



Dense linear algebra : direct methods

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Linear Algebra Basics

- Gaussian Elimination and LU factorization

- LU Factorization with partial pivoting

- Symmetric matrices

- Cholesky Factorization

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System of linear equations ?

Example:

$$\begin{array}{rrcrcl} 2 x_1 & - & 1 x_2 & + & 3 x_3 & = & 13 \\ -4x_1 & + & 6 x_2 & - & 5 x_3 & = & -28 \\ 6 x_1 & + & 13 x_2 & + & 16 x_3 & = & 37 \end{array}$$

can be written under the form:

$$\mathbf{Ax} = \mathbf{b},$$

$$\text{with } \mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ -4 & 6 & -5 \\ 6 & 13 & 16 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} 13 \\ -28 \\ 37 \end{pmatrix}$$

Gaussian Elimination

Example:

$$2x_1 - x_2 + 3x_3 = 13 \quad (1)$$

$$-4x_1 + 6x_2 - 5x_3 = -28 \quad (2)$$

$$6x_1 + 13x_2 + 16x_3 = 37 \quad (3)$$

With $2 * (1) + (2) \rightarrow (2)$ and $-3*(1) + (3) \rightarrow (3)$ we obtain:

$$2x_1 - x_2 + 3x_3 = 13 \quad (4)$$

$$0x_1 + 4x_2 + x_3 = -2 \quad (5)$$

$$0x_1 + 16x_2 + 7x_3 = -2 \quad (6)$$

Thus x_1 is eliminated from (5) and (6). With $-4*(5) + (6) \rightarrow (6)$:

$$2x_1 - x_2 + 3x_3 = 13$$

$$0x_1 + 4x_2 + x_3 = -2$$

$$0x_1 + 0x_2 + 3x_3 = 6$$

The linear system is then solved by backward ($x_3 \rightarrow x_2 \rightarrow x_1$) substitution: $x_3 = \frac{6}{3} = 2$, $x_2 = \frac{1}{4}(-2 - x_3) = -1$, and finally $x_1 = \frac{1}{2}(13 - 3x_3 + x_2) = 3$

LU Factorization

- ▶ Find **L** unit lower triangular and **U** upper triangular such that:
A = **L** × **U**

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 6 & -5 \\ 6 & 13 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

- ▶ Procedure to solve **Ax** = **b**
 - ▶ **A** = **LU**
 - ▶ Solve **Ly** = **b** (*forward elimination, down*)
 - ▶ Solve **Ux** = **y** (*backward substitution, up*)

$$\mathbf{Ax} = (\mathbf{LU})\mathbf{x} = \mathbf{L}(\mathbf{Ux}) = \mathbf{Ly} = \mathbf{b}$$

From Gaussian Elimination to LU Factorization

$\mathbf{A} = \mathbf{A}^{(1)}$, $\mathbf{b} = \mathbf{b}^{(1)}$, $\mathbf{A}^{(1)}\mathbf{x} = \mathbf{b}^{(1)}$:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \begin{array}{l} 2 \leftarrow 2 - 1 \times a_{21}/a_{11} \\ 3 \leftarrow 3 - 1 \times a_{31}/a_{11} \end{array}$$

$\mathbf{A}^{(2)}\mathbf{x} = \mathbf{b}^{(2)}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^{(2)} \\ b_3^{(2)} \end{pmatrix} \quad \begin{array}{l} b_2^{(2)} = b_2 - a_{21}b_1/a_{11} \\ a_{32}^{(2)} = a_{32} - a_{31}a_{12}/a_{11} \end{array}$$

Finally $3 \leftarrow 3 - 2 \times a_{32}/a_{22}$ gives $\mathbf{A}^{(3)}\mathbf{x} = \mathbf{b}^{(3)}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2^{(2)} \\ b_3^{(3)} \end{pmatrix} \quad a_{33}^{(3)} = a_{33}^{(2)} - a_{32}^{(2)}a_{23}^{(2)}/a_{22}^{(2)}$$

