Profesor Blaga Mirela-Gabriela

Calculul limitelor de funcții

$$\frac{\infty}{\infty}$$

Exerciții rezolvate

1. $\lim\limits_{x\to\infty}\frac{x^2-2x-3}{x^3+x^2+1}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\frac{x^2}{x^3}=\lim\limits_{x\to\infty}\frac{1}{x}=\frac{1}{\infty}=0$

2. $\lim\limits_{x\to\infty}\frac{2x+1}{\sqrt{4x^2-1}}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\frac{2x}{2x}=1$

3. $\lim\limits_{x\to-\infty}\frac{2x+1}{\sqrt{4x^2-1}}=\frac{\infty}{\infty}=\lim\limits_{x\to-\infty}\frac{2x}{|2x|}=\lim\limits_{x\to-\infty}\frac{2x}{-2x}=-1$

4. $\lim\limits_{x\to\infty}\frac{2^x+3^x}{3^x-4^x}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\frac{3^x}{-4^x}=\lim\limits_{x\to\infty}\left[-\left(\frac{3}{4}\right)^x\right]=0,\begin{cases}3>2>1\\4>3>1\end{cases}\Rightarrow\begin{cases}3^x>2^x\\4^x>3^x\end{cases},\ \frac{3}{4}\in(0,1)\Rightarrow\left(\frac{3}{4}\right)^x\xrightarrow[x\to\infty]{}0$

5. $\lim\limits_{x\to\infty}\frac{ln(x^3+x^2+1)}{ln(x^2+x+1)}=\frac{\infty}{\infty}=\lim\limits_{x\to\infty}\frac{lnx^3}{lnx^2}=\lim\limits_{x\to\infty}\frac{3lnx}{2lnx}=\frac{3}{2}$

Exerciții propuse

1. $\lim\limits_{x\to-\infty}\frac{x^3+x^2+1}{x^2-2x-3}$

2. $\lim\limits_{x\to\infty}\frac{5x+1}{\sqrt{5x^2+1}}$

3. $\lim\limits_{x\to-\infty}\frac{3x+1}{\sqrt{16x^2+1}}$

4. $\lim\limits_{x\to\infty}\frac{2^x+3^{x+1}}{3^x-2^x}$

5. $\lim\limits_{x\to\infty}\frac{ln(2x^2+1)}{ln(x^2+1)}$

6. $\lim\limits_{x\to\infty}\frac{ln(e^x+1)}{ln(e^{2x}+1)}$

7. $\lim\limits_{x\to-\infty}\frac{7x+1}{|x-1|}$

8. $\lim\limits_{x\to-\infty}\sqrt[3]{\frac{x^2}{7x+1}}$

9. $\lim\limits_{x\to\infty}\frac{ln(e^x+1)}{3x+1}$

10. $\lim\limits_{x\to-\infty}\frac{x+1}{\sqrt{2}x^2+1}$

11.Determinați $a,b,c\in\mathbb{R}$ astfel încât $\lim\limits_{x\to\infty}\frac{ax^2+bx+c}{x+1}=1.$

1) $-\infty$

2) $\sqrt{5}$

3) $-\frac{3}{4}$

4) 3

5) 1

6) $\frac{1}{2}$

7) -7

8) $-\infty$

9) $\frac{1}{3}$

10) 0

11) $a=0,b=1,c\in\mathbb{R}$

Profesor Blaga Mirela-Gabriela

Calculul limitelor de funcții

$\infty-\infty$

Exerciții rezolvate

1. $\lim_{x\to\infty}\left(\frac{x^2-2x-3}{x+1}-\frac{x^2-2}{x}\right)=\infty-\infty=\lim_{x\to\infty}\frac{-3x^2-x+2}{x^2+x}=\lim_{x\to\infty}\frac{-3x^2}{x^2}=-3$

2. $\lim_{x\to\infty}\left(\sqrt{4x^2-x+1}-2x\right)=\infty-\infty=\lim_{x\to\infty}\frac{4x^2-x+1-4x^2}{\sqrt{4x^2-x+1}+2x}=\lim_{x\to\infty}\frac{-x}{4x}=-\frac{1}{4}$

3. $\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)=\infty-\infty=\lim_{x\to-\infty}\frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}-x}=\lim_{x\to-\infty}\frac{x}{|x|-x}=\lim_{x\to-\infty}\frac{x}{-2x}=-\frac{1}{2}$

4. $\lim_{x\to\infty}\left(x-\sqrt[3]{x^3+x^2+x+1}\right)=\infty-\infty=\lim_{x\to\infty}\frac{x^3-(x^3+x^2+x+1)}{x^2+x\sqrt[3]{x^3+x^2+x+1}+\sqrt[3]{x^3+x^2+x+1}^2}=$

$=\lim_{x\to\infty}\frac{-x^2}{3x^2}=-\frac{1}{3}$

5. $\lim_{x\to\infty}\left(ln(x^3+x^2+1)-ln(x^2+x+1)\right)=\infty-\infty=\lim_{x\to\infty}ln\frac{x^3+x^2+1}{x^2+x+1}=\lim_{x\to\infty}lnx=\infty$

Exerciții propuse

1. $\lim_{x\to-\infty}\left(\sqrt{9x^2-3x+1}+3x\right)$
2. $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-x\right)$
3. $\lim_{x\to-\infty}\left(\sqrt[3]{x^3+x^2+x+1}-x\right)$
4. $\lim_{x\to\infty}\left(\sqrt{x^2+3x+1}-\sqrt{x^2+1}\right)$
5. $\lim_{x\to\infty}\left(\sqrt{x^2-x+1}-x+1\right)$
6. $\lim_{x\to\infty}\left(ln(4x+1)-ln(2x+1)\right)$
7. $\lim_{x\to-\infty}\left(x-2+\sqrt{x^2+4x+1}\right)$
8. $\lim_{\substack{x\to-2\\x>-2}}\left(\frac{x}{x+2}-\frac{1}{x^2-4}\right)$
9. $\lim_{x\to\infty}\left(x-ln(e^x+1)\right)$
10. $\lim_{x\to\infty}\left(ln(e^x+1)-ln(e^{2x}+1)\right)$

11.Determinați $a,b\in\mathbb{R}$ astfel încât $\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-ax+b\right)=\frac{3}{2}$.

