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*Short Note*

# An Integer Programming Formulation of the Steiner Problem in Graphs

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*Abstract:* A succinct integer linear programming model for the Steiner problem in networks is presented.

*Key words:* graph, network, integer programming model, Steiner problem, Steiner tree.

Given a connected graph $G$ (undirected, without loops and multiple edges) with positive edge costs (lengths) and a set $Z \subseteq V(G)$ of special (distinguished) vertices, the Steiner problem in graphs (networks) asks for a minimum cost tree spanning $Z$ in $G$. In this paper $V(G)$ and $E(G)$ denote the vertex set and edge (arc) set of a graph or digraph $G$, respectively. Their cardinalities are $n := |V(G)|$ and $m := |E(G)|$; we also put $p := |Z|$. The cost of an edge or arc $ij$ is denoted by $c_{ij}$. The cost of a subgraph is the sum of all its edge (arc) costs.

The Steiner problem has an extensive literature and numerous applicatons, such as the design of road networks, integrated circuits and telephone networks. Many exact and approximation methods have been developed for this NP-hard problem. For good surveys see e.g. [3, 7].

The aim of this note is to present a new integer linear programming (ILP) formulation of the Steiner problem. Several such formulations are known [7]. Any ILP problem can be relaxed from the integrality of the variables and solved as an LP problem (by the classical simplex or recent polynomial time algorithms). The obtained minimum value of the criterion function is evidently a lower bound for the minimum cost of a Steiner tree. We do not study the quality of this bound but note that some models provide only weak approximation of the optimal value [4]. Our formulation has an additional advantage because of its small size as to the number of variables, denoted by $\alpha$, as well as to the number of constraints, say $\beta$. Recall that ILP models in the literature use only binary variables and have the following parameters:

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Beasley [2]: $\alpha = 2mp$

$$\beta = (2m + n)(p - 1) + 2$$

Claus and Maculan (see [6, p. 269]):

$$\alpha = 2mp$$

$$\beta = (2m + n)(p - 1)$$

Wong [8]: $\alpha = 2m(p + 1)$

$$\beta = (2m + n)(p - 1)$$

(see also Magnanti and Wong [5] and Jain [4])

Aneja [1]: $\alpha = m$

$\beta =$ the number of all edge cuts separating $W$ and $V(G) - W$ with $Z \cap W \neq \emptyset$ and $Z \cap (V(G) - W) \neq \emptyset$; hence $\beta$ depends exponentially on $m$, $n$ and $p$.

We shall present an ILP formulation with:

$\alpha = 2m + n - 1$ variables ($2m$ binary and $n - 1$ integer variables)

and

$\beta = 4m + 3n + p - 4$ constraints .

Thus, for typical instances, our model is smaller than the others.

Our formulation is based on the following ideas. Pick up a vertex $v_0 \in Z$. Let $\vec{G}$ denote the digraph obtained from $G$ by replacing each edge $ij$ by two oppositely directed arcs $ij$ and $ji$, both with cost $c_{ij}$, and deleting the arcs entering $v_0$. If $T$ is a tree spanning $Z$ in $G$, orient all edges in $T$ such that $v_0$ is a source in the obtained directed outtree $\vec{T}$ (i.e. each vertex in $\vec{T}$ is reachable from $v_0$ along a directed path). Thus we have to describe $\vec{T}$ as a subgraph of $\vec{G}$.

We shall assume that $V(G) = \{1, 2, \ldots, n\}$, $Z = \{1, 2, \ldots, p\}$ and $v_0 = 1$. Further we denote $V^-(j) := \{i \in V(\vec{G}) | ij \in E(\vec{G})\}$. For a better understanding of our construction Fig. 1 can be useful where $n = 9$, $p = 4$ ($Z$-vertices are depicted as

