Shadowing, asymptotic shadowing and s-limit shadowing

by

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Abstract. We study three notions of shadowing: classical shadowing, limit (or asymptotic) shadowing, and s-limit shadowing. We show that classical and s-limit shadowing coincide for tent maps and, more generally, for piecewise linear interval maps with constant slopes, and are further equivalent to the linking property introduced by Chen in 1991.

We also construct a system which exhibits shadowing but not limit shadowing, and we study how shadowing properties transfer to maximal transitive subsystems and inverse limits (sometimes called natural extensions).

Where practicable, we show that our results are best possible by means of examples.

It seems that the first time *shadowing* appeared in the literature was in a paper [33] by Sinaĭ where it is shown that Anosov diffeomorphisms have shadowing and furthermore that any pseudo-orbit has a unique point shadowing it. This type of shadowing lemma subsequently appears in all sorts of hyperbolic systems, especially in their relation to Markov partitions and symbolic dynamics. We refer the reader to monographs by Palmer [27] and Pilyugin [29, 31].

Over time different variations of shadowing appeared in the literature driven by different problems people tried to solve using it. Our interest in shadowing comes from its relation to a type of attractors called ω-limit sets. Recall that the *ω-limit set of a point $x \in X$* is the set of limit points of its forward orbit. It can be shown that each ω-limit set is also *internally chain transitive* (ICT), meaning that under the given dynamics and allowing small perturbations one can form a pseudo-orbit between any two points of that set (precise definitions are given in Section 1).

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As the ICT condition is more operational, we are interested in systems for which these two notions coincide. It is easy to see that a form of shadowing introduced by Pilyugin et al. [18], called *limit (or asymptotic) shadowing*, suffices.

Over the years it was shown that $\omega_f = \mathrm{ICT}(f)$ holds for many other systems. Bowen [13] proved it for Axiom A diffeomorphisms; and in a series of papers [2, 6, 1, 4, 3, 5] Barwell, Davies, Good, Knight, Oprocha, and Raines proved it for shifts of finite type and Julia sets for certain quadratic maps, amongst others. It soon became apparent that most of these systems satisfy both the classical and the asymptotic notion of shadowing. This led Barwell, Davies, and Good [2, Conjectures 1.2 and 1.3] to conjecture that classical shadowing alone will imply $\omega_f = \mathrm{ICT}(f)$. Recently, Meddaugh and Raines [25] confirmed this for interval maps ([1]), and Good and Meddaugh [19] have shown that, for general compact metric X, $\omega_f = \mathrm{ICT}(f)$ if and only if f satisfies Pilyugin's notion of *orbital limit shadowing* [30]. In Section 3 we disprove the other conjecture by constructing a system with shadowing but for which $\omega_f \subsetneq ICT(f)$ and thus without limit shadowing. This is the content of Theorem 3.3.

THEOREM 3.3. *There exists a dynamical system (X, f) on a compact metric space which exhibits shadowing but for which $\omega_f \neq \mathrm{ICT}(f)$.*

Interestingly, Meddaugh and Raines's result did not answer whether for interval maps classical shadowing implies limit shadowing. Trying to resolve this question forms the second line of inquiry in this paper. Sadly, we have not yet been able to provide a definite answer for all interval maps, but a great deal can be said for systems given by piecewise linear maps with constant slopes. Indeed, Chen [15] showed that the *linking condition* is strong enough to completely resolve whether such a system has shadowing.

Using the same condition we have been able to show that for these systems shadowing and linking are equivalent to a condition tightly related to limit shadowing, called *s-limit shadowing*. This notion, stronger than either classical or limit shadowing, was introduced in [32] where Sakai extended the definition of limit shadowing to account for the fact that many systems have limit shadowing but not shadowing [23, 29]. Below, we quote the two main results we obtain.

THEOREM 2.2. *Let $f\colon I \to I$ be a continuous piecewise linear map with a constant slope $s > 1$. Then the following are equivalent:*

(1) *f has s-limit shadowing,*
(2) *f has shadowing,*

([1]) It is worth noting here that neither classical nor limit shadowing implies the other one—more on this in Section 3.

(3) f has the linking property.

If furthermore the map is transitive, all of the above are additionally equivalent to

(4) f has limit shadowing.

THEOREM 2.7. *Let* $f_s\colon [0,1] \to [0,1]$ *be a tent map with* $s \in (\sqrt{2}, 2]$ *and denote its core by* $C = [f_s^2(1/2), f_s(1/2)]$. *The following conditions are equivalent:*

(1) f_s *has s-limit shadowing,*
(2) f_s *has shadowing,*
(3) f_s *has limit shadowing,*
(4) $f_s|_C$ *has s-limit shadowing,*
(5) $f_s|_C$ *has shadowing,*
(6) $f_s|_C$ *has limit shadowing.*

The article is organised as follows. In Section 1 we introduce all the basic notions we use. In Section 2 we prove our main results, Theorems 2.2 and 2.7. Section 3 clarifies the distinction between three notions of shadowing: limit, s-limit, and classical shadowing; we also construct a system which exhibits shadowing but not limit shadowing (Theorem 3.3). In Section 4 we include a few results providing sufficient conditions for shadowing on maximal limit sets of interval maps. Finally, Section 5 contains a result on shadowing of the shift map in inverse limit spaces.

1. Preliminaries. An interval map $f\colon [a,b] \to [a,b]$ is *piecewise monotone* if there are $p_0 = a < p_1 < \cdots < p_k < p_{k+1} = b$ such that $f|_{[p_i,p_{i+1}]}$ is monotone for $i = 0, \ldots, k$; and f is *piecewise linear* if moreover, for each i, $f|_{[p_i,p_{i+1}]}(x) = a_i + b_i x$ for some $a_i, b_i \in \mathbb{R}$. If further there is $s \geq 0$ such that $|b_i| = s$ for all i, we say that f is *piecewise linear with constant slope s*.

A point $x \in [a,b]$ is a *critical point* of f if $x = a$, or $x = b$, or f is not differentiable at x, or $f'(x) = 0$. The set of critical points of f is denoted by $\mathrm{Crit}(f)$.

For every $s \in (1,2]$ define a *tent map* $f_s\colon [0,1] \to [0,1]$ by

$$f_s(x) = \begin{cases} sx & \text{if } x \in [0, 1/2], \\ s(1-x) & \text{if } x \in [1/2, 1]. \end{cases}$$

DEFINITION 1.1 (Pseudo-orbit). A sequence $\langle x_0, x_1, \ldots \rangle$ is said to be a *δ-pseudo-orbit* for some $\delta > 0$ if $d(f(x_i), x_{i+1}) < \delta$ for each $i \in \mathbb{N}_0$. A *finite δ-pseudo-orbit of length* $l \geq 1$ is a finite sequence $\langle x_0, x_1, \ldots, x_l \rangle$ satisfying $d(f(x_i), x_{i+1}) < \delta$ for $0 \leq i < l$. We also say that it is a *δ-pseudo-orbit between x_0 and x_l*.

We say that the sequence $\langle x_0, x_1, \ldots \rangle$ is an *asymptotic pseudo-orbit* if $\lim_{i \to \infty} d(f(x_i), x_{i+1}) = 0$.

DEFINITION 1.2. A point $z \in X$ is said to ε-*shadow* a sequence $\langle x_0, x_1, \ldots \rangle$ for some $\varepsilon > 0$ if $d(x_i, f^i(z)) < \varepsilon$ for each $i \in \mathbb{N}_0$; and z *asymptotically shadows* the sequence if $\lim_{i \to \infty} d(x_i, f^i(z)) = 0$.

DEFINITION 1.3 (Shadowing). A dynamical system $f \colon X \to X$ is said to have *shadowing* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every δ-pseudo-orbit is ε-shadowed by some point in X.

If $\langle x_n \rangle_{n \in \mathbb{N}}$ is a δ-pseudo-orbit and an asymptotic pseudo-orbit then we simply say that it is an *asymptotic δ-pseudo-orbit*. Similarly, if a point z ε-traces and asymptotically traces a pseudo-orbit $\langle x_n \rangle_{n \in \mathbb{N}}$, then we say that z *asymptotically ε-traces* $\langle x_n \rangle_{n \in \mathbb{N}}$.

DEFINITION 1.4 (s-limit shadowing). Let $f \colon X \to X$ be a continuous map on a compact metric space X. We say that f *has s-limit shadowing* if for every $\varepsilon > 0$ there is $\delta > 0$ such that

(1) for every δ-pseudo-orbit $\langle x_n \rangle_{n \in \mathbb{N}} \subset X$ of f, there is $y \in X$ that ε-shadows $\langle x_n \rangle_{n \in \mathbb{N}}$, and
(2) for every asymptotic δ-pseudo-orbit $\langle z_n \rangle_{n \in \mathbb{N}} \subset X$ of f, there is $y \in X$ that asymptotically ε-shadows $\langle z_n \rangle_{n \in \mathbb{N}}$.

REMARK 1.5. Note that s-limit shadowing implies both classical and limit shadowing.

DEFINITION 1.6. Let $f \colon X \to X$ be continuous and let $\varepsilon > 0$. A point $x \in X$ is ε-*linked to a point* $y \in X$ *by* f if there exists an integer $m \geq 1$ and a point z such that $f^m(z) = y$ and $d(f^j(x), f^j(z)) \leq \varepsilon$ for $j = 0, \ldots, m$.

