n(x-a))|\left\|D_q^a f\right\|_\infty^{(a,b]}.$$

Proof. For $n=0$ the statement is trivial. When $n>0$, one can write

$$\begin{aligned}\frac{f(x)-f(a+q^n(x-a))}{(1-q)(x-a)}&=\sum_{i=0}^{n-1}\frac{f(a+q^i(x-a))-f(a+q^{i+1}(x-a))}{(1-q)(x-a)}\\&=\sum_{i=0}^{n-1}q^i\cdot\frac{f(a+q^i(x-a))-f(a+q^{i+1}(x-a))}{(q^i-q^{i+1})(x-a)}\\&=\sum_{i=0}^{n-1}q^i\cdot D_q^a f(a+q^i(x-a)).\end{aligned}$$

Dividing this by $1+q+q^2+\cdots+q^{n-1}=\frac{1-q^n}{1-q}$ we obtain

$$\frac{f(x)-f(a+q^n(x-a))}{(1-q^n)(x-a)}=\frac{\sum_{i=0}^{n-1}q^i\cdot D_q^a f(a+q^i(x-a))}{\sum_{i=0}^{n-1}q^i}.$$

On the right hand side, we have a weighted average of numbers

$$D_q^a f(x),\ D_q^a f(a+q(x-a)),\ \ldots,\ D_q^a f(a+q^{n-1}(x-a)),$$

with weights $1,q,q^2,\ldots q^{n-1}$ respectively. This average must therefore be in-between $\min\limits_{0\leqslant i<n} D_q^a f(a+q^i(x-a))$ and $\max\limits_{0\leqslant i<n} D_q^a f(a+q^i(x-a))$ and in particular

$$\left|\frac{f(x)-f(a+q^n(x-a))}{(1-q^n)(x-a)}\right|\leqslant\|D_q^a f\|_\infty^{(a,b]}.$$

Multiplying both sides by $|(1-q^n)(x-a)|$, we obtain the claim. □

Corollary 6 ([24] Corollary 3.2)**.** *] Let $f:[a,b]\to\mathbb{R}$ be an arbitrary function. Then*

$$\left|f(a+q^m(b-a))-f\big(a+q^k(b-a)\big)\right|\leqslant(b-a)\,|q^m-q^k|\,\left\|D_q^a f\right\|_\infty^{(a,b]}$$

holds for all $m,k\in\mathbb{N}\cup\{0\}$.

Proof. Without loss of generality, we can assume that $k\geqslant m$. If we now set $x=a+q^m(b-a)$ and $n=k-m$, and then apply Theorem 5, the statement follows. □

Remark 7. The results in Theorem 5 and Corollary 6 are not new, and have been proved in [24]. We restate them here for reference, and keep their proofs for completeness.

If we additionally assume that the function in Theorem 5 is continuous, we can obtain a q-calculus version of Lagrange's mean value theorem.

Theorem 8 (Lagrange's mean value theorem for. q-calculus) *Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let $x, y \in (a, b]$ be such that $y = a + q^n(x-a)$, for some $n \in \mathbb{N} \cup \{0\}$. Then there exists some $c \in [y, x]$ such that*

$$f(x) - f(y) = D_q^a f(c)(x-y).$$

Proof. For $x = y$ ($n = 0$) the statement is trivial, so assume $x \neq y$ ($n > 0$). From the proof of Theorem 5 we have

$$\min_{0\leqslant i<n} D_q^a f(a+q^i(x-a)) \leqslant \frac{f(x)-f(y)}{x-y} \leqslant \max_{0\leqslant i<n} D_q^a f(a+q^i(x-a)).$$

The function f is continuous on $[a, b]$, hence, its q-derivative

$$D_q^a f(t) = \frac{f(t) - f(a+q(t-a))}{(1-q)(t-a)}$$

is also continuous on $(a, b]$. Therefore, it is also continuous on $[y, x]$. Since

$$\min_{t\in[y,x]} D_q^a f(t) \leqslant \min_{0\leqslant i<n} D_q^a f(a+q^i(x-a)) \leqslant \max_{0\leqslant i<n} D_q^a f(a+q^i(x-a)) \leqslant \max_{t\in[y,x]} D_q^a f(t),$$

there must exist some $c \in [y, x]$ such that

$$\frac{f(x)-f(y)}{x-y} = D_q^a f(c),$$

which proves the claim. □

3. q-Ostrowski inequality for points on q-lattice

In the following theorem, we give a correct proof of q-Ostrowski inequality for the points on the q-lattice of the form $x = a + q^m(b-a)$. For these values of x, the estimate we obtain here is matching that from Tariboon and Ntouyas [22], but the proof given there is incorrect.