From Gaussian Elimination to LU Factorization

Typical Gaussian elimination at step k :

$$\left(\begin{array}{cccccc} a_{11}^{(1)} & \dots & \dots & \dots & \dots & \dots & a_{1n}^{(1)} \\ 0 & \ddots & & & & & \vdots \\ \vdots & \ddots & a_{k-1,k-1}^{(k-1)} & \dots & \dots & \dots & a_{k-1,n}^{(k-1)} \\ \vdots & & 0 & a_{kk}^{(k)} & a_{k,k+1}^{(k)} & \dots & a_{kn}^{(k)} \\ \vdots & & \vdots & a_{k+1,k}^{(k)} & a_{k+1,k+1}^{(k)} & \dots & a_{k+1,n}^{(k)} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & a_{nk}^{(k)} & a_{n,k+1}^{(k)} & \dots & a_{nn}^{(k)} \end{array} \right) \quad \left(\begin{array}{c} b_1^{(1)} \\ \vdots \\ b_{k-1}^{(k-1)} \\ b_k^{(k)} \\ b_{k+1}^{(k)} \\ \vdots \\ b_n^{(k)} \end{array} \right)$$

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} a_{kj}^{(k)}, \text{ for } i > k$$

(and $a_{ij}^{(k+1)} = a_{ij}^{(k)}$ for $i \leq k$)

From Gaussian Elimination to **LU** factorization

$$\begin{cases} a_{ij}^{(k+1)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} a_{kj}^{(k)}, & \text{for } i > k \\ a_{ij}^{(k+1)} = a_{ij}^{(k)}, & \text{for } i \leq k \end{cases}$$

- One step of Gaussian elimination can be written:

$$\mathbf{A}^{(k+1)} = \mathbf{L}^{(k)} \mathbf{A}^{(k)} \quad (\text{and } b^{(k+1)} = \mathbf{L}^{(k)} b^{(k)}), \text{ with}$$

$$\mathbf{L}^k = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & -l_{k+1,k} & \ddots & \\ & & \vdots & & \ddots \\ & & -l_{n,k} & & & 1 \end{pmatrix} \quad \text{and } l_{ik} = \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}}.$$

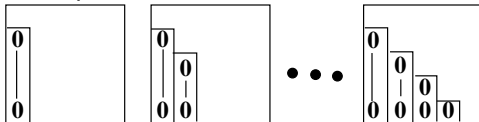
- After $n - 1$ steps, $\mathbf{A}^{(n)} = \mathbf{U} = \mathbf{L}^{(n-1)} \dots \mathbf{L}^{(1)} \mathbf{A}$ gives **$\mathbf{A} = \mathbf{LU}$** ,
with $\mathbf{L} = [\mathbf{L}^{(1)}]^{-1} \dots [\mathbf{L}^{(n-1)}]^{-1} =$

$$\begin{pmatrix} 1 & & & & \\ l_{21} & 1 & & & \\ \vdots & & \ddots & & \\ l_{n1} & & & 1 & \end{pmatrix} \dots \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & 0 & & \\ l_{21} & 1 & & & \\ \vdots & & \ddots & & \\ l_{n1} & \dots & & l_{n,n-1} & 1 \end{pmatrix}$$

LU Factorization Algorithm

- ▶ Overwrite matrix **A**: we store $a_{ij}^{(k)}, k = 2, \dots, n$ in $A(i, j)$
- ▶ In the end, $\mathbf{A} = \mathbf{A}^{(n)} = \mathbf{U}$

```
do k=1, n-1
  L(k, k) = 1
  do i=k+1, n
    L(i, k) = A(i, k)/A(k, k)
    do j=k, n    ! (better than: do j=1, n)
      A(i, j) = A(i, j) - L(i, k) * A(k, j)
    end do
  enddo
enddo
L(n, n)=1
```
- ▶ Matrix **A** at each step:



