



On a variant of Flory model

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ABSTRACT

We consider a one-dimensional variant of a recently introduced settlement planning problem in which houses can be built on finite portions of the rectangular integer lattice subject to certain requirements on the amount of insolation they receive. In our model, each house occupies a unit square on a $1 \times n$ strip, with the restriction that at least one of the neighboring squares must be free. We are interested mostly in situations in which no further building is possible, i.e. in maximal configurations of houses in the strip. We reinterpret the problem as a problem of restricted packing of vertices in a path graph and then apply the transfer matrix method in order to compute the bivariate generating functions for the sequences enumerating all maximal configurations of a given length with respect to the number of houses. This allows us to determine the asymptotic behavior of the enumerating sequences and to compute some interesting statistics. Along the way, we establish close connections between our maximal configurations and several other types of combinatorial objects, including restricted permutations and walks on certain small oriented graphs. We then generalize our results in several directions by considering multi-story houses, by varying the insolation restrictions, and, finally, by considering strips of width 2 and 3. At the end we comment on several possible directions of future research.

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1. Introduction

Many problems of practical importance can be formulated in terms of packings. Intuitively, a packing is any arrangement of non-overlapping copies from a (usually finite) collection of objects \mathcal{P} within a prescribed part of a large(r) set \mathcal{E} . It often happens that both the large set and the objects being packed can be naturally endowed with the same type of discrete structure. In such cases, general packings can be successfully modeled by packings of graphs. If, for example, \mathcal{E} can be represented by a graph G and elements of \mathcal{P} by graphs H_1, \dots, H_k , then a \mathcal{P} -packing of G is a collection of vertex-disjoint subgraphs of G such that each of them is isomorphic to some H_i , $i = 1, \dots, k$. When \mathcal{P} consists of a single element H , one simply speaks of H -packings of G .

Clearly, one would expect that the difficulty of packing problems increases with the increase of the complexity of H . Indeed, the simplest non-trivial case, $H = K_2$, is well researched, while the results on larger H are much less abundant. This does not prevent graph packings from being a very versatile tool; even the simplest case of packing dimers ($H = K_2$) into a larger graph is one of most commonly used models in several areas of physics and chemistry. It suffices to mention the

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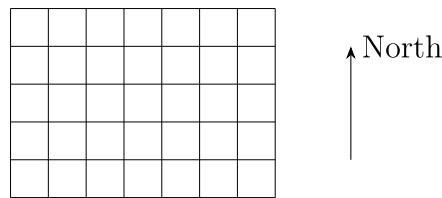


Fig. 1. An example of a tract of land ($m = 5$, $n = 7$).

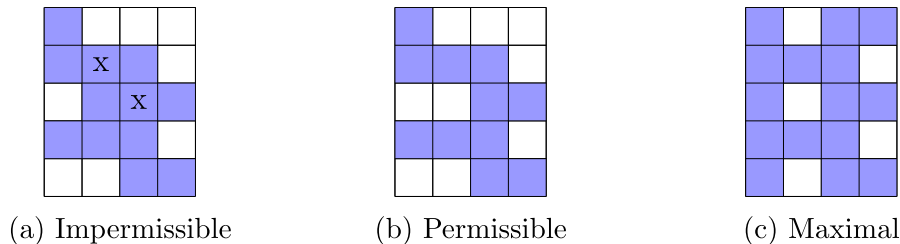


Fig. 2. Examples of impermissible, permissible and maximal configuration on a 5×4 tract of land. 'x' marks a house blocked from the sunlight.

Ising model of magnetic materials and the concept of the topological resonant energy, crucial for stability of conjugated molecules. Both models employ perfect matchings, i.e. packings of dimers covering all vertices of the underlying graph. For a very brief introduction to both topics we refer the reader to [22, §8.7] and references therein. For some recent results on packing larger H see, for example, [7,30].

In this paper we look at a problem which can be modeled by packing even simpler graphs, the copies of K_1 , into finite portions of regular rectangular lattice. Without further restrictions this problem would be trivial, but our problem imposes restrictions that arise quite naturally in the context of settlement growth and planning. It turns out that with those restrictions even packing the simplest possible graphs, K_1 , into finite pieces of the square lattice gives rise to very interesting behavior and exhibits often surprising relations with several other classes of combinatorial objects. (Another non-trivial problem which can be reduced to restricted packings of trivial graph K_1 is the problem of finding a large independent set in a given graph.)

All the aforementioned packing problems can be studied in a static or in a dynamic variant. In the present work we focus only on the static models with the aim to enumerate all configurations that arise in such models and that satisfy certain additional requirements. The study of the dynamic variant of the same models aims to find the distribution of configurations constructed by a random process in which the pieces from \mathcal{P} arrive sequentially and are placed randomly onto available locations in \mathcal{E} until saturation. These kinds of models that evolve over time are extensively studied, see [19, §7] for introduction, and [14–16,18] for some recent results in this direction. The efforts to extend our results in this direction are currently underway.

In [25], three of the present authors introduced the following settlement model. A rectangular $m \times n$ tract of land, with sides oriented north-south and east-west, is divided into mn unit squares, see Fig. 1. Each square lot can be either occupied (by a house) or left vacant. An arrangement of houses on such a tract of land is called a *configuration* and can be encoded as an $m \times n$ matrix C , where $c_{i,j} = 1$ if the lot (i, j) is occupied, and $c_{i,j} = 0$ otherwise.

A configuration C is *permissible* if no occupied lot (i, j) borders simultaneously with three other occupied lots to its east, south and west – in other words – the house on the position (i, j) receives the sunlight during at least one part of the day (be it in the morning from the east, or during the midday from the south, or in the evening from the west).

As is the case with other packings, one is, naturally, interested in *large* permissible configurations, since the small ones tend to be trivial and easy to construct. One way of being large is to have the largest possible number of occupied lots, hence the largest possible size. We call such configurations *maximum* configurations (in [25,26] these were called *efficient*). Another, more interesting, way of being large is in the sense of set inclusion. A permissible configuration C is *maximal* if no additional houses can be added to the configuration without rendering it impermissible, see Fig. 2. Unlike the maximum configurations, the maximal ones usually come in a range of different sizes, and it is of interest to know the exact distribution of sizes. Moreover, here also the smallest such configurations are interesting (in [25,26] these were called *inefficient*), as they describe either the worst possible outcome if we are interested in packing as many elements as possible, or the best possible outcome if we are trying to satisfy certain needs by the smallest possible number of packed objects.

The authors in [25] found maximal configurations with the lowest *occupancy* (the number of houses in a configuration) among all the maximal configurations on an $m \times n$ grid. They also obtained bounds on the highest occupancy possible. A natural next step would be to find the total number of all maximal configurations for a given grid and to refine the

enumeration by the number of occupied lots. The problem seems to be too hard in the general $m \times n$ case. Hence, in this paper we consider its restriction to the one-dimensional case $1 \times n$ to which we can apply the transfer matrix method. This method allows us to obtain a complete solution of the one-dimensional case by computing and analyzing the bivariate generating functions for the corresponding enumerating sequences. Our results could be, in principle, generalized to larger grids; indeed, for grids of size $2 \times n$ and $3 \times n$ we derive (bivariate) generating functions counting the number of maximal configurations by the same transfer matrix method used in the $1 \times n$ case. However, the calculations get increasingly infeasible for larger strips and we decided not to pursue it beyond $m = 3$.

The transfer matrix method, see [29, §4.7] or [8, §V], and also [21, §2–4], is a well known method for counting words of a regular language. Applicability of this method to our setting relies on the fact that permissibility as well as maximality of a configuration can be verified by inspecting only finite size patches of a given configuration. The limitation, however, is that the method deals with, essentially, one dimensional objects, so we first consider a modification of the settlement model on the $1 \times n$ grid. This modification, defined later in text, we call the Riviera model. It turns out that the Riviera model can be seen as a variant of Flory polymer model [9] which is in turn related to Page–Rényi parking process [11,24].

The main contribution of this paper is in computing the following statistics: the expected occupancy, i.e., the expected number of buildings among maximal configurations in a strip of length n , which is equal to $0.577203n$ (see Proposition 2.9); the expected efficiency of a maximal configuration (defined as the ratio of the number of buildings in the configuration and the maximum number of buildings that can appear in a maximal configuration), which is equal to 0.865804 (see Proposition 2.9); the expected length of a maximal configuration with k buildings, which is equal to $1.758283k$ (see Remark 2.10). Moreover, we provide a lower bound, an upper bound and the maximum of the complexity function (see Remark 2.11), which for large n is related to the number of configurations of length n with prescribed building density (fraction of occupied lots). For the two considered generalizations, namely multiple stories (see Remark 3.4) and two-dimensional grids of size $2 \times n$ (see Remark 4.1) and $3 \times n$ (see Remark 4.2) we also compute the above mentioned statistics.

The paper is organized as follows. In Section 2 we introduce the Riviera model. We find the bivariate generating function counting the number of maximal configurations of length n with precisely k houses ($n, k \in \mathbb{N}$). Furthermore, we relate the Riviera model with some other combinatorial objects that were already studied in the literature. In Section 3 we generalize the Riviera model introduced in Section 2 in the sense that we allow houses to have multiple stories. Additionally, we comment on the close relation between the Riviera model and the famous Flory model. In Section 4 we deal with configurations on $m \times n$ grids with $m = 2$ and $m = 3$. Finally, in Section 5 we recapitulate our findings and indicate several possible directions of future research. Some lengthy formulas are relegated to the Appendix in order to improve legibility.

A note on notation: Whenever a non-integer decimal is encountered in the text, its value should be interpreted as an approximation of the true value rounded to six decimal places. By $a_n \sim b_n$ (as $n \rightarrow \infty$) we mean $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$. In several places in the text we use the same name for different functions. Most prominently, the generating function for almost every model is denoted as $F(x)$, $F(x, y)$, or $F(x, y, z)$. This should not lead to any confusion, as it is always clear from the context to which function the text refers.

2. Riviera model

We introduce a 1D-modification of the above settlement planning model which ignores the possibility of obtaining sunlight from the south, but instead retains only the constraints pertaining to the east and west directions. As this is a model on a strip of land, it resembles a Mediterranean settlement along the coast (riviera), hence the name. The configuration of built houses is represented with a row vector¹ $C = (c_k)$, where $c_k = 1$ if the lot k is occupied and $c_k = 0$ otherwise. Similarly as before, a configuration is said to be permissible if every occupied lot has at least one neighboring lot unoccupied (except maybe for the first and the last lot which receive sunlight from the boundary) so that it is not blocked from the sunlight. Among permissible configurations, we are interested in the maximal ones, namely configurations such that any addition of a house on an unoccupied lot would result in an impermissible configuration.

The properties of maximality and permissibility are locally verifiable in a sense that, if one wants to check whether a state of a certain lot (occupied or unoccupied) has caused the configuration to be impermissible or not maximal, one only needs to check the situation on the lots in a certain finite radius of the observed lot, where that radius is uniform for each lot on the tract of land.

More precisely, to verify that a configuration is permissible, one needs to check that no occupied lot has both of its neighboring lots occupied as well. This can be done by inspecting all the length 3 substrings of a configuration. And to verify that a configuration is maximal, one needs to check, additionally, that no unoccupied lots can be built on. This can be done by observing the eastern two and western two lots around the unoccupied lot, i.e. by inspecting all the length 5 substrings of a configuration. In order to be able to verify this condition at the left and right boundaries of a configuration, we will identify the configuration with an infinite string obtained by adding infinitely many zeros to the left and right of the configuration (see also Remark 2.2). This procedure is often referred to as *padding* in computer science literature.

This property of local verifiability of a constraint which describes the model is a recurring motif throughout our analysis of related models in this paper.

¹ We write configurations as strings of 0's and 1's, and we refer to any consecutive sequence of letters in a configuration as a substring or a (sub)word in that configuration.