Theorem 9 (. q-Ostrowski inequality on q-lattice) *Let $f : [a, b] \to \mathbb{R}$ be a q-integrable function over $[a, b]$. Then, for every $m \in \mathbb{N} \cup \{0\}$, the following inequality holds:*

$$\left| f(a+q^m(b-a)) - \frac{1}{b-a}\int_a^b f(t) d_q^a t \right| \leqslant (b-a)\left(\frac{1+2q^{2m+1}}{1+q} - q^m\right) \left\| D_q^a f \right\|_\infty^{(a,b]}. \tag{4}$$

Proof. Let $k \in \mathbb{N} \cup \{0\}$ be arbitrary. From Corollary 6 we have

$$\left| f(a+q^m(b-a)) - f\big(a+q^k(b-a)\big) \right| \leqslant \left\| D_q^a f \right\|_\infty^{(a,b]} |q^m - q^k| \cdot (b-a).$$

Note that

$$f(a+q^m(b-a)) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t = \sum_{k=0}^{\infty} \big[f(a+q^m(b-a)) - f(a+q^k(b-a)) \big](1-q)q^k,$$

and therefore

$$\begin{aligned} \left| f(a+q^m(b-a)) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t \right| &\leqslant (1-q)\sum_{k=0}^{\infty} \left| f(a+q^m(b-a)) - f(a+q^k(b-a)) \right| q^k \\ &\leqslant (1-q)(b-a)\| D_q^a f\|_\infty^{(a,b]} \sum_{k=0}^{\infty} \left| q^m - q^k \right| q^k. \end{aligned}$$

Finally, since

$$\begin{aligned} \sum_{k=0}^{\infty} q^k \left| q^m - q^k \right| &= \sum_{k=0}^{m-1} q^k\left(q^k - q^m\right) + \sum_{k=m}^{\infty} q^k\left(q^m - q^k\right) \\ &= \frac{1-q^{2m}}{1-q^2} - q^m\frac{1-q^m}{1-q} + q^m\frac{q^m}{1-q} - q^{2m}\frac{1}{1-q^2} = \frac{1}{1-q}\left(\frac{1+2q^{2m+1}}{1+q} - q^m\right), \end{aligned} \tag{5}$$

we obtain the desired inequality (4). □

Remark 10. Note that q-Ostrowski inequality in the previous theorem, assuming q-differentiability of f, can, equivalently, be written as

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t \right| \leqslant (b-a)\left(\frac{1+2q\left(\frac{x-a}{b-a}\right)^2}{1+q} - \frac{x-a}{b-a}\right)\|D_q^a f\|_\infty, \tag{6}$$

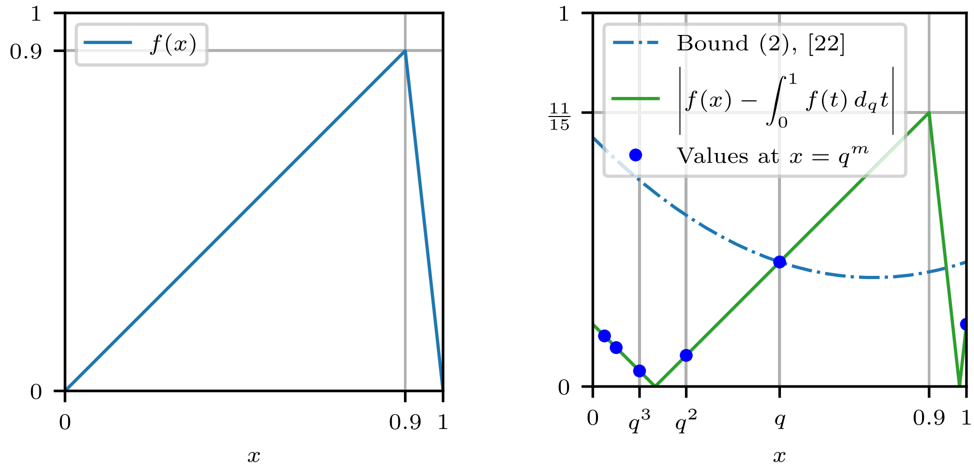


Fig. 1. A counterexample showing that inequality (2) does not hold for every $x \in [a, b]$. Note, however, that it does hold for every x of the form $a + q^m(b - a)$, as is claimed in Theorem 9. (Here $q = \frac{1}{2}$).

where $x = a + q^m(b - a)$. This is misleading, since Theorem 9 claims that inequality (6) holds only for $x \in [a, b]$ of the form $x = a + q^m(b - a)$.

After some algebraic manipulations, it can be checked that inequality (6) is exactly the same as inequality (2) from Tariboon and Ntouyas [22, Theorem 3.5], where it is claimed that it holds for all $x \in [a, b]$.