- ▶ Avoid building the zeros under the diagonal
- ▶ Before

```

L(n, n)=1
do k=1, n-1
  L(k, k) = 1
  do i=k+1, n
    L(i, k) = A(i, k)/A(k, k)
    do j=k, n
      A(i, j) = A(i, j) - L(i, k) * A(k, j)
    
```

- ▶ After

```

L(n, n)=1
do k=1, n-1
  L(k, k) = 1
  do i=k+1, n
    L(i, k) = A(i, k)/A(k, k)
    do j=k+1, n
      A(i, j) = A(i, j) - L(i, k) * A(k, j)
    
```

- ▶ Use lower triangle of array **A** to store $L_{i,k}$ multipliers

- ▶ Before:

```
L(n, n) = 1
do k = 1, n-1
  L(k, k) = 1
  do i = k+1, n
    L(i, k) = A(i, k) / A(k, k)
    do j = k+1, n
      A(i, j) = A(i, j) - L(i, k) * A(k, j)
```

- ▶ After (diagonal 1 of L is not stored):

```
do k = 1, n-1
  do i = k+1, n
    A(i, k) = A(i, k) / A(k, k)
    do j = k+1, n
      A(i, j) = A(i, j) - A(i, k) * A(k, j)
```

- More compact array syntax (Matlab, scilab):

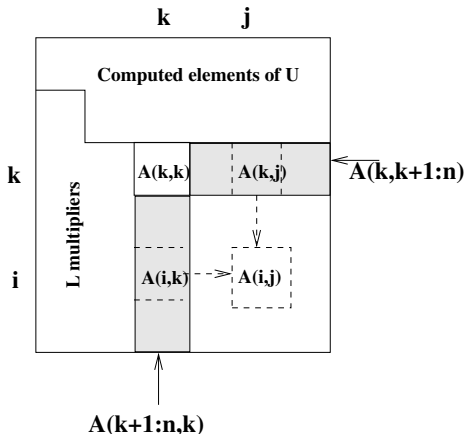
do $k=1, n-1$

$$A(k+1:n, k) = A(k+1:n, k) / A(k, k)$$

$$A(k+1:n, k+1:n) = A(k+1:n, k+1:n) - A(k+1:n, k) * A(k, k+1:n)$$

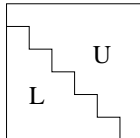
end do

- corresponds to a rank-1 update:



What we have computed

- ▶ we have stored the **L** and **U** factors in **A**:
 - ▶ $\mathbf{A}_{i,j}$, $i > j$ corresponds to l_{ij}
 - ▶ $\mathbf{A}_{i,j}$, $i \leq j$ corresponds to u_{ij}
 - ▶ with $l_{ii} = 1, i = 1, n$
- ▶ Finally,



after factorization: $\mathbf{A} = \mathbf{L} + \mathbf{U} - I$

LU factorization : summary

- ▶ Step by step columns of \mathbf{A} are set to zero and \mathbf{A} is updated
 $\mathbf{L}^{(n-1)} \dots \mathbf{L}^{(1)} \mathbf{A} = \mathbf{U}$ leading to
 $\mathbf{A} = \mathbf{L}\mathbf{U}$ where $\mathbf{L} = [\mathbf{L}^{(1)}]^{-1} \dots [\mathbf{L}^{(n-1)}]^{-1}$
- ▶ At each step $\mathbf{A}(k, k)$ is referred to as the **pivot**
 - zero entries in column of \mathbf{A} can be replaced by entries in \mathbf{L}
 - row entries of \mathbf{U} are stored in corresponding locations of \mathbf{A}

Algorithm 1 LU factorization

```
for  $k = 1, n - 1$  do
  if  $|\mathbf{A}(k, k)|$  too small then
    exit (small pivots are not allowed)
  end if
   $\mathbf{A}(k+1:n, k) = \mathbf{A}(k+1:n, k) / \mathbf{A}(k, k)$ 
   $\mathbf{A}(k+1:n, k+1:n) = \mathbf{A}(k+1:n, k+1:n) - \mathbf{A}(k+1:n, k) * \mathbf{A}(k, k+1:n)$ 
end for
```

When $|\mathbf{A}(k, k)|$ is too small, one could consider other pivots: **numerical pivoting strategies** will be introduced later.