1) $\frac{1}{2}$

2) $\frac{1}{2}$

3) $\frac{1}{3}$

4) $\frac{3}{2}$

5) $\frac{1}{2}$

6) $ln2$

7) -4

8) $-\infty$

9) 0

10) $-\infty$

11) $a=1,b=1$

Profesor Blaga Mirela-Gabriela

Calculul limitelor de funcții

1^{∞}

Exerciții rezolvate

1. $\lim\limits_{x\to\infty}\left(1+\frac{1}{2x+1}\right)^{x}=1^{\infty}=\lim\limits_{x\to\infty}\left[\left(1+\frac{1}{2x+1}\right)^{2x+1}\right]^{\frac{1}{2x+1}\cdot x}=e^{\lim\limits_{x\to\infty}\frac{x}{2x+1}}=e^{\frac{1}{2}}$

2. $\lim\limits_{x\to\infty}\left(\frac{x^2-2x-3}{x^2+x+1}\right)^{2x-1}=1^{\infty}=\lim\limits_{x\to\infty}\left(1+\frac{x^2-2x-3}{x^2+x+1}-1\right)^{2x-1}=\lim\limits_{x\to\infty}\left(1+\frac{-3x-4}{x^2+x+1}\right)^{2x-1}=$

$=\lim\limits_{x\to\infty}\left[\left(1+\frac{-3x-4}{x^2+x+1}\right)^{\frac{x^2+x+1}{-3x-4}}\right]^{\frac{-3x-4}{x^2+x+1}\cdot(2x-1)}=e^{\lim\limits_{x\to\infty}\frac{(-3x-4)(2x-1)}{x^2+x+1}}=e^{\lim\limits_{x\to\infty}\frac{-6x^2}{x^2}}=e^{-6}$

3. $\lim\limits_{x\to\infty}\left(\frac{2x+1}{\sqrt{4x^2-1}}\right)^{x}=1^{\infty}=\lim\limits_{x\to\infty}\left(1+\frac{2x+1}{\sqrt{4x^2-1}}-1\right)^{x}=\lim\limits_{x\to\infty}\left(1+\frac{2x+1-\sqrt{4x^2-1}}{\sqrt{4x^2-1}}\right)^{x}=$

$=\lim\limits_{x\to\infty}\left(1+\frac{4x^2+4x+1-4x^2+1}{\sqrt{4x^2-1}\left(2x+1+\sqrt{4x^2-1}\right)}\right)^{x}=\lim\limits_{x\to\infty}\left(1+\frac{4x+2}{\sqrt{4x^2-1}\left(2x+1+\sqrt{4x^2-1}\right)}\right)^{x}=$

$=\lim\limits_{x\to\infty}\left[\left(1+\frac{4x+2}{\sqrt{4x^2-1}\left(2x+1+\sqrt{4x^2-1}\right)}\right)^{\frac{\sqrt{4x^2-1}\left(2x+1+\sqrt{4x^2-1}\right)}{4x+2}}\right]^{\frac{4x+2}{\sqrt{4x^2-1}\left(2x+1+\sqrt{4x^2-1}\right)}\cdot x}=$

$=e^{\lim\limits_{x\to\infty}\frac{(4x+2)x}{\sqrt{4x^2-1}\left(2x+1+\sqrt{4x^2-1}\right)}}=e^{\lim\limits_{x\to\infty}\frac{4x^2}{2x\cdot 4x}}=e^{\frac{1}{2}}$

4. $\lim\limits_{x\to 0}(1+2x)^{\frac{1}{x}}=1^{\infty}=\lim\limits_{x\to 0}\left[(1+2x)^{\frac{1}{2x}}\right]^{2x\cdot\frac{1}{x}}=e^{2}$

5. $\lim\limits_{x\to 1}x^{\frac{1}{x-1}}=1^{\infty}=\lim\limits_{x\to 1}(1+x-1)^{\frac{1}{x-1}}=$e

Exerciții propuse

1. $\lim\limits_{x\to-\infty}\left(\frac{x^2-x+1}{x^2-2x-3}\right)^{x}$

2. $\lim\limits_{x\to\infty}\left(\frac{5x+1}{5x-3}\right)^{\frac{x^2-1}{x}}$

3. $\lim\limits_{x\to\infty}\left(\frac{3x+1}{\sqrt{9x^2+1}}\right)^{x}$

4. $\lim\limits_{x\to\infty}\left(\frac{2^x+3^x}{3^x-2^x}\right)^{x}$

5. $\lim\limits_{x\to\infty}\left(\frac{x+\sqrt{x}}{x-\sqrt{x}}\right)^{5\sqrt{x}}$

6. $\lim\limits_{x\to 0}(1-6x)^{\frac{x+1}{x}}$

7. $\lim\limits_{x\to -1}(2x^2-1)^{\frac{1}{x+1}}$

8. $\lim\limits_{x\to 2}(x-1)^{\frac{1}{x^2-4}}$

9. $\lim\limits_{x\to 0}(1+sinx)^{\frac{1}{tgx}}$

10. $\lim\limits_{x\to -3}\left(\frac{x^2+2x-6}{x}\right)^{\frac{1}{3+x}}$

11. Determinați $a, b, c \in \mathbb{R}$ astfel încât $\lim_{x\to\infty}\left(\frac{ax^2+bx+c}{x+1}\right)^x = e$.

1) e

2) $e^{\frac{4}{5}}$

3) $e^{\frac{1}{3}}$

4) 1

5) e^{10}

6) e^{-6}

7) e^{-4}

8) $e^{\frac{1}{4}}$

9) e

10) $e^{\frac{5}{3}}$

11) $a = 0, b = 1, c = 2$

Calculul limitelor de funcții

$$\frac{0}{0}$$

Exerciții rezolvate

1. $\lim\limits_{x\to 1}\frac{x^2-2x+1}{x^3-1}=\frac{0}{0}=\lim\limits_{x\to 1}\frac{(x-1)^2}{(x-1)(x^2+x+1)}=\lim\limits_{x\to 1}\frac{x-1}{x^2+x+1}=0$