We stress again that inequality (2) (i.e. (6)) does not hold for all $x \in [a, b]$. It is not hard to find examples invalidating it. For example, set $a = 0$, $b = 1$, $q = \frac{1}{2}$ and take function $f : [0, 1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x, & \text{if } x \in [0, \frac{9}{10}], \\ -9x + 9, & \text{if } x \in [\frac{9}{10}, 1]. \end{cases}$$

We leave it to the reader to confirm that $\int_0^1 f(t)\,d_qt = \frac{q^2}{1+q} = \frac{1}{6}$, that D_qf is a continuous function, and that $\|D_qf\|_\infty = 1$. Plugging all this into (2) (or (6)), gives an obvious contradiction at $x = \frac{9}{10}$:

$$\left|\frac{9}{10} - \frac{1}{6}\right| \not\leqslant \frac{23}{75}.$$

Function f, as well as the bound it surpasses, is shown in Fig. 1.

Theorem 11. *Ostrowski inequality for q-calculus (4) is sharp for every $q \in (0, 1)$.*

Proof. In order to simplify notation, we prove sharpness in case $a = 0$ and $b = 1$. Pre-composing the examples below with the affine transformation $t \mapsto \frac{t-a}{b-a}$ will produce examples that work for any a and b.

We will now show that for the function

$$f(x) = |x - q^m|,\ x \in [0, 1],$$

equality in (4) is obtained at $x = q^m$, i.e.

$$\left|f(q^m) - \int_0^1 f(t)\,d_qt\right| = \left(\frac{1 + 2q^{2m+1}}{1+q} - q^m\right)\|D_qf\|_\infty^{[0,1]}.$$

We have $f(q^m) = 0$ and

$$\int_0^1 f(t)\,d_qt = (1-q)\sum_{k=0}^{\infty} q^k f\left(q^k\right) = (1-q)\sum_{k=0}^{\infty} q^k|q^k - q^m| = \frac{1 + 2q^{2m+1}}{1+q} - q^m,$$

where we used the previously computed sum (5).

Since it is obvious that

$$\begin{aligned} &D_qf(t) = -1, && 0 \leqslant t \leqslant q^m, \\ &D_qf(t) \in (-1, 1), && q^m < t < q^{m-1}, \\ &D_qf(t) = 1, && q^{m-1} \leqslant t \leqslant 1, \end{aligned}$$

we have $\|D_qf\|_\infty^{[0,1]} = 1$, and the equality holds. □

Corollary 12. *Let $f:[a,b]\to\mathbb{R}$ be a q-integrable function over $[a,b]$. Then, the following inequality holds*

$$\left| f(b) - \frac{1}{b-a}\int_a^b f(t)d_q^a t \right| \leqslant \frac{q(b-a)}{1+q}\left\| D_q^a f \right\|_\infty^{(a,b]}.$$

Proof. Take $m=0$ in Theorem 9. □

Corollary 13. *Let $f:[a,b]\to\mathbb{R}$ be a q-integrable function over $[a,b]$. Additionally, assume that f is continuous at $x=a$. Then, the following inequality holds*

$$\left| f(a) - \frac{1}{b-a}\int_a^b f(t)d_q^a t \right| \leqslant \frac{b-a}{1+q}\left\| D_q^a f \right\|_\infty^{(a,b]}.$$

Proof. Let $m\to\infty$ in Theorem 9. The claim follows because f is continuous at $x=a$, hence $\lim\limits_{m\to\infty} f(a+q^m(b-a)) = f(a)$. □

The tight bound in the classical Ostrowski inequality is obtained for $x=\frac{a+b}{2}$. In that case (1) reduces to midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \frac{b-a}{4}\left\| f' \right\|_\infty.$$

We now argue that for q-Ostrowski inequality the tight bound is obtained for $m=\left\lfloor \log_q \frac{1}{2} \right\rfloor$. For a fixed $q\in(0,1)$, we want to find

$$\min_{m\in\mathbb{N}\cup\{0\}} \left\{ \frac{1+2q^{2m+1}}{1+q} - q^m \right\}.$$

It is easy to check that the function

$$f(x) = \frac{1+2q^{2x+1}}{1+q} - q^x, \quad x\in[0,\infty)$$

has only one critical point $x=\log_q \frac{1+q}{4q}$, at which a strict global minimum occurs. So, we want to find the largest $m\in\mathbb{N}$ for which the following inequality holds

$$\frac{1+2q^{2m+1}}{1+q} - q^m \leqslant \frac{1+2q^{2(m-1)+1}}{1+q} - q^{m-1}.$$

In case this does not hold for any $m\in\mathbb{N}$, the minimum is clearly attained at $m=0$.

From the inequality above we get

$$q^{m-1}(1-2q^m)(1-q) \leqslant 0,$$

$$q^m \geqslant \frac{1}{2},$$

hence

$$m \leqslant \log_q \frac{1}{2},$$

and therefore $m=\left\lfloor \log_q \frac{1}{2} \right\rfloor$.

The following corollary is the midpoint inequality for q-calculus.