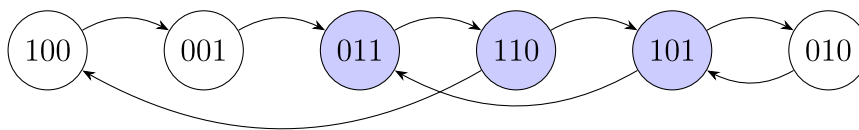


Fig. 3. Transfer digraph \mathcal{G}_R for the Riviera model. For example, a maximal configuration 110011010110 is represented with a walk: $110 \rightarrow 100 \rightarrow 001 \rightarrow 011 \rightarrow 110 \rightarrow 101 \rightarrow 010 \rightarrow 101 \rightarrow 011 \rightarrow 110$. Each walk must start and end at shaded nodes.

Lemma 2.1. Let $n \in \mathbb{N}$. A configuration $C \in \{0, 1\}^n$ in the Riviera model is maximal if and only if, when padded with zeros, it does not contain any of the following (decorated) substrings:

$$\underline{111}, \quad \underline{000}, \quad \underline{0100}, \quad \underline{0010}. \quad (2.1)$$

Remark 2.2. Throughout the paper, unless stated otherwise, we assume that the lots on the boundary can get sunlight from the boundary side, i.e. we assume that our configurations are padded with zeros. When inspecting whether a configuration $c_1 \dots c_n$ contains a decorated word $d_1 \dots d_k \dots d_l$, we check against a padded word $\dots 000c_1 \dots c_n 000 \dots$ but with the underlined letter of the decorated word aligned with c_i for $i = 1, \dots, n$. This is necessary as e.g. the configuration 10011 would otherwise be considered allowed (not containing any of the forbidden substrings), although it is not maximal.

Proof of Lemma 2.1. A configuration C is maximal if and only if it is permissible and for each $k = 1, \dots, n$ one has

$$c_k = 0 \implies (c_{k-1} = 1 \text{ and } c_{k+1} = 1) \text{ or } (c_{k-1} = 1 \text{ and } c_{k-2} = 1) \\ \text{or } (c_{k+1} = 1 \text{ and } c_{k+2} = 1).$$

The contrapositive of the above implication reads

$$(c_{k-1} = 0 \text{ and } c_{k+1} = 0) \text{ or } (c_{k-1} = 0 \text{ and } c_{k+2} = 0) \\ \text{or } (c_{k+1} = 0 \text{ and } c_{k+2} = 0) \implies c_k = 1. \quad (2.2)$$

From here we can compile the list of forbidden words. We include $\underline{111}$ to ensure permissibility. \square

Remark 2.3. An alternative approach for constructing the set of forbidden words, once we know that it suffices checking substrings of length 5, is to consider all 2^5 binary words of length 5 and, out of those, take words that do not appear in any finite maximal configuration to be the forbidden set of words. This approach is more amenable for use in a computer algorithm and we will make use of it later on.

Using this approach one would come up with the set of forbidden length 5 words

$$\{*\underline{111}*, \quad *\underline{000}*, \quad \underline{01000}, \underline{01001}, \quad \underline{00010}, \underline{10010}\},$$

which again can be reduced to $\{\underline{111}, \underline{000}, \underline{0010}, \underline{0100}\}$. Here, $*$ stands for any symbol, and e.g. the string $*\underline{111}*$ actually accounts for 4 different (decorated) words.

2.1. Counting maximal configurations

Upon examining Lemma 2.1, one sees that it is possible to encode each maximal configuration as a walk on the directed graph in Fig. 3 whose vertices represent all allowed substrings of length 3 and the directed edges represent allowed transitions (namely, transitions which comply with the condition stated in Lemma 2.1, see [21, §2.3] for more details on this construction. There is an edge from the word $u_1 u_2 u_3$ to $v_1 v_2 v_3$ if they *overlap progressively*, meaning that $u_2 u_3 = v_1 v_2$, and if the word $u_1 u_2 u_3 v_3 = u_1 v_1 v_2 v_3$ is not forbidden. (Our graph is therefore a subgraph of the 3-dimensional de Bruijn graph over symbols $\{0, 1\}$. Not all edges are present, since the transitions that correspond to forbidden 4 letter words must be deleted.) Thus, a transition simply represents the addition of a new lot to the right of the configuration, state of which is described with the last letter of the string of the target node.

Depending on the choice of boundary conditions, we are left with a constrained subset of vertices which may serve as a starting point or an ending point of the walk which encodes the configuration. The default boundary condition states that the first and the last lot in a configuration obtain sunlight from the boundary. One can easily check that in this situation, the only allowed starting and ending vertices are 110, 101, 011 (Otherwise, one would not obtain a maximal configuration from the walk). An alternative to this boundary condition is the setting where the first and the last lot do not obtain sunlight from the boundary. In this situation, the allowed starting vertices are 100, 011, 101, 010, while the allowed ending vertices are 001, 110, 101, 010. A third option which one may consider is the periodic boundary condition in which case, one simply searches for closed walks.

One can count the number of walks of fixed length n on the graph in Fig. 3 by examining the powers of the transfer matrix A associated with that graph:

$$\begin{matrix} & 100 & 001 & 011 & 110 & 101 & 010 \\ \begin{matrix} 100 \\ 001 \\ 011 \\ 110 \\ 101 \\ 010 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} & =: A. \end{matrix} \quad (2.3)$$

By definition of matrix multiplication, we have (see [29, page 385, above Theorem 3.21.12])

$$\begin{matrix} \text{\# of walks of length } n \\ \text{starting with the node } i \text{ and ending in the node } j \end{matrix} = [A^n]_{i,j}. \quad (2.4)$$

This gives us a neat way of counting the maximal configurations since

$$\begin{matrix} \text{\# of walks of length } n \\ \text{starting with 110 or 101 or 011} \\ \text{and ending with 110 or 101 or 011} \end{matrix} = \begin{matrix} \text{\# of maximal configurations} \\ \text{of length } n+3. \end{matrix} \quad (2.5)$$

This is due to the fact that each transition adds another lot to the configuration, which at the beginning of the walk had 3 lots. See Fig. 3 for an example of a walk of length 9 producing a configuration of length 12.

By introducing the vector $a = (0, 0, 1, 1, 1, 0)^T$, we have the following:

$$\text{\# of maximal configurations of length } n =: a_n = a^T \cdot A^{n-3} \cdot a, \quad n \geq 3. \quad (2.6)$$

Remark 2.4. A straightforward asymptotic formula for a_n can be obtained by calculating the Perron–Frobenius eigenvalue λ of the matrix A , namely the largest real eigenvalue of A . There exist a constant C such that

$$a_n \sim C \lambda^n, \text{ as } n \rightarrow \infty. \quad (2.7)$$

This is due to the fact that vector a possesses a nontrivial component in the direction of the Perron–Frobenius eigenvector. The Perron–Frobenius eigenvalue of our matrix A is equal to $\lambda = 1.401268$. The numerical value of the constant C will be calculated further in the text (see Remark 2.8).

Remark 2.5. If one would study the alternative boundary conditions of no sun from the boundary, instead of (2.6), one would obtain:

$$\begin{matrix} \text{\# of maximal configurations of length } n \\ \text{with no-sun boundary condition} \end{matrix} =: b_n = b^T \cdot A^{n-3} \cdot d, \quad n \geq 3, \quad (2.8)$$

where $b = (1, 0, 1, 0, 1, 1)$, $d = (0, 1, 0, 1, 1, 1)$. This sequence appears on the OEIS [28] under the number A253412.

In the case of periodic boundary conditions, there is a clear 1 to 1 correspondence between the maximal configurations of length n and closed walks on the graph in Fig. 3. Thus, we have:

$$\begin{matrix} \text{\# of maximal configurations of length } n \\ \text{with periodic boundary conditions} \end{matrix} =: d_n = \text{tr}(A^n), \quad n \geq 1. \quad (2.9)$$

This sequence appears on the OEIS under the number A253413.

2.1.1. Generating functions for the Riviera model

From the structure of the sequence (a_n) , one can easily calculate its generating function $f = f(y)$ by calculating the resolvent $(I - yA)^{-1}$, where y is a formal variable. This is a somewhat standard calculation for which we explicitly need to determine the first 3 values of (a_n) . We have:

$$\begin{aligned} f(y) &= 1 + y + y^2 + \sum_{n=3}^{\infty} a^T \cdot A^{n-3} \cdot a \cdot y^n \\ &= 1 + y + y^2 + a^T \cdot \left(\sum_{n=0}^{\infty} (yA)^n \right) \cdot a \cdot y^3 \\ &= 1 + y + y^2 + a^T \cdot (I - yA)^{-1} \cdot a \cdot y^3 \\ &= \frac{1 + y + y^3 - y^5}{1 - y^2 - y^3 - y^4 + y^6}. \end{aligned} \quad (2.10)$$

For inverting matrix functions, we have used the software for symbolic calculation, Maxima [23].

Remark 2.6. The generating function f encodes the infinite sequence (a_n) into a simple rational function. Immediately we deduce that the sequence (a_n) satisfies the following 6th order linear recurrence relation:

$$a_n = a_{n-2} + a_{n-3} + a_{n-4} - a_{n-6}, \quad n \geq 7, \quad (2.11)$$

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 3, \quad a_4 = 3, \quad a_5 = 4, \quad a_6 = 6.$$

The sequence (a_n) cannot easily be represented with an explicit formula as it would involve the roots of the polynomial $p(y) = y^6 - y^4 - y^3 - y^2 + 1$.

The information on the number of maximal configurations of fixed length is already useful, but our aim is to determine the precise number of maximal configurations of length $n \in \mathbb{N}$ with a fixed number of houses $k \in \mathbb{N}$, which we denote by $J_{k,n}$. This information gives us insight into the distribution of the *occupancy* $|C| := \sum_{i=1}^n c_i$ among maximal configurations of length n , for all values of $n \in \mathbb{N}$. Knowing this quantity would lead to determining the so-called *complexity* (see Remark 2.11) of our model, namely a distribution of occupancy (or associated *building density* which is defined as $\frac{|C|}{n}$) when the length of configurations n grows large.

To this end, we calculate the bivariate generating function $g(x, y)$, where x is a formal variable associated with the occupancy of a configuration, while y remains a formal variable associated with the length of the configuration. We define the following matrix function:

$$\begin{matrix} & \begin{matrix} 100 & 001 & 011 & 110 & 101 & 010 \end{matrix} \\ \begin{matrix} 100 \\ 001 \\ 011 \\ 110 \\ 101 \\ 010 \end{matrix} & \begin{bmatrix} 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & x & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & x & 0 \end{bmatrix} \end{matrix} =: A(x). \quad (2.12)$$

The purpose of this matrix function is to keep track when a transition on the graph in Fig. 3 results in the increase of the number of occupied lots. Namely:

$$i \rightarrow j \text{ is a transition which adds an occupied lot} \iff [A(x)]_{i,j} = x, \quad (2.13)$$

while the rest of the transitions which do not contribute an occupied lot are denoted with $1 = x^0$. The powers of $A(x)$, namely $(A(x))^n$, encode the distribution of occupancies for the configurations of length n . We have:

$$[(A(x))^n]_{i,j} = p_0^{i,j} + p_1^{i,j}x + p_2^{i,j}x^2 + \dots + p_n^{i,j}x^n, \quad (2.14)$$

where

$$p_k^{i,j} = \begin{matrix} \# \text{ of walks of length } n \text{ on the graph in Fig. 3} \\ \text{starting with node } i \text{ and ending with } j \\ \text{where the number of occupied lots was increased by } 1, k \text{ times.} \end{matrix} \quad (2.15)$$

In order to take into account the number of occupied lots with which we start the walk, we define vectors:

$$a(x) = (0, 0, x^2, x^2, x^2, 0)^T, \quad b = (0, 0, 1, 1, 1, 0)^T. \quad (2.16)$$

The x^2 terms in the vector $a(x)$ encode the fact that all the starting nodes have precisely 2 occupied lots. By plugging this into the familiar formula and determining the first few terms, we obtain:

$$\begin{aligned} g(x, y) &= 1 + xy + x^2y^2 + \sum_{n=3}^{\infty} a(x)^T \cdot (A(x))^{n-3} \cdot b \cdot y^n \\ &= \frac{1 + xy - (x - x^2)y^2 + x^2y^3 - x^3y^5}{1 - xy^2 - x^2y^3 - x^2y^4 + x^3y^6} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} J_{k,n} x^k y^n, \end{aligned}$$

where $J_{k,n}$ is precisely a number of maximal configurations of length n with k occupied lots.