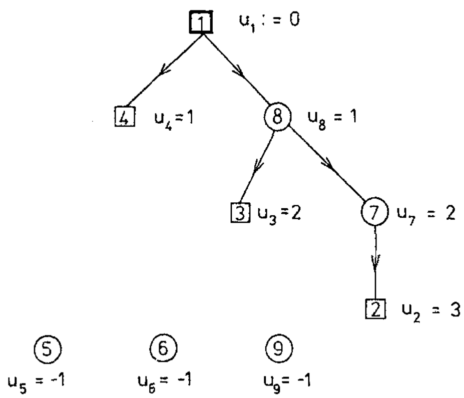


**Fig. 1**

squares), and only the edges of $\vec{T}$ are drawn. Each arc $ij$ is assigned a variable $x_{ij} \in \{0, 1\}$. An arc of $\vec{G}$ is called 1-arc or 0-arc if $x_{ij} = 1$ or $x_{ij} = 0$, respectively. We want to have $x_{ij} = 1$ if and only if $ij$ belongs to $\vec{T}$. Thus each $j \in V(\vec{T}) - \{1\}$ is entered by exactly one 1-arc. Moreover, each vertex $j \in V(\vec{G})$ is assigned a variable $u_j$, which should express the distance of $j$ from source 1 in $\vec{T}$ if $j \in V(\vec{T})$ ($u_j$ is the number of arcs in the unique directed $1 - j$ path). We put $u_j := -1$ if $j$ does not belong to $\vec{T}$. Thus, every $u_j \leq n - 1$. We have arrived to the following model.

$$\text{minimize} \quad \sum_{ij \in E(\vec{G})} c_{ij} x_{ij} \tag{1}$$

subject to

$$\sum_{i \in V^-(j)} x_{ij} \leq 1 \qquad \forall j \in V(\vec{G}) - \{1\} \tag{2}$$

$$n \sum_{i \in V^-(j)} x_{ij} \geq u_j + 1 \qquad \forall j \in V(\vec{G}) - \{1\} \tag{3}$$

$$(n + 1) \sum_{i \in V^-(j)} x_{ij} \leq n(u_j + 1) \qquad \forall j \in V(\vec{G}) - \{1\} \tag{4}$$

$$1 - n(1 - x_{ij}) \leq u_j - u_i \leq 1 + n(1 - x_{ij}) \qquad \forall ij \in E(\vec{G}) \tag{5}$$

$$x_{ij} \in \{0, 1\} \qquad \forall ij \in E(\vec{G}) \tag{6}$$

$$u_1 = 0, \qquad u_i \geq 0 \qquad \forall i \in Z - \{1\} \tag{7}$$

Since it is clear that any minimum tree $\vec{T}$ spanning $Z$ fulfils (1)–(7) it remains to show the converse, i.e. that any solution of (1)–(7) corresponds to a minimum tree $\vec{T}$ spanning $Z$. Let $\vec{H}$ denote the subgraph of $\vec{G}$ formed by the 1-arcs. We examine the properties of $\vec{H}$.

First recall that $\vec{G}$ has no $i1$ arcs and thus no 1-arc of $\vec{H}$ enters vertex 1. By (2) at most one 1-arc enters any vertex. According to (7) and (3) each $Z$-vertex $j \neq 1$ is entered by an 1-arc $ij$. Then by (4) $u_j \geq 1$ and by (5) $u_i = u_j - 1$. These considerations can be repeated for $i$ instead of $j$ (unless $u_i = 0$) and thus we can prove that for each $Z$-vertex $j$ there is a vertex $i$ with $u_i = 0$ and a directed $i - j$ path consisting of 1-arcs. By (4) such a vertex $i$ cannot be entered by any 1-arc. On the other hand, if $i \neq 1$ then by (3) $i$ must be entered by a 1-arc. Consequently, $i = 1$ and hence all $Z$-vertices are reachable from 1 by a path of 1-arcs.

Since (1) excludes endvertices which are not $Z$-vertices, $\vec{H}$ must be a minimum cost tree spanning $Z$, as desired.

Evidently, the numbers $u_j$ are restricted to be real ones with $-1 \leq u_j \leq n - 1$ (cf. (3) and (4)). Moreover, as we have seen, $u_j$ can be required to be integers. In fact, model (1)–(7) provides only integer solutions. (Let $u_j > -1$ be minimal noninteger. Then by (3) and (5) we arrive to another $u_i = u_j - 1$, a contradiction.) Hence (1)–(7) is an ILP model.

*Remark:* Our ILP model can be easily transformed to a fully binary model as follows. Since $-1 \leq u_j \leq n - 1$, we can put $\bar{u}_j := u_j + 1$ ($j \in V(G) - Z$) and then express $\bar{u}_j$ by binary variables $y_{j0}, \ldots, y_{jt} \in \{0, 1\}$:

$$\bar{u}_j = 2^0 y_{j0} + 2^1 y_{j1} + \cdots + 2^t y_{jt}$$

where $t = \lfloor \log_2 n \rfloor$. The variables $u_j$ ($j \in Z - \{1\}$) can be expressed immediately and their nonnegativity constraints (7) are omitted. Thus we receive a model with at most $2m + (n - 1)(\lfloor \log_2 n \rfloor + 1)$ $0 - 1$ variables and at most $4m + 3n - 3$ constraints, where $m$ and $n$ are the numbers of edges and vertices of the original graph $G$, respectively. Also this model remains smaller than the models in the literature.

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