We say $x \in X$ is *linked to* $y \in X$ *by* f if x is ε-linked to y by f for every $\varepsilon > 0$. A set $A \subset X$ is *linked by* f if every $x \in A$ is linked to some $y \in A$ by f.

DEFINITION 1.7. Let $f \colon [0, 1] \to [0, 1]$ be a continuous piecewise monotone map and let $\mathrm{Crit}(f)$ be the finite set of critical points of f. We say f has the *linking property* if $\mathrm{Crit}(f)$ is linked by f.

The following theorem is the main result in [15].

THEOREM 1.8 (Chen [15]). *Suppose $f \colon [0, 1] \to [0, 1]$ is a map that is conjugate to a continuous piecewise linear map with constant slope ([2]) $s > 1$. Then f has shadowing if and only if it has the linking property.*

In [15, proof of Proposition 15], Chen shows the following implication.

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([2]) Note that the slope is actually $\pm s$ but throughout the text for definiteness we always take s to be the positive value.

LEMMA 1.9 (Chen [15]). *Let* $f \colon [0,1] \to [0,1]$ *be a map with constant slope* $s > 1$. *If* f *has the linking property then there is* $\hat{\varepsilon} > 0$ *such that for every* $\varepsilon \in (0,\hat{\varepsilon})$ *there is* $N = N(\varepsilon) > 0$ *such that for every* $x \in X$ *there is an integer* $n = n(x,\varepsilon) < N$ *such that*

$$B(f^n(x), s\varepsilon) \subset f^{n+1}\big(B(x,\varepsilon) \cap f^{-1}(B_{n-1}(f(x), s^2\varepsilon))\big)$$

where $B_k(x,\varepsilon) = \{y \in [0,1] : |f^i(x) - f^i(y)| < \varepsilon \text{ for } i = 0, \ldots, k\}.$

Finally, we define two important notions whose relation to shadowing will soon become apparent.

DEFINITION 1.10 (internally chain transitive sets). An f-invariant and closed set $A \subseteq X$ is said to be *internally chain transitive* if given any $\delta > 0$ there exists a δ-pseudo-orbit between any two points of A that is completely contained inside A. We denote by $\mathrm{ICT}(f)$ the minimal set containing all ICT subsets of X as its elements. This is a subset of the hyperspace 2^X of all closed non-empty subsets of X.

DEFINITION 1.11 (ω-limit sets). The ω-*limit set* of a point $x \in X$ is the set of limit points of its orbit:

$$\omega_f(x) = \bigcap_{i=1}^{\infty} \overline{\bigcup_{j=i}^{\infty} \{f^j(x)\}}.$$

Each ω-limit set is a non-empty, closed, f-invariant subset of X (see e.g. [8, Chapter IV]). In particular, all ω-limit sets belong to 2^X. Let

$$\omega_f = \{\omega_f(x) \mid x \in X\} \subseteq 2^X$$

denote the set of all ω-limit sets in the system.

It is known (see e.g. [5]) that any ω-limit set is also internally chain transitive. We thus have the following inclusion of sets in the hyperspace 2^X:

$$\omega_f \subseteq \mathrm{ICT}(f).$$

For some systems this is a strict inclusion and it is not hard to find such examples. It is much more interesting to try to characterise systems in which ω_f and $\mathrm{ICT}(f)$ coincide. This would be useful as it is easier to check if a given set is ICT than if it is an ω-limit set.

2. Shadowing condition

LEMMA 2.1. *Let* $f \colon X \to X$ *be a continuous map on the compact metric space* X. *Assume that there exist constants* $\lambda \geq 1$ *and* $\hat{\varepsilon} > 0$ *such that for every* $\varepsilon \in (0,\hat{\varepsilon})$, *there exist a positive integer* $N = N(\varepsilon)$ *and* $\eta = \eta(\varepsilon) > 0$ *such that for each* $x \in X$, *there exists a positive integer* $n = n(x,\varepsilon) \leq N$

satisfying

(2.1) $\quad f(B(f^n(x), \varepsilon + \eta))$
$$\subset \{f^{n+1}(y) : d(x, y) \le \varepsilon, \, d(f^i(x), f^i(y)) \le \lambda\varepsilon, \, 1 \le i \le n\}.$$

Then f has s-limit shadowing.

Proof. By [17, Lemma 2.4] (see also [17, Lemma 2.3]) the assumptions are sufficient for shadowing, therefore it is enough to prove that for every sufficiently small $\varepsilon > 0$ (we may take $\varepsilon < \hat\varepsilon$) there is $\delta > 0$ such that every asymptotic δ-pseudo-orbit is asymptotically ε-traced.

Fix any $\varepsilon > 0$, put $\varepsilon_0 = \varepsilon/(3\lambda)$, let $\eta_0 = \min\langle\eta(\varepsilon_0), \varepsilon_0\rangle$ and δ be such that $d(f^k(x_0), x_k) < \eta_0$ for any δ-pseudo-orbit $\langle x_i\rangle_{i=0}^k$, where $k = 1, \ldots, N(\varepsilon_0)$. We will show that the δ is as desired.

Fix any asymptotic δ-pseudo-orbit $\langle x_i\rangle_{i=0}^\infty$. Denote $\delta_0 = \delta$ and for every $m > 0$ define $\varepsilon_m = \varepsilon_0/2^m$, let $\eta_m = \min\langle\eta(\varepsilon_m), \varepsilon_m\rangle$ and let $0 < \delta_m < \varepsilon_m$ be such that $d(f^k(x_0), x_k) < \eta_m$ for any δ_m-pseudo-orbit $\langle x_i\rangle_{i=0}^k$, where $k = 1, \ldots, N(\varepsilon_m)$.

For any $x \in X$ denote
$$A(x, n, \gamma) = \{y : d(x, y) \le \gamma, \, d(f^i(x), f^i(y)) \le \lambda\gamma, \, 1 \le i \le n\}.$$

We define integers m_k, n_k, j_k, and sets W_k for all $k \ge 0$ as follows:
$$m_0 = 0, \quad j_0 = 0, \quad n_0 = n(x_0, \varepsilon_{j_0}), \quad W_0 = A(x_{m_0}, n_0, \varepsilon_0).$$

Next, for $k \ge 1$ we put
$$m_k = m_{k-1} + n_{k-1} = n_0 + \cdots + n_{k-1}.$$

By the definition of j_{k-1} the sequence $\langle x_i\rangle_{i=m_k}^\infty$ is an asymptotic $\delta_{j_{k-1}}$-pseudo-orbit. If it is also a $\delta_{j_{k-1}+1}$-pseudo-orbit then we put $j_k = j_{k-1} + 1$; otherwise we put $j_k = j_{k-1}$. Finally, let
$$n_k = n(x_{m_k}, \varepsilon_{j_k}), \quad W_k = W_{k-1} \cap f^{-m_k-1}\big(f(A(x_{m_k}, n_k, \varepsilon_{j_k}))\big).$$

First, we claim that for every $k \ge 0$ we have

(2.2) $\qquad\qquad f^{m_k+1}(W_k) \subset f(A(x_{m_k}, n_k, \varepsilon_{j_k})).$

We will prove the claim by induction on k. For $k = 0$ the claim holds just by definition, since
$$f^{m_0+1}(W_0) = f(W_0) = f(A(x_{m_0}, n_0, \varepsilon_0)).$$

Next, fix any $s \ge 0$ and suppose that (2.2) holds for all $0 \le k \le s$. Since, by definition,
$$f^{m_{s+1}+1}(W_{s+1}) = f^{m_{s+1}+1}\big(W_s \cap f^{-m_{s+1}-1}(f(A(x_{m_{s+1}}, n_{s+1}, \varepsilon_{j_{s+1}})))\big)$$
$$= f^{m_{s+1}+1}(W_s) \cap f(A(x_{m_{s+1}}, n_{s+1}, \varepsilon_{j_{s+1}})),$$

it remains to prove that
$$f(A(x_{m_{s+1}}, n_{s+1}, \varepsilon_{j_{s+1}})) \subset f^{m_{s+1}+1}(W_s).$$
By the choice of j_s we see that $\langle x_i \rangle_{i=m_s}^{\infty}$ is a δ_{j_s}-pseudo-orbit and hence
$$d(f^j(x_{m_s}), x_{m_s+j}) < \eta_{j_s} \quad \text{for every } 0 \le j \le N(\varepsilon_{j_s}).$$
In particular $d(f^{n_s}(x_{m_s}), x_{m_s+n_s} = x_{m_{s+1}}) < \eta_{j_s}$ and thus
$$B(x_{m_{s+1}}, \varepsilon_{j_s}) \subset B(f^{n_s}(x_{m_s}), \varepsilon_{j_s} + \eta_{j_s}).$$
By the definition of n_s we also have
$$\begin{aligned} f\big(B(f^{n_s}(x_{m_s}), \varepsilon_{j_s} + \eta_{j_s})\big) \subset \{f^{n_s+1}(y) : d(x_{m_s}, y) \le \varepsilon_{j_s}, \\ d(f^i(x_{m_s}), f^i(y)) \le \lambda\varepsilon_{j_s}, \, 1 \le i \le n_s\} \\ = f^{n_s+1}(A(x_{m_s}, n_s, \varepsilon_{j_s})). \end{aligned}$$
Combining the above two observations with the induction assumption we obtain
$$\begin{aligned} f(A(x_{m_{s+1}}, n_{s+1}, \varepsilon_{j_{s+1}})) &\subset f(B(x_{m_{s+1}}, \varepsilon_{j_s})) \subset f(B(f^{n_s}(x_{m_s}), \varepsilon_{j_s} + \eta_{j_s})) \\ &\subset f^{n_s+1}(A(x_{m_s}, n_s, \varepsilon_{j_s})) = f^{n_s}(f^{m_s+1}(W_s)) \\ &= f^{m_{s+1}+1}(W_s). \end{aligned}$$
This completes the induction, so (2.2) is proved.