The bivariate generating function $g(x, y)$ encodes, among other things, the information on the asymptotic behavior of our sequence (a_n) (the same we have recovered from the Perron–Frobenius eigenvalue of the transfer matrix A), and also enables us to determine the expected values of two quantities of interest – the expected number of buildings in a maximal configuration of a given length, and the expected length of a maximal configuration with a given number of buildings. The starting point is the following classical result, a version of Darboux's theorem as formulated in [4]. For more information on obtaining the asymptotics of a sequence from its generating function we refer the reader to [4,32].

Theorem 2.7. If the generating function $f(x) = \sum_{n \geq 0} a_n x^n$ of a sequence (a_n) can be written in the form $f(x) = \left(1 - \frac{x}{w}\right)^\alpha h(x)$, where w is the smallest modulus singularity of f and h is analytic in w , then $a_n \sim \frac{h(w)n^{-\alpha-1}}{\Gamma(-\alpha)w^n}$, where Γ denotes the gamma function.

Remark 2.8. It can be shown that the Perron–Frobenius eigenvalue λ of the matrix A can be related to w from Theorem 2.7 as $\lambda = \frac{1}{w}$. The constant C from Remark 2.4 can be therefore be computed from the generating function (2.10) as

$$C = \frac{\lambda^6 + \lambda^5 + \lambda^3 - \lambda}{2\lambda^4 + 3\lambda^3 + 4\lambda^2 - 6} = 0.803796.$$

Now the expected number of built sites in a maximal configuration of length n can be computed (see [32]) as

$$\frac{[y^n] \frac{\partial g(x, y)}{\partial x} |_{x=1}}{[y^n] g(x, y) |_{x=1}},$$

where $[y^n]F(y)$ denotes the coefficient of y^n in the expansion of $F(y)$.

Since $g(1, y) = \frac{p(1, y)}{q(1, y)}$ is rational, its smallest modulus singularity is the smallest (by absolute value) root of its denominator $q(1, y)$. While it does not have a closed-form expression, its approximate numerical value is readily computed as $w = 0.713639$, the reciprocal value of the Perron–Frobenius eigenvalue 1.401268. Now we can write

$$g(x, y) |_{x=1} = \left(1 - \frac{y}{w}\right)^{-1} g_1(y)$$

and

$$\frac{\partial g(x, y)}{\partial x} |_{x=1} = \left(1 - \frac{y}{w}\right)^{-2} g_2(y),$$

where

$$g_1(y) = \frac{p(1, y)(w - y)}{wq(1, y)} \text{ and } g_2(y) = \frac{q(1, y) \frac{\partial p}{\partial x}(x, y) |_{x=1} - p(1, y) \frac{\partial q}{\partial x}(x, y) |_{x=1}}{\left(w \frac{q(1, y)}{w - y}\right)^2}.$$

By evaluating the ratio $r(y) = \frac{g_2(y)}{g_1(y)}$ at w we finally obtain

$$r(w) = \frac{1}{w} \frac{\frac{\partial q}{\partial x}(x, w) |_{x=1}}{\frac{\partial q}{\partial y}(1, y) |_{y=w}} = 0.577203$$

and the expected number $\langle k(n) \rangle$ of buildings in a maximal configuration of length n is given as

$$\langle k(n) \rangle = 0.577203 n.$$

If one defines the *efficiency* of a maximal configuration as the ratio of the actual number of occupied lots and the largest possible number of occupied lots, which is $\lceil \frac{2n}{3} \rceil$, one gets the *expected efficiency* of a maximal configuration as

$$\varepsilon = \frac{0.577203 n}{\lceil \frac{2n}{3} \rceil} = 0.865804.$$

This efficiency is higher than the efficiency of unrestricted P_m -packings of P_n for small m (cf. [30]). Thus, we have proved the following proposition.

Proposition 2.9. The expected number $\langle k(n) \rangle$ of buildings in a maximal configuration of length n in the Riviera model is given as $\langle k(n) \rangle = 0.577203 n$. The expected efficiency of a maximal configuration (the ratio of the actual number of occupied lots and the largest possible number of occupied lots) in the Riviera model is $\varepsilon = 0.865804$.

Remark 2.10. Clearly, by choosing $x = 1$ we obtain $f(y)$. However, by choosing $y = 1$, we obtain $h(x) = g(x, 1)$ which is a generating function for the sequence (h_k) which counts the number of maximal configurations with a fixed number of occupied lots (with variable length). One easily computes this function as

$$h(x) = \frac{1 + 2x^2 - x^3}{1 - x - 2x^2 + x^3}. \quad (2.17)$$

Straight away we read the recurrence relation for the sequence (h_k) :

$$\begin{aligned} h_k &= h_{k-1} + 2h_{k-2} - h_{k-3}, \quad k \geq 4, \\ h_1 &= 1, \quad h_2 = 5, \quad h_3 = 5. \end{aligned} \quad (2.18)$$

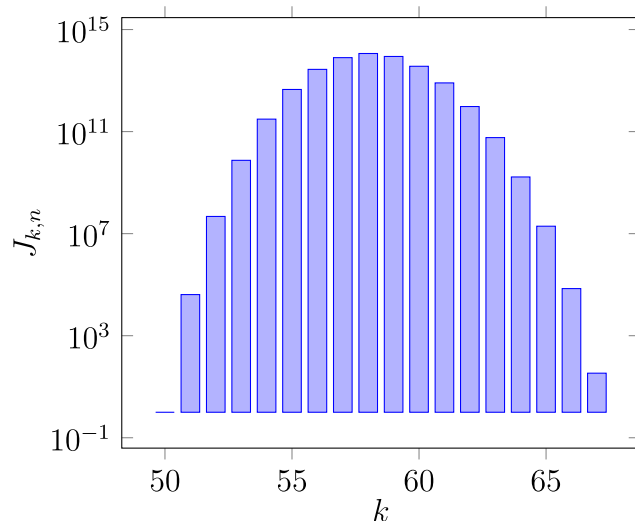


Fig. 4. Precise values of $J_{k,n}$ (log-scale) for $n = 100$.

By a completely analogous procedure, with switched roles of x and y , one can now compute the expected length $\langle n(k) \rangle$ of a maximal configuration with k buildings as

$$\langle n(k) \rangle = 1.758283 k.$$

We omit the details.

It is, perhaps, interesting to note that $\frac{1}{1.758283} = 0.568737$ which is, as one might hope, close to previously computed 0.577203. Although this is what one might expect, there is no reason, in general, why should $\frac{\langle n(k) \rangle}{k}$ be equal (or even close) to $\frac{n}{\langle k(n) \rangle}$.

Remark 2.11. For a building density $\rho \in [0, 1]$ one can introduce the number $J_n(\rho) = J_{\lfloor \rho n \rfloor, n}$ which counts the number of configurations of length n with exactly $\lfloor \rho n \rfloor$ occupied lots. The function

$$f(\rho) = \lim_{n \rightarrow \infty} \frac{\ln J_n(\rho)}{n} \quad (2.19)$$

is called the complexity and it represents the exponential growth rate of the number of configurations with building density ρ as their length n increases. With this function in hand, one can express the asymptotic behavior of the sequence $(J_n(\rho))$ for large n as:

$$J_n(\rho) \sim h(n) e^{nf(\rho)} = h(n) (e^{f(\rho)})^n, \quad n \rightarrow \infty, \quad \rho \in [0, 1] \quad (2.20)$$

where $h(n)$ is some function with subexponential growth. One can easily verify that $\text{supp } f = [\frac{1}{2}, \frac{2}{3}]$. Also, by comparing (2.20) with Remark 2.4 we infer:

$$\max_{\rho} f(\rho) = \ln \lambda = \ln 1.401268 = 0.337377. \quad (2.21)$$

However, determining the precise formula for f remains an open problem.²

In Fig. 4 the precise values of $J_{k,100}$ are plotted on the log-scale. For large n , one expects the shape of this bar plot to approximate the true shape of the complexity function $f(\rho)$.

2.2. Connections to combinatorial optimization

In this subsection, we explore the possibility of using combinatorial optimization techniques to approach our model. We are greatly indebted to the anonymous reviewer who suggested the following reformulation of our problem.

Consider the set of vectors

$$P = \{x \in [0, 1]^n : x_i + x_{i+1} + x_{i+2} \leq 2 \text{ for all } i = 1, 2, \dots, n-2\}.$$

² This was solved recently by Krapivsky and Luck in [17].

This is a closed convex polyhedron and the corresponding constraint matrix has the consecutive-ones property which in turn implies that it is a totally unimodular matrix and therefore the vertices of this polyhedron are precisely binary 0–1 vectors satisfying the constraints, see [10,13]. Note that the constraints prevent three consecutive 1's from occurring which means that the vertices of P precisely correspond to permissible Riviera configurations.

If we now relax the constraint on entries of x and require only $x_i \leq 1$ instead of $0 \leq x_i \leq 1$ for all $i = 1, 2, \dots, n-2$, we get an unbounded convex polyhedron

$$P' = \{x \in (-\infty, 1]^n : x_i + x_{i+1} + x_{i+2} \leq 2 \text{ for all } i = 1, 2, \dots, n-2\}$$

such that $P' \supset P$ and whose vertices are still binary 0–1 vectors which in addition to being permissible are also maximal Riviera configurations. Notice also that permissible configurations that are not maximal are still feasible, but after removing the non-negativity constraints $x_i \geq 0$, they are no longer vertices. Hence, counting maximal configurations in the Riviera model is equivalent to counting the vertices of the unbounded convex polyhedron P' defined above.

In addition to formulating the problem using packing constraints, we can express everything in terms of set covering, by complementing the variables, i.e., setting $y_i = 1 - x_i$ for all $i = 1, 2, \dots, n$. This corresponds to an affine map of P and P' . More precisely, we obtain

$$Q = \{y \in [0, 1]^n : y_i + y_{i+1} + y_{i+2} \geq 1 \text{ for all } i = 1, 2, \dots, n-2\},$$

and

$$Q' = \{y \in [0, \infty)^n : y_i + y_{i+1} + y_{i+2} \geq 1 \text{ for all } i = 1, 2, \dots, n-2\}.$$

In this case, the vertices of Q' are again associated (up to complementing) to maximal configurations of the Riviera model. The combinatorial optimization literature on set covering is very rich, and in fact set covering problems with the consecutive ones property have been studied, see e.g. [12]. The problems of packing and covering arise in the context of shift scheduling [1,3], and in problems related to interval graphs and the minimum cost network flow problem [31]. However, we are not aware of any results dealing with counting the vertices of polyhedron of feasible points.