Existence and uniqueness of **LU** decomposition

Theorem 1

$\mathbf{A} \in \mathbb{R}^{n \times n}$ has an LU factorization (where \mathbf{L} is unit lower triangular and \mathbf{U} is upper triangular) if $\det(\mathbf{A}(1:k, 1:k)) \neq 0$ for all $k \in \{1 \dots n-1\}$. If the LU factorization exists, then it is unique and $\det(\mathbf{A}) = u_{11} \dots u_{nn}$.

Theorem 2

For each nonsingular matrix \mathbf{A} , there exists a permutation matrix \mathbf{P} such that \mathbf{PA} possesses an LU factorization $\mathbf{PA} = \mathbf{LU}$.

Definition 1

$\mathbf{A} \in \mathbb{R}^{n \times n}$ is **strictly diagonally dominant** iff $|a_{ii}| > \sum_{j=1, j \neq i}^n |a_{ij}|$ for all $i = 1, \dots, n$

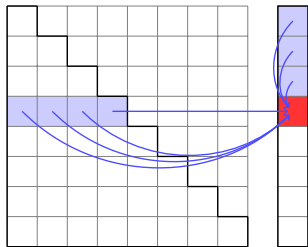
Theorem 3

If \mathbf{A}^T is strictly diagonally dominant then \mathbf{A} is non-singular and \mathbf{A} has an LU factorization and $l_{ij} \leq 1$

Solution phase : $\mathbf{Lx} = \mathbf{b}$ (Left-Looking and Right-looking)

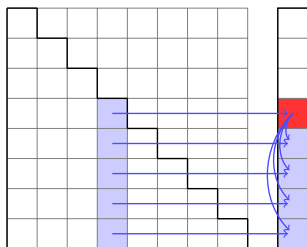
Algorithm 2 LL (sans report)

```
x = b
for j = 1, n do
  for i = 1, j - 1 do
     $x_j = x_j - l_{ji}x_i$ 
  end for
   $x_j = \frac{x_j}{l_{jj}}$ 
end for
```



Algorithm 3 RL (avec report)

```
x = b
for j = 1, n do
   $x_j = \frac{x_j}{l_{jj}}$ 
  for i = j + 1, n do
     $x_i = x_i - l_{ij}x_j$ 
  end for
end for
```



Blocked **LU** and Schur decomposition

Exercise 1 (Blocked **LU** and Schur decomposition)

Let \mathbf{A} be a non singular matrix of order n for which $\exists \mathbf{P}$ permutation matrix such that \mathbf{PA} can be factored without pivoting and consider the block form

$$\mathbf{PA} = \begin{pmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{pmatrix}. \text{ We define the so called } \textcolor{red}{\text{Schur matrix } \mathbf{S}} \text{ as}$$

$$\textcolor{red}{\mathbf{S}} = \mathbf{A}_{2,2} - \mathbf{A}_{2,1} (\mathbf{A}_{1,1})^{-1} \mathbf{A}_{1,2}$$

1. Explain how to adapt the **LU** factorisation algorithm to obtain the following

$$\text{decomposition of } \mathbf{PA}. \mathbf{PA} = \begin{pmatrix} \mathbf{L}_{1,1} & 0 \\ \mathbf{L}_{2,1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{U}_{1,1} & \mathbf{U}_{1,2} \\ 0 & \mathbf{S} \end{pmatrix} =$$
$$\begin{pmatrix} \mathbf{L}_{1,1} & 0 \\ \mathbf{L}_{2,1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{S} \end{pmatrix} \begin{pmatrix} \mathbf{U}_{1,1} & \mathbf{U}_{1,2} \\ 0 & \mathbf{I} \end{pmatrix}$$

2. Prove that $\det(\mathbf{A}) = \det(\mathbf{P})\det(\mathbf{A}_{1,1})\det(\mathbf{S})$
3. We assume that we know how to compute \mathbf{Y} such that $\mathbf{Y} = \mathbf{S}^{-1}\mathbf{Z}$. Describe how to use previous incomplete blocked factorization to solve $\mathbf{AX} = \mathbf{B}$.