Note that since every set $A(x, n, \gamma)$ is closed, we have a nested sequence of closed non-empty sets $W_0 \supset W_1 \supset \cdots$, and therefore there is at least one point $z \in \bigcap_{k=0}^{\infty} W_k$. We claim that for $k = 0$ and $0 \le i \le m_1$, and also for $k > 0$ and each $m_k < i \le m_{k+1}$, we have
$$d(f^i(z), x_i) \le 2\lambda\varepsilon_{j_k}.$$
Again we use induction on k. First, let $k = 0$ and fix any $0 \le i \le n_0 = n(x_0, \varepsilon_0) < N(\varepsilon_0)$. Since $z \in W_0 = A(x_0, n_0, \varepsilon_0)$ we see that $d(f^i(z), f^i(x_0)) \le \lambda\varepsilon_0$ and additionally, by the definition of $\delta_{j_0} = \delta_0$, we conclude that $d(f^i(x_0), x_i) < \varepsilon_0$. Hence $d(f^i(z), x_i) \le (\lambda + 1)\varepsilon_0 \le 2\lambda\varepsilon_0$. The base case of induction is complete.

Now fix any $k > 0$ and any $m_k < i \le m_{k+1}$. Denote $t = i - m_k - 1$ and observe that $0 \le t < n_k = n(x_{m_k}, \varepsilon_{j_k})$. Observe that by (2.2),
$$f^i(z) = f^t(f^{m_k+1}(z)) \in f^t(f^{m_k+1}(W_k)) \subset f^{t+1}(A(x_{m_k}, n_k, \varepsilon_{j_k}))$$
and so $d(f^i(z), f^{t+1}(x_{m_k})) < \lambda\varepsilon_{j_k}$. Additionally, by the choice of j_k and δ_{j_k}, we have $d(f^{t+1}(x_{m_k}), x_{m_k+t+1}) < \varepsilon_{j_k}$. Combining these two inequalities we obtain
$$d(f^i(z), x_i) \le \lambda\varepsilon_{j_k} + \varepsilon_{j_k} \le 2\lambda\varepsilon_{j_k}.$$
The claim is proved.

Observe that since $\langle x_i \rangle_{i=0}^{\infty}$ is an asymptotic pseudo-orbit, the sequence $\langle j_k \rangle_{k=0}^{\infty}$ is unbounded (i.e. $\lim_{k \to \infty} \varepsilon_{j_k} = 0$) and hence z asymptotically traces $\langle x_i \rangle_{i=0}^{\infty}$. Additionally,

$$d(f^i(z), x_i) \leq 2\lambda \varepsilon_{j_k} \leq 2\lambda \varepsilon_0 < \varepsilon$$

and so, in fact, z asymptotically ε-traces $\langle x_i \rangle_{i=0}^{\infty}$, which ends the proof. ∎

The equivalence of shadowing and linking for piecewise linear maps with constant slope was demonstrated by Chen [15]. In fact, in [15, proof of Proposition 15] it is shown that for such maps the linking property implies the stronger condition (2.1) with $\lambda = s^2$ and $\eta = (s-1)\varepsilon$ where s is the slope of the map. This observation together with Lemma 2.1 immediately gives the following theorem.

THEOREM 2.2. *Let $f \colon I \to I$ be a continuous piecewise linear map with a constant slope $s > 1$. Then the following are equivalent:*

(1) *f has s-limit shadowing,*
(2) *f has shadowing,*
(3) *f has the linking property.*

If furthermore f is transitive, all of the above are additionally equivalent to

(4) *f has limit shadowing.*

REMARK 2.3. It is worth noting that this is the best one could hope for, as one can construct a (non-transitive) constant slopes piecewise linear map with limit shadowing but without the other properties. Indeed, take a tent map $f = f_s$ for which the critical point $c = 1/2$ forms a three-cycle $c \mapsto f(c) \mapsto f^2(c) \mapsto c$. This corresponds to the slope being equal to the golden ratio $s \approx 1.6180$; the map is shown in Figure 1. Both the map f over $[0, 1]$ and the restriction $f|_{[f^2(c), f(c)]}$ have the linking property and thus all the shadowing properties we are considering in this article.

Now take $d = 1/(s + s^2) \in (0, f^2(c))$, which is a preimage of the interior fixed point under two iterates. The map $f|_{[d, f(c)]}$ no longer has linking as d is clearly not ε-linked to any of the critical points $\{d, 1/2, f(c)\}$ for sufficiently small ε. This implies that this restriction fails shadowing. We claim, however, that it has limit shadowing.

To see this, take any asymptotic pseudo-orbit $\langle x_i \rangle_{i=0}^{\infty}$ in $[d, f(c)]$. Replace any $x_i \in [d, f^2(c))$ in this sequence with $f^2(c)$ and denote the new sequence in $[f^2(c), f(c)]$ by $\langle y_i \rangle_{i=0}^{\infty}$. It is easy to check that this is still an asymptotic pseudo-orbit for $f|_{[f^2(c), f(c)]}$, which, as established above, has s-limit shadowing. If we now take $z \in [f^2(c), f(c)]$ which asymptotically shadows the pseudo-orbit $\langle y_i \rangle_{i=0}^{\infty}$, it is easy to show that it must also asymptotically shadow the original pseudo-orbit $\langle x_i \rangle_{i=0}^{\infty}$.
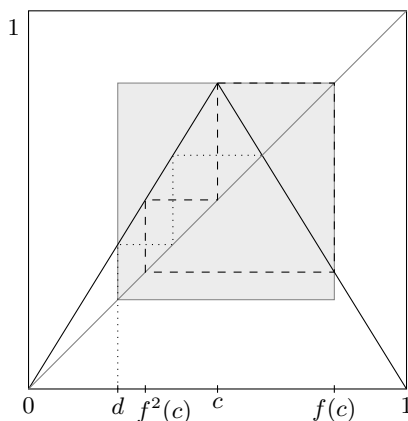
Fig. 1. A map with limit shadowing but without the classical shadowing property

To conclude, $f|_{[d,f(c)]}$ is indeed a piecewise linear map with constant slopes which has limit shadowing but fails linking, and consequently does not have shadowing. ∎

We now proceed to prove Theorem 2.2.

Proof of Theorem 2.2. Equivalence of (2) and (3) is provided by Theorem 1.8. If f has linking then the assumptions of Lemma 2.1 are satisfied, which shows that (1) is a consequence of (3). The implication (1)⇒(2) is trivial.

In the transitive case the implication (4)⇒(2) follows from a result by Kulczycki, Kwietniak, and Oprocha [22, Theorem 7.3] whereas (1)⇒(4) is immediate by definition. ∎

Parry [28] showed that any transitive continuous piecewise monotone map of a compact interval is conjugate to a piecewise linear map with a constant slope of the same entropy. By a result of Blokh [9] (for a proof see [7]) the entropy h of a transitive interval map is strictly positive (actually $h \geq \log \sqrt{2}$) and thus this slope is $e^h > 1$. As each of the four properties in the theorem above is preserved under conjugations, this argument allows us to replace the constant slopes assumption by transitivity.

COROLLARY 2.4. *For every transitive continuous piecewise monotone map of a compact interval, all the four properties: s-limit shadowing, limit shadowing, classical shadowing, and linking are equivalent.*

It is natural to ask at this point if the characterisation in Theorem 2.2 extends to maps with countably many monotone pieces. It turns out that the answer is no, as the following example shows.

EXAMPLE 2.5. We construct a piecewise linear map with constant slope $s > 1$ and countably many pieces of monotonicity which has the linking property but does not have shadowing.

The key steps of the construction are represented in Figure 2. We first construct a nucleus of the map depicted in Figure 2(a) where each critical point is mapped to another critical point in the rescaled version of the map. To be precise, the critical point denoted by C_1 in Figure 2(a) is mapped to C_2^+, the rescaled (by a factor $\mu < 1$) copy of the critical point C_2 which is in turn mapped to the original point C_2. The slope is everywhere the same and is denoted by $s > 3$. If we denote by a the length of the first (and third) piece of monotonicity of the nucleus, one easily deduces that the parameters a, s, μ have to satisfy

$$sa = 1 + \mu(1 - a), \quad s\mu a = \mu + a, \quad 4sa = s + 1.$$

This system has a solution $a \approx 0.301696, \mu \approx 0.657298, s \approx 4.83598$.

Once the nucleus is constructed, one takes two μ-scaled copies of it and glues them to both ends of the nucleus. Then another two μ^2-scaled copies are added, and so on ad infinitum. This process gives the map depicted in Figure 2(b) which we denote by $f\colon [0,1] \to [0,1]$ after rescaling to the unit interval.

Note that each critical point of f (except 0 and 1 which are fixed) is pre-periodic and is eventually mapped onto the cycle $C_1 \mapsto C_2^+ \mapsto C_2 \mapsto C_1^- \mapsto C_1$ on the nucleus. Thus f has linking. It is not hard to see that f is also topologically mixing. Namely, its slope exceeds 2 and so every open interval $U \subset [0,1]$ must eventually cover two critical points (i.e. $f^n(U)$ contains two consecutive critical points), which in turn implies that for every $\varepsilon > 0$ there is $m > 0$ such that $[\varepsilon, 1 - \varepsilon] \subset f^m(U)$. This proves topological mixing. Unfortunately, it is not completely clear whether f has shadowing, and to ensure this we make a simple modification.