It seems that when it comes to general grids and/or multiple stories, the same polyhedron P' would lose the property that its vertices are exactly the permissible maximal configurations. The problem is that the resulting packing polyhedron might not have the consecutive ones property, meaning that we would be counting maximal binary vectors in the polyhedron, rather than the vertices of the polyhedron. For example, in the two-story model in Section 3.1.1 it is sufficient to enforce permissibility by constraints

$$x_i + x_{i+1} + 2x_{i+2} + x_{i+3} + x_{i+4} \leq 3 \text{ for all } i = 1, 2, \dots, n-4,$$

and to account for the extremes of the strip we additionally require

$$2x_1 + x_2 + x_3 \leq 3,$$

$$x_1 + 2x_2 + x_3 + x_4 \leq 3,$$

$$x_{n-3} + x_{n-2} + 2x_{n-1} + x_n \leq 3,$$

$$x_{n-2} + x_{n-1} + 2x_n \leq 3.$$

2.3. Surprising relationship between the Riviera model and other combinatorial objects

The integer sequence (a_n) associated with the generating function $f = f(y)$ can be evaluated for any $n \in \mathbb{N}$ simply by expanding f into the formal power series in powers of y . Even more, by expanding the bivariate generating function $g = g(x, y)$, we obtain the precise distribution of the occupancies of maximal configurations relative to their length. The first few coefficients in the expansion of $g(x, y)$ are given in Table 1. By inspecting the non-zero coefficients in the table, we see that the ratio $\frac{k}{n}$ is in-between $\frac{1}{2}$ and $\frac{2}{3}$, for large n .

The first several values of (a_n) can be read as column sums: 1, 1, 3, 3, 4, 6, 9, 12, 16, 24, 33, 46, 64, ... On the other hand, one might do the same for the generating function $h = h(x)$ and the associated sequence (h_k) . The first several values of (h_k) can be read as row sums: 1, 5, 5, 14, 19, 42, 66, 131, ... Also, the expansion of g gives insight into the distribution of lengths of maximal configurations relative to the number of occupied lots.

Upon consulting The On-Line Encyclopedia of Integer Sequences [28], we have come across the fact that these sequences were studied in quite different settings (cf. A080013 & A096976). We illustrate these connections in the following subsections.

2.3.1. Relation to strongly restricted permutations

The notion of strongly restricted permutations was introduced by Lehmer in [20]. If W is some fixed subset of integers, one would like to count the number of all the permutations $\pi \in S_n^3$ such that $\pi(i) - i \in W$, for all $i \in [n]$.

³ S_n denotes the set of all permutations of the set $[n] = \{1, \dots, n\}$.

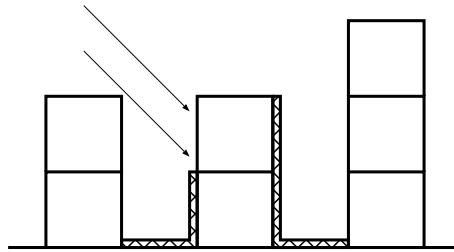


Fig. 6. Both stories of the middle house are blocked from the sunlight from the east, while only the first story is blocked from the west.

The permissibility constraint in such models can be concisely stated in the following way: Every story of each house must obtain sunlight from at least one side (east or west) during the day, namely, not to be blocked from sunlight from both sides by other buildings. Naturally, we are again interested in maximal configurations. The maximality of the configuration $C \in \mathbb{N}_0^n$ would mean that for each lot i , $1 \leq i \leq n$, no additional stories can be built on it. In other words, if an addition of a single story to any lot in C would still result in a permissible configuration, then it is not maximal.

In this section we also comment a logical counterpart of a multi-story Riviera model, namely a multi-story Flory model. As already stated, in the Riviera model we require that each story obtains sunlight from the east **or** the west. However, one could alternatively require that each story obtains sunlight from the east **and** from the west. In this case, we would obtain models of Flory type, which in the single-story case comes down to the classical Flory model introduced in [9]. We comment on both of these variants, which we can refer to as the **AND** and the **OR** variant.

3.1. OR variant

Our intention here is again to count the number of maximal configurations. To this end, we apply the transfer matrix method, for which we are required to construct a digraph which allows us to encode our configurations in terms of walks on that graph in the same way as in Section 2.1. It is somewhat obvious that, on this level of generality, this is impossible, due to the fact that the minimal set of impermissible substrings of maximal configurations cannot be reduced to a finite set. However, by fixing a maximal number of stories allowed, we denote it by k , we are still in the domain of transfer matrix method.

Once we fix k , in order to construct a digraph, we must know the minimal length of the nodal strings which would guarantee the permissibility and maximality of the configurations associated with walks. Next, we must find all of the possible substrings of that length which occur in maximal configurations. One way of doing this is by an exhaustive search among all the maximal configurations until one is sure that all the substrings have been found.

This, however, is quite involved, which can be deduced from the following two lemmas.

Lemma 3.1. *Let $k \in \mathbb{N}$ be the maximal number of stories allowed. It is enough to take substrings of length $4k + 1$ to form nodes of the graph \mathcal{G} to be certain that this graph can be used in the transfer matrix method for counting the corresponding maximal configurations.*

Proof. As already mentioned, applicability of the transfer matrix method relies on the fact that permissibility, as well as maximality of a configuration, can be verified by inspecting only finite size patches of a given configuration. More precisely, if one wants to check whether a state of a certain lot (occupied or unoccupied) has caused the configuration to be impermissible or not maximal, one only needs to check the situation on the lots in a certain finite radius of the observed lot, where that radius is uniform for each lot on the tract of land. What we are claiming in this lemma is that, when the maximal number of stories allowed is k , then this radius is $2k$, i.e. it is enough to check the $2k$ immediate neighbors to the west of a lot and the $2k$ immediate neighbors to the east of a lot to be certain whether this central lot violates permissibility or maximality.

Let us first discuss what needs to be checked to be sure that building an additional story on that central lot will not violate permissibility and then we will discuss what needs to be checked to ensure that not building an additional story on this lot will not violate maximality. There are two ways in which permissibility can be violated. On one hand, it can happen that the built story itself will not be exposed to sunlight and on the other hand, the built story can block a story of another house from the sunlight. The radius that needs to be checked is the biggest in the case when we want to build as many stories as allowed, so we assume from now that we want to build a k -story house and that its first neighbors to the east and to the west are also k -story houses. To check whether the central lot is already blocked from the sunlight, we only need to check k lots to the east and k lots to the west. Notice now that if the first neighbor to the east of the central lot is more than k empty lots away, then this neighbor is definitely getting sunlight from the west and building a house on a central lot will not block the sunlight to that neighboring house. If that neighbor is less than k empty lots away, then building a k -story house on the central lot would block the sunlight from the west at least to the first story of

the mentioned neighbor. In that case, the question is whether this first neighbor to the east is exposed to sunlight from the east. To verify that, we again need to check what happens on k immediate neighboring lots to the east of it. Hence, the worst-case scenario is that we need to check $2k$ lots to the east of the central lot. Due to the symmetry, the same holds while checking lots to the west of the central one.

It is now easy to see that applying the same logic and checking $2k$ lots immediately to the east and immediately to the west of a central lot is enough to verify whether we can build additional stories on that central lot and in this way we check maximality. \square

Lemma 3.2. *Let $k \in \mathbb{N}$ be the maximal number of stories allowed. Then the maximal configurations of length $8k + 1$ contain as substrings all the strings of length $4k + 1$ that can appear in any maximal configuration of arbitrary length.*

Moreover, if such a string is at the beginning/end of a maximal configuration then it is possible to choose a maximal configuration of length $8k + 1$ in which it also appears at the beginning/end.

Remark 3.3. Notice that Lemma 3.2 tells us that if we find all the maximal configurations of length $8k + 1$ (which can be done by exhaustive search using a computer), and then go through those maximal configurations with the window of size $4k + 1$, we will find all the vertices that we need for the graph \mathcal{G} . Moreover, using these maximal configurations of length $8k + 1$ we can easily extract the vertices with which the maximal configuration can start and end (see Algorithm 1).

Proof of Lemma 3.2. It is clear from Lemma 3.1 that the state of a certain lot in a maximal configuration, padded with zeros, is a function of the states of $2k$ lots immediately to the east and $2k$ lots immediately to the west of it. More precisely, the state s of that lot is the largest number $0 \leq s \leq k$ which keeps the configuration permissible. This means that it is impossible to have two nodes in the graph \mathcal{G} such that they only differ in the state of the central lot.

The above mentioned function $f(u_1 \dots u_{2k}, v_1 \dots v_{2k})$, which to each pair of permissible words of length $2k$ assigns the largest number of stories that could be built on the central lot of a configuration obtained by concatenating the word $u_1 \dots u_{2k}$ to the left and the word $v_1 \dots v_{2k}$ to the right of that central lot whilst not violating permissibility, is monotonic in the sense that if $u_i \leq u'_i$ and $v_i \leq v'_i$ for all $1 \leq i \leq 2k$ then $f(u_1 \dots u_{2k}, v_1 \dots v_{2k}) \geq f(u'_1 \dots u'_{2k}, v'_1 \dots v'_{2k})$.

What we need to show here is that every substring of length $4k + 1$ that can appear in some maximal configuration of arbitrary length, also appears in some maximal configuration of length $8k + 1$. We prove this by showing that any substring of length $4k + 1$ that can appear in a maximal configuration can be expanded to a maximal configuration by adding additional $2k$ lots to the east and to the west of it and building additional houses only on those $4k$ added lots.

As announced, we start from a substring $u_1 u_2 \dots u_{4k+1}$ of length $4k + 1$ which appears in some maximal configuration. We then extract this substring along with $2k$ lots to the east of that substring within this maximal configuration in which it appears and also $2k$ lots to the west of it, padding with zeros when necessary. This gives us a string $u_{-2k+1} \dots u_0 u_1 \dots u_{4k+1} \dots u_{6k+1}$ of length $8k + 1$ which we further pad with zeros to the left and right to obtain an infinite word $(u_i)_{i \in \mathbb{Z}}$.

Note that the maximality of the initial configuration in which the substring $u_1 u_2 \dots u_{4k+1}$ appeared implies that the relation $u_i = f(u_{i-2k} \dots u_{i-1}, u_{i+1} \dots u_{i+2k})$ holds for any $1 \leq i \leq 4k + 1$.

We now start changing the entries $u_0, u_{-1}, \dots, u_{-2k+1}$, and $u_{4k+2}, u_{4k+3}, \dots, u_{6k+1}$ one-by-one, in this order, according to the output of f on the current state of the $2k$ lots to the left and right of the observed block. We first change u_0 to $u'_0 = f(u_{-2k} \dots u_{-1}, u_1 u_2 \dots u_{2k})$, then u_{-1} to $u'_{-1} = f(u_{-2k-1} \dots u_{-2}, u'_0 u_1 \dots u_{2k-1})$ and so on. Continuing in this way, we will construct the configuration

$$u'_{-2k+1} \dots u'_0 u_1 u_2 \dots u_{4k+1} u'_{4k+2} \dots u'_{6k+1}$$

of length $8k + 1$ which, we claim, is maximal.

Observe that since we have started with a permissible configuration, the updates provided by f can only ever increase the number of the stories already present at the inspected lot. Also note that the configuration we obtain after performing these updates is also permissible as the updates provided by the function f never violate permissibility.

Now, to verify maximality note that, by the definition of function f , the number u'_i , where $-2k + 1 \leq i \leq 0$ or $4k + 2 \leq i \leq 6k + 1$, was as large as permitted at the time the update $u_i \mapsto u'_i$ was performed. Taking into account the monotonicity of f , it is clear that u'_i is as large as possible in the final configuration obtained. Finally note that, again, because of the monotonicity of f , u_i for $1 \leq i \leq 4k + 1$ is still as large as possible.