Blocked factorization and null space

Exercise 2 (Null space)

We suppose that after $n - r$ steps of **LU** factorization we have

$$\mathbf{PA} = \begin{pmatrix} \mathbf{L}_{1,1} & 0 \\ \mathbf{L}_{2,1} & I_r \end{pmatrix} \begin{pmatrix} \mathbf{U}_{1,1} & \mathbf{U}_{1,2} \\ 0 & \mathbf{S}_r \end{pmatrix} \text{ where } \mathbf{S}_r \text{ is the Schur}$$

complement matrix of order r . We also suppose that $\mathbf{S}_r = 0$ (in practice one could also assume that $\|\mathbf{S}_r\| \leq \varepsilon \|\mathbf{A}\|$ for some matrix norm). Finally we assume that $\det(\mathbf{U}_{11}) \neq 0$.

Prove that the dimension of the null-space is r and compute a basis of the null space.

Number of floating-point operations (flops)

- ▶ In forward elimination ($Ly = b$), computing the k^{th} unknown

$$y_k = b_k - \sum_{j=1}^{k-1} L_{kj}y_j$$

leads to $(k-1)$ multiplications and $(k-1)$ additions, for $1 \leq k \leq n$
 $n^2 - n$ flops overall

- ▶ Idem for $Ux = y$ and at worst n divisions ($U_{kk} \neq 1$).
- ▶ Number of flops during factorization:
 - ▶ $n - k$ divisions
 - ▶ $(n - k)^2$ multiplications, $(n - k)^2$ additions
 - ▶ $k = 1, 2, \dots, n - 1$
 - ▶ total: $\approx \frac{2 \times n^3}{3}$
(Strassen's algorithm can reduce this to $\Theta(n^{\log_2 7}) \simeq \Theta(n^{2.8})$)

Exercise 3 (How to compute \mathbf{x} such that $\mathbf{x} = (\mathbf{A}^2)^{-1} \mathbf{b}$)

Let \mathbf{A} be a non singular matrix of order n (i.e. it exists \mathbf{L} , \mathbf{U} and \mathbf{P} such that $\mathbf{PA} = \mathbf{LU}$ (note that \mathbf{A}^2 is also non singular).

1. Compare the computational complexity of solving $\mathbf{A}^2 \mathbf{x} = \mathbf{b}$ with the following two algorithms:
 - 1.1 Compute $\mathbf{B} = \mathbf{A}^2$, factor \mathbf{B} and solve $\mathbf{Bx} = \mathbf{b}$
 - 1.2 Factor \mathbf{A} and use the factored form to solve $\mathbf{A}^2 \mathbf{x} = \mathbf{b}$
2. Explain why computing directly $\mathbf{C} = (\mathbf{A}^2)^{-1}$ and performing $\mathbf{x} = \mathbf{Cb}$ is not a method of choice.

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Consider $A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{\varepsilon} & 1 \end{bmatrix} \times \begin{bmatrix} \varepsilon & 1 \\ 0 & 1 - \frac{1}{\varepsilon} \end{bmatrix}$

$$\kappa_2(A) = \frac{\lambda_{\max}}{\lambda_{\min}} = \frac{1 + \varepsilon + \sqrt{5 + \varepsilon^2 - 2\varepsilon}}{-1 - \varepsilon + \sqrt{5 + \varepsilon^2 - 2\varepsilon}} \simeq 2.6$$

If one solves:

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + \varepsilon \\ 2 \end{bmatrix}$$

Exact solution $x^* = (1, 1)$.

$$A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{\varepsilon} & 1 \end{bmatrix} \times \begin{bmatrix} \varepsilon & 1 \\ 0 & 1 - \frac{1}{\varepsilon} \end{bmatrix}$$

ε	$\frac{\ x^* - x\ }{\ x^*\ }$	$\kappa_2(A)$
10^{-3}	6×10^{-16}	2.621
10^{-6}	2×10^{-11}	2.618
10^{-9}	9×10^{-8}	2.618
10^{-12}	9×10^{-5}	2.618
10^{-15}	7×10^{-2}	2.618

Table: Relative error as a function of ε .