Let $g\colon [-1,1] \to [-1,1]$ be given by

$$g(x) = \begin{cases} -f(x) & \text{if } x \geq 0, \\ f(-x) & \text{if } x < 0. \end{cases}$$

It is clear that g also has linking and constant slopes, but it cannot have shadowing. The reason is that g is transitive but not mixing, as g^2 has invariant intervals $[-1,0]$ and $[0,1]$, thus no point can have a dense orbit in $[-1,1]$ under g^2. On the other hand, g^2 is chain transitive on $[-1,1]$ because for any δ-pseudo-orbit, jumping to the other side of the origin poses no problem. If g had shadowing then so would g^2, and this would contradict a result from [25, Corollary 6] which says that for any interval map with shadowing, any ICT set is also an ω-limit set. ◁

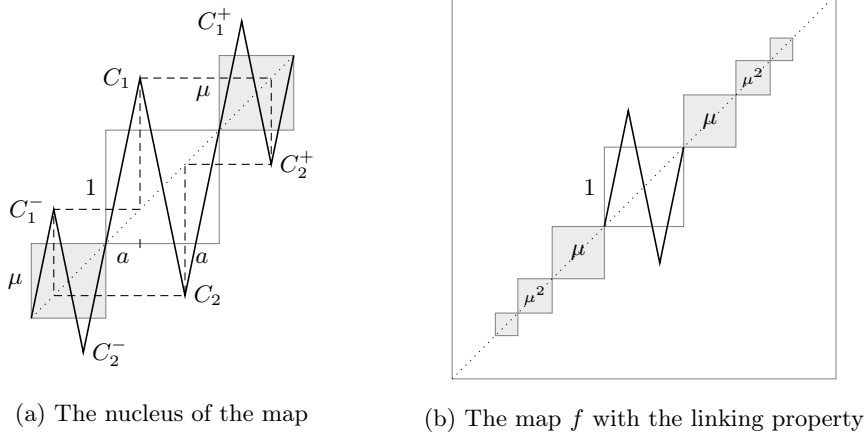(a) The nucleus of the map

(b) The map f with the linking property

Fig. 2. Construction of the map which has linking but not shadowing

EXAMPLE 2.6. By [21] the standard tent map f_2 can be perturbed to a map $f\colon [0,1] \to [0,1]$ such that:

(1) f has shadowing and is topologically mixing,
(2) the inverse limit of $[0,1]$ with f as a unique bonding map is the pseudo-arc, and hence f has infinite topological entropy (see [26]).

By the above f is not conjugate to a piecewise-linear map with constant slope, since all these maps have finite entropy.

The answer to the question whether f has s-limit shadowing is unknown to the authors. ◁

THEOREM 2.7. *Let $f_s\colon [0,1] \to [0,1]$ be a tent map with $s \in (\sqrt{2}, 2]$ and denote its core by $C = [f_s^2(1/2), f_s(1/2)]$. The following conditions are equivalent:*

(1) *f_s has s-limit shadowing,*
(2) *f_s has shadowing,*
(3) *f_s has limit shadowing,*
(4) *$f_s|_C$ has s-limit shadowing,*
(5) *$f_s|_C$ has shadowing,*
(6) *$f_s|_C$ has limit shadowing.*

Proof. The equivalence (1)⇔(2) is provided by Theorem 2.2. Similarly, since $f_s|_C$ is a unimodal map with constant slope, we obtain (4)⇔(5). In fact by Theorem 2.2 we know that for f_s and $f_s|_C$, shadowing is equivalent to linking.

Let $c = 1/2$ be the unique critical point of f_s in $(0,1)$. Note that $\mathrm{Crit}(f_s) = \{0, c, 1\}$ and $\mathrm{Crit}(f_s|_C) = \{f_s^2(c), c, f_s(c)\}$. As $f_s(1) = 0$ and $f_s(0) = 0$, both 0 and 1 are linked to 0 by f_s. Similarly, both c and $f_s(c)$

are linked to $f_s^2(c)$ by $f_s|_C$. Thus for $s < 2$ checking the linking property of f_s (resp. $f_s|_C$) boils down to checking whether c is linked to itself (resp. whether $f_s^2(c)$ is linked to itself).

But if c is linked to itself then clearly $f_s^2(c)$ is also linked to c, and thus to $f_s^2(c)$. For the converse, note that if $s < 2$, the only preimage of $f_s^2(c)$ under $f_s|_C$ is $f_s(c)$ and in turn the only preimage of $f_s(c)$ is c. It is now straightforward to check that if $f_s^2(c)$ is linked to itself then c must also be linked to itself. Thus we have showed (1)⇔(2)⇔(4)⇔(5) (note that for $s = 2$ this is trivially satisfied).

If f has s-limit shadowing then by definition it has limit shadowing, so the implications (1)⇒(3) and (4)⇒(6) are trivially satisfied.

It is proved in [22] that if a map is transitive and has limit shadowing then it also has shadowing. But it is well known (see e.g. [14, Remark 3.4.17]) that each $f_s|_C$ is transitive, hence (6)⇒(5) is also valid.

To close the circle of implications it now remains to show (3)⇒(6). To this end we fix any asymptotic pseudo-orbit $\langle x_i \rangle_{i=0}^{\infty} \subset C$ and let $z \in [0,1]$ be a point which asymptotically traces it under the action of f_s. Observe that the set $\Lambda = \omega(z, f_s)$ is f_s-invariant and thus $\Lambda \subset C$. Note that for $\sqrt{2} < s < 2$ (again for $s = 2$ the implication (3)⇒(6) trivially holds) the f_s-invariant set Λ cannot be contained in $\{f_s^2(c), f_s(c)\}$ but must intersect the interior of C. Hence there exists an integer $n \geq 0$ such that $f^n(z) \in C$ and so $f^{n+j}(z) \in C$ for every $j \geq 0$. If we fix any point $y \in C$ such that $f^n(y) = f^n(z)$ then clearly y asymptotically traces $\langle x_i \rangle_{i=0}^{\infty}$. This completes the proof. ∎

3. How limit shadowing, s-limit shadowing, and classical shadowing relate to each other. In this section we describe the distinction between three notions of shadowing: limit ($LmSP$), s-limit (s-$LmSP$), and classical shadowing property (SP).

Pilyugin [29, Theorem 3.1.3] showed that for circle homeomorphisms, SP implies $LmSP$. In fact, he gave efficient characterisations of both SP and $LmSP$ for orientation preserving circle homeomorphisms which fix a nowhere dense set containing at least two points (see [29, Theorems 3.1.1 and 3.1.2]).

Pilyugin's results roughly state that such a homeomorphism with a hyperbolic (either repelling or attracting) fixed point has $LmSP$, and it further has SP if the repelling and attracting fixed points alternate. We abstain from stating the full characterisation precisely and instead refer the interested reader to [29].

Using this, it is not hard to construct maps that land in areas denoted by (a), (b), and (c) in Figure 3. The corresponding graphs are depicted in Figure 4. (Note that the circle is represented as the interval $[-1,1]$ with endpoints identified.)
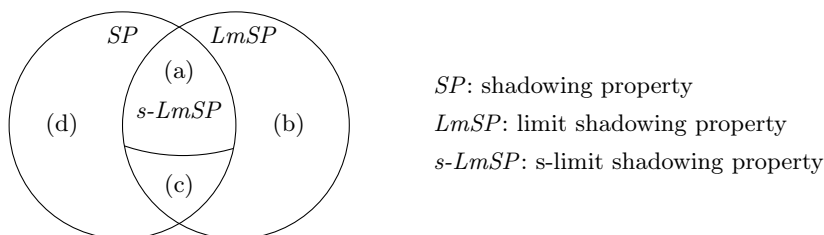
Fig. 3. Shadowing properties

SP: shadowing property

LmSP: limit shadowing property

s-LmSP: s-limit shadowing property

The map in Figure 4(a) has exactly one repelling fixed point (at ± 1) and one attracting fixed point (at 0). They are alternating and thus this circle homeomorphism has *SP* and *LmSP*. It trivially has *s-LmSP* as any point other than ± 1 is attracted to 0, and therefore for $0 < \varepsilon < 1/2$ any orbit that ε-shadows an asymptotic δ-pseudo-orbit (where $\delta = \delta(\varepsilon)$ is implied by *SP*) in fact asymptotically shadows it.

The map in Figure 4(b) has a repelling fixed point at ± 1 and so it has *LmSP*. But it does not have *SP* because the fixed point at 0 is non-hyperbolic.

The graph in Figure 4(c) is an extension to $[-1, 1]$ of the map given by Barwell, Good, and Oprocha [4, Example 3.5]. Our map is given by

$$\Phi(x) = \begin{cases} x + \frac{1}{2\pi\sqrt{2}} x \sin(2\pi \ln|x|) & \text{if } x \in [-1, 0), \\ x^3 & \text{if } x \in [0, 1]. \end{cases}$$

One readily checks that this is indeed a strictly increasing map on $[-1, 1]$. It has a sequence of fixed points converging to the fixed point at 0 and each of them is hyperbolic (by convention, 0 is also hyperbolic as the limit of hyperbolic fixed points). The repelling and attracting fixed points alternate, thus this homeomorphism (call it ϕ) has both *SP* and *LmSP*. However, it does not have *s-LmSP*, and the argument is essentially the one in [4]. We present it below for completeness.