2. $\lim\limits_{x\to 0}\frac{sin3x}{x}=\frac{0}{0}=\underbrace{\lim\limits_{x\to 0}\frac{sin3x}{3x}}_{1}\cdot 3=3$

3. $\lim\limits_{x\to 0}\frac{5^x-3^x}{x}=\frac{0}{0}=\lim\limits_{x\to 0}\frac{5^x-1+1-3^x}{x}=\lim\limits_{x\to 0}\left(\frac{5^x-1}{x}-\frac{3^x-1}{x}\right)=ln5-ln3=ln\frac{5}{3}$

4. $\lim\limits_{x\to 0}\frac{ln(x^2+x+1)}{x}=\frac{0}{0}=\lim\limits_{x\to 0}\frac{ln(x^2+x+1)}{x^2+x}\cdot\frac{x^2+x}{x}=$

$=\lim\limits_{x\to 0}\underbrace{\frac{ln(x^2+x+1)}{x^2+x}}_{1}\cdot\frac{x(x+1)}{x}=\lim\limits_{x\to 0}(x+1)=1$

5. $\lim\limits_{x\to 1}\frac{\sqrt[7]{2-x}-1}{x-1}=\frac{0}{0}=\lim\limits_{x\to 1}\frac{\left(1+(1-x)\right)^{\frac{1}{7}}-1}{x-1}=-\frac{1}{7}$

Exerciții propuse

1. $\lim\limits_{x\to 0}\frac{x+x^2}{arcsin\,x}$

2. $\lim\limits_{x\to 2}\frac{\sqrt{x^2+5}-3}{x-2}$

3. $\lim\limits_{x\to -1}\frac{x^2-8x-9}{3x^2+2x-1}$

4. $\lim\limits_{x\to 2}\frac{4-x^2}{3^x-9}$

5. $\lim\limits_{x\to 0}\frac{cosx-cos3x}{x^2}$

6. $\lim\limits_{x\to 0}\frac{ln(1+tg5x)}{x}$

7. $\lim\limits_{x\to 1}\frac{x^7-7x+6}{(x-1)^2}$

8. $\lim\limits_{x\to -1}\frac{\sqrt{3-x}-2}{x+\sqrt{x+2}}$

9. $\lim\limits_{\substack{x\to 1\\ x<1}}\frac{\sqrt[3]{x^3-3x+2}}{x-1}$

10. $\lim\limits_{x\to 1}\frac{x-1}{x+x^2+\cdots+x^n-n},n\in\mathbb{N}^*$

11. Determinați $a,b\in\mathbb{R}$ astfel încât $\lim\limits_{x\to 1}\frac{ax^2+bx+1}{x-1}=1$.

1) 1

2) $\frac{2}{3}$

3) $\frac{5}{2}$

4) $-\frac{4}{9ln3}$

5) 4

6) 5

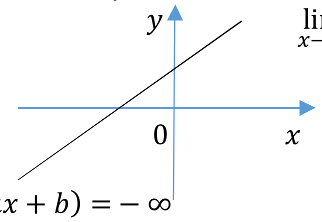
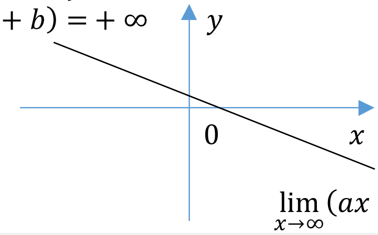
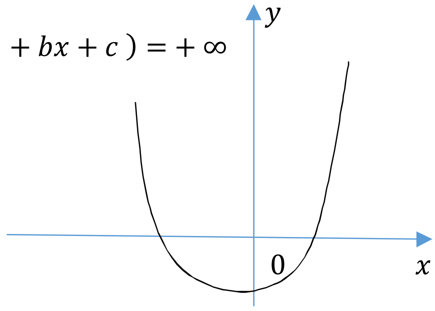
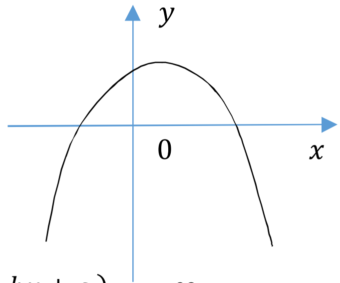
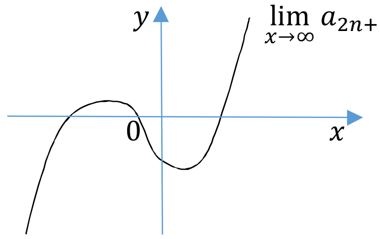
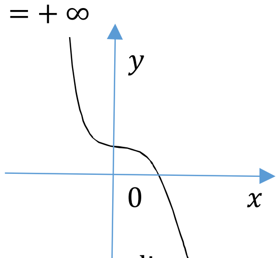
7) 21

