Fundamentals of Image Processing

Lectures 09 and 10

Introduction to Signal Theory
and
The Sampling Theorem

Introduction to Signal Theory

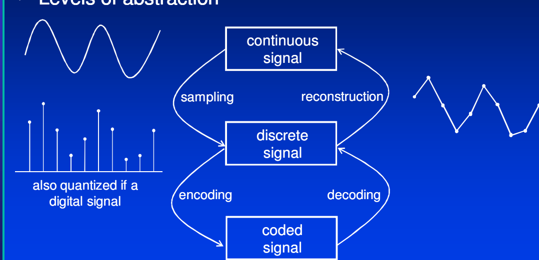
- Signals
- Sampling and Reconstruction
- Aliasing
- Mathematical Models for Signals
- Spatial and Frequency Domains
- Convolution and Comb Filters
- The Sampling Theorem
- Filters: Overview

Signals

- Signal: a detectable physical quantity arising from the variation of some physical magnitude (e.g., voltage, magnetic field strength, color, etc.) over time or space
- Examples
  - image: variation of color in space
  - video: variation of color both in space and time
- Sampling: discretization of the domain of a signal
- Quantization: discretization of the range of a signal
- Digital Signal: both sampled and quantized

Signal Representation

- Levels of abstraction



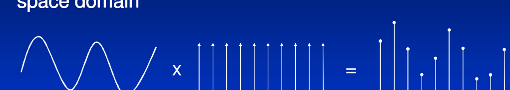
encoding: discrete to finite representation (lossy *vs* lossless)

Comments

- Continuous signals are known as analog signals in engineering
- Signal reconstruction is based on interpolation
- If the interpolation method recovers the original signal the reconstruction is said to be ideal or exact
- Recovering the original signal requires sampling at the appropriate rate (to avoid aliasing)

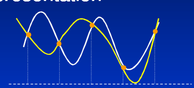
Sampling (Space Domain)

- Sampling is equivalent to multiplication by a comb filter in space domain



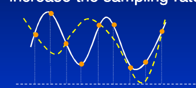
Aliasing

- Distinct signals can have the same point sampling representation

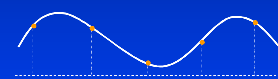


Avoiding Aliasing

- In order to avoid aliasing one can
  - Increase the sampling rate (to avoid ambiguity)



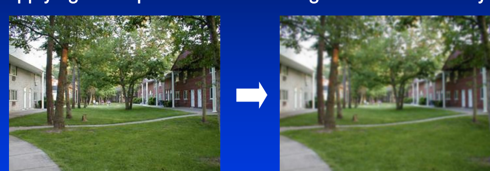
  - Remove high frequencies (low-pass filtering) before sampling



- It is not possible to avoid aliasing after sampling
  - if high frequencies get mixed with low frequencies we cannot reconstruct the original signal anymore

Low Pass Filters

- Smooth a signal by removing rapidly changing information
- Applying a low-pass filter to an image will make it blurry



- To understand filtering we need to look at the signal in the frequency domain (need appropriate tools)

Mathematical Models for Signals

- A signal is represented by a mathematical object that records the variation of the physical magnitude
- Functional model: deterministic variations
  - Spatial Model
  - Spectral (Frequency) Model

  Dual representations
- Stochastic model: non-deterministic variations

Spatial *vs* Frequency Domain

- Dual representations

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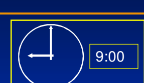
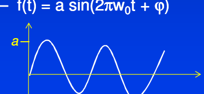


- Space or Time Domain: a signal is represented by the variation of a physical magnitude over space or time
- Frequency Domain: a signal is characterized by its amplitude *a*, its frequency w_0 and its phase φ
  - $f(t) = a \sin(2\pi w_0 t + \varphi)$



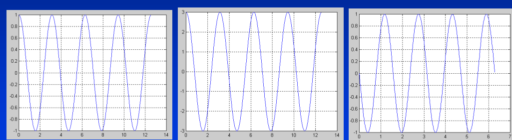
Spatial Model

- $f : U \subset \Re^m \rightarrow V$, where V is a vector space
  - U represents a region in space or a time interval
  - U is called the space domain or time domain
- Represents the signal variation in space or time domain
  - Relates to the signal magnitude of the physical world
- Dimension of the signal = dimension of the domain U
  - Audio: 1D signal over time $(f : U' \subset \Re \rightarrow \Re^n)$
  - Photograph: 2D signal over a subset of $\Re^2$ $(f : U \subset \Re^2 \rightarrow \Re^n)$
  - Video: 3D signal - varies over time $(f : (U \times \Re) \subset \Re^3 \rightarrow \Re^n)$

Spectral Model

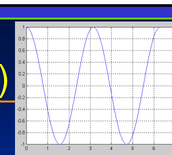
- A signal is characterized by its amplitude *a*, its frequency w_0, and its phase φ

  $f(t) = a \cos(2\pi w_0 t + \varphi)$



a = 1; w_0 = 4; φ = 0   a = 3; w_0 = 4; φ = 0   a = 1; w_0 = 4; φ = π/2

Spectral Model (Cont.)