For $\varepsilon = 1/2$ and any small $\delta > 0$ one can find a fixed point $x_\delta \in (0, \delta)$ and an integer $N \in \mathbb{N}$ large enough such that

$$\langle 1/2, \phi(1/2), \ldots, \phi^N(1/2), 0, x_\delta, x_\delta, \ldots \rangle$$

is an asymptotic δ-pseudo-orbit. Any point that could potentially ε-shadow this pseudo-orbit would have to be in $(0, 1)$. The orbit of such a point would eventually converge to 0 and would not asymptotically shadow the pseudo-orbit above.

We have already said that Pilyugin ruled out the possibility of a circle homeomorphism in the region marked (d) in Figure 3. If one drops the bijectivity requirement and asks only for continuity, the answer seems to be unknown. Until recently it was unknown whether such a system exists on any compact metric space.
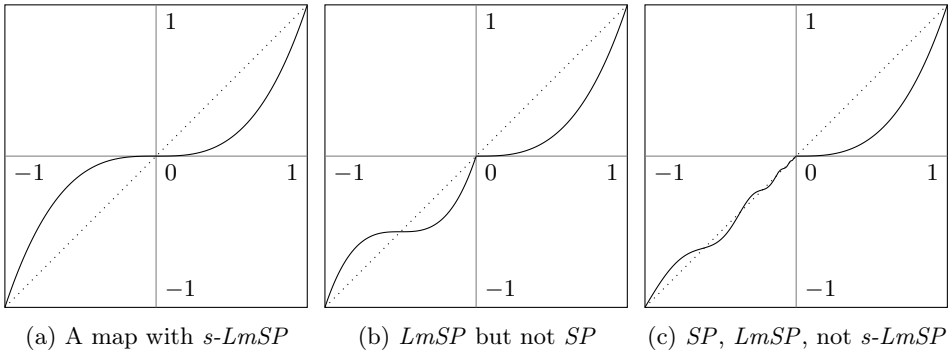
(a) A map with *s-LmSP* (b) *LmSP* but not *SP* (c) *SP*, *LmSP*, not *s-LmSP*

Fig. 4. Circle homeomorphisms with various shadowing properties

To see why this might be interesting, recall that any ω-limit set is also an ICT set. As any ICT set can be traced in the limit by an asymptotic pseudo-orbit, for systems with *LmSP*, ICT sets and ω-limit sets coincide. If *SP* necessarily implied *LmSP* then this would give a positive resolution to a problem posed by Barwell, Davies, and Good [2, Conjectures 1.2 and 1.3]. They asked if $\mathrm{ICT}(f) = \omega_f$ for every tent map (or more generally any dynamical system on a compact metric space) with *SP*.

The answer to [2, Conjecture 1.2] was given by Meddaugh and Raines [25]. They showed that the set $\mathrm{ICT}(f)$ must be closed in 2^X, the hyperspace of closed non-empty subsets of X furnished with the Hausdorff distance. Assuming *SP* they further showed that $\mathrm{ICT}(f)$ is the closure of ω_f:

$$\overline{\omega_f} = \mathrm{ICT}(f).$$

When combined with a result by Blokh, Bruckner, Humke, and Smítal [12] that ω_f is already closed for any continuous map $f\colon [0,1] \to [0,1]$, this yields the positive answer for all interval maps.

Their resolution did not, however, resolve whether *SP* implies *LmSP* even for interval maps. Indeed, we believe that this question is still open.

QUESTION 3.1. Is it true that for interval maps, *SP* implies *LmSP*?

Surprisingly, the answer to [2, Conjecture 1.3] is negative, as we now show. We shall construct a system (Theorem 3.3) on the Cantor set with *SP* but for which $\mathrm{ICT}(f) \neq \omega_f$. This in particular implies that this system does not have *LmSP* and thus lies in region (d). Independently, Gareth Davies found a similar example which remains unpublished.

3.1. A system in $SP \cap (LmSP)^c$. Let $\mathcal{A}$ be an alphabet, i.e. a finite discrete set of symbols. Recall that a *word* over $\mathcal{A}$ is a finite sequence of elements of $\mathcal{A}$. If one can find a finite collection W of words such that a shift space X is precisely the set of sequences in which no word from W, then the shift space is said to be of *finite type*. Walters [34] showed that

the shifts of finite type are precisely those shift spaces with the shadowing property.

PROPOSITION 3.2 (Walters [34]). *A shift space over a finite alphabet is of finite type if and only if it has shadowing.*

We can now proceed with the construction. We fix $\mathcal{A} = \{0, 1\}$ and we let Σ_2 be the full one-sided shift over $\mathcal{A}$. Now for each $k \in \mathbb{N}_0$ we set

$X_k = \{\xi \in \Sigma_2 \,|\, \text{any two 1s in } \xi \text{ are separated by at least } (k+1) \text{ 0s}\},$

$X_\infty = \{\xi \in \Sigma_2 \,|\, \xi \text{ has at most one symbol } 1\}.$

Note that each X_k is in fact a shift space of finite type where the set of forbidden words is exactly

$$\{1\underbrace{0 \ldots 0}_{l \text{ zeros}} 1 \,|\, 0 \leq l \leq k\} = \{11, 101, 1001, \ldots, 1\underbrace{0 \ldots 0}_{k \text{ zeros}} 1\}.$$

We also set $N = \{1/2^k \,|\, k \in \mathbb{N} \cup \{0, \infty\}\}$ where $1/2^\infty = 0$ by convention. The topology on N is taken to be inherited from the real line, and the observant reader might realise that N and X_∞ are in fact homeomorphic. The space $N \times \Sigma_2$ is a compact metric space equipped with the max-distance

$$d((a_1, \xi_1), (a_2, \xi_2)) = \max\{|a_1 - a_2|, d_{\Sigma_2}(\xi_1, \xi_2)\},$$

where d_{Σ_2} is the standard metric on the full shift space Σ_2.

On $N \times \Sigma_2$ we define a continuous map f as the product of the identity on N and the shift map σ on Σ_2:

$$f(a, \xi) = (a, \sigma(\xi)).$$

This is easily seen to be continuous. Finally, we take

$X = \big\{(a, \xi) \in N \times \Sigma_2 \,\big|\, a = 2^{-k} \text{ and } \xi \in X_k, \text{ for some } k \in \mathbb{N} \cup \{0, \infty\}\big\},$

or equivalently

$$X = \{0\} \times X_\infty \cup \bigcup_{k=0}^{\infty} \{1/2^k\} \times X_k.$$

This is clearly an f-invariant subset of $N \times \Sigma_2$. Below we show that X is also closed and that f restricted to X provides the counter-example we have been looking for.

THEOREM 3.3. (X, f) *is a dynamical system on a compact metric space which exhibits shadowing but for which* $\omega_f \neq \mathrm{ICT}(f)$.

Let us briefly describe the idea behind the construction. The map f on each $\{1/2^k\} \times X_k$ is conjugate to a shift of finite type and hence, by Proposition 3.2, f has shadowing on those subspaces. The space $\{0\} \times X_\infty$ on the other hand is not of finite type and does not have shadowing. In the construction we exploit the fact that the sequence $\langle \{1/2^k\} \times X_k \rangle_{k \in \mathbb{N}_0}$ of

spaces converges to $\{0\} \times X_\infty$ in 2^X as $k \to \infty$. This allows us to shadow pseudo-orbits in $\{0\} \times X_\infty$ using real orbits in $\{1/2^k\} \times X_k$ for k large enough. In this way we succeed (Lemma 3.5) to impose shadowing on f in the whole space by having shadowing on a family of proper subspaces approximating X.

It remains to check that $\omega_f \neq \mathrm{ICT}(f)$ for this system. The counter-example is the set $\{0\} \times X_\infty$ which is not the ω-limit set of any point in X (Theorem 3.3). Yet, it is the limit of the sequence of subspaces $\{1/2^k\} \times X_k$ as $k \to \infty$, each of which is the ω-limit set of a point in X. This, when combined with the result of Meddaugh and Raines, implies that $\{0\} \times X_\infty$ is an ICT set but not an ω-limit set.

We shall now proceed to prove these claims.

LEMMA 3.4. *X is a closed and hence a compact subset of $N \times \Sigma_2$.*

Proof. Let $(a, \xi) \in N \times \Sigma_2 \setminus X$. If $a = 1/2^k > 0$, this means that $\xi \in \Sigma_2 \setminus X_k$. Since X_k is closed in Σ_2, there is an open set V around ξ that does not intersect X_k. Since $U = \{a\}$ is an open (and closed) set in N, $U \times V$ is an open neighbourhood containing (a, ξ) that does not intersect X.