8) $-\frac{1}{6}$

9) $-\infty$

10) $\frac{2}{n(n+1)}$

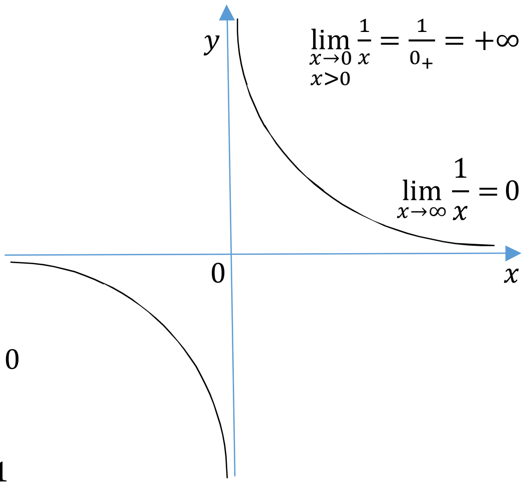
11) $a=2,b=-3$

Lectura grafică și determinarea limitelor de funcții – 1 –	
Funcția de gradul întâi $f:\mathbb{R}\to\mathbb{R}, f(x)=ax+b, a\in\mathbb{R}^*, b\in\mathbb{R}$	
$a>0$ f este crescătoare $\lim\limits_{x\to\infty}(ax+b)=+\infty$  $\lim\limits_{x\to-\infty}(ax+b)=-\infty$	$a<0$ f este descrescătoare $\lim\limits_{x\to-\infty}(ax+b)=+\infty$  $\lim\limits_{x\to\infty}(ax+b)=-\infty$
$2\cdot\infty=\infty$	$-2\cdot\infty=-\infty$
Funcția de gradul al doilea $f:\mathbb{R}\to\mathbb{R}, f(x)=ax^2+bx+c, a\in\mathbb{R}^*, b,c\in\mathbb{R}$	
$a>0$ f este convexă $\lim\limits_{x\to\infty}(ax^2+bx+c)=+\infty$ $\lim\limits_{x\to-\infty}(ax^2+bx+c)=+\infty$ 	$a<0$ f este concavă  $\lim\limits_{x\to-\infty}(ax^2+bx+c)=-\infty$ $\lim\limits_{x\to\infty}(ax^2+bx+c)=-\infty$
$\infty^2=\infty$	$(-\infty)^2=\infty$
Funcția polinomială de grad impar $f:\mathbb{R}\to\mathbb{R}, f(x)=a_{2n+1}x^{2n+1}+\dots+a_0, a_{2n+1}\in\mathbb{R}^*, a_i\in\mathbb{R}, i=\overline{0,2n}$	
$a_{2n+1}>0$ $\lim\limits_{x\to\infty}a_{2n+1}x^{2n+1}=+\infty$  $\lim\limits_{x\to-\infty}a_{2n+1}x^{2n+1}=-\infty$	$a_{2n+1}<0$ $\lim\limits_{x\to-\infty}a_{2n+1}x^{2n+1}=+\infty$  $\lim\limits_{x\to\infty}a_{2n+1}x^{2n+1}=-\infty$
$\infty^3=\infty$	$(-\infty)^3=-\infty$

Funcția rațională $f:\mathbb{R}^* \to \mathbb{R}\,, f(x) = \frac{1}{x}$

$\lim\limits_{\substack{x\to 0\\ x>0}} \frac{1}{x} = \frac{1}{0_+} = +\infty$

$\lim\limits_{x\to\infty} \frac{1}{x} = 0$



$\lim\limits_{x\to -\infty} \frac{1}{x} = 0$

$\lim\limits_{\substack{x\to 0\\ x<0}} \frac{1}{x} = \frac{1}{0_-} = -\infty$

$\frac{1}{-\infty} = 0$ $\qquad \frac{1}{\infty} = 0$

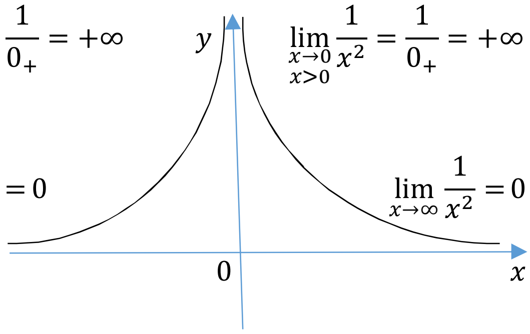
Funcția rațională $f:\mathbb{R}^* \to \mathbb{R}\,, f(x) = \frac{1}{x^2}$

$\lim\limits_{\substack{x\to 0\\ x<0}} \frac{1}{x^2} = \frac{1}{0_+} = +\infty$

$\lim\limits_{\substack{x\to 0\\ x>0}} \frac{1}{x^2} = \frac{1}{0_+} = +\infty$

$\lim\limits_{x\to -\infty} \frac{1}{x^2} = 0$

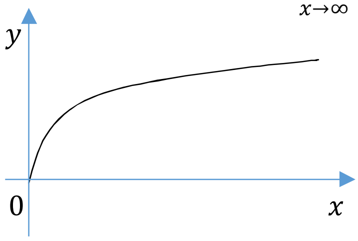
$\lim\limits_{x\to \infty} \frac{1}{x^2} = 0$



$\frac{1}{0_-} = -\infty$ $\qquad \frac{1}{0_+} = +\infty$

Funcția radical $f:[0,\infty)\to\mathbb{R}\,, f(x) = \sqrt[2n]{x}\,, n \in \mathbb{N}^*$

$\lim\limits_{x\to\infty} \sqrt[2n]{x} = +\infty$

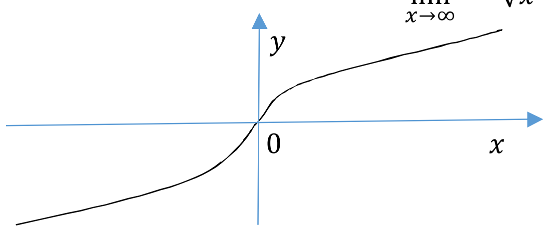


$f(0) = 0$

$\sqrt{\infty} = +\infty$

Funcția radical $f:\mathbb{R}\to\mathbb{R}\,, f(x) = \sqrt[2n+1]{x}\,, n \in \mathbb{N}^*$

$\lim\limits_{x\to\infty} \sqrt[2n+1]{x} = +\infty$



$\lim\limits_{x\to-\infty} \sqrt[2n+1]{x} = -\infty$

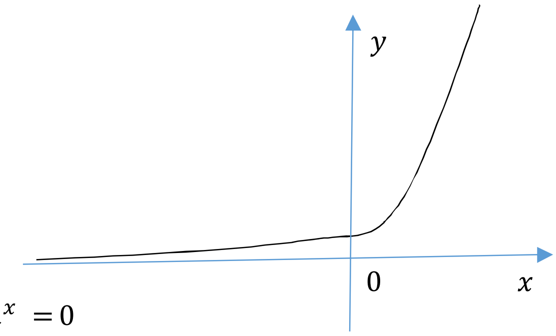
$\sqrt[3]{-\infty} = -\infty$ $\qquad \sqrt[3]{\infty} = +\infty$