To see that the ‘moreover’ part of the statement is true, note that an analogous procedure can be followed if the initial substring appears at the beginning/end. In this case we extract the substring from its maximal configuration along with $4k$ additional lots to its right/left and then pad with zeros. The updates via f are again performed only on these extra lots and similar arguments show that the obtained configuration of length $8k + 1$ which starts/ends with the chosen substring is maximal. \square

Now we propose an algorithm (see Algorithm 1) for calculating the $(k + 1)$ -variate generating function related to a Riviera model where it is possible to have houses with different number of stories, but the maximal number of stories allowed is a fixed number $k \in \mathbb{N}$. Variables x_r , $r \in \{1, 2, \dots, k\}$, are formal variables associated with the number of r -story

houses in the maximal configuration and y is a formal variable associated with the length of a maximal configuration. The proposed algorithm is written in a very general way and it works even in situations where all the r -story houses appear ($r \in \{1, 2, \dots, k\}$), but in the case when we allow only k -story houses for one fixed $k \in \mathbb{N}$, we use slightly simpler notation than the one in Algorithm 1.

Algorithm 1 The calculation of the $(k + 1)$ -variate generating function

MaximalConfigurations = all the maximal configurations of length $8k + 1$

▷ this is found by an exhaustive search

AllowedNodes = \emptyset , StartNodes = \emptyset , EndNodes = \emptyset

for $mc \in \text{MaximalConfigurations}$ **do**

 add substring(mc , start = 1, end = $4k + 1$) to StartNodes

 add substring(mc , start = $4k + 1$, end = $8k + 1$) to EndNodes

for $1 \leq i \leq 4k + 1$ **do**

 add substring(mc , start = i , end = $4k + i$) to AllowedNodes

end for

end for

$n = \text{length}(\text{AllowedNodes})$

$A = \text{null matrix}(nrow = n, ncol = n)$

for $1 \leq i \leq n$ **do**

for $1 \leq j \leq n$ **do**

if substring(AllowedNodes[i], start = 2, end = $4k + 1$) = \ \

 substring(AllowedNodes[j], start = 1, end = $4k$) **then**

▷ if substrings overlap progressively

$r = \text{substring}(\text{AllowedNodes}[j], \text{start} = 4k + 1, \text{end} = 4k + 1)$

if $r = 0$ **then**

$A[i, j] = 1$

else

$A[i, j] = x_r$

end if

end if

end for

end for

$a = \text{null vector}(\text{length} = n)$, $b = \text{null vector}(\text{length} = n)$

for $1 \leq i \leq n$ **do**

if AllowedNodes[i] \in StartNodes **then**

$a[i] = 1$

for $1 \leq r \leq k$ **do**

$c = \text{count}(r \text{ in AllowedNodes}[i])$

$a[i] = a[i] \cdot x_r^c$

end for

end if

if AllowedNodes[i] \in EndNodes **then**

$b[i] = 1$

end if

end for

$a_{c_1, c_2, \dots, c_k, j}$ = the number of maximal configurations of length j with precisely c_r r -story houses ▷ this is found by an exhaustive search
for $0 \leq j \leq 4k$

$$F_1(x_1, x_2, \dots, x_k, y) = \sum_{j=0}^{4k} \sum_{c_1, c_2, \dots, c_k=0}^j a_{c_1, c_2, \dots, c_k, j} \prod_{r=1}^k x_r^{c_r} y^j$$

$$F_2(x_1, x_2, \dots, x_k, y) = \sum_{j=4k+1}^{\infty} a^T \cdot A^{j-4k} \cdot b \cdot y^j = a^T \cdot (I - Ay)^{-1} \cdot A \cdot b \cdot y^{4k+1}$$

return $F(x_1, x_2, \dots, x_k, y) = F_1(x_1, x_2, \dots, x_k, y) + F_2(x_1, x_2, \dots, x_k, y)$

3.1.1. Two-story Riviera model

In the two-story Riviera model, all the houses on our $1 \times n$ tract of land have precisely two stories. This means that, to ensure permissibility of the configuration, each house needs to have at least two empty lots immediately to the east or at least two empty lots immediately to the west of itself (we again assume that there is no obstruction to sunlight on the east and west boundary of the tract of land). Maximal configurations are, as before, those that are permissible, but become impermissible as soon as we add one more house on any of the empty lots. Representing configurations of length n with row vectors $C = (c_1, \dots, c_n) \in \mathbb{N}_0^n$ where $c_i = 2$ denotes that on the lot i ($i \in \{1, 2, \dots, n\}$) there is a house with 2 stories (and $c_i = 0$ denotes that the lot i is not occupied by a house), one example of maximal configuration in the two-story Riviera model is 0202000022. It is easy to check that each of the two stories of every house in this configuration is exposed

Table 2
The first few coefficients in the expansion of the bivariate generating function $F(x, y)$.

$k \backslash n$	1	y	y^2	y^3	y^4	y^5	y^6	y^7	y^8	y^9	y^{10}	y^{11}	y^{12}	y^{13}	y^{14}	y^{15}
1	1															
x		1														
x^2			1	3	6	5	3	1								
x^3						2	4	5	2							
x^4							1	5	16	27	31	24	13	5	1	
x^5										3	12	28	36	29	14	3
x^6											1	7	31	80	142	177
x^7														4	24	82

to sunlight from at least one side (east or west), and that adding another two-story house on any of the empty lots would turn this configuration into an impermissible one. The goal now is to repeat the same procedure as for the original Riviera model and to obtain bivariate generating function which will give us information not only about the number of maximal configurations with a fixed length, but also about the number of maximal configurations with a fixed number of houses. Using Algorithm 1, we can construct the digraph that enables us to encode maximal configurations with walks on that digraph. Analogously as in Section 2.1.1 we define the matrix function $A(x)$, which is not only the adjacency matrix of the mentioned digraph, but additionally encodes whether a transition from one node of the graph to another one results in the increase of the total number of occupied lots. Due to the dimension of the matrix $A(x)$ and the technicalities in developing bivariate generating function, we omit the details. The precise shape of the generating function $F(x, y)$ (given in Appendix – see (A.1)) was obtained following Algorithm 1 and using R programming language [27] for creating the necessary objects (matrix A , vectors a and b) and Maxima software to perform the calculation of the generating function using objects created in R. The first few coefficients in the expansion of $F(x, y)$ are given in Table 2. By inspecting the non-zero coefficients in the table, we see that the ratio $\frac{k}{n}$ is in-between $\frac{2}{7}$ and $\frac{1}{2}$, for large n .

From Table 2 we read that there are 44 maximal configurations of length 10, out of which 31 configurations have precisely 4 two-story houses built on them. This means that, apart from the maximal configuration that we gave as an example (0202000022), there are 43 more maximal configurations with the same length and 30 more maximal configurations with the same length and the same number of houses.

The first several values of the integer sequence that counts the number of maximal configurations in the two-story Riviera model of length n can be read as column sums: 1, 1, 3, 6, 7, 8, 11, 18, 30, 44, . . . and this sequence is still not a part of the On-Line Encyclopedia of Integer Sequences. On the other hand, by expanding $F(x, y)$ into the formal power series in powers of x , (i.e. taking row sums) we can see that, beside the mentioned maximal configuration with precisely 4 two-story houses, there are 122 more maximal configurations with the same number of houses. The first several values of the integer sequence that counts the number of maximal configurations in the two-story Riviera model with precisely k houses can be read as row sums: 1, 19, 13, 123, 125, 811, 1069, 5435, 8605, 36939, This sequence is also not yet included in the OEIS.

Remark 3.4. By performing the same analysis as in the case of one-story houses, one can obtain the expected number of occupied lots in a maximal configuration of length n as $\langle k_2(n) \rangle = 0.388957 n$, and the expected length of a maximal configuration with k two-story houses as $\langle n_2(k) \rangle = 2.706054 k$. Since the largest possible number of buildings in a maximal configuration is, roughly, $n/2$, one can see that the expected efficiency ε_2 is lower than in the case of one-story buildings: $\varepsilon_2 = 0.777914$. However, by counting each house as two basic units (two flats), it follows that on a maximal configuration of the same length n one can build, on average, $2 \cdot 0.388957/0.577203 = 1.34773$ times more basic units by opting for two-story houses. It is reassuring that our model, although very simple, captures the common wisdom of planners and builders. It would be interesting to investigate how this gain in efficiency is affected by further increase in the number of stories.

3.1.2. Mixed Riviera model

By the mixed Riviera model, we refer to the model where one-story houses and two-story houses can be built on the same $1 \times n$ tract of land. Notice that, even though we cannot add additional two-story house on any of the empty lots in the configuration 0202000022, we could add one-story house without making the configuration impermissible. One possible position to add one-story house to this configuration is $i = 6$. If we set $c_6 = 1$ we get the configuration 0202010022. This one-story house is not blocking the sun to any of the stories of already built houses, and the one-story house itself is exposed to sunlight. It is easy to check that this new configuration with one-story house is maximal since we cannot add additional stories to any of the empty lots nor to the lot where there is a one-story house. In this model, words that represent configurations have three letters (0, 1 and 2). However, the transfer matrix method and Algorithm 1 can again be used to obtain the digraph through which we encode all the maximal configurations in this mixed Riviera model. Once we have the digraph, we have the adjacency matrix, but as before, we use this adjacency matrix for more than just counting the number of maximal configurations of fixed length. Depending on whether a transition from node i to node j results in addition of an empty lot, a lot on which one-story house is built or a lot on which two-story house

is built, matrix function A has 1, x or y on the position (i, j) , respectively. Hence, A is a function of two variables (x and y) where x , rather than x_1 , is a formal variable associated with the number of one-story houses in the configuration and y , rather than x_2 , is a formal variable associated with the number of two-story houses in the configuration. Denoting the formal variable associated with the length of the configuration by z , rather than y , and using similar calculations as in Section 2.1.1 we obtain trivariate generating function. The precise shape of this generating function is again obtained by R and Maxima and it is given in the Appendix (see (A.2)). By expanding $F(x, y, z)$ into the formal power series in powers of z , we get

$$\begin{aligned} F(x, y, z) = & 1 \\ & + yz \\ & + y^2 z^2 \\ & + 3y^2 z^3 \\ & + (2x + 4)y^2 z^4 \\ & + (2y^3 + (x^2 + 6x + 1)y^2)z^5 \\ & + (y^4 + 4y^3 + (7x^2 + 6x)y^2)z^6 \\ & + (5y^4 + (2x^2 + 2x + 3)y^3 + (2x^3 + 11x^2 + 2x)y^2)z^7 \\ & + ((4x + 13)y^4 + (10x^2 + 4x)y^3 + (x^4 + 8x^3 + 5x^2)y^2)z^8 \\ & + (3y^5 + (3x^2 + 18x + 15)y^4 + (4x^3 + 18x^2 + 2x)y^3 + (7x^4 + 8x^3 + x^2)y^2)z^9 \\ & + (y^6 + 12y^5 + (30x^2 + 34x + 8)y^4 + (4x^4 + 16x^3 + 10x^2)y^3 + (2x^5 + 11x^4 + 2x^3)y^2)z^{10} \\ & + \dots \end{aligned}$$

Our example of a maximal configuration in the mixed Riviera model has length 10, 1 one-story house and 4 two-story houses. From this expansion it is easy to read that there are 33 more maximal configurations of length 10 with 1 one-story house and 4 two-story houses. Notice that plugging $y = x$ in the trivariate generating function $F(x, y, z)$ gives bivariate generating function in variables x and z where formal variable x is associated with the number of houses (regardless of the number of stories) and formal variable z is still associated with the length of the maximal configuration. Similarly, setting $y = x^2$ we get bivariate generating function where formal variable x is associated with the number of stories on maximal configurations. The most natural sequence related to this model is the one that counts the number of maximal configurations of length n . The first several values of this sequence are 1, 1, 3, 6, 10, 18, 27, 45, 79, 130, ... This sequence is not found on the OEIS.