If $a = 0$ and $\xi \notin X_\infty$ then there exists a $k \in \mathbb{N}_0$ such that the word $10 \ldots 01$ (with k zeros) occurs somewhere in ξ. Take V to be the set of all 0-1 sequences in Σ_2 which have this word at the same position as ξ does. This set is easily seen to be clopen. It is indeed what is called a cylinder set in Σ_2 (see e.g. [24]). Setting $U = [0, 1/2^k) \cap N$ one readily checks that $U \times V$ is an open neighbourhood containing (a, ξ) but not intersecting X. ∎

LEMMA 3.5. *(X, f) has shadowing.*

Proof. Let $\varepsilon > 0$ and additionally assume $\varepsilon < 1$. Choose a $k \in \mathbb{N}$ so that $\varepsilon/2 \leq 1/2^k < \varepsilon$. Set $\delta = \min\{\varepsilon/4, \delta_1(\varepsilon), \ldots, \delta_k(\varepsilon)\} > 0$, where each $\delta_j(\varepsilon)$ for $1 \leq j \leq k$ is a positive number chosen so that every $\delta_j(\varepsilon)$-pseudo-orbit in X_j is ε-shadowed. This can be done by Proposition 3.2. We claim that for this δ, every δ-pseudo-orbit in X is ε-shadowed by a real orbit.

Let $\langle (a_n, \xi_n) \rangle_{n \in \mathbb{N}_0}$ be a δ-pseudo-orbit in X. We distinguish two cases.

CASE 1: $a_0 > \varepsilon/2$. If $a_1 > a_0$ then $a_1 \geq 2a_0$ and hence $|a_1 - a_0| > \varepsilon/2 > \delta$. On the other hand, if $a_1 < a_0$ then $2a_1 \leq a_0$ and hence $|a_0 - a_1| \geq a_0/2 > \varepsilon/4 \geq \delta$. Therefore, $a_1 = a_0$ and inductively $a_n = a_0$ for all $n \in \mathbb{N}$. This means that the whole pseudo-orbit is actually contained in the same subspace $\{1/2^m\} \times X_m$ where $a_0 = 1/2^m$.

Clearly $m \leq k$. Since $\delta \leq \delta_m(\varepsilon)$, we find that $\langle \xi_n \rangle_{n \in \mathbb{N}_0}$ is a $\delta_m(\varepsilon)$-pseudo-orbit in X_m, hence we can choose a point ξ^* that ε-shadows it. But then (a_0, ξ^*) clearly ε-shadows the initial pseudo-orbit.

CASE 2: $a_0 \leq \varepsilon/2$. A similar argument to the one above shows that $a_n \leq \varepsilon/2$, and hence $a_n \leq 1/2^k$ for all $n \in \mathbb{N}$. Since $(X_n)_{n \in \mathbb{N}_0}$ is a decreasing

sequence of sets, each ξ_n is in X_k. The sequence $\langle \xi_n \rangle_{n \in \mathbb{N}_0}$ is a $\delta_k(\varepsilon)$-pseudo-orbit in X_k, hence there exists a point ξ^* that ε-shadows it. Again, it is readily checked that $(1/2^k, \xi^*)$ ε-shadows the initial pseudo-orbit. ∎

Proof of Theorem 3.3. It suffices to show that $\{0\} \times X_\infty$ is an ICT set that is not the ω-limit set of any point of X. If it were the ω-limit set of some $(a, \xi) \in X$, then necessarily $a = 0$. But it is not hard to see that the ω-limit set of any point in $\{0\} \times X_\infty$ is the singleton $\{(0, 0^\infty)\}$ as each point in $\{0\} \times X_\infty$ is eventually mapped to the fixed point $(0, 0^\infty)$. Here we denote by 0^∞ the sequence in Σ_2 consisting only of zeros. Therefore $\{0\} \times X_\infty$ is not in ω_f.

It remains to show that $\{0\} \times X_\infty$ is in $\mathrm{ICT}(f)$. To simplify notation we shall instead show that X_∞ is ICT under the shift map σ. This is clearly an equivalent statement. Let $\delta > 0$ and let $\xi, \eta \in X_\infty$. We can always choose $k \in \mathbb{N}$ such that $\sigma^k(\xi) = 0^\infty$. If $\eta = 0^\infty$ we are done, since

$$\langle \xi, \sigma(\xi), \ldots, \sigma^k(\xi) = \eta \rangle$$

is a δ-pseudo-orbit between ξ and η.

If otherwise $\eta = 0^m 10^\infty$ for some $m \in \mathbb{N}_0$, choose $n > m$ so large that $\zeta = 0^n 10^\infty$ is δ-close to 0^∞. Then

$$\langle \xi, \sigma(\xi), \ldots, \sigma^k(\xi) = 0^\infty, \zeta, \sigma(\zeta), \ldots, \sigma^{n-m}(\zeta) = \eta \rangle$$

is a δ-pseudo-orbit between ξ and η. ∎

The dynamical system (X, f) in Theorem 3.3 is not transitive but it is possible to modify it and produce a transitive system on the Cantor set with *SP* but not *LmSP*.

THEOREM 3.6. *There exists a topologically transitive dynamical system on a compact metric space which exhibits shadowing but for which $\omega_f \neq \mathrm{ICT}(f)$ and which, in particular, fails limit shadowing.*

Below we shall give a sketch of the proof but the details are left to the reader. We shall need the following lemma.

LEMMA 3.7. *Let (X, f) be a dynamical system and let $F_1 \subset F_2 \subset \cdots \subset X$ be an increasing sequence of closed f-invariant subsets converging to X, $\overline{\bigcup_{i=1}^\infty F_i} = X$. Further, assume that for each $n \in \mathbb{N}$ there exists a continuous map $\pi_n \colon X \to F_n$ which is:*

(1) *non-expanding, i.e. $d(\pi_n(x), \pi_n(y)) \leq d(x, y)$,*
(2) *commuting with f, i.e. $\pi_n \circ f = f \circ \pi_n$,*
(3) *a nearest point projection, i.e. $d(x, \pi_n(x)) = \min\{d(x, y) : y \in F_n\}$ (note that such a π_n is a retraction of X onto F_n).*

If $(F_n, f|_{F_n})$ has shadowing for each $n \in \mathbb{N}$ then so does (X, f).

Proof. Let $\varepsilon > 0$. Choose $n \in \mathbb{N}$ large enough that F_n and X are $\varepsilon/2$-close in Hausdorff distance on 2^X. As π_n is a nearest point projection, this implies that

$$(3.1) \qquad\qquad d(x, \pi_n(x)) \leq \varepsilon/2 \quad \text{for all } x \in X.$$

Now let $\delta = \delta_n(\varepsilon/2) > 0$ be provided by the shadowing property in $(F_n, f|_{F_n})$. We claim that this δ suffices.

To this end, let $\langle x_0, x_1, \dots \rangle$ be a δ-pseudo-orbit in X. The properties of π_n ensure that $\langle \pi_n(x_0), \pi_n(x_1), \dots \rangle$ is still a δ-pseudo-orbit in F_n. This pseudo-orbit can be $\varepsilon/2$-shadowed by a point $y \in F_n$. But now using (3.1) one easily checks that y ε-shadows the original pseudo-orbit $\langle x_0, x_1, \dots \rangle$. This completes the proof. ∎

This lemma can be seen as a generalisation of Lemma 3.5. Each of the sets $F_n = \bigcup_{k=0}^{n-1} \{1/2^k\} \times X_k$ is a disjoint union of n shifts of finite type and therefore has shadowing. It is not hard to check that the projections $\pi_n \colon (a, \xi) \mapsto (\max\{a, 1/2^{n-1}\}, \xi)$ have all the required properties.

Let us now represent the system from Theorem 3.3 in another way. Let $a_k = 1/3^{k+1}$ and $b_k = 1 - 1/3^{k+1}$ (any other sequences converging to 0 and 1 respectively would work equally well). For each $k \in \mathbb{N}_0$ the system $(\{1/2^k\} \times X_k, f|_{\{1/2^k\} \times X_k})$ is naturally isomorphic to a shift X_k of finite type, which in turn is isomorphic to a shift of finite type where all the occurrences of 0 are replaced by a_k, and all the occurrences of 1 are replaced by b_k. Now the disjoint union $F_n = \bigcup_{k=0}^{n-1} \{1/2^k\} \times X_k$ can be simply represented as a shift of finite type over $\{a_0, \dots, a_{n-1}; b_0, \dots, b_{n-1}\}$ where any word containing a_k/b_k and a_l/b_l with $k \neq l$ is forbidden, and furthermore any b_k must be preceded (if possible) and followed by at least $(k+1)$ a_ks. The set X is then the closure of the union $\bigcup_{n=1}^{\infty} F_n$ in the countable product $\{a_0, \dots; 0; b_0, \dots; 1\}^{\mathbb{N}}$ where the base set inherits the topology from $[0, 1]$. The dynamics is provided by the shift map σ.

So far we have only given a different description of the example from Theorem 3.3, the advantage being that we have disposed of the first coordinate and are now able to represent our system as a shift space, albeit over a countable alphabet. The next step is to make the system transitive. We introduce an extra symbol 2 and for each $n \in \mathbb{N}$ we let F_n be a shift of finite type over $\{2; a_0, \dots, a_{n-1}; b_0, \dots, b_{n-1}\}$ where as before any b_k must be preceded (if possible) and followed by at least $(k+1)$ a_ks, and any two symbols a_k and a_l with $k \neq l$ need to be separated by at least $(k+l)$ 2s. Now taking the closure of $\bigcup_{n=1}^{\infty} F_n$ in $\{2; a_0, \dots; 0; b_0, \dots; 1\}^{\mathbb{N}}$ gives a transitive example with shadowing where the ICT set coinciding with X_∞ is not an ω-limit set for the shift map. We leave the details to the reader.

In relation to these results, Good and Meddaugh [19] very recently obtained a characterisation of systems in which $\omega_f = \mathrm{ICT}(f)$. They show that

this is equivalent to another technical property named *orbital limit shadowing*, defined in their paper. Our example thus also shows that a system can exhibit shadowing without having orbital limit shadowing.