Lectura grafică și determinarea limitelor de funcții – 2 –

Funcția exponențială $f:\mathbb{R}\to(0,\infty)$, $f(x)=a^x$, $a\in(0,\infty)\setminus\{1\}$ $\quad a^x>0$

$a\in(1,\infty)$
f este crescătoare
f este convexă

$\lim\limits_{x\to\infty} a^x=+\infty$

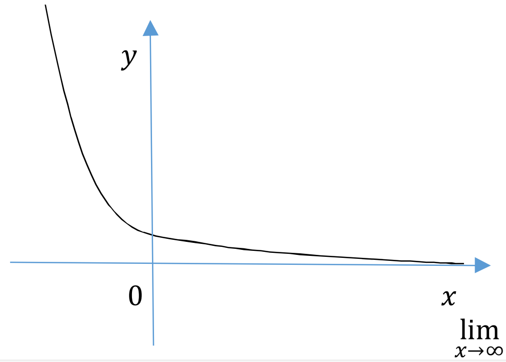


$\lim\limits_{x\to-\infty} a^x=0$

$e^{-\infty}=0 \qquad e^{\infty}=\infty$

$a\in(0,1)$
f este descrescătoare
f este convexă

$\lim\limits_{x\to-\infty} a^x=+\infty$



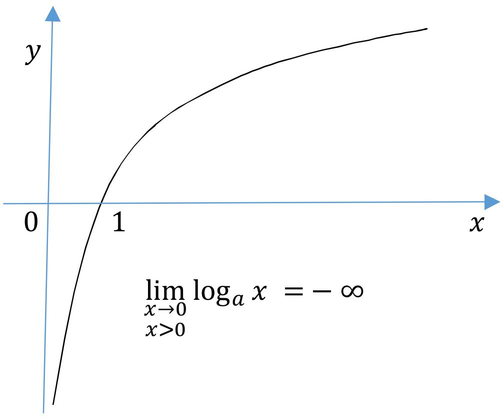
$\lim\limits_{x\to\infty} a^x=0$

$\left(\frac{1}{2}\right)^{-\infty}=\infty \qquad \left(\frac{1}{2}\right)^{\infty}=0$

Funcția logaritmică $f:(0,\infty)\to\mathbb{R}$, $f(x)=\log_a x$, $a\in(0,\infty)\setminus\{1\}$

$a\in(1,\infty)$
f este crescătoare
f este concavă

$\lim\limits_{x\to\infty} \log_a x=+\infty$

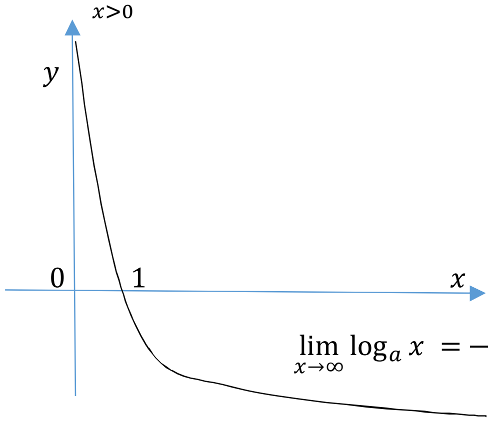


$\lim\limits_{\substack{x\to0\\x>0}} \log_a x=-\infty$

$ln\, 0_+=-\infty \qquad ln\,\infty=\infty$

$a\in(0,1)$
f este descrescătoare
f este convexă

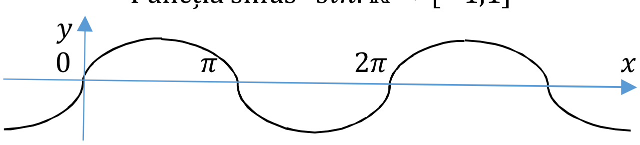
$\lim\limits_{\substack{x\to0\\x>0}} \log_a x=+\infty$



$\lim\limits_{x\to\infty} \log_a x=-\infty$

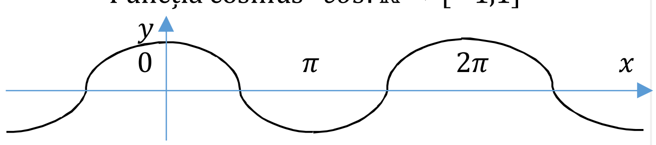
$\log_{\frac{1}{2}} 0_+=\infty \qquad \log_{\frac{1}{2}}\infty=-\infty$

Funcția sinus $sin:\mathbb{R}\to[-1,1]$



$\nexists \lim\limits_{x\to-\infty} sinx \qquad \nexists \lim\limits_{x\to+\infty} sinx$

Funcția cosinus $cos:\mathbb{R}\to[-1,1]$



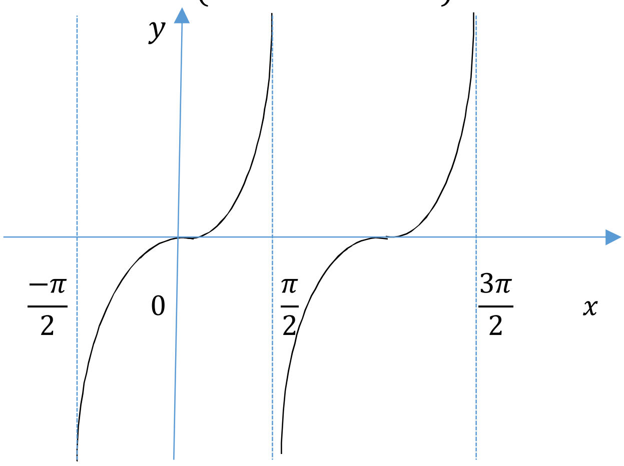
$\nexists \lim\limits_{x\to-\infty} cosx \qquad \nexists \lim\limits_{x\to+\infty} cosx$

$$\lim_{x\to+\infty}\frac{\sin x}{x}=0$$

$$\lim_{x\to0}\frac{\sin x}{x}=1$$

Funcția tangentă

$$tg:\mathbb{R}\setminus\left\{\frac{(2k+1)\pi}{2},k\in\mathbb{Z}\right\}\to\mathbb{R}$$