- Even if A is well conditioned, Gaussian elimination may introduce errors

$$A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{\varepsilon} & 1 \end{bmatrix} \times \begin{bmatrix} \varepsilon & 1 \\ 0 & 1 - \frac{1}{\varepsilon} \end{bmatrix}$$

ε	$\frac{\ x^* - x\ }{\ x^*\ }$	$\kappa_2(A)$
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10^{-12}	9×10^{-5}	2.618
10^{-15}	7×10^{-2}	2.618

Table: Relative error as a function of ε .

- ▶ Even if A is well conditioned, Gaussian elimination may introduce errors
- ▶ Explanation: pivot ε is too small and leads to a large element growth (**growth factor**) in L and U : $\frac{1}{\varepsilon}$ in L leads to a loss of information/accuracy in $1 - \frac{1}{\varepsilon}$

$$A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{\varepsilon} & 1 \end{bmatrix} \times \begin{bmatrix} \varepsilon & 1 \\ 0 & 1 - \frac{1}{\varepsilon} \end{bmatrix}$$

- Let us try to exchange rows 1 and 2 of A and b :

$$\begin{bmatrix} 1 & 1 \\ \varepsilon & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 + \varepsilon \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ \varepsilon & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & \textcolor{red}{1} - \varepsilon \end{bmatrix}$$

→ Multipliers are bounded: $\forall i = k + 1 : n, \frac{|a_{i,k}^{(k)}|}{|a_{k,k}^{(k)}|} \leq 1$

→ terms of original matrix remain significant in LU factors

→ perfect accuracy obtained !

Partial Pivoting

- ▶ Partial pivoting: choose at each step the largest element of the column as the pivot
- avoids large elements in factors matrix (**growth factor**)
- ▶ Then (P : permutation), $PA = LU$, $Ly = Pb$, $Ux = y$
- ▶ LU with partial pivoting is practically *backward stable*

$$\frac{\|Ax - b\|}{\|A\| \times \|x\| + \|b\|} \approx \varepsilon \quad (1)$$

$$\frac{\|x - x^*\|}{\|x^*\|} \approx \varepsilon \times \kappa(A) \quad (2)$$

- (1) small backward error (and small residual) independently of the conditioning
- (2) accuracy depends on conditioning
if $\varepsilon \approx 10^{-q}$ et $\kappa(A) \approx 10^p$ then x has approximatively $(q - p)$ correct digits

LU factorization with partial pivoting

Next algorithm computes \mathbf{L} and \mathbf{U} such that $\mathbf{PA} = \mathbf{LU}$, and computes \mathbf{Pb} .

Algorithm 4 LU factorization with partial pivoting

for $k = 1, n - 1$ **do**

Pivot search : Find index i of largest entry in $\mathbf{A}(k : n, k)$

if $|A(i, k)| \leq \varepsilon \|\mathbf{A}\|$ **then**

 exit since \mathbf{A} is numerically singular

end if

 Swap rows i and k of \mathbf{A} and \mathbf{b}

$\mathbf{A}(k+1:n, k) = \mathbf{A}(k+1:n, k) / A(k, k)$

$\mathbf{A}(k+1:n, k+1:n) = \mathbf{A}(k+1:n, k+1:n) - \mathbf{A}(k+1:n, k) * \mathbf{A}(k, k+1:n)$

end for

To solve $\mathbf{Ax} = \mathbf{b}$ one should thus perform $\mathbf{LU} = \mathbf{Pb}$. Permutation \mathbf{P} is either applied on \mathbf{b} during Algorithm 4 or needs to be saved to be applied later.