4. Shadowing on maximal w-limit sets. In Theorem 2.7 we saw that for tent maps, shadowing implies shadowing on the core. For $s \in (\sqrt{2}, 2]$ the core of a tent map f_s is its maximal ω-limit set. It is an ω-limit set as it is clearly an invariant set and the map f_s is transitive on its core (see e.g. [14, Remark 3.4.17]), hence there exists a point whose orbit is dense in the core. The core is a maximal ω-limit set as the trajectory of any point whose ω-limit set contains the core would eventually have to enter the interior of the core, and therefore its ω-limit set would after all have to be contained in the core.

The results we prove below can thus be seen as generalising the implication (2)$\Rightarrow$(5) of Theorem 2.7.

THEOREM 4.1. *Let $f \colon [0,1] \to [0,1]$ be an interval map with shadowing and assume that there are only finitely many maximal ω-limit sets for f. Then the restriction of f to any of these maximal ω-limit sets also has shadowing.*

Proof. Let $A_1, \ldots, A_k$ be the collection of all maximal ω-limit sets for f. By a result of Meddaugh and Raines [25, Corollary 6] we know that for a map with shadowing, ω-limit sets and internally chain transitive (ICT) sets coincide. It is clear that any two maximal ICT sets must be disjoint, as otherwise their union would again be an ICT set, contradicting maximality. One can also see this directly as follows.

Assume that $A_i \cap A_j \neq \emptyset$ for some $i \neq j$. Then there exists an asymptotic pseudo-orbit $\langle z_i \rangle_{i=1}^{\infty}$ whose set of accumulation points contains $A_i \cup A_j$. But then constructing appropriate periodic pseudo-orbits it is not hard to see that for every n there is a point y_n such that $B(\omega(y_n, f), 1/n) \supset A_i \cup A_j$. Since the space of all ω-limit sets of an interval map is closed by a result of Blokh, Bruckner, Humke, and Smítal [12], and since the hyperspace 2^X is compact, this would imply that there exists an ω-limit set $A \supset A_i \cup A_j$, which is a contradiction. Thus indeed $A_i \cap A_j = \emptyset$ for $i \neq j$.

Denote $\eta = \min_{i \neq j} \text{dist}(A_i, A_j)/2 > 0$, fix any $\varepsilon > 0$ and let $\delta = \delta(\hat{\varepsilon})$ be provided by the shadowing property of f for $\hat{\varepsilon} = \min\{\varepsilon/2, \eta\}$. Fix any A_i and any finite δ-pseudo-orbit $\langle x_0, \ldots, x_n \rangle \subset A_i$. Since A_i is an ω-limit set, it is internally chain transitive and therefore there are $x_{n+1}, \ldots, x_{m-1} \in A_i$ such that the sequence $\langle x_0, \ldots, x_{m-1}, x_0 \rangle$ is a (periodic) δ-pseudo-orbit. Set $y_i = x_{(i \bmod m)}$ for all $i \geq 0$ and let z be a point that $\varepsilon/2$-traces the δ-pseudo-orbit $\langle y_i \rangle_{i=0}^{\infty}$. The set $\omega(z, f)$ must be contained in some maximal ω-limit set, and from our choice of η it is not hard to see that this must be A_i.

Now let $q \in \omega(z, f) \subset A_i$ be any limit point of $\langle f^{nm}(z)\rangle_{n=0}^{\infty}$. It is readily checked that q ε-traces the pseudo-orbit $\langle y_i \rangle_{i=0}^{\infty}$ and in particular the first portion of it, $\langle x_0, \ldots, x_n \rangle$. This shows that any finite δ-pseudo-orbit consisting of points in A_i can be ε-traced by a point from A_i. This shows that $f|_{A_i}$ has shadowing, which completes the proof. ∎

In [11, Theorem 5.4] Blokh proved that for interval maps the set $\omega(f) = \bigcup_{x \in X} \omega(x, f)$ has a spectral decomposition into a family of maximal ω-limit sets:

$$\omega(f) = X_f \cup \left(\bigcup_\alpha S_\omega^{(\alpha)} \right) \cup \left(\bigcup_i B_i \right)$$

where X_f is a collection of periodic orbits that are also maximal ω-limit set (sets of genus 0 in Blokh's notation); each of $S_\omega^{(\alpha)}$s is a solenoidal limit set and none of them contains a periodic orbit (genus 1); finally each B_i is a so-called basic set, an infinite maximal limit set containing a periodic orbit (genus 2). It turns out [11, Theorem PM6] (also [10]) that for piecewise linear maps with a constant slope $s > 1$ there are no sets of genus 1 in this decomposition, and the number of maximal ω-limit sets of genus 0 and 2 is finite. Combining this with the previous theorem we obtain the following result.

COROLLARY 4.2. *Let $f \colon [0, 1] \to [0, 1]$ be a piecewise linear map with constant slope $s > 1$ and with shadowing. Then the restriction of f to any of its maximal ω-limit sets has shadowing.*

REMARK 4.3. The assumption of maximality here is crucial as the ICT set $W = \bigcup_{i=0}^{\infty} \{1/2^i\} \cup \{0\}$ is an ω-limit set for the full tent map f_2. Note that f_2 has shadowing but $f_2|_W$ does not.

THEOREM 4.4. *Let $f \colon [0, 1] \to [0, 1]$ be a piecewise monotone map with shadowing. Then the restriction of f to any of its ω-limit sets with non-empty interior has s-limit shadowing.*

Proof. We first observe that any such limit set $\omega(x)$ must in fact be a cycle of disjoint closed intervals (see e.g. [8, IV, Lemma 5]). Namely, $\omega(x) = \bigcup_{i=0}^{p-1} J_i$ where $p \in \mathbb{N}$, the sets J_i are p pairwise disjoint segments contained in $[0, 1]$, and $f(J_i) = J_{(i+1 \bmod p)}$. Furthermore, $f^p|_{J_0} \colon J_0 \to J_0$ is transitive (as it has a dense orbit) and also piecewise monotone. The result will now follow from Corollary 2.4 if we could only show that $f^p|_{J_0}$ has shadowing. We remind the reader that both shadowing and s-limit shadowing hold for a power f^p of the map f if and only if they hold for the map itself.

Let $\varepsilon > 0$ and without loss of generality assume that $\varepsilon < \operatorname{diam}(J_0)/2$. Let $\delta = \delta(\varepsilon) > 0$ be provided by the shadowing property of f^p over $[0, 1]$. We claim that this δ will suffice. To see this, take a finite δ-pseudo-orbit $\langle x_0, \ldots, x_n \rangle$ contained in J_0. As f^p is transitive on J_0, we can extend this

pseudo-orbit to a longer δ-pseudo-orbit $\langle x_{-m}, x_{-m+1}, \ldots, x_0, \ldots, x_n \rangle$ where $x_{-m} \in J_0$ denotes the midpoint of J_0. Let $z \in I$ be a point that ε-traces this pseudo-orbit. Since $d(z, x_{-m}) < \varepsilon$, it must be that $z \in J_0$. Therefore the f^p-orbit of z is actually contained in J_0 and the point $f^{mp}(z)$ ε-traces $\langle x_0, \ldots, x_n \rangle$ under the action of f^p. This completes the proof. ∎

REMARK 4.5. The converse to Theorem 4.4 does not hold, as the double tent map $f_{2,2} \colon [-1, 1] \to [-1, 1]$ shows. It is defined as a symmetric extension of the standard tent map f_2:

$$f_{2,2}(x) = \begin{cases} f_2(x) & \text{if } x \in [0, 1], \\ -f_2(-x) & \text{if } x \in [-1, 0). \end{cases}$$

This map is piecewise monotone with constant slope $s = 2$ and does not have shadowing as the set of critical points $\{-1, -1/2, 1/2, 1\}$ is not linked (1 is mapped to a fixed point 0). The restrictions $f_{2,2}|_{[0,1]}$ and $f_{2,2}|_{[-1,0]}$ to its two maximal ω-limit sets both have linking (as 0 is now also a critical point) and thus by Theorem 2.2 both have s-limit shadowing.

5. Inverse limit space

THEOREM 5.1. *Let X be a compact metric space and $f \colon X \to X$ a continuous onto map. Let σ be the shift map on $\varprojlim (X, f)$. Then:*

(1) *σ has shadowing if and only if f has shadowing,*
(2) *σ has limit shadowing if and only if f has limit shadowing,*
(3) *σ has s-limit shadowing if and only if f has s-limit shadowing.*

Proof. The first equivalence was proven by Chen and Li [16, Theorem 1.3], the second by Gu and Sheng [20, Theorem 3.2], and we shall now see that the third one also holds. While the idea is clear, one needs to be careful with indexing everything properly.

First we will show that if σ has s-limit shadowing then f has s-limit shadowing. Without loss of generality we may assume that $\text{diam}(X) = 1$. Fix any $\varepsilon > 0$ and let $\delta > 0$ be assigned to $\varepsilon/2$ by s-limit shadowing of σ. Let N be such that $\sum_{i=N}^{\infty} 2^{-i} < \delta/2$ and let $\gamma \in (0, \varepsilon/(4N))$ be such that if $x, y \in X$ satisfy $d(x, y) < \gamma$ then $d(f^i(x), f^i(y)) < \min\{\delta/4, \varepsilon/(4N)\}$ for $i = 0, \ldots, N$. We claim that this γ suffices.