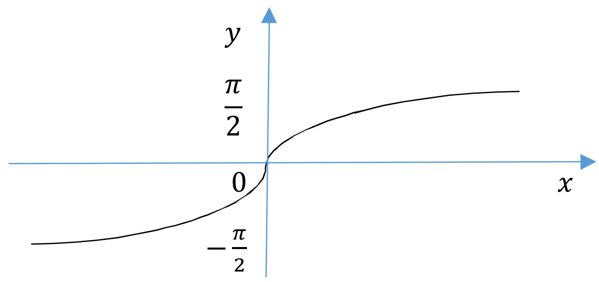


$$\lim_{\substack{x\to\frac{\pi}{2}\\ x<\frac{\pi}{2}}} tgx=+\infty$$

$$\lim_{\substack{x\to\frac{\pi}{2}\\ x>\frac{\pi}{2}}} tgx=-\infty$$

Funcția arctangentă

$$arctg:\mathbb{R}\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

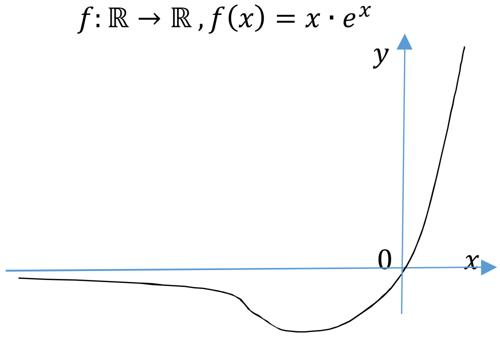


$$\lim_{x\to+\infty} arctg\,x=\frac{\pi}{2}$$

$$\lim_{x\to-\infty} arctg\,x=-\frac{\pi}{2}$$

$$arctg\,\infty=\frac{\pi}{2}$$

$$f:\mathbb{R}\to\mathbb{R},f(x)=x\cdot e^x$$

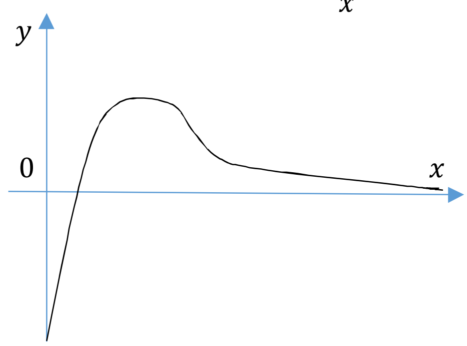


$$\lim_{x\to-\infty} x\cdot e^x=-\infty\cdot0=\lim_{x\to-\infty}\frac{x}{e^{-x}}=\lim_{x\to+\infty}\frac{-x}{e^x}=0$$

Obs. La ∞ funcția exponențială crește mai repede decât funcția polinomială.

$$\lim_{x\to+\infty} x\cdot e^x=\infty\cdot\infty=\infty$$

$$f:(0,\infty)\to\mathbb{R},f(x)=\frac{\ln x}{x}$$



$$\lim_{\substack{x\to0\\ x>0}}\frac{\ln x}{x}=\frac{\ln 0_+}{0_+}=-\infty\cdot\infty=-\infty$$

$$\lim_{x\to+\infty}\frac{\ln x}{x}=0$$

Obs. La ∞ funcția polinomială crește mai repede decât funcția logaritmică.

Calculul limitelor funcţiilor polinomiale

$$f:\mathbb{R}\to\mathbb{R},f(x)=a_nx^n+a_{n-1}x^{n-1}+\ \dots\ +a_1x+a_0\quad ,a_n\neq 0,a_i\in\mathbb{R},i=\overline{0,n},n\in\mathbb{N}^*$$

1) $\lim\limits_{x\to\infty} x^n=\infty\, ,n\in\mathbb{N}^*$

2) $\lim\limits_{x\to\infty}\dfrac{1}{x^n}=\dfrac{1}{\infty}=0\, ,n\in\mathbb{N}^*$

Pentru a calcula $\lim\limits_{x\to\infty} f(x)$ dăm factor comun forțat termenul de grad maxim al funcției f, adică pe a_nx^n. Prin factor comun forțat înțelegem factorul comun care nu se divide cu toți termenii funcției și teoretic avem de impus condiția $x\neq 0$, pe care de cele mai multe ori o ignorăm, deoarece limita funcției o calculăm pentru x tinde la ∞. Analog calculăm $\lim\limits_{x\to-\infty} f(x)$.

3) $\lim\limits_{x\to\infty}(a_nx^n+a_{n-1}x^{n-1}+\ \dots\ +a_1x+a_0)=\lim\limits_{x\to\infty} a_nx^n\left(1+\dfrac{a_{n-1}x^{n-1}}{a_nx^n}+\ \dots\ +\dfrac{a_1x}{a_nx^n}+\dfrac{a_0}{a_nx^n}\right)$

$$=\lim_{x\to\infty} a_nx^n\left(1+\overbrace{\frac{a_{n-1}}{a_nx}}^{\nearrow 0}+\ \dots\ +\overbrace{\frac{a_1}{a_nx^{n-1}}}^{\nearrow 0}+\overbrace{\frac{a_0}{a_nx^n}}^{\nearrow 0}\right)=\lim_{x\to\infty} a_nx^n=\begin{cases}\infty, a_n>0\\ -\infty, a_n<0\end{cases},a_n\in\mathbb{R}^*$$

În continuare pentru a calcula limita spre $+\infty/-\infty$ a unei funcții polinomiale folosim regula:
$\lim\limits_{x\to\infty}(a_nx^n+a_{n-1}x^{n-1}+\ \dots\ +a_1x+a_0)=\lim\limits_{x\to\infty} a_nx^n\ ,a_n\in\mathbb{R}^*,n\in\mathbb{N}^*$