Extensions to **LU** factorization with partial pivoting

Exercise 4 (LU with pivoting)

1. *Explain how to modify algorithm 4 to factor singular matrices in the form proposed at exercise 2 (so that using exercise 2 one could then also compute the null-space of \mathbf{A})*
2. *Let us suppose then that in algorithm 4, we want at each step of **Pivot search** step to find the largest entry not only in the column ($\mathbf{A}(k : n, k)$) both also ($\mathbf{A}(k : n, k : n)$), so called **total pivoting**. Describe how algorithm 4 should be modified.*
3. *Compare algorithm proposed at questions 1 and 2.*

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Symmetric matrices

- ▶ Assumption: A has a LU factorization
- ▶ A symmetric: only store lower or upper triangle
- ▶ $A = LU$ and $A^T = A \Rightarrow LU = U^T L^T$,
thus $LU(L^T)^{-1} = U^T \Rightarrow (U)(L^T)^{-1} = L^{-1}U^T = D$ diagonal
and $U = DL^T$,
finally $A = L(DL^T) = LDL^T$, with $D = \text{Diag}(U)$ and
- ▶ Example:

$$\begin{bmatrix} 4 & -8 & -4 \\ -8 & 18 & 14 \\ -4 & 14 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

- ▶ Solution of $Ax = b$: with $A = LDL^T$:
 1. $Ly = b$ then $Dz = y$ followed by
 2. $L^T x = z$

Properties of LDL^T Algorithm

We have shown that if \mathbf{A} symmetric and $A = LU$ exists then $\exists L$ and $D(= \text{Diag}(U))$ such that $A = LDL^T$. **LU** Algorithm 1 thus already computes all we need: L and D .

Proposition 1 (LDL^T Algorithm)

Given a symmetric matrix A for which an LU factorisation exists, the LU algorithm 1 can be adapted to compute LDL^T factorization.

Proposition 2 (Complexity of LDL^T factorization)

If only the lower triangular part of the matrix (including diagonal) is used/updated and if only L and D matrices are stored, then the cost of LDL^T factorisation is $\approx \frac{n^3}{3}$

LDL^T Algorithm for symmetric matrices

- ▶ let l_k be column k of L and u_k be row k of U then in Algorithm 1,
$$A(k+1:n, k+1:n) = A(k+1:n, k+1:n) - l_k * u_k^T$$
- ▶ since $l_k = u_k / u_{kk}$, when U is not stored, one must temporarily save u_k to perform the update.

Algorithm 5 LDT^T factorization

```
for  $k = 1, n - 1$  do
  if  $|A(k, k)|$  too small exit (small pivots)
   $\mathbf{v}_k = A(k+1:n, k)$  (corresponds to  $u_k$  in LU Algorithm)
   $A(k+1:n, k) = A(k+1:n, k) / A(k, k)$ 
  for  $j = k + 1, n$  do
     $A(j:n, j) = A(j:n, j) - A(j:n, k) * \mathbf{v}_k(j)$ 
  end for
end for
```

Complexity of LDL^T factorization

```
for  $k = 1, n - 1$  do
  if  $|\mathbf{A}(k, k)|$  too small exit (small pivots)
   $\mathbf{v}_k = \mathbf{A}(k+1:n, k)$  (corresponds to  $u_k$  in LU Algorithm)
   $\mathbf{A}(k+1:n, k) = \mathbf{A}(k+1:n, k) / \mathbf{A}(k, k)$ 
  for  $j = k + 1, n$  do
     $\mathbf{A}(j:n, j) = \mathbf{A}(j:n, j) - \mathbf{A}(j:n, k) * \mathbf{v}_k(j)$ 
  end for
end for
```

$$\text{flops}(LDL^T) = 2 \sum_{k=1}^{n-1} \left(\sum_{i=1}^{n-k} i \right) = 2 \sum_{k=1}^{n-1} \left(\frac{(n-k)(n-k+1)}{2} \right)$$