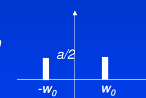


a = 1; w_0 = 2; φ = 0

- $\text{Freq}(f)(s) = \begin{cases} a, & \text{if } s = w_0 \\ 0, & \text{if } s \neq w_0 \end{cases}$

- Since $\cos(t) = 0.5*(e^{it} + e^{-it})$

- $\text{Freq}(f)(s) = \begin{cases} 0.5*a, & \text{if } s = w_0 \text{ or } s = -w_0 \\ 0, & \text{otherwise} \end{cases}$



Fourier Transform

- Maps a space domain representation of a signal onto its corresponding frequency domain representation

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t s} dt$$

- Intensity of each frequency *s* in the signal *f*
- Inverse Fourier Transform

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(s) e^{2\pi i t s} ds$$

Convolution

- An integral that expresses the amount of overlap of one function *g* as it is reversed and shifted over another function *f* *(more often over an infinite range)*

$$f * g = \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) d\tau$$

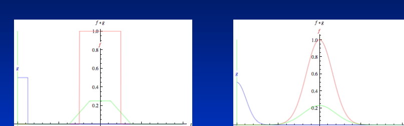
Convolution Applet

- *Applet available at: http://www.jhu.edu/~signals/convolve/*
- *Reference: Weisstein, Eric W. "Convolution." From MathWorkd --A Wolfram Web Resource. http://mathworld.wolfram.com/Convolution.html*

The Convolution Procedure

1. Flip (reverse) the function $g(\tau)$ in time yielding $g(-\tau)$
2. Shift $g(-\tau)$ by an amount $t=t_1$ giving $g(t_1-\tau)$
3. Multiply the shifted $g(t_1-\tau)$ by $f(\tau)$ obtaining $f(\tau)g(t_1-\tau)$
4. Integrate $f(\tau)g(t_1-\tau)$ to find the area under the product to obtain the value of the convolution at the single time t_1
5. Complete steps 1-4 for all values of t_1 running from -infinity to +infinity

Convolution in Action



*"The animations above graphically illustrate the convolution of two rectangle functions (left) and two Gaussians (right). In the plots, the green curve shows the convolution of the blue and red curves as a function of **t**, the position indicated by the vertical green line. The gray region indicates the product as a function of **t**, so its area as a function of **t** is precisely the convolution".*

Animation: courtesy of MathWorld, copyright © http://mathworld.wolfram.com/Convolution.html

Convolution Properties

Commutativity: $f*g = g*f$

Associativity: $f*(g*h) = (f*g)*h$

Distributivity: $f*(g+h) = (f*g) + (f*h)$

Identity Element: $f*\delta = \delta*f = f$ (δ *is the Dirac Delta*)

Associativity with Scalar Multip.: $a(f*g) = (af)*g = f*(ag)$

Differentiation Rule: $D(f*g) = Df*g = f*Dg$

Convolution Theorem: $\mathfrak{F}(f*g) = \mathfrak{F}(f) \cdot \mathfrak{F}(g)$

convolution in one domain equals point-wise multiplication in the other domain

Discrete Convolution

- A discrete version of the convolution operator

$$f * g = \sum_{m=-\infty}^{+\infty} f(m)g(n-m)$$

Discrete Convolution Applet

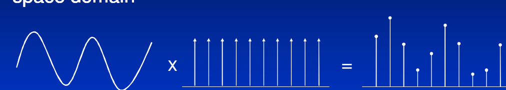
- *Applet available at: http://www.jhu.edu/~signals/discreteconv2/index.html*

Convolution

- Example: continuous signal * discrete signal

Remember: Sampling (Space Domain)

- Sampling is equivalent to multiplication by a comb filter in space domain

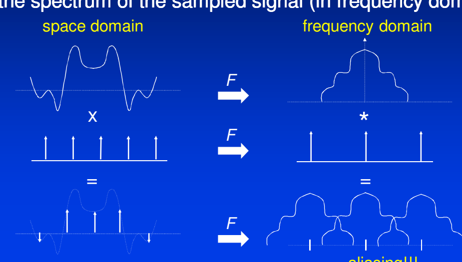


Convolution and Comb Filter

- Multiplication in space domain is equivalent to convolution in frequency domain and vice versa
- The Fourier transform of a comb filter is another comb filter: $F(\delta_{\Delta t}(t)) = (1/\Delta t)\ \delta_{1/\Delta t}(s)$

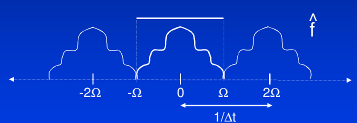
Sampling (Frequency Domain)

- Sampling is equivalent to convolution of a comb filter with the spectrum of the sampled signal (in frequency domain)



The Sampling Theorem

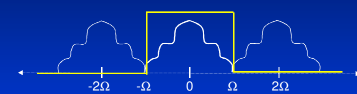
- Let f be a bandlimited signal and let Ω be the smallest frequency such that supp $\hat{f} \subset [-\Omega, \Omega]$. Then, f can be exactly recovered from a uniform sample sequence if $\Delta t < 1/(2\,\Omega)$



If $\Delta t < 1/(2\,\Omega)$ for the comb filter, its spectrum will have spikes spaced more than 2 Ω apart!