Exemple

1. $\lim\limits_{x\to\infty}(x^2+2x+4)=\lim\limits_{x\to\infty}x^2=\infty$
2. $\lim\limits_{x\to\infty}(1+x^2-2x^3)=\lim\limits_{x\to\infty}(-2x^3)=-2\cdot\infty=-\infty$
3. $\lim\limits_{x\to-\infty}(1+x^2-2x^3)=\lim\limits_{x\to-\infty}(-2x^3)=-2\cdot(-\infty)=+\infty$

Valoarea limitei în $x_0\in\mathbb{R}$ se obține calculând valoarea funcției polinomiale în acel punct.
$\lim\limits_{x\to x_0} f(x)=f(x_0)$

Exemple

1. $\lim\limits_{x\to 1}(x^2+2x+4)=1+2+4=7$
2. $\lim\limits_{x\to 0}(1+x^2-2x^3)=1+0-0=1$

Calculul limitelor funcţiilor raţionale

$$f:\mathbb{R}\to\mathbb{R},f(x)=a_nx^n+a_{n-1}x^{n-1}+\ \dots\ +a_1x+a_0\quad ,a_n\neq 0,a_i\in\mathbb{R},i=\overline{0,n},n\in\mathbb{N}^*$$
$$g:\mathbb{R}\to\mathbb{R},g(x)=b_mx^m+b_{m-1}x^{m-1}+\ \dots\ +b_1x+b_0\, ,b_m\neq 0,b_j\in\mathbb{R},j=\overline{0,m},m\in\mathbb{N}^*$$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\ \dots\ +a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\ \dots\ +b_1x+b_0}=\lim_{x\to\infty}\frac{a_nx^n}{b_mx^m}=\begin{cases}\dfrac{a_n}{b_m}\cdot\infty & ,n>m\\ \dfrac{a_n}{b_m} & ,n=m\\ 0 & ,n<m\end{cases}$$

Exemple

1. $\lim\limits_{x\to\infty}\dfrac{-x^3+2x-3}{x^2+8x+1}=\dfrac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{-x^3}{x^2}=\lim\limits_{x\to\infty}(-x)=-\infty$

2. $\lim\limits_{x\to\infty}\dfrac{x^2-2x-3}{x^2+8x+1}=\dfrac{\infty}{\infty}=\lim\limits_{x\to\infty}\dfrac{x^2}{x^2}=1$

3. $\lim\limits_{x\to-\infty}\dfrac{x^2-2x-3}{x^3+x^2+1}=\dfrac{\infty}{\infty}=\lim\limits_{x\to-\infty}\dfrac{x^2}{x^3}=\lim\limits_{x\to-\infty}\dfrac{1}{x}=\dfrac{1}{-\infty}=0$

Valoarea limitei funcției raționale în $x_0\in\mathbb{R}$, punct în care nu se anulează numitorul, se obține calculând valoarea funcției polinomiale în acel punct, $\lim\limits_{x\to x_0}\dfrac{f(x)}{g(x)}=\dfrac{f(x_0)}{g(x_0)}$.

Exemple

1. $\lim\limits_{x\to1}\dfrac{x^3+2x-3}{x^2+8x+1}=\dfrac{1+2-3}{1+8+1}=0$

2. $\lim\limits_{x\to-1}\dfrac{x^2+2x-3}{x^2+8x+1}=\dfrac{1-2-3}{1-8+1}=\dfrac{2}{3}$

Dacă $x_0\in\mathbb{R}$ este rădăcină a funcției g, $g(x_0)=0$, distingem cazurile:

Exemple

1. $\lim\limits_{x\to1}\dfrac{x^3-x^2-x+1}{x^2+2x-3}=\dfrac{0}{0}=\lim\limits_{x\to1}\dfrac{(x-1)^{\cancel{2}}(x+1)}{\cancel{(x-1)}(x+3)}=\dfrac{0}{4}=0$

2. $\lim\limits_{x\to1}\dfrac{x^2+2x-3}{x^2-6x+5}=\dfrac{1+2-3}{1-6+5}=\dfrac{0}{0}=\lim\limits_{x\to1}\dfrac{\cancel{(x-1)}(x+3)}{\cancel{(x-1)}(x-5)}=\dfrac{4}{-4}=-1$

3. $\lim\limits_{x\to1}\dfrac{2x-3}{x^3-x^2-x+1}=\dfrac{-1}{0}=\lim\limits_{x\to1}\dfrac{2x-3}{(x-1)^2(x+1)}=\dfrac{-1}{0_+}=-\infty$

4. $\lim\limits_{x\to1}\dfrac{x^2+2x-3}{x^3-x^2-x+1}=\dfrac{1+2-3}{1-1-1+1}=\dfrac{0}{0}=\lim\limits_{x\to1}\dfrac{(x-1)(x+3)}{(x-1)^2(x+1)}=\lim\limits_{x\to1}\dfrac{x+3}{(x-1)(x+1)}=\dfrac{4}{0}\ \nexists$

Nu există limita, deoarece $\lim\limits_{\substack{x\to1\\x<1}}\dfrac{x+3}{(x-1)(x+1)}=\dfrac{4}{0_-}=-\infty\neq\lim\limits_{\substack{x\to1\\x>1}}\dfrac{x+3}{(x-1)(x+1)}=\dfrac{4}{0_+}=+\infty$.

Dacă $x_0\in\mathbb{R}$ este rădăcină de ordin p a funcției f, atunci descompunem $f(x)=(x-x_0)^pf_1(x)$, iar dacă x_0 este rădăcină de ordin q a funcției g, atunci avem $g(x)=(x-x_0)^qg_1(x)$.

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{(x-x_0)^pf_1(x)}{(x-x_0)^qg_1(x)}=\begin{cases}0 & \text{pentru } p>q\\ \dfrac{f_1(x_0)}{g_1(x_0)} & \text{pentru } p=q\\ \infty\cdot\dfrac{f_1(x_0)}{g_1(x_0)} & \text{pentru } p<q,\ q-p \text{ număr par}\\ \nexists & \text{pentru } p<q,\ q-p \text{ număr impar}\end{cases}$$

Profesor Blaga Mirela-Gabriela

Calculul limitelor de funcții
$0 \cdot \infty$

Acest caz se transformă în cazul $\frac{\infty}{\infty}$ sau în cazul $\frac{0}{0}$, astfel $f \cdot g = \frac{f}{\frac{1}{g}} = \frac{g}{\frac{1}{f}}$.