3.2. AND variant

As explained at the beginning of this section, the AND variant, unlike the OR variant, requires that each story of each house gets sunlight from the east and from the west, and in the one-story case this comes down to the classical Flory polymer model introduced by Flory [9] already in 1939. The sequence (a_n) that counts the number of maximal configurations of length n in the one-story Flory model is the famous Padovan sequence (see A000931) with offset 6 or, equivalently, the number of compositions (ordered partitions) of number $n + 3$ into parts 2 and 3 (i.e. A182097 with offset 3). Moreover, if we are interested in the asymptotic formula for a_n , we can obtain it as explained in Remark 2.4. It holds that $a_n \sim C \lambda^n$ as n tends to infinity where λ is the well-known plastic number $\lambda = \frac{1}{w} = 1.324718$. As before, the numerical value of $C = \frac{\lambda^3 + \lambda^2 + \lambda}{2\lambda + 3} = 0.956611$ can be determined by using Theorem 2.7. In the next two subsections we discuss variants of Flory model where we can have houses that have more than one story.

Remark 3.5. As might be expected, there are more maximal configurations in the Riviera model than in the Flory model, since the Perron–Frobenius eigenvalue in the Riviera model is greater than the plastic number (which is the Perron–Frobenius eigenvalue in the Flory model). It is also interesting to compare these constants with 2 since there are 2^n binary sequences of length n when we do not impose any restrictions on those sequences.

3.2.1. Multi-story Flory models

By using the same technique as in the last two subsections, we could readily obtain generating functions related to the multi-story Flory models. However, the sequences counting the maximal configurations (in multi-story Flory models) with fixed length or with fixed number of houses turn out to be some well-known sequences. Therefore, they can be related to our structures by establishing the appropriate bijections.

The simplest way to explain the connection of the sequences that arise from the multi-story Flory models and some already known sequences is by using an example. Let us consider the two-story Flory model. In this model, all the houses have precisely two stories and each story of each house needs to get sunlight from both east and west (we assume that there is no obstruction to sunlight on the east and west boundary of the tract of land). This implies that each house needs to have at least two empty lots immediately to the east and immediately to the west of it (except

those houses that are near the boundary). Let us have a look at all the maximal configurations of length 6. Those are 200200, 200020, 020020, 200002, 020002 and 002002. Clearly, between each two houses we can have 2, 3 or 4 empty lots. Less than 2 empty lots would mean that those houses are not getting the sunlight from both east and west (hence, the permissibility would be violated) and more than 4 empty lots would mean that we can add additional house between those two houses without violating permissibility (hence, the maximality would be violated). This implies that if we take a block of lots that includes all the empty lots to the west of the house and the house itself, this block can have length 3, 4 or 5. This is true for all the houses except maybe the first one since this one can get the sunlight from outside of the tract of land. For the first house this block including all the empty lots to the west of it and the first house itself can be of size 1, 2 or 3. For that reason, we artificially add 2 empty lots in front of our maximal configurations. Also, if the configuration does not end with a house, empty lots at the end of the configuration will not be a part of any block. To solve this problem, we artificially add 002 at the end of each maximal configuration since in this way we will create a block at the end that has length 3, 4 or 5. After adding these 5 additional lots, we ended up with strings of length 11. Each of these strings is different and they can be divided into blocks that contain one house and all the empty lots preceding that house. As explained, these blocks will be of length 3, 4 and 5 and hence will give as a representation of number 11 as a sum of numbers 3, 4 and 5. In our example we have

$$\begin{aligned}
 200200 &\rightarrow 00|200200|002 \rightarrow \underbrace{002}_3 \underbrace{002}_3 \underbrace{00002}_5 \\
 200020 &\rightarrow 00|200020|002 \rightarrow \underbrace{002}_3 \underbrace{0002}_4 \underbrace{0002}_4 \\
 020020 &\rightarrow 00|020020|002 \rightarrow \underbrace{0002}_4 \underbrace{002}_3 \underbrace{0002}_4 \\
 200002 &\rightarrow 00|200002|002 \rightarrow \underbrace{002}_3 \underbrace{00002}_5 \underbrace{002}_3 \\
 020002 &\rightarrow 00|020002|002 \rightarrow \underbrace{0002}_4 \underbrace{0002}_4 \underbrace{002}_3 \\
 002002 &\rightarrow 00|002002|002 \rightarrow \underbrace{00002}_5 \underbrace{002}_3 \underbrace{002}_3
 \end{aligned}$$

It is also trivial to go in the other direction. If we are given one possible ordered partition of number 11 into parts 3, 4 and 5, we can just concatenate the blocks whose lengths will correspond to those parts and in the end we just remove the first two and the last three lots. For example

$$11 = 3 + 5 + 3 \rightarrow \underbrace{002}_3 \underbrace{00002}_5 \underbrace{002}_3 \rightarrow 00|200002|002 \rightarrow 200002.$$

It is clear from here that there is a bijection between maximal configurations of the two-story Flory model of length n and compositions (ordered partitions) of number $n + 5$ into parts 3, 4 and 5. The sequence that counts the number of ordered partitions of n into parts 3, 4 and 5 can be found on the OEIS under name [A017818](#).

Of course, there is nothing special about the two-story Flory model and the same reasoning can be applied to any multi-story Flory model. If we consider k -story Flory model, then we get maximal configurations with at least k empty lots and at most $2k$ empty lots between each two occupied lots. Adding k empty lots in front of each maximal configuration and k empty lots and one occupied lot at the end of each configuration we get a string of length $n + 2k + 1$. Splitting this string into blocks containing an occupied lot and all the empty lots to the west of it, we get decomposition of the number $n + 2k + 1$ into parts $k + 1, k + 2, \dots, 2k + 1$. On the other hand, if we start from a composition of the number $n + 2k + 1$ into parts $k + 1, k + 2, \dots, 2k + 1$, we can trivially reconstruct the corresponding maximal configuration. Hence, we have the following proposition.

Proposition 3.6. *The number of maximal configurations with fixed length $n \in \mathbb{N}$ in the k -story Flory model ($k \in \mathbb{N}$) is equal to the number of compositions (ordered partitions) of number $n + 2k + 1$ into parts $k + 1, k + 2, \dots, 2k + 1$.*

For $k \in \{3, \dots, 9\}$ those sequences can be found on the OEIS under the following names

- $k = 3$ – compositions of n into parts p where $4 \leq p \leq 7$ – [A017829](#),
- $k = 4$ – compositions of n into parts p where $5 \leq p \leq 9$ – [A017840](#),
- $k = 5$ – compositions of n into parts p where $6 \leq p \leq 11$ – [A017851](#),
- $k = 6$ – compositions of n into parts p where $7 \leq p \leq 13$ – [A017862](#),
- $k = 7$ – compositions of n into parts p where $8 \leq p \leq 15$ – [A017873](#),
- $k = 8$ – compositions of n into parts p where $9 \leq p \leq 17$ – [A017884](#),
- $k = 9$ – compositions of n into parts p where $10 \leq p \leq 19$ – [A017895](#).

Since maximal configurations of length $n = 1$ are in bijection with the ordered partitions of the number $2k + 2$, we have to look at all the above sequences with the offset of $2k + 1$.

Note that it is straightforward to obtain the recurrence relation for these sequences (and from that recurrence relation also the generating function) regardless of the value of k . Hence, even though the sequences for $k \geq 10$ do not appear on the OEIS, we can easily calculate their elements. Let us explain how to get the recurrence relation and the generating function in the case $k = 3$ and then formulate the general result. Denote by a_n the number of compositions of n into parts 4, 5, 6 and 7. Clearly, $a_0 = 1$ since there is one way to get 0 from parts 4, 5, 6 and 7, we do not take any of the parts. For integers smaller than the smallest available part there are zero compositions, hence $a_1 = a_2 = a_3 = 0$. Since available parts are integers between $k + 1$ and $2k + 1$, they can be composed in only one way, therefore $a_4 = a_5 = a_6 = a_7 = 1$. For any $n \geq 8$ the logic is that we can first compose $n - 4$ from available parts and then just add 4 and similar for other available parts. Hence, we have the following recurrence relation

$$a_n = a_{n-4} + a_{n-5} + a_{n-6} + a_{n-7}.$$

From this recurrence relation, we clearly have that the generating function of the sequence a_n is given with

$$F(x) = \frac{1}{1 - x^4 - x^5 - x^6 - x^7}.$$

It is clear now that we can follow the same logic in the general case with k -story houses.

Proposition 3.7. Fix $k \in \mathbb{N}$. Let $(a_n)_{n \geq 0}$ be the integer sequence where

$$a_n = \text{the number of compositions of } n \text{ into parts } k + 1, k + 2, \dots, 2k + 1.$$

Then for every $n \geq 2k + 2$ the sequence $(a_n)_{n \geq 0}$ satisfies the recurrence relation

$$a_n = \sum_{i=k+1}^{2k+1} a_{n-i},$$

with initial conditions

$$a_0 = 1,$$

$$a_i = 0, \quad 1 \leq i \leq k$$

$$a_i = 1, \quad k + 1 \leq i \leq 2k + 1.$$

Furthermore, the generating function $F(x)$ of the sequence $(a_n)_{n \geq 0}$ is given with

$$F(x) = \frac{1}{1 - \sum_{i=k+1}^{2k+1} x^i}.$$

The situation with maximal configurations with fixed number of houses in the k -story Flory model is even simpler. Let us again describe what happens in the case $k = 2$ and then formulate the general result. To find all the maximal configurations with exactly 3 houses, we start with the maximal configuration 2002002 which has 3 houses. Now we can obtain all the maximal configurations with exactly 3 houses from this one in the following way: for each house, we choose whether we want to put 0, 1 or 2 additional empty lots to the west of it and for the last house, we also choose whether we want to put 0, 1 or 2 additional empty lots to the east of it. Since we have 3 houses, we need to make 4 decisions and we have 3 options in each of those 4 decisions. Hence, to each ordered quadruple of elements 0, 1 and 2, we can assign a unique maximal configuration with precisely 3 houses. For example

$$(1, 2, 2, 1) \rightarrow 0|200|00|200|00|2|0,$$

$$(0, 1, 2, 1) \rightarrow 200|0|200|00|2|0,$$

$$(1, 1, 2, 0) \rightarrow 0|200|0|200|00|2,$$

$$(0, 0, 0, 0) \rightarrow 2002002.$$

Since we start with 2 empty lots between each two houses, by adding 0, 1 or 2 more empty lots, we end up with 2, 3 or 4 empty lots between each two houses which still guarantees maximality and we obviously cannot violate permissibility with adding empty lots. Also, since we started from a configuration that has houses on the easternmost and on the westernmost lots, adding 0, 1 or 2 additional empty lots before the first house or after the last house, will not violate maximality. On the other hand, if we are given a maximal configuration in the two-story Flory model with precisely 3 houses, we can easily obtain the corresponding ordered quadruple of elements 0, 1 and 2. Each maximal configuration has to have 0, 1 or 2 empty lots before the first house and after the last one and it has to have at least 2 and at most 4 empty lots between each two houses. First element of the ordered quadruple corresponds to the number of empty lots in front of the first house. Counting the number of empty lots between each two houses and subtracting 2 gives as all the other elements of our ordered quadruple, except for the last one. The last one corresponds to the number of empty lots after the last house. For example

$$02000200200 \rightarrow \underbrace{0}_1 \underbrace{2 \ 000}_3 \underbrace{2 \ 00}_2 \underbrace{2 \ 00}_2 \rightarrow (1, 3 - 2, 2 - 2, 2) = (1, 1, 0, 2).$$

Hence, the total number of maximal configurations with precisely 3 houses in the two-story Flory model is 3^4 . Again, there is nothing special about the two-story Flory model and the same logic can be applied to any k -story Flory model. If we want to count the number of maximal configurations with precisely n houses in the k -story Flory model, we start with the configuration

$$\overbrace{k \underbrace{00 \dots 00}_{k \text{ zeros}} k \underbrace{00 \dots 00}_{k \text{ zeros}} k 0 \dots 0k}_{n \text{ houses}}$$

and then we choose ordered $(n+1)$ -tuple of elements $0, 1, \dots, k$. Denote this $(n+1)$ -tuple with (k_0, k_1, \dots, k_n) . To this $(n+1)$ -tuple we assign the maximal configuration

$$\overbrace{\underbrace{00 \dots 00}_{k_0 \text{ zeros}} k \underbrace{00 \dots 00}_{k \text{ zeros}} \underbrace{00 \dots 00}_{k_1 \text{ zeros}} k \underbrace{00 \dots 00}_{k \text{ zeros}} \underbrace{00 \dots 00}_{k_2 \text{ zeros}} \dots \underbrace{00 \dots 00}_{k_{n-1} \text{ zeros}} k \underbrace{00 \dots 00}_{k_n \text{ zeros}}}_{n \text{ houses}}$$

In this way we obtain a configuration where in front of the first house and after the last house we have at most k empty lots, and between each two houses we have between k and $2k$ empty lots. Such configurations are obviously maximal. On the other hand, if we start from a maximal configuration with precisely n houses, we can trivially reconstruct the corresponding $(n+1)$ -tuple of elements $0, 1, \dots, k$.