$$\text{flops}(LDL^T) \approx \sum_{k=1}^{n-1} (n-k)^2 \quad (\text{thus } \tfrac{1}{2} \text{flops}(\mathbf{LU}))$$

$$LDL^T \approx \frac{n^3}{3} \text{floating point operations}$$

Symmetric matrices and pivoting

- ▶ Diagonal pivoting preserves symmetry but is insufficient for stability
- ▶ In general one looks for a permutation P such that:

$$PAP^T = LDL^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x & 0 & 1 & 0 \\ x & x & x & 1 \end{bmatrix} \times \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & x & 0 \\ 0 & x & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix} \times \begin{bmatrix} 1 & x & x & x \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- ▶ D : matrix of diagonal 1×1 and 2×2 blocks

Examples of 2×2 pivots: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} \varepsilon_1 & 1 \\ 1 & \varepsilon_2 \end{bmatrix}$

- ▶ Pivot choice more complex: 2 columns at each step Let

$PAP^T = \begin{bmatrix} E & C^T \\ C & B \end{bmatrix}$. If E is a 2×2 pivot, form E^{-1} to get:

$$PAP^T = \begin{bmatrix} I & 0 \\ CE^{-1} & I \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & B - CE^{-1}C^T \end{bmatrix} \begin{bmatrix} I & E^{-1}C^T \\ 0 & I \end{bmatrix}$$

Linear Algebra Basics

Gaussian Elimination and LU factorization

LU Factorization with partial pivoting

Symmetric matrices

Cholesky Factorization

Cholesky Factorization

- ▶ **A** positive definite if $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0 \quad \forall \mathbf{x} \neq 0$
- ▶ **A** symmetric positive definite \Rightarrow Cholesky factorization
 $\mathbf{A} = \mathbf{L} \mathbf{L}^T$ with L lower triangular, **L** is unique
- ▶ By identification :

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \times \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

- ▶ It follows: $a_{11} = l_{11}^2$ $a_{21} = l_{21} \times l_{11}$ $a_{31} = l_{31} \times l_{11}$
 $a_{22} = l_{21}^2 + l_{22}^2$ $a_{32} = l_{31} \times l_{21} + l_{32} \times l_{22}$ $a_{33} = l_{31}^2 + l_{32}^2 + l_{33}^2$

Thus:

$$\begin{aligned} l_{11} &= \sqrt{a_{11}} & l_{21} &= a_{21}/l_{11} & l_{31} &= a_{31}/l_{11} \\ a_{22}^{(1)} &= a_{22} - l_{21}^2 & a_{32}^{(1)} &= a_{32} - l_{31} \times l_{21} & a_{33}^{(1)} &= a_{33} - l_{31}^2 \\ l_{22} &= \sqrt{a_{22}^{(1)}} & l_{32} &= a_{32}^{(1)}/l_{22} & a_{33}^{(2)} &= a_{33}^{(1)} - l_{32}^2 \\ l_{33} &= \sqrt{a_{33}^{(2)}} \end{aligned}$$

Cholesky Factorization

Cholesky Factorization

```
do k=1, n
  A(k,k)=sqrt(A(k,k))
  A(k+1:n,k) = A(k+1:n,k)/A(k,k)
  do j=k+1, n
    A(j:n,j) = A(j:n,j) - A(j:n,k) A(j,k)
  end do
end do
```

- ▶ Cholesky is backward stable (without pivoting)
- ▶ Factorization: $\approx \frac{n^3}{3}$ flops
- ▶ Similar to LU, but only on the lower triangle. **LU** factorization:

$$\begin{aligned} A(k+1:n,k) &= A(k+1:n,k)/A(k,k) \\ A(k+1:n,k+1:n) &= A(k+1:n,k+1:n) - & \\ &A(k+1:n,k) * A(k,k+1:n) \end{aligned}$$