Exerciții rezolvate

1. $\lim\limits_{x\to\infty} x \cdot e^{-x} = \infty \cdot 0 = \lim\limits_{x\to\infty} \frac{x}{e^x} \underbrace{=}_{\frac{\infty}{\infty}} 0$, deoarece funcția exponențială e^x crește mai repede decât funcția polinomială x

2. $\lim\limits_{x\to 0} x \cdot ctg3x = 0 \cdot \infty = \lim\limits_{x\to 0} \frac{x}{tg3x} = \frac{0}{0} = \lim\limits_{x\to 0} \frac{x}{\frac{tg3x}{3x} \cdot 3x} = \lim\limits_{x\to 0} \frac{x}{3x} = \frac{1}{3}$

3. $\lim\limits_{x\to\infty} sin\frac{1}{x} \cdot x = 0 \cdot \infty = \lim\limits_{x\to\infty} \frac{sin\frac{1}{x}}{\frac{1}{x}} \underbrace{=}_{\frac{0}{0}} 1$

O altă variantă de calcul a limitei presupune utilizarea notației $\frac{1}{x} = y$, unde $x \to \infty$ implică $y \to 0$ și avem $\lim\limits_{x\to\infty} sin\frac{1}{x} \cdot x = 0 \cdot \infty = \lim\limits_{y\to 0} \frac{siny}{y} \underbrace{=}_{\frac{0}{0}} 1$.

Exerciții propuse

1) $\lim\limits_{x\to\infty} x \cdot tg\frac{3}{x}$

2) $\lim\limits_{x\to\infty} x \cdot \left(e^{\frac{1}{x}} - 1\right)$

3) $\lim\limits_{x\to 3} (x-3) \cdot tg\frac{\pi x}{6}$

4) $\lim\limits_{x\to\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \cdot tg3x$

1) 3

2) 1

3) $-\frac{6}{\pi}$

4) $\frac{1}{3}$

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Calculul limitelor de funcții

0^0

$f^g = e^{g\cdot lnf}$

Exerciții rezolvate

1. $\lim\limits_{\substack{x\to 0\\ x>0}} x^x = 0^0 = \lim\limits_{\substack{x\to 0\\ x>0}} e^{x\cdot lnx} = e^{\lim\limits_{\substack{x\to 0\\ x>0}} x\cdot lnx} = e^0 = 1$

$\lim\limits_{\substack{x\to 0\\ x>0}} x\cdot lnx = 0\cdot\infty = \lim\limits_{\substack{x\to 0\\ x>0}} \frac{lnx}{\frac{1}{x}} = \frac{\infty}{\infty} = \lim\limits_{x\to\infty} \frac{ln\frac{1}{x}}{x} = \lim\limits_{x\to\infty} \frac{-lnx}{x} = 0$, deoarece funcția polinomială x crește mai repede decât funcția logaritmică lnx

În cazul în care aplicăm l'Hospital pentru calcularea limitei $\lim\limits_{\substack{x\to 0\\ x>0}} x\cdot lnx$, avem

$$\lim\limits_{\substack{x\to 0\\ x>0}} x\cdot lnx = 0\cdot\infty = \lim\limits_{\substack{x\to 0\\ x>0}} \frac{lnx}{\frac{1}{x}} \overset{l'H}{=} \lim\limits_{\substack{x\to 0\\ x>0}} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0}(-x) = 0$$

2. $\lim\limits_{x\to\infty} \left(\frac{1}{x}\right)^{\frac{1}{lnx}} = 0^0 = \lim\limits_{x\to\infty} e^{\frac{1}{lnx}\cdot ln\frac{1}{x}} = e^{\lim\limits_{x\to\infty}\frac{-lnx}{lnx}} = e^{-1} = \frac{1}{e}$

Exerciții propuse

1) $\lim\limits_{\substack{x\to 0\\ x>0}} (sinx)^{tgx}$

2) $\lim\limits_{\substack{x\to 0\\ x>0}} x^{\sqrt{x}}$

3) $\lim\limits_{\substack{x\to 4\\ x<4}} \left(2-\sqrt{x}\right)^{x-4}$

4) $\lim\limits_{\substack{x\to 0\\ x>0}} \left(\frac{x}{1+x}\right)^x$

1) 1

2) 1

3) 1

4) 1

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Calculul limitelor de funcții

$$\infty^0$$

$$f^g = e^{g\cdot lnf}$$

Exerciții rezolvate

1. $\lim\limits_{\substack{x\to 0\\ x>0}} \left(\frac{1}{x}\right)^x = \infty^0 = \lim\limits_{\substack{x\to 0\\ x>0}} e^{x\cdot ln\frac{1}{x}} = e^{\lim\limits_{\substack{x\to 0\\ x>0}}(-x\cdot lnx)} = e^0 = 1$

$\lim\limits_{\substack{x\to 0\\ x>0}}(-x\cdot lnx) = 0\cdot\infty = \lim\limits_{\substack{x\to 0\\ x>0}} \frac{-lnx}{\frac{1}{x}} = \frac{\infty}{\infty} \underbrace{=}_{x\to\frac{1}{x}} \lim\limits_{x\to\infty} \frac{-ln\frac{1}{x}}{x} = \lim\limits_{x\to\infty}\frac{lnx}{x} = 0$, deoarece funcția polinomială x crește mai repede decât funcția logaritmică lnx

2. $\lim\limits_{x\to\infty} x^{\frac{1}{lnx}} = \infty^0 = \lim\limits_{x\to\infty} e^{\frac{1}{lnx}\cdot lnx} = e^1 = e$

Exerciții propuse

1) $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$

2) $\lim\limits_{x\to\infty} (x+sinx)^{\frac{1}{x}}$

3) $\lim\limits_{\substack{x\to 0\\ x>0}} \left(\frac{1}{x}\right)^{tgx}$

1) 1

2) 1

3) 1