Proposition 3.8. *The number of maximal configurations with fixed number of houses $n \in \mathbb{N}$ in the k -story Flory model ($k \in \mathbb{N}$) is equal to the number of ordered $(n+1)$ -tuples of elements $0, 1, \dots, k$ which is equal to $(k+1)^{n+1}$.*

3.2.2. Mixed Flory model

Just as in the case of the mixed Riviera model, mixed Flory model refers to the model where one-story and two-story houses can be built on the same $1 \times n$ tract of land. For this model we again obtain trivariate generating function $F(x, y, z)$, where x is a formal variable associated with the number of one-story houses in the configuration, y is a formal variable associated with the number of two-story houses in the configuration and z is a formal variable associated with the length of the configuration. The explicit form of this trivariate generating function was obtained using R and Maxima and it is given in the [Appendix](#) (see (A.3)). By expanding $F(x, y, z)$ into formal power series in powers of z , we get

$$\begin{aligned} F(x, y, z) = & 1 \\ & + yz \\ & + 2yz^2 \\ & + (3y + x^2)z^3 \\ & + (y^2 + 2y + 2x^2)z^4 \\ & + (3y^2 + y + x^3 + x^2)z^5 \\ & + (6y^2 + 2x^2y + 2x^3)z^6 \\ & + (y^3 + 7y^2 + 6x^2y + x^4 + x^3)z^7 \\ & + (4y^3 + 6y^2 + (2x^3 + 8x^2)y + 3x^4)z^8 \\ & + (10y^3 + (3x^2 + 3)y^2 + (6x^3 + 6x^2)y + x^5 + 3x^4)z^9 \\ & + (y^4 + 16y^3 + (12x^2 + 1)y^2 + (2x^4 + 8x^3 + 2x^2)y + 4x^5 + x^4)z^{10} + \dots \end{aligned}$$

The first several elements of the sequence that counts the number of different maximal configurations of length n are 1, 2, 4, 5, 6, 10, 16, 23, 32, 47, ... This sequence does not appear on the OEIS.

4. One-story model on an $m \times n$ grid for $m = 2, 3$ and $n \in \mathbb{N}$

In this section we come back to the original settlement model introduced in [25] in which we have a rectangular $m \times n$ tract of land where $m > 1$ and the sun can come from the east, south and west. The question that we were interested in is what is the total number of all maximal configurations for all grid sizes $m \times n$. This is exactly the question posed in [26, Question 2.2]. We give a partial answer to this question when the considered tract of land is narrow. As already announced in the introduction, we were unable to obtain a closed-form formula for all grid sizes, but for grids of size $2 \times n$ and $3 \times n$ we derive the (bivariate) generating functions counting the number of maximal configurations. The method we use is again the transfer matrix method which could be similarly adopted for bigger m 's, but the calculations soon get increasingly infeasible. In this section we only consider one-story houses and we do not look at the Flory's version of the problem where each house would need to get sunlight from east, south and west. Of course, those models could be studied with the same method.

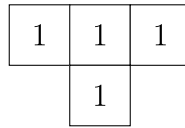


Fig. 7. The forbidden pattern.

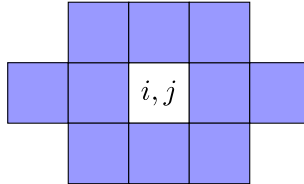


Fig. 8. Lots that need to be checked to verify maximality of a configuration.

Recall that every configuration on an $m \times n$ tract of land can be encoded as a $m \times n$ matrix C , where $c_{i,j} = 1$ if the lot (i, j) is occupied and $c_{i,j} = 0$ otherwise. We now treat all the possible different columns of such matrices as letters in our alphabet. Notice that those are just binary numbers of length m written as column vectors. Once we interpret each column of such a matrix as a letter, every configuration can be viewed as a word of length n . As already explained in the introduction, the transfer matrix method can be applied if the properties of permissibility and maximality of a configuration can be verified by inspecting only finite size patches of a given configuration. Luckily, in our model, the properties of maximality and permissibility are locally verifiable, i.e. if one wants to check whether a state of a certain lot (occupied or unoccupied) has caused the configuration to be impermissible or not maximal, one only needs to check the situation on the lots in a finite radius of the observed lot. These verifications whether some configuration is permissible and maximal are done directly on the matrix C , but as soon as we find a forbidden pattern, we forbid the corresponding word. More precisely, to verify that a configuration is permissible, one needs to check that no occupied lot (i, j) borders simultaneously with three other occupied lots to its east, south and west. This can be done by inspecting whether the forbidden pattern shown in Fig. 7 appears anywhere in the matrix C . To verify that a configuration is maximal, one needs to additionally check that no unoccupied lots can be built on. Since it holds that

$$c_{i,j} = 0 \implies \begin{aligned} &(c_{i,j-2} = 1 \text{ and } c_{i,j-1} = 1 \text{ and } c_{i+1,j-1} = 1) \text{ or} \\ &(c_{i,j+1} = 1 \text{ and } c_{i,j+2} = 1 \text{ and } c_{i+1,j+1} = 1) \text{ or} \\ &(c_{i-1,j-1} = 1 \text{ and } c_{i-1,j} = 1 \text{ and } c_{i-1,j+1} = 1) \text{ or} \\ &(c_{i,j-1} = 1 \text{ and } c_{i,j+1} = 1 \text{ and } c_{i+1,j} = 1), \end{aligned}$$

this can be done by inspecting all the lots surrounding the lot (i, j) that appear in the above expression. Those are precisely the shaded lots in Fig. 8.

Therefore, the transfer matrix method can be applied in a completely analogous way as before, but as the number of rows grows, the alphabet becomes bigger so we only consider the cases $m = 2$ and $m = 3$.

4.1. $2 \times n$

In this subsection, we derive the bivariate generating function from which we can extract the sequence that counts the total number of different maximal configurations of fixed length n (and width 2) and the sequence that counts the total number of different maximal configurations with precisely k houses on all the $2 \times n$ grids ($n \in \mathbb{N}$). One example of a maximal configuration on a 2×6 tract of land is

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Since we can apply the transfer matrix method, we can again use Algorithm 1 to construct the digraph with which we can encode all the maximal configurations. Once we have the digraph, we can define the matrix function $A(x)$. This matrix is obtained from the adjacency matrix of the corresponding digraph, but instead of putting 1 on the position (i, j) when the transition from node i to node j is possible, we put x if the transition from node i to node j adds one house to the configuration and we put x^2 if the transition from node i to node j adds two houses to the configuration (notice that it is impossible that the transition from node i to node j adds zero houses to the configuration because the word $(0, 0)^T$ never appears since it would cause that the resulting configuration is not maximal). As before, the bivariate generating function is obtained using R and Maxima and the precise shape can be found in the Appendix (see (A.4)). The first few coefficients in the expansion of $F(x, y)$ are given in Table 3. By inspecting the non-zero coefficients in the table, we see that the ratio $\frac{k}{n}$ is in-between 1 and $\frac{5}{3}$, for large n .

Table 3The first few coefficients in the expansion of the bivariate generating function $F(x, y)$.

$n \backslash k$	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}
1	1																	
y			1															
y^2				1														
y^3					1													
y^4						4												
y^5							4	2										
y^6								4	5	1								
y^7									4	4	8							
y^8										4	4	18	3					
y^9											4	4	25	16	1			
y^{10}												4	4	33	31	13		
													4	4	41	42	50	4

Table 4The first few coefficients in the expansion of the bivariate generating function $F(x, y)$.

$n \backslash k$	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}	x^{13}	x^{14}	x^{15}	x^{16}	x^{17}
1	1																	
y				1														
y^2					1													
y^3						1		9	1									
y^4									4	8	7							
y^5											12	8	20	1				
y^6													24	12	65	4		
y^7														24	12	84	122	27
y^8															4	40	40	228
y^9																	24	44

Notice that there are 16 maximal configurations on 2×6 tract of land and 8 of those have 10 houses. This means that in addition to our example, there are 15 more maximal configurations on 2×6 tract of land and 7 more on that tract of land that have precisely 10 houses. The first several elements of the integer sequence that counts the number of maximal configurations of fixed length n (and width 2) can be read as row sums: 1, 1, 4, 6, 10, 16, 29, 50, 85, 145, ... and this sequence is still not included in the OEIS. On the other hand, by expanding $F(x, y)$ into the formal power series in powers of x (i.e. taking column sums), we can see that, beside the mentioned maximal configuration with precisely 10 houses, there are 15 more maximal configurations with the same number of houses. The first several elements of the integer sequence that counts the number of maximal configurations with precisely k houses on all the $2 \times n$ grids ($n \in \mathbb{N}$) can be read as column sums: 0, 1, 0, 1, 4, 4, 6, 9, 9, 16, ... This sequence is not found on the OEIS.

Remark 4.1. Here again we compute the expected number of occupied lots in a maximal $2 \times n$ configuration and the expected length n of a maximal $2 \times n$ configuration with k houses. The obtained values are

$$\langle k_{2,n}(n) \rangle = 1.437496 n \quad \text{and} \quad \langle n_{2,n}(k) \rangle = 0.724696 k.$$

4.2. $3 \times n$

In this subsection, we derive the bivariate generating function from which we can extract the sequence that counts the total number of different maximal configurations of fixed length n (and width 3) and the sequence that counts the total number of different maximal configurations with precisely k houses on all the $3 \times n$ grids ($n \in \mathbb{N}$). One example of a maximal configuration on a 3×5 tract of land is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly as in the case $2 \times n$ we obtain the bivariate generating function (see (A.5)). The first few coefficients in the expansion of $F(x, y)$ are given in Table 4. By inspecting the non-zero coefficients in the table, we see that the ratio $\frac{k}{n}$ is in-between $\frac{3}{2}$ and $\frac{7}{3}$, for large n .

Notice that there are 41 maximal configurations on 3×5 tract of land and 12 of those have 10 houses. This means that in addition to our example, there are 40 more maximal configurations on 3×5 tract of land and 11 more on that tract of land that have precisely 10 houses. The first several elements of the integer sequence that counts the number of maximal configurations of fixed length n (and width 3) can be read as row sums: 1, 1, 10, 19, 41, 105, 269, 651, 1560, 3861, ... and this sequence is still not included in the OEIS. On the other hand, by expanding $F(x, y)$ into the formal power series in powers of x (i.e. taking column sums), we can see that beside the mentioned maximal configuration with precisely

10 houses, there are 18 more maximal configurations with the same number of houses. The first several elements of the integer sequence that counts the number of maximal configurations with precisely k houses on all the $3 \times n$ grids ($n \in \mathbb{N}$) can be read as column sums: 0, 0, 1, 0, 0, 1, 9, 5, 8, 19. This sequence cannot be found on the OEIS.