Corollary 14 (Midpoint inequality for. *q-calculus*) *Let $f:[a,b]\to\mathbb{R}$ be a q-integrable function over $[a,b]$. Then the following inequality holds*

$$\left| f\left(a + q^{\left\lfloor \log_q \frac{1}{2} \right\rfloor}(b-a)\right) - \frac{1}{b-a}\int_a^b f(t)d_q^a t \right|$$
$$\leqslant (b-a)\left(\frac{1+2q^{2\left\lfloor \log_q \frac{1}{2} \right\rfloor+1}}{1+q} - q^{\left\lfloor \log_q \frac{1}{2} \right\rfloor} \right)\left\| D_q^a f \right\|_\infty^{(a,b]}.$$

Proof. Take $m=\left\lfloor \log_q \frac{1}{2} \right\rfloor$ in Theorem 9. □

4. *q*-Ostrowski inequality

We will now derive the correct bound for all the other $x \in [a,b]$ that are not of the form $x = a + q^m(b-a)$.

Theorem 15. *Let $f : [a,b] \to \mathbb{R}$ be q-integrable over $[a,b]$, and further assume that f is continuous at $x = a$. Then, for all $x \in [a,b]$, the following inequality holds:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t \right| \leqslant (b-a)\left(\frac{x-a}{b-a} + \frac{1}{1+q}\right) \|D_q^a f\|_\infty^{(a,b]}. \tag{7}$$

Moreover, this bound is sharp whenever x is not of the form $a + q^m(b-a)$.

Remark 16. Note that the bound given in this theorem is strictly worse than the one in Theorem 9. Unlike that bound, this one holds for all $x \in [a,b]$.

Proof of Theorem 15.. In Theorem 5 we proved a version of mean value inequality for q-calculus:

$$|f(x) - f(a + q^n(x-a))| \leqslant |x - (a + q^n(x-a)|\, \|D_q^a f\|_\infty^{(a,b]},$$

that holds for all $x \in [a,b]$, and $n \in \mathbb{N} \cup \{0\}$. Fixing x and letting $n \to \infty$ we get

$$|f(x) - f(a)| \leqslant (x-a)\, \|D_q^a f\|_\infty^{(a,b]}, \text{ for all } x \in [a,b],$$

where we made use of the continuity of f at $x = a$.

From Corollary 13 we know that

$$\left| f(a) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t \right| \leqslant \frac{b-a}{1+q} \|D_q^a f\|_\infty^{(a,b]}.$$

Putting these two inequalities together, we obtain the claim

$$\begin{aligned}\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t \right| &\leqslant |f(x) - f(a)| + \left| f(a) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t \right| \\ &\leqslant \left(x - a + \frac{b-a}{1+q}\right)\|D_q^a f\|_\infty^{(a,b]} = (b-a)\left(\frac{x-a}{b-a} + \frac{1}{1+q}\right)\|D_q^a f\|_\infty^{(a,b]}.\end{aligned}$$

□

4.1. Sharpness of the inequality

In order to simplify notation in this subsection, we set $a = 0$ and $b = 1$. All the claims hold in full generality for any shifted domain $[a,b]$ and corresponding q-derivative D_q^a and q-integral.

Our goal is to show that for any $q \in (0,1)$, and for each $x \in [0,1]$ not of the form $x = q^n$, there exists a function $f_x : [0,1] \to \mathbb{R}$ that attains the bound

$$\left| f(x) - \int_0^1 f(t)\, d_q t \right| \leqslant \left(x + \frac{1}{1+q}\right) \|D_q f\|_\infty^{(0,1]}. \tag{8}$$

We assume that $q \in (0,1)$. Once it was chosen, it is taken to be fixed.

Observe that inequality (8) is scale invariant, so it is sufficient to look for examples f with $\|D_q f\|_\infty^{(0,1]} = 1$. Setting $\tilde{f} = Mf$, for any $M \geqslant 0$, produces examples with $\|D_q \tilde{f}\|_\infty^{(0,1]} = M$, that also attain the bound.

We now fix $x \in [0,1]$, not of the form $x = q^n$, and proceed with the construction of f_x. We further assume that $x \in (q,1)$. We will show how to treat other x afterwards.

Let $f_x : [0,1] \to \mathbb{R}$ be a function such that $f_x(q^n) = -q^n$ and $f_x(xq^n) = xq^n$ for all $n \in \mathbb{N} \cup \{0\}$. Furthermore, let $f_x(0) = 0$, and for all other $t \in [0,1]$, let $f_x(t)$ be defined as the linear interpolation of the previously defined points. More precisely, let f_x be defined as:

$$f_x(t) = \begin{cases} -\frac{1+x}{1-x}t + \frac{2x}{1-x}q^n, & \text{if } xq^n \leqslant t \leqslant q^n, \\ \frac{x+q}{x-q}t - \frac{2qx}{x-q}q^n, & \text{if } q^{n+1} \leqslant t \leqslant xq^n, \end{cases}$$

where $n \in \mathbb{N} \cup \{0\}$ and, additionally, $f_x(0) = 0$ (see Fig. 2).

Note that f_x is continuous on [0,1], and it is not q-differentiable at $t = 0$. Later, we will show how to alter these examples to produce q-differentiable examples.

Claim 1.. f_x is q-differentiable on $(0,1]$ with $\|D_q f\|_\infty^{(0,1]} = 1$.

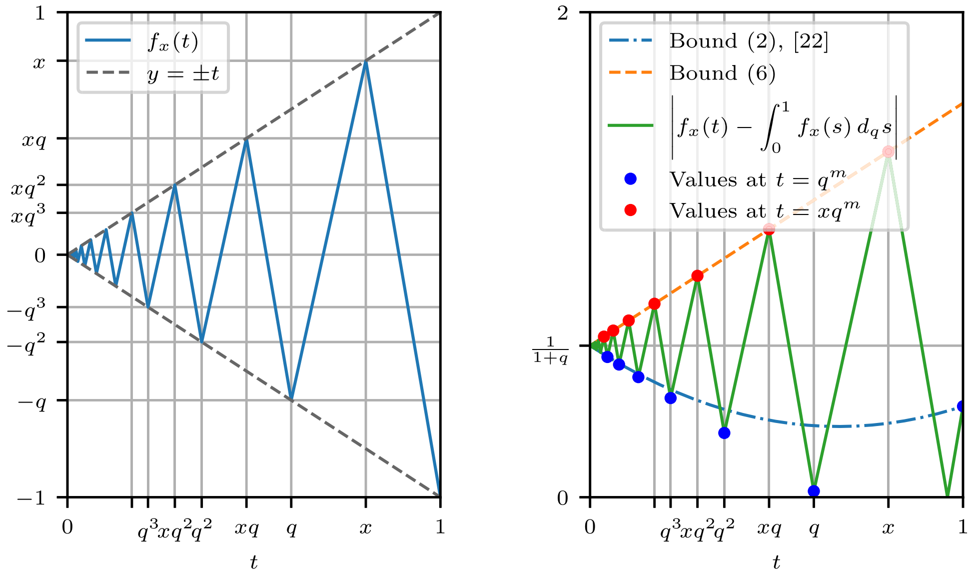


Fig. 2. An example showing that the bound in Theorem 15 is sharp. Here $q = 0.6$, $x = 0.8$, and f_x attains the bound at all $t = xq^m$. Note that f_x is not q-differentiable at $t = 0$, but somewhat surprisingly, its q-derivative over (0,1] never exceeds 1 in absolute value. Also, the incorrect bound (2) does hold for t of the form $t = q^m$.

Note that f_x is self-similar. To be precise, f_x over $(q^2, q^1]$ is scaled version of f_x over $(q, 1]$. Indeed, it is readily checked that $f_x(qt) = qf_x(t)$, for all $t \in (0, 1]$. This allows us to easily compute $D_q f_x$:

$$D_q f_x(t) = \frac{f_x(t) - f_x(qt)}{t(1-q)} = \frac{f_x(t) - qf_x(t)}{t(1-q)} = \frac{f_x(t)}{t}.$$

Therefore,

$$|D_q f_x(t)| = \left|\frac{f_x(t)}{t}\right| \leqslant 1.$$

□

Claim 2.. f_x is q-integrable on [0,1] with $\int_0^1 f_x(t)\, d_q t = -\frac{1}{1+q}$.

We calculate:

$$\int_0^1 f_x(t)\, d_q t = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n = (1-q)\sum_{n=0}^{\infty} -q^n q^n = (1-q)\sum_{n=0}^{\infty} -q^{2n} = -\frac{1}{1+q}.$$

□

Lastly, note that

$$\left|f_x(x) - \int_0^1 f_x(t)\, d_q t\right| = \left|x - \left(-\frac{1}{1+q}\right)\right| = x + \frac{1}{1+q},$$

which completes the proof that f_x attains the bound at x for $x \in (q, 1)$.

For $x \notin (q, 1)$, we first find $n \in \mathbb{N}$ such that $q^{n+1} < x < q^n$ and set $\tilde{x} = \frac{x}{q^n}$. Now $\tilde{x} \in (q, 1)$ and for function $f_{\tilde{x}}$ as above we have:

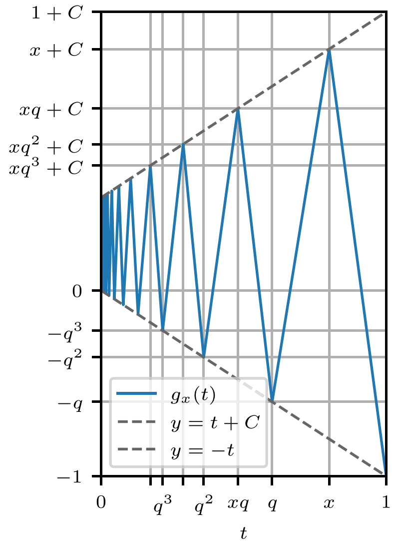
$$\left|f_{\tilde{x}}(x) - \int_0^1 f_{\tilde{x}}(t)\, d_q t\right| = \left|f_{\tilde{x}}(\tilde{x}q^n) - \left(-\frac{1}{1+q}\right)\right| = \tilde{x}q^n + \frac{1}{1+q} = x + \frac{1}{1+q}.$$

Therefore, $f_{\tilde{x}}$ attains the bound at x for $x \in \left(q^{n+1}, q^n\right)$.

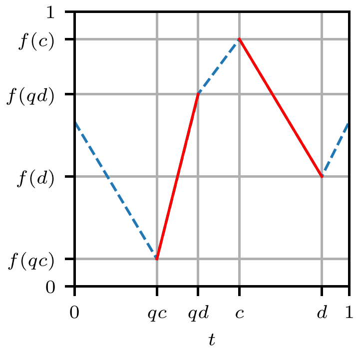
Putting together Theorems 9 and 15 and taking into account the examples above, we obtain the following corollary.

Corollary 17 (Full. *q*-Ostrowski inequality) *Let $f : [a, b] \to \mathbb{R}$ be q-integrable over $[a, b]$, and assume that f is continuous at $x = a$. Then, for all $x \in [a, b]$, the following sharp inequality holds*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\, d_q^a t\right| \leqslant M(x)\|D_q^a f\|_\infty^{(a,b)},$$



(a)



(b) An example of a function for which Lemma 19 applies. Note that the values of f corresponding to dashed lines are irrelevant and f is not required to be piecewise linear there.

Fig. 3. The function constructed in Remark 18 (a) and an example of function from Lemma 19 (b).

where $M(x)$ is a discontinuous function:

$$M(x)=\begin{cases}(b-a)\left(\frac{1+2q\left(\frac{x-a}{b-a}\right)^2}{1+q}-\frac{x-a}{b-a}\right), & \text{if } x=a+q^m(b-a) \text{ for } m\in\mathbb{N}\cup\{0\},\\ (b-a)\left(\frac{x-a}{b-a}+\frac{1}{1+q}\right), & \text{otherwise.}\end{cases}$$

Remark 18. It is reasonable to ask whether the hypothesis in Theorem 15, and consequently in the corollary above, could be further relaxed. It is immediately clear that, instead of requiring f to be continuous at $x=a$, it is sufficient to ask for the existence of limit $\lim_{x\to a+} f(x)$. In that case, the claim only holds for $x\in(a,b]$. Alternatively, one can redefine f at $x=a$, in order to make it continuous.

Relaxing the hypothesis even further is not possible, as the following example shows. Assuming $a=0$, $b=1$, construct a piecewise linear map $g_x:(0,1]\to\mathbb{R}$ in a similar fashion as f_x was constructed before, but set $g_x(q^n)=-q^n$ and $g_x(xq^n)=xq^n+C$, for some fixed $C>0$, and all $n\in\mathbb{N}\cup\{0\}$ (see Fig. 3 a). Note that $\lim_{x\to 0+} f(x)$ does not exist. Now, using Lemma 19 below, it can be checked that $|D_q g_x(t)|\leqslant 1$ for $t\in(0,1]$, and $\left|f(x)-\int_0^1 f(t)\,d_qt\right|$ can be made arbitrarily large by choosing a large C.

4.2. q-Differentiable examples

One might wonder whether imposing more restrictions on f in Theorem 15 could lead to a better bound. We will show that asking for q-differentiability (at a) does not improve things. We do this by constructing q-differentiable examples, that show the bound in Theorem 15 is best possible, even if one considers a class of q-differentiable functions. The examples we will construct will be piecewise linear functions with finitely many pieces. The following lemma will give us a tool to bound the q-derivative of such a function.

Lemma 19. *Let $c,d\in\mathbb{R}$ be such that $0<qc<qd<c<d\leqslant 1$, and assume that $f:[0,1]\to\mathbb{R}$ is a function whose values over $[qc,qd]$ are obtained as linear interpolation of values of f at endpoints qc and qd. Similarly, the values over $[c,d]$ are linear interpolation of values of f at endpoints c and d (see Fig. 3b).*

More precisely, assume that $f(\alpha c+(1-\alpha)d)=\alpha f(c)+(1-\alpha)f(d)$ and $f(\alpha qc+(1-\alpha)qd)=\alpha f(qc)+(1-\alpha)f(qd)$, for all $\alpha\in[0,1]$.

Then, the q-derivative of f over $[c,d]$ satisfies the following inequality:

$$\min(D_qf(c),D_qf(d))\leqslant D_qf(t)\leqslant\max(D_qf(c),D_qf(d)),\qquad \text{for all } t\in[c,d].$$

Proof. We first write $t\in[c,d]$ as $t=\alpha c+(1-\alpha)d$, for some $\alpha\in[0,1]$. We then calculate

$$D_qf(t)=\frac{f(t)-f(qt)}{(1-q)t}=\frac{f(\alpha c+(1-\alpha)d)-f(\alpha qc+(1-\alpha)qd)}{(1-q)(\alpha c+(1-\alpha)d)}$$

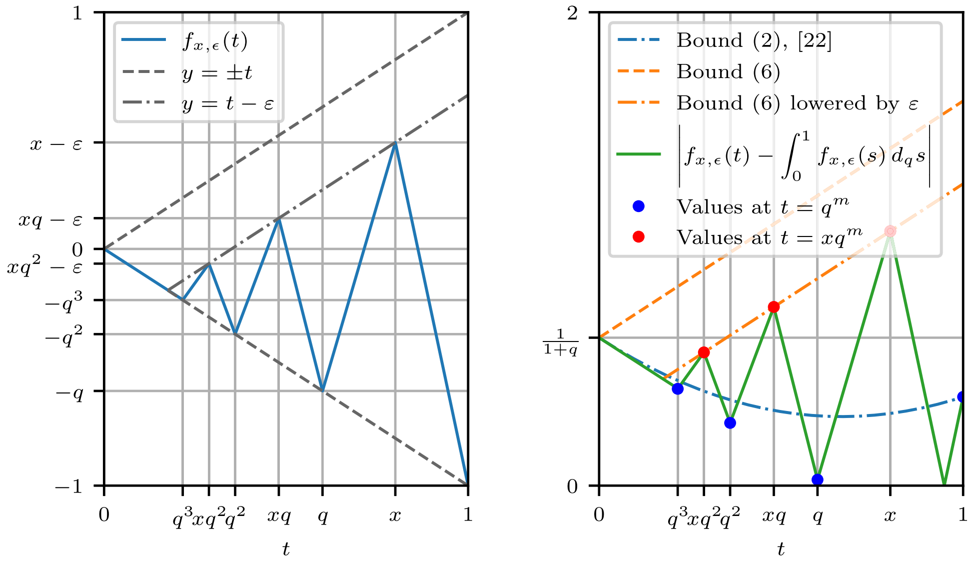


Fig. 4. A q-differentiable example showing that the bound in Theorem 15 is best possible. Here $q = 0.6$, $x = 0.8$, and $f_{x,\varepsilon}$ comes ε-close to the bound at all $t = xq^m$. Again, the incorrect bound (2) does hold for t of the form $t = q^m$.

$$= \frac{\alpha f(c) + (1-\alpha) f(d) - \alpha f(qc) - (1-\alpha) f(qd)}{(1-q)(\alpha c + (1-\alpha) d)}$$
$$= \frac{\alpha (f(c) - f(qc)) + (1-\alpha)(f(d) - f(qd))}{(1-q)(\alpha c + (1-\alpha) d)}$$
$$= \frac{\alpha c D_q f(c) + (1-\alpha) d D_q f(d)}{(\alpha c + (1-\alpha) d)}.$$

Since the last expression is a weighted average of $D_q f(c)$ and $D_q f(d)$, the statement of the lemma follows. □

We will now show that for any x not of the form $x = q^m$, and any $\varepsilon > 0$, we can find a q-differentiable function $f_{x,\varepsilon}$ that comes ε-close to the bound in Theorem 15. This will show that the bound in the theorem is the best possible.

As before, we will first show that we can do this for $x \in (q, 1)$. Once x is fixed, let $\varepsilon > 0$ be sufficiently small ($\varepsilon \leqslant 2x$), and let $f_{x,\varepsilon} : [0,1] \to \mathbb{R}$ be a function such that $f_{x,\varepsilon}(q^n) = -q^n$, and $f_{x,\varepsilon}(xq^n) = \max(xq^n - \varepsilon, -xq^n)$ for all $n \in \mathbb{N} \cup \{0\}$. Note that $f_{x,\varepsilon}(x) = x - \varepsilon$.

If we denote by $m \in \mathbb{N} \cup \{0\}$ the integer such that $xq^{m+1} < \frac{\varepsilon}{2} \leqslant xq^m$, i.e. $m = \lfloor \log_q \frac{\varepsilon}{2x} \rfloor$, then $f_{x,\varepsilon}(xq^n) = xq^n - \varepsilon$, for all $n \leqslant m$ and $f_{x,\varepsilon}(xq^n) = -xq^n$, for all integers $n > m$.

Furthermore, let $f_{x,\varepsilon}(0) = 0$, and for all other $t \in [0,1]$ let $f_{x,\varepsilon}(t)$ be defined as the linear interpolation of the previously defined points. Graph of $f_{x,\varepsilon}(xq^n)$ is shown in Fig. 4.

Claim 1.. $f_{x,\varepsilon}$ is continuously q-differentiable over [0,1].

Note that $f_{x,\varepsilon}(t) = -t$ for all $t \in [0, q^{m+1}]$, and $f_{x,\varepsilon}$ is therefore differentiable at $t = 0$. This further means that $f_{x,\varepsilon}$ is continuously q-differentiable over [0,1] as $f_{x,\varepsilon}$ is continuous. □

Claim 2.. $f_{x,\varepsilon}$ is q-integrable on [0,1] with $\int_0^1 f_{x,\varepsilon}(t)\, d_q t = -\frac{1}{1+q}$.

The values $f_{x,\varepsilon}$ match those of f_x at $t = q^n$ for all $n \in \mathbb{N} \cup \{0\}$. Hence,

$$\int_0^1 f_{x,\varepsilon}(t)\, d_q t = \int_0^1 f_x(t)\, d_q t = -\frac{1}{1+q}.$$

□

Claim 3.. $|D_q f_{x,\varepsilon}(t)| \leqslant 1$ for all $t \in [0,1]$.

To show this we employ Lemma 19. It follows immediately, from the definition of $f_{x,\varepsilon}$, that $D_q f_{x,\varepsilon}(q^n) = -1$, for all $n \in \mathbb{N} \cup \{0\}$. Also, $D_q f_{x,\varepsilon}(xq^n) = 1$, for all $n \leqslant m-1$, and $D_q f_{x,\varepsilon}(xq^n) = -1$, for $n \geqslant m+1$. The only problematic point is $t = xq^m$, so we calculate

$$D_q f_{x,\varepsilon}(xq^m) = \frac{f_{x,\varepsilon}(xq^m) - f_{x,\varepsilon}(xq^{m+1})}{(1-q)xq^m} = \frac{xq^m - \varepsilon - (-xq^{m+1})}{(1-q)xq^m} = \frac{1 + q - \frac{\varepsilon}{xq^m}}{1-q}.$$

Taking into account that $xq^{m+1} < \frac{\varepsilon}{2} \leqslant xq^m$, it follows that $2q < \frac{\varepsilon}{xq^m} \leqslant 2$, and hence $D_q f_{x,\varepsilon}(xq^m) \in [-1, 1]$.

To sum up, we have shown that $D_q f_{x,\varepsilon}(t) \in [-1, 1]$, for all t that are equal to q^n or xq^n, for some $n \in \mathbb{N} \cup \{0\}$. Since any other $t \in (0, 1)$ lies inside some interval $[c = xq^n, d = q^n]$ or inside $[c = q^{n+1}, d = xq^n]$, for some n, and since the endpoints of interval $[qc, qd]$ are again two neighboring points of the same form, and since $f_{x,\varepsilon}$ is defined to be linear interpolation of values at those endpoints — all the conditions of Lemma 19 are met. We can, therefore, conclude that $|D_q f_{x,\varepsilon}(t)| \leqslant 1$ for all $t \in [0, 1]$. □

It now only remains to note that

$$\left| f_{x,\varepsilon}(x) - \int_0^1 f_{x,\varepsilon}(t)\, d_q t \right| = \left| x - \varepsilon - \left(-\frac{1}{1+q} \right) \right| = x + \frac{1}{1+q} - \varepsilon,$$

which completes the proof that there are q-differentiable functions $f_{x,\varepsilon}$ that come arbitrarily close to the bound at x for any $x \in (q, 1)$.

As before, if $x \notin (q, 1)$, we first find $n \in \mathbb{N}$ such that $q^{n+1} < x < q^n$ and set $\tilde{x} = \frac{x}{q^n}$, so that $\tilde{x} \in (q, 1)$. Afterwards, choose $\varepsilon > 0$ sufficiently small, such that $\max(\tilde{x}q^n - \varepsilon, -\tilde{x}q^n) = \tilde{x}q^n - \varepsilon$, i.e. $\varepsilon \leqslant 2x$. Now for the function $f_{\tilde{x},\varepsilon}$, as above, we have

$$\left| f_{\tilde{x},\varepsilon}(x) - \int_0^1 f_{\tilde{x},\varepsilon}(t)\, d_q t \right| = \left| f_{\tilde{x},\varepsilon}(\tilde{x}q^n) - \left(-\frac{1}{1+q} \right) \right| = \tilde{x}q^n - \varepsilon + \frac{1}{1+q} = x + \frac{1}{1+q} - \varepsilon.$$

Therefore, $f_{\tilde{x},\varepsilon}$ comes arbitrarily close to the bound at x.

Remark 20. In the light of the previous examples, one could ask whether there are any q-differentiable functions that achieve the bound (7) exactly, at some x not of the form $x = a + q^m(b - a)$. We do not know of such examples and conjecture that any such an example will fail to be q-differentiable at a.

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