Recovering the Original Signal

- After isolating one copy of the replicated spectrum, the original signal can be recovered



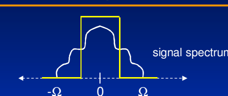
- A copy is isolated with the use of a lowpass filter
- The original signal in space domain is recovered with the use of an inverse Fourier transform

Filters: Overview

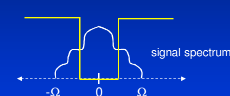
- Change the relative amplitudes of the frequency components of a signal (possibly eliminating some of them)
- Very important image processing tools
- Filters can be used to (among other things):
  - Blur images, thus reducing noise (lowpass filters, *e.g.*, Box, Gaussian)
  - Find edges, thus identifying silhouettes (highpass filters, *e.g.*, Laplacian)
  - Enlarge images, by interpolating the original samples

Examples of Filters (frequency domain)

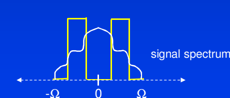
- Lowpass
  - Remove high frequencies



- Highpass
  - Remove low frequencies



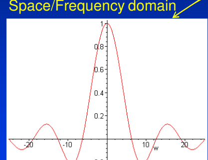
- Bandpass/ Bandstop
  - Keeps only specific frequency ranges



The Ideal Lowpass Filter

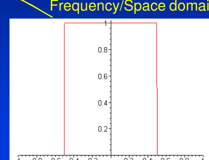
- *Box filter* in frequency domain
- Its space domain counterpart is called the *sinc filter*

Space/Frequency domain



$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

Frequency/Space domain



2-D Sinc Filter

- An image is a 2-D signal
- Separable kernel: sinc(x,y) = sinx(x) *x* sinc(y)

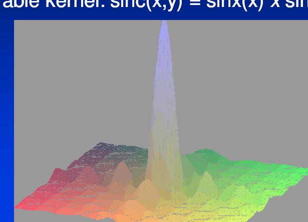
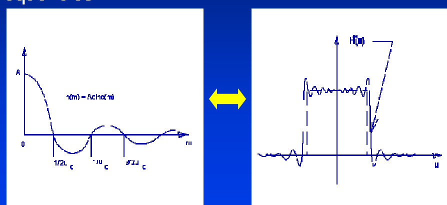


Image by Dana Jacobsen (http://www.ecst.csuchico.edu/~jacobsd/wave/wavepic.html)

The Practical Lowpass Filter

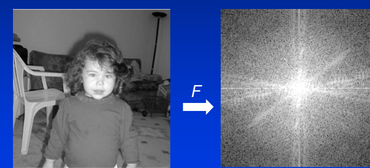
- Truncation of the sinc function produces some ripples (Gibbs phenomenon), thus introducing undesirable high frequencies



Images by Carl G. Looney
Applet (http://cnyack.homestead.com/files/afourse/fsgibbs.htm)

Filtering in Frequency Domain

- Example

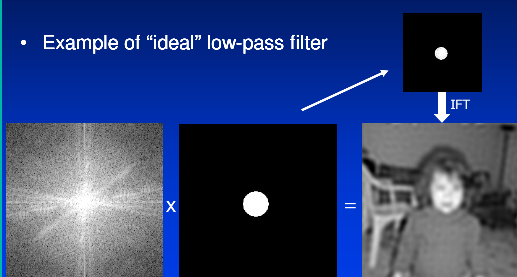


original image

amplitude spectrum (after shift)

Filtering in Frequency Domain

- Example of “ideal” low-pass filter



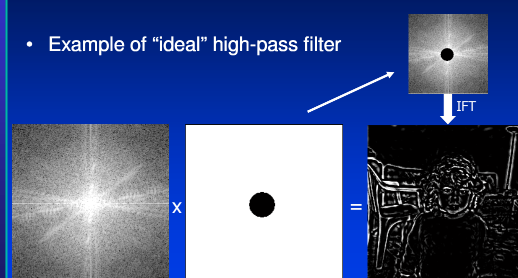
amplitude spectrum (after shift)

mask (spectrum of low-pass filter)

filtered image

Filtering in Frequency Domain

- Example of “ideal” high-pass filter



amplitude spectrum (after shift)

mask (spectrum of high-pass filter)

filtered image

Cause of the Ripples

- Gibbs Phenomenon
  - According to the Convolution Theorem, multiplication by these masks in frequency domain is equivalent to convolution by the FT of the mask in space domain