Remark 4.2. In this case, the expected number of occupied lots in a maximal $3 \times n$ configuration and the expected length n of a maximal $3 \times n$ configuration with k houses are given as

$$\langle k_{3,n}(n) \rangle = 2.071886 n \quad \text{and} \quad \langle n_{3,n}(k) \rangle = 0.503345 k.$$

Since the largest possible number of occupied lots is, roughly, of the order $5n/3$ in the $2 \times n$ strip and $7n/3$ in the $3 \times n$ strip for large n , the respective efficiencies are given by

$$\varepsilon_{2 \times n} = 0.862498 \quad \text{and} \quad \varepsilon_{3 \times n} = 0.887951.$$

One can see that the efficiency of $2 \times n$ strip is lower than for the Riviera model. This low value is easily explained by the effects of the fully built lower row. As expected, its effects decrease with the increasing strip width.

5. Concluding remarks

In this paper we have considered a one-dimensional toy-model of a settlement planning problem introduced recently by three of the present authors. In particular, we studied maximal configurations of buildings in a narrow strip of land oriented east–west subject to the condition that each building must receive sunlight from either east or west or from both sides. We have formulated the problem of enumerating such maximal configurations as a problem of counting binary words of a given length satisfying certain additional conditions on the allowed patterns. By reducing the new problem to counting certain types of walks on a small directed graph with only six vertices we were able to employ the transfer matrix method which yielded generating functions for the enumerating sequences, and hence also their asymptotics. Along the way we discovered that our maximal configurations are equinumerous with certain type of restricted permutations.

We have also considered some generalizations of the original problem, such as allowing the buildings to have more than one floor and varying the type of restrictions on the sunlight direction. The methods developed on the toy model were successfully adapted to the more complex settings and allowed us to obtain multivariate generating functions for the corresponding enumerating sequences. We have also obtained generating functions for the sequences enumerating maximal configurations on strips of width 2 and 3, in most cases obtaining sequences not (yet) in the OEIS.

There are other ways, not explored here, to formulate the original problem and hence to extend our results. For example, one could consider our problem “in negative” and consider unoccupied places instead of occupied ones. In that case, the unoccupied places in a maximal configuration must form a dominating set. For our Riviera model, the maximality condition implies that such a dominating set would induce a graph of maximum degree one, hence a dissociation. (Dissociations interpolate between matchings and independent sets; see [5,6] for definition and some basic properties.) We are not aware of any results on such dominating sets. However, closely related (and less general) independent dominating sets are being intensely studied, along with other types of domination in graphs.

Our problem could be also formulated and studied on other types of lattices. Some of them, say the hexagonal one, could be better suited to modeling real planning applications. On the other hand, the triangular lattice might prove more tractable and might lead to closed-form results for the considered quantities.

Another problem, closely related to the present one, is to study temporal evolution of built configurations subject to given rules. It is known that the jamming density of the static and dynamic variant of the Flory model is not the same; in the static cases, all configurations are considered to be equally likely, while in the dynamic case some of them are less likely to evolve than some others. It would be interesting to explore how additional restrictions (with respect to the Flory model) affect the difference.

Finally, it would be interesting to model the evolution of configurations in terms of antagonistic games. It seems plausible that the interest of a developer is not aligned with the interests of inhabitants — one would prefer more buildings, hence more profit, while the others might prefer more sunshine, hence less profit for the former. Some toy-model simulations of several variants on dynamics are currently under way.

Data availability

No data was used for the research described in the article.

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Appendix

Here we give explicit expressions for generating functions related to models discussed in Sections 3 and 4.

- Bivariate generating function for the two-story Riviera model is given by

$$F(x, y) = -\frac{p(x, y)}{q(x, y)}, \quad (\text{A.1})$$

$$p(x, y) = x^3y^9 + 3x^3y^8 + 3x^3y^7 + (x^3 - x^2)y^6 - 3x^2y^5 - 4x^2y^4 + (x - 3x^2)y^3 - x^2y^2 - xy - 1,$$

$$q(x, y) = x^3y^{10} + 2x^3y^9 + x^3y^8 - x^2y^7 - 2x^2y^6 - 2x^2y^5 - x^2y^4 - xy^3 + 1,$$

where x is a formal variable associated with the number of two-story houses in the configuration and y is a formal variable associated with the length of the configuration.

- Trivariate generating function for the mixed Riviera model is given by

$$F(x, y, z) = -\frac{p(x, y, z)}{q(x, y, z)}, \quad (\text{A.2})$$

$$p(x, y, z) = x^6y^3z^{17} + ((x^4 - x^5)y^3 + x^6y^2)z^{15} + ((2x^4 - x^5)y^3 - x^6y^2)z^{14}$$

$$+ ((-3x^4 - x^3)y^3 - x^5y^2)z^{13} + ((-2x^4 - 3x^3 + 2x^2)y^3 + (x^4 - 2x^5)y^2)z^{12}$$

$$+ ((-3x^3 - x^2)y^3 - 4x^4y^2)z^{11} + ((-x^3 - 8x^2)y^3 + x^3y^2)z^{10}$$

$$+ ((-4x^2 - 2x + 1)y^3 + (x^3 - x^2)y^2)z^9 + ((2x^3 + x^2)y^2 - 4xy^3)z^8$$

$$+ ((-2x - 2)y^3 + 6x^2y^2)z^7 + ((2x^2 + 4x - 1)y^2 - y^3)z^6$$

$$+ ((6x - 1)y^2 - 2x^2y)z^5 + ((2x + 2)y^2 - x^2y)z^4$$

$$+ (3y^2 - y - x^2)z^3 + y^2z^2 + yz + 1,$$

$$q(x, y, z) = x^6y^3z^{17} + ((x^5 + x^4)y^3 + x^6y^2)z^{15} + 4x^4y^3z^{14} + (x^3y^3 + x^5y^2)z^{13}$$

$$+ ((3x^3 + 3x^2)y^3 + 2x^4y^2)z^{12} + (4x^2y^3 - x^4y^2)z^{11} + (2xy^3 + x^3y^2)z^{10}$$

$$+ ((2x + 1)y^3 + (-x^3 - 2x^2)y^2)z^9 + (y^3 - 3x^2y^2)z^8 + (-x^2 - 2x)y^2z^7$$

$$+ (-2x - 1)y^2z^6 + (-2y^2 - 2x^2y)z^5 - y^2z^4 + (-y - x^2)z^3 + 1,$$

where x is a formal variable associated with the number of one-story houses in the configuration, y is a formal variable associated with the number of two-story houses in the configuration and z is a formal variable associated with the length of the configuration.

- Trivariate generating function for the mixed Flory model is given by

$$F(x, y, z) = -\frac{p(x, y, z)}{q(x, y, z)}, \quad (\text{A.3})$$

$$p(x, y, z) = x^2yz^8 + x^2yz^7 - xyz^6 - 2xyz^5 + ((1 - 2x)y + 2x^2)z^4$$

$$+ ((2 - x)y + x^2)z^3 + (2y - x)z^2 + yz + 1,$$

$$q(x, y, z) = x^2yz^9 - xyz^7 - xyz^6 + ((1 - x)y + x^2)z^5 + yz^4 + yz^3 + xz^2 - 1,$$

where x is a formal variable associated with the number of one-story houses in the configuration, y is a formal variable associated with the number of two-story houses in the configuration and z is a formal variable associated with the length of the configuration.

- Bivariate generating function for the $2 \times n$ model is given by

$$F(x, y) = -\frac{p(x, y)}{q(x, y)}, \quad (\text{A.4})$$

$$p(x, y) = x^8y^5 - (x^5 + x^4)y^3 + (2x^3 - x^4)y^2 + (x - x^2)y - 1,$$

$$q(x, y) = x^9y^6 - x^6y^4 + (x^4 - x^5)y^3 - x^3y^2 - xy + 1,$$

where x is a formal variable associated with the number of houses in the configuration and y is a formal variable associated with the length of the configuration.

- Bivariate generating function for the $3 \times n$ model is given by

$$F(x, y) = -\frac{p(x, y)}{q(x, y)}, \quad (\text{A.5})$$

$$p(x, y) = (x^{40} - x^{39})y^{19} + (2x^{38} - x^{37})y^{18} + (-2x^{37} + 3x^{36} - 2x^{35} + x^{34} - x^{33})y^{17}$$

$$+ (-5x^{35} + x^{34} + 4x^{33} - x^{32} - x^{31})y^{16} + (-3x^{33} - x^{32} + 5x^{31} - x^{30} - 2x^{29})y^{15}$$

$$\begin{aligned}
& + (4x^{31} - 9x^{30} + 12x^{29} - 7x^{28} + x^{26})y^{14} \\
& + (15x^{29} - 13x^{28} + 4x^{27} - 3x^{26} - 2x^{25} + x^{24})y^{13} + (9x^{27} - 9x^{26} + 3x^{25} + 2x^{22})y^{12} \\
& + (11x^{24} - 11x^{23} + 12x^{22} - 5x^{21})y^{11} + (-15x^{23} + 34x^{22} - 11x^{21} + 9x^{20} - 7x^{19})y^{10} \\
& + (-9x^{21} + 30x^{20} - 14x^{19} + 7x^{18} - 9x^{17} + x^{16})y^9 \\
& + (-4x^{19} + 14x^{18} + 4x^{16} - x^{14} + x^{13})y^8 \\
& + (5x^{17} - 23x^{16} + 10x^{15} + x^{14} + x^{13} - x^{12})y^7 \\
& + (3x^{15} - 21x^{14} + 20x^{13} - 2x^{12} + x^{11} - x^{10} - x^9)y^6 \\
& + (2x^{13} - 15x^{12} + 15x^{11} + 2x^{10})y^5 + (-3x^{10} - 3x^9 + x^8 - x^7 + x^6)y^4 \\
& + (-x^8 - 5x^7 + x^6 + x^5)y^3 + (-x^6 + x^4 + x^3)y^2 - x^3y - 1, \\
q(x, y) = & (x^{41} - x^{40})y^{20} + (2x^{39} - x^{38})y^{19} + (-2x^{38} + 3x^{37} - 2x^{36})y^{18} \\
& + (-5x^{36} + x^{35} + 3x^{34} - 2x^{33})y^{17} + (-3x^{34} - x^{33} + 5x^{32} - 2x^{31})y^{16} \\
& + (4x^{32} - 7x^{31} + 10x^{30} - 4x^{29})y^{15} + (15x^{30} - 8x^{29} + 4x^{28} - x^{26})y^{14} \\
& + (9x^{28} - 6x^{27} - 2x^{26} + x^{25} - x^{24})y^{13} + (7x^{25} - 11x^{24} + 7x^{23} - 2x^{22})y^{12} \\
& + (-15x^{24} + 19x^{23} - 12x^{22} + 5x^{21} - 3x^{20})y^{11} + (-9x^{22} + 21x^{21} - 6x^{20} + 6x^{19})y^{10} \\
& + (-4x^{20} + 14x^{19} + 2x^{18} + 6x^{17} + 3x^{16})y^9 \\
& + (5x^{18} - 8x^{17} + 6x^{16} - x^{15} + x^{14} - x^{13})y^8 + (3x^{16} - 12x^{15} + 4x^{14} - x^{13})y^7 \\
& + (2x^{14} - 11x^{13} - x^{12} + 3x^{11} + x^{10} + x^9)y^6 + (-8x^{11} - 6x^{10} + x^9)y^5 \\
& + (-4x^9 - 4x^8 + x^7 - x^6)y^4 + (-3x^7 - x^5)y^3 + (-x^4 - x^3)y^2 + 1,
\end{aligned}$$

where x is a formal variable associated with the number of houses in the configuration and y is a formal variable associated with the length of the configuration.

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