Fix any asymptotic γ-pseudo-orbit $\langle x_n \rangle_{n=0}^{\infty}$ for f. There exists a strictly increasing sequence $\langle u_k \rangle_{k=1}^{\infty}$ of positive integers such that for all $n \geq u_k - N$ we have $d(f^{i+1}(x_n), f^i(x_{n+1})) < \gamma/(k+1)^2$ for $i = 0, \ldots, k + N - 1$. For technical reasons we put $u_0 = 0$. Define a sequence $\langle z_n \rangle_{n=0}^{\infty} \subset X$ by taking any $y \in f^{-N}(x_0)$, putting $x_{-i} = f^{N-i}(y)$ for $0 \leq i \leq N$, and next

$$z_n = \begin{cases} x_{n-N} & \text{if } n \leq u_1, \\ f^k(x_{n-k-N}) & \text{if } u_k + k \leq n \leq u_{k+1} + k. \end{cases}$$

We define $\langle y^{(i)} \rangle_{i=0}^{\infty} \subset \varprojlim(X,f)$ by the following rule. For $n \leq N$ and $i \geq 0$ we put $y_n^{(i)} = f^{N-n}(z_i)$. For $n > N$ we distinguish two cases. If $i \in [u_k + k, u_{k+1} + k]$ for some $k \geq 0$ and if $n \leq N + k$, then we put $y_n^{(i)} = f^{k-n+N}(x_{i-k-N})$; in the other case we take any $y_n^{(i)} \in f^{-1}(y_{n-1}^{(i)})$, which is possible since f is onto. Note that if $n \leq N + k$ then we have already defined $y_N^{(i)} = z_i = f^k(x_{i-k-N})$, and we also put $y_{N+1}^{(i)} = f^{k-1}(x_{i-k-N})$ and therefore $f(y_{n+1}^{(i)}) = y_n^{(i)}$ for every $n \geq 0$, so with this definition indeed $y^{(i)} \in \varprojlim(X,f)$. We claim that the sequence $y^{(i)}$ so constructed is an asymptotic δ-pseudo-orbit for σ.

Firstly, we claim that $\langle z_n \rangle_{n=0}^{\infty} \subset X$ is an asymptotic γ-pseudo-orbit. Fix $n \in \mathbb{N}$ and assume that both n and $n+1$ belong to $[u_k + k, u_{k+1} + k]$. Then

$$d(f(z_n), z_{n+1}) = d(f^{k+1}(x_{n-k-N}), f^k(x_{n-k-N+1})) < \gamma/(k+1)^2.$$

If otherwise $n = u_{k+1} + k$, then

$$d(f(z_n), z_{n+1}) = d(f^{k+1}(x_{n-k-N}), f^{k+1}(x_{n-k-N})) = 0.$$

Thus, the claim holds.

By the choice of γ for $n \leq N$ we have

$$d(f(y_n^{(i)}), y_n^{(i+1)}) = d(f^{N-n}(f(z_i)), f^{N-n}(z_{i+1})) < \delta/4,$$

and so

$$d(\sigma(y^{(i)}), y^{(i+1)}) = \sum_{n=0}^{\infty} 2^{-n} d(f(y_n^{(i)}), y_n^{(i+1)}) \leq \sum_{n=0}^{N} 2^{-n-2}\delta + \sum_{n=N+1}^{\infty} 2^{-n}$$
$$< \delta/2 + \delta/2,$$

showing that $y^{(i)}$ is indeed a δ-pseudo-orbit for σ. To show that it is an asymptotic pseudo-orbit it suffices to show that $\langle y_n^{(i)} \rangle_{i=0}^{\infty}$ is an asymptotic pseudo-orbit for any sufficiently large n. To this end, fix any $n > N$. Then if k is such that $k \geq n - N$, then for all $i \in [u_k + k, u_{k+1} + k)$ we have $i - k - N \geq u_k - N$, and so

$$d(f(y_n^{(i)}), y_n^{(i+1)}) = d\big(f^{k-n+N}(f(x_{i-k-N})), f^{k-n+N}(x_{i-k-N+1})\big) < \frac{\gamma}{(k+1)^2},$$

while if $i = u_{k+1} + k$ then

$$d(f(y_n^{(i)}), y_n^{(i+1)}) = d\big(f^{k-n+N}(f(x_{i-k-N})), f^{k-n+N+1}(x_{i-k-N})\big) = 0.$$

This immediately implies that $\langle y^{(i)} \rangle_{i=0}^{\infty}$ is an asymptotic pseudo-orbit for σ.

Now, let $p \in \varprojlim(X,f)$ be a point asymptotically $\varepsilon/2$-tracing a pseudo-orbit $\langle y^{(i)} \rangle_{i=0}^{\infty} \subset \varprojlim(X,f)$. Then by the definition of the metric in $\varprojlim(X,f)$ the coordinate $p_0 \in X$ of p asymptotically $\varepsilon/2$-traces $\langle y_0^{(i)} \rangle_{i=0}^{\infty} \subset X$. Then,

by the definition of γ,

$$d(f^i(p_0), z_{i+N}) \le d(f^i(p_0), f^N(z_i)) + \sum_{j=0}^{N-1} d(f^{N-j}(z_{i+j}), f^{N-j-1}(z_{i+j+1}))$$

$$\le d(f^i(p_0), y_0^{(i)}) + N\frac{\varepsilon}{4N} < 3\varepsilon/4.$$

The same calculation also yields $\lim_{i\to\infty} d(f^i(p_0), z_{i+N}) = 0$. Furthermore, if $i \in [u_k + k - N, u_{k+1} + k - N]$ then $z_{i+N} = f^k(x_{i-k})$ and so

$$d(x_i, z_{i+N}) = d(x_i, f^N(z_i)) = d(x_i, f^k(x_{i-k}))$$

$$\le \sum_{j=0}^{k-1} d(f^j(x_{i-j}), f^{j+1}(x_{i-j-1}))$$

$$\le k\frac{\gamma}{(k+1)^2} \le \frac{\gamma}{k+1} < \varepsilon/4.$$

The same calculation, in particular the bound $\frac{\gamma}{k+1}$, yields $\lim_{i\to\infty} d(x_i, z_{i+N}) = 0$. Putting together these two calculations shows that p_0 asymptotically ε-traces the pseudo-orbit $\langle x_n \rangle_{n=0}^\infty \subset X$ and so the proof of the first implication is complete.

Next we prove that if f has s-limit shadowing then σ has s-limit shadowing. Fix any $\varepsilon > 0$ and let N be such that $\sum_{i=N}^\infty 2^{-i} < \varepsilon/2$. There is $\delta > 0$ such that if $d(x, y) < \delta$ then

$$d(f^i(x), f^i(y)) < \varepsilon/(2N) \quad \text{for } i = 0, \ldots, N.$$

There is $\gamma > 0$ such that if $\langle x_n \rangle_{n=0}^\infty$ is an asymptotic γ-pseudo-orbit then it is δ-traced. Let $\langle y^{(i)} \rangle_{i=0}^\infty \subset \varprojlim(X, f)$ be an asymptotic $2^{-N}\gamma$-pseudo-orbit. Then $\langle y_N^{(i)} \rangle_{i=0}^\infty$ is an asymptotic γ-pseudo-orbit, so let $z \in X$ be a point which asymptotically δ-traces it. Define $q \in \varprojlim(X, f)$ by putting $q_i = f^{N-i}(z)$ for $0 \le i \le N$, and for $i > N$ let q_i be any point in $f^{-1}(q_{i-1})$. We claim that q asymptotically ε-traces $\langle y^{(i)} \rangle_{i=0}^\infty$. Note that ε-tracing is almost obvious, because for $n \le N$ we have

$$d(y_n^{(i)}, f^i(q_n)) = d(f^{N-n}(y_N^{(i)}), f^{N-n}(f^i(q_N))) < \frac{\varepsilon}{2N}$$

and therefore

$$d(y^{(i)}, \sigma^i(q)) \le \varepsilon/2 + \sum_{n=0}^N d(y_n^{(i)}, f^i(q_n)) < \varepsilon.$$

By the same argument $\lim_{i\to\infty} d(y_n^{(i)}, f^i(q_n)) = 0$ for every $n \le N$. But if we

fix any $n > N$ then for $i \geq n - N$ we have

$$d(y_n^{(i)}, f^i(q_n)) = d(y_n^{(i)}, f^{i-(n-N)}(q_N))$$

$$\leq \sum_{j=1}^{n-N} d(y_{n-(j-1)}^{(i-(j-1))}, y_{n-j}^{(i-j)}) + d(y_N^{(i-(n-N))}, f^{i-(n-N)}(q_N))$$

$$= \sum_{j=1}^{n-N} d(y_{n-j+1}^{(i-j+1)}, f(y_{n-j+1}^{(i-j)})) + d(y_N^{(i-(n-N))}, f^{i-(n-N)}(q_N))$$

and so also in this case $\lim_{i \to \infty} d(y_n^{(i)}, f^i(q_n)) = 0$, where we use the fact that $z = q_N$ asymptotically traces $\langle y_N^{(i)} \rangle_{i=0}^\infty$, and $\langle y_k^{(i)} \rangle_{i=0}^\infty$ is an asymptotic pseudo-orbit for each $k \geq 0$. Thus, q asymptotically traces $\langle y^{(i)} \rangle_{i=0}^\infty$, which completes the proof